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# A MULTILEVEL SCHWARZ PRECONDITIONER BASED ON A HIERARCHY OF ROBUST COARSE SPACES\*

HUSSAM AL DAAS<sup>†</sup>, LAURA GRIGORI<sup>‡</sup>, PIERRE JOLIVET<sup>§</sup>, AND PIERRE-HENRI  
TOURNIER<sup>¶</sup>

**Abstract.** In this paper we present a multilevel preconditioner based on overlapping Schwarz methods for symmetric positive definite (SPD) matrices. Robust two-level Schwarz preconditioners exist in the literature to guarantee fast convergence of Krylov methods. As long as the dimension of the coarse space is reasonable, that is, exact solvers can be used efficiently, two-level methods scale well on parallel architectures. However, the factorization of the coarse space matrix may become costly at scale. An alternative is then to use an iterative method on the second level, combined with an algebraic preconditioner, such as a one-level additive Schwarz preconditioner. Nevertheless, the condition number of the resulting preconditioned coarse space matrix may still be large. One of the difficulties of using more advanced methods, like algebraic multigrid or even two-level overlapping Schwarz methods, to solve the coarse problem is that the matrix does not arise from a partial differential equation (PDE) anymore. We introduce in this paper a robust multilevel additive Schwarz preconditioner where at each level the condition number is bounded, ensuring a fast convergence for each nested solver. Furthermore, our construction does not require any additional information than for building a two-level method, and may thus be seen as an algebraic extension.

**Key words.** domain decomposition, multilevel, elliptic problems, subspace correction

**AMS subject classifications.** 65F08, 65F10, 65N55

**1. Introduction.** We consider the solution of a linear system of equations

$$(1.1) \quad Ax = b,$$

where  $A \in \mathbb{R}^{n \times n}$  is a symmetric positive definite (SPD) matrix,  $b \in \mathbb{R}^n$  is the right-hand side, and  $x \in \mathbb{R}^n$  is the vector of unknowns. To enhance convergence, it is common to solve the preconditioned system

$$M^{-1}Ax = M^{-1}b.$$

Standard domain decomposition preconditioners such as block Jacobi, additive Schwarz, and restricted additive Schwarz methods are widely used [31, 9, 8]. In a parallel framework, such preconditioners have the advantage of relatively low communication costs. However, their role in lowering the condition number of the system typically deteriorates when the number of subdomains increases. Multilevel approaches have shown a large impact on enhancing the convergence of Krylov methods [32, 12, 7, 25, 20, 10, 21, 1, 15, 23]. In multigrid and domain decomposition communities, multilevel methods have proven their capacity of scaling up to large numbers of processors and tackling ill-conditioned systems [35, 4, 19]. While some preconditioners are purely algebraic [7, 20, 10, 26, 29, 16, 1], several multilevel methods are based on hierarchical meshing in both multigrid and domain decomposition communities [33, 9, 25, 15, 23]. Mesh coarsening depends on the geometry of the problem. One

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39 has to be careful when choosing a hierarchical structure since it can have a signifi-  
 40 cant impact on the iteration count [23, 25]. In [23], the authors propose a multilevel  
 41 Schwarz domain decomposition solver for the elasticity problem. Based on a heuristic  
 42 approach and following the maximum independent set method [2], they coarsen the  
 43 fine mesh while preserving the boundary in order to obtain a two-level method. This  
 44 strategy is repeated recursively to build several levels. However, they do not provide a  
 45 bound on the condition number of the preconditioned matrix of the multilevel method.  
 46 Multilevel domain decomposition methods are mostly based on non-overlapping ap-  
 47 proaches [33, 9, 25, 23, 35, 4]. Two-level overlapping domain decomposition methods  
 48 are well studied and provide robust convergence estimates [32, 12, 5]. However, ex-  
 49 tending such a construction to more than two levels while preserving robustness is not  
 50 straightforward. In [6], the authors propose an algebraic multilevel additive Schwarz  
 51 method. Their approach is inspired by algebraic multigrid strategies. One drawback  
 52 of it is that it is sensitive to the number of subdomains. In [15], the authors suggest  
 53 applying the two-level Generalized Dryja–Smith–Widlund preconditioner recursively  
 54 to build a multilevel method. In this case, the condition number bound of the two-  
 55 level approach depends on the width of the overlap, the diameter of discretization  
 56 elements, and the diameter of the subdomains. They focus on the preconditioner for  
 57 the three-level case. One drawback of their approach is that the three-level precon-  
 58 ditioner requires more iterations than the two-level variant. In this paper, the only  
 59 information from the PDE needed for the construction of the preconditioner consists  
 60 of the local Neumann matrices at the fine level. These matrices correspond to the  
 61 integration of the bilinear form in the weak formulation of the studied PDE on the  
 62 subdomain-decomposed input mesh. No further information is necessary: except on  
 63 the fine level, our method is algebraic and does not depend on any coarsened mesh or  
 64 auxiliary discretized operator.

65 Our preconditioner is based on a hierarchy of coarse spaces and is defined as fol-  
 66 lowing. At the first level, the set of unknowns is partitioned into  $N_1$  subdomains and  
 67 each subdomain has an associated matrix  $A_{1,j} = R_{1,j} A R_{1,j}^\top$  obtained by using appro-  
 68 priate restriction and prolongation operators  $R_{1,j}$  and  $R_{1,j}^\top$  respectively, defined in the  
 69 following section. The preconditioner is formed as an additive Schwarz preconditioner  
 70 coupled with an additive coarse space correction, defined as,

$$71 \quad M^{-1} = M_1^{-1} = V_1 A_2^{-1} V_1^\top + \sum_{j=1}^{N_1} R_{1,j}^\top A_{1,j}^{-1} R_{1,j},$$

72 where  $V_1$  is a tall-and-skinny matrix spanning a coarse space obtained by solving for  
 73 each subdomain  $j = 1$  to  $N_1$  a generalized eigenvalue problem involving the matrix  
 74  $A_{1,j}$  and the Neumann matrix associated with subdomain  $j$ . The coarse space matrix  
 75 is  $A_2 = V_1^\top A V_1$ . This is equivalent to the GenEO preconditioner, and is described  
 76 in detail in [32] and recalled briefly in section 2. The dimension of the coarse space  
 77 is proportional to the number of subdomains  $N_1$ . When it increases, factorizing  $A_2$   
 78 by using a direct method becomes prohibitive, and hence the application of  $A_2^{-1}$  to a  
 79 vector should also be performed through an iterative method.

80 Our multilevel approach defines a hierarchy of coarse spaces  $V_i$  and coarse space  
 81 matrices  $A_i$  for  $i = 2$  to any depth  $L + 1$ , and defines a preconditioner  $M_i^{-1}$  such that  
 82 the condition number of  $M_i^{-1} A_i$  is bounded. The depth  $L + 1$  is chosen such that the  
 83 coarse space matrix  $A_{L+1}$  can be factorized efficiently by using a direct method. At  
 84 each level  $i$ , the graph of the coarse space matrix  $A_i$  is partitioned into  $N_i$  subdomains,  
 85 and each subdomain  $j$  is associated with a local matrix  $A_{i,j} = R_{i,j} A_i R_{i,j}^\top$  obtained by

86 using appropriate restriction and prolongation operators  $R_{i,j}$  and  $R_{i,j}^\top$ , respectively.  
 87 The preconditioner at level  $i$  is defined as,

$$88 \quad M_i^{-1} = V_i A_{i+1}^{-1} V_i^\top + \sum_{j=1}^{N_i} R_{i,j}^\top A_{i,j}^{-1} R_{i,j},$$

89 where the coarse space matrix is  $A_{i+1} = V_i^\top A_i V_i$ .

90 One of the main contributions of the paper concerns the construction of the  
 91 hierarchy of coarse spaces  $V_i$  for levels  $i$  going from 2 to  $L$ , that are built algebraically  
 92 from the coarse space of the previous level  $V_{i-1}$ . This construction is based on the  
 93 definition of local symmetric positive semi-definite (SPSD) matrices associated with  
 94 each subdomain  $j$  at each level  $i$  that we introduce in this paper. These matrices are  
 95 obtained by using the local SPSP matrices of the previous level  $i-1$  and the previous  
 96 coarse space  $V_{i-1}$ . They are then involved, with the local matrices  $A_{i,j}$ , in concurrent  
 97 generalized eigenvalue problems solved for each subdomain  $j$  that allows to compute  
 98 the local eigenvectors contributing to the coarse space  $V_i$ .

99 We show in [Theorem 5.3, section 5](#), that the condition number of  $M_i^{-1} A_i$  is  
 100 bounded and depends on the maximum number of subdomains at the first level that  
 101 share an unknown, the number of distinct colors required to color the graph of  $A_i$  so  
 102 that  $\{span\{R_{i,j}^\top\}\}_{1 \leq j \leq N_i}$  of the same color are mutually  $A_i$ -orthogonal, and a user  
 103 defined tolerance  $\tau$ . It is thus independent of the number of subdomains  $N_i$ .

104 The main contribution of this paper is based on the combination of two previous  
 105 works on two-level additive Schwarz methods [\[3, 32\]](#). The coarse space proposed in  
 106 [\[32\]](#) guarantees an upper bound on the condition number that can be prescribed by  
 107 the user. The SPSP splitting in the context of domain decomposition presented in  
 108 [\[3\]](#) provides an algebraic view for the construction of coarse spaces. The combination  
 109 of these two works leads to a robust multilevel additive Schwarz method. Here,  
 110 robustness refers to the fact that at each level, an upper bound on the condition  
 111 number of the associated matrix can be prescribed by the user a priori. The rest  
 112 of the paper is organized as follows. In the next section, we present the notations  
 113 used throughout the paper. In [section 2](#), we present a brief review of the theory of  
 114 one- and two-level additive Schwarz methods. We extend in [section 3](#) the class of  
 115 SPSP splitting matrices presented in [\[3\]](#) in order to make it suitable for multilevel  
 116 methods. Afterwards, we define the coarse space at level  $i$  based on the extended  
 117 class of local SPSP splitting matrices associated with this level. [Section 4](#) describes  
 118 the partitioning of the domain at level  $i+1$  from the partitioning at level  $i$ . In  
 119 [Section 5](#), we explain the computation of the local SPSP matrices associated with each  
 120 subdomain at level  $i+1$ . We compute them using those associated with subdomains  
 121 at level  $i$ . [Section 6](#) presents numerical experiments on highly challenging diffusion  
 122 and linear elasticity problems in two- and three-dimensional problems. We illustrate  
 123 the theoretical robustness and practical usage of our proposed method by performing  
 124 strong scalability tests up to 8,192 processes.

