Quasi-Complementary Foliations and the Mather-Thurston Theorem

Gaël Meigniez

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GAËL MEIGNIEZ

We establish a form of the h–principle for the existence of foliations of codimension at least 2 which are quasicomplementary to a given one. Roughly, "quasicomplementary" means that they are complementary except on the boundaries of some kind of Reeb components. The construction involves an adaptation of W Thurston's "inflation" process. The same methods also provide a proof of the classical Mather–Thurston theorem.

57R30, 57R32, 58H10

1 Introduction

1.1 Quasicomplementary foliations

Given, on a manifold M, a dimension-q foliation $\mathscr{F}$, the existence of a foliation $\mathscr{G}$ complementary to $\mathscr{F}$ (that is, $\mathscr{G}$ is of codimension q and transverse to $\mathscr{F}$) is of course in general an intractable problem. In this paper, we weaken the transversality condition, prescribing a simple (and classical) model for the tangentialities between $\mathscr{F}$ and $\mathscr{G}$ which implies that $\mathscr{G}$ is a limit of plane fields complementary to $\mathscr{F}$ but themselves not necessarily integrable. We establish, when $q \geq 2$, a form of Gromov's h–principle for such "quasicomplementary" foliations.

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Here is a very elementary example similar to what we call quasicomplementarity, although $q = 1$. Consider the Hopf foliation $\mathscr{F}$ of the 3–sphere $\mathbb{S}^3$ by circles. The classical geometric theory of foliations shows that $\mathscr{F}$ admits no complementary foliation $\mathscr{G}$: indeed, by the Novikov closed leaf theorem (see for example Camacho and Lins Neto [2] or Candel and Conlon [3]), $\mathscr{G}$ would have a compact leaf which would separate $\mathbb{S}^3$, in contradiction to the transversality to $\mathscr{F}$; alternatively, one can argue that the Hopf fibration would then be a foliated bundle [2, pages 99–100; 3, Example 2.1.5] over a simply connected base $\mathbb{S}^2$; hence all leaves of $\mathscr{G}$ would be diffeomorphic to the base; and by the Reeb global stability theorem [2, Chapter IV, Theorem 4; 3, Theorem 6.1.5], the total space would be $\mathbb{S}^2 \times \mathbb{S}^1$, not $\mathbb{S}^3$.

However, it is easily verified that the sphere has a Reeb foliation $\mathscr{G}$ which is complementary to the circles except on its unique compact leaf, which is tangential to them. Moreover, $\mathscr{G}$ is a limit of 2–plane fields complementary to the circles, provided that one makes its two Reeb components "turbulize" in appropriate directions: precisely, the holonomy of $\mathscr{G}$ along any circle fibre in the compact leaf must be contracting on one side of the leaf, and expanding on the other.

The models for the tangentialities are classical, being nothing but W Thurston's construction to fill holes in codimensions 2 and more [36, Section 4]. To fix ideas, the smooth (C^∞) differentiability class is understood everywhere, unless otherwise specified. On the interval $\mathbb{I} := [0,1]$, fix a smooth real function $r \mapsto u(r)$ such that $u'(r) > 0$ for $0 < r < 1$, and $u(r)$ and all its successive derivatives vanish at $r = 0$, and $u(r) + u(1-r) = 1$. Write $\mathbb{D}^n$ (resp. $\mathbb{S}^{n-1}$) for the compact unit ball (resp. sphere) in $\mathbb{R}^n$; endow $\mathbb{D}^2$ with the polar coordinates ρ, θ; endow $\mathbb{S}^1$ with the coordinate s; on $\mathbb{D}^2 \times \mathbb{S}^1$, one has the smooth 1–parameter family $(\omega_r)_{r\in\mathbb{I}}$ of smooth nonsingular 1–forms defined for $0 \le r \le \frac{1}{2}$ by

$$\omega_r := u(1-2\rho)\,ds + u(2\rho)\,d\rho$$

$\left(\text{on } \left\{\rho \le \frac{1}{2}\right\}\right)$ and

$$\omega_r := u(2-2\rho)\,d\rho + u(2\rho-1)(ds - u(1-2r)\,d\theta)$$

$\left(\text{on } \left\{\rho \ge \frac{1}{2}\right\}\right)$; while, for $\frac{1}{2} \le r \le 1$,

$$\omega_r := u(2-2r)\omega_{1/2} + u(2r-1)\,ds.$$

For every r, the form ω_r is integrable (it is immediately verified that $\omega_r \wedge d\omega_r = 0$); hence one gets on $\mathbb{D}^2 \times \mathbb{S}^1$ a 1–parameter family of codimension-1 foliations, seen in Figure 1.

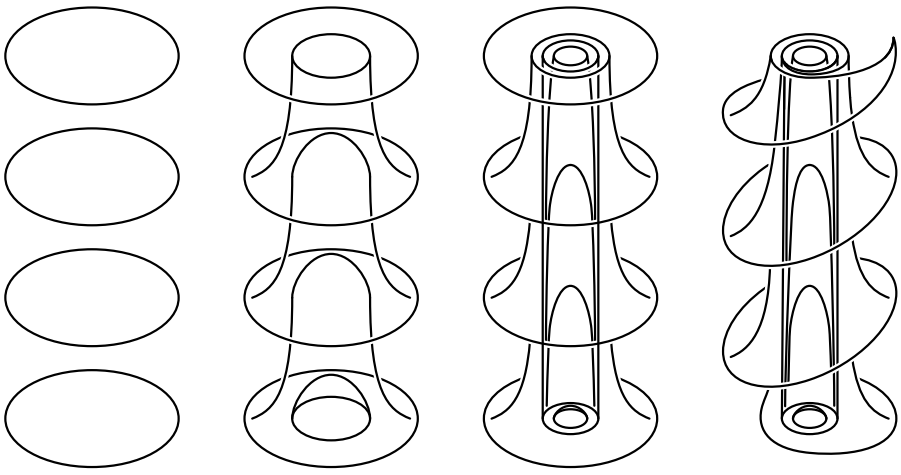

Figure 1: Thurston's construction to fill holes in codimensions 2 and more.

Fix $p, q \geq 2$. For every closed $(p-2)$–fold Σ, we define the *multifold Reeb component of core* Σ as the $(p+q)$–fold

$$C_\Sigma := \Sigma \times \mathbb{D}^2 \times \mathbb{D}^{q-1} \times \mathbb{S}^1$$

(whose projection to the i^{th} factor for $1 \leq i \leq 4$ will be denoted by pr_i) endowed with two foliations:

- The dimension-q foliation $\mathscr{F}_\Sigma$ parallel to the factor $\mathbb{D}^{q-1} \times \mathbb{S}^1$.
- The codimension-q foliation $\mathscr{G}_\Sigma$ obtained by endowing, for every $a \in \mathbb{D}^{q-1}$, the fibre $\mathrm{pr}_3{}^{-1}(a)$ with its codimension-1 foliation pullback of the 1–form $\omega_{|a|}$ under $\mathrm{pr}_2 \times \mathrm{pr}_4$.

Definition 1.1 On a $(p+q)$–fold M, the codimension-q foliation $\mathscr{G}$ is *quasicomplementary* to the dimension-q foliation $\mathscr{F}$ if they are transverse except maybe for finitely many disjoint multifold Reeb components $C_\Sigma \hookrightarrow M$, in which $\mathscr{F}$ (resp. $\mathscr{G}$) coincides with $\mathscr{F}_\Sigma$ (resp. $\mathscr{G}_\Sigma$).

The components may have different cores; one can of course also consider the union of the components as one component whose core may be not connected. Note that $\mathscr{G}$ is almost everywhere complementary to $\mathscr{F}$: precisely, everywhere on M, except on the subset defined in each component C_Σ by $\rho \circ \mathrm{pr}_2 = \frac{1}{2}$ and $\|\mathrm{pr}_3\| \leq \frac{1}{2}$, which

is a hypersurface tangential to $\mathscr{F}$. Also, it is easily verified that $\mathscr{G}$ is a limit of codimension-q plane fields complementary to $\mathscr{F}$ on M.

For the elements on Haefliger structures, namely normal bundle, *differential*, concordances, regularity and relations to foliations, see Section 2. Recall the foliation theorem in codimensions 2 and more, also known as the "h–principle for foliations", on closed manifolds [36]; see also Eliashberg and Mishachev [6] and Mitsumatsu and Vogt [30]. We use the notation $\mathscr{O}p_X(Y)$ for "some open neighbourhood of Y in X".

Theorem 1.2 (Thurston) *On a compact manifold M, let ν be a real vector bundle of dimension $q \geq 2$, let γ be a Γ_q–structure whose normal bundle is ν, and let ω be a 1–form valued in ν of constant rank q; assume that $d\gamma = \omega$ on $\mathscr{O}p_M(\partial M)$.*

Then M admits a ***regular*** *Γ_q–structure γ' of normal bundle ν such that*:

- *$\gamma' = \gamma$ on $\mathscr{O}p_M(\partial M)$, and γ' is concordant to γ on M (rel ∂M).*
- *$d\gamma'$ is homotopic to ω on M (rel ∂M) among the 1–forms valued in ν of constant rank q.*

Our main result is a refinement of this classical one for a manifold already foliated:

Theorem A *On a compact manifold M, let $\mathscr{F}$ be a foliation of dimension $q \geq 2$, let γ be a Γ_q–structure whose normal bundle is $\tau\mathscr{F}$, and let ω be a 1–form valued in $\tau\mathscr{F}$ such that $\omega|_{\tau\mathscr{F}}$ is of constant rank q; assume that $d\gamma = \omega$ on $\mathscr{O}p_M(\partial M)$.*

Then M admits a ***regular*** *Γ_q–structure γ' of normal bundle $\tau\mathscr{F}$ such that*:

- *$\gamma' = \gamma$ on $\mathscr{O}p_M(\partial M)$, and γ' is concordant to γ on M (rel ∂M).*
- *$d\gamma'$ is homotopic to ω on M (rel ∂M) among the 1–forms valued in $\tau\mathscr{F}$ of constant rank q.*
- *The foliation induced by γ' is quasicomplementary to $\mathscr{F}$ on M.*

Remark 1.3 We put no restriction on the position of $\mathscr{F}$ with respect to ∂M.

Remark 1.4 The construction does not use sophisticated results on groups of diffeomorphisms. For every integer $r \geq 1$, Theorem A holds as well in the differentiability class C^r, with the same proof.

Remark 1.5 This result and our proof are also valid for $p = 1$, in which case there exist no multifold Reeb components at all, hence "quasicomplementary" means "complementary". Of course, it is not a great deal to produce a foliation complementary to a given codimension-1 foliation; however, the point is that there is one *in every concordance class* of Γ_q–structures whose normal bundle is $\tau\mathscr{F}$, and this goes in every differentiability class (Remark 1.4), including C^{q+1}.

Remark 1.6 By construction, for $p := \operatorname{codim}(\mathscr{F}) \geq 2$, the core Σ of each multifold Reeb component resulting from our construction will be a product of two spheres $\mathbb{S}^{i-1} \times \mathbb{S}^{p-i-1}$, with $1 \leq i \leq p-1$; see Proposition 4.21. One can if one likes better, by a trick due to Thurston [37, beginning of Section 5], arrange that Σ is the $(p-2)$–torus, or, for $p \geq 3$, that $\Sigma = \mathbb{S}^1 \times \mathbb{S}^{p-3}$ — see Meigniez [27, Sections 3.3.2 and 3.3.3].

Denote, as usual, by $B\Gamma_q^r$ (resp. $B\overline{\Gamma}_q^r$) the Haefliger space classifying the Γ–structures (resp. parallelized Γ–structures) of codimension q and differentiability class C^r. One has in particular the following corollaries of Theorem A, since $B\overline{\Gamma}_q^1$ is contractible — see Tsuboi [38] — and since $B\overline{\Gamma}_q^\infty$ is $(q+1)$–connected; see Thurston [35].

Corollary 1.7 *Let $\mathscr{F}$ be a C^∞ foliation of dimension at least 2 on a closed manifold. Then $\mathscr{F}$ admits a quasicomplementary foliation of class C^1.*

If moreover $\operatorname{codim}(\mathscr{F}) = 2$, *or if the bundle $\tau\mathscr{F}$ is trivializable, then $\mathscr{F}$ admits a quasicomplementary foliation of class C^∞.*

1.1.1 About the proof Theorem A is better established under a version (Theorem A′) producing a *cleft* foliation $\mathscr{G}$ with fissures (see Section 3) instead of multifold Reeb components; the fissures are some kind of discontinuities in the foliation that have a product structure $\Sigma \times \mathbb{D}^q$ where Σ is a compact $(p-2)$–fold; in that frame, quasicomplementarity means that outside the fissures, $\mathscr{G}$ is complementary to $\mathscr{F}$, and that on the fissures, every slice parallel to $\mathbb{D}^q$ is a plaque of $\mathscr{F}$. For $q \geq 2$, Theorems A and A′ are straightforwardly equivalent to each other through Thurston's method to fill the holes, but Theorem A′ also holds in codimension $q = 1$.

To prove Theorem A′, the problem is translated, using the Gromov–Phillips–Haefliger parametric foliation theorem on open manifolds, into an extension problem whose proof falls to an adaptation of the original "inflation" process that Thurston introduced to prove Theorem 1.2.

We feel that the present work illustrates the power and the accuracy, in the framework of Gromov's h–principle, of the tools that Thurston left to us after his early works on foliations.

1.2 A proof of the Mather–Thurston theorem

A second application of our method deals with the construction of foliated products, and more generally of foliated bundles. We begin with discussing foliated products.

In the case where the given foliation $\mathscr{F}$ is a product foliation, we can get full complementarity at the price of modifying the base factor of the product by some surgeries; in other words we give a proof of the classical Mather–Thurston theorem [35], by means of a geometric construction pertaining to the h–principle.

See J Mather's proof in codimension 1 in [21]; for the general codimensions, Mather [23, pages 79–80] mentions that Thurston had three different proofs — the first seems to be lost; see Mather [22; 23] McDuff [24; 25] and Segal [34] for the two other proofs; see also Kupers [18] for a modern one.

Precisely, fix a manifold X of dimension $q \geq 1$, without boundary, not necessarily compact. "At infinity" means as usual "except maybe on some compact subset". For a compact oriented manifold V of dimension $p \geq 0$ maybe with smooth boundary, on $V \times X$, consider the slice foliation $\mathscr{F}_V$ parallel to X; and the *horizontal* codimension-q foliation parallel to V, or equivalently the *horizontal* Γ_q–structure whose differential is the projection

$$\tau(V \times X) \to \tau\mathscr{F}_V$$

parallel to τV.

Recall that a *foliated X–product* over V means a codimension-q foliation on $V \times X$ complementary to $\mathscr{F}_V$ on $V \times X$ and horizontal at infinity. Equivalently, this amounts to a Γ_q–structure on $V \times X$ whose normal bundle is $\tau\mathscr{F}_V$, whose differential induces the identity on $\tau\mathscr{F}_V$, and horizontal at infinity.

We use the notation $\widehat{V} \subset V \times \mathbb{I}$ for the union of $V \times 0$ with $\partial V \times \mathbb{I}$. An *oriented cobordism* (V, W, V^*) *rel* ∂V means as usual a compact oriented p–fold V^* together with an oriented diffeomorphism $\partial V^* \cong \partial V$, and a compact oriented $(p+1)$–fold W bounded by $-\widehat{V} \cup_{\partial V} V^*$. We write $\widehat{\mathrm{pr}}$ for the restriction to $\widehat{V} \times X \subset V \times \mathbb{I} \times X$ of the projection $V \times \mathbb{I} \times X \to V \times X$. We shall prove:

Theorem 1.8 (Mather–Thurston, version "for geometrically minded topologists") *Let V be a compact oriented manifold; let γ be a Γ_q–structure on $V \times X$, of normal bundle $\tau\mathscr{F}_V$, horizontal at infinity, and restricting to a foliated X–product over ∂V.*

Then there are an oriented cobordism (V, W, V^*) rel ∂V and a Γ_q–structure on $W \times X$, of normal bundle $\tau\mathscr{F}_W$, horizontal at infinity, coinciding with $\widehat{\mathrm{pr}}^*(\gamma)$ on $\widehat{V} \times X$, and restricting to a foliated X–product over V^*.

This interpretation of the Mather–Thurston theorem was introduced by D B Fuchs [9]. Let us now recall how the classical version of the theorem can be deduced.

Consider the group $\mathrm{Diff}_c(X)$ of the compactly supported diffeomorphisms of X, endowed with the smooth topology; the same group $\mathrm{Diff}_c(X)_\delta$ with the discrete topology; the identity map

$$\mathrm{Diff}_c(X)_\delta \to \mathrm{Diff}_c(X)$$

and the topological group $\overline{\mathrm{Diff}_c(X)}$ that is the homotopy-theoretic fibre of this map. Recall that its classifying space $B\overline{\mathrm{Diff}_c(X)}$ also classifies the foliated X–products.

Realize $B\Gamma_q$ as a fibred space over BO_q with fibre $B\overline{\Gamma}_q$. Denote by $\mathscr{X}$ the foliation of X by points (regarded as a Γ_q–structure on X whose normal bundle is τX and whose differential is the identity of τX). Consider the space $\Gamma_c(\tau X)$ of the maps

$$f : X \to B\Gamma_q$$

such that f lifts the map $X \to BO_q$ classifying τX, and such that f classifies $\mathscr{X}$ at infinity. One can view $\Gamma_c(\tau X)$ as the space of the Γ_q–structures on X whose normal bundle is τX and coinciding with $\mathscr{X}$ at infinity. For $X = \mathbb{R}^q$, the space $\Gamma_c(\tau\mathbb{R}^q)$ coincides with the q^{th} loop space $\Omega^q(B\overline{\Gamma}_q)$.

Consider the map

$$B\overline{\mathrm{Diff}_c(X)} \xrightarrow{\alpha} \Gamma_c(\tau X)$$

adjoint to the map

$$X \times B\overline{\mathrm{Diff}_c(X)} \to B\Gamma_q$$

that classifies the Γ_q–structure of the total space of the universal foliated X–product [21; 35].

Theorem 1.8 amounts to saying that α induces an isomorphism in oriented bordism. Equivalently, by the "Hurewicz theorem for bordism groups" (see Atiyah and Hirzebruch [1] and Eliashberg, Galatius and Mishachev [4, Appendix B]), α *induces*

an isomorphism in integral homology. That last wording is the classical one. The map α is actually a homology equivalence: indeed, $B\overline{\Gamma}_q$ being $(q+1)$–connected in class C^∞ [35], the space $\Gamma_c(\tau X)$ is simply connected.

There is also a well-known generalization of the Mather–Thurston theorem for foliated bundles rather than foliated products.

