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Broadcasts on Paths and Cycles

Sabrina BOUCHOUIKA * Isma BOUCHEMAKH * Éric SOPENA †

January 23, 2020

Abstract

A broadcast on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, \dots, \text{diam}(G)\}$ such that $f(v) \leq e_G(v)$ for every vertex $v \in V$, where $\text{diam}(G)$ denotes the diameter of G and $e_G(v)$ the eccentricity of v in G . The cost of such a broadcast is then the value $\sum_{v \in V} f(v)$. Various types of broadcast functions on graphs have been considered in the literature, in relation with domination, irredundance, independence or packing, leading to the introduction of several broadcast numbers on graphs.

In this paper, we determine these broadcast numbers for all paths and cycles, thus answering a question raised in [D. Ahmadi, G.H. Fricke, C. Schroeder, S.T. Hedetniemi and R.C. Laskar, Broadcast irredundance in graphs. *Congr. Numer.* 224 (2015), 17–31].

Keywords: Broadcast; Dominating broadcast; Irredundant broadcast; Independent broadcast; Packing broadcast; Path; Cycle.

MSC 2010: 05C12, 05C69.

1 Introduction

Let $G = (V, E)$ be a graph of order $n = |V|$ and size $m = |E|$. The *open neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{u : uv \in E\}$ of vertices adjacent to v . Each vertex $u \in N_G(v)$ is a *neighbor* of v in G . The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ of vertices is $N_G(S) = \cup_{v \in S} N_G(v)$, while the *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$. The *degree* of a vertex v in G , denoted $\deg_G(v)$, is the size of the open neighborhood of v .

A (u, v) -*geodesic* in a graph G is a shortest path joining u and v . We denote by $d_G(u, v)$ the *distance* between the vertices u and v in G , that is, the length of a (u, v) -geodesic in G . The *eccentricity* $e_G(v)$ of a vertex v in G is the maximum distance from v to any other vertex of G . The *radius* $\text{rad}(G)$ and the *diameter* $\text{diam}(G)$ of a graph G are the minimum and the maximum eccentricity among the vertices of G , respectively.

A function $f : V \rightarrow \{0, \dots, \text{diam}(G)\}$ is a *broadcast* on a graph $G = (V, E)$ if $f(v) \leq e_G(v)$ for every vertex $v \in V$. The value $f(v)$ is called the *f-value* of v . An *f-broadcast vertex* (or an *f-dominating vertex*) is a vertex v for which $f(v) > 0$. The set of all *f-broadcast vertices* is denoted $V_f^+(G)$. If $v \in V_f^+$ is an *f-broadcast vertex*, $u \in V$ and $d_G(u, v) \leq f(v)$, then the vertex u *hears*

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a broadcast from v and v *broadcasts to* (or *f-dominates*) u . Note that, in particular, each vertex $v \in V_f^+$ hears a broadcast from itself and *f-dominates* itself.

The *f-broadcast neighborhood* of a vertex $v \in V_f^+$ is the set of vertices that hear v , that is

$$N_f(v) = \{u : d_G(u, v) \leq f(v)\},$$

and the *broadcast neighborhood* of f is the set

$$N_f(V_f^+) = \cup_{v \in V_f^+} N_f(v).$$

The set of *f-broadcast vertices* that a vertex $u \in V$ can hear is the set

$$H_f(u) = \{v \in V_f^+ : d_G(u, v) \leq f(v)\}.$$

For a vertex $v \in V_f^+$, the *private f-neighborhood* of v is the set of vertices that hear only v , that is

$$PN_f(v) = \{u \in V : H_f(u) = \{v\}\},$$

and every vertex $u \in PN_f(v)$ is a *private f-neighbor* of v . Moreover, the *private f-border* of v is either the set of private *f-neighbors* of v that are at distance $f(v)$ from v , or the singleton $\{v\}$ if $f(v) = 1$ and $PN_f(v) = \{v\}$, that is

$$PB_f(v) = \begin{cases} \{v\} & \text{if } f(v) = 1 \text{ and } PN_f(v) = \{v\}, \\ \{u \in PN_f(v) : d_G(u, v) = f(v)\} & \text{otherwise.} \end{cases}$$

Every vertex in $PB_f(v)$ is a *bordering private f-neighbor* of v . In particular, if $f(v) = 1$ and $PN_f(v) = \{v\}$, then v is its own bordering private *f-neighbor*.

The *cost* of a broadcast f on a graph G is

$$\sigma(f) = \sum_{v \in V_f^+} f(v).$$

A broadcast f on G of some type is *minimal* (resp. *maximal*) if there does not exist any broadcast $g \neq f$ on G of the same type such that $g(u) \leq f(u)$ (resp. $g(u) \geq f(u)$) for all $u \in V$. Several types of broadcasts have been defined in the literature, in relation with domination, irredundence, independence or packing, leading to the introduction of several broadcast numbers on graphs, corresponding to the minimum or maximum possible cost of a maximal or minimal broadcast of the corresponding type, respectively. For any such parameter, say $q(G)$, a broadcast f on G of the corresponding type with $\sigma(f) = q(G)$ will be simply called a *q-broadcast*. We will also say that such a broadcast is *optimal*.

We now introduce the various types of broadcasts we will consider in this paper.

Dominating broadcasts. A broadcast f on G is a *dominating broadcast* if every vertex in $V - V_f^+$ is *f-dominated* by some vertex in V_f^+ or, equivalently, if for every vertex $v \in V$, $|H_f(v)| \geq 1$. The *broadcast domination number* $\gamma_b(G)$ of G is the minimum cost of a dominating broadcast on G . The *upper broadcast domination number* $\Gamma_b(G)$ of G is the maximum cost of a minimal dominating broadcast on G . If f is a minimal dominating broadcast on G such that $f(v) = 1$ for each $v \in V_f^+$, then V_f^+ is a *minimal dominating set* in G , and the minimum (resp. maximum) cost of such a broadcast is the *domination number* $\gamma(G)$ (resp. the *upper domination number* $\Gamma(G)$) of G .

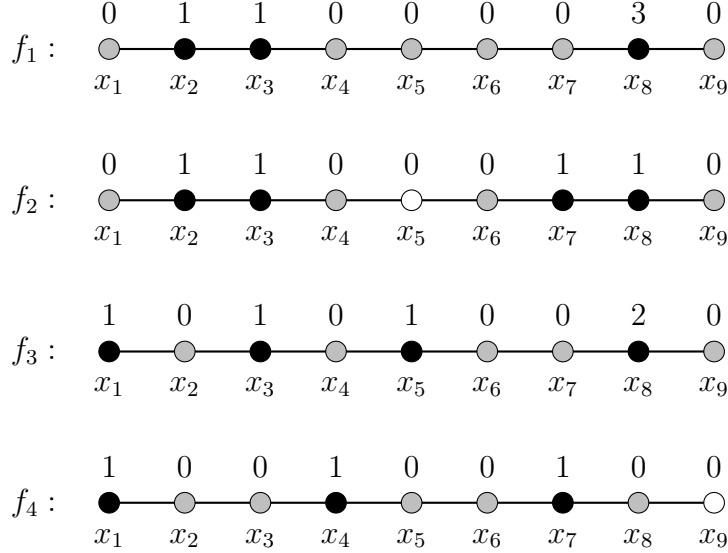


Figure 1: Sample broadcasts on the path P_9 .

Irredundant broadcasts. A broadcast f on G is an *irredundant broadcast* if $PB_f(v) \neq \emptyset$ for every vertex $v \in V_f^+$. Stated equivalently, a broadcast f is irredundant if the following two conditions are satisfied : (i) for every f -broadcast vertex v with $f(v) \geq 2$, there exists a vertex u such that $H_f(u) = \{v\}$ and $d_G(u, v) = f(v)$, and (ii) for every f -broadcast vertex v with $f(v) = 1$, there exists a vertex $u \in N_G[v]$ such that $H_f(u) = \{v\}$ (note that, in this case, we can have $u = v$). The *upper broadcast irredundance number* $IR_b(G)$ of G is the maximum cost of an irredundant broadcast on G . The *broadcast irredundance number* $ir_b(G)$ of G is the minimum cost of a maximal irredundant broadcast on G . If f is a maximal irredundant broadcast on G such that $f(v) = 1$ for each $v \in V_f^+$, then V_f^+ is a *maximal irredundant set* in G , and the minimum (resp. the maximum) cost of such a broadcast is the *irredundance number* $ir(G)$ (resp. the *upper irredundance number* $IR(G)$) of G .

Independent broadcasts. A broadcast f is an *independent broadcast* if no broadcast vertex f -dominates any other broadcast vertex or, equivalently, if for every $v \in V_f^+$, $|H_f(v)| = 1$. The *broadcast independence number* $\beta_b(G)$ of G is the maximum cost of an independent broadcast on G . The *lower broadcast independence number* $i_b(G)$ of G is the minimum cost of a maximal independent broadcast on G . If f is a maximal independent broadcast such that $f(v) = 1$ for each $v \in V_f^+$, then V_f^+ is a *maximal independent set* in G , and the maximum (resp. minimum) cost of such a broadcast is the *vertex independence number* $\beta_0(G)$ (resp. the *independent domination number* $i(G)$) of G .

Packing broadcasts. A broadcast f is a *packing broadcast* if every vertex hears at most one broadcast, that is, for every vertex $v \in V$, $|H_f(v)| \leq 1$. The *broadcast packing number* $P_b(G)$ of G is the maximum cost of a packing broadcast on G . The *lower broadcast packing number* $p_b(G)$ of G is the minimum cost of a maximal packing broadcast on G . If f is a maximal packing broadcast such that $f(v) = 1$ for each $v \in V_f^+$, then V_f^+ is a *maximal packing set* in G , and the maximum (resp. the minimum) cost of such a broadcast is the *packing number* $P(G)$ (resp. the *lower packing number* $p(G)$) of G .

These four different types of broadcasts are illustrated in Figure 1 (broadcast vertices are drawn as black vertices, non-broadcast dominated vertices as gray vertices, and non dominated vertices as white vertices): f_1 is a dominating broadcast, f_2 is an irredundant broadcast (the f_2 -broadcast vertices x_2 ,

x_3 , x_7 and x_8 all have a bordering private f_2 -neighbor, namely x_1 , x_4 , x_6 and x_9 , respectively), f_3 is an independent broadcast, and f_4 is a packing broadcast. Moreover, observe the following:

- f_1 is a *minimal* dominating broadcast and also a maximal irredundant broadcast. However, f_1 is neither an independent broadcast (the f_1 -broadcast vertices x_2 and x_3 are both f_1 -dominated twice), nor a packing broadcast (x_2 and x_3 both hear two f_1 -broadcast vertices).
- f_2 is a *maximal* irredundant broadcast, but is neither a dominating broadcast (x_5 is not f_2 -dominated), nor an independent broadcast (the f_2 -broadcast vertices x_2 , x_3 , x_7 and x_8 are f_2 -dominated twice), nor a packing broadcast (x_2 , x_3 , x_7 and x_8 all hear two f_2 -broadcast vertices).
- f_3 is a *maximal* independent broadcast and a dominating broadcast, but is neither an irredundant broadcast (the f_3 -broadcast vertex x_8 has no bordering private f_3 -neighbor), nor a packing broadcast (vertices x_2 , x_4 and x_6 are f_3 -dominated twice).
- f_4 is a *maximal* packing broadcast, and also an irredundant broadcast and an independent broadcast. However, f_4 is neither a dominating broadcast (x_9 is not f_4 -dominated), nor a maximal independent broadcast (we can increase the cost of f_4 by setting $f_4(x_1) = f_4(x_4) = f_4(x_7) = 2$), nor a maximal irredundant broadcast (we can increase the cost of f_4 by setting $f_4(x_7) = 2$, so that x_4 has still a bordering private f_4 -neighbor, namely x_3 , and x_9 is now the bordering private f_4 -neighbor of x_7).

Directly from the definitions of these four types of broadcasts, we get the following observations.

Observation 1.1.

- (1) *Every maximal independent broadcast is a dominating broadcast, and thus $\gamma_b(G) \leq i_b(G)$ for every graph G .*
- (2) *Every packing broadcast is an independent broadcast, and thus $P_b(G) \leq \beta_b(G)$ for every graph G .*
- (3) *Every packing broadcast is an irredundant broadcast, and thus $P_b(G) \leq IR_b(G)$ for every graph G .*
- (4) *Every dominating maximal irredundant broadcast is a minimal dominating broadcast.*

Broadcast domination was introduced by Erwin [20] in his Ph.D. thesis, in which he discussed several types of broadcast parameters and the relationships between them. Many of these results appeared later in [19]. Since then, several papers have been published on various aspects of broadcasts in graphs, including the algorithmic complexity [7, 25, 26], the determination of the broadcast domination number for several classes of graphs [10, 12, 18, 24, 32, 33, 34], and a characterization of the classes of trees for which the broadcast domination number equals the radius [27] or equals the domination number [17, 28, 31]. The upper broadcast domination number is studied in [1, 11, 19, 21, 22, 29], the broadcast irredundance number is studied in [1, 29], and the broadcast independence number is studied in [2, 3, 4, 8, 9, 13]. Broadcast domination and multipacking are considered in [5, 6, 15, 16, 23, 30].

In this paper, we determine all the above defined numbers for paths and cycles. Ahmadi *et al.* observed in [1] that very little is known concerning these parameters. We first recall some preliminary results in Section 2, and prove our main results in Section 3. These results are summarized in Table 1. They confirm the conjectures given in [1] for $\gamma_b(P_n)$, $\gamma_b(C_n)$ and $\Gamma_b(P_n)$, but disprove all other conjectures.