125 **Context and notation.** By convention, the finest level, on which [\(1.1\)](#) is  
 126 defined, is the first level. A subscript index is used in order to specify which level  
 127 an entity is defined on. In the case where additional subscripts are used, the first  
 128 subscript always denotes the level. For the sake of clarity, we omit the subscript cor-  
 129 responding to level 1 when it is clear from context, e.g., matrix  $A$ . Furthermore, the  
 130 subscripts  $i$  and  $j$  always refer to a specific level  $i$  and its subdomain  $j$ , respectively.  
 131 The number of levels is  $L+1$ . Let  $A_i \in \mathbb{R}^{n_i \times n_i}$  denote symmetric positive definite

132 matrices, each corresponding to level  $i = 1, \dots, L+1$ . We suppose that a direct solver  
 133 can be used at level  $L+1$  to compute an exact factorization of  $A_{L+1}$ .

134 Let  $B \in \mathbb{R}^{p \times q}$  be a matrix. Let  $P \subset \llbracket 1; p \rrbracket$  and  $Q \subset \llbracket 1; q \rrbracket$  be two sets of  
 135 indices. The concatenation of  $P$  and  $Q$  is represented by  $[P, Q]$ . We note that the  
 136 order of the concatenation is important.  $B(P, :)$  is the submatrix of  $B$  formed by  
 137 the rows whose indices belong to  $P$ .  $B(:, Q)$  is the submatrix of  $B$  formed by the  
 138 columns whose indices belong to  $Q$ .  $B(P, Q) = (B(P, :))(:, Q)$ . The identity matrix  
 139 of size  $p$  is denoted  $I_p$ . We suppose that the graph of  $A_i$  is partitioned into  $N_i$  non-  
 140 overlapping subdomains, where  $N_i \ll n_i$  and  $N_{i+1} \leq N_i$  for  $i = 1, \dots, L$ . We note that  
 141 partitioning at level 1 can be performed by using a graph partitioning library such as  
 142 ParMETIS [22] or PT-SCOTCH [11]. Partitioning at greater levels will be described  
 143 later in section 4. In the following, we define for each level  $i = 1, \dots, L$  notations  
 144 for subsets and restriction operators that are associated with the partitioning. Let  
 145  $\Omega_i = \llbracket 1; n_i \rrbracket$  be the set of unknowns at level  $i$  and let  $\Omega_{i,j,I}$  for  $j = 1, \dots, N_i$  be the  
 146 subset of  $\Omega_i$  that represents the unknowns in subdomain  $j$ . We refer to  $\Omega_{i,j,I}$  as the  
 147 *interior unknowns* of subdomain  $j$ . Let  $\Gamma_{i,j}$  for  $j = 1, \dots, N_i$  be the subset of  $\Omega_i$  that  
 148 represents the neighbor unknowns of subdomain  $j$ , i.e., the unknowns at distance 1  
 149 from subdomain  $j$  through the graph of  $A_i$ . We refer to  $\Gamma_{i,j}$  as the *overlapping*  
 150 *unknowns* of subdomain  $j$ . We denote  $\Omega_{i,j} = [\Omega_{i,j,I}, \Gamma_{i,j}]$ , for  $j = 1, \dots, N_i$ , the  
 151 concatenation of interior and overlapping unknowns of subdomain  $j$ . We denote  
 152  $\Delta_{i,j}$ , for  $j = 1, \dots, N_i$ , the complementary of  $\Omega_{i,j}$  in  $\Omega_i$ , i.e.,  $\Delta_{i,j} = \Omega_i \setminus \Omega_{i,j}$ . In  
 153 Figure 1.1, a triangular mesh is used to discretize a square domain. The set of  
 154 nodes of the mesh is partitioned into 16 disjoint subsets  $\Omega_{1,j,I}$ , which represent a  
 155 non-overlapping decomposition, for  $j = 1, \dots, 16$  (left). On the left, a matrix  $A_1$   
 156 whose connectivity graph corresponds to the mesh is illustrated. The submatrix  
 157  $A_1(\Omega_{1,j,I}, \Omega_{1,j,I})$  is associated with the non-overlapping subdomain  $j$ . Each submatrix  
 158  $A_1(\Omega_{1,j,I}, \Omega_{1,j,I})$  is colored with a distinct color. The same color is used to color the  
 159 region that contains the nodes in the non-overlapping subdomain  $\Omega_{1,j,I}$ . Note that  
 160 if two subdomains  $j_1, j_2$  are neighbors, the submatrix  $A_1(\Omega_{1,j_1,I}, \Omega_{1,j_2,I})$  has nonzero  
 161 elements. For  $j = 1, \dots, N_i$ , we denote by  $n_{i,j,I}$ ,  $\gamma_{i,j}$  and  $n_{i,j}$  the cardinality of  $\Omega_{i,j,I}$ ,  
 162  $\Gamma_{i,j}$  and  $\Omega_{i,j}$  respectively.  
 163 Let  $R_{i,j,I} \in \mathbb{R}^{n_{i,j,I} \times n_i}$  be defined as  $R_{i,j,I} = I_{n_i}(\Omega_{i,j,I}, :)$ .  
 164 Let  $R_{i,j,\Gamma} \in \mathbb{R}^{\gamma_{i,j} \times n_i}$  be defined as  $R_{i,j,\Gamma} = I_{n_i}(\Gamma_{i,j}, :)$ .  
 165 Let  $R_{i,j} \in \mathbb{R}^{n_{i,j} \times n_i}$  be defined as  $R_{i,j} = I_{n_i}(\Omega_{i,j}, :)$ .  
 166 Let  $R_{i,j,\Delta} \in \mathbb{R}^{(n_i - n_{i,j}) \times n_i}$  be defined as  $R_{i,j,\Delta} = I_{n_i}(\Delta_{i,j}, :)$ .  
 167 Let  $\mathcal{P}_{i,j} = I_{n_i}([\Omega_{i,j,I}, \Gamma_{i,j}, \Delta_{i,j}], :) \in \mathbb{R}^{n_i \times n_i}$ , be a permutation matrix associated  
 168 with the subdomain  $j$ , for  $j = 1, \dots, N_i$ . The matrix of the overlapping subdomain  $j$ ,  
 169  $R_{i,j} A_i R_{i,j}^\top$ , is denoted  $A_{i,j}$ . We denote  $D_{i,j} \in \mathbb{R}^{n_{i,j} \times n_{i,j}}$ ,  $j = 1, \dots, N_i$ , any set of  
 170 non-negative diagonal matrices such that

$$171 \quad I_{n_i} = \sum_{j=1}^{N_i} R_{i,j}^\top D_{i,j} R_{i,j}.$$

172 We refer to  $\{D_{i,j}\}_{1 \leq j \leq N_i}$  as the algebraic partition of unity. Let  $V_i \in \mathbb{R}^{n_i \times n_{i+1}}$  be  
 173 a tall-and-skinny matrix of full rank. We denote  $\mathcal{S}_i$  the subspace spanned by the  
 174 columns of  $V_i$ . This subspace will stand for the coarse space associated with level  $i$ .  
 175 By convention, we refer to  $\mathcal{S}_i$  as subdomain 0 at level  $i$ . Thus, we have  $n_{i,0} = n_{i+1}$ .

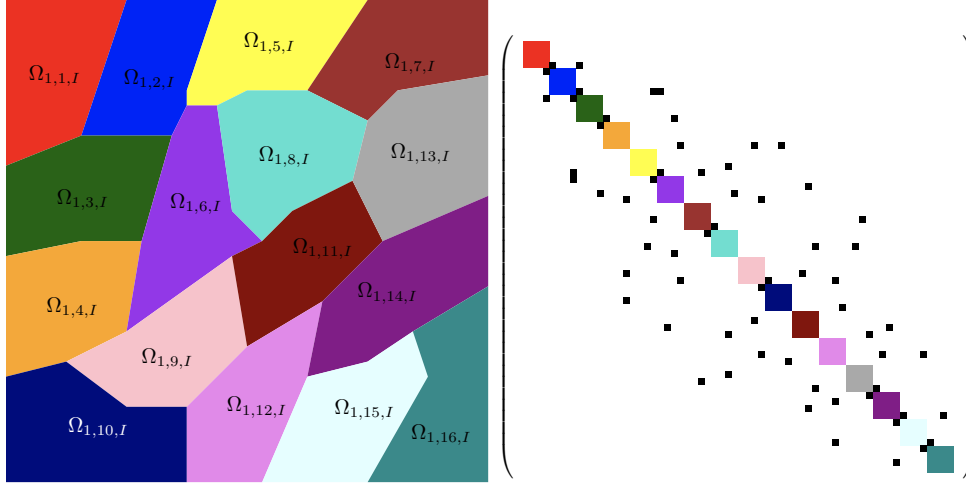


FIG. 1.1. *Left: a triangular mesh is used to discretize the unit square. The set of nodes of the mesh is partitioned into 16 disjoint subsets, non-overlapping subdomains,  $\Omega_{1,j,I}$  for  $j = 1, \dots, 16$ . Right: Illustration of the matrix  $A_1$  whose connectivity graph corresponds to the mesh on the left. The diagonal block  $j$  of  $A_1$  corresponds to the non-overlapping subdomain  $\Omega_{1,j,I}$ . Each submatrix  $A_1(\Omega_{1,j,I}, \Omega_{1,j,I})$  is colored with a distinct color. The same color is used to color the region of the square that contains nodes in  $\Omega_{1,j,I}$ .*

176 The interpolation operator at level  $i$  is defined as:

$$\begin{aligned}
 & \mathcal{R}_{i,2}: \prod_{j=0}^{N_i} \mathbb{R}^{n_{i,j}} \rightarrow \mathbb{R}^{n_i} \\
 & (u_j)_{0 \leq j \leq N_i} \mapsto \sum_{j=0}^{N_i} R_{i,j}^\top u_j.
 \end{aligned}
 \tag{1.2}$$

178  
 179 Finally, we denote  $\mathcal{V}_{i,j}$  the set of neighboring subdomains of each subdomain  $j$  at  
 180 level  $i$  for  $(i, j) \in \llbracket 1; L \rrbracket \times \llbracket 1; N_i \rrbracket$ .

$$\mathcal{V}_{i,j} = \{k \in \llbracket 1; N_i \rrbracket : \Omega_{i,j} \cap \Omega_{i,k} \neq \emptyset\}.$$

182 As previously mentioned, partitioning at level 1 can be performed by graph parti-  
 183 tioning libraries such as ParMETIS [22] or PT-SCOTCH [11]. Partitioning at further  
 184 levels will be defined later: the sets  $\Omega_{i,j,I}$ ,  $\Omega_{i,j,\Gamma}$ ,  $\Omega_{i,j}$ , and  $\Delta_{i,j}$  for  $i > 1$  are defined  
 185 in subsection 4.2. The coarse spaces  $\mathcal{S}_i$  as well as the projection and prolongation  
 186 operators  $V_i^\top$  and  $V_i$  are defined in subsection 3.2. We suppose that the connectivity  
 187 graph between the subdomains on each level is sparse. This assumption is not true in  
 188 general, however, it is valid in structures based on locally constructed coarse spaces  
 189 in domain decomposition as we show in this paper, see [18, Section 4.1 p.81] for the  
 190 case of two levels.