Still consider a manifold X of dimension $q \geq 1$; to simplify, assume that X is closed. As usual, an *X–bundle* (E, π) over a compact manifold V is a smooth locally trivial bundle map $\pi : E \to V$ whose fibres are diffeomorphic with X; a *foliated X–bundle* means moreover a codimension-q foliation on E complementary to the fibres of π. Equivalently, this amounts to a Γ_q–structure on E whose normal bundle is $\ker(d\pi)$ and whose differential induces the identity on $\ker(d\pi)$.

The generalized Mather–Thurston theorem can be stated as an isomorphism in integral homology between the space $B\mathrm{Diff}^\infty(X)_\delta$ that classifies the foliated X–bundles, and the classifying space for Γ_q–structures on X–bundles whose normal bundle is tangential to the fibres. For a construction of this last classifying space and a proof of the generalized theorem as a corollary of the usual Mather–Thurston theorem, see Nariman [31, Section 1.2.2].

Here is the version "for geometrically minded topologists" of the generalized Mather–Thurston theorem. The notation $\widehat{V}$ has the same meaning as above.

Theorem 1.9 *Let (E, π) be an X–bundle over a compact oriented manifold V; let γ be a Γ_q–structure on E, of normal bundle $\ker(d\pi)$, and restricting to a foliated X–bundle over ∂V.*

Then there are

- *an oriented cobordism (V, W, V^*) rel ∂V;*
- *an X–bundle (E_W, π_W) over W, coinciding over $\widehat{V}$ with the pullback of (E, π) through the projection $V \times \mathbb{I} \to V$;*
- *a Γ_q–structure on E_W, of normal bundle $\ker(d\pi_W)$, coinciding over $\widehat{V}$ with the pullback of γ through the projection $E \times \mathbb{I} \to E$, and restricting to a foliated X–bundle over V^*.*

Our method to prove Theorem 1.8 actually also gives Theorem 1.9 after a very few straightforward changes which are left to the reader.

Note 1.10 (other differentiability classes) The Mather–Thurston theorem and its generalization for foliated bundles are well known to hold for every differentiability class C^r for r an integer ≥ 1. Our proof goes without change in these classes. Three other interesting classes are C^0, Lipschitz and PL. The validity of the Mather–Thurston theorem is classical in the classes C^0 and Lipschitz, and also in the class PL in codimension 1, but it remains open in the class PL for $q \geq 2$. Our methods cannot handle the C^0 class, but do handle the Lipschitz and PL classes after some technicalities and adaptations; this material will be covered in a forthcoming paper.

Let us illustrate Theorem 1.9 by two known corollaries. Since $B\overline{\Gamma}_q^1$ is contractible [38] and since $B\overline{\Gamma}_q^\infty$ is $(q+1)$–connected [35]:

Corollary 1.11 (i) *Every X–bundle over a closed oriented manifold is cobordant to a C^1–foliated X–bundle.*

(ii) *Every X–bundle over a closed oriented* surface *is cobordant to a C^∞–foliated X–bundle.*

For (ii), when X itself is a surface, see also [31, Remark 3.16].

Jenseits des Homotopieprinzips One could maybe speak of a "c–principle", with a c for "cobordism". Recall that in situations where Gromov's famous *h–principle* holds, every "formal" object is *homotopic* to a genuine object through the formal objects. In the same way, let us say that the *c–principle* holds when every formal object is *cobordant* to a genuine object through the formal objects. Examples of results pertaining to the c–principle are the Mather–Thurston theorem for foliated products or bundles, the Madsen–Weiss theorem for fibrations whose fibres are surfaces (see [4]), and the realization of taut compactly generated pseudogroups by foliations of dimension 2 and codimension 1; see Meigniez [26]. In the same spirit, see Fuchs' early paper [9] and Kupers' recent one [18].

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2 Haefliger structures

In this section, we recall A Haefliger's notion of Γ_q*–structure* [12; 13; 14; 15], under the form of microfoliated bundle (this form was introduced in [13]; see also [29]).

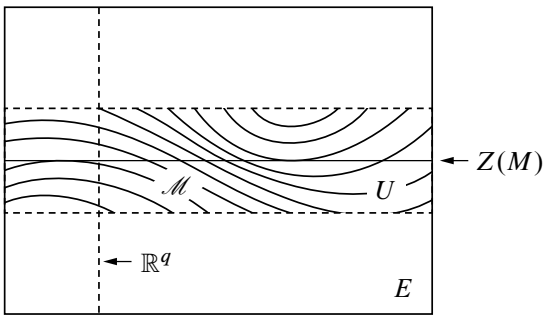

Figure 2: A Γ_q–structure.

We fix some vocabulary, point out a few elementary facts and prove (again) the two parametric forms of the classical foliation theorem on open manifolds.

We write every real vector bundle ν over a manifold M in the form $\nu = (E, \pi, Z)$ where E is the total space, $\pi : E \to M$ is the projection and $Z : M \to E$ is the zero section.

2.0.1 Definition A Γ_q–structure γ on M is given by

- a real vector bundle $\nu = (E, \pi, Z)$ of dimension q over M;
- an open neighbourhood U of $Z(M)$ in E;
- on U, a codimension-q foliation $\mathscr{M}$ transverse to every fibre.

One calls ν the *normal bundle*, and $\mathscr{M}$ the *microfoliation* (Figure 2). One regards two Γ_q–structures as equal if they have the same normal bundle and if their microfoliations coincide on some open neighbourhood of the zero section; in other words, γ is actually the germ of $\mathscr{M}$ along $Z(M)$; we shall also denote γ by $[\nu, U, \mathscr{M}]$.

A $\overline{\Gamma}_q$–*structure* means a Γ_q–structure whose normal bundle is $M \times \mathbb{R}^q$.

2.0.2 Canonical form and differential Consider a Γ_q–structure $\gamma = [\nu, U, \mathscr{M}]$ on a manifold M, where $\nu = (E, \pi, Z)$.

On the manifold U, let Ω be the differential 1–form valued in $\pi^*(\nu)$, defined at every point $v \in U$ as the projection of $\tau_v E$ onto $\ker(d_v\pi) = \nu_{\pi(v)}$ parallel to $\tau_v\mathscr{M}$. If one likes better, Ω is the unique 1–form defining the foliation $\mathscr{M}$ (in the sense that $\tau\mathscr{M} = \ker\Omega$) and whose restriction to every fibre of π is the identity. We call Ω the *canonical* form of the Γ_q–structure γ. Let us define the *differential* $d\gamma$ of the Γ_q–structure γ as $Z^*(\Omega)$: a differential 1–form on M valued in ν.

Note 2.1 The notion of differential for a Haefliger structure does not seem to appear in the literature. In the case where $d\gamma$ is of rank q, of course $d\gamma$ admits a convex set of left inverse vector bundle monomorphisms $\nu \hookrightarrow \tau M$, which are the objects that the authors have considered instead. The differential exists for every Haefliger structure at every point, not only at the regular ones (see Section 2.0.6). It is functorial with respect to pullbacks (see Section 2.0.3). We feel that, speaking in terms of $d\gamma$, the analogy between the h–principle for foliations and several other *avatars* of Gromov's h–principle becomes more transparent. From our viewpoint, the foliation theorems, Theorems 1.2 and 2.13, deal with homotoping a given differential form of maximal rank to one which is somehow integrable.

2.0.3 Pullback Given a Γ_q–structure $\gamma = [\nu, U, \mathscr{M}]$ on M, and given a smooth mapping $f: N \to M$, one defines a pullback Γ_q–structure $f^*(\gamma)$ on N whose normal bundle is the usual pullback vector bundle $f^*(\nu)$, and whose microfoliation is the pullback of the foliation $\mathscr{M}$ under the canonical bundle morphism $f^*(\nu) \to \nu$.

Example 2.2 We give, as an example, a useful construction allowing one to regard the microfoliation of any Haefliger structure as another Haefliger structure.

Given $\gamma = [\nu, U, \mathscr{M}]$ as above, where $\nu = (E, \pi, Z)$, consider the vector bundle $\widehat{\nu} := \pi^*(\nu)$ of rank q over the base E; consider its total space

$$\widehat{E} = E \times_M E = \{(v, w) \in E \times E \mid \pi(v) = \pi(w)\}$$

and the exponential map

$$\exp: \widehat{E} \to E, \quad (v, w) \mapsto v + w.$$

One has *on the manifold* U the Γ_q–structure $\widehat{\gamma}$ whose normal bundle is $\widehat{\nu}|_U$ and whose microfoliation is the pullback of the foliation $\mathscr{M}$ under exp. Clearly,

$$Z^*(\widehat{\gamma}) = \gamma. \tag{1}$$

2.0.4 Isomorphisms J Milnor's notion of *microbundle* (here smooth) is the natural one for the bundle normal to a Haefliger structure. For simplicity, one rather speaks in terms of vector bundle; but this underlying fact is reflected in the morphisms that one admits. It is enough for us to consider isomorphisms.

Definition 2.3 A *microisomorphism* between two real vector bundles $\nu = (E, \pi, Z)$ and $\nu' = (E', \pi', Z')$ over the same manifold M is a germ of diffeomorphism $\varphi: E \to E'$ along $Z(M)$ such that $Z' = \varphi \circ Z$ and $\pi = \pi' \circ \varphi$.

If moreover $\nu' = \nu$ and if the differential of φ at every point of $Z(M)$ is the identity, we call φ a *special microautomorphism* of ν.

Clearly, the group $\mathrm{Maut}(\nu)$ of the microautomorphisms of ν splits as the semidirect product of the group $\mathrm{Smaut}(\nu)$ of the special microautomorphisms by the group $\mathrm{Aut}(\nu)$ of the linear automorphisms.

Note that $\mathrm{Smaut}(\nu)$ is convex, hence contractible.

Definition 2.4 Given two Γ_q–structures $\gamma := [\nu, U, \mathscr{M}]$ and $\gamma' := [\nu', U', \mathscr{M}']$ on the same manifold M, an *isomorphism* between them is a microisomorphism $\varphi\colon \nu \to \nu'$ such that $\mathscr{M} = \varphi^*(\mathscr{M}')$ (as germs of foliations along $Z(M)$).

2.0.5 Concordance Let γ_0 and γ_1 be two Γ_q–structures on the same manifold M which coincide in restriction to some submanifold $N \subset M$ (maybe empty).

Definition 2.5 A *concordance,* also known as a *homotopy,* between γ_0 and γ_1 (rel N) is a Γ_q–structure γ on $M \times \mathbb{I}$ such that

- $\gamma|_{M\times i} = \gamma_i$ for $i = 0, 1$;
- γ coincides with $\mathrm{pr}_1^*(\gamma_0)$ in restriction to $N \times \mathbb{I}$.

Lemma 2.6 *If γ_0 and γ_1 are isomorphic (rel N), then they are concordant (rel N).*

Proof Put $\gamma_i = [\nu_i, U_i, \mathscr{M}_i]$ for $i = 0, 1$. Let φ be a microisomorphism between γ_0 and γ_1 such that $\varphi|_N = \mathrm{id}$. Without loss of generality, shrinking U_0 and U_1, arrange that $U_1 = \varphi(U_0)$.

Endow $\overline{U}_0 := U_0 \times \mathbb{I}$ with the foliation $\overline{\mathscr{M}}_0 := \mathrm{pr}_1^*(\mathscr{M}_0)$; in the disjoint union $\overline{U}_0 \sqcup U_1$, identify each point $(v, 1) \in \overline{U}_0$ with $\phi(v) \in U_1$. The quotient space is diffeomorphic to an open neighbourhood of the zero section in a vector bundle over $M \times \mathbb{I}$; and on this quotient space, the image of $\overline{\mathscr{M}}_0$ is the microfoliation of a concordance (rel N) between γ_0 and γ_1. □

Proposition 2.7 (concordance extension property) *Given a Γ_q–structure γ on M and a submanifold $N \subset M$, every concordance of $\gamma|_N$ extends to a concordance of γ.*

This is obvious from Haefliger's original viewpoint on the Γ–structures [12], obvious if one sees Γ_q–structures as maps to $B\Gamma_q$, and almost as obvious from the geometric viewpoint adopted here. See also [21, page 199].

2.0.6 Regular Haefliger structures and foliations The following remarks are intended to make clear the relations between two notions: foliations on the one hand, regular Haefliger structures on the other hand.

Vocabulary 2.8 Given a codimension-q foliation $\mathscr{G}$ on a manifold M, we define as usual the bundle *normal* to $\mathscr{G}$ as $\nu\mathscr{G} := \tau M/\tau\mathscr{G}$; and we define the *differential* of $\mathscr{G}$ as the canonical projection $d\mathscr{G}\colon \tau M \to \nu\mathscr{G}$.

If $\mathscr{G}$ is moreover complementary to a dimension-q foliation $\mathscr{F}$ on M, we often identify $\nu\mathscr{G}$ with $\tau\mathscr{F}$; then, $d\mathscr{G}$ becomes a *linear retraction* $\tau M \to \tau\mathscr{F}$.

A Γ_q–structure $\gamma = [\nu, U, \mathscr{M}]$ on M, where $\nu = (E, \pi, Z)$, is called *regular* at a point $x \in M$ if $d\gamma$ is of maximal rank q at x; in other words, Z is transverse to $\mathscr{M}$ at x.

Definition 2.9 If γ is regular on M, one says that γ *induces* the codimension-q foliation $\mathscr{G} := Z^*(\mathscr{M})$ on M. If moreover

- the bundle ν normal to γ *equals* the bundle $\nu\mathscr{G}$ normal to $\mathscr{G}$;
- the differential $d\gamma$ *equals* the differential $d\mathscr{G}$ of the foliation $\mathscr{G}$;

then we say that γ *specially* induces $\mathscr{G}$, and that γ is a *special* regular Haefliger structure.

Lemma 2.10 (a) *Every codimension-q foliation is specially induced by some regular Γ_q–structure.*

(b) *Two regular Γ_q–structures are isomorphic if and only if they induce the same foliation.*

(c) *The identity is the unique automorphism of a regular Γ_q–structure.*

Proof Fix an auxiliary Riemannian metric on M; denote by $D(x, r) \subset M$ the open ball of centre x and radius r.

(a) Given a codimension-q foliation $\mathscr{G}$ on M, let $\exp\colon \nu\mathscr{G} \to M$ be the exponential map $\tau M \to M$ restricted to the subbundle $\tau\mathscr{G}^\perp \cong \nu\mathscr{G}$. Clearly, on a neighbourhood of the zero section, $\exp^*(\mathscr{G})$ is the microfoliation of a Γ_q–structure specially inducing $\mathscr{G}$.

(b) One can assume that M is compact. Let $\gamma := [E, \pi, Z, U, \mathscr{M}]$ and $\gamma' := [E', \pi', Z', U', \mathscr{M}']$ be two regular Γ_q–structures on M inducing the same foliation $\mathscr{G} = Z^*(\mathscr{M}) = Z'^*(\mathscr{M}')$.

Since $\mathscr{M}'$ is complementary in U' to the fibres of π', there is an $r > 0$ so small that for every $x \in M$, the projection π' admits a local section $\sigma'_x : D(x,r) \to U'$ tangential to $\mathscr{M}'$ and such that $\sigma'_x(x) = Z'(x)$. Clearly, $\sigma'_x(y)$ depends smoothly on the pair (x,y).

On the other hand, since γ is regular, after shrinking U, one has for every $v \in U$ a point $Z(\mathrm{pr}(v))$ of the zero section in the leaf of $\mathscr{M}$ through v; and one can arrange that $\mathrm{pr} : U \to M$ is smooth, and that $\mathrm{pr} \circ Z = \mathrm{id}$. One has $\mathscr{M} = \mathrm{pr}^*(\mathscr{G})$. Moreover, after shrinking U again, one can arrange that for every $x \in M$, the diameter of $\pi(\mathrm{pr}^{-1}(x))$ in M is less than r. One defines, for $v \in U$,

$$\varphi(v) := \sigma'_{\mathrm{pr}(v)}(\pi(v))$$

So, φ is smooth, and maps every leaf of $\mathscr{M}$ into a leaf of $\mathscr{M}'$, and $\pi' \circ \varphi = \pi$, and $\varphi \circ Z = Z'$.

It remains to verify that φ induces a diffeomorphism between $\mathscr{O}p_E(Z(M))$ and $\mathscr{O}p_{E'}(Z'(M))$. To this end, permuting the roles of γ and γ', one makes a similar map

$$\varphi' : \mathscr{O}p_{E'}(Z'(M)) \to \mathscr{O}p_E(Z(M))$$

and one considers

$$\psi := \varphi' \circ \varphi : \mathscr{O}p_E(Z(M)) \to \mathscr{O}p_E(Z(M)).$$

So, ψ maps every leaf of $\mathscr{M}$ into a leaf of $\mathscr{M}$, and $\pi \circ \psi = \pi$, and $\psi \circ Z = Z$. We claim that such a map ψ is necessarily the identity on some neighbourhood of $Z(M)$.

Indeed, shrinking U again, one can arrange that ψ is defined on U. Since γ is regular, shrinking U again, the saturation of $Z(M)$ with respect to the foliation $\mathscr{M}$ is U. Consider any leaf L of $\mathscr{M}$. In particular, L is connected. Since L meets $Z(M)$ and ψ is the identity there, one has $\psi(L) \subset L$. Since $\pi|_L$ is an étale map preserved by ψ, the fixed-point set of $\psi|_L$ is open and closed in L, hence $\psi|_L = \mathrm{id}$. The claim is proved.

(c) This is a particular case of the claim. □

In other words:

(i) A codimension-q foliation on M amounts to an *isomorphism class* of regular Γ_q–structures on M.

(ii) For every codimension-q foliation $\mathscr{G}$ on M, the regular Γ_q–structures on M *specially* inducing $\mathscr{G}$ form a principal space under the group $\mathrm{Smaut}(\nu\mathscr{G})$.

Conclusion 2.11 (a) In view of Lemma 2.6 and of (i) above, one can speak of the concordance class of a codimension-q foliation $\mathscr{G}$, defined as the concordance class of any Γ_q–structure inducing $\mathscr{G}$.

(b) In view of (ii) above and $\mathrm{Smaut}(\nu\mathscr{G})$ being contractible, there is no inconvenience in identifying a codimension-q foliation $\mathscr{G}$ with any regular Γ_q–structure *specially* inducing $\mathscr{G}$.

2.0.7 The parametric foliation theorem on open manifolds The classical foliation theorem on open manifolds admits two parametric versions, that we respectively call "nonintegrable" and "integrable". The second will be used repeatedly in the proof of Theorem A′. It does not seem to appear explicitly in the literature, although all proofs of the nonintegrable version (for example the one in [20]) actually prove the integrable one as well.

The space of parameters will be a compact manifold A; fix a compact submanifold $B \subset A$, maybe empty. Consider, over a manifold M, a real vector bundle $\nu = (E, \pi, Z)$ of dimension q, and its pullback $\widetilde{\nu}$ over $M \times A$.