	$\gamma_b = ir_b$	i_b	p_b	$\Gamma_b = IR_b$	β_b	P_b
P_n	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{2n}{5} \rceil$, $n \neq 3$	$\frac{n}{4}$ if $n \equiv 0 \pmod{8}$ $2 \lfloor \frac{n}{8} \rfloor + 1$ if $n \equiv 1, 2, 3 \pmod{8}$ $2 \lfloor \frac{n}{8} \rfloor + 2$ otherwise	$n - 1$, $n \geq 2$	$2n - 4$, $n \geq 3$	$n - 1$
	Th. 3.12	Th. 3.7	Th. 3.16	Th. 3.1	[20]	Th. 3.14
C_n	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{2n}{5} \rceil$, $n \neq 3$	$\frac{n}{4}$ if $n \equiv 0 \pmod{8}$ $2 \lfloor \frac{n}{8} \rfloor + 1$ if $n \equiv 1, 2, 3 \pmod{8}$ $2 \lfloor \frac{n}{8} \rfloor + 2$ otherwise	$2(\lfloor \frac{n}{2} \rfloor - 1)$, $n \geq 4$	$n - 2$, $n \geq 3$	$\lfloor \frac{n}{2} \rfloor$
	Th. 3.13	Th. 3.8	Th. 3.17	Th. 3.3, 3.4	Th. 3.5	Th. 3.14

Table 1: Broadcast parameters of paths and cycles

2 Preliminary results

The characterization of minimal dominating broadcasts was first given by Erwin in [21], and then restated in terms of private borders¹ by Mynhardt and Roux in [29].

Proposition 2.1 (Erwin [21], restated in [29]). *A dominating broadcast f is a minimal dominating broadcast if and only if $PB_f(v) \neq \emptyset$ for each $v \in V_f^+$.*

Dunbar *et al.* proved in [19] the following bound on the upper broadcast domination number of graphs.

Theorem 2.2 (Dunbar *et al.* [19]). *For every graph G with size m , $\Gamma_b(G) \leq m$. Moreover, $\Gamma_b(G) = m$ if and only if G is a nontrivial star or path.*

This upper bound was later improved in [11].

Theorem 2.3 (Bouchemakh and Fergani [11]). *If G is a graph of order n with minimum degree $\delta(G)$, then $\Gamma_b(G) \leq n - \delta(G)$, and this bound is sharp.*

From Proposition 2.1 and the definition of a maximal irredundant broadcast, one gets the following result.

Corollary 2.4 (Ahmadi *et al.* [1]). *Every minimal dominating broadcast is a maximal irredundant broadcast.*

Since the characteristic function of a minimal dominating set in a graph is a minimal dominating broadcast, Corollary 2.4 implies the following chain of inequalities.

Corollary 2.5 (Ahmadi *et al.* [1]). *For every graph G ,*

$$ir_b(G) \leq \gamma_b(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_b(G) \leq IR_b(G).$$

Moreover, Dunbar *et al.* [19] proved the following.

¹In their paper, Mynhardt and Roux used a slightly different definition of the set $PB_f(v)$ when $f(v) = 1$ and $N_f(v) \neq \{v\}$, by including the vertex v in $PB_f(v)$. Moreover, they called the set $PB_f(v)$ the *private f -boundary* of v . We here use the term *private f -border* to avoid confusion between these two definitions. However, it is easy to check that the private f -boundary of v is empty if and only if the private f -border of v is empty, so that Proposition 2.1 is still valid in our setting.

Proposition 2.6 (Dunbar *et al.* [19]). *For every graph G ,*

$$\gamma_b(G) \leq i_b(G) \leq \beta_b(G) \geq i(G) \geq \gamma(G) \geq \gamma_b(G).$$

However, $\beta_b(G)$ and $\Gamma_b(G)$ are in general incomparable.

It is worth pointing out that the difference $IR_b(G) - \Gamma_b(G)$ can be arbitrarily large. Indeed, Mynhardt and Roux [29] constructed a family of graphs $\{G_r\}_{r \geq 3}$, where each G_r is obtained by joining two copies of K_{r+1} by r independent edges, and proved the relation $\Gamma_b(G_r) = 3 \leq IR_b(G_r) = r$ for every $r \geq 3$. Nevertheless, we may have $IR_b(G) = \Gamma_b(G)$, as we will prove in Subsection 3.1 when G is a path or a cycle.

Dunbar *et al.* observed in [19] that, for any graph G , neither $P(G)$ nor $p(G)$ is comparable with $p_b(G)$, while we have $p(G) \leq P(G) \leq P_b(G)$ and $p_b(G) \leq \text{rad}(G) \leq \text{diam}(G) \leq P_b(G) \leq \beta_b(G)$. For paths and cycles, we will prove in Section 3 that the lower bound $\text{diam}(G)$ for $P_b(G)$ is achieved, while the difference between $\text{rad}(G)$ and $p_b(G)$ can be arbitrarily large.

3 Broadcast numbers of paths and cycles

As mentioned by Ahmadi *et al.* in [1], it is quite surprising that the values of several broadcast parameters have not been determined yet for paths or cycles. Moreover, in the same paper, they conjecture the values of these parameters. In this section, we will determine the exact values of these parameters, which in some cases, but not all, correspond to their conjecture.

Throughout this section, we will denote by $P_n = x_1x_2 \dots x_n$, $n \geq 2$, the path of order n , and by $C_n = x_0x_1 \dots x_{n-1}$, $n \geq 3$, the cycle of order n . Moreover, we assume throughout this section that subscripts of vertices of C_n are taken modulo n , and that the vertices x_1, \dots, x_n of P_n are “ordered” from left to right, so that by the *leftmost* (resp. the *rightmost*) vertex in P_n satisfying any property, we mean the vertex with minimum (resp. maximum) subscript satisfying this property.

3.1 Upper broadcast domination number and upper broadcast irredundance number

We first consider the case of paths.

Theorem 3.1. *For every integer $n \geq 2$, $\Gamma_b(P_n) = IR_b(P_n) = \text{diam}(P_n) = n - 1$.*

Proof. Theorem 2.2 directly gives $\Gamma_b(P_n) = n - 1$. By Corollary 2.5, we have $n - 1 = \Gamma_b(P_n) \leq IR_b(P_n)$. We now prove the opposite inequality. Let f be an irredundant broadcast on P_n , and let $V_f^+ = \{x_{i_1}, \dots, x_{i_t}\}$, $i_1 < \dots < i_t$, $t \geq 1$. From the definition of an irredundant broadcast, we get that for every vertex $x_{i_j} \in V_f^+$ with $f(x_{i_j}) \geq 2$, there exists a vertex $x_{i_j}^p$ such that $H_f(x_{i_j}^p) = \{x_{i_j}\}$ and $d_{P_n}(x_{i_j}, x_{i_j}^p) = f(x_{i_j})$ and, for every vertex $x_{i_j} \in V_f^+$ with $f(x_{i_j}) = 1$, there exists a vertex $x_{i_j}^p \in N_{P_n}[x_{i_j}]$ such that $H_f(x_{i_j}^p) = \{x_{i_j}\}$.

Let t' , $0 \leq t' \leq t$, denote the number of f -broadcast vertices x_{i_j} that are their own bordering private f -neighbor, that is, such that $x_{i_j}^p = x_{i_j}$ (which implies $f(x_{i_j}) = 1$), and suppose that these vertices are $\{x_{i_1}, \dots, x_{i_{t'}}\}$. We thus have

$$|V(P_n)| \geq t' + \sum_{j=t'+1}^t (d_{P_n}(x_{i_j}, x_{i_j}^p) + 1) \geq \sum_{j=1}^{t'} f(x_{i_j}) + \sum_{j=t'+1}^t (f(x_{i_j}) + 1) = IR_b(P_n) + t - t',$$

which gives $IR_b(P_n) \leq n - t + t'$. If $t' < t$, then $IR_b(P_n) \leq n - 1$ and we are done. Otherwise, every f -broadcast vertex is its own bordering private f -neighbor, which implies that V_f^+ is either the set

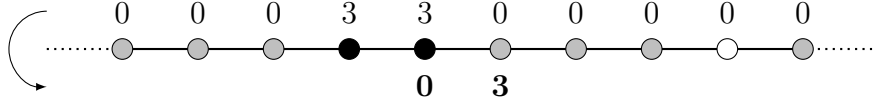


Figure 2: Irredundant broadcast for the proof of Theorem 3.3.

of all vertices with even subscript, or the set of all vertices with odd subscript. In both cases, we get $\sigma(f) = IR_b(P_n) \leq \lceil \frac{n}{2} \rceil \leq n - 1$, as required. \square

We now consider the case of cycles. For that, we will use the following lemma.

Lemma 3.2. *Let f be an IR_b -broadcast on C_n . If $H_f(x_i) = \emptyset$ for some vertex x_i , then $H_f(x_{i-1}) \neq \emptyset$ and $H_f(x_{i+1}) \neq \emptyset$.*

Proof. Assume to the contrary that we have $H_f(x_{i-1}) = \emptyset$, so that x_{i-1} is not f -dominated, which implies in particular $f(x_{i-2}) = 0$. Therefore, there exists an f -broadcast vertex x_j , $j < i - 2$, which f -dominates x_{i-2} , for otherwise we could set $f(x_{i-1}) = 1$, contradicting the optimality of f .

We then necessarily have $d_{C_n}(x_j, x_{i-2}) = f(x_j)$. But, in that case, the function g obtained from f by setting $g(x_j) = 0$ and $g(x_{j+1}) = f(x_j) + 1$ would be an irredundant broadcast on C_n with cost $\sigma(g) = \sigma(f) + 1 > \sigma(f)$, again a contradiction.

It follows that we have $H_f(x_{i-1}) \neq \emptyset$ and, by symmetry, that we also have $H_f(x_{i+1}) \neq \emptyset$. \square

We can now prove the following result.

Theorem 3.3. *For every integer $n \geq 3$, $IR_b(C_n) = \Gamma_b(C_n)$.*

Proof. By Corollary 2.5, we only need to prove the inequality $IR_b(C_n) \leq \Gamma_b(C_n)$. For this, it is enough to construct, from any non-dominating IR_b -broadcast on C_n , a dominating irredundant broadcast (which is then a minimal dominating broadcast, by Observation 1.1(4)) with the same cost $IR_b(C_n)$.

Let f be a non-dominating IR_b -broadcast on C_n . Then, there exists $i \in \{0, \dots, n - 1\}$ such that $H_f(x_i) = \emptyset$. By Lemma 3.2, we have $H_f(x_{i-1}) \neq \emptyset$ and $H_f(x_{i+1}) \neq \emptyset$. Since x_i is not f -dominated, we know that x_{i-1} is f -dominated by a unique f -broadcast vertex, say x_j , $j < i - 1$, such that $f(x_j) = d_{C_n}(x_j, x_{i-1})$, which implies $|PB_f(x_j)| \geq 1$. We claim that we have $|PB_f(x_j)| = 1$. Indeed, if $|PB_f(x_j)| = 2$, then we could set $f(x_{j+1}) = f(x_j)$, contradicting the optimality of f .

Now, observe that the function g obtained from f by setting $g(x_j) = 0$ and $g(x_{j+1}) = f(x_j)$ is an irredundant broadcast with $\sigma(g) = \sigma(f)$, such that x_i is g -dominated, and all vertices that were f -dominated remain g -dominated (see Figure 2). Repeating the same transformation for each vertex which is not dominated, we eventually produce a minimal dominating broadcast on C_n with cost $IR_b(C_n)$. \square

We obviously have $\Gamma_b(C_3) = 1$. For $n \geq 4$, the value of $\Gamma_b(C_n)$ is given by the following result.

Theorem 3.4. *For every integer $n \geq 4$, $\Gamma_b(C_n) = 2(\lfloor \frac{n}{2} \rfloor - 1)$.*

Proof. From Theorem 2.3, we directly get $\Gamma_b(C_n) \leq n - \delta(C_n) = n - 2$. Let now f be the function defined by

$$f(x_i) = \begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{if } i \in \{ \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1 \}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is a minimal dominating broadcast on C_n with cost

$$\sigma(f) = \begin{cases} n - 2 & \text{if } n \text{ is even,} \\ n - 3 & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, $\sigma(f) \leq \Gamma_b(C_n)$. Combining this inequality with the previous one, we already infer that we have $\Gamma_b(C_n) = n - 2$ if n is even, and $n - 3 \leq \Gamma_b(C_n) \leq n - 2$ if n is odd.

It remains to discuss the case n odd. For $n = 5$, it is not difficult to check that we have $\Gamma_b(C_5) = 2$. Suppose $n \geq 7$ and let g be any Γ_b -broadcast on C_n such that $V_g^+ = \{x_{i_1}, \dots, x_{i_t}\}$, $i_1 < \dots < i_t$, $t \geq 1$. We first prove the following claim.

Claim A. $t = 2$.

Proof. If $t = 1$, then there is a unique g -broadcast vertex $x_i \in V_g^+$, and thus $\Gamma_b(C_n) = g(x_i) = e_{C_n}(x_i) = \frac{n-1}{2} < n - 3$, a contradiction. Hence, $t \geq 2$. We know by Proposition 2.1 that each g -broadcast vertex $x_i \in V_g^+$ has a bordering private g -neighbor, say x_i^p , with possibly $x_i^p = x_i$ (and, in that case, $g(x_i) = 1$). Let Q_{x_i} be the set of edges defined as follows:

- if $g(x_i) \geq 2$, then Q_{x_i} is the set of edges of the unique (x_i, x_i^p) -geodesic,
- if $g(x_i) = 1$ and $x_i^p \in \{x_{i-1}, x_{i+1}\}$ is a bordering private g -neighbor of x_i , then Q_{x_i} is the singleton $\{x_i x_i^p\}$,
- if $g(x_i) = 1$ and x_i is its own bordering private g -neighbor, then Q_{x_i} is the singleton $\{x_i x_{i+1}\}$.

Clearly, $g(x_i) = |Q_{x_i}|$ for every $x_i \in V_g^+$, and $Q_{x_i} \cap Q_{x_j} = \emptyset$ for every $x_i, x_j \in V_g^+$.

We now claim that for every $Q_{x_i} = \{x_i x_{i+1}, \dots, x_{i+t-1} x_{i+t}\}$ (resp. $Q_{x_i} = \{x_{i-t} x_{i-t+1}, \dots, x_{i-1} x_i\}$), $t \geq 1$, the edge $x_{i+t} x_{i+t+1}$ (resp. $x_i x_{i+1}$) that “follows” Q_{x_i} does not belong to any $Q_{x_{i'}}$, with $x_{i'} \in V_g^+$. Indeed, this directly follows from the following observations.