191 **2. Background.** In this section, we review briefly several theoretical results  
 192 related to additive Schwarz preconditioners. We introduce them for the sake of com-  
 193 pleteness.

194 LEMMA 2.1 (fictitious subspace lemma). *Let  $A \in \mathbb{R}^{n_A \times n_A}$ ,  $B \in \mathbb{R}^{n_B \times n_B}$  be two*

195 symmetric positive definite matrices. Let  $\mathcal{R}$  be an operator defined as

$$\begin{aligned} 196 \quad & \mathcal{R}: \mathbb{R}^{n_B} \rightarrow \mathbb{R}^{n_A} \\ 197 \quad & v \mapsto \mathcal{R}v, \end{aligned}$$

198 and let  $\mathcal{R}^\top$  be its transpose. Suppose that the following conditions hold:

- 199 1. The operator  $\mathcal{R}$  is surjective.
- 200 2. There exists  $c_u > 0$  such that

$$201 \quad (\mathcal{R}v)^\top A (\mathcal{R}v) \leq c_u v^\top B v, \quad \forall v \in \mathbb{R}^{n_B}.$$

- 202 3. There exists  $c_l > 0$  such that for all  $v_{n_A} \in \mathbb{R}^{n_A}, \exists v_{n_B} \in \mathbb{R}^{n_B} | v_{n_A} = \mathcal{R}v_{n_B}$
- 203 and

$$204 \quad c_l v_{n_B}^\top B v_{n_B} \leq (\mathcal{R}v_{n_B})^\top A (\mathcal{R}v_{n_B}) = v_{n_A}^\top A v_{n_A}.$$

205 Then, the spectrum of the operator  $\mathcal{R}B^{-1}\mathcal{R}^\top A$  is contained in the segment  $[c_l, c_u]$ .

206 *Proof.* We refer the reader to [12, Lemma 7.4 p.164] or [28, 27, 13] for a detailed  
207 proof.  $\square$

208 LEMMA 2.2. The operator  $\mathcal{R}_{i,2}$  as defined in (1.2) is surjective.

209 *Proof.* The proof follows from the definition of  $\mathcal{R}_{i,2}$  (1.2).  $\square$

210 LEMMA 2.3. Let  $k_{i,c}$  for  $i = 1, \dots, L$  be the minimum number of distinct colors  
211 so that  $\{\text{span}\{R_{i,j}^\top\}\}_{1 \leq j \leq N_i}$  of the same color are mutually  $A_i$ -orthogonal. Then, we  
212 have

$$\begin{aligned} 214 \quad & (\mathcal{R}_{i,2}u_{\mathcal{B}_i})^\top A_i (\mathcal{R}_{i,2}u_{\mathcal{B}_i}) \\ 215 \quad & \leq (k_{i,c} + 1) \sum_{j=0}^{N_i} u_j^\top (R_{i,j} A_i R_{i,j}^\top) u_j, \quad \forall u_{\mathcal{B}_i} = (u_j)_{0 \leq j \leq N_i} \in \prod_{j=0}^{N_i} \mathbb{R}^{n_{i,j}}. \end{aligned}$$

217 *Proof.* We refer the reader to [9, Theorem 12 p.93] for a detailed proof.  $\square$

218 We note that at level  $i$ , the number  $k_{i,c}$  is smaller than the maximum number of  
219 neighbors over the set of subdomains  $\llbracket 1; N_i \rrbracket$

$$220 \quad k_{i,c} \leq \max_{1 \leq j \leq N_i} \#\mathcal{V}_{i,j}.$$

221 Due to the sparse structure of the connectivity graph between the subdomains at  
222 level  $i$ , the maximum number of neighbors over the set of subdomains  $\llbracket 1; N_i \rrbracket$  is  
223 independent of the number of subdomains  $N_i$ . Then, so is  $k_{i,c}$ .

224 LEMMA 2.4. Let  $u_{A_i} \in \mathbb{R}^{n_{A_i}}$  and  $u_{\mathcal{B}_i} = \{u_j\}_{0 \leq j \leq N_i} \in \prod_{j=0}^{N_i} \mathbb{R}^{n_{i,j}}$  such that  $u_{A_i} =$   
225  $\mathcal{R}_{i,2}u_{\mathcal{B}_i}$ . The additive Schwarz operator without any other restriction on the coarse  
226 space  $\mathcal{S}_i$  verifies the following inequality

$$227 \quad \sum_{j=0}^{N_i} u_j^\top (R_{i,j} A_i R_{i,j}^\top) u_j \leq 2u_{A_i}^\top A_i u_{A_i} + (2k_{i,c} + 1) \sum_{j=1}^{N_i} u_j^\top R_{i,j} A_i R_{i,j}^\top u_j,$$

228 where  $k_{i,c}$  is defined in Lemma 2.3.

229 *Proof.* We refer the reader to [12, Lemma 7.12, p. 175] to view the proof in  
 230 detail.  $\square$

231 **LEMMA 2.5.** *Let  $A, B \in \mathbb{R}^{m \times m}$  be two symmetric positive semi-definite matrices.*  
 232 *Let  $\ker(A)$ ,  $\text{range}(A)$  denote the null space and the range of  $A$  respectively.*  
 233 *Let  $\ker(B)$  denote the kernel of  $B$ . Let  $L = \ker(A) \cap \ker(B)$ , we denote  $L^{\perp_{\ker(A)}}$*   
 234 *the orthogonal complementary of  $L$  in  $\ker(A)$ . Let  $P_0$  be an orthogonal projection*  
 235 *on  $\text{range}(A)$ . Let  $\tau$  be a positive real number. Consider the generalized eigenvalue*  
 236 *problem,*

$$237 \quad P_0 B P_0 u_k = \lambda_k A u_k,$$

$$238 \quad (u_k, \lambda_k) \in \text{range}(A) \times \mathbb{R}.$$

239 Let  $P_\tau$  be an orthogonal projection on the subspace

$$240 \quad Z = L^{\perp_{\ker(A)}} \oplus \text{span}\{u_k | \lambda_k > \tau\},$$

241 then, the following inequality holds:

$$242 \quad (2.1) \quad (u - P_\tau u)^\top B (u - P_\tau u) \leq \tau u^\top A u, \quad \forall u \in \mathbb{R}^m.$$

243 Furthermore,  $Z$  is the subspace of smallest dimension such that (2.1) holds.

244 *Proof.* We refer the reader to [3, Lemma 2.4] for a detailed proof.  $\square$

245 **2.1. GenEO coarse space.** In [32, 12] the authors present the GenEO coarse  
 246 space which relies on defining appropriate symmetric positive semi-definite (SPSD)  
 247 matrices  $\tilde{A}_j \in \mathbb{R}^{n \times n}$  for  $j = 1, \dots, N$ . These are the unassembled Neumann matrices,  
 248 corresponding to the integration on each subdomain of the operator defined in the  
 249 variational form of the PDE. These matrices are local, i.e.,  $R_{j,\Delta} \tilde{A}_j = 0$ . Furthermore,  
 250 they verify the relations

$$251 \quad u^\top \tilde{A}_j u \leq u^\top A u, \quad \forall u \in \mathbb{R}^n,$$

$$252 \quad u^\top \sum_{j=1}^N \tilde{A}_j u \leq k_{\text{GenEO}} u^\top A u, \quad \forall u \in \mathbb{R}^n,$$

253 where  $k_{\text{GenEO}} \leq N$  is the maximum number of subdomains that share an unknown.

254 **2.2. Local SPSP splitting of an SPD matrix.** In [3], the authors present  
 255 the local SPSP splitting of an SPD matrix. Given the permutation matrix  $\mathcal{P}_j$ , a local  
 256 SPSP splitting matrix  $\tilde{A}_j$  of  $A$  associated with subdomain  $j$  is defined as

$$257 \quad (2.2) \quad \mathcal{P}_j \tilde{A}_j \mathcal{P}_j^\top = \begin{pmatrix} R_{j,I} A R_{j,I}^\top & R_{j,I} A R_{j,\Gamma}^\top \\ R_{j,\Gamma} A R_{j,I}^\top & \tilde{A}_\Gamma^j \\ & & 0 \end{pmatrix},$$

258 where  $\tilde{A}_\Gamma^j \in \mathbb{R}^{\gamma_j \times \gamma_j}$  satisfies the two following conditions: For all  $u \in \mathbb{R}^{\gamma_j}$ ,

- 259 •  $u^\top (R_{j,\Gamma} A R_{j,I}^\top) (R_{j,I} A R_{j,I}^\top)^{-1} (R_{j,I} A R_{j,\Gamma}^\top) u \leq u^\top \tilde{A}_\Gamma^j u$
- 260 •  $u^\top \tilde{A}_\Gamma^j u \leq u^\top \left( (R_{j,\Gamma} A R_{j,\Gamma}^\top) - (R_{j,\Gamma} A R_{j,\Delta}^\top) (R_{j,\Delta} A R_{j,\Delta}^\top)^{-1} (R_{j,\Delta} A R_{j,\Gamma}^\top) \right) u.$



261 The authors prove that the matrices  $\tilde{A}_j$  defined in such a way verify the following  
 262 relations:

$$263 \quad (2.3) \quad R_{j,\Delta}\tilde{A}_j = 0,$$

$$264 \quad (2.4) \quad u^\top \tilde{A}_j u \leq u^\top A u, \quad \forall u \in \mathbb{R}^n,$$

$$265 \quad (2.5) \quad u^\top \sum_{j=1}^N \tilde{A}_j u \leq k u^\top A u, \quad \forall u \in \mathbb{R}^n,$$

266  
 267 where  $k$  is a number that depends on the local SPSD splitting matrices and can be  
 268 at most equal to the number of subdomains  $k \leq N$ . The authors also show that the  
 269 local matrices defined in GenEO [32, 12] can be seen as a local SPSD splitting.