Definition 2.12 By a *family* of Haefliger structures $(\gamma(a))_{a\in A}$ whose normal bundle is ν, one means, for every $a \in A$, a Γ_q–structure γ_a on M whose normal bundle is ν — denote its microfoliation by $\mathscr{M}(a)$ — such that the (germ of the) plane field $\tau\mathscr{M}(a)$ on $\mathscr{O}p_E(Z(M))$ depends smoothly on a.

Call the family $(\gamma(a))_{a\in A}$ *integrable* (with respect to the parameter) if, moreover, there is a global Γ_q–structure $\widetilde{\gamma}$ on $M \times A$ whose normal bundle is $\widetilde{\nu}$ and whose restriction to the slice $M \times a$ is $\gamma(a)$ for every $a \in A$.

Fix a compact submanifold $N \subset M$ (maybe empty) such that the pair (M, N) is *open,* in the sense, classical in the h–principle, that every connected component of the complement $M \setminus N$ which is relatively compact in M meets ∂M. Assume that M carries a (smooth) parametric family $(\omega(a))_{a\in A}$ of ν–valued differential 1–forms of constant rank q such that $d\gamma(a) = \omega(a)$ holds on $\mathscr{O}p_M(N)$ for every $a \in A$, and on M for every $a \in \mathscr{O}p_A(B)$. Consider the projection $\mathrm{pr}_1 : (x, t) \mapsto x$ and the embedding $\iota_t : x \mapsto (x, t)$ for $x \in M$ and $t \in \mathbb{I}$.

Theorem 2.13 *Let $(\gamma(a))_{a\in A}$ be a family of Γ_q–structures on M.*

Then, under the above hypotheses:

(i) *There is a smooth family* $(\overline{\gamma}(a))_{a\in A}$ *of* Γ_q*–structures on* $M\times\mathbb{I}$ *such that for every* $a\in A$:
   (1) $\iota_0^*(\overline{\gamma}(a))=\gamma(a)$.
   (2) $\overline{\gamma}(a)=\mathrm{pr}_1^*(\gamma(a))$ *on* $\mathcal{O}p_M(N)\times\mathbb{I}$.
   (3) *If* $a\in B$, *then* $\overline{\gamma}(a)=\mathrm{pr}_1^*(\gamma(a))$ *on* $M\times\mathbb{I}$.
   (4) $\gamma'(a):=\iota_1^*(\overline{\gamma}(a))$ *is regular.*

(ii) *Moreover, the family* $(d\gamma'(a))_{a\in A}$ *is homotopic on* M, *rel* $(N\times A)\cup(M\times B)$, *to the family* $(\omega(a))_{a\in A}$ *among the families of* ν*–valued* 1*–forms of constant rank* q.

(iii) *If the family* $(\gamma(a))_{a\in A}$ *is moreover integrable, then one can choose the family* $(\overline{\gamma}(a))_{a\in A}$ *to be also integrable.*

Proof Haefliger's original proof of the nonparametric foliation theorem on open manifolds [13; 14] is a direct application of the Gromov–Phillips transversality theorem in the frame of Γ–structures: the transversality theorem is applied, in the total space of the normal bundle of the structure, to the zero section, and provides a homotopy that puts it transverse to the microfoliation. This argument goes with parameters and thus proves both parametric versions, the nonintegrable and the integrable. Here are the details, for the sake of completeness.

Choose a smooth plateau function χ on $M\times A$ such that

- $\chi=1$ on a neighbourhood of $P:=(N\times A)\cup(M\times B)$;
- $\omega(a)_x=d\gamma(a)_x$ for every (x,a) in the support $\mathrm{spt}(\chi)$.

Let $V\subset E$ be an open neighbourhood of the zero section $Z(M)$, so small that $\mathscr{M}(\gamma(a))$ is defined on V for every parameter a, and that the plane field $\tau\mathscr{M}(a)$ on V depends smoothly on a. Let $\Omega(a)$ be, on V, the canonical, $\pi^*(\nu)$–valued 1–form defining $\mathscr{M}(a)$ (recall Section 2.0.2).

For a fixed $a\in A$, consider, over the map $Z\colon M\to E$, the bundle morphism

$$\zeta(a)\colon \tau M\to\tau E$$

defined for every $x\in M$ and $u\in\tau_x M$ as

$$\zeta(a)_x u:=\chi(x,a)u\oplus(1-\chi(x,a))\omega(a)_x u$$

(where the tangent space $\tau_{Z(x)}E$ is decomposed as $\tau_x M \oplus \nu_x$). In view of the choice of χ and of the definitions of $d\gamma$ and ζ,

$$\Omega(a)_{Z(x)} \circ \zeta(a)_x = \omega(a)_x. \tag{2}$$

Hence, $\zeta(a)_x$ is transverse to $\tau_{Z(x)}\mathscr{M}(a)$ in $\tau_{Z(x)}E$. By the Gromov–Phillips transversality theorem [10; 32; 11, pages 53 and 102] (which, if one likes, one can today obtain as an immediate application of the Eliashberg–Mishachev holonomic approximation theorem [7]), one has a map

$$H : M \times A \times \mathbb{I} \to V$$

and, over H, a homotopy of parametric families of bundles morphisms

$$\eta : \tau M \times A \times \mathbb{I} \to \tau E$$

such that for every $x \in M$, $u \in \tau_x M$, $a \in A$ and $t \in \mathbb{I}$:

(I) $H(x,a,t) = Z(x)$ if $(x,a) \in P$ or $t = 0$.

(II) $\eta(u,a,t) = \zeta(a)_x u$ if $(x,a) \in P$ or $t = 0$.

(III) The map $u \mapsto \eta(u,a,t)$ maps $\tau_x M$ linearly into $\tau_{H(x,a,t)}E$ transversely to $\tau_{H(x,a,t)}\mathscr{M}(a)$.

(IV) $\eta(u,a,1) = (\partial H/\partial x)(x,a,1)u$.

For every parameter a, define the Γ_q–structure $\overline{\gamma}(a)$ on $M \times \mathbb{I}$ as the pullback of $\widehat{\gamma}(a)$ (recall Example 2.2) through the map $(x,t) \mapsto H(x,a,t)$. The properties (1)–(3) of Theorem 2.13 follow from (I) above and from (1) in Example 2.2, while (4) follows from (III) and (IV).

(ii) Consider on M the 1–parameter family of A–parametrized families of rank-q, ν–valued 1–forms:

$$\varpi(a,t)_x u := \Omega_{H(x,a,t)}\eta(u,a,t)$$

(where $a \in A$, $t \in \mathbb{I}$, $x \in M$ and $u \in \tau_x M$). By (2) above and by (I) and (II), the identity $\varpi(a,t)_x = \omega(a)_x$ is met for $(x,a) \in P$ or $t = 0$. On the other hand, by the definitions of Ω and $\widehat{\gamma}$ and by (IV), one has $\varpi(a,1) = d\gamma'(a)$.

(iii) Assume moreover that every $\gamma(a)$ is the restriction to the slice $M \times a$ of a global Γ_q–structure $\widetilde{\gamma}$ on $M \times A$. Then every $\overline{\gamma}(a)$ is the restriction to the slice $M \times a \times \mathbb{I}$ of the global Γ_q–structure on $M \times A \times \mathbb{I}$ that is the pullback of $\widehat{\widetilde{\gamma}}$ through the map

$$H \times \mathrm{pr}_2 : M \times A \times \mathbb{I} \to V \times A. \qquad \square$$

3 Cleft Haefliger structures and cleft foliations

We shall actually prove first a form (Theorem A′) of Theorem A dealing with "fissures" instead of multifold Reeb components; this form goes as well for $q = 1$. Our fissures are some kind of discontinuities in a foliation which in the compact case amount to the same as the classical "holes", ubiquitous in the theory of foliations since Thurston's works [36; 37]. The advantage of fissures over holes is that fissures are more functorial.

Consider a compact manifold Q of dimension $q \geq 1$ with smooth boundary (we are essentially interested in the case $Q = \mathbb{D}^q$).

Notation 3.1 One denotes by $\mathrm{Diff}(Q)$ the group of the diffeomorphisms of Q *whose support is contained in the interior of* Q.

One denotes by $\mathrm{Diff}(Q)^{\mathbb{I}}$ the topological group (for the smooth topology) of the smooth pointed paths in $\mathrm{Diff}(Q)$: in other words, the families of diffeomorphisms $\varphi := (\varphi_t)_{t\in\mathbb{I}}$ such that

- $\varphi_t \in \mathrm{Diff}(Q)$;
- $\varphi_0 = \mathrm{id}$;
- $\varphi_t(y)$ depends smoothly on the pair $(t, y) \in \mathbb{I} \times Q$.

It is convenient to add that

- φ_t coincides with id (resp. φ_1) for t close to 0 (resp. 1).

Then, one considers as usual on $\mathbb{S}^1 \times Q$ the 1–dimensional foliation $\mathscr{S}_\varphi$, called the *suspension* of φ, spanned by the vector field $(\partial/\partial t, \partial\varphi/\partial t)$ (here $\mathbb{S}^1$ is the quotient of the interval $\mathbb{I}$ by $0 = 1$).

Fix a $\varphi \in \mathrm{Diff}(\mathbb{D}^q)^{\mathbb{I}}$. Denote by θ the angle coordinate $z \mapsto z/|z|$ on $\mathbb{C}^*$. The *model cleft of monodromy* φ is the manifold $\mathbb{C} \times \mathbb{R}^q$ endowed on the complement of $0 \times \mathbb{D}^q$ with the codimension-q foliation $\mathscr{C}_\varphi$ equal to $(\theta \times \mathrm{id}_{\mathbb{D}^q})^*(\mathscr{S}_\varphi)$ on $\mathbb{C}^* \times \mathbb{D}^q$ and horizontal (parallel to $\mathbb{C}$) on $\mathbb{C} \times (\mathbb{R}^q \setminus \mathbb{D}^q)$.

Let M be a manifold of dimension ≥ 0.

Definition 3.2 A (q–)*fissure* (or cleft) in M is a pair $(C, [c])$ where

(1) $C \subset M$ is a proper (ie topologically closed) codimension-2 submanifold, which may have a boundary;

(2) $[c]$ is the germ along C of a submersion

$$c\colon \mathscr{O}p_M(C) \to \mathbb{C}\times\mathbb{R}^q$$

such that $C = c^{-1}(0\times\mathbb{D}^q)$;

(3) at every point of $C\cap\partial M$, the rank of $c|_{\partial M}$ is also $q+2$.

Note:

- The image $c(C)$ is not necessarily all of $0\times\mathbb{D}^q$.
- We admit the case where $\dim(M)<2$, in which case C is necessarily empty.
- Sometimes, we regard the connected components of the fissure as several distinct fissures.
- By (3), if C intersects ∂M, then $(C,[c])$ induces a fissure $(C\cap\partial M,[c|_{\partial M}])$ in ∂M.

Definition 3.3 A *cleft Γ_q–structure of monodromy φ and normal bundle ν* on M is a triple $\Gamma := (C,[c],\gamma)$ such that

- $(C,[c])$ is a fissure in M;
- $\nu = (\mathrm{pr}_2\circ c)^*(\tau\mathbb{R}^q)$ on $\mathscr{O}p_M(C)$;
- γ is on $M\setminus C$ a Γ_q–structure of normal bundle $\nu|_{M\setminus C}$;
- γ specially induces $c^*(\mathscr{C}_\varphi)$ on $\mathscr{O}p_M(C)\setminus C$.

(Recall Definition 2.9.) The fissure may be empty, and must be empty if $\dim(M)<2$.

If M has a boundary, Γ induces on ∂M a cleft Γ_q–structure

$$\Gamma|_{\partial M} := (C\cap\partial M,[c|_{\partial M}],\gamma|_{\partial M}).$$

Definition 3.4 Let $\Gamma := (C,[c],\gamma)$ be a cleft Γ_q–structure on M. If moreover γ is regular on $M\setminus C$, and *specially* induces a foliation $\mathscr{G}$ there, then one calls Γ, or (recalling Conclusion 2.11(b)) the triple $G := (C,[c],\mathscr{G})$, a *cleft foliation*.

The two following notions will be crucial in the proof of Theorem A′. Let G be a cleft foliation on a manifold M.

Definition 3.5 One calls a vector field ∇ on M *tangential* to G if

- $\nabla(x)\in\tau_x\mathscr{G}$ at every point $x\in M\setminus C$;
- $\nabla(x)\in\ker(d_xc)$ at every $x\in\mathscr{O}p_M(C)$.

Definition 3.6 One calls a foliation $\mathscr{X}$ on M *tangential* to G if

- $\tau_x\mathscr{X} \subset \tau_x\mathscr{G}$ at every point $x \in M \setminus C$;
- $\tau_x\mathscr{X} \subset \ker(d_x c)$ at every $x \in \mathscr{O}p_M(C)$.

Now, let $\mathscr{F}$ be a dimension-q foliation on M.

Definition 3.7 The q–fissure $(C,[c])$ in M is *in standard position* with respect to $\mathscr{F}$ if for every leaf L of $\mathscr{F}|_{\mathscr{O}p_M(C)}$ there is a $z \in \mathbb{C}$ such that c induces a rank-q map from L into the slice $z \times \mathbb{R}^q$.

In particular, C is then tangential to $\mathscr{F}$, in the sense that $\tau_x\mathscr{F} \subset \tau_x C$ at every $x \in C$. One thus gets a dimension-q foliation $\mathscr{F}_C$ on C.

Definition 3.8 The cleft foliation $G := (C,[c],\mathscr{G})$ on M is *quasicomplementary* to $\mathscr{F}$ if $(C,[c])$ is in standard position with respect to $\mathscr{F}$, and if $\mathscr{G}$ is complementary to $\mathscr{F}$ on $M \setminus C$.

Recall that $\overline{M}$ denotes $M \times \mathbb{I}$ and that $\widehat{M} \subset \overline{M}$ is the union of $M \times 0$ with $\partial M \times \mathbb{I}$; also, pr_1 denotes the first projection $\overline{M} \to M$.

Theorem A′ *Fix a $\varphi \in \mathrm{Diff}(\mathbb{D}^q)^{\mathbb{I}}$ such that $\varphi_1 \neq \mathrm{id}$. On a compact manifold M, let $\mathscr{F}$ be a foliation of dimension $q \geq 1$, and γ be a Γ_q–structure whose normal bundle is $\tau\mathscr{F}$. Let $\overline{\mathscr{F}}$ denote the dimension-q foliation on $\overline{M}$ parallel to $\mathscr{F}$.*

Assume that over $\mathscr{O}p_M(\partial M)$, the differential $d\gamma$ is a linear retraction $\tau M \to \tau\mathscr{F}$. Then:

(i) *$\overline{M}$ admits a cleft Γ_q–structure $\Gamma := (C,[c],\overline{\gamma})$ of normal bundle $\tau\overline{\mathscr{F}}$ and of monodromy φ such that*
- *$(C,[c])$ is disjoint from $\widehat{M}$ and in standard position with respect to $\overline{\mathscr{F}}$;*
- *$\overline{\gamma} = \mathrm{pr}_1^*(\gamma)$ on $\mathscr{O}p_{\overline{M}}(\widehat{M})$;*
- *$\Gamma|_{M\times 1}$ is a cleft foliation quasicomplementary to $\mathscr{F}$.*

(ii) *Moreover, in the case where M is a product $V \times X$ and where $\mathscr{F}$ is its slice foliation parallel to X, one can arrange that the projection $V \times X \times \mathbb{I} \to V \times \mathbb{I}$ is one-to-one in restriction to each connected component of $c^{-1}(0,0)$.*

Note 3.9 The hypothesis that, over $\mathscr{O}p_M(\partial M)$, the differential $d\gamma$ is a linear retraction $\tau M \to \tau\mathscr{F}$, amounts to saying that the Γ_q–structure γ is regular there, complementary to $\mathscr{F}$, and special (recall the note after Vocabulary 2.8 and Definition 2.9).

The proof of Theorem A′ will be given in Section 4. The proof of Theorem A as a corollary of Theorem A′ will be given in Section 3.2. The proof of the Mather–Thurston Theorem 1.8 as a corollary of Theorem A′ will be given in Section 5.

3.1 Toolbox

We shall use the following tools, among which Section 3.1.5 is classical; the others are more or less obvious.

3.1.1 Structure of compact fissures Let $(C,[c])$ be a q–fissure in a manifold M, and $\Sigma := c^{-1}(0,0)$. Note that if M is compact, then C is compact.

Lemma 3.10 *Assume that C is compact. Then*:

(a) *There are a small $\epsilon > 0$ and an equidimensional embedding*

$$h\colon \Sigma \times \mathbb{D}^2 \times \mathbb{D}^q \hookrightarrow M$$

whose image H is a thin compact neighbourhood of C in M, and such that

$$c \circ h\colon (\sigma, z, y) \mapsto (\epsilon z, (1+\epsilon)y).$$

(b) *If $(C,[c])$ is moreover in standard position (Definition 3.7) with respect to a dimension-q foliation $\mathscr{F}$, then one can moreover arrange that $h^*(\mathscr{F})$ is the slice foliation parallel to $\mathbb{D}^q$.*

After the lemma, the compact fissure C is diffeomorphic with $\Sigma \times \mathbb{D}^q$. We call Σ the *core* of the fissure.

Proof (a) Since C is compact and $\mathbb{D}^q$ is compact connected, the submersion c induces a proper map from $\mathscr{O}p_M(C)$ onto $\mathscr{O}p_{\mathbb{C}\times\mathbb{R}^q}(0\times\mathbb{D}^q)$. For $\epsilon > 0$ small enough, after Ehresmann's lemma, c induces a locally trivial fibration of some neighbourhood H of C in M over the bidisk

$$\{(z,y) \in \mathbb{C}\times\mathbb{R}^q \mid |z| \le \epsilon, |y| \le 1+\epsilon\}.$$

Of course, the bidisk being contractible, the fibration is actually trivial.

(b) If $(C,[c])$ is moreover in standard position with respect to a dimension-q foliation $\mathscr{F}$, consider the (trivial) fibration $c|_C$ of C over $\mathbb{D}^q$. The leaves of $\mathscr{F}_C$ (Definition 3.7), being complementary in C to the fibres of this fibration, are mapped

diffeomorphically onto the simply connected base. By the Reeb local stability theorem [2, Chapter IV, paragraph 4], shrinking H if necessary, all leaves of $\mathscr{F}|_H$ are compact q–disks, and they are the fibres of a fibration $p\colon H \to \Sigma\times\mathbb{D}^2$. The wanted embedding h is the inverse of the diffeomorphism

$$p \times (\mathrm{pr}_2 \circ c)\colon H \to \Sigma \times \mathbb{D}^2 \times \mathbb{D}^q. \qquad \square$$

3.1.2 Horizontal perturbation of a compact fissure Let $(C,[c])$ be a compact q–fissure in a manifold M.

Choose a neighbourhood $H = h(\Sigma \times \mathbb{D}^2 \times \mathbb{D}^q)$ of C in M as in Lemma 3.10(a). For any self-diffeomorphism $f \in \mathrm{Diff}(\Sigma \times \mathbb{D}^2)$ (Notation 3.1), consider the self-diffeomorphism $\bar{f}$ of H defined by

$$\bar{f}(h(\sigma, z, y)) := h(f(\sigma, z), y).$$

Then $(\bar{f}^{-1}(C), [c \circ \bar{f}])$ is a new q–fissure in M.