- (a) Every g -broadcast vertex x_i belongs to exactly one such path, namely Q_{x_i} .
- (b) Every bordering private g -neighbor belongs to at most one such path.
- (c) An end-vertex x_j of such a path Q is neither a g -broadcast vertex nor a bordering private g -neighbor if and only if $Q = Q_{x_{j-1}}$, x_{j-1} is a g -broadcast vertex, $g(x_{j-1}) = 1$ and x_{j-1} is its own bordering private g -neighbor.

Hence, we have

$$\Gamma_b(C_n) = \sum_{x_i \in V_g^+} g(x_i) = \sum_{x_i \in V_g^+} |Q_{x_i}| = |\cup_{x_i \in V_g^+} Q_{x_i}| \leq n - |V_g^+| = n - t.$$

If $t > 3$, then $\Gamma_b(C_n) < n - 3$, a contradiction. Assume now that $t = 3$ and let $V_g^+ = \{x_a, x_b, x_c\}$, with $a < b < c$. Since $\Gamma_b(C_n) = n - 3$, any two of the paths Q_{x_a} , Q_{x_b} and Q_{x_c} are separated by exactly one edge.

Suppose first that one of these g -broadcast vertices, say x_c , is such that $g(x_c) = 1$ and x_c is its own private g -neighbor, so that $Q_{x_c} = x_c x_{c+1}$. In that case, x_{c-1} is neither a g -broadcast vertex nor a bordering private g -neighbor, which implies, by observation (c) above, that $x_b = x_{c-2}$, $g(x_b) = 1$ and x_b is its own bordering private g -neighbor. Using the same argument, we get that $x_a = x_{b-2}$, $g(x_a) = 1$, x_a is its own bordering private g -neighbor, and $x_c = x_{a-2}$, leading to $n = 6$, a contradiction since we assumed $n \geq 7$.

Hence, each end-vertex of any of the paths Q_{x_a} , Q_{x_b} and Q_{x_c} is either a g -broadcast vertex or a bordering private g -neighbor of the g -broadcast vertex belonging to the same path. But since we have three paths, one of x_a , x_b or x_c must be adjacent to a bordering private g -neighbor of another g -broadcast vertex, a contradiction. \square

By Claim A, we can assume $V_g^+ = \{x_a, x_b\}$, which implies $|PB_f(x_a)| = |PB_f(x_b)|$. If $|PB_f(x_a)| = |PB_f(x_b)| = 2$, then $n = 2f(x_a) + 1 + 2f(x_b) + 1$, a contradiction since n is odd. We thus have $|PB_f(x_a)| = |PB_f(x_b)| = 1$, and the bordering private g -neighbors x_a^p of x_a , and x_b^p of x_b are adjacent. Moreover, if we assume, without loss of generality, that $x_a x_{a+1} \dots x_b$ is a (x_a, x_b) -geodesic, then $b - a \leq 2$ for otherwise the function h obtained from g by setting $h(x_a) = 0$, $h(x_b) = 0$, $h(x_{a+1}) = g(x_a) + 1$ and $h(x_{b-1}) = g(x_b) + 1$, would be a minimal dominating broadcast with cost $\Gamma_b(h) = \Gamma_b(g) + 2$, contradicting the optimality of g . Hence, we have $d_{C_n}(x_a, x_b) \leq 2$. If x_a and x_b are joined by an edge, we necessarily have $g(x_a) = g(x_b)$, implying that n is even, contrary to our assumption. We thus have $d_{C_n}(x_a, x_b) = 2$, and thus $g(x_a) = g(x_b) = \frac{n-3}{2}$, which gives $\sigma(g) = n - 3$. \square

3.2 Broadcast independence number and lower broadcast independence number

Dunbar *et al.* [19] noted that the upper broadcast domination number $\Gamma_b(G)$ and the broadcast independence number $\beta_b(G)$ of a graph G are in general incomparable. Erwin gave in [20] the exact value of the broadcast independence number of paths. He proved that for every integer $n \geq 3$, $\beta_b(P_n) = 2(n - 2)$, so that, by Theorem 3.1, $\beta_b(P_n) > \Gamma_b(P_n)$ for every $n > 3$.

Our next result proves that the equality $\beta_b(C_n) = \Gamma_b(C_n)$ holds for every cycle C_n , $n \geq 3$ (recall Theorem 3.4).

Theorem 3.5. *For every integer $n \geq 3$, $\beta_b(C_n) = 2(\lfloor \frac{n}{2} \rfloor - 1)$.*

Proof. It is easy to check that we have $\beta_b(C_3) = 1$ and $\beta_b(C_4) = 2$. Assume thus $n \geq 5$. Clearly, the function f defined by

$$f(x_i) = \begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{if } i \in \{0, \lfloor \frac{n}{2} \rfloor\}, \\ 0 & \text{otherwise,} \end{cases}$$

is an independent broadcast on C_n with cost $\sigma(f) = 2(\lfloor \frac{n}{2} \rfloor - 1)$, which implies $\beta_b(C_n) \geq 2(\lfloor \frac{n}{2} \rfloor - 1)$.

We now prove the opposite inequality. For this, let g be any β_b -broadcast on C_n . If $|V_g^+| = 1$, say $V_g^+ = \{x_i\}$, then

$$\sigma(g) = g(x_i) = e_{C_n}(x_i) = \lfloor \frac{n}{2} \rfloor \leq 2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right),$$

and, if $|V_g^+| = 2$, then the two vertices of V_g^+ are antipodal, which gives $\sigma(g) = 2(\lfloor \frac{n}{2} \rfloor - 1)$.

Assume now that we have $|V_g^+| \geq 3$, and let $V_g^+ = \{x_{i_0}, \dots, x_{i_{k-1}}\}$, $k \geq 3$. Since g is an independent broadcast, we have $d_{C_n}(x_{i_j}, x_{i_{j+1}}) \geq g(x_{i_j}) + 1$ for every j , $0 \leq j \leq k - 1$ (subscripts of i are taken modulo k). We thus get

$$\beta_b(C_n) = \sigma(g) = \sum_{x_{i_j} \in V_g^+} g(x_{i_j}) \leq \sum_{x_{i_j} \in V_g^+} (d_{C_n}(x_{i_j}, x_{i_{j+1}}) - 1) = n - |V_g^+| \leq n - 3 \leq 2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right),$$

as required. \square

We now determine the value of the lower broadcast independence number of paths and cycles. For that, we will use the following lemma.

Lemma 3.6. *If f is an i_b -broadcast on P_n , $n \geq 3$, with $V_f^+ = \{x_{i_1}, \dots, x_{i_t}\}$, $i_1 < \dots < i_t$, $t \geq 2$, then we have*

1. $f(x_{i_1}) \geq f(x_{i_2})$ and $f(x_{i_t}) \geq f(x_{i_{t-1}})$,

2. $d_{P_n}(x_1, x_{i_1}) \leq f(x_{i_1})$ and $d_{P_n}(x_{i_t}, x_n) \leq f(x_{i_t})$,
3. for every j , $1 \leq j \leq t-1$, $\max\{f(x_{i_j}), f(x_{i_{j+1}})\} + 1 \leq d_{P_n}(x_{i_j}, x_{i_{j+1}}) \leq f(x_{i_j}) + f(x_{i_{j+1}}) + 1$,
4. $d_{P_n}(x_{i_1}, x_{i_2}) = f(x_{i_1}) + 1$ and $d_{P_n}(x_{i_{t-1}}, x_{i_t}) = f(x_{i_t}) + 1$.

Proof. If Item 1 is not satisfied, then we can increase by 1 the value of $f(x_1)$ or $f(x_n)$, contradicting the maximality of f .

If Item 2 is not satisfied, then we can set $f(x_1) = 1$, or $f(x_n) = 1$, contradicting the maximality of f .

For Item 3, the inequality $\max\{f(x_{i_j}), f(x_{i_{j+1}})\} + 1 \leq d_{P_n}(x_{i_j}, x_{i_{j+1}})$ directly follows from the definition of an independent broadcast. Finally, if $d_{P_n}(x_{i_j}, x_{i_{j+1}}) > f(x_{i_j}) + f(x_{i_{j+1}}) + 1$, then we can set $f(x_{i_j+f(x_{i_j})+1}) = 1$, contradicting the maximality of f .

Consider now Item 4. By items 1 and 3, we have $f(x_{i_1}) + 1 \leq d_{P_n}(x_{i_1}, x_{i_2})$ and $f(x_{i_t}) + 1 \leq d_{P_n}(x_{i_{t-1}}, x_{i_t})$. If any of these inequalities is strict, then we can increase by 1 the f -value of the involved f -broadcast vertex, again contradicting the maximality of f . \square

In the rest of the paper, for convenience, we will often define a broadcast function f on the path P_n (resp. on the cycle C_n) by the word $f(x_1) \dots f(x_n)$ (resp. $f(x_0) \dots f(x_{n-1})$), using standard notation from Formal Language Theory. In particular, recall that when we write $(a_1 \dots a_k)^q$ for some integer $q \geq 0$, we mean that the sequence $a_1 \dots a_k$ is repeated exactly q times (in particular, if $q = 0$, then $(a_1 \dots a_k)^q$ is the empty word ε).

Theorem 3.7. *For every integer $n \geq 2$, $i_b(P_n) = \lceil \frac{2n}{5} \rceil$.*

Proof. Let $n = 5q + r$, with $q \geq 0$, $0 \leq r \leq 4$ and $5q + r \geq 2$. It is easy to check that the functions 10, 010 and 0101 are i_b -broadcasts on P_n with cost $\lceil \frac{2n}{5} \rceil$ when $n = 2, 3$ and 4 , respectively.

Assume now $n \geq 5$. According to the value of r , we define the broadcasts f_r on P_n for each r , $0 \leq r \leq 4$, as follows:

$$f_0(P_n) = (01010)^q, \quad f_1(P_n) = (01010)^q 1, \quad f_2(P_n) = (01010)^q 10, \\ f_3(P_n) = (01010)^q 101, \quad \text{and} \quad f_4(P_n) = (01010)^q 0101.$$

It is not difficult to check that each f_r , $0 \leq r \leq 4$, is a maximal independent broadcast on C_{5q+r} , for each $q \geq 1$, with cost $\sigma(f_r) = \lceil \frac{2n}{5} \rceil$, which gives $i_b(P_n) \leq \lceil \frac{2n}{5} \rceil$.

Let us now consider the opposite inequality. Let f be an i_b -broadcast on P_n . If $|V_f^+| = 1$, then we have $i_b(P_n) = \text{rad}(P_n) = \lfloor \frac{n}{2} \rfloor$, which implies $n \in S = \{1, \dots, 9, 11, 13\}$, since otherwise we would have $i_b(P_n) = \lfloor \frac{n}{2} \rfloor > \lceil \frac{2n}{5} \rceil$, contradicting the inequality we have established before. Observe also that we have $\lfloor \frac{n}{2} \rfloor = \lceil \frac{2n}{5} \rceil$ for every $n \in S$, so that we are done.

The remaining case we have to consider is thus $n \notin S$, which implies $|V_f^+| \geq 2$. Let $V_f^+ = \{x_{i_1}, \dots, x_{i_t}\}$, $i_1 < \dots < i_t$, with $t \geq 2$. The two following claims will prove that we can always choose f such that $f(x_{i_j}) = 1$ for every f -broadcast vertex $v_{i_j} \in V_f^+$.

Claim B. *There exists an i_b -broadcast g on P_n such that $g(x_{i_1}) = g(x_{i_2}) = g(x_{i_{t-1}}) = g(x_{i_t}) = 1$.*

Proof. We first prove that there exists an i_b -broadcast g_0 on P_n such that $g_0(x_{i_1}) = g_0(x_{i_2}) = 1$. If $f(x_{i_1}) = f(x_{i_2}) = 1$, then we set $g_0 := f$ and we are done. So, suppose that we have $f(x_{i_1}) + f(x_{i_2}) \geq 3$.

If $|V_f^+| = 2$, then, by Lemma 3.6(4), we have $d_{P_n}(x_{i_1}, x_{i_2}) = f(x_{i_1}) + 1 = f(x_{i_2}) + 1$. Using Lemma 3.6(2) and (3), we then get

$$n = d_{P_n}(x_1, x_{i_1}) + d_{P_n}(x_{i_1}, x_{i_2}) + d_{P_n}(x_{i_2}, x_n) + 1 \leq 3f(x_{i_1}) + 2,$$

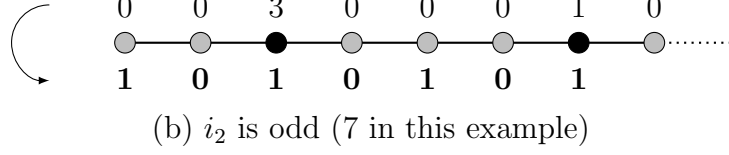
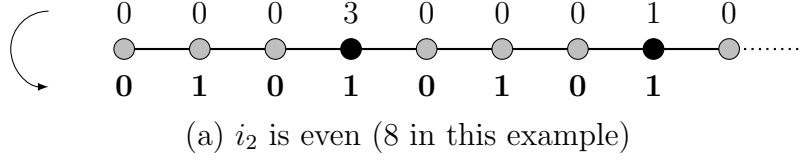


Figure 3: Maximal independent broadcast for the proof of Claim B, Case 1.

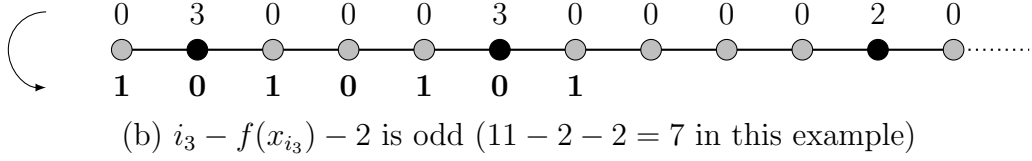
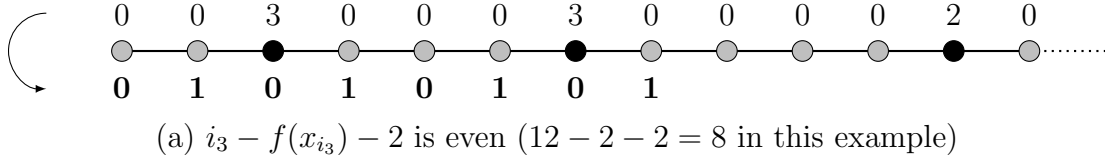


Figure 4: Maximal independent broadcast for the proof of Claim B, Case 2.

and thus $\sigma(f) = 2f(x_{i_1}) \geq \frac{2(n-2)}{3}$. Now, recall that the above defined maximal independent broadcast f_r , with $r = n \bmod 5$, is such that $\sigma(f_r) = \lceil \frac{2n}{5} \rceil$. Since $n \geq 10$, this contradicts the optimality of f .