270 In [3], the authors highlight that the key idea to construct a coarse space relies  
 271 on the ability to identify the so-called local SPSD splitting matrices. They present  
 272 a class of algebraically constructed coarse spaces based on the local SPSD splitting  
 273 matrices. Moreover, this class can be extended to a larger variety of local SPSD  
 274 matrices. This extension has the advantage of allowing to construct efficient coarse  
 275 spaces for a multilevel structure in a practical way. This is discussed in the following  
 276 section.

277 **3. Extension of the class of coarse spaces.** In this section we extend the  
 278 class of coarse spaces presented in [3]. To do so, we present a class of matrices, that is  
 279 larger than the class of local SPSD splitting matrices. This will be our main building  
 280 block in the construction of efficient coarse spaces. Furthermore, this extension can  
 281 lead to a straightforward construction of hierarchical coarse spaces in a multilevel  
 282 Schwarz preconditioner setting.

283 **3.1. Extension of the class of local SPSD splitting matrices.** Regarding  
 284 the two-level additive Schwarz method, the authors of [3] introduced the local SPSD  
 285 splitting related to a subdomain as defined in (2.2). As it can be seen from the theory  
 286 presented in that paper, it is not necessary to have the exact matrices  $R_{j,I}AR_{j,I}^\top$ ,  
 287  $R_{j,I}AR_{j,\Gamma}^\top$ , and  $R_{j,\Gamma}AR_{j,I}^\top$  in the definition of the local SPSD splitting in order to  
 288 build an efficient coarse space. Indeed, the one and only necessary condition is to  
 289 define for each subdomain  $j$  an SPSD matrix  $\tilde{A}_j$  for  $j = 1, \dots, N$  such that:

$$290 \quad (3.1) \quad \begin{aligned} & R_{j,\Delta}\tilde{A}_j = 0, \\ & u^\top \sum_{j=1}^N \tilde{A}_j u \leq k u^\top A u, \forall u \in \mathbb{R}^n, \end{aligned}$$

291  
 292 where  $k$  is a number that depends on the local SPSD matrices  $\tilde{A}_j$  for  $j = 1, \dots, N$ .  
 293 The first condition means that  $\tilde{A}_j$  has the local SPSD structure associated with sub-  
 294 domain  $j$ , i.e., it has the following form:

$$295 \quad \mathcal{P}_j \tilde{A}_j \mathcal{P}_j^\top = \begin{pmatrix} \tilde{A}_{I,\Gamma}^j & 0 \\ 0 & 0 \end{pmatrix},$$

296 where  $\tilde{A}_{I,\Gamma}^j \in \mathbb{R}^{n_j \times n_j}$ . The second condition is associated with the stable decom-  
 297 position property [34, 12]. Note that with regard to the local SPSD matrices, the  
 298 authors in [32] only use these two conditions. That is to say, with matrices that verify  
 299 conditions (3.1) the construction of the coarse space is straightforward through the

theory presented in either [32] or [3]. To this end, we define in the following the local  
 SPSPD (LSPSPD) matrix associated with subdomain  $j$  as well as the associated local  
 filtering subspace that contributes to the coarse space.

DEFINITION 3.1 (local SPSPD matrices). *An SPSPD matrix  $\tilde{A}_{i,j} \in \mathbb{R}^{n_i \times n_i}$  is called  
 local SPSPD (LSPSPD) with respect to subdomain  $j$  if*

- $R_{i,j,\Delta} \tilde{A}_{i,j} = 0$ ,
  - $u^\top \sum_{j=1}^{N_i} \tilde{A}_{i,j} u \leq k_i u^\top A_i u$ ,
- where  $k_i > 0$ .

We note that the local SPSPD splitting matrices form a subset of the local SPSPD  
 matrices.

**3.2. Multilevel coarse spaces.** This section summarizes the steps to be per-  
 formed in order to construct the coarse space at level  $i$  once we have the LSPSPD  
 matrices associated with each subdomain at that level.

DEFINITION 3.2 (coarse space based on LSPSPD matrices). *Let  $\tilde{A}_{i,j} \in \mathbb{R}^{n_i \times n_i}$  for  
 $j = 1, \dots, N_i$  be LSPSPD matrices. Let  $D_{i,j} \in \mathbb{R}^{n_i \times j}$  for  $j = 1, \dots, N_i$  be the partition  
 of unity. Let  $\tau_i > 0$  be a given number. For a subdomain  $j \in \llbracket 1; N_i \rrbracket$ , let*

$$G_{i,j} = D_{i,j} (R_{i,j} A_i R_{i,j}^\top) D_{i,j}.$$

Let  $\tilde{P}_{i,j}$  be the projection on  $\text{range}(R_{i,j} \tilde{A}_j R_{i,j}^\top)$  parallel to  $\ker(R_{i,j} \tilde{A}_j R_{i,j}^\top)$ . Let  $K_{i,j} =$   
 $\ker(R_{i,j} \tilde{A}_{i,j} R_{i,j}^\top)$ ,  $L_{i,j} = \ker(G_{i,j}) \cap K_{i,j}$ , and  $L_{i,j}^{\perp K_{i,j}}$  the orthogonal complementary  
 of  $L_{i,j}$  in  $K_{i,j}$ . Consider the generalized eigenvalue problem:

$$(3.2) \quad \begin{aligned} \tilde{P}_{i,j} G_{i,j} \tilde{P}_{i,j} u_{i,j,k} &= \lambda_{i,j,k} R_{i,j} \tilde{A}_{i,j} R_{i,j}^\top u_{i,j,k}, \\ (u_{i,j,k}, \lambda_{i,j,k}) &\in \text{range}(R_{i,j} \tilde{A}_{i,j} R_{i,j}^\top) \times \mathbb{R}. \end{aligned}$$

Set

$$(3.3) \quad Z_{i,j} = L_{i,j}^{\perp K_{i,j}} \oplus \text{span}\{u_{i,j,k} \mid \lambda_{i,j,k} > \tau_i\}.$$

Then, the coarse space associated with LSPSPD matrices  $\tilde{A}_{i,j}$  for  $j = 1, \dots, N_i$  at level  $i$   
 is defined as:

$$(3.4) \quad \mathcal{S}_i = \bigoplus_{j=1}^{N_i} R_{i,j}^\top D_{i,j} Z_{i,j}.$$

Following notations from section 1, the columns of  $V_i$  span the coarse space  $\mathcal{S}_i$ . The  
 matrix  $A_{i+1}$  is defined as:

$$(3.5) \quad A_{i+1} = V_i^\top A_i V_i.$$

The local SPSPD splitting matrices at level 1 will play an important role in the  
 construction of the LSPSPD matrices at subsequent levels. In the following, we present  
 an efficient approach for computing LSPSPD matrices for levels greater than 1.

**4. Partitioning for levels strictly greater than 1.** In this section, we explain  
 how to obtain the partitioning sets  $\Omega_{i,j,I}$  for  $(i,j) \in \llbracket 2; L \rrbracket \times \llbracket 1; N_i \rrbracket$ . Once the sets  
 $\Omega_{i,j,\Gamma}$  for  $j = 1, \dots, N_i$  are defined at level  $i$ , the following elements are readily  
 available: sets  $\Omega_{i,j,\Gamma}$ ,  $\Omega_{i,j,\Delta}$ , and  $\Omega_{i,j}$ ; restriction operators  $R_{i,j,I}$ ,  $R_{i,j,\Gamma}$ ,  $R_{i,j,\Delta}$ , and  
 $R_{i,j}$ ; permutation matrices  $\mathcal{P}_{i,j}$  for  $j = 1, \dots, N_i$ .

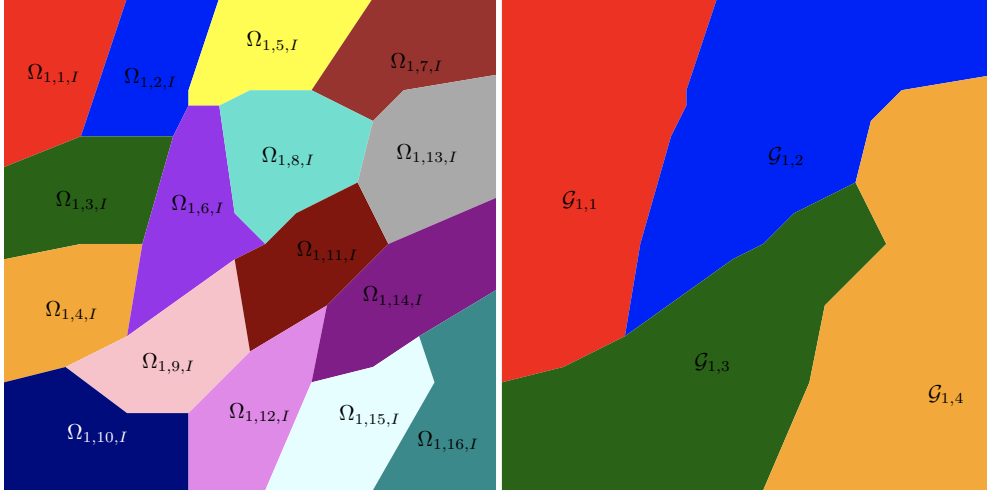


FIG. 4.1. Left: 16 subdomains at level 1. Right: 4 superdomains at level 1.  $\mathcal{G}_{1,j} = \llbracket 4(j-1) + 1; 4(j-1) + 4 \rrbracket$ .