(a) If moreover $(C,[c])$ is in standard position with respect to a foliation $\mathscr{F}$ on M, then, after Lemma 3.10(b), for a proper choice of the embedding h, for every $f \in \mathrm{Diff}(\Sigma \times \mathbb{D}^2)$, the perturbed fissure $(\bar{f}^{-1}(C), [c \circ \bar{f}])$ is also in standard position with respect to $\mathscr{F}$.

(b) Given a cleft Γ_q–structure $\Gamma := (C, [c], \gamma)$ on M, provided that H is thin enough, $\bar{f}$ preserves the germ of $c^*(\gamma)$ along ∂H. We thus get on M a *perturbed* cleft Γ_q–structure,

$$\Gamma_{h,f} := (\bar{f}^{-1}(C), [c \circ \bar{f}], \gamma_{h,f}),$$

where $\gamma_{h,f}$ is the Γ_q–structure on $M \setminus \bar{f}^{-1}(C)$ defined on $H \setminus \bar{f}^{-1}(C)$ by $\gamma' := \bar{f}^*(\gamma)$, and on $M \setminus \mathrm{Int}(H)$ by $\gamma' := \gamma$.

(c) If moreover Γ is a cleft foliation quasicomplementary to $\mathscr{F}$, then, after Lemma 3.10(b), for a proper choice of h, for every $f \in \mathrm{Diff}(\Sigma \times \mathbb{D}^2)$, the perturbed cleft foliation $\Gamma_{h,f}$ is also quasicomplementary to $\mathscr{F}$.

3.1.3 Pulling back a fissure, a cleft Haefliger structure or a cleft foliation Let $u\colon M' \to M$ be a smooth map between two manifolds, and $(C,[c])$ be a q–fissure in M.

Definition 3.11 The fissure $(C,[c])$ is *pullable* through u if at every point of $u^{-1}(C)$, the map $c \circ u$ is of maximal rank $q+2$.

Then one gets in M' a *pullback* fissure $(u^{-1}(C), [c \circ u])$. If u is an inclusion $M' \subset M$, we say "restrictable" instead of "pullable" and "restriction" instead of "pullback". The following remarks (a)–(e) will be useful:

(a) If moreover

(1) one has dimension-q foliations $\mathscr{F}$ on M and $\mathscr{F}'$ on M';

(2) u is *leafwise étale* with respect to $\mathscr{F}'$ and $\mathscr{F}$, in the sense that at every $x \in M'$, the differential $d_x u$ maps isomorphically $\tau_x \mathscr{F}'$ onto $\tau_{u(x)} \mathscr{F}$;

(3) the fissure $(C, [c])$ is in standard position with respect to $\mathscr{F}$ (Definition 3.7);

then obviously the pullback fissure $(u^{-1}(C), [c \circ u])$ is in standard position with respect to $\mathscr{F}'$.

(b) The pullability condition is actually generic in this frame, in the following sense:

Under the hypotheses of Definition 3.11 and (1)–(3) above, assume moreover that C is compact. Choose a neighbourhood $H = h(\Sigma \times \mathbb{D}^2 \times \mathbb{D}^q)$ of C in M as in Lemma 3.10(a)–(b). Then:

Lemma 3.12 *For $f \in \mathrm{Diff}(\Sigma \times \mathbb{D}^2)$ generic, the perturbed fissure $(\bar{f}^{-1}(C), [c \circ \bar{f}])$ (Section 3.1.2) is pullable through u.*

Proof Apply the Thom transversality theorem under, if one likes, Poénaru's version "with constraints" (see [19, paragraph 5.4]). □

(c) Under the hypothesis of Definition 3.11, any cleft Γ_q–structure $\Gamma := (C, [c], \gamma)$ on M admits a *pullback* cleft Γ_q–structure of the same monodromy,

$$u^*(\Gamma) := (u^{-1}(C), [c \circ u], u^*(\gamma)).$$

If u is an inclusion $M' \subset M$, we rewrite $u^*(\Gamma)$ as $\Gamma|_{M'}$.

(d) Let ∇ be on M a vector field tangential to a cleft foliation G (Definition 3.5).

Then, the (maybe only partially defined) flow of ∇ preserves G, in the following sense: for every $t \in \mathbb{R}$ and $x \in \mathrm{Int}(M)$, if $\nabla^t(x)$ is defined and also lies in $\mathrm{Int}(M)$, then $G = (\nabla^t)^*(G)$ on $\mathscr{O}p_M(x)$.

(e) Under the hypothesis of Definition 3.11 and (1)–(2) above, the pullback through u of any cleft foliation quasicomplementary to $\mathscr{F}$ on M is a cleft foliation quasicomplementary to $\mathscr{F}'$ on M'.

3.1.4 Holes

Definition 3.13 For a compact manifold Σ, a compact manifold Q of dimension $q \geq 1$ and an element $\varphi \in \mathrm{Diff}(Q)^{\mathbb{I}}$, the *hole* of core Σ, fibre Q and monodromy φ consists of the manifold $H := \Sigma \times \mathbb{D}^2 \times Q$ together with, along ∂H, the germ of a codimension-q foliation $\mathscr{H}_\varphi$ such that

- $\mathscr{H}_\varphi$ is the slice foliation parallel to $\mathbb{D}^2$ on some neighbourhood of $\Sigma \times \mathbb{D}^2 \times \partial Q$;
- $\mathscr{H}_\varphi$ restricts, on $\Sigma \times \partial\mathbb{D}^2 \times Q$, to the preimage of the suspension $\mathscr{S}_\varphi$ under the projection $(\sigma, z, y) \mapsto (z, y)$.

Let $(C, [c], \gamma)$ be a cleft Γ_q–structure of monodromy φ on a manifold M, whose fissure is compact. Choose a neighbourhood H as in Lemma 3.10. One can then forget γ in the interior of H, remembering only its germ along ∂H. One has thus *enlarged* the fissure into a hole of the same core, of fibre $\mathbb{D}^q$ and of the same monodromy.

Conversely, any hole $(\Sigma \times \mathbb{D}^2 \times \mathbb{D}^q, \mathscr{H}_\varphi)$ of fibre $\mathbb{D}^q$ can be *horizontally shrunk* into a fissure, by extending radially the germ of foliation $\mathscr{H}_\varphi$ from the vertical boundary $\Sigma \times \partial\mathbb{D}^2 \times \mathbb{D}^q$ and thus foliating the subset $\Sigma \times (\mathbb{D}^2 \setminus 0) \times \mathbb{D}^q$.

3.1.5 Filling a hole whose monodromy is a multirotation (after [36, Section 4])

Definition 3.14 Let Q be a compact manifold of dimension $q \geq 2$. One calls $\varphi := (\varphi_t)_{t\in\mathbb{I}} \in \mathrm{Diff}(Q)^{\mathbb{I}}$ a *multirotation* if there are a function u on $\mathbb{I}$ as in Section 1.1 and an equidimensional embedding

$$F : \mathbb{D}^{q-1} \times \mathbb{S}^1 \hookrightarrow \mathrm{Int}(Q)$$

such that, for every $t \in \mathbb{I}$,

- $\varphi_t(F(a,s)) = F\big(a, s + tu(1-2|a|)\big)$ for every $a \in \mathbb{D}^{q-1}$ such that $|a| \leq \frac{1}{2}$ and every $s \in \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$;
- φ_t is the identity on the rest of Q.

For any (compact) manifold Σ, on $H := \Sigma \times \mathbb{D}^2 \times Q$, denote by $\mathscr{F}$ (resp. $\mathscr{H}_0$) the slice foliation parallel to Q (resp. to $\Sigma \times \mathbb{D}^2$). Assuming that φ is the above multirotation, consider the foliation $\mathscr{H}$ of $\Sigma \times \mathbb{D}^2 \times Q$ equal to the foliation $\mathscr{G}_\Sigma$ of Section 1.1 in $\Sigma \times (\mathbb{D}^2 \times \mathrm{Im}(F)) \cong C_\Sigma$ (see Figure 1), and equal to $\mathscr{H}_0$ in the complement. The germ of $\mathscr{H}$ along ∂H matches Definition 3.13. One has thus *filled*, by means of the foliation $\mathscr{H}$, the hole of core Σ, fibre Q and monodromy φ.

Remark 3.15 (on the differential of the filling foliation) Recall the definitions (Vocabulary 2.8) of the normal bundle $\nu\mathscr{H}$ and of the differential $d\mathscr{H}$ for the foliation $\mathscr{H}$. The 1–parameter family (ω_r) of Section 1.1 provides a homotopy between the codimension-q plane fields $\tau\mathscr{H}_0$ and $\tau\mathscr{H}$ on H, rel ∂H. Hence, the family (ω_r) also provides, uniquely up to a homotopy, a linear isomorphism of vector bundles

$$h\colon \nu\mathscr{H} \to \nu\mathscr{H}_0 = \tau\mathscr{F}$$

which is the identity on ∂H, and such that the image $h \circ d\mathscr{H}$ of the differential $d\mathscr{H}$ is homotopic to $d\mathscr{H}_0$ among the $(\tau\mathscr{F})$–valued 1–forms of rank q on H, rel ∂H.

3.1.6 Vertical reparametrization of a hole

Definition 3.16 Given a locally trivial fibration between some manifolds $\pi\colon E \to B$ and a 1–parameter family $(f_s)_{s\in\mathbb{I}}$ of self-diffeomorphisms of E such that $f_0 = \mathrm{id}$ and that $\pi \circ f_s = \pi$ for every s, one calls the family $(f_s)_{s\in\mathbb{I}}$ (or f_1 alone) an isotopy *vertical* with respect to π.

The vertical isotopies enjoy an obvious extension property.

Now, let Q be a compact q–manifold; consider the identity component $\mathrm{Diff}(Q)_0$ and its universal cover $\widetilde{\mathrm{Diff}}(Q)_0$, which can be realized as the quotient of $\mathrm{Diff}(Q)^{\mathbb{I}}$ by the relation of homotopy rel $\partial\mathbb{I}$. We denote by $[\varphi]$ the class of the 1–parameter family of diffeomorphisms φ. The projection $\widetilde{\mathrm{Diff}}(Q)_0 \to \mathrm{Diff}(Q)_0$ is $[\varphi] \mapsto \varphi_1$.

The following is classical; we recall the short proof for completeness.

Lemma 3.17 *The suspension foliation $\mathscr{S}_\varphi$ only depends, up to an isotopy in $\mathbb{S}^1 \times Q$ vertical with respect to the projection to $\mathbb{S}^1$ and which is the identity close to $\mathbb{S}^1 \times \partial Q$, on the conjugation class of $[\varphi]$ in the group $\widetilde{\mathrm{Diff}}(Q)_0$.*

Proof (Here $\mathbb{S}^1$ is the quotient of the interval $\mathbb{I}$ by the relation $0 = 1$.)

One the one hand, a homotopy of the family $(\varphi_t)_{t\in\mathbb{I}}$ rel $\partial\mathbb{I}$ obviously amounts to a vertical isotopy of $\mathscr{S}_\varphi$ rel $0 \times Q$.

On the other hand, let $\varphi' = \psi\varphi\psi^{-1}$ be a conjugation in the group $\mathrm{Diff}(Q)^{\mathbb{I}}$. Put, for $s \in \mathbb{I}$,

$$f_s\colon \mathbb{S}^1 \times Q \to \mathbb{S}^1 \times Q, \quad (t, y) \mapsto \big(t, \varphi'_{st}\big(\psi_s(\varphi_{st}{}^{-1}(y))\big)\big).$$

This family (f_s) is a vertical isotopy of $\mathbb{S}^1 \times Q$, and $f_1^*(\mathscr{S}_{\varphi'}) = \mathscr{S}_\varphi$. □

It follows that the notion of hole of monodromy φ does actually not depend on φ itself, but only on the conjugation class of $[\varphi]$ in $\widetilde{\mathrm{Diff}}(Q)_0$. Indeed, any isotopy in $\Sigma\times\mathbb{S}^1\times Q$ vertical with respect to the projection to $\Sigma\times\mathbb{S}^1$, and which is the identity close to $\Sigma\times\mathbb{S}^1\times\partial Q$, extends into an isotopy in $\Sigma\times\mathbb{D}^2\times Q$ vertical with respect to the projection to $\Sigma\times\mathbb{D}^2$, and which is the identity close to $\Sigma\times\mathbb{D}^2\times\partial Q$. (The same holds of course for the monodromy of cleft Haefliger structures).

3.1.7 Splitting a hole into two holes Given a compact manifold Q and a factorization $\varphi=\varphi'\varphi''$ in the group $\mathrm{Diff}(Q)^{\mathbb{I}}$, let S be the 2–sphere minus the interiors of three disjoint compact disks. Hence, $\pi_1(S)$ is the free group on two generators γ' and γ'', which one chooses so that the three components of ∂S, oriented as such, are respectively conjugate to γ'^{-1} and γ''^{-1} and $\gamma'\gamma''$. Suspending the representation $\pi_1(S)\to\mathrm{Diff}(Q)^{\mathbb{I}}$ that maps γ' to φ' and γ'' to φ'', one gets on $S\times Q$ a codimension-q foliation $\mathscr{G}$ such that

(1) $\mathscr{G}$ is complementary to the slice foliation parallel to Q;

(2) $\mathscr{G}$ is parallel to S on a neighbourhood of $S\times\partial Q$;

(3) $\mathscr{G}$ induces the suspensions of φ^{-1}, φ'^{-1} and φ'' on the three components of $\partial S\times Q$.

Given also a manifold Σ, pulling back $\mathscr{G}$ into $\Sigma\times S\times Q$, one obtains a partial filling of the hole H of core Σ, fibre Q and monodromy φ, leaving two holes $H'\subset H$ and $H''\subset H$ of core Σ, fibre Q and respective monodromies φ' and φ''. Note that the core of H' (resp. H'') *embeds* into the base $\Sigma\times\mathbb{D}^2$ of H through the projection $H\mapsto\Sigma\times\mathbb{D}^2$.

3.1.8 Vertical shrinking of a hole If $Q'\subset Q$ is a domain containing in its interior the support of every φ_t, where $\varphi=(\varphi_t)_{t\in\mathbb{I}}\in\mathrm{Diff}(Q)^{\mathbb{I}}$, then in the hole $\Sigma\times\mathbb{D}^2\times Q$ of core Σ, fibre Q and monodromy φ, we can foliate the subset $\Sigma\times\mathbb{D}^2\times(Q\setminus Q')$ by the horizontal slice foliation parallel to $\Sigma\times\mathbb{D}^2$, leaving a smaller hole $\Sigma\times\mathbb{D}^2\times Q'$ of the same core, fibre Q' and monodromy $\varphi|_{Q'}$.

3.2 Proof of Theorem A as a corollary of Theorem A′

Let M, $\mathscr{F}$, γ and ω be as in the hypotheses of Theorem A. After pushing the microfoliation of γ by the linear automorphism $(\omega|_{\tau\mathscr{F}})^{-1}$ of the vector bundle $\nu\gamma=\tau\mathscr{F}$, one is reduced to the case where $\omega|_{\tau\mathscr{F}}=\mathrm{id}$. Choose φ to be a multirotation (Definition 3.14). Applying Theorem A′(i), one obtains a cleft Γ_q–structure $(C,[c],\overline{\gamma})$. Enlarge the

fissure into a hole $H \subset \overline{M}$ of fibre $\mathbb{D}^q$ and monodromy φ (Section 3.1.4). Fill up this hole according to Section 3.1.5; there results on $\overline{M}$ a (noncleft) Γ_q–structure $\widetilde{\gamma}$ whose normal bundle is $\tau\overline{\mathscr{F}}$, which coincides with $\overline{\gamma}$ on the complement of H, and whose restriction $\gamma' := \widetilde{\gamma}|_{M\times 1}$ to $M \times 1 \cong M$ is a (noncleft) foliation quasicomplementary to $\mathscr{F}$. Finally, γ' being special (Definition 2.9) on $(M \times 1) \setminus (H \cap (M \times 1))$, and after Remark 3.15 applied in $H \cap (M \times 1)$, the differential $d\gamma'$ is homotopic to ω, rel $\partial(M \times 1)$, among the $(\tau\mathscr{F})$–valued 1–forms on M whose restriction to $\tau\mathscr{F}$ is of constant rank q.

Note 3.18 (on the hypothesis $q \geq 2$) In the proof of Theorem A, the hypothesis that $q \geq 2$ is crucial, allowing one to choose the diffeomorphism φ of $\mathbb{D}^q$ to be a multirotation, and consequently, after enlarging the fissure, to fill the resulting hole of monodromy φ (following Thurston) with the foliation described in the introduction. On the contrary, in codimension $q = 1$, there exist no multirotations, since $\mathbb{S}^1$ does not embed in the interval $\mathbb{D}^1$! Filling a hole in codimension 1 requires in general a complicated construction where one first has to enlarge the hole by a "worm gallery" [37] (see also [27]). For this reason, there seems to be no general result for the existence of codimension-1 quasicomplementary foliations, like Theorem A in the higher codimensions.

4 Proof of Theorem A′

4.1 Foliating a neighbourhood of a codimension-1 skeleton transversely to $\mathscr{F}$

Following a classical scheme, the first part of the proof of Theorem A′ will solve the problem on a small neighbourhood of the codimension-1 skeleton of a triangulation of a large domain in M; this part is somehow standard, pertaining to Gromov's h–principle for Diff–invariant open differential relations on open manifolds, with the help of Thurston's jiggling lemma.

Let M, $\mathscr{F}$ and γ be as in Theorem A′. Put $p := \dim(M) - q$. By the jiggling lemma [36, Section 5], one has

- a compact domain $D \subset \mathrm{Int}(M)$, large enough that $d\gamma$ is a linear retraction $\tau M \to \tau\mathscr{F}$ over some open neighbourhood of $M \setminus \mathrm{Int}(D)$;
- a triangulation K of D which is in *general position*, in Thurston's sense, with respect to $\mathscr{F}$.