We thus have $|V_f^+| \geq 3$. We consider two cases, depending on the value of $d_{P_n}(x_{i_2}, x_{i_3})$.

1. $f(x_{i_3}) + 1 \leq d_{P_n}(x_{i_2}, x_{i_3}) \leq f(x_{i_3}) + 2$.

Let g be the mapping obtained from f by replacing the f -values $0^{i_1-1}f(x_{i_1})0^{i_2-i_1-1}f(x_{i_2})$ of $x_1 \dots x_{i_1} \dots x_{i_2}$ by $(01)^{\frac{i_2}{2}}$ if i_2 is even, or by $1(01)^{\frac{i_2-1}{2}}$ if i_2 is odd (see Figure 3). In both cases, we have $g(x_{i_2}) = 1$, which implies that g is a maximal independent broadcast on P_n . We then have

$$\sigma(g) - \sigma(f) = \left\lceil \frac{i_2}{2} \right\rceil - f(x_{i_1}) - f(x_{i_2}).$$

By Lemma 3.6(2) and (4), we have

$$i_2 = d_{P_n}(x_1, x_{i_1}) + d_{P_n}(x_{i_1}, x_{i_2}) + 1 \leq 2f(x_{i_1}) + 2,$$

and thus

$$\sigma(g) - \sigma(f) \leq f(x_{i_1}) + 1 - f(x_{i_1}) - f(x_{i_2}) = 1 - f(x_{i_2}).$$

The optimality of f then implies $f(x_{i_2}) = 1$, so that we have $\sigma(g) = \sigma(f)$ and we can set $g_0 := g$.

2. $d_{P_n}(x_{i_2}, x_{i_3}) \geq f(x_{i_3}) + 3$.

Let g be the mapping obtained from f by replacing the f -values

$$0^{i_1-1}f(x_{i_1})0^{i_2-i_1-1}f(x_{i_2})0^{i_3-i_2-f(x_{i_3})-2}$$

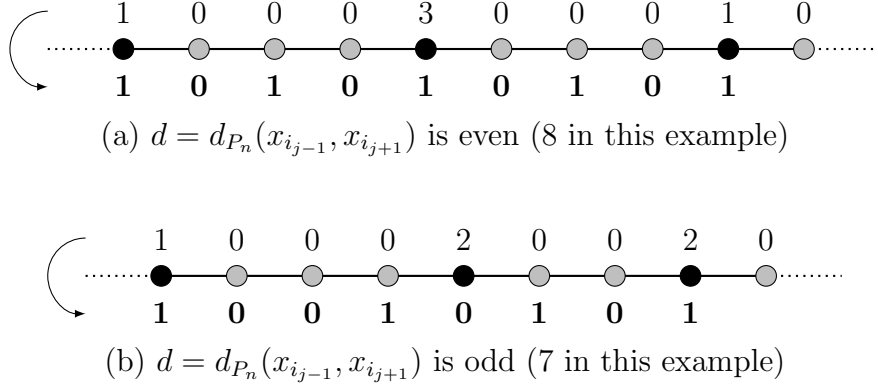


Figure 5: Maximal independent broadcast for the proof of Claim C, Case 1.

of $x_1 \dots x_{i_1} \dots x_{i_2} \dots x_{i_3 - f(x_{i_3}) - 2}$ by $(01)^{\frac{i_3 - f(x_{i_3}) - 2}{2}}$ if $i_3 - f(x_{i_3}) - 2$ is even, or by $1(01)^{\frac{i_3 - f(x_{i_3}) - 3}{2}}$ if $i_3 - f(x_{i_3}) - 2$ is odd (see Figure 4).

In both cases, all vertices are g -dominated and g is clearly a maximal independent broadcast on P_n . We then have

$$\sigma(g) - \sigma(f) = \left\lceil \frac{i_3 - f(x_{i_3}) - 2}{2} \right\rceil - f(x_{i_1}) - f(x_{i_2}).$$

By Lemma 3.6(2) and (4), we have

$$i_2 = d_{P_n}(x_1, x_{i_1}) + d_{P_n}(x_{i_1}, x_{i_2}) + 1 \leq 2f(x_{i_1}) + 2,$$

and thus $i_3 - f(x_{i_3}) - 2 \leq 2f(x_{i_1}) + f(x_{i_2}) + 1$. We then get

$$\sigma(g) - \sigma(f) \leq f(x_{i_1}) + \left\lceil \frac{f(x_{i_2}) + 1}{2} \right\rceil - f(x_{i_1}) - f(x_{i_2}) = \left\lceil \frac{1 - f(x_{i_2})}{2} \right\rceil.$$

The optimality of f then implies $f(x_{i_2}) \leq 2$, so that we have $\sigma(g) = \sigma(f)$ and we can set $g_0 := g$.

Observe now that in both of the above cases we have $g_0(x_{i_j}) = f(x_{i_j})$ for every j , $3 \leq j \leq t$. Therefore, using symmetry and starting from g_0 instead of f , we can similarly construct an i_b -broadcast g on P_n with $g(x_{i_1}) = g(x_{i_2}) = g(x_{i_{t-1}}) = g(x_{i_t}) = 1$, as required. \square

Claim C. *There exists an i_b -broadcast g on P_n such that $g(x_i) = 1$ for every vertex $x_i \in V_g^+$.*

Proof. By Claim B, we can suppose that $f(x_{i_1}) = f(x_{i_2}) = f(x_{i_{t-1}}) = f(x_{i_t}) = 1$. If $f(x_i) = 1$ for every vertex $x_i \in V_f^+$, there is nothing to prove. Suppose thus that this is not the case, which implies $t \geq 5$, and let x_{i_j} , $3 \leq j \leq t - 2$, be the leftmost f -broadcast vertex for which $f(x_{i_j}) \geq 2$.

We will prove that we can always construct an i_b -broadcast f' on P_n such that the number of broadcast vertices with f' -value at least 2 is strictly less than the number of broadcast vertices with f -value at least 2.

We consider two cases, depending on the value of $d_{P_n}(x_{i_{j+1}}, x_{i_{j+2}})$.

1. $f(x_{i_{j+2}}) + 1 \leq d_{P_n}(x_{i_{j+1}}, x_{i_{j+2}}) \leq f(x_{i_{j+2}}) + 2$.

Let $d = i_{j+1} - i_{j-1} = d_{P_n}(x_{i_{j-1}}, x_{i_{j+1}}) = d_{P_n}(x_{i_{j-1}}, x_{i_j}) + d_{P_n}(x_{i_j}, x_{i_{j+1}})$, and f' be the mapping obtained from f by replacing the f -values

$$f(x_{i_{j-1}})0^{i_j - i_{j-1} - 1}f(x_{i_j})0^{i_{j+1} - i_j - 1}f(x_{i_{j+1}})$$

of $x_{i_{j-1}} \dots x_{i_j} \dots x_{i_{j+1}}$ by $1(01)^{\frac{d}{2}}$ if d is even, or by $10(01)^{\frac{d-1}{2}}$ if d is odd (see Figure 5).

Observe that we have $f'(x_{i_{j-1}}) = 1$ and $f'(x_{i_{j+1}}) = 1$ in both cases, which implies that f' is a maximal independent broadcast on P_n . Moreover, in both cases, we have

$$\sigma(f') - \sigma(f) = \left\lfloor \frac{d}{2} \right\rfloor - f(x_{i_j}) - f(x_{i_{j+1}}).$$

Since $f(x_{i_{j-1}}) = 1 < f(x_{i_j})$, Lemma 3.6(3) gives

$$f(x_{i_j}) + 1 \leq d_{P_n}(x_{i_{j-1}}, x_{i_j}) \leq f(x_{i_j}) + 2.$$

We thus consider two subcases, depending on the value of $d_{P_n}(x_{i_{j-1}}, x_{i_j})$.

(a) $d_{P_n}(x_{i_{j-1}}, x_{i_j}) = f(x_{i_j}) + 1$.

By Lemma 3.6(3), we have

$$d = d_{P_n}(x_{i_{j-1}}, x_{i_j}) + d_{P_n}(x_{i_j}, x_{i_{j+1}}) \leq f(x_{i_j}) + 1 + f(x_{i_j}) + f(x_{i_{j+1}}) + 1 = 2f(x_{i_j}) + f(x_{i_{j+1}}) + 2,$$

and thus

$$\sigma(f') - \sigma(f) \leq f(x_{i_j}) + \left\lfloor \frac{f(x_{i_{j+1}})}{2} \right\rfloor + 1 - f(x_{i_j}) - f(x_{i_{j+1}}) = \left\lfloor \frac{2 - f(x_{i_{j+1}})}{2} \right\rfloor \leq 0.$$

The optimality of f then implies the optimality of f' , and the number of broadcast vertices with f' -value at least 2 is strictly less than the number of broadcast vertices with f -value at least 2, as required.

(b) $d_{P_n}(x_{i_{j-1}}, x_{i_j}) = f(x_{i_j}) + 2$.

Since f is maximal, we necessarily have $d_{P_n}(x_{i_j}, x_{i_{j+1}}) = f(x_{i_j}) + 1$, since otherwise we could increase $f(x_{i_j})$ by 1. This gives

$$d = d_{P_n}(x_{i_{j-1}}, x_{i_j}) + d_{P_n}(x_{i_j}, x_{i_{j+1}}) = 2f(x_{i_j}) + 3,$$

and thus

$$\sigma(f') - \sigma(f) = f(x_{i_j}) + 1 - f(x_{i_j}) - f(x_{i_{j+1}}) = 1 - f(x_{i_{j+1}}) \leq 0.$$

Again, the optimality of f then implies the optimality of f' , and the number of broadcast vertices with f' -value at least 2 is strictly less than the number of broadcast vertices with f -value at least 2, as required.

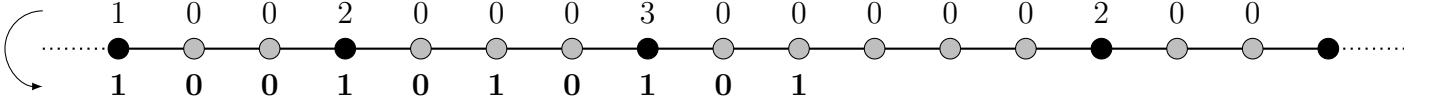
2. $d_{P_n}(x_{i_{j+1}}, x_{i_{j+2}}) \geq f(x_{i_{j+2}}) + 3$.

Let $d' = d_{P_n}(x_{i_{j+2}-f(x_{i_{j+2}})-2}, x_{i_{j-1}+1}) = i_{j+2} - f(x_{i_{j+2}}) - 2 - i_{j-1} - 1 = i_{j+2} - i_{j-1} - f(x_{i_{j+2}}) - 3$, and f' be the mapping obtained from f by replacing the f -values

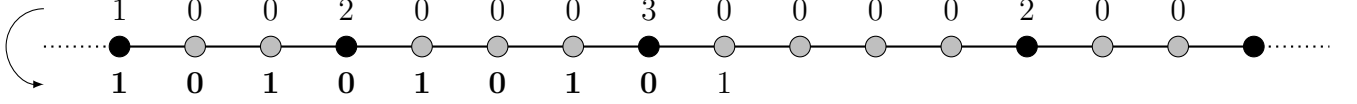
$$f(x_{i_{j-1}})0^{i_j-i_{j-1}-1}f(x_{i_j})0^{i_{j+1}-i_j-1}f(x_{i_{j+1}})0^{i_{j+2}-f(x_{i_{j+2}})-2-i_{j+1}}$$

of $x_{i_{j-1}} \dots x_{i_j} \dots x_{i_{j+1}} \dots x_{i_{j+2}-f(x_{i_{j+2}})-2}$ by $10(01)^{\frac{d'}{2}}$ if d' is even, or by $1(01)^{\frac{d'+1}{2}}$ if d' is odd (see Figure 6).

Since $i_{j+2} - i_{j+1} \leq f(x_{i_{j+1}}) + f(x_{i_{j+2}}) + 1$ and $i_{j+1} - i_{j-1} \leq 2f(x_{i_j}) + f(x_{i_{j+1}}) + 2$, we get $i_{j+2} - i_{j-1} \leq 2f(x_{i_j}) + 2f(x_{i_{j+1}}) + f(x_{i_{j+2}}) + 3$ and thus $d' \leq 2f(x_{i_j}) + 2f(x_{i_{j+1}})$.



(a) $d' = i_{j+2} - i_{j-1} - f(x_{i_{j+2}}) - 3$ is even ($13 - 2 - 3 = 8$ in this example)



(b) $d' = i_{j+2} - i_{j-1} - f(x_{i_{j+2}}) - 3$ is odd ($12 - 2 - 3 = 7$ in this example)

Figure 6: Maximal independent broadcast for the proof of Claim C, Case 2.

Hence, the mapping f' is a maximal independent broadcast on P_n and we have

$$\sigma(f') - \sigma(f) = \left\lceil \frac{d'}{2} \right\rceil - f(x_{i_j}) - f(x_{i_{j+1}}) \leq \left\lceil \frac{2f(x_{i_j}) + 2f(x_{i_{j+1}})}{2} \right\rceil - f(x_{i_j}) - f(x_{i_{j+1}}) = 0.$$

The optimality of f then implies the optimality of f' , and the number of broadcast vertices with f' -value at least 2 is strictly less than the number of broadcast vertices with f -value at least 2, as required.