338 **4.1. Superdomains as unions of several subdomains.** In this section, we  
 339 introduce the notion of *superdomain*. It refers to the union of several neighboring  
 340 subdomains. Let  $\mathcal{G}_{i,1}, \dots, \mathcal{G}_{i,N_{i+1}}$  be disjoint subsets of  $\llbracket 1; N_i \rrbracket$ , where  $\bigcup_{j=1}^{N_{i+1}} \mathcal{G}_{i,j} =$   
 341  $\llbracket 1; N_i \rrbracket$ . We call the union of the subdomains  $\{k \in \llbracket 1; N_i \rrbracket : k \in \mathcal{G}_{i,j}\}$  superdomain  $j$ ,  
 342 for  $j = 1, \dots, N_{i+1}$ . Figure 4.1 gives an example of how to set superdomains. Though  
 343 this definition of superdomains may look somehow related to the fine mesh, it is in  
 344 practice done at the algebraic level, as explained later on. Note that the indices of  
 345 columns and rows of  $A_{i+1}$  are associated with the vectors contributed by the subdo-  
 346 mains at level  $i$  in order to build the coarse space  $\mathcal{S}_i$ , see Figure 4.2. Hence, defining  
 347 subdomains on the structure of  $A_{i+1}$  is natural once we have the subsets  $\mathcal{G}_{i,j}$ , for  
 348  $j = 1, \dots, N_{i+1}$ .

349 **4.2. Heritage from superdomains.** Let  $e_{i,j}$  be the set of indices of the vectors  
 350 that span  $R_{i,j}^\top D_{i,j} Z_{i,j}$  in the matrix  $V_i$  for some  $(i, j) \in \llbracket 1; L-1 \rrbracket \times \llbracket 1; N_i \rrbracket$ , see  
 351 Figure 4.2. We define  $\Omega_{i+1,j,I} = \cup_{k \in \mathcal{G}_{i,j}} e_{i,k}$ , for  $j = 1, \dots, N_{i+1}$ . We denote  $\Omega_{i+1,j,\Gamma}$   
 352 the subset of  $\llbracket 1; n_{i+1} \rrbracket \setminus \Omega_{i+1,j,I}$  whose elements are at distance 1 from  $\Omega_{i+1,j,I}$  through  
 353 the graph of  $A_{i+1}$ . We note that

$$354 \quad \Omega_{i+1,j,\Gamma} \subset \bigcup_{p \in \mathcal{G}_{i,j}} \bigcup_{k \in \mathcal{V}_{i,p}} e_{i,k},$$

355 where  $\mathcal{V}_{i,j}$  represents the set of subdomains that are neighbors of subdomain  $j$  at  
 356 level  $i$  for  $j = 1, \dots, N_i$ . The overlapping subdomain  $j$  is defined by the set  $\Omega_{i+1,j} =$   
 357  $[\Omega_{i+1,j,I}, \Omega_{i+1,j,\Gamma}]$ . The rest of the sets, restriction, and prolongation operators can  
 358 be defined as given in section 1.

359 **5. LSPSD matrices for levels strictly greater than 1.** In [32, 12, 3], differ-  
 360 ent methods are suggested to obtain local SPSP splitting matrices at level 1. These  
 361 matrices are used to construct efficient two-level additive Schwarz preconditioners.  
 362 Here in this section, we do not discuss the construction of these matrices at level 1. We  
 363 suppose that we have the local SPSP matrices  $\hat{A}_{1,j} \in \mathbb{R}^{n_1 \times n_1}$  for  $j = 1, \dots, N_1$ . We

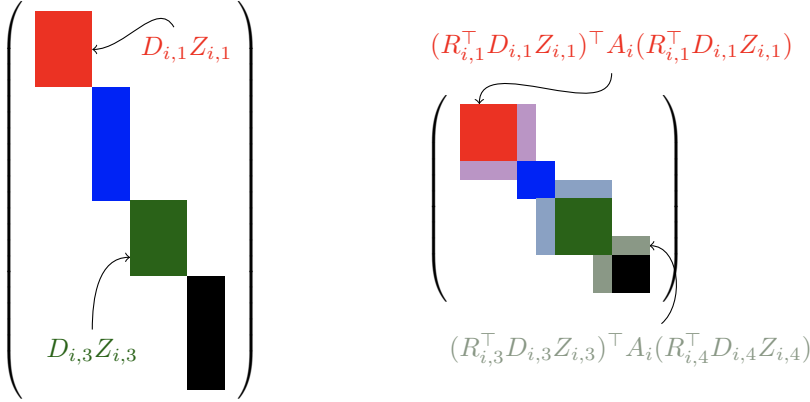


FIG. 4.2. Illustration of the correspondence of indices between the columns of  $V_i$  (left) and the rows and columns of  $A_{i+1}$  (right). Having no overlap in  $V_i$  is possible through a non-overlapping partition of unity.

364 focus on computing LSPSD matrices  $\tilde{A}_{i,j} \in \mathbb{R}^{n_i \times n_i}$  for  $(i,j) \in \llbracket 2; L \rrbracket \times \llbracket 1; N_i \rrbracket$ . We also  
 365 suppose that the coarse space  $\mathcal{S}_1$  is available, i.e., the matrices  $V_1$  and  $A_2 = V_1^\top A_1 V_1$   
 366 are known explicitly.

367 PROPOSITION 5.1. Let  $i$  be a fixed level index, and let  $\tilde{A}_{i,j}$  be an LSPSD of  $A_i$ ,  
 368 (see Definition 3.1), associated with subdomain  $j$ , for  $j = 1, \dots, N_i$ . Let  $\mathcal{G}_{i,1}, \dots, \mathcal{G}_{i,N_{i+1}}$   
 369 be a set of superdomains at level  $i$  associated with the partitioning at level  $i + 1$ , see  
 370 subsection 4.1. Let  $V_i^\top$  be the restriction matrix to the coarse space at level  $i$ . Then,  
 371 the matrix  $\tilde{A}_{i+1,j}$  which is defined as:

$$372 \quad \tilde{A}_{i+1,j} = \sum_{k \in \mathcal{G}_{i,j}} V_i^\top \tilde{A}_{i,k} V_i,$$

373 satisfies the conditions in Definition 3.1. That is,  $\tilde{A}_{i+1,j}$  is LSPSD of  $A_{i+1}$  with  
 374 respect to subdomain  $j$  for  $j = 1, \dots, N_{i+1}$ .

375 *Proof.* To prove that  $\tilde{A}_{i+1,j}$  is LSPSD of  $A_{i+1}$  with respect to subdomain  $j$ , we  
 376 have to prove the following:

- 377 •  $R_{i+1,j,\Delta} \tilde{A}_{i+1,j} = 0$
- 378 •  $u^\top \sum_{j=1}^{N_{i+1}} \tilde{A}_{i+1,j} u \leq k_{i+1} u^\top A_{i+1} u$  for all  $u \in \mathbb{R}^{n_{i+1}}$ .

379 First, note that  $R_{i,k} \tilde{A}_{i,j} = 0$  for all non-neighboring subdomains  $k$  of subdomain  $j$ .  
 380 This yields  $Z_{i,k}^\top D_{i,k} R_{i,k} \tilde{A}_{i,j} = 0$  for these subdomains  $k$ .

381 Now, let  $m \in \llbracket 1; n_{i+1} \rrbracket \setminus \Omega_{i+1,j}$ . We will show that the  $m$ th row of  $\tilde{A}_{i+1,j}$  is zero.  
 382 Following the partitioning of subdomains at level  $i + 1$ , there exists a subdomain  $\Omega_{p_0}$   
 383 such that the  $m$ th column of  $V_i$  is part of  $R_{i,p_0}^\top D_{i,p_0} Z_{i,p_0}$ . We denote this column  
 384 vector by  $v_m$ . Furthermore, the subdomain  $p_0$  is not a neighbor of any subdomain  
 385 that is a part of the superdomain  $\mathcal{G}_{i,j}$ . Hence,  $v_m^\top \tilde{A}_{i,k} = 0$  for  $k \in \mathcal{G}_{i,j}$ . The  $m$ th row  
 386 of  $\tilde{A}_{i+1,j}$  is given as  $v_m^\top \sum_{k \in \mathcal{G}_{i,j}} \tilde{A}_{i,k} V_i$ . Then,  $v_m^\top \sum_{k \in \mathcal{G}_{i,j}} \tilde{A}_{i,k} = 0$ , and the  $m$ th row  
 387 of  $\tilde{A}_{i+1,j}$  is zero.

388 To prove the second condition, we have

$$389 \quad u^\top \sum_{j=1}^{N_{i+1}} \tilde{A}_{i+1,j} u = u^\top \sum_{j=1}^{N_{i+1}} \sum_{k \in \mathcal{G}_{i,j}} V_i^\top \tilde{A}_{i,k} V_i u.$$

390 Since  $\{\mathcal{G}_{i,j}\}_{1 \leq j \leq N_{i+1}}$  form a disjoint partitioning of  $\llbracket 1; N_i \rrbracket$ , we can write

$$391 \quad u^\top \sum_{j=1}^{N_{i+1}} \tilde{A}_{i+1,j} u = u^\top \sum_{k=1}^{N_i} V_i^\top \tilde{A}_{i,k} V_i u,$$

$$392 \quad \leq u^\top V_i^\top \sum_{k=1}^{N_i} \tilde{A}_{i,k} V_i u.$$

393  $\tilde{A}_{i,k}$  is an LSPSD matrix of  $A_i$  for  $k = 1, \dots, N_i$ . Hence, we have

$$394 \quad u^\top \sum_{j=1}^{N_{i+1}} \tilde{A}_{i+1,j} u \leq k_i u^\top V_i^\top A_i V_i u,$$

$$395 \quad \leq k_i u^\top A_{i+1} u.$$

396 We finish the proof by setting  $k_{i+1} = k_i$ .  $\square$

397 **Figure 5.1** gives an illustration of the LSPSD construction provided by **Proposition 5.1**. **Figure 5.1** (top left) represents the matrix  $A_1$ . The graph of  $A_1$  is partitioned into 16 subdomains. Each subdomain is represented by a different color. **Figure 5.1** (top right) represents the matrix  $V_1$  whose column vectors form a basis of the coarse space  $\mathcal{S}_1$ . Colors of columns of  $V_1$  correspond to those of subdomains in  $A_1$ . **Figure 5.1** (bottom left) represents the matrix  $A_2 = V_1^\top A_1 V_1$ . Note that column and row indices of  $A_2$  are associated with column indices of  $V_1$ . Four subdomains are used at level 2. The partitioning at level 2 is related to the superdomain  $\mathcal{G}_{1,j} = \llbracket 4(j-1)+1; 4(j-1)+4 \rrbracket$  for  $j = 1, \dots, 4$ . **Figure 5.1** (bottom right) represents an LSPSD matrix of  $A_2$  with respect to subdomain 1 at level 2.