From Thurston's definition of "general position", we only need to recall that every cell S of K is transverse to $\mathscr{F}$ (when $\dim(S) < p$, "transverse" means that there is no tangency) and that, when $\dim(S) \geq p$, the foliation $\mathscr{F}|_{\mathrm{Int}(S)}$ is conjugate to the standard linear codimension-p foliation on $\mathbb{R}^{\dim(S)}$. Consider the $(p+q-1)$–skeleton $K^{(p+q-1)}$ of K.

Lemma 4.1 *After a first concordance of γ on M rel ∂M (no cleft is necessary at this stage), one can moreover arrange that $d\gamma$ is also a linear retraction $\tau M \to \tau\mathscr{F}$ over some open neighbourhood U of $K^{(p+q-1)}$ in M.*

(About the differential being a linear retraction, recall Note 3.9.)

Proof Choose a filtration (K_n) for $0 \leq n \leq N$ of $K^{(p+q-1)}$ by subcomplexes such that $K_0 = K|_{\partial D}$, that $K_N = K^{(p+q-1)}$ and that K_n is the union of K_{n-1} with a single cell S_n. By induction on n, assume that $d\gamma$ is already a linear retraction $\tau M \to \tau\mathscr{F}$ over some open neighbourhood U_{n-1} of K_{n-1} in M. For convexity reasons, $d\gamma|_{U_{n-1}}$ extends over M to a global linear retraction $\omega\colon \tau M \to \tau\mathscr{F}$. There are two cases, depending on $d := \dim(S_n)$.

In the case $0 \leq d \leq p$, one has an equidimensional embedding

$$e\colon S_n \times \mathbb{D}^{p-d} \times \mathbb{D}^q \hookrightarrow M$$

whose image is contained in a small neighbourhood of S_n, and such that

- $e(x,0,0) = x$ for every $x \in S_n$;
- e maps $\partial S_n \times \mathbb{D}^{p-d} \times \mathbb{D}^q$ into U_{n-1};
- $e^*(\mathscr{F})$ is the slice foliation on $S_n \times \mathbb{D}^{p-d} \times \mathbb{D}^q$ parallel to the $\mathbb{D}^q$ factor.

One applies the integrable parametric foliation theorem on open manifolds (Theorem 2.13 of Section 2.0.7) to the manifold $\mathbb{D}^q$, with $N := \varnothing$, the space of parameters being $A := S_n \times \mathbb{D}^{p-d}$, while $B := \partial S_n \times \mathbb{D}^{p-d}$.

In the case $p \leq d \leq p+q-1$, one has an equidimensional embedding

$$e\colon \mathbb{D}^p \times \mathbb{D}^{d-p} \times \mathbb{D}^{p+q-d} \hookrightarrow M$$

whose image is contained in a small neighbourhood of S_n, and such that

- $e^{-1}(S_n) = \mathbb{D}^p \times \mathbb{D}^{d-p} \times 0$;
- e maps $\partial(\mathbb{D}^p \times \mathbb{D}^{d-p}) \times \mathbb{D}^{p+q-d}$ into U_{n-1};

- $U_{n-1} \cup e(\mathbb{D}^p \times \mathbb{D}^{d-p} \times \mathbb{D}^{p+q-d})$ is a neighbourhood of S_n;
- $e^*(\mathscr{F})$ is the slice foliation on $\mathbb{D}^p \times \mathbb{D}^{d-p} \times \mathbb{D}^{p+q-d}$ that is parallel to the $\mathbb{D}^{d-p} \times \mathbb{D}^{p+q-d}$ factor.

One applies Theorem 2.13 to $\mathbb{D}^{d-p} \times \mathbb{D}^{p+q-d}$, with $N := \partial\mathbb{D}^{d-p} \times \mathbb{D}^{p+q-d}$, the space of parameters being $A := \mathbb{D}^p$, while $B := \partial\mathbb{D}^p$.

In both cases, one gets on some open neighbourhood U_n of K_n in M a regular Γ_q–structure γ' such that

(1) $\nu\gamma' = \tau\mathscr{F}|_{U_n}$;

(2) γ' is concordant with γ rel $\mathscr{O}p_M(K_{n-1})$;

(3) the restriction $d\gamma'|_{\tau\mathscr{F}}$ is of constant rank q over U_n;

(4) $d\gamma'|_{\tau\mathscr{F}}$ is homotopic with $\mathrm{id}_{\tau\mathscr{F}}$ among the linear automorphisms of the vector bundle $\tau\mathscr{F}|_{U_n}$.

Over U_n, push the microfoliation of γ' through the linear automorphism $(d\gamma'|_{\tau\mathscr{F}})^{-1}$ of the vector bundle $\tau\mathscr{F}$, and thus get an isomorphic Γ_q–structure γ'' whose differential is a linear retraction of τU_n onto $\tau\mathscr{F}|_{U_n}$. To verify that γ'' is concordant with γ', and hence with γ, rel $\mathscr{O}p_M(K_{n-1})$, one can refer to the general Lemma 2.6; but here it is even more obvious, because of (4). Finally, one calls to the concordance extension property (Proposition 2.7) to complete the induction. □

4.2 Preparing the niche for inflation

The next step in the proof of Theorem A′ has antecedents in Poénaru's flexibility for folded equidimensional maps [33], in Gromov's proof of the microcompressibility lemma [11, page 81], in Lemma 2.3A in [5], etc.

According to Lemma 4.1, one can define $\overline{\gamma}$ on $U \times \mathbb{I}$ as $\mathrm{pr}_1^*(\gamma)$. Then, vaguely, it remains, in the wall $M \times \mathbb{I}$, to fill the niches $\mathrm{Int}(S) \times (0, 1]$, where S runs over the $(p+q)$–cells of K. The present step will somehow normalize the Haefliger structure along the boundary of the niche, in order that, further down, we can solve the new extension problem by the "inflation" method.

Precisely, for each $(p+q)$–cell S of K, one has a large equidimensional embedding $j : \mathbb{I}^p \times \mathbb{D}^q \hookrightarrow \mathrm{Int}(S)$ containing $S \setminus (S \cap U)$ in its interior, and such that $j^*(\mathscr{F})$ is the slice foliation on $\mathbb{I}^p \times \mathbb{D}^q$ parallel to $\mathbb{D}^q$. In this way, the proof of Theorem A′ is

reduced to the case where $M := \mathbb{I}^p \times \mathbb{D}^q$, and where $\mathscr{F}$ is the slice foliation parallel to $\mathbb{D}^q$. We restrict the attention to this case.

Note: the choice of the cube $\mathbb{I}^p$ here, rather than the ball $\mathbb{D}^p$, will be of no importance for most of the construction, but it will make civilization easier in Section 4.3.2.

Recall that $\overline{M} := M \times \mathbb{I}$, that $\widehat{M} := (M \times 0) \cup (\partial M \times \mathbb{I}) \subset \overline{M}$, that $\overline{\mathscr{F}}$ is on $\overline{M}$ the dimension-q slice foliation parallel to $\mathbb{D}^q$, and that $\mathrm{pr}_1 : M \times \mathbb{I} \to M$ denotes the first projection. Consider on $\overline{M}$ the $\overline{\Gamma}_q$–structure $\mathrm{pr}_1^*(\gamma)$.

Put

$$X := (\partial \mathbb{I}^p \times \mathbb{D}^q \times \mathbb{I}) \cup (\mathbb{I}^p \times \partial \mathbb{D}^q \times 1) \subset \widehat{M} \subset \overline{M}.$$

Note that $X \cap (M \times 1) = \partial M \times 1$.

Lemma 4.2 *There is on $\mathcal{O}p_{\overline{M}}(\widehat{M})$ a foliation $\mathscr{G}$ such that*

- *$\mathscr{G}$ is complementary to $\overline{\mathscr{F}}$ on $\mathcal{O}p_{\overline{M}}(\widehat{M})$;*
- *$\mathscr{G}$ is specially induced by $\mathrm{pr}_1^*(\gamma)$ on $\mathcal{O}p_{\overline{M}}(X)$;*
- *$\mathscr{G}$ is concordant to $\mathrm{pr}_1^*(\gamma)$ on $\mathcal{O}p_{\overline{M}}(\widehat{M})$ rel $\mathcal{O}p_{\overline{M}}(X)$* (recall Conclusion 2.11(a)).

Proof Let ω be, over $\overline{M}$, the identity automorphism of the bundle $\tau\overline{\mathscr{F}}$.

First, regard $\gamma|_{M\times 0}$ as an integrable parametric family of Γ_q–structures on $\mathbb{D}^q$, with space of parameters $\mathbb{I}^p$. The integrable parametric foliation theorem, Theorem 2.13, is first applied to γ and ω on the open manifold $\mathbb{D}^q$, the pair of parameters spaces being

$$(A, B) := (\mathbb{I}^p, \partial \mathbb{I}^p).$$

One gets on $M \cong M \times 0$ a foliation $\mathscr{G}_0$ complementary to $\mathscr{F}$ there, specially induced by γ on $\mathcal{O}p_{M\times 0}(\partial \mathbb{I}^p \times \mathbb{D}^q \times 0)$, and concordant to γ (rel $\partial \mathbb{I}^p \times \mathbb{D}^q \times 0$).

Next, by the concordance extension property for Γ_q–structures (see Section 2.0.5), $\mathscr{G}_0$ can be extended over $\mathcal{O}p_{\overline{M}}(\widehat{M})$ by a $\overline{\Gamma}_q$–structure $\widehat{\gamma}$, coinciding with $\mathrm{pr}_1^*(\gamma)$ on $\mathcal{O}p_{\overline{M}}(X)$, and concordant to $\mathrm{pr}_1^*(\gamma)$ rel X.

Consider some small collar neighbourhood $T \cong \mathbb{S}^{q-1} \times \mathbb{I}$ of $\partial \mathbb{D}^q$ in $\mathbb{D}^q$, and

$$N := \mathbb{I}^p \times T \times \mathbb{I} \subset \overline{M}.$$

Regard $\widehat{\gamma}|_N$ as an integrable family of Γ_q–structures on T, parametrized by $\mathbb{I}^{p+1}$. One obtains $\mathscr{G}$ on N by again applying the integrable parametric foliation, Theorem 2.13, this time to $\widehat{\gamma}$ and ω on the open manifold T, the pair of parameters spaces being

$$(A, B) := (\mathbb{I}^{p+1}, \partial \mathbb{I}^{p+1}). \qquad \square$$

Fix a compact collar neighbourhood Col of ∂M in M, small enough that $d\gamma$ is a linear retraction $\tau M \to \tau\mathscr{F}$ over Col. Put $\Delta := M \setminus \mathrm{Int}(\mathrm{Col})$ and

$$W := \big(M \times \big[0, \tfrac{1}{2}\big]\big) \cup \big(\mathrm{Col} \times \big[\tfrac{1}{2}, 1\big]\big) \subset \overline{M} \quad \text{and} \quad \partial_1 W := \big(\Delta \times \big(\tfrac{1}{2}\big)\big) \cup \big(\partial\Delta \times \big[\tfrac{1}{2}, 1\big]\big).$$

As a paraphrase of Lemma 4.2, there is on $\mathscr{O}p_{\overline{M}}(W)$ a $\overline{\Gamma}_q$–structure $\overline{\gamma}$ such that:

- On $\mathscr{O}p_{\overline{M}}(\widehat{M} \cup (\mathrm{Col} \times 1))$, one has $\overline{\gamma} = \mathrm{pr}_1^*(\gamma)$.
- On $\mathscr{O}p_{\overline{M}}(\partial_1 W)$, the $\overline{\Gamma}_q$–structure $\overline{\gamma}$ is regular and specially induces a foliation $\mathscr{G}$ complementary to $\overline{\mathscr{F}}$.

To prove Theorem A′(i), it remains to extend $\overline{\gamma}$ through the smaller niche $\overline{M} \setminus W$. Hence, one is reduced to Proposition 4.3 below.

4.3 Inflation

Recall that $M = \mathbb{I}^p \times \mathbb{D}^q$ ($p \geq 0$, $q \geq 1$), that $\overline{M} = M \times \mathbb{I}$, that $\widehat{M} = (M \times 0) \cup (\partial M \times \mathbb{I}) \subset \overline{M}$, that $\overline{\pi}$ is the projection

$$\overline{\pi} \colon \overline{M} = \mathbb{I}^p \times \mathbb{D}^q \times \mathbb{I} \to \mathbb{I}^{p+1}, \quad (x, y, t) \mapsto (x, t),$$

and that $\overline{\mathscr{F}}$ is the q–dimensional slice foliation of $\overline{M}$ parallel to the $\mathbb{D}^q$ factor. Let φ be as in Theorem A′.

Proposition 4.3 *Let $\mathscr{G}$ be along $\widehat{M}$ a germ of foliation complementary to $\overline{\mathscr{F}}$ there. Then $\mathscr{G}$ extends to all of $\overline{M}$ as a* cleft *foliation of monodromy φ quasicomplementary to $\overline{\mathscr{F}}$.*

Note: here, the fact that cubes and disks are cubes and disks is actually unimportant; the same result would hold for any two compact manifolds instead of $\mathbb{I}^p$ and $\mathbb{D}^q$, with almost the same proof.

The rest of the present Section 4.3 is to prove Proposition 4.3. The difficulty lies of course in the arbitrary position of $\mathscr{G}$ with respect to $\mathbb{I}^p \times \partial\mathbb{D}^q \times \mathbb{I}$. The "inflation" method introduced by Thurston [36; 37] to prove the foliation theorem on closed manifolds will fit to solve this difficulty, after some adjustments. We give the details for three reasons. First, our framework is not exactly the same as Thurston's: he foliated simplices, we foliate prisms (by which we mean the product of a simplex by a smooth disk). Secondly, some think that the argument in [36] is difficult; some have even believed that it was not fully convincing — of course, this is not the case. Third, we feel that "inflation" deserves to be more widely used as a general method in the h–principle, which it has not been since the fundamental papers [36; 37], except in [20].

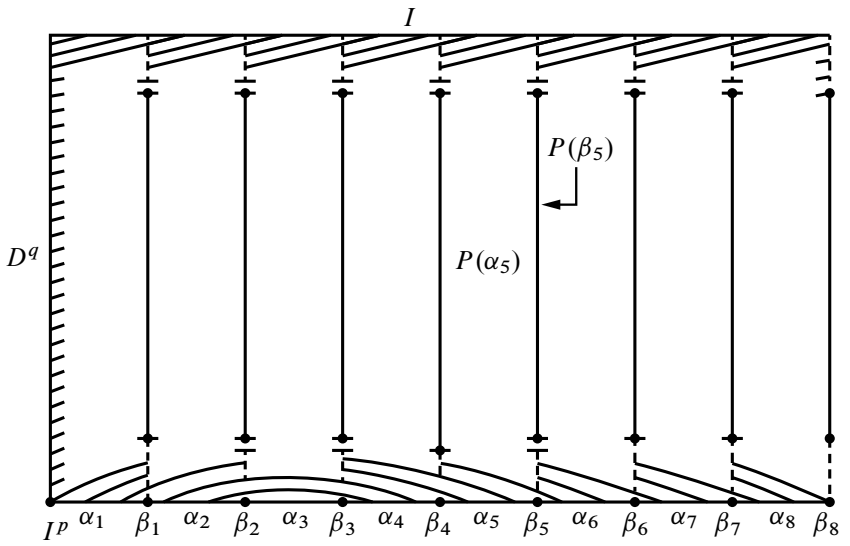

Figure 3: Decomposition of the most part of $\overline{M}$ into prisms, in the case $p = 0$ and $q = 1$.

4.3.1 Building the prisms Let $\mathrm{Dom}(\mathscr{G})$ be a small open neighbourhood of $\widehat{M}$ in $\overline{M}$, on which $\mathscr{G}$ is defined and complementary to $\overline{\mathscr{F}}$. We shall first decompose the domain to be foliated, precisely $\overline{M}$ minus some smaller open neighbourhood of $\widehat{M}$, into many thin vertical prisms (Figure 3).

Endow $\mathbb{I}^{p+1}$ with a triangulation K; choose it to be linear for convenience. The choice of K will be made more precise further down.

Lemma 4.4 *Provided that K is fine enough, one can choose, for each cell α of K, an embedding*

$$e_\alpha : \alpha \times \mathbb{D}^q \hookrightarrow \overline{M}$$

such that, for every $x \in \alpha$,

- $e_\alpha(x \times \mathbb{D}^q) \subset \mathrm{Int}(\overline{\pi}^{-1}(x))$;
- $\overline{\pi}^{-1}(x) \subset e_\alpha(x \times \mathrm{Int}(\mathbb{D}^q)) \cup \mathrm{Dom}(\mathscr{G})$;
- *if x lies on some proper face $\beta \subsetneq \alpha$, then*

$$e_\beta(x \times \mathbb{D}^q) \subset e_\alpha(x \times \mathrm{Int}(\mathbb{D}^q));$$

and that, for every $y \in \partial\mathbb{D}^q$, $e_\alpha(\alpha \times y)$ is contained in a leaf of $\mathscr{G}$.

Proof Regard $\overline{M}$ as $\mathbb{I}^{p+1}\times\mathbb{D}^q$; endow both factors with the Euclidean metrics. Fix $\epsilon>0$ so small that $(x,y)\in\mathrm{Dom}(\mathscr{G})$ holds for every $x\in\mathbb{I}^{p+1}$ and every $y\in\mathbb{D}^q$ such that

$$|y|\geq 1-(p+3)\epsilon.$$

Put $r_d := 1-(p+2-d)\epsilon$ for every integer $0\leq d\leq p+1$.

Consider the partially defined monodromy of $\mathscr{G}$ close to $\partial M\times\mathbb{I}$. Precisely, for a linear path $u\colon\mathbb{I}\to\mathbb{I}^{p+1}$ and for $y\in\mathbb{D}^q$, denote by $(u(1),h_u(y))$ the extremity of the path (if any) in $\overline{M}$ originated at $(u(0),y)$, covering u through $\overline{\pi}$, and tangential to $\mathscr{G}$. Fix $\delta>0$ small enough that for every linear path u whose length $|u|$ is at most δ, and every $y\in\mathbb{D}^q$ with $r_0\leq|y|\leq r_{p+1}$, the monodromy $h_u(y)$ is defined, and

$$|h_u(y)-y|<\tfrac{1}{2}\epsilon.$$

Choose the linear triangulation K such that every cell α of K is of diameter at most δ. For every point $x\in\alpha$ and every $y\in\partial\mathbb{D}^q$, put

$$e_\alpha(x,y) := h_{[\flat(\alpha),x]}(r_{\dim(\alpha)}y).$$

This is a smooth family of embeddings $\partial\mathbb{D}^q\hookrightarrow\mathbb{D}^q$, parametrized by the points of α. By the isotopy extension property, this family extends to a smooth family of embeddings $\mathbb{D}^q\hookrightarrow\mathbb{D}^q$, also denoted by e_α. The demanded properties hold by definition or follow from the triangle inequality. □

We write

$$P(\alpha) := e_\alpha(\alpha\times\mathbb{D}^q)\quad\text{and}\quad R(\alpha) := (\alpha\times\mathbb{D}^q)\setminus e_\alpha(\alpha\times\mathrm{Int}(\mathbb{D}^q)).$$

Let $R\subset\overline{M}$ be the union of the $R(\alpha)$'s for all cells α of K. Hence, R is a compact neighbourhood of $\mathbb{I}^p\times\partial\mathbb{D}^q\times\mathbb{I}$ in $\overline{M}$. The foliation $\mathscr{G}$ is defined on $\mathscr{O}p_{\overline{M}}(R\cup\overline{M}_0)$ and complementary to $\overline{\mathscr{F}}$ there.