In each case, we were able to construct an i_b -broadcast f' such that the number of broadcast vertices with f' -value at least 2 is strictly less than the number of broadcast vertices with f -value at least 2, as required. Repeating this construction until no such broadcast vertex exists, we eventually get an i_b -broadcast g such that $g(x_{i_j}) = 1$ for every g -broadcast vertex x_{i_j} . This completes the proof of Claim C. \square

By Claim C, we can thus now assume that the i_b -broadcast f is such that $f(x_{i_j}) = 1$ for every f -broadcast vertex x_{i_j} . It remains to prove that, for every $n \geq 5$, $\sigma(f) = \lceil \frac{2n}{5} \rceil$.

For that, we first prove the following claim. Let w_f denote the word on the alphabet $\{0, 1\}$ defined by $w_f = f(x_1) \dots f(x_n)$.

Claim D. *There exists an i_b -broadcast f on P_n , $n \geq 5$, such that $f(x_{i_j}) = 1$ for every f -broadcast vertex x_{i_j} , that satisfies the following properties:*

- (1) w_f does not contain the factor 000, and
- (2) w_f does not contain the factor 1001001.
- (3) 0101 is a prefix of w_f ,
- (4) either 101 or 1010 is a suffix of w_f .

Proof. If $f(x_{i-1})f(x_i)f(x_{i+1}) = 000$, then we can set $f(x_i) = 1$, contradicting the maximality of f , which proves Item (1). Similarly, if $f(x_{i-3}) \dots f(x_i) \dots f(x_{i+3}) = 1001001$, then we can set $f(x_i) = 2$, again contradicting the maximality of f , which proves Item (2).

Now, observe that we can have neither $f(x_1)f(x_2) = 00$, since otherwise we could set $f(x_1) = 1$, neither $f(x_1) \dots f(x_4) = 0100$, since otherwise we could set $f(x_2) = 2$, nor $f(x_1) \dots f(x_4) = 1001$, since otherwise we could set $f(x_1) = 2$, contradicting in each case the maximality of f . Therefore, either 0101 or 1010 is a prefix of w_f . In the former case we are done, so let us assume that 1010

is a prefix of w_f . Suppose that w_f contains the factor 100 and consider its first occurrence, that is, suppose that $(10)^k 100$ is a prefix of w_f for some $k \geq 1$. In that case, we can replace the f -values $(10)^k 100$ of $x_1 \dots x_{2k+3}$ by $0(10)^k 10$ and we are done. If w_f does not contain the factor 100, then we necessarily have either $w_f = (10)^{\frac{n}{2}}$ or $w_f = (10)^{\frac{n-1}{2}} 1$, which gives $\sigma(f) = \lceil \frac{n}{2} \rceil > \lceil \frac{2n}{5} \rceil$, a contradiction. This proves Item (3).

By symmetry, using the same argument as for the previous item, we get that none of 00, 0010, or 1001 can be a suffix of w_f . Hence, either 1010 or 0101 is a suffix of w_f , which proves Item (4). \square

By Item (3) of Claim D, we let w'_f be the word defined by $w_f = 0101w'_f$. We will now “split” w'_f in factors (or blocks) w_1, \dots, w_q , $q \geq 0$ ($q = 0$ meaning that no such block appeared), each of length 2 or 5, inductively defined as follows.

- If 01 is a prefix of w'_f , then $w_1 = 01$,
- If 00101 is a prefix of w'_f , then $w_1 = 00101$,
- If $w'_f = w_1 \dots w_{k-1} w''$ and 01 is a prefix of w'' , then $w_k = 01$,
- If $w'_f = w_1 \dots w_{k-1} w''$ and 00101 is a prefix of w'' , then $w_k = 00101$.

We then have either $w_f = 0101w''$ and $q = 0$, or $w_f = 0101w_1 \dots w_q w''$, for some $q \geq 1$, with w'' being either empty or 0 (observe that w'' cannot start with a 1, and recall that 00 cannot be a suffix of w_f), and, if $q > 0$, then $w_i \in \{01, 00101\}$ for every i , $1 \leq i \leq q$.

Observe that if $w_i = 01$ and $w_{i+1} = 00101$ for some i , $1 \leq i \leq q-1$, then the mapping g defined by $w_g = 0101w_1 \dots w_{i-1}.00101.01.w_{i+2} \dots w''$ is still an i_b -broadcast on P_n . Therefore, f can be chosen in such a way that there exists some $q_0 \leq q$ such that $w_i = 00101$ if and only if $i \leq q_0$.

We now claim that f can be chosen in such a way that we have at most two blocks equal to 01, that is, $q - q_0 \leq 2$. Indeed, if we add at least three such blocks, then either 1010101 or 10101010 is a suffix of w_f . In the former case, 1010101 could be replaced by 1001010, contradicting the optimality of f . In the latter case, we may replace 10101010 by 10010101, so that $q - q_0 = 2$.

Finally, we get that the structure of w_f (recall that $n \geq 5$) is either

$$01010, 010101, 0101010, 01010101, \text{ or } 0101w_1 \dots w_{q_0} w',$$

with $q_0 \geq 1$, $w_i = 00101$ for every i , $1 \leq i \leq q_0$, and $w' \in \{\varepsilon, 0, 01, 010, 0101, 01010\}$. It is now routine to check that $\sigma(f) = \lceil \frac{2n}{5} \rceil$ in each case, which gives $i_b(P_n) = \lceil \frac{2n}{5} \rceil$ for every $n \geq 3$.

This completes the proof. \square

Using Theorem 3.7, we can also prove a similar result for cycles.

Theorem 3.8. *For every integer $n \geq 3$, $i_b(C_n) = \lceil \frac{2n}{5} \rceil$.*

Proof. Observe first that 010 and 0101 are i_b -broadcasts on C_n with cost $\lceil \frac{2n}{5} \rceil$, when $n = 3, 4$, respectively, and that the five functions f_0, \dots, f_4 , defined in the proof of Theorem 3.7, are also i_b -broadcasts on C_n , $n \geq 5$, with cost $\lceil \frac{2n}{5} \rceil$. We thus have $i_b(C_n) = \lceil \frac{2n}{5} \rceil$ for $n = 3, 4$, and $i_b(C_n) \leq \lceil \frac{2n}{5} \rceil$ for every $n \geq 5$.

We now prove the opposite inequality when $n \geq 5$. For that, let f be any i_b -broadcast on C_n , $n \geq 5$. Suppose first that $|V_f^+| = 1$, which implies $i_b(C_n) = \text{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$. As observed in the proof of Theorem 3.7, according to the inequality we established before, this situation can only happen if $n \in S = \{1, \dots, 9, 11, 13\}$.

Suppose now that $n \notin S$, which implies $|V_f^+| \geq 2$ and, in particular, $n \geq 10$. We now claim that we necessarily have $|V_f^+| \geq 3$. Indeed, suppose to the contrary that $|V_f^+| = 2$, and let $V_f^+ = \{x_{i_1}, x_{i_2}\}$.

Denote by $Q_1 = x_{i_1}x_{i_1+1}\dots x_{i_2}$, and $Q_2 = x_{i_2}x_{i_2+1}\dots x_{i_1}$ the two paths joining x_{i_1} and x_{i_2} . Since f is maximal, we can assume, without loss of generality, that $|Q_1| = d_{C_n}(x_{i_1}, x_{i_2}) = f(x_{i_1}) + 1 = f(x_{i_2}) + 1$, which gives $f(x_{i_1}) = f(x_{i_2})$. Since Item 3 of Lemma 3.6 also holds for cycles, we have $|Q_2| \leq f(x_{i_1}) + f(x_{i_2}) + 1 = 2f(x_{i_1}) + 1$, which gives

$$n = |Q_1| + |Q_2| \leq f(x_{i_1}) + 1 + 2f(x_{i_1}) + 1 = 3f(x_{i_1}) + 2.$$

We then get $f(x_{i_1}) \geq \lceil \frac{n-2}{3} \rceil$, and thus

$$i_b(C_n) = 2f(x_{i_1}) \geq 2 \cdot \left\lceil \frac{n-2}{3} \right\rceil,$$

a contradiction with the inequality we established before since $2 \lceil \frac{n-2}{3} \rceil > \lceil \frac{2n}{5} \rceil$ when $n \geq 10$.

We can thus assume $|V_f^+| \geq 3$, and let $V_f^+ = \{x_{i_0}, \dots, x_{i_{t-1}}\}$, $t \geq 3$. We first claim that there exists some j , $0 \leq j \leq t-1$, such that $d_{C_n}(x_{i_j}, x_{i_{j+1}}) = f(x_{i_j}) + f(x_{i_{j+1}}) + 1$ (subscripts are taken modulo t). Indeed, if this is not the case, we get

$$n = \sum_{0 \leq j \leq t-1} d_{C_n}(x_{i_j}, x_{i_{j+1}}) \leq 2 \sum_{0 \leq j \leq t-1} f(x_{i_j}) = 2i_b(C_n),$$

which gives $i_b(C_n) \geq \lceil \frac{n}{2} \rceil$, in contradiction with the inequality $i_b(C_n) \leq \lceil \frac{2n}{5} \rceil$ we established before, since $n \notin S$.

We can thus suppose, without loss of generality, that $d_{C_n}(x_{i_1}, x_{i_2}) = f(x_{i_1}) + f(x_{i_2}) + 1$, which implies that $d_{C_n}(x_{i_2}, x_{i_3}) = f(x_{i_2}) + 1$ (we may have $x_{i_3} = x_{i_0}$) and, similarly, that $d_{C_n}(x_{i_0}, x_{i_1}) = f(x_{i_1}) + 1$. To avoid confusion, let us denote $P_n = y_0y_1\dots y_{n-1}$, with $y_{f(x_{i_2})} = x_{i_2}$, and let g be the function defined by $g(y_j) = f(x_{j-f(x_{i_2})+i_2})$ for every j , $0 \leq j \leq n-1$ (subscripts are taken modulo n). Observe that both y_0 and y_{n-1} are g -dominated, since all vertices lying between x_{i_1} and x_{i_2} were f -dominated in C_n . Moreover, we cannot increase the g -value of $y_{f(x_{i_2})}$ (the leftmost g -broadcast vertex in P_n) since we had $d_{C_n}(x_{i_2}, x_{i_3}) = f(x_{i_2}) + 1$, neither the value of $y_{n-f(x_{i_1})-1}$ (the rightmost g -broadcast vertex in P_n) since we had $d_{C_n}(x_{i_0}, x_{i_1}) = f(x_{i_1}) + 1$.

Since f was a maximal independent broadcast on C_n , we thus get that g is a maximal independent broadcast on P_n , which gives $i_b(P_n) \leq i_b(C_n)$, and thus $i_b(C_n) \geq \lceil \frac{2n}{5} \rceil$, as required. \square

3.3 Broadcast irredundance number and broadcast domination number

Erwin proved in [21] that $\gamma_b(P_n) = \gamma(P_n) = \lceil n/3 \rceil$. Knowing the value of $\gamma_b(P_n)$, we can infer the value of $\gamma_b(C_n)$. Indeed, Brešar and Špacapan proved in [14] that, for every connected graph G , there is a spanning tree T of G such that $\gamma_b(G) = \gamma_b(T)$. Since spanning trees of the cycle C_n are all isomorphic to the path P_n , we get the following result.

Proposition 3.9. *For every integer $n \geq 3$, $\gamma_b(C_n) = \gamma_b(P_n) = \lceil \frac{n}{3} \rceil$.*

We now consider the broadcast irredundance number of paths. For that, we first prove the two following lemmas.

Lemma 3.10. *Let f be a maximal irredundant broadcast on P_n . If $H_f(x_i) = \emptyset$ for some vertex x_i , then $N_{P_n}(x_i) \cap N_f(V_f^+) \neq \emptyset$.*

Proof. Assume to the contrary that we have $N_{P_n}(x_i) \cap N_f(V_f^+) = \emptyset$. In that case, we could set $f(x_i) = 1$, contradicting the maximality of f . \square

Lemma 3.11. *For every integer $n \geq 3$, the following statements hold.*

1. If f is a maximal irredundant broadcast on P_n , then $H_f(x_2) \neq \emptyset$ and $H_f(x_{n-1}) \neq \emptyset$.
2. There exists an ir_b -broadcast f on P_n such that $H_f(x_1) \neq \emptyset$ and $H_f(x_n) \neq \emptyset$.

Proof. We prove the two statements separately.

1. If $H_f(x_2) = \emptyset$, then x_2 is not f -dominated, and consequently x_1 is also not f -dominated. We then have $N_{P_n}(x_1) \cap N_f(V_f^+) = \emptyset$, in contradiction with Lemma 3.10. The case $H_f(x_{n-1}) = \emptyset$ is similar.
2. Let g be an ir_b -broadcast on P_n . If $H_g(x_1) \neq \emptyset$ and $H_g(x_n) \neq \emptyset$, then we let $f := g$ and we are done.

Suppose that we have $H_g(x_1) = \emptyset$. By the previous item, we know that x_2 is g -dominated by some vertex x_i , $i > 2$, such that $f(x_i) = d_{P_n}(x_2, x_i)$. We then necessarily have $|PB_g(x_i)| = 1$ if $g(x_i) \geq 2$, and $x_i \notin PN_g(x_i)$ if $g(x_i) = 1$, since otherwise we could set $g(x_1) = 1$, contradicting the optimality of g .

Now, observe that the function h obtained from g by setting $h(x_i) = 0$ and $h(x_{i-1}) = g(x_i)$ is a maximal irredundant broadcast on P_n , with cost $\sigma(h) = \sigma(g) = ir_b(P_n)$, that satisfies $H_h(x_1) \neq \emptyset$.

If $H_h(x_n) \neq \emptyset$, then we let $f := h$ and we are done. Otherwise, using the same reasoning (by symmetry), we claim that can produce an ir_b -broadcast f on P_n such that $H_f(x_1) \neq \emptyset$ and $H_f(x_n) \neq \emptyset$. Observe first that we cannot have $|V_g^+| = 2$ if $H_g(x_1) = \emptyset$ and $H_g(x_n) = \emptyset$, since in that case we could increase by 1 the g -values of x_{i_1} and x_{i_2} , contradicting the maximality of g . Therefore, if g was such that $H_g(x_1) = \emptyset$ and $H_g(x_n) = \emptyset$, then the optimality of g implies $|V_g^+| \geq 3$, so that the modification of h does not affect $h(x_i)$, and thus x_1 is still f -dominated.