398 **Theorem 5.2** shows that the third condition of the fictitious subspace lemma **Lemma 2.1** holds at level  $i$  for  $i = 1, \dots, L$ .

399 **THEOREM 5.2.** *Let  $\tilde{A}_{i,j}$  be an LSPSD of  $A_i$  associated with subdomain  $j$ , for  $(i, j) \in \llbracket 1; L \rrbracket \times \llbracket 1; N_i \rrbracket$ . Let  $\tau_i > 0$ ,  $Z_{i,j}$  be the subspace associated with  $\tilde{A}_{i,j}$ , and  $P_{i,j}$  be the projection on  $Z_{i,j}$  as defined in **Lemma 2.5**. Let  $u_i \in \mathbb{R}^{n_i}$  and let  $u_{i,j} = (D_{i,j} (I_{n_{i,j}} - P_{i,j}) R_{i,j} u_i)$  for  $(i, j) \in \llbracket 1; L \rrbracket \times \llbracket 1; N_i \rrbracket$ . Let  $u_{i,0}$  be defined as,*

$$400 \quad u_{i,0} = (V_i^\top V_i)^{-1} V_i^\top \left( \sum_{j=1}^{N_i} R_{i,j}^\top D_{i,j} P_{i,j} R_{i,j} u_i \right).$$

401 Let  $m_i = (2 + (2k_{i,c} + 1)k_i \tau_i)^{-1}$ . Then,

$$402 \quad u_i = \sum_{j=0}^{N_i} R_{i,j}^\top u_{i,j},$$

403 and

$$404 \quad (5.1) \quad m_i \sum_{j=0}^{N_i} u_{i,j}^\top R_{i,j} A_i R_{i,j}^\top u_{i,j} \leq u_i^\top A_i u_i.$$

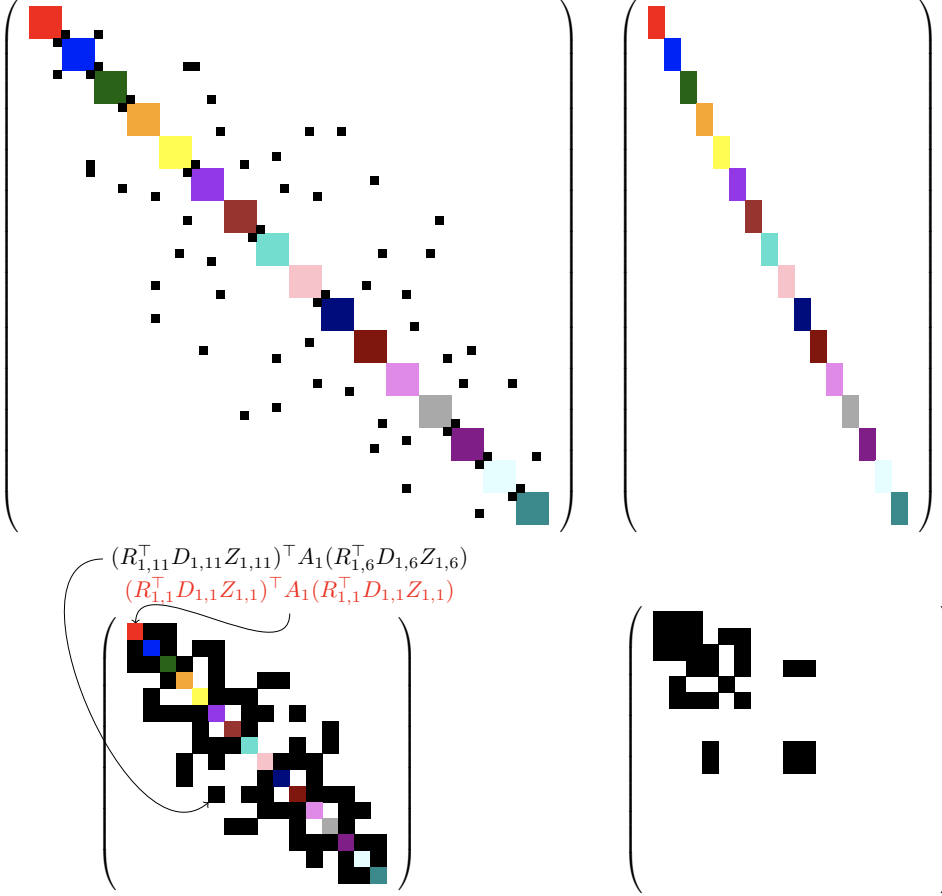


FIG. 5.1. Illustration of the LSPSD construction provided by Proposition 5.1. Top left: the matrix  $A_1$ , top right:  $V_1$ , bottom left: the matrix  $A_2 = V_1^\top A_1 V_1$ , bottom right:  $\tilde{A}_{2,1} = \sum_{j \in \mathcal{G}_{1,1}} V_1^\top \tilde{A}_{1,j} V_1$ , where  $\mathcal{G}_{1,1} = 1, \dots, 4$

421 *Proof.* We have

$$422 \quad \sum_{j=0}^{N_i} R_{i,j}^\top u_{i,j} = V_i (V_i^\top V_i)^{-1} V_i^\top \left( \sum_{j=1}^{N_i} R_{i,j}^\top D_{i,j} P_{i,j} R_{i,j} u_i \right) + \sum_{j=1}^{N_i} R_{i,j}^\top u_{i,j}$$

423  
424 Since for all  $y \in \mathcal{S}_i$ ,  $V_i (V_i^\top V_i)^{-1} V_i^\top y = y$ , we have

$$425 \quad \sum_{j=0}^{N_i} R_{i,j}^\top u_{i,j} = \sum_{j=1}^{N_i} R_{i,j}^\top D_{i,j} P_{i,j} R_{i,j} u_i + \sum_{j=1}^{N_i} R_{i,j}^\top (D_{i,j} (I_{n_{i,j}} - P_{i,j}) R_{i,j} u_i),$$

$$426 \quad = \sum_{j=1}^{N_i} R_{i,j}^\top D_{i,j} R_{i,j} u_i,$$

$$427 \quad = u_i.$$

429 To prove the inequality (5.1), we start with the inequality from Lemma 2.4. We

430 have

$$431 \quad (5.2) \quad \sum_{j=0}^{N_i} u_{i,j}^\top R_{i,j} A_i R_{i,j}^\top u_{i,j} \leq 2u_i^\top A_i u_i + (2k_{i,c} + 1) \sum_{j=1}^{N_i} u_{i,j}^\top R_{i,j} A_i R_{i,j}^\top u_{i,j},$$

433 where we chose  $u_{\mathcal{B}_i}$  in [Lemma 2.4](#) to be  $(u_{i,j})_{j=0,\dots,N_i}$  and  $u_{A_i} = u_i$ . In [Definition 3.2](#),  
434 we defined  $Z_{i,j}$ , such that for all  $w \in \mathbb{R}^{n_{i,j}}$  we have

$$435 \quad ((I_{n_{i,j}} - P_{i,j})w)^\top (D_{i,j} R_{i,j} A_i R_{i,j}^\top D_{i,j}) ((I_{n_{i,j}} - P_{i,j})w) \leq \tau_i w^\top (R_{i,j} \tilde{A}_{i,j} R_{i,j}^\top) w.$$

437 Hence, in the special case  $w = R_{i,j} u_i$ , we can write

$$438 \quad ((I_{n_{i,j}} - P_{i,j})R_{i,j} u_i)^\top (D_{i,j} R_{i,j} A_i R_{i,j}^\top D_{i,j}) ((I_{n_{i,j}} - P_{i,j})R_{i,j} u_i) \\ 439 \quad \leq \tau_i (R_{i,j} u_i)^\top (R_{i,j} \tilde{A}_{i,j} R_{i,j}^\top) (R_{i,j} u_i).$$

442 Equivalently,

$$443 \quad u_{i,j}^\top R_{i,j} A_i R_{i,j}^\top u_{i,j} \leq \tau_i (R_{i,j} u_i)^\top R_{i,j} \tilde{A}_{i,j} R_{i,j}^\top (R_{i,j} u_i).$$

445 Plugging this inequality in (5.2) gives

$$446 \quad \sum_{j=0}^{N_i} u_{i,j}^\top R_{i,j} A_i R_{i,j}^\top u_{i,j} \leq 2u_i^\top A_i u_i + (2k_{i,c} + 1) \tau_i \sum_{j=1}^{N_i} (R_{i,j} u_i)^\top R_{i,j} \tilde{A}_{i,j} R_{i,j}^\top (R_{i,j} u_i).$$

448 Since  $\tilde{A}_{i,j}$  is local, we have

$$449 \quad (R_{i,j} u_i)^\top R_{i,j} \tilde{A}_{i,j} R_{i,j}^\top (R_{i,j} u_i) = u_i^\top \tilde{A}_{i,j} u_i, \text{ for } j = 1, \dots, N_i.$$

450 By using the fact that  $\tilde{A}_{i,j}$  is LSPSD of  $A_i$  for  $j = 1, \dots, N_i$ , we obtain the following:

$$451 \quad \sum_{j=0}^{N_i} u_{i,j}^\top R_{i,j} A_i R_{i,j}^\top u_{i,j} \leq 2u_i^\top A_i u_i + (2k_{i,c} + 1) k_i \tau_i u_i^\top A_i u_i.$$

453 Multiplying both sides with  $m_i$  ends the proof, i.e.,

$$454 \quad m_i \sum_{j=0}^{N_i} u_{i,j}^\top R_{i,j} A_i R_{i,j}^\top u_{i,j} \leq u_i^\top A_i u_i. \quad \square$$

456 [Theorem 5.3](#) provides an upper bound on the condition number of the preconditioned matrix  $M_i^{-1} A_i$  for  $i = 1, \dots, L$ .