4.3.2 Civilizing the prisms Thurston [36] introduced the tool of "civilization" in order to guarantee that some microextensions of foliations defined on the cells of a triangulation are compatible and thus give a global smooth foliation; we slightly depart from his vocabulary.

Definition 4.5 Let X be a manifold and $Y\subset X$ be a compact submanifold, maybe with boundary and corners. A *civilization* for Y in X is a foliation $\mathscr{C}$ of codimension $\dim(Y)$ on $\mathscr{O}p_X(Y)$, transverse to Y.

Actually, the object of interest is the *germ* of $\mathscr{C}$ along Y. The utility of civilization lies in the following obvious remark. Consider the local retraction along the leaves of $\mathscr{C}$,

$$\mathrm{pr}\colon \mathscr{O}p_X(\mathrm{Int}(Y)) \to \mathrm{Int}(Y).$$

Remark 4.6 (recall Section 3.1.3(c) and Definitions 3.6 and 3.11)

- Let G be a cleft foliation on $\mathrm{Int}(Y)$. Then $\mathrm{pr}^*(G)$ is a cleft foliation on $\mathscr{O}p_X(\mathrm{Int}(Y))$ and $\mathscr{C}$ is tangential to $\mathrm{pr}^*(G)$ there.
- Conversely, let G be a cleft foliation on $\mathscr{O}p_X(Y)$ to which $\mathscr{C}$ is tangential. Then G is restrictable to $\mathrm{Int}(Y)$, and $G = \mathrm{pr}^*(G|_{\mathrm{Int}(Y)})$ on $\mathscr{O}p_X(\mathrm{Int}(Y))$.

Notation 4.7 Given two foliations $\mathscr{A}$ and $\mathscr{B}$ on the same manifold, we briefly write $\mathscr{A} \subset \mathscr{B}$ to mean that $\tau_x\mathscr{A} \subset \tau_x\mathscr{B}$ at every point x.

The advantage of having chosen especially a *polyhedral* base $\mathbb{I}^{p+1} \subset \mathbb{R}^{p+1}$ and a *linear* triangulation K of this base is that each cell α of K admits in $\mathbb{R}^{p+1}$ an obvious civilization $\mathscr{N}_\alpha$, namely the parallel linear foliation of $\mathbb{R}^{p+1}$ orthogonal to α for the Euclidean metric, such that for every face $\beta \subset \alpha$, the compatibility condition $\mathscr{N}_\alpha \subset \mathscr{N}_\beta$ holds on $\mathscr{O}p_{\mathbb{R}^{p+1}}(\beta)$.

Consider the inclusion

$$\overline{M} = \mathbb{I}^{p+1} \times \mathbb{D}^q \subset \mathbb{R}^{p+1} \times \mathbb{D}^q$$

and the first projection $\pi\colon \mathbb{R}^{p+1} \times \mathbb{D}^q \to \mathbb{R}^{p+1}$.

Lemma 4.8 *One can choose, for every cell α of K, a civilization $\mathscr{C}_\alpha$ for $\alpha \times \mathbb{D}^q$ in $\mathbb{R}^{p+1} \times \mathbb{D}^q$ such that*

(1) $\mathscr{C}_\alpha \subset \mathscr{G}$ *on* $\mathscr{O}p_{\overline{M}}(R(\alpha))$ *and, if* $\alpha \subset K_0$, *on* $\mathscr{O}p_{\overline{M}}(\alpha \times \mathbb{D}^q)$;

(2) $\mathscr{C}_\alpha \subset \pi^*(\mathscr{N}_\alpha)$ *on* $\mathscr{O}p_{\mathbb{R}^{p+1}}(\alpha) \times \mathbb{D}^q$;

(3) $\mathscr{C}_\alpha \subset \mathscr{C}_\beta$ *on* $\mathscr{O}p_{\mathbb{R}^{p+1}}(\beta) \times \mathbb{D}^q$ *for every face* $\beta \subset \alpha$.

Proof Denote by $\mathscr{F}$ the slice foliation of $\mathbb{R}^{p+1} \times \mathbb{D}^q$ parallel to $\mathbb{D}^q$. It is easy to extend $\mathscr{G}$ to a codimension-q foliation, still denoted by $\mathscr{G}$, complementary to $\mathscr{F}$ on $\mathscr{O}p_{\mathbb{R}^{p+1}\times\mathbb{D}^q}(R \cup \overline{M}_0)$. Then, make a second, not necessarily integrable extension: let $\xi \subset \tau(\mathbb{R}^{p+1} \times \mathbb{D}^q)$ be a $(p+1)$–plane field complementary to $\mathscr{F}$, and coinciding with $\tau\mathscr{G}$ over $\mathscr{O}p_{\mathbb{R}^{p+1}\times\mathbb{D}^q}(R \cup \overline{M}_0)$. Consider a cell α of K. On $\mathscr{O}p_{\mathbb{R}^{p+1}\times\mathbb{D}^q}(R(\alpha))$,

define $\mathscr{C}_\alpha$ as the pullback of $\mathscr{N}_\alpha$ through π tangentially to $\mathscr{G}$. Then, extend $\mathscr{C}_\alpha$ over $\mathscr{O}p_{\mathbb{R}^{p+1}\times\mathbb{D}^q}(\alpha\times\mathbb{D}^q)$ as follows. Denote by $\langle\alpha\rangle\subset\mathbb{R}^{p+1}$ the affine subspace spanned by α. At every point $(x,y)\in\langle\alpha\rangle\times\mathbb{D}^q$, define the leaf of $\mathscr{C}_\alpha$ through (x,y) as the union of the paths in $\mathbb{R}^{p+1}\times\mathbb{D}^q$ starting from (x,y), tangential to ξ, and whose projection through π is a linear path in $\mathbb{R}^{p+1}$ orthogonal to $\langle\alpha\rangle$ at x. The properties (1)–(3) are immediate. □

4.3.3 Foliating the prisms We can choose the fine linear triangulation K of $\mathbb{I}^p\times\mathbb{I}$ such that, moreover, K collapses onto its subcomplex

$$K_0 := K|_{(\mathbb{I}^p\times 0)\cup(\partial\mathbb{I}^p\times\mathbb{I})}.$$

Such a collapse amounts to a filtration of K by subcomplexes (K_n) for $0\le n\le N$ such that $K_N=K$, and such that for every $1\le n\le N$, exactly two cells α_n and β_n lie in K_n but not in K_{n-1}; moreover, β_n is a hyperface of α_n. We write

$$\overline{M}_n := K_n\times\mathbb{D}^q\subset\overline{M}.$$

The inflation process is an induction: the prisms will be foliated (by cleft foliations quasicomplementary to the fibres of $\overline{\pi}$) one after the other, in the order given by the collapse.

Vocabulary 4.9 (a) A *smooth partial triangulation* in a manifold Σ is a geometric simplicial complex which is topologically embedded in Σ, and such that each cell is smoothly embedded in Σ.

(b) By a *system of spheres* in Σ, we mean a finite disjoint union of smoothly embedded spheres, whose dimensions are not necessarily the same and may vary between 0 and $\dim(\Sigma)$, and whose normal bundles are trivial.

The induction hypotheses are the following. Recall Definitions 3.8 and 3.6.

Property 4.10 There is on $\mathscr{O}p_{\overline{M}}(R\cup\overline{M}_n)$ a cleft foliation

$$G_n := (C_n,[c_n],\mathscr{G}_n)$$

of monodromy φ such that

(1) G_n is quasicomplementary to $\overline{\mathscr{F}}$ on $\mathscr{O}p_{\overline{M}}(R\cup\overline{M}_n)$;

(2) G_n coincides with $\mathscr{G}$ on $\mathscr{O}p_{\overline{M}}(R\cup\overline{M}_0)$;

(3) for each cell γ of K_n, the civilization $\mathscr{C}_\gamma$ is tangential to G_n on $\mathscr{O}p_{\overline{M}}(\gamma\times\mathbb{D}^q)$;

(4) $\overline{M}_n \cap \Sigma_n$ (where $\Sigma_n := {c_n}^{-1}(0,0)$) admits a triangulation Δ_n which is a smooth partial triangulation of Σ_n and collapses onto a system of spheres in Σ_n;

(5) $\bar{\pi}$ is one-to-one in restriction to each connected component of $\overline{M}_n \cap \Sigma_n$.

The object of interest is actually the *germ* of G_n along $R \cup \overline{M}_n$.

For $n = 0$, the (noncleft) foliation $G_0 := \mathscr{G}$ satisfies Property 4.10(1)–(5), in view of Lemma 4.8(1).

Now, for some $1 \leq n \leq N$, assume that the above induction hypotheses are met for $n-1$.

For short, write α, β and i, instead of α_n, β_n and $\dim(\beta_n)$. We work in $\alpha \times \mathbb{D}^q$. "Horizontal", for a vector field or a foliation, means parallel to the α factor. Denote by $\widehat{\partial}\alpha$ the union of the hyperfaces of α other than β, by π_α the first projection $\alpha \times \mathbb{D}^q \to \alpha$, and by $\mathscr{F}_\alpha$ the slice foliation of $\alpha \times \mathbb{D}^q$ parallel to $\mathbb{D}^q$.

The extension of the cleft foliation G_{n-1}, defined on $\mathscr{O}p_{\overline{M}}(R \cup \overline{M}_{n-1})$, to a cleft foliation G_n on $\mathscr{O}p_{\overline{M}}(R \cup \overline{M}_n)$, will be in two steps: first, an extension only to $P(\alpha)$; then, a microextension from $P(\alpha)$ to $\mathscr{O}p_{\overline{M}}(P(\alpha))$, by means of the civilizations $\mathscr{C}(\alpha)$ and $\mathscr{C}(\beta)$.

Lemma 4.11 *For each cell γ of K, the cleft foliation G_{n-1} is pullable through the map e_γ restricted to $\mathscr{O}p_\gamma(\gamma \cap K_{n-1}) \times \mathbb{D}^q$, and defines on such a neighbourhood a pullback cleft foliation $e_\gamma^*(G_{n-1})$ quasicomplementary to the slices parallel to $\mathbb{D}^q$.*

Proof Consider any point $(x, y) \in (\gamma \cap K_{n-1}) \times \mathbb{D}^q$. Let $\delta \subset \gamma \cap K_{n-1}$ be the smallest face containing x. By Property 4.10(3) applied to δ at order $n-1$ and by Lemma 4.8(3), $\mathscr{C}_\gamma$ is tangential to G_{n-1} on $\mathscr{O}p_{\overline{M}}(x, y)$. Since the embedding e_γ is transverse to $\mathscr{C}_\gamma$ at the point (x, y), the pullability and the quasicomplementarity follow (see Section 3.1.3(e)). □

Lemma 4.12 *There is on $\alpha \times \mathbb{D}^q$ a cleft foliation G_α such that*

(1) *G_α is quasicomplementary to $\mathscr{F}_\alpha$ on $\alpha \times \mathbb{D}^q$;*

(2) *G_α coincides with $e_\alpha^*(G_{n-1} = \mathscr{G})$ on $\mathscr{O}p_{\alpha \times \mathbb{D}^q}(\alpha \times \partial\mathbb{D}^q)$;*

(3) *G_α coincides with $e_\alpha^*(G_{n-1})$ on $\mathscr{O}p_{\alpha \times \mathbb{D}^q}(\widehat{\partial}\alpha \times \mathbb{D}^q)$;*

(4) *G_α coincides with $e_\alpha^*(G_{n-1} = \mathscr{G})$ on $\mathscr{O}p_{\alpha \times \mathbb{D}^q}\big({e_\alpha}^{-1}(R(\beta))\big)$;*

(5) *$e_\alpha^*(\mathscr{C}_\beta)$ is tangential to G_α on $\mathscr{O}p_{\alpha \times \mathbb{D}^q}(\beta \times \mathbb{D}^q)$.*

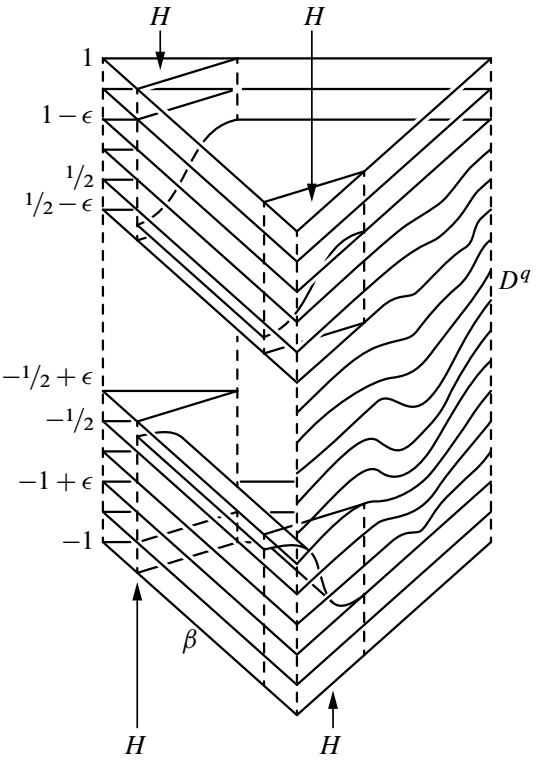

Figure 4: Foliating the prism, in the case $i = q = 1$.

Proof The construction of G_α is the main part of the inflation process (Figure 4). Informally, after Thurston's image [36], we can think of the prism $\alpha \times \mathbb{D}^q$, which we have to foliate, as a room. The back wall is $\widehat{\partial}\alpha \times \mathbb{D}^q$; the ceiling and floor are $\alpha \times \partial\mathbb{D}^q$ (here of course the image is less realistic for $q > 1$); the front wall $\beta \times \mathbb{D}^q$ splits as the union of a window ${e_\alpha}^{-1}(P(\beta))$ and a corona ${e_\alpha}^{-1}(R(\beta))$; there are no side walls in our prismatic version.

A cleft foliation $e_\alpha^*(G_{n-1})$ is already defined close to the back wall, the ceiling and floor, and the corona. Close to the ceiling, floor and corona, the foliation $e_\alpha^*(G_{n-1})$ is not cleft; actually it is essentially horizontal (parallel to α) there. On the contrary, close to the back wall, $e_\alpha^*(G_{n-1})$ is in general cleft and complicated, being produced by the previous steps of the induction.

The general idea is to pull back this foliation through the room by means of a flow crossing the room from the front wall to the back wall and tangential to the ceiling and to the floor.

The difficulty is that, in general, it is not possible to make the pullback foliation match the horizontal foliation on the corona. This is solved at the price of a new cleft (or hole), which will allow us to somehow divert the complicated part of the back wall foliation from the corona, and make it exit, instead, through the window.

Some technicalities arise from the fact that the foliation to be pulled back is cleft, that the flow must be tangential to this foliation where it is already defined, that the extended cleft foliation must be quasicomplementary to the verticals (condition (1) above) and tangential to the civilization of the front wall $\beta \times \mathbb{D}^q$ (condition (5)), this last condition being crucial in view of the subsequent microextension from $\alpha \times \mathbb{D}^q \cong P(\alpha)$ to $\mathscr{O}p_{\overline{M}}(P(\alpha))$.

First, some normalizations will fix the ideas and simplify the notation. We have some choice in the parametrization e_α of $P(\alpha)$: we can change e_α to $e_\alpha \circ F$, where F is any isotopy of $\alpha \times \mathbb{D}^q$ vertical with respect to π_α (Definition 3.16).

(I) The boundary component $\alpha \times \partial\mathbb{D}^q$ is saturated for the foliation $e_\alpha^*(\mathscr{G})$ (recall Lemma 4.4), which on $\alpha \times \partial\mathbb{D}^q$ restricts to the slice foliation parallel to α. Hence, after a first vertical isotopy, we can arrange, on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\alpha \times \partial\mathbb{D}^q)$, that $e_\alpha^*(\mathscr{G})$ (which coincides with $e_\alpha^*(\mathscr{G}_{n-1})$ there by Property 4.10(2) at order $n-1$) is horizontal there.

(II) In $\mathbb{R}^q$, denote by D_r (resp. A_r) the compact disk (resp. corona) defined by $|y| \le r$ (resp. $r \le |y| \le 1$). After a second vertical isotopy, relative to $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\alpha \times \partial\mathbb{D}^q)$, we can arrange that

$$e_\alpha{}^{-1}(P(\beta)) = \beta \times D_{1/2} \quad \text{and} \quad e_\alpha{}^{-1}(R(\beta)) = \beta \times A_{1/2}.$$

(III) Since $e_\alpha^*(\mathscr{G})|_{\beta\times A_{1/2}}$ is a foliated product whose base β is simply connected, this foliated product is trivial. Hence, after a third vertical isotopy, relative to $\beta \times D_{1/2}$ and to $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\alpha \times \partial\mathbb{D}^q)$, we can moreover arrange that $e_\alpha^*(\mathscr{G}_{n-1} = \mathscr{G})$ is horizontal on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\beta \times A_{1/2})$.

In the case $i = 0$, Lemma 4.12 is trivial: α is an interval whose endpoints are β and $\widehat{\alpha}$. The foliation $\mathscr{C}(\alpha)$ being of codimension 1 in $\overline{M}$, tangentiality, Property 4.10(3), at order $n-1$ simply means that the prism $P(\alpha)$ does not meet C_{n-1}. Hence, $\alpha \times \mathbb{D}^q$ is endowed along $\widehat{\alpha} \times \mathbb{D}^q$, along $\alpha \times \partial\mathbb{D}^q$ and along $\beta \times A_{1/2}$ with a germ of 1–dimensional foliation $e_\alpha^*(\mathscr{G}_{n-1})$ (not cleft), complementary to $\mathscr{F}_\alpha$ and horizontal on $\alpha \times \partial\mathbb{D}^q$ (Property 4.10(1)–(2) at order $n-1$). One has an obvious extension of $e_\alpha^*(\mathscr{G}_{n-1})$ by a 1–dimensional foliation complementary to $\mathscr{F}_\alpha$ on all of $\alpha \times \mathbb{D}^q$.

For the rest of the proof of Lemma 4.12 , we assume that $i \ge 1$.