This concludes the proof. \square

We are now able to determine the value of the broadcast irredundance number and of the broadcast domination number of paths.

Theorem 3.12. *For every integer $n \geq 2$, $ir_b(P_n) = \gamma_b(P_n) = \lceil \frac{n}{3} \rceil$.*

Proof. By Corollary 2.5 and Proposition 3.9, we only need to prove that $\gamma_b(P_n) \leq ir_b(P_n)$. For this, it is enough to construct, from any non-dominating ir_b -broadcast, a dominating ir_b -broadcast.

Let f be an ir_b -broadcast on P_n . By Lemma 3.11, we can assume that x_1, x_2, x_{n-1} and x_n are f -dominated. If f is dominating, then we are done. Thus suppose that f is non-dominating, and let x_i , $3 \leq i \leq n-2$, be the leftmost non-dominated vertex. We will prove that there exists a maximal irredundant broadcast g on P_n , with $\sigma(g) = \sigma(f) = ir_b(P_n)$, such that the number of vertices that are not g -dominated is strictly less than the number of vertices that are not f -dominated.

Let x_j , $j \leq i-2$, denote the f -broadcast vertex that dominates x_{i-1} . Since x_i is not f -dominated, we necessarily have $x_{i-1} \in PB(x_j)$. Observe that we have $|PB(x_j)| = 1$, that is, the bordering private f -neighbor of x_j is x_{i-1} , since otherwise we could set $f(x_{i-1}) = 1$, contradicting the maximality of f . Note also that x_1 is not f -dominated by x_j , that is, x_j is not the leftmost f -broadcast vertex, since otherwise we could increase the f -value of x_j by 1, again contradicting the maximality of f . Let then $x_{j'}$, $j' < j$, denote the closest f -broadcast vertex to the left of x_j , and x_{j^p} , $j^p < j'$, denote the bordering private f -neighbor of $x_{j'}$.

Since x_{j^p} is the bordering private f -neighbor of $x_{j'}$, we necessarily have $d_{P_n}(x_{j^p}, x_j) \geq f(x_j) + 1$. Moreover, we necessarily have $d_{P_n}(x_{j^p}, x_j) = f(x_j) + 1$, since otherwise we could increase the value

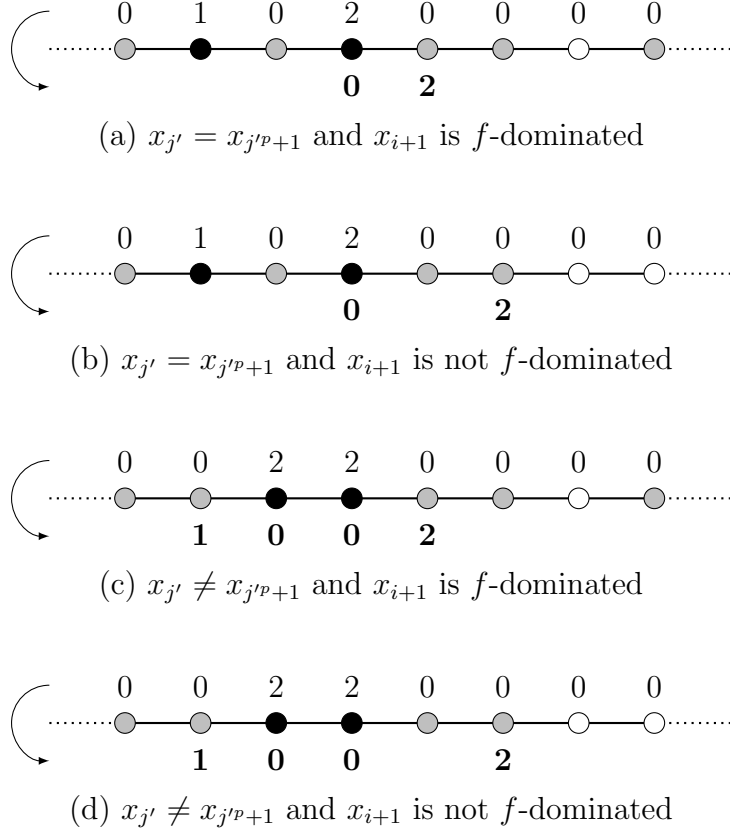


Figure 7: Maximal irredundant broadcast for the proof of Theorem 3.12.

of $f(x_j)$ by 1 (x_i becoming the bordering private f -neighbor of x_j), contradicting the maximality of f . Hence, we have $d_{P_n}(x_{j^p}, x_j) = f(x_j) + 1$, and thus

$$d_{P_n}(x_{j^p}, x_i) = d_{P_n}(x_{j^p}, x_j) + d_{P_n}(x_j, x_i) = f(x_j) + 1 + f(x_j) + 1 = 2f(x_j) + 2.$$

Let now g be the function obtained from f by setting

- $g(x_j) = 0$,
- $g(x_{j^p+1}) = 1$ and $g(x_{j'}) = 0$ if $x_{j'} \neq x_{j^p+1}$, and
- $g(x_{j+1}) = f(x_j)$ if x_{i+1} is f -dominated, or $g(x_{j+2}) = f(x_j)$ if x_{i+1} is not f -dominated (see Figure 7).

Observe that x_{j^p} is a bordering private g -neighbor of x_{j^p+1} , and that either x_i is a bordering private g -neighbor of x_{j+1} (if x_{i+1} is f -dominated), or x_{i+1} is a bordering private g -neighbor of x_{j+2} (if x_{i+1} is not f -dominated). Moreover, all vertices x_1, \dots, x_i are g -dominated. Hence, g is a maximal irredundant broadcast with cost

$$\sigma(g) = \sigma(f) - f(x_j) - f(x_{j'}) + f(x_j) + 1 = \sigma(f) - f(x_{j'}) + 1.$$

The optimality of f then imply $f(x_{j'}) = 1$, so that g is an ir_b -broadcast on P_n such that the number of vertices that are not g -dominated is strictly less than the number of vertices that are not f -dominated.

By repeating this modification while they remain non-dominated vertices, we eventually get a dominating ir_b -broadcast on P_n , which concludes the proof. \square

Using Theorem 3.12, we can prove a similar result for cycles.

Theorem 3.13. *For every integer $n \geq 3$, $ir_b(C_n) = \gamma_b(C_n) = \lceil \frac{n}{3} \rceil$.*

Proof. We already know by Proposition 3.9 that $\gamma_b(C_n) = \lceil \frac{n}{3} \rceil$, so that we only need to prove that $ir_b(C_n) = \lceil \frac{n}{3} \rceil$. By Corollary 2.5, we only need to prove that $ir_b(C_n) \geq \gamma_b(C_n)$ or, by Proposition 3.9 and Theorem 3.12, that $ir_b(C_n) \geq ir_b(P_n)$.

Observe that 010, 0200 and 00200 are dominating ir_b -broadcasts for C_3 , C_4 and C_5 , respectively, with cost $\lceil \frac{n}{3} \rceil$. It thus remains to consider the case $n \geq 6$.

Let f be an ir_b -broadcast on C_n , $n \geq 6$. If f is dominating, then we are done. Thus suppose that f is non-dominating. We claim that $|V_f^+| \geq 2$. Indeed, if $|V_f^+| = 1$, then the maximality of f implies $\sigma(f) = \lfloor \frac{n}{2} \rfloor$, which gives $ir_b(C_n) = \lfloor \frac{n}{2} \rfloor > \lceil \frac{n}{3} \rceil = \gamma_b(C_n)$, in contradiction with Corollary 2.5 since $n \geq 6$.

We thus have $|V_f^+| \geq 2$. Since f is maximal, we cannot have three consecutive vertices that are not f -dominated. Moreover, since f is non-dominating, we can assume, without loss of generality, that x_0 is not f -dominated, and that x_1 is f -dominated by some f -broadcast vertex x_{i_1} , $i_1 > 1$. Note that x_{n-1} may be f -dominated or not.

To avoid confusion, let $P_n = y_0 y_1 \dots y_{n-1}$. Let then g be the mapping defined by $g(y_i) = f(x_i)$ for every i , $0 \leq i \leq n-1$. As in C_n , x_0 is not g -dominated, x_1 is g -dominated by x_{i_1} , and $g(x_{i_1})$ cannot be increased since otherwise $f(x_{i_1})$ could be increased, contradicting the maximality of f . Similarly, the g -value of the rightmost g -broadcast vertex in P_n cannot be increased, since otherwise its f -value could be increased (since x_0 is not f -dominated, while x_{n-1} may be f -dominated or not), again contradicting the maximality of f . We can apply the same reasoning if x_{n-1} is not g -dominated, which implies that x_{n-1} is not f -dominated. Hence, since f is an ir_b -broadcast on C_n , we get that g is also a maximal irredundant broadcast on P_n , which gives $ir_b(C_n) = \sigma(f) = \sigma(g) \geq ir_b(P_n)$ as required. \square

3.4 Packing broadcast number and lower packing broadcast number

We first determine the broadcast packing number of paths and cycles.

Theorem 3.14. *For every $n \geq 2$, $P_b(P_n) = \text{diam}(P_n) = n - 1$, and, for every $n \geq 3$, $P_b(C_n) = \text{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$.*

Proof. We first consider the case of the path P_n , $n \geq 2$. Observe first that the function f defined by $f(x_1) = n - 1$ and $f(x_i) = 0$ for every i , $2 \leq i \leq n$ is a maximal broadcast packing with cost $n - 1 = \text{diam}(P_n)$, which gives $\text{diam}(P_n) \leq P_b(P_n)$. The opposite inequality directly follows from Observation 1.1(3) and Theorem 3.1.

Let us now consider the case of the cycle C_n , $n \geq 3$. Observe first that the function f defined by $f(x_0) = \lfloor \frac{n}{2} \rfloor$ and $f(x_i) = 0$ for every i , $1 \leq i \leq n - 1$ is a maximal broadcast packing with cost $\lfloor \frac{n}{2} \rfloor = \text{diam}(C_n)$, which gives $\text{diam}(C_n) \leq P_b(C_n)$.

Again, to establish the opposite inequality, it suffices to prove that, for every P_b -broadcast f on C_n , $|V_f^+| = 1$. Similarly as above, if we suppose that f is a P_b -broadcast on C_n with $|V_f^+| \geq 2$, we get

$$\sum_{x_i \in V_f^+} (2f(x_i) + 1) \leq n,$$

which gives

$$2P_b(C_n) + |V_f^+| \leq n,$$

and thus

$$P_b(C_n) \leq \frac{n - |V_f^+|}{2} \leq \frac{n-2}{2} < \left\lfloor \frac{n}{2} \right\rfloor = \text{diam}(C_n),$$

again a contradiction. \square

In order to determine the values of $p_b(P_n)$, $n \geq 1$, we first prove the following lemma.

Lemma 3.15. *For every integer $n \geq 2$, there exists a p_b -broadcast f on P_n such that $f(x_i) = 1$ for every f -broadcast vertex x_i .*

Proof. Observe first that 10, 010, 1001 and 10010 define p_b -broadcasts on P_n , $n = 2, 3, 4, 5$, respectively, that satisfy the statement of the lemma. Suppose thus $n \geq 6$ and let g be any p_b -broadcast on P_n . If $g(x_i) = 1$ for every g -broadcast vertex x_i , then we set $f := g$ and we are done.

Otherwise, let $V_g^+ = \{x_{i_1}, \dots, x_{i_t}\}$, $i_1 < \dots < i_t$, $t \geq 1$. We first claim that we necessarily have $t \geq 2$. Indeed, if $t = 1$, we get

$$p_b(P_n) = g(x_{i_1}) = \text{rad}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor,$$

while the function g' defined by $g'(x_i) = 1$, $1 \leq i \leq n$, if and only if $i \equiv 1 \pmod{3}$ is a maximal packing broadcast with cost $\sigma(g') = \left\lfloor \frac{n}{3} \right\rfloor < \left\lfloor \frac{n}{2} \right\rfloor$, a contradiction.

We thus have $|V_g^+| \geq 2$. Let x_{i_j} , $1 \leq j \leq t$, be a g -broadcast vertex with minimum subscript such that $g(x_{i_j}) \geq 2$. We will consider three cases, depending on the value of i_j . In each case, we will prove either that the case cannot occur, or that we can produce a p_b -broadcast g' on P_n , with $\sigma(g') = \sigma(g)$, such that the subscript of the leftmost g' -broadcast vertex with g' -value at least 2, if any, is strictly greater than the subscript of the leftmost g -broadcast vertex with g -value at least 2.

1. $i_j = i_1$ or $i_j = i_t$.

Assume $i_j = i_1$, the case $i_j = i_t$ being similar, by symmetry. We first claim that we have $g(x_{i_1}) \in \{i_1 - 2, i_1 - 1\}$. Indeed, if $g(x_{i_1}) \geq i_1$, then the function h obtained from g by setting $h(x_{i_1}) = 0$ and $h(x_{i_1+1}) = g(x_{i_1}) - 1$ is clearly a maximal packing broadcast with cost $\sigma(h) = \sigma(g) - 1$, contradicting the optimality of g . Now, if $g(x_{i_1}) \leq i_1 - 3$, then we could set $g(x_1) = 1$, contradicting the maximality of g . We thus have two cases to consider.

- (a) $g(x_{i_1}) = i_1 - 1$, and thus $i_1 \geq 3$.

Let g_1 be the function obtained from g by modifying the g -values of $x_1, \dots, x_{2g(x_{i_1})+1}$ as follows:

- $g_1(x_1 \dots x_{2g(x_{i_1})+1}) = (010)^\alpha$, if $2g(x_{i_1}) + 1 \equiv 0 \pmod{3}$,
- $g_1(x_1 \dots x_{2g(x_{i_1})+1}) = 0(010)^\alpha$, if $2g(x_{i_1}) + 1 \equiv 1 \pmod{3}$,
- $g_1(x_1 \dots x_{2g(x_{i_1})+1}) = 10(010)^\alpha$, if $2g(x_{i_1}) + 1 \equiv 2 \pmod{3}$,

where $\alpha = \left\lfloor \frac{2g(x_{i_1})+1}{3} \right\rfloor$ (see Figure 8(a)). It is then not difficult to check that g_1 is a maximal packing broadcast such that

$$\sigma(g_1) - \sigma(g) = \left(\left\lfloor \frac{2g(x_{i_1}) - 1}{3} \right\rfloor + 1 \right) - g(x_{i_1}) = \left\lfloor \frac{2 - g(x_{i_1})}{3} \right\rfloor,$$

which gives $\sigma(g_1) - \sigma(g) < 0$ whenever $g(x_{i_1}) \neq 2$. Consequently, $g(x_{i_1}) = g(x_3) = 2$ and we can then define g' from g by setting $g'(x_1 \dots x_5) = 10010$.