458 **THEOREM 5.3.** *Let  $M_i$  be the additive Schwarz preconditioner at level  $i$  combined*  
459 *with the coarse space correction induced by  $\mathcal{S}_i$  defined in (3.4). The following inequality*  
460 *holds,*

$$461 \quad \kappa(M_i^{-1} A_i) \leq (k_{i,c} + 1) (2 + (2k_{i,c} + 1) k_i \tau_i).$$

462 *Proof.* [Lemma 2.2](#), [Lemma 2.3](#), and [Theorem 5.2](#) prove that the multilevel preconditioner verifies the conditions in [Lemma 2.1](#) at each level  $i$ . Hence, the spectrum of the preconditioned matrix  $M_i^{-1} A_i$  is contained in the interval  $[(2 + (2k_{i,c} + 1) k_i \tau_i)^{-1}, k_{i,c} + 1]$ . Equivalently, the condition number of the preconditioned matrix at level  $i$  verifies the following inequality

$$467 \quad \kappa(M_i^{-1} A_i) \leq (k_{i,c} + 1) (2 + (2k_{i,c} + 1) k_i \tau_i). \quad \square$$

468 **Proposition 5.1** shows that the constant  $k_i$  associated with the LSPSD matrices at  
 469 level  $i$  is independent of the number of levels and bounded by the number of subdo-  
 470 mains at level 1. Indeed,

$$471 \quad k_1 \geq k_i \text{ for } i = 2, \dots, L.$$

472 Furthermore, in the case where the LSPSD matrices at the first level are the Neumann  
 473 matrices,  $k_i$  is bounded by the maximum number of subdomains at level 1 that share  
 474 an unknown.

475 The constant  $k_{i,c}$  for  $i = 1, \dots, L$  is the minimum number of distinct colors so that  
 476  $\{\text{span}\{R_{i,j}^\top\}\}_{1 \leq j \leq N_i}$  of the same color are mutually  $A_i$ -orthogonal. Both constants  
 477  $k_i$  and  $k_{i,c}$  are independent of the number of subdomains for each level  $i$ .

478 The constant  $\tau_i$  can be chosen such that the condition number of the preconditioned  
 479 system at level  $i$  is upper bounded by a prescribed value. Hence, this allows  
 480 to have a robust convergence of the preconditioned Krylov solver at each level.

481 Algorithm 5.1 presents the construction of the multilevel additive Schwarz method  
 482 by using GenEO. The algorithm iterates over the levels. At each level, three main  
 483 operations are performed. First, the construction of the LSPSD matrices. At level 1,  
 484 the LSPSD matrices are the Neumann matrices, otherwise, **Proposition 5.1** is used  
 485 to compute them. Once the LSPSD matrix is available, the generalized eigenvalue  
 486 problem in (3.2) has to be solved concurrently. Given the prescribed upper bound on  
 487 the condition number,  $Z_{i,j}$  can be set. Finally, the coarse space is available and the  
 coarse matrix is assembled.

---

**Algorithm 5.1** Multilevel GenEO

---

**Require:**  $A_1 = A \in \mathbb{R}^{n \times n}$  SPD,  $L + 1$  number of levels,  $N_i$  number of subdomains  
 at each level,  $\mathcal{G}_{i,j}$  sets of superdomains

**Ensure:** preconditioner at each level  $i$ ,  $M_i^{-1}$  with bounded condition number of  
 $M_i^{-1}A_i$

- 1: **for**  $i = 1, \dots, L$  **do**
- 2:   **for** each subdomain  $j = 1, \dots, N_i$  **do**
- 3:      $A_{i,j} = R_{i,j}A_iR_{i,j}^\top$  (*local matrix associated with subdomain  $j$* )
- 4:     **if**  $i = 1$  **then**
- 5:       local SPSD  $\tilde{A}_{i,j}$  is Neumann matrix of subdomain  $j$
- 6:     **else**
- 7:       compute local SPSD matrix as

$$\tilde{A}_{i,j} = \sum_{k \in \mathcal{G}_{i,j}} V_{i-1}^\top \tilde{A}_{i-1,k} V_{i-1}$$

- 8:     **end if**
  - 9:     solve the generalized eigenvalue problem (3.2), set  $Z_{i,j}$  as in (3.3)
  - 10:   **end for**
  - 11:    $\mathcal{S}_i = \bigoplus_{j=1}^{N_i} D_{i,j}R_{i,j}^\top Z_{i,j}$ ,  $V_i$  basis of  $\mathcal{S}_i$
  - 12:   coarse matrix  $A_{i+1} = V_i^\top A_i V_i$ ,  $A_{i+1} \in \mathbb{R}^{n_{i+1} \times n_{i+1}}$
  - 13: **end for**
  - 14:  $M_i^{-1} = V_i A_{i+1}^{-1} V_i^\top + \sum_{j=1}^{N_i} R_{i,j}^\top A_{i,j}^{-1} R_{i,j}$
- 

488

489 **6. Numerical experiments.** In this section, the developed theory is validated  
 490 numerically with FreeFEM [14] for finite element discretizations and HPDDM [19]



491 for domain decomposition methods. We present numerical experiments on two highly  
 492 challenging problems illustrating the efficiency and practical usage of the proposed  
 493 method. For both problems, we use  $N_1 = 2,048$  MPI processes (equal to the number  
 494 of subdomains at level 1). We compare the two-level GenEO preconditioner and its  
 495 multilevel extension by varying  $N_2$  between 4 and 256. For the two-level method,  
 496  $N_2$  corresponds to the number of MPI processes that solve the coarse problem in a  
 497 distributed fashion using MKL CPARDISO [17]. For the multilevel method,  $N_3$  is  
 498 set to 1, i.e., a three-level method is used. The goal of these numerical experiments  
 499 is to show that when one switches from a two-level method with an exact coarse  
 500 solver, to our proposed multilevel method, the number of outer iterations is not im-  
 501 pacted. Thus, three levels are sufficient. As an outer solver, since all levels but the  
 502 coarsest are solved approximately, the flexible GMRES [30] is used. It is stopped  
 503 when relative unpreconditioned residuals are lower than  $10^{-6}$ . Subdomain matrices  
 504  $\{A_{i,j}\}_{1 \leq i \leq 2, 1 \leq j \leq N_i}$  are factorized concurrently using MKL PARDISO, and eigenvalue  
 505 problems are solved using ARPACK [24]. In both, two- and three-level GenEO, we  
 506 factorize the local matrices  $A_{1,j}$  for  $j \in \llbracket 1; N_1 \rrbracket$  and solve the generalized eigenvalue  
 507 problems concurrently at the first level. For this reason, we do not take into account  
 508 the time needed for these two steps which are performed without any communication  
 509 between MPI processes. We compare the time needed to assemble and factorize  $A_2$   
 510 in the two-level approach against the time needed to assemble  $A_2$  and local SPSD  
 511 matrices  $\tilde{A}_{2,j}$  for  $j \in \llbracket 1; N_2 \rrbracket$ , solve the generalized eigenvalue problems concurrently  
 512 on the second level, assemble, and factorize the matrix  $A_3$  in the three-level approach.  
 513 We also compare the time spent in the outer Krylov solver during the solution phase.  
 514 Readers interested by a comparison of the efficiency of GenEO and multigrid methods  
 515 such as GAMG [1] are referred to [18]. FreeFEM scripts used to produce the following  
 516 results are available at the following URL: <https://github.com/prj-/aldaas2019multi><sup>1</sup>.

517 **6.1. Diffusion test cases.** The scalar diffusion equation with highly heteroge-  
 518 neous coefficient  $\kappa$  is solved in  $[0, 1]^d$  ( $d = 2$  or  $3$ ). The strong formulation of the  
 519 equation is:

$$\begin{aligned} -\nabla \cdot (\kappa \nabla u) &= 1 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_N. \end{aligned}$$

522 The exterior normal vector to the boundary of  $\Omega$  is denoted  $n$ .  $\Gamma_D$  is the subset  
 523 of the boundary of  $\Omega$  corresponding to  $x = 0$  in 2D and 3D.  $\Gamma_N$  is defined as the  
 524 complementary of  $\Gamma_D$  with respect to the boundary of  $\Omega$ . We discretize the equation  
 525 using  $\mathbb{P}_2$  and  $\mathbb{P}_4$  finite elements in the 3D and 2D test cases, respectively. The number  
 526 of unknowns is  $441 \times 10^6$  and  $784 \times 10^6$ , with approximately 28 and 24 nonzero elements  
 527 per row in the 3D and 2D cases, respectively. The heterogeneity is due to the jumps  
 528 in the diffusion coefficient  $\kappa$ , see Figure 6.1, which is modeled with the following  $\mathbb{P}_0$   
 529 function:

$$\kappa = \begin{cases} 10^5([9y]) & \text{if } [9x] \equiv [9y] \equiv 0 \pmod{2}, \\ 1 & \text{elsewhere.} \end{cases}$$

531 The results in two dimensions are reported in Table 6.1. The number of outer itera-  
 532 tions for both two- and three-level GenEO is 32. The size of the level 2 operator is

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<sup>1</sup>note to reviewers: the repository is not yet public but will be once the paper is accepted

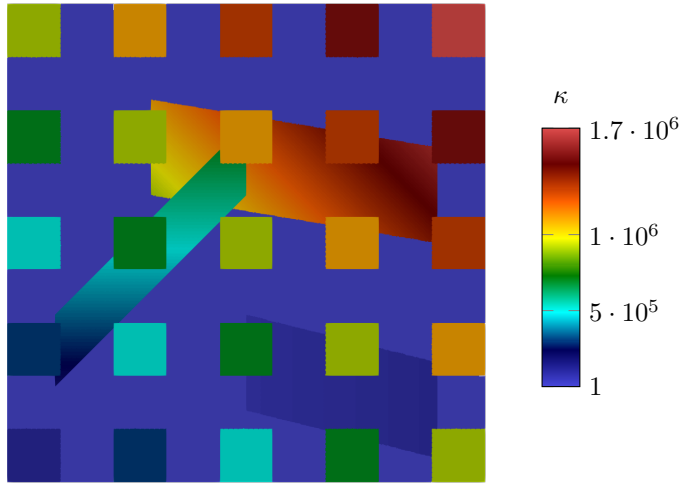


FIG. 6.1. Variation of the coefficient  $\kappa$  used for the diffusion test case