Claim 4.13 *There are*

- *on α, a nonsingular vector field ∇;*
- *on $\alpha \times \mathbb{D}^q$, a nonsingular vector field $\widetilde{\nabla}$;*

such that

(1) *∇ is transverse to every hyperface of α;*

(2) *every orbit of ∇ enters α through β and exits α through $\widehat{\partial}\alpha$;*

(3) *$\widetilde{\nabla}$ lifts ∇ through π_α;*

(4) *$\widetilde{\nabla}$ is tangential to $e_\alpha^*(\mathscr{G})$ on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\alpha \times \partial\mathbb{D}^q)$;*

(5) *$\widetilde{\nabla}$ is tangential to $e_\alpha^*(\mathscr{G})$ on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\beta \times A_{1/2})$;*

(6) *$\widetilde{\nabla}$ is tangential to $e_\alpha^*(G_{n-1})$ on $\mathscr{O}p_\alpha(\widehat{\partial}\alpha) \times \mathbb{D}^q$.*

For (6), recall Definition 3.5.

The construction requires a little care because α is an arbitrary linear $(i+1)$–simplex in the Euclidean space $\mathbb{R}^{p+1}$; for example, some of the dihedral angles of α along the hyperfaces of β can be obtuse.

Proof For simplicity, in the proof of this claim, denote by $\mathscr{C}_\gamma$, $\mathscr{G}$ and G_{n-1} the pullbacks of $\mathscr{C}_\gamma$, $\mathscr{G}$ and G_{n-1} in $\alpha \times \mathbb{D}^q$ through e_α.

For each face $\gamma \subseteq \alpha$, its barycenter $\flat(\gamma)$ is a vertex of the first barycentric subdivision $\mathrm{Bar}(\alpha)$ of α. Let $S_\gamma \subset \alpha$ be the open star of $\flat(\gamma)$ with respect to $\mathrm{Bar}(\alpha)$ (the interior of the union of the cells of $\mathrm{Bar}(\alpha)$ containing $\flat(\gamma)$). The interest of these stars is that for every two faces $\gamma, \delta \subseteq \alpha$, one obviously has

(a) $\mathrm{Int}(\gamma) \subset S_\gamma$;

(b) S_γ intersects δ if and only if $\gamma \subset \delta$;

(c) S_γ intersects S_δ if and only if $\gamma \subset \delta$ or $\delta \subset \gamma$.

Subclaim A *There is, for every face $\gamma \subseteq \alpha$, an open subset $U_\gamma \subset \alpha$ such that, for every two faces $\gamma, \gamma' \subseteq \alpha$,*

(i) $\mathrm{Int}(\gamma) \subset U_\gamma \subset S_\gamma$;

(ii) *$\mathscr{C}_\gamma$ is defined on $U_\gamma \times \mathbb{D}^q$, and $\mathscr{C}_\gamma \subset \mathscr{G}$ on $U_\gamma \times \mathscr{O}p_{\mathbb{D}^q}(\partial\mathbb{D}^q)$;*

(iii) *every leaf of $\mathscr{C}_\gamma|_{U_\gamma\times\mathbb{D}^q}$ is mapped diffeomorphically onto a leaf of $\mathscr{N}_\gamma|_{U_\gamma}$ through π_α;*

(iv) *if* $\gamma \neq \alpha, \beta$, *then* G_{n-1} *is defined on* $U_\gamma \times \mathbb{D}^q$, *and* $\mathscr{C}_\gamma$ *is tangential to* G_{n-1} *on* $U_\gamma \times \mathbb{D}^q$;

(v) $\mathscr{C}_\beta \subset \mathscr{G}$ *on* $U_\beta \times \mathscr{O}p_{\mathbb{D}^q}(A_{1/2})$;

(vi) *if* $\gamma \subset \gamma'$ *then* $\mathscr{C}_{\gamma'} \subset \mathscr{C}_\gamma$ *on* $(U_\gamma \cap U_{\gamma'}) \times \mathbb{D}^q$.

The proof of Subclaim A is straightforward by means of a descending induction on $\dim(\gamma)$, using Lemma 4.8, and (a) above, and the induction, Property 4.10(2)–(3), at the order $n-1$.

We now resume the proof of Claim 4.13. By (i), the family $(U_\gamma)_{\gamma \subseteq \alpha}$ is an open cover of α. Let $(u_\gamma)_{\gamma \subseteq \alpha}$ be on α a partition of the unity subordinate to this cover.

Definition of ∇ Consider the vector $v_\alpha := \flat(\alpha) - \flat(\beta)$, and, for each $\gamma \subset \partial\alpha$, the orthogonal projection v_γ of v_α into the linear subspace $\gamma^\perp \subset \mathbb{R}^{p+1}$ orthogonal to γ. Put

$$\nabla := \sum_{\gamma \subseteq \alpha} u_\gamma v_\gamma$$

Verification of (1) Given a hyperface $\eta \subset \alpha$, let $n_\eta \in \mathbb{R}^{p+1}$ be the unit vector parallel to α and normal to η, pointing inwards α if $\eta = \beta$, and outwards if $\eta \neq \beta$. Since α is a linear simplex, $\langle v_\alpha, n_\eta \rangle > 0$ (Euclidean scalar product in $\mathbb{R}^{p+1}$). Consider any point $x \in \eta$. For every face $\gamma \subseteq \alpha$ such that $x \in U_\gamma$, by (i) and (b) one has $\gamma \subset \eta$, hence $n_\eta \in \gamma^\perp$, hence $\langle v_\gamma, n_\eta \rangle = \langle v_\alpha, n_\eta \rangle$. Finally,

$$\langle \nabla(x), n_\eta \rangle = \langle v_\alpha, n_\eta \rangle > 0.$$

Verification of (2) For every face $\gamma \subset \partial\alpha$, clearly v_α is not parallel to γ, hence $v_\gamma \neq 0$ and

$$\langle v_\alpha, v_\gamma \rangle = |v_\gamma|^2 > 0.$$

It follows, in view of the definition of ∇, that $\langle v_\alpha, \nabla \rangle > 0$ at every point of α. So, every orbit of ∇ is proper in α (a compact segment with endpoints on $\partial\alpha$). Since $\langle \nabla(x), n_\eta \rangle > 0$ at every point x of every hyperface η, necessarily every orbit goes across α from β to $\widehat{\partial}\alpha$.

Definition of $\widetilde{\nabla}$ The constant vector field on α parallel to v_α lifts through π_α to a vector field $\widetilde{v}_\alpha$ on $\alpha \times \mathbb{D}^q$, which we can choose tangential to $\mathscr{G}$ close to $\alpha \times \partial\mathbb{D}^q$. On the other hand, for each face $\gamma \subset \partial\alpha$, by (iii), the constant vector field on U_γ parallel

to v_γ, being tangential to $\mathscr{N}_\gamma$, lifts in $\alpha \times \mathbb{D}^q$ through π_α to a unique vector field $\widetilde{v}_\gamma$ on $U_\gamma \times \mathbb{D}^q$ tangential to $\mathscr{C}_\gamma$. Put

$$\widetilde{\nabla} := \sum_{\gamma \subseteq \alpha} (u_\gamma \circ \pi_\alpha) \widetilde{v}_\gamma.$$

(3) holds by definition; (4) follows from (ii). We are left to verify (5) and (6).

First, note that for every $x \in \alpha$, the set of the faces $\gamma \subseteq \alpha$ such that $x \in \operatorname{spt}(u_\gamma)$ has a smallest element $\delta(x)$: indeed, this finite set is *totally* ordered by the inclusion relation, because of (i) and (c). Moreover, for every $x' \in \alpha$ close enough to x, one has $\delta(x) \subset \delta(x')$ (since the supports $\operatorname{spt}(u_\gamma)$ for $\gamma \subseteq \alpha$ are compact and finite in number.) Consequently, for $x \in \partial\alpha$, the set

$$N_x := \{x' \in U_{\delta(x)} \mid x' \notin \operatorname{spt}(u_\alpha) \text{ and } \delta(x) \subset \delta(x')\}$$

is an open neighbourhood of x in α.

Subclaim B *For every $x \in \partial\alpha$, $x' \in N_x$ and $y \in \mathbb{D}^q$,*

$$\widetilde{\nabla}(x', y) \in \tau_{(x',y)} \mathscr{C}_{\delta(x)}.$$

Indeed, for each proper face $\gamma \subset \partial\alpha$ such that $x' \in \operatorname{spt}(u_\gamma)$, one has

$$\delta(x) \subset \delta(x') \subset \gamma;$$

hence, thanks to (vi),

$$\widetilde{v}_\gamma(x', y) \in \tau_{(x',y)} \mathscr{C}_\gamma \subset \tau_{(x',y)} \mathscr{C}_{\delta(x)}.$$

Subclaim B follows by definition of $\widetilde{\nabla}$.

Verification of (5) Fix $x \in \beta$. There are two cases.

In the case $\delta(x) = \beta$, for every $x' \in N_x$, by definition of N_x, necessarily $\delta(x') = \beta$ also; hence, by definition of $\widetilde{\nabla}$ and $\widetilde{v}_\beta$, for every $y \in \mathbb{D}^q$,

$$\widetilde{\nabla}(x', y) = \widetilde{v}_\beta(x', y) \in \tau_{(x',y)} \mathscr{C}_\beta.$$

Thus, provided that y is close enough to $A_{1/2}$ in $\mathbb{D}^q$, after (v),

$$\widetilde{\nabla}(x', y) \in \tau_{(x',y)} \mathscr{G}.$$

Now, consider the second case $\delta(x) \neq \beta$. On the one hand, by Subclaim B and by (iv) applied to $\gamma = \delta(x)$, the vector field $\widetilde{\nabla}$ is tangential to G_{n-1} on $N_x \times \mathbb{D}^q$.

On the other hand, by the induction, Property 4.10(2), G_{n-1} coincides with $\mathscr{G}$ on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\beta\times A_{1/2})$. So, we also get (5) in that case.

Verification of (6) For every $x\in\widehat{\partial}\alpha$, by Subclaim B and by (iv) applied to $\gamma:=\delta(x)$, the tangency property (6) holds on $N_x\times\mathbb{D}^q$. The proof of Claim 4.13 is complete. □

Let us resume the proof of Lemma 4.12. By (4) and (5) of Claim 4.13, the vector field $\widetilde{\nabla}$ is horizontal on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\alpha\times\partial\mathbb{D}^q)$ and on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\beta\times A_{1/2})$. By a fourth vertical isotopy relative to $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\alpha\times\partial\mathbb{D}^q)$ and to $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\beta\times A_{1/2})$ and to $\beta\times\mathbb{D}^q$, we can moreover arrange that

(a) $\widetilde{\nabla}$ is horizontal everywhere on $\alpha\times\mathbb{D}^q$.

Denoting by x_0 the barycentric coordinate function on α that restricts to 0 on β and to 1 on the opposite vertex, fix $\epsilon>0$ small enough that

(b) the domains $\alpha\times A_{1-\epsilon}$ and ${x_0}^{-1}([0,\epsilon])\times A_{1/2-\epsilon}$ do not intersect C_{n-1}, and, on some neighbourhood of these domains, the foliations $e_\alpha^*(\mathscr{G}_{n-1})$ and $e_\alpha^*(\mathscr{G})$ are both defined, coincide, and are horizontal.

Since ∇ enters α through β (recall (1) and (2) of Claim 4.13), provided that ϵ is small enough, after rescaling ∇ (and accordingly rescaling $\widetilde{\nabla}$, so that (3) of Claim 4.13 still holds), we can moreover arrange that

(c) the derivative $\nabla\cdot x_0$ equals ϵ on ${x_0}^{-1}[0,\epsilon]$.

Let $\mathrm{pr}_\nabla:\alpha\to\beta$ denote the projection along the flowlines of ∇. Decompose α into three subsets α', α'', α''' and β into two subsets β' and β'', as follows (Figure 5):

- $\alpha':={x_0}^{-1}[\epsilon,1]$.
- $\beta'=\mathrm{pr}_\nabla(\alpha')$.

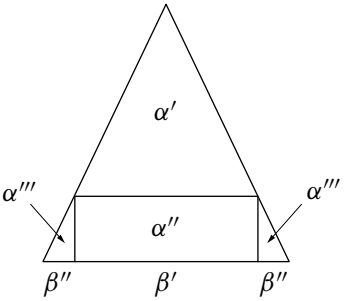

Figure 5: Decomposition of the simplices α and β.

- $\alpha'' := {x_0}^{-1}[0,\epsilon] \cap {\mathrm{pr}_\nabla}^{-1}(\beta')$.
- $\beta'' :=$ topological closure of $\beta \setminus \beta'$.
- $\alpha''' := {\mathrm{pr}_\nabla}^{-1}(\beta'')$.

Fix a self-diffeomorphism v of $\mathbb{I}$, supported in the open interval $\left(\frac{1}{2}-\epsilon, 1\right)$ and such that

(d) $v(1-\epsilon) < \frac{1}{2}$.

Define the self-diffeomorphism ψ_1 of $\mathbb{D}^q$ as

$$\psi_1 \colon y \mapsto v(|y|)y/|y|$$

for $y \neq 0$, and $\psi_1(0) = 0$. Fix a 1–parameter family $\psi := (\psi_t)_{t\in\mathbb{I}}$ of self-diffeomorphisms of $\mathbb{D}^q$, supported in the interior of $A_{1/2-\epsilon}$, such that $\psi_t = \mathrm{id}$ for t close to 0 and $\psi_t = \psi_1$ for t close to 1. Let $(Y_t)_{t\in\mathbb{I}}$ be the 1–parameter family of vector fields on $\mathbb{D}^q$ whose flow is (ψ_t). Define a nonsingular vector field V on $\alpha \times \mathbb{D}^q$ by

(e) $V(x,y) := \nabla(x) \oplus 0$ if $x \in \alpha'$;

(f) $V(x,y) := \nabla(x) \oplus -Y_{1-\epsilon^{-1}x_0(x)}(y)$ if $x \in \alpha'' \cup \alpha'''$.

Note that

(g) V lifts ∇ through π_α.

The vector fields $\widetilde{\nabla}$ and V being both horizontal on $\mathscr{O}p_\alpha(\widehat{\partial}\alpha) \times \mathbb{D}^q$ (after (a), (e) and (f)), and both of them lifting ∇ there (Claim 4.13(3) and (g)), let U be an open neighbourhood of $\widehat{\partial}\alpha$ in α so small that $V = \widetilde{\nabla}$ on $U \times \mathbb{D}^q$. Provided that U is small enough, one has, on $U \times \mathbb{D}^q$ (after the induction, Property 4.10(1), at the order $n-1$ and after Claim 4.13(6)),

(h) $e_\alpha^*(G_{n-1})$ is quasicomplementary to $\mathscr{F}_\alpha$ on $U \times \mathbb{D}^q$;

(i) V is tangential to $e_\alpha^*(G_{n-1})$ on $U \times \mathbb{D}^q$.

Recalling Claim 4.13(1)–(2), let χ be on α a nonnegative real function such that

- χ vanishes on a neighbourhood of $\widehat{\partial}\alpha$;
- the time $t = 1$ of the flow (∇'^t) of the vector field $\nabla' := \chi\nabla$ embeds α into U.

Put $V' := (\chi \circ \pi_\alpha)V$ and consider the time $t = 1$ of the flow (V'^t).

Obviously, since $V'^1(\alpha\times\mathbb{D}^q)\subset U\times\mathbb{D}^q$ and since V is tangential to $e_\alpha^*(G_{n-1})$ there ((g), (i)), the cleft foliation $e_\alpha^*(G_{n-1})$ is pullable (Section 3.1.3(c)) through V'^1. Consider on $\alpha\times\mathbb{D}^q$ the pullback cleft foliation

$$G^0 := (e_\alpha\circ V'^1)^*(G_{n-1})$$

and write $G^0=(C^0,[c^0],\mathscr{G}^0)$. Put $H:=\alpha'''\times A_{1-\epsilon}$.

Claim 4.14 (1) *The cleft foliation G^0 is quasicomplementary to $\mathscr{F}_\alpha$ on $\alpha\times\mathbb{D}^q$.*

(2) *The fissure C^0 does not intersect H.*

(3) *The fissure C^0 does not intersect $\alpha'\times A_{1-\epsilon}$ nor $\alpha\times\partial\mathbb{D}^q$, and $\mathscr{G}^0$ is horizontal on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\alpha'\times A_{1-\epsilon})$ and on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\alpha\times\partial\mathbb{D}^q)$.*

(4) *G^0 and $e_\alpha^*(G_{n-1})$ coincide on $\mathscr{O}p_\alpha(\widehat{\partial}\alpha)\times\mathbb{D}^q$.*

(5) *V is tangential to G^0 on $\alpha\times\mathbb{D}^q$.*

(6) *G^0 is invariant by the flow of V.*

(7) *The fissure C^0 does not meet $\beta'\times A_{1/2}$, and $\mathscr{G}^0$ is horizontal on some open neighbourhood of $\beta'\times A_{1/2}$ in $\alpha\times\mathbb{D}^q$.*

(8) *The foliation $e_\alpha^*(\mathscr{C}_\beta)$ is tangential to G^0 on $\mathscr{O}p_\alpha(\partial\beta)\times\mathbb{D}^q$.*

Proof (1) The quasicomplementarity that holds, (h), on $U\times\mathbb{D}^q$ extends to all of $\alpha\times\mathbb{D}^q$, since, the vector field V' being projectable (g) through π_α, its flow preserves the foliation $\mathscr{F}_\alpha$.

(2) Since, by (e)–(f), the domain H is saturated for the flow of V, and since $e_\alpha^*(G_{n-1})$ is not cleft in H (recall (b)).

(3) Clear by (b), (e) and (f).

(4)–(6) After the very definition of G^0.

(7) On the one hand, by (3) of the present Claim 4.14, for $\eta<\epsilon$ close enough to ϵ, in restriction to ${x_0}^{-1}(\eta)\times A_{1-\epsilon}$, the foliation G^0 is not cleft, and $\mathscr{G}^0$ is horizontal (ie parallel to β) there. On the other hand, by (c) and (f), the projection pr_V of the hypersurface ${x_0}^{-1}(\eta)\times\mathbb{D}^q$ into $\beta\times\mathbb{D}^q$ along the flowlines of V is

$$\mathrm{pr}_V : (x,y)\mapsto(\mathrm{pr}_\nabla(x),\psi_1(y)).$$

Since moreover $\mathscr{G}^0$ is V^t–invariant by (6), it follows that $\mathscr{G}^0$ is horizontal in restriction to the image $\mathrm{pr}_V({x_0}^{-1}(\eta)\times A_{1-\epsilon})$, which is an open neighbourhood of $\beta'\times A_{1/2}$ in

$\beta\times\mathbb{D}^q$, since $v(1-\epsilon)<\frac{1}{2}$ by (d). The horizontality holds in fact on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\beta'\times A_{1/2})$, since V is horizontal on $\mathscr{O}p_\alpha(\beta)\times\mathbb{D}^q$ (in view of (e) and (f)).