- (b) $g(x_{i_1}) = i_1 - 2$, and thus $i_1 \geq 4$.

Similarly to the previous case, let g_2 be the function obtained from g by modifying the g -values of $x_1, \dots, x_{2g(x_{i_1})+2}$ as follows:

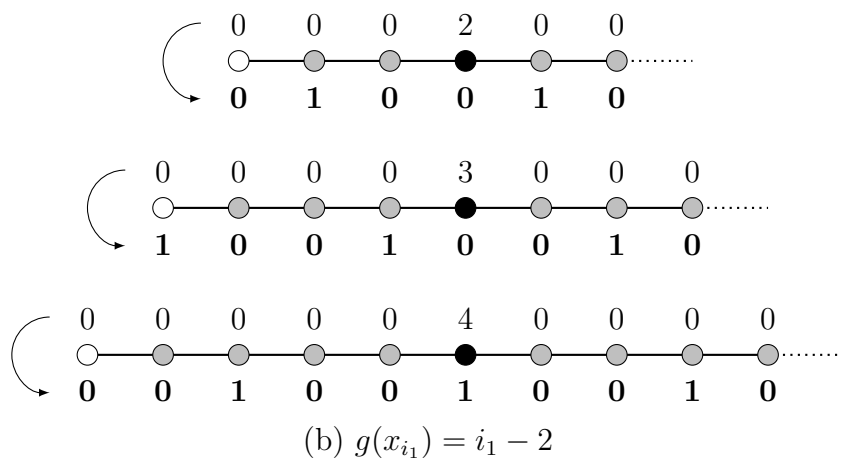
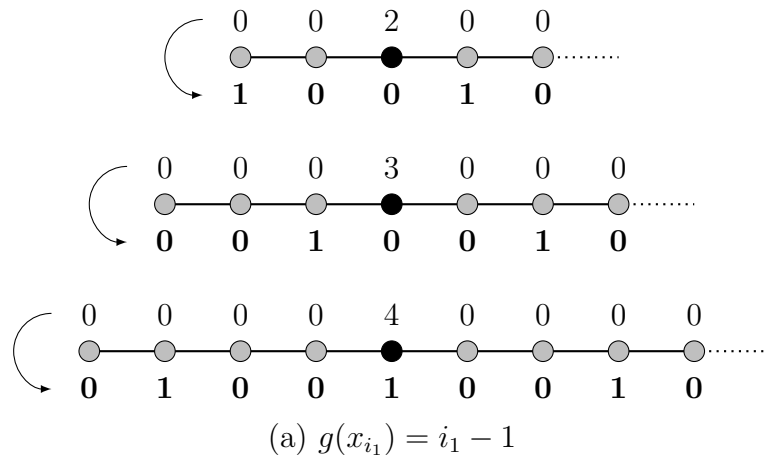


Figure 8: Packing broadcasts for the proof of Lemma 3.15, Case 1.

- $g_2(x_1 \dots x_{2g(x_{i_1})+2}) = (010)^\alpha$, if $2g(x_{i_1}) + 2 \equiv 0 \pmod{3}$,
- $g_2(x_1 \dots x_{2g(x_{i_1})+2}) = 0(010)^\alpha$, if $2g(x_{i_1}) + 2 \equiv 1 \pmod{3}$,
- $g_2(x_1 \dots x_{2g(x_{i_1})+2}) = 10(010)^\alpha$, if $2g(x_{i_1}) + 2 \equiv 2 \pmod{3}$,

where $\alpha = \left\lfloor \frac{2g(x_{i_1})+2}{3} \right\rfloor$ (see Figure 8(b)). Again, it is not difficult to check that g_2 is a maximal packing broadcast such that

$$\sigma(g_2) - \sigma(g) = \left(\left\lfloor \frac{2g(x_{i_1})}{3} \right\rfloor + 1 \right) - g(x_{i_1}) = \left\lfloor \frac{3 - g(x_{i_1})}{3} \right\rfloor,$$

which gives $\sigma(g_2) - \sigma(g) < 0$ whenever $g(x_{i_1}) \neq 2$ and $g(x_{i_1}) \neq 3$. Consequently, either $g(x_{i_1}) = g(x_4) = 2$ and we can then define g' from g by setting $g'(x_1 \dots x_6) = 010010$, or $g(x_{i_1}) = g(x_5) = 3$ and we can then define g' from g by setting $g'(x_1 \dots x_8) = 10010010$.

2. $i_j \in \{i_2, \dots, i_{t-1}\}$.

Since g is a maximal packing broadcast, we necessarily have

$$1 \leq d_{P_n}(x_{i_{j-1}}, x_{i_j}) - g(x_{i_{j-1}}) - g(x_{i_j}) \leq 3.$$

Moreover, we also have either

$$d_{P_n}(x_{i_{j-1}}, x_{i_j}) = g(x_{i_{j-1}}) + g(x_{i_j}) + 1, \quad \text{or} \quad d_{P_n}(x_{i_j}, x_{i_{j+1}}) = g(x_{i_j}) + g(x_{i_{j+1}}) + 1.$$

We consider four subcases, depending on the value of $2g(x_{i_j}) + 1 \pmod{3}$, and on the number p of vertices lying between $x_{i_{j-1}}$ and x_{i_j} , or between x_{i_j} and $x_{i_{j+1}}$, that are not g -dominated. Note that since g is a maximal packing broadcast, we have either (i) $p = 0$, or (ii) $p = 1$ and either $x_{i_j - g(x_{i_j}) - 1}$ or $x_{i_j + g(x_{i_j}) + 1}$ is not g -dominated, or (iii) $p = 2$ and either $x_{i_j - g(x_{i_j}) - 2}$ and $x_{i_j - g(x_{i_j}) - 1}$, or $x_{i_j + g(x_{i_j}) + 1}$ and $x_{i_j + g(x_{i_j}) + 2}$, are not g -dominated.

(a) $2g(x_{i_j}) + 1 \equiv 0 \pmod{3}$.

Let g_0 be the function obtained from g by setting $g_0(x_{i_j - g(x_{i_j})} \dots x_{i_j + g(x_{i_j})}) = (010)^\alpha$, where $\alpha = \frac{2g(x_{i_j})+1}{3}$ (see Figure 9(a)). Since g is maximal, g_0 is also maximal and we have

$$\sigma(g_0) - \sigma(g) = \frac{2g(x_{i_j}) + 1}{3} - g(x_{i_j}) = \frac{1 - g(x_{i_j})}{3} < 0,$$

which contradicts the optimality of g .

(b) $2g(x_{i_j}) + 1 \equiv 1 \pmod{3}$ and $p = 2$, or $2g(x_{i_j}) + 1 \equiv 2 \pmod{3}$ and $1 \leq p \leq 2$.

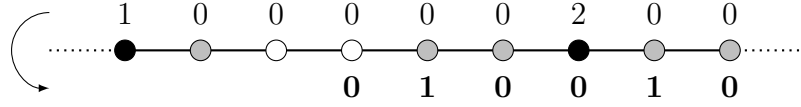
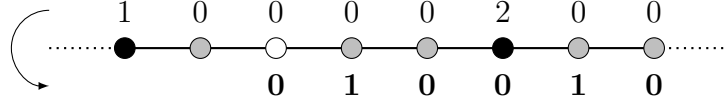
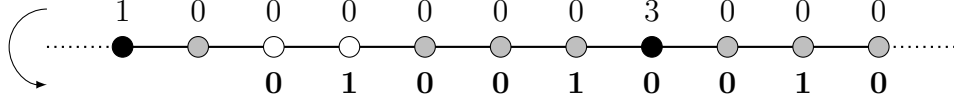
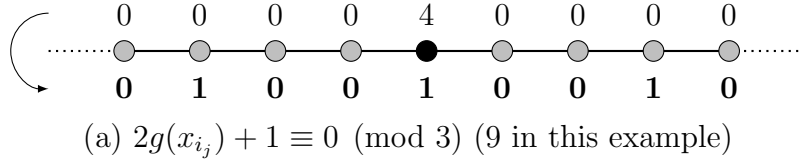
Suppose that $2g(x_{i_j}) + 1 \equiv 1 \pmod{3}$, and $x_{i_j - g(x_{i_j}) - 2}$, $x_{i_j - g(x_{i_j}) - 1}$ are not g -dominated, or that $2g(x_{i_j}) + 1 \equiv 2 \pmod{3}$, and $x_{i_j - g(x_{i_j}) - 1}$ is not g -dominated. (The other cases are similar.)

Let g' be the function obtained from g by setting

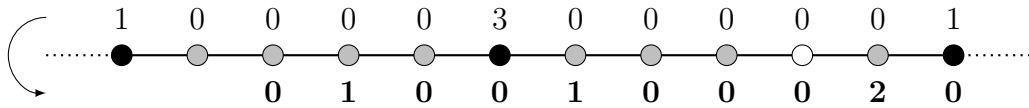
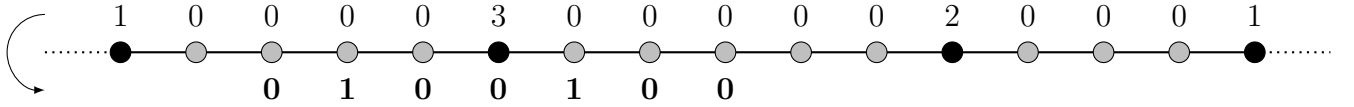
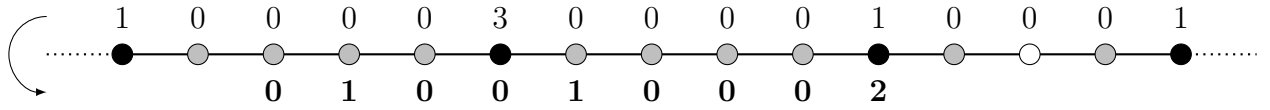
- $g'(x_{i_j - g(x_{i_j}) - 2} \dots x_{i_j + g(x_{i_j})}) = (010)^\alpha$ if $2g(x_{i_j}) + 1 \equiv 1 \pmod{3}$,
- $g'(x_{i_j - g(x_{i_j}) - 1} \dots x_{i_j + g(x_{i_j})}) = (010)^\alpha$ if $2g(x_{i_j}) + 1 \equiv 2 \pmod{3}$,

where $\alpha = \left\lfloor \frac{2g(x_{i_j})+p+1}{3} \right\rfloor$ (see Figure 9(b)). Since g is maximal, g' is also maximal and we have

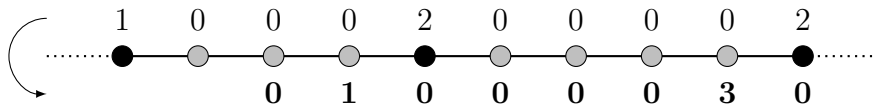
$$\sigma(g') - \sigma(g) = \frac{2g(x_{i_j}) + p + 1}{3} - g(x_{i_j}) = \frac{p + 1 - g(x_{i_j})}{3} \leq 0.$$



(b) $2g(x_{i_j}) + 1 \equiv 1 \pmod{3}$ (7 in this example) and $p = 2$,
or $2g(x_{i_j}) + 1 \equiv 2 \pmod{3}$ (5 in this example) and $p = 1$,
or $2g(x_{i_j}) + 1 \equiv 2 \pmod{3}$ (5 in this example) and $p = 2$



(c) $2g(x_{i_j}) + 1 \equiv 1 \pmod{3}$ (7 in this example) and $p = 0$,
or $2g(x_{i_j}) + 1 \equiv 1 \pmod{3}$ (7 in this example), $p = 0$,
and both $x_{i_j+g(x_{i_j})+1}$ and $x_{i_{j+1}+g(x_{i_{j+1})}+1}$ are g -dominated,
or $2g(x_{i_j}) + 1 \equiv 1 \pmod{3}$ and $p = 1$



(d) $2g(x_{i_j}) + 1 \equiv 2 \pmod{3}$ (5 in this example) and $p = 0$

Figure 9: Packing broadcasts for the proof of Lemma 3.15, Case 2.

The optimality of g then implies either $p = 1$ and $g(x_{i_j}) = 2$, or $p = 2$ and $g(x_{i_j}) = 3$. In each case, g' is also optimal and the subscript of the leftmost g' -broadcast vertex with g' -value at least 2, if any, is strictly greater than the subscript of the leftmost g -broadcast vertex with g -value at least 2, as required.

(c) $2g(x_{i_j}) + 1 \equiv 1 \pmod{3}$ and $0 \leq p \leq 1$.

In that case, $x_{i_j-g(x_{i_j})-2}$ and $x_{i_j+g(x_{i_j})+2}$ are g -dominated, and at most one vertex among $x_{i_j-g(x_{i_j})-1}$ and $x_{i_j+g(x_{i_j})+1}$ is not g -dominated. Let g' be the function obtained from g (see Figure 9(c)) by setting $g'(x_{i_j-g(x_{i_j})} \dots x_{i_j+g(x_{i_j})}) = (010)^\alpha 0$, where $\alpha = \frac{2g(x_{i_j})}{3}$, and

- $g'(x_{i_{j+1}}) = 0$, $g'(x_{i_{j+1}-1}) = g(x_{i_{j+1}}) + 1$, if $x_{i_j+g(x_{i_j})+1}$ is not g -dominated,
- $g'(x_{i_{j+1}}) = g(x_{i_{j+1}}) + 1$, if $i_j = i_{t-1}$ or $x_{i_j+g(x_{i_j})+1}$ is g -dominated, and $x_{i_{j+1}+g(x_{i_{j+1})}+1}$ is not g -dominated,

Note that we necessarily have one of the above cases, since otherwise g' would be a maximal packing broadcast with $\sigma(g') < \sigma(g)$, contradicting the optimality of g . Since g is maximal, g' is also maximal and we have

$$\sigma(g') - \sigma(g) = \frac{2g(x_{i_j})}{3} + 1 - g(x_{i_j}) = \frac{3 - g(x_{i_j})}{3} \leq 0.$$

(Recall that since $2g(x_{i_j}) + 1 \equiv 1 \pmod{3}$, we have $g(x_{i_j}) \geq 3$.) The optimality of g then implies $g(x_{i_j}) = 3$. We then get that g' is also optimal and the subscript of the leftmost g' -broadcast vertex with g' -value at least 2, if any, is strictly greater than the subscript of the leftmost g -broadcast vertex with g -value at least 2, as required.