$N_2$	two-level GenEO		three-level GenEO		
	CS	solve	CS	solve	inner iter
4	2.4	11.9	6.5	27.4	14
16	1.8	11.3	3.6	15.4	15
64	1.9	12.1	3.0	16.7	14
256	2.4	18.4	2.8	13.9	13

TABLE 6.1

Diffusion 2D test case, comparison between two- and three-level GenEO

533  $n_2 = 25 \times 2,048 = 51,200$ . It is striking that the multilevel method does not deteri-  
 534 orate the numerical performance of the outer solver. For the two-level method, the  
 535 first column corresponds to the time needed to assemble the Galerkin operator  $A_2$   
 536 from (3.5) (assuming  $V_1$  has already been computed by ARPACK), and to factorize it  
 537 using  $N_2$  MPI processes. For the three-level method, the first column corresponds to  
 538 the time needed to assemble level 2 local subdomain matrices  $\{A_{2,j}\}_{1 \leq j \leq N_2}$ , level 2  
 539 local SPSD matrices, solve the generalized eigenvalue problem (3.2) concurrently, as-  
 540 semble the Galerkin operator  $A_3$  and factorize it on a single process. The size of  
 541 the level 3 operator is  $n_3 = 20 \times N_2$ . For both two- and three-level methods, the  
 542 second column is the time spent in the outer Krylov solver once the preconditioner  
 543 has been set up. In the last column of the three-level method, the number of inner  
 544 iterations for solving systems involving  $A_2$ , which is not inverted exactly anymore,  
 545 is reported. Another important numerical property of our method is that, thanks to  
 546 fully controlled bounds at each level, the number of inner iterations is low, indepen-  
 547 dently of the number of superdomains  $N_2$ . Because this problem is not large enough,  
 548 it is still tractable by a two-level method, for which HPDDM was highly optimized  
 549 for. Thus, there is no performance gain to be expected at this scale. However, one  
 550 can notice that the construction of the coarse operator(s) scales nicely with  $N_2$  for  
 551 the three-level method, whereas the performance of the direct solver MKL CPARDISO  
 552 quickly plateaus because of the finer and finer parallel workload granularity.

553 The results in three dimensions are reported in Table 6.2. The number of outer

$N_2$	two-level GenEO		three-level GenEO		
	CS	solve	CS	solve	inner iter
4	7.0	20.9	16.9	43.6	17
16	5.0	19.8	7.7	26.7	17
64	5.1	20.1	5.8	32.7	15
256	5.2	24.1	5.3	22.6	14

TABLE 6.2

*Diffusion 3D test case, comparison between two- and three-level GenEO*

554 iterations for both the two- and three-level GenEO is 19. The observations made  
 555 in two dimensions still hold, and the dimensions of  $A_2$  and  $A_3$  are the same. Once  
 556 again, it is important to note that the number of outer iterations is the same for both  
 557 methods.

558 **6.2. Linear elasticity test cases.** The system of linear elasticity with highly  
 559 heterogeneous elastic moduli is solved in 2D and 3D. The strong formulation of the  
 560 equation is given as:

$$\begin{aligned}
 (6.1) \quad & \operatorname{div} \sigma(u) + f = 0 \quad \text{in } \Omega, \\
 & u = 0 \quad \text{on } \Gamma_D, \\
 & \sigma(u) \cdot n = 0 \quad \text{on } \Gamma_N.
 \end{aligned}$$

563 The physical domain  $\Omega$  is a beam of dimensions  $[0, 10] \times [0, 1]$ , extruded for  $z \in$   
 564  $[0, 1]$  in 3D. The Cauchy stress tensor  $\sigma(\cdot)$  is given by Hooke's law: it can be expressed  
 565 in terms of Young's modulus  $E$  and Poisson's ratio  $\nu$ .

$$\sigma_{ij}(u) = \begin{cases} 2\mu\varepsilon_{ij}(u) & i \neq j, \\ 2\mu\varepsilon_{ii}(u) + \lambda\operatorname{div}(u) & i = j, \end{cases}$$

567 where

$$\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \mu = \frac{E}{2(1+\nu)}, \text{ and } \lambda = \frac{E\nu}{1-2\nu}.$$

569 The exterior normal vector to the boundary of  $\Omega$  is denoted  $n$ .  $\Gamma_D$  is the subset  
 570 of the boundary of  $\Omega$  corresponding to  $x = 0$  in 2D and 3D.  $\Gamma_N$  is defined as the  
 571 complementary of  $\Gamma_D$  with respect to the boundary of  $\Omega$ . We discretize (6.1) using  
 572 the following vectorial finite elements:  $(\mathbb{P}_2, \mathbb{P}_2, \mathbb{P}_2)$  in 3D and  $(\mathbb{P}_3, \mathbb{P}_3)$  in 2D. The  
 573 number of unknowns is  $146 \times 10^6$  and  $847 \times 10^6$ , with approximately 82 and 34  
 574 nonzero elements per row in the 3D and 2D cases, respectively. The heterogeneity is  
 575 due to the jumps in  $E$  and  $\nu$ . We consider discontinuous piecewise constant values  
 576 for  $E$  and  $\nu$ :  $(E_1, \nu_1) = (2 \times 10^{11}, 0.25)$ ,  $(E_2, \nu_2) = (10^7, 0.45)$ , see Figure 6.2.

577 Results in two (resp. three) dimensions are reported in Table 6.3 (resp. Table 6.4).  
 578 The number of outer iterations are 73 and 45 respectively. For these test cases, we  
 579 slightly relaxed the criterion for selecting eigenvectors in coarse spaces, which explains  
 580 why the iteration counts increase. However, the same observations as for the diffusion  
 581 test cases still hold. The dimension of the level 2 matrix is  $n_2 = 50 \times 2,048 = 1.02 \cdot 10^5$ ,  
 582 while for the level 3 matrix it is  $n_3 = 20 \times N_2$ . We observe that the number of iterations  
 583 of the inner solver increases slowly when increasing the number of subdomains from  
 584 4 to 256 in the 2D case and remains almost constant in the 3D case. In terms of

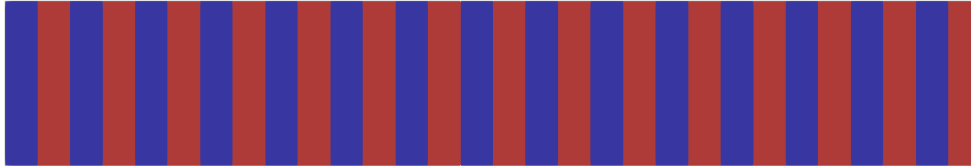


FIG. 6.2. Variation of the structure coefficients used for the elasticity test case

$N_2$	two-level GenEO		three-level GenEO		
	CS	solve	CS	solve	inner iter
4	4.8	52.7	22.5	179.3	31
16	3.9	50.3	9.3	124.9	57
64	4.0	53.1	7.2	71.5	34
256	4.8	63.2	6.8	71.2	44

TABLE 6.3

Elasticity 2D test case, comparison between two- and three-level GenEO

585 runtime, the two-level GenEO is faster than three-level GenEO for these matrices of  
 586 medium dimensions.

587 To show the potential of our method at larger scales, a three-dimensional linear  
 588 elasticity problem of size  $616 \times 10^6$  is now solved on  $N_1 = 8,192$  processes and  $N_2 = 256$   
 589 superdomains. With the two-level method,  $A_2$  is assembled and factorized in 27.5  
 590 seconds. With the three-level method, this step now takes 13.2 seconds, see Table 6.5.  
 591 Once again, as before, the number of outer iterations remains constant and equal to  
 592 53 for both methods. Not taking into account the preconditioner setup, the problem is  
 593 solved in 52 seconds in both cases. At this regime, it is clear that there are important  
 594 gains for the setup phase. At even greater scales, gains for the solution phase are  
 595 also expected. Moreover, another interesting fact to note regarding computation time  
 596 is that the generalized eigenvalue problems solved concurrently at the first level to  
 597 obtain  $V_1$  actually represents a significant part of the total time of 152.8 seconds (resp.  
 598 138.5 seconds) with the two- (resp. three-)level method: 113.3 seconds. This cost can  
 599 be reduced by taking a larger number of (smaller) subdomains, with the drawback of  
 600 increasing the size of  $V_1$  and thus  $A_2$ . This drawback represents a clear bottleneck  
 601 for the two-level method but is alleviated by using the three-level method, making it  
 602 a good candidate for problems at greater scales.

603 **7. Conclusion.** In this paper, we reviewed generalities of overlapping Schwarz  
 604 preconditioners and presented a framework for its multilevel extension. We generalized  
 605 the local SPSD splitting presented in [3] to cover a larger set of matrices leading to  
 606 more flexibility for building robust coarse spaces. Based on local SPSD matrices  
 607 on the first level, we presented how to compute local SPSD matrices for coarser  
 608 levels. The multilevel solver based on hierarchical local SPSD matrices is robust and  
 609 guarantees a bound on the condition number of the preconditioned matrix at each  
 610 level depending on predefined values. Numerical experiments illustrate the theory and  
 611 prove the efficiency of the method on challenging problems of large size arising from  
 612 heterogeneous linear elasticity and diffusion problems with jumps in the coefficients  
 613 of multiple orders of magnitude.

$N_2$	two-level GenEO		three-level GenEO		
	CS	solve	CS	solve	inner iter
4	28.5	46.9	78.9	296.7	23
16	17.3	35.4	24.5	124.5	23
64	15.0	33.2	15.4	62.2	21
256	13.6	40.7	10.6	50.7	23

TABLE 6.4

Elasticity 3D test case, comparison between two- and three-level GenEO

$N_2$	two-level GenEO		three-level GenEO		
	CS	solve	CS	solve	inner iter
256	27.5	52.0	13.2	52.0	25

TABLE 6.5

Elasticity 3D test case, comparison between two- and three-level GenEO

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