(8) Fix any point $x\in\partial\beta$. Let γ be the smallest face of β containing x. By Lemma 4.8(3), one has $e_\alpha^*(\mathscr{C}_\beta)\subset e_\alpha^*(\mathscr{C}_\gamma)$ on $\mathscr{O}p_\alpha(x)\times\mathbb{D}^q$. By the induction, Property 4.10(3), at the order $n-1$, the civilization $e_\alpha^*(\mathscr{C}_\gamma)$ is tangential to $e_\alpha^*(G_{n-1})$ on $\mathscr{O}p_\alpha(x)\times\mathbb{D}^q$. By (4), the cleft foliations G_{n-1} and G^0 coincide on $\mathscr{O}p_\alpha(x)\times\mathbb{D}^q$. □

Note that $\mathscr{G}^0$ is *not* horizontal on $\beta''\times A_{1/2}$; in other words, it does *not* match $\mathscr{G}_{n-1}$ there. This is why we need to modify G^0 in H, introducing a new fissure there.

Decompose $\partial\alpha''$ as

$$\partial\alpha''=\partial_1\alpha''\cup\partial_2\alpha''\cup\beta',$$

where

$$\partial_1\alpha'':=\alpha''\cap\alpha'\quad\text{and}\quad\partial_2\alpha'':=\alpha''\cap\alpha'''.$$

First, we define a cleft foliation $G^1=(C^1,[c^1],\mathscr{G}^1)$ on $(\alpha\times\mathbb{D}^q)\setminus H$, and we also define it as a germ along ∂H inside H. Let

(j) G^1 coincide with G^0 on $(\alpha\times\mathbb{D}^q)\setminus H$;

(k) G^1 coincide with G^0 on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\partial_1\alpha''\times A_{1/2-\epsilon})$ and $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\alpha''\times\partial A_{1/2-\epsilon})$; in other words, $\mathscr{G}^1$ is horizontal there;

(l) G^1 coincide with G^0 on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\partial_2\alpha''\times A_{1/2-\epsilon})$;

(m) C^1 not intersect $\beta''\times A_{1/2-\epsilon}$, and $\mathscr{G}^1$ be horizontal on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\beta''\times A_{1/2-\epsilon})$.

This turns H into a hole (Definition 3.13) of core $\mathbb{S}^{i-1}$, fibre $A_{1/2-\epsilon}$ and monodromy ψ (in view of (f)).

Claim 4.15 *Provided that we take some extra care in the choice of ψ, the cleft foliation G^1 extends over H, giving on $\alpha\times\mathbb{D}^q$ a cleft foliation, still denoted by $G^1=(C^1,[c^1],\mathscr{G}^1)$, of monodromy φ and quasicomplementary to $\mathscr{F}_\alpha$. Moreover, C^1 has a component interior to H, whose core is a finite disjoint union of spheres $\mathbb{S}^{i-1}$, each of which embeds into* $\mathrm{Int}(\alpha)$ *through π_α.*

Proof In every codimension $q\geq 2$, consider the diffeomorphism $\psi_1':=\psi_1|_{A_{1/2-\epsilon}}$. Fix an arbitrary equidimensional embedding of $\mathbb{D}^q$ in the interior of $A_{1/2-\epsilon}$. Thus, φ becomes a family of self-diffeomorphisms of $A_{1/2-\epsilon}$. It is easy, in the above construction of V, to choose the diffeomorphism $v|_{[1/2-\epsilon,1]}$ in the group $\mathrm{Diff}\big(\big[\frac{1}{2}-\epsilon,1\big]\big)$ as a product of commutators. Consequently, so is ψ_1' in the group $\mathrm{Diff}(A_{1/2-\epsilon})_0$.

Since φ_1 is not the identity, and since $A_{1/2-\epsilon}$ is connected, by Epstein's perfectness theorem [8, Proposition 1.2 and Theorem 1.4], ψ'_1 belongs to the normal subgroup of $\mathrm{Diff}(A_{1/2-\epsilon})_0$ generated by φ_1; hence some lift $[\psi'] \in \mathrm{Diff}(A_{1/2-\epsilon})^{\mathbb{I}}$ of ψ'_1 (recall Notation 3.1) lies in the normal subgroup of $\mathrm{Diff}(A_{1/2-\epsilon})^{\mathbb{I}}$ generated by φ. In other words, one has in the group $\mathrm{Diff}(A_{1/2-\epsilon})^{\mathbb{I}}$ a splitting

$$\psi' = \chi_1 \cdots \chi_\ell,$$

where $\chi_1, \ldots, \chi_\ell$ are conjugate to φ in $\mathrm{Diff}(A_{1/2-\epsilon})^{\mathbb{I}}$. In the above construction of V, we choose the family (ψ_t) to represent this particular lift ψ'.

Then, by Section 3.1.7 and an obvious induction on ℓ, we partially fill the hole H by a foliation complementary to $\mathscr{F}_\alpha$ (thanks to the property (1) in 3.1.7), leaving some smaller holes H_j $(1 \le j \le \ell)$ of core $\mathbb{S}^{i-1}$, fibre $A_{1/2-\epsilon}$, and respective monodromies $\chi_1, \ldots, \chi_\ell$. Equivalently, each hole H_j is of monodromy φ (Section 3.1.6). Then, each hole H_j is vertically shrunk (Section 3.1.8) into a hole H'_j of core $\mathbb{S}^{i-1}$, fibre $\mathbb{D}^q$ and monodromy φ. Finally (Section 3.1.4), each H'_j is horizontally shrunk into a fissure of core $\mathbb{S}^{i-1}$, fibre $\mathbb{D}^q$ and monodromy φ.

This core does embed into $\mathrm{Int}(\alpha)$ through π_α, after the note that ends Section 3.1.7.

The case $q = 1$ is similar, but we need *two* arbitrary equidimensional embeddings of $\mathbb{D}^1$ in the interior of $A_{1/2-\epsilon} \cong \mathbb{S}^0 \times \mathbb{I}$, one in each connected component. □

In view of Claim 4.14(1), (3), (4) and (7), and of (j), (l) and (m), the cleft foliation G^1 on $\alpha \times \mathbb{D}^q$ matches the properties (1) through (4) of Lemma 4.12 with G^1 instead of G_α. However (Claim 4.14(8)), the property (5), namely the tangentiality of $\mathscr{C}_\beta$ to G^1, only holds on $W \times \mathbb{D}^q$ for some open neighbourhood W of $\partial\beta$ in α. To rectify this, consider on α the unit vector field B normal to β and that enters α through β; consider on $\mathscr{O}p_\alpha(\beta) \times \mathbb{D}^q$ the lift $\widetilde{B}$ of B tangentially to $e_\alpha^*(\mathscr{C}_\beta)$ (Lemma 4.8, (2)). Let $(u, 1-u)$ be on α a partition of the unity subordinate to the open cover $(W, \alpha \setminus \partial\beta)$. Define vector fields Z on α and $\widetilde{Z}$ on $\alpha \times \mathbb{D}^q$ by

$$Z := (1-u)\nabla + uB \quad \text{and} \quad \widetilde{Z} := (1-u\circ\pi_\alpha)V + (u\circ\pi_\alpha)\widetilde{B}.$$

Claim 4.16 *$\widetilde{Z}$ is tangential to G^1 on $\mathscr{O}p_\alpha(\beta) \times \mathbb{D}^q$.*

Proof On the one hand, V is tangential to G^1 on $\mathscr{O}p_\alpha(\beta) \times \mathbb{D}^q$: indeed, outside H, this follows from (5) of Claim 4.14 together with (j); while on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\beta'' \times A_{1/2-\epsilon})$, both V and $\mathscr{G}^1$ are horizontal (by (e) and (m)). On the other hand, $\widetilde{B}$ is tangential to G^1 on $U \times \mathbb{D}^q$, by the above definition of U. □

Since B and Z both enter α through β and coincide on $\mathcal{O}p_\alpha(\partial\beta)$, there is an isotopy $f=(f_t)_{t\in\mathbb{I}}$ of self-diffeomorphisms of α (with $f_0=\mathrm{id}_\alpha$) such that, for every $t\in\mathbb{I}$,

- the support of f_t is contained in a small neighbourhood of β and disjoint from $\hat{\partial}\alpha$;
- f_t induces the identity on β;
- $f_t^*(Z)=(1-t)Z+tB$ on $\mathcal{O}p_\alpha(\beta)$.

Claim 4.17 *There is an isotopy $\widetilde{f}=(\widetilde{f}_t)_{t\in\mathbb{I}}$ of self-diffeomorphisms of $\alpha\times\mathbb{D}^q$ (with $f_0=\mathrm{id}_{\alpha\times\mathbb{D}^q}$) such that, for every $t\in\mathbb{I}$,*

(1) $\pi_\alpha\circ\widetilde{f}_t=f_t\circ\pi_\alpha$;

(2) *the vector field $\partial\widetilde{f}_t/\partial t$ is at the time t horizontal, ie tangential to $e_\alpha^*(\mathscr{G})$, on $\alpha\times\mathcal{O}p_{\mathbb{D}^q}(\partial\mathbb{D}^q)$ and on $\mathcal{O}p_{\alpha\times\mathbb{D}^q}(\beta\times A_{1/2})$;*

(3) *$\widetilde{f}_t$ is supported in $(\alpha\setminus\hat{\partial}\alpha)\times\mathbb{D}^q$;*

(4) *$\widetilde{f}_t$ induces the identity on β;*

(5) *$\widetilde{f}_t^*(\widetilde{Z})=(1-t)\widetilde{Z}+t\widetilde{B}$ on $\mathcal{O}p_\alpha(\beta)\times\mathbb{D}^q$.*

Proof On $\mathcal{O}p_\alpha(\beta)\times\mathbb{D}^q$, the isotopy $\widetilde{f}$ is uniquely determined by (4) and (5). There, (1) is satisfied, since $\widetilde{Z}$ and $\widetilde{B}$ respectively lift Z and B through π_α. Also, on $\mathcal{O}p_\alpha(\beta)\times\mathbb{D}^q$, (2) is satisfied, since V, $\widetilde{B}$ and hence $\widetilde{Z}$ are horizontal there (after (e), (f), Lemma 4.8(1) applied to β, and after (III)). Also, on $\mathcal{O}p_\alpha(\beta)\times\mathbb{D}^q$, (3) is satisfied, since $\widetilde{Z}=\widetilde{B}$ on $\mathcal{O}p_\alpha(\partial\beta)$. Finally, by the standard extension property for vertical isotopies, $\widetilde{f}$ admits an extension to all of $\alpha\times\mathbb{D}^q$ which obeys the conditions (1), (2) and (3). □

Define the cleft foliation G_α on $\alpha\times\mathbb{D}^q$ as $\widetilde{f}_1^*(G^1)$; let us verify that the conditions (1)–(5) of Lemma 4.12 are obeyed:

(1) By Claim 4.15 and by Claim 4.17(1).

(2) By (I), by Claim 4.14(3), by (j), by (k), and by Claim 4.17(2).

(3) By Claim 4.14(4), by (j), by (k), and by Claim 4.17(3).

(4) By Claim 4.14(7), by (j), by (m), and by Claim 4.17(2).

(5) By Claim 4.16 and by Claim 4.17(5) applied for $t=1$.

Lemma 4.12 is proved. □

Recall that the notation α, β and i holds for α_n, β_n and $\dim(\beta_n)$.

It remains to us, in order to complete the inflation induction, to microextend G_α from $\alpha \times \mathbb{D}^q \cong P(\alpha)$ to $\mathcal{O}p_{\overline{M}}(P(\alpha))$ so as to match G_{n-1} on the intersection with $\mathcal{O}p_{\overline{M}}(R \cup \overline{M}_{n-1})$, and to satisfy Property 4.10(1)–(5). To this end, we use the civilizations of the prisms.

For every cell γ of K, denote by $\langle\gamma\rangle \subset \mathbb{R}^{p+1}$ the affine subspace spanned by γ, and consider the local retraction along the leaves of $\mathscr{C}_\gamma$,

$$\widetilde{\mathrm{pr}}_\gamma : \mathcal{O}p_{\overline{M}}(P(\gamma)) \to \mathcal{O}p_{\langle\gamma\rangle\times\mathbb{D}^q}(P(\gamma)).$$

Actually, the object of interest is the germ of $\widetilde{\mathrm{pr}}_\gamma$ along $P(\gamma)$. Also, denote by pr_γ the orthogonal projection $\mathbb{R}^{p+1} \to \langle\gamma\rangle$. Obviously, by Lemma 4.8:

Corollary 4.18 (1) $\mathscr{G} = \widetilde{\mathrm{pr}}_\gamma^*(\mathscr{G}|_{\langle\gamma\rangle\times\mathbb{D}^q})$ *on* $\mathcal{O}p_{\overline{M}}(e_\gamma(\gamma \times \partial\mathbb{D}^q))$.

(2) $\overline{\pi} \circ \widetilde{\mathrm{pr}}_\gamma = \mathrm{pr}_\gamma \circ \overline{\pi}$ *on* $\mathcal{O}p_{\overline{M}}(P(\gamma))$.

(3) $\widetilde{\mathrm{pr}}_\delta \circ \widetilde{\mathrm{pr}}_\gamma = \widetilde{\mathrm{pr}}_\delta$ *on* $\mathcal{O}p_{\overline{M}}(P(\delta))$ *for every face* $\delta \subset \gamma$.

Also, Lemma 4.11 admits the following complement:

Lemma 4.19 *For each cell* γ *of* K, *the cleft foliation* G_{n-1} *coincides with the pullback* $\widetilde{\mathrm{pr}}_\gamma^*(G_{n-1}|_{P(\gamma)})$ *on*

$$\mathcal{O}p_{\overline{M}}\big(\mathrm{Int}(P(\gamma))\big) \cap \mathcal{O}p_{\overline{M}}((R \cup \overline{M}_{n-1}) \cap P(\gamma)).$$

Proof Consider any point $(x, y) \in (R \cup \overline{M}_{n-1}) \cap P(\gamma)$. Let $\delta \subset \gamma$ be the smallest face containing x. We claim that $\mathscr{C}_\delta$ is tangential to G_{n-1} on $\mathcal{O}p_{\overline{M}}(x, y)$.

Indeed, $\delta \subset K_{n-1}$ or $y \in R(\delta)$. In the case where $\delta \subset K_{n-1}$, the claim follows from the induction, Property 4.10(3), applied to δ at the order $n-1$. In the case where $y \in R(\delta)$, on $\mathcal{O}p_{\overline{M}}(x, y)$, the foliation G_{n-1} coincides with $\mathscr{G}$ (induction, Property 4.10(2), at the order $n-1$), while $\mathscr{C}_\delta \subset \mathscr{G}$ (after Lemma 4.8(1)). The claim is proved.

In view of Lemma 4.8(3) , $\mathscr{C}_\gamma$ is also tangential to G_{n-1} on $\mathcal{O}p_{\overline{M}}(x, y)$. Since $P(\gamma)$ is transverse to $\mathscr{C}_\gamma$, the wanted coincidence follows on

$$\mathcal{O}p_{\overline{M}}\big(\mathrm{Int}(P(\gamma))\big) \cap \mathcal{O}p_{\overline{M}}(x, y)$$

(see Section 3.1.3 and Remark 4.6). □

Claim 4.20 (1) *G_{n-1} coincides with $(e_\alpha{}^{-1}\circ\widetilde{\mathrm{pr}}_\alpha)^*(G_\alpha)$ on*

$$\mathscr{O}p_{\overline{M}}(R\cup\overline{M}_{n-1})\cap\mathscr{O}p_{\overline{M}}\big(\mathrm{Int}(P(\alpha))\big).$$

(2) *G_{n-1} coincides with $(e_\alpha{}^{-1}\circ\widetilde{\mathrm{pr}}_\beta)^*(G_\alpha)$ on*

$$\mathscr{O}p_{\overline{M}}(R\cup\overline{M}_{n-1})\cap\mathscr{O}p_{\overline{M}}\big(\mathrm{Int}(P(\beta))\big).$$

(3) *$(e_\alpha{}^{-1}\circ\widetilde{\mathrm{pr}}_\alpha)^*(G_\alpha)$ coincides with $(e_\alpha{}^{-1}\circ\widetilde{\mathrm{pr}}_\beta)^*(G_\alpha)$ on*

$$\mathscr{O}p_{\overline{M}}\big(\mathrm{Int}(P(\alpha))\big)\cap\mathscr{O}p_{\overline{M}}\big(\mathrm{Int}(P(\beta))\big).$$

Proof (1) One has

$$e_\alpha{}^{-1}(R\cup\overline{M}_{n-1})=(\alpha\times\partial\mathbb{D}^q)\cup(\hat{\partial}\alpha\times\mathbb{D}^q)\cup(\beta\times A_{1/2}).$$

After Lemma 4.12(2)–(4), on some open neighbourhood of this subset in $\alpha\times\mathbb{D}^q$, one has $G_\alpha=e_\alpha^*(G_{n-1})$. Hence, the claim (1) amounts to Lemma 4.19 applied to α.

(2) One has

$$e_\alpha{}^{-1}(R\cup\overline{M}_{n-1})\cap(\beta\times D_{1/2})=\partial(\beta\times D_{1/2}).$$

After Lemma 4.12(3)–(4), on some open neighbourhood of this subset in $\alpha\times\mathbb{D}^q$, one has $G_\alpha=e_\alpha^*(G_{n-1})$. After Lemma 4.12(5), this cleft foliation is restrictable to $\beta\times D_{1/2}$. Hence, the claim (2) amounts to Lemma 4.19 applied to β.

(3) After Lemma 4.12(5), on $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\beta\times\mathbb{D}^q)$, the cleft foliation G_α coincides with the pullback of $G_\alpha|_{\beta\times\mathbb{D}^q}$ through the local retraction of $\mathscr{O}p_{\alpha\times\mathbb{D}^q}(\beta\times\mathbb{D}^q)$ onto $\beta\times\mathbb{D}^q$ along the leaves of $e_\alpha^*(\mathscr{C}_\beta)$. Hence, the claim (3) follows from Corollary 4.18(3) applied to the pair $\beta\subset\alpha$. □

Globally, we have got a cleft foliation G_n on $\mathscr{O}p_{\overline{M}}(R\cup\overline{M}_n)$ such that

(i) $G_n=G_{n-1}$ on $\mathscr{O}p_{\overline{M}}(R\cup\overline{M}_{n-1})$;

(ii) $G_n=(e_\alpha{}^{-1}\circ\widetilde{\mathrm{pr}}_\alpha)^*(G_\alpha)