(d) $2g(x_{i_j}) + 1 \equiv 2 \pmod{3}$ and $p = 0$.

Let g' be the function obtained from g by setting $g'(x_{i_j-g(x_{i_j})} \dots x_{i_j+g(x_{i_j})}) = (010)^\alpha 00$, where $\alpha = \frac{2g(x_{i_j})-1}{3}$, $g'(x_{i_{j+1}}) = 0$, and $g'(x_{i_{j+1}-1}) = g(x_{i_{j+1}}) + 1$ (see Figure 9(d)). Since g is maximal, g' is also maximal and we have

$$\sigma(g') - \sigma(g) = \frac{2g(x_{i_j}) - 1}{3} + 1 - g(x_{i_j}) = \frac{2 - g(x_{i_j})}{3} \leq 0.$$

The optimality of g then implies $g(x_{i_j}) = 2$. Therefore, g' is also optimal and the subscript of the leftmost g' -broadcast vertex with g' -value at least 2, if any, is strictly greater than the subscript of the leftmost g -broadcast vertex with g -value at least 2, as required.

Repeating the same transformation for each vertex with g -value at least 2, we eventually produce a p_b -broadcast g' on P_n all of whose broadcast vertices have g' -value 1, as claimed in the statement of the lemma. This concludes the proof. \square

We are now able to determine the lower broadcast packing number of paths.

Theorem 3.16. *For every integer $n \geq 2$,*

$$p_b(P_n) = \begin{cases} \frac{n}{4} & \text{if } n \equiv 0 \pmod{8}, \\ 2 \lfloor \frac{n}{8} \rfloor + 1 & \text{if } n \equiv 1, 2, 3 \pmod{8}, \\ 2 \lfloor \frac{n}{8} \rfloor + 2 & \text{if } n \equiv 4, 5, 6, 7 \pmod{8}. \end{cases}$$

Proof. Observe first that 10, 010, 1001, 01001, 001001, 0010010 and 00100100 define optimal p_b -broadcasts on P_n for $n = 2, \dots, 8$, respectively, whose costs are the values claimed by the theorem.

Suppose now $n \geq 9$ and let $n = 8q + r$, with $q \geq 1$ and $0 \leq r \leq 7$. According to the value of r , we define the broadcasts f_r , $0 \leq r \leq 7$, on P_n as follows:

$$\begin{aligned} f_0(P_n) &= (00100100)^q, & f_1(P_n) &= (00100100)^q 1, & f_2(P_n) &= (00100100)^q 10, \\ f_3(P_n) &= (00100100)^q 100, & f_4(P_n) &= (00100100)^q 1001, & f_5(P_n) &= (00100100)^q 01001, \\ f_6(P_n) &= (00100100)^q 001001, & \text{and } f_7(P_n) &= (00100100)^q 0010010. \end{aligned}$$

It is not difficult to check that each f_r is indeed a maximal packing broadcast on P_{8q+r} , $8q+r \geq 9$, with cost

$$\sigma(f_r) = \begin{cases} \frac{n}{4} & \text{if } n \equiv 0 \pmod{8}, \\ 2 \lfloor \frac{n}{8} \rfloor + 1 & \text{if } n \equiv 1, 2, 3 \pmod{8}, \\ 2 \lfloor \frac{n}{8} \rfloor + 2 & \text{if } n \equiv 4, 5, 6, 7 \pmod{8}, \end{cases}$$

which gives $p_b(P_n) \leq \sigma(f_r)$, that is, $p_b(P_n)$ is not greater than the value claimed by the theorem.

We now prove the opposite inequality. By Lemma 3.15, we know that there exists a p_b -broadcast all of whose broadcast vertices have broadcast value 1. Let g be such a broadcast. For every k , $0 \leq k \leq q-1$, let

$$\sigma_k = g(x_{8k+1}) + \cdots + g(x_{8k+8}).$$

From the definition of a packing broadcast, we get that the distance between any two consecutive g -broadcast vertices (with g -value 1) is 3, 4 or 5, which implies $2 \leq \sigma_k \leq 3$ for every k , $0 \leq k \leq q-1$. Observe also that x_2 and x_{n-1} must be g -dominated, since otherwise we could set $g(x_1) = 1$ or $g(x_n) = 1$, contradicting the maximality of g , and that $d_{P_n}(x_{i_1}, x_{i_2}) = d_{P_n}(x_{i_{t-1}}, x_{i_t}) = 3$, since otherwise we could increase the value of $g(x_{i_1})$ or $g(x_{i_t})$, contradicting the maximality of g .

We now consider three cases, depending on the value of r , and prove in each case that the cost of g is the value claimed in the statement of the theorem.

1. $r = 0$.

In that case, we have

$$p_b(P_n) = \sigma(g) = \sum_{k=0}^{q-1} \sigma_k \geq 2q = \frac{2n}{8} = \frac{n}{4}.$$

2. $r \in \{1, 2, 3\}$.

If $\sigma_k = 3$ for some k , $0 \leq k \leq q-1$, then

$$p_b(P_n) = \sigma(g) \geq \sum_{k=0}^{q-1} \sigma_k \geq 2(q-1) + 3 = 2q + 1 = 2 \lfloor \frac{n}{8} \rfloor + 1,$$

and we are done.

Suppose now that $\sigma_k = 2$ for every k , $0 \leq k \leq q-1$, which implies $p_b(P_n) \geq 2 \lfloor \frac{n}{8} \rfloor$. Suppose, contrary to the statement of the theorem, that $p_b(P_n) = 2 \lfloor \frac{n}{8} \rfloor$. This implies $g(x_{n-r+1}) = \cdots = g(x_n) = 0$. If $r = 3$, then we have a contradiction since x_{n-1} must be g -dominated.

We thus have $r \in \{1, 2\}$. Since x_{n-1} must be g -dominated and $d_{P_n}(x_{i_{t-1}}, x_{i_t}) = 3$, we necessarily have $\sigma_{q-1} \in \{00001001, 00010010\}$ if $r = 1$, and $\sigma_{q-1} = 00001001$ if $r = 2$. Since g is maximal, we cannot have three consecutive vertices that are not g -dominated, which gives $\sigma_j \in \{00001001, 00010010\}$ if $r = 1$, and $\sigma_j = 00001001$ if $r = 2$, for every j , $0 \leq j \leq q-1$, a contradiction since, in each case, this would imply that x_2 is not g -dominated.

Hence we have $p_b(P_n) = 2 \lfloor \frac{n}{8} \rfloor + 1$, as required.

3. $r \in \{4, 5, 6, 7\}$.

Note first that we necessarily have $g(x_{n-r+1}) + \cdots + g(x_n) \geq 1$. Hence, if $\sigma_k = 3$ for some k , $0 \leq k \leq q-1$, then

$$p_b(P_n) = \sigma(g) \geq \sum_{k=0}^{q-1} \sigma_k + 1 \geq 2(q-1) + 3 + 1 = 2q + 2 = 2 \left\lfloor \frac{n}{8} \right\rfloor + 2,$$

and we are done.

Suppose now that $\sigma_k = 2$ for every k , $0 \leq k \leq q-1$, and, contrary to the statement of the theorem, that $p_b(P_n) \leq 2 \left\lfloor \frac{n}{8} \right\rfloor + 1$, which implies $g(x_{n-r+1}) + \cdots + g(x_n) = 1$ since x_{n-1} must be g -dominated. If $r = 6$ or $r = 7$, then we have a contradiction since we must have $d_{P_n}(x_{i_{t-1}}, x_{i_t}) = 3$.

We thus have $r \in \{4, 5\}$. Again, since x_{n-1} must be g -dominated and $d_{P_n}(x_{i_{t-1}}, x_{i_t}) = 3$, we necessarily have

$$\sigma_{q-1} \in \{00100001, 00001001, 01000010, 00100010, 00010010\}$$

if $r = 4$, and

$$\sigma_{q-1} \in \{00100001, 00001001\}$$

if $r = 5$. Now, since g is maximal, every g -broadcast vertex must be at distance 3 from another g -broadcast vertex, and thus we get

$$\sigma_j \in \{00100001, 00001001, 01000010, 00100010, 00010010\}$$

for every j , $0 \leq j \leq q-1$. We then get a contradiction since x_2 must be g -dominated and we must have $d(x_{i_1}, x_{i_2}) = 3$.

Hence, in this case also, we have $p_b(P_n) = 2 \left\lfloor \frac{n}{8} \right\rfloor + 2$, as required.

This completes the proof. □

Using Theorem 3.16, we can also prove a similar result for cycles.

Theorem 3.17. *For every integer $n \geq 3$,*

$$p_b(C_n) = \begin{cases} \frac{n}{4} & \text{if } n \equiv 0 \pmod{8}, \\ 2 \left\lfloor \frac{n}{8} \right\rfloor + 1 & \text{if } n \equiv 1, 2, 3 \pmod{8}, \\ 2 \left\lfloor \frac{n}{8} \right\rfloor + 2 & \text{if } n \equiv 4, 5, 6, 7 \pmod{8}. \end{cases}$$

Proof. Observe first that the broadcasts f_0, \dots, f_7 , defined in the proof of Theorem 3.16, are also maximal packing broadcast on C_n , with $n \geq 3$, $n = 8q + r$ and $0 \leq r \leq 7$, which gives $p_b(C_n) \leq \sigma(f_r)$, that is, $p_b(C_n)$ is not greater than the value claimed by the theorem.

We now prove the opposite inequality. Observe first that 010, 2000, 20000 and 001001 are optimal solutions for C_3 , C_4 , C_5 and C_6 , respectively, and that their cost is exactly the value claimed by the theorem. We can thus assume $n \geq 7$. Let f be a p_b -broadcast on C_n , $n \geq 7$.

We first claim that we necessarily have $|V_f^+| \geq 2$. Indeed, suppose to the contrary that $|V_f^+| = 1$ and that $V_f^+ = \{x_0\}$, without loss of generality. Since f is maximal, we necessarily have $\sigma(f) = f(x_0) = \left\lfloor \frac{n-1}{2} \right\rfloor$, and thus $p_b(C_n) = \left\lfloor \frac{n-1}{2} \right\rfloor$, which contradicts the inequality we previously established when $n \geq 7$.

We thus have $|V_f^+| \geq 2$. Let $V_f^+ = \{x_{i_0}, \dots, x_{i_{t-1}}\}$, $t \geq 2$. Since f is maximal, we necessarily have

$$f(x_{i_j}) + f(x_{i_{j+1}}) + 1 \leq d_{C_n}(x_{i_j}, x_{i_{j+1}}) \leq f(x_{i_j}) + f(x_{i_{j+1}}) + 3$$

for every j , $0 \leq j \leq t-1$. Moreover, if $d_{C_n}(x_{i_j}, x_{i_{j+1}}) \geq f(x_{i_j}) + f(x_{i_{j+1}}) + 2$, then we necessarily have $d_{C_n}(x_{i_{j+1}}, x_{i_{j+2}}) = f(x_{i_{j+1}}) + f(x_{i_{j+2}}) + 1$ (we may have $i_j = i_{j+2}$), since otherwise we could increase the value of $f(x_{i_{j+1}})$ by 1.

We consider the two following cases.

1. If all vertices in C_n are f -dominated, that is, $d_{C_n}(x_{i_j}, x_{i_{j+1}}) = f(x_{i_j}) + f(x_{i_{j+1}}) + 1$ for every j , $0 \leq j \leq t-1$, then, to avoid confusion, we let $P_n = y_0 y_1 \dots y_{n-1}$ with $y_0 = x_{i_0+f(x_{i_0})+1}$. Let now g be the function defined by $g(y_j) = f(x_{j+i_0+f(x_{i_0})+1})$ for every j , $0 \leq j \leq n-1$. Clearly, both y_0 and y_{n-1} are g -dominated. Since f was a maximal packing broadcast on C_n , we get that g is also a maximal packing broadcast on P_n , which gives $p_b(P_n) \leq p_b(C_n)$.
2. If C_n contains at least one vertex which is not f -dominated, then we can assume, without loss of generality, that $x_{i_1+f(x_{i_1})+1}$ is not f -dominated, which implies $d_{C_n}(x_{i_0}, x_{i_1}) = f(x_{i_0}) + f(x_{i_1}) + 1$ and $d_{C_n}(x_{i_2}, x_{i_3}) = f(x_{i_2}) + f(x_{i_3}) + 1$ (we may have $x_{i_0} = x_{i_2}$ and $x_{i_1} = x_{i_3}$). Again, to avoid confusion, we let $P_n = y_0 y_1 \dots y_{n-1}$ with $y_0 = x_{i_1+f(x_{i_1})+2}$. Let now g be the function defined by $g(y_j) = f(x_{j+i_1+f(x_{i_1})+2})$ for every j , $0 \leq j \leq n-1$. Since $d_{C_n}(x_{i_0}, x_{i_1}) = f(x_{i_0}) + f(x_{i_1}) + 1$ and $d_{C_n}(x_{i_2}, x_{i_3}) = f(x_{i_2}) + f(x_{i_3}) + 1$, the values of both the leftmost and the rightmost g -broadcast vertex in P_n cannot be increased. Since f was a maximal packing broadcast on C_n , we thus get that g is also a maximal packing broadcast on P_n , which gives $p_b(P_n) \leq p_b(C_n)$.

We thus have $p_b(P_n) \leq p_b(C_n)$ in all cases and the result follows. \square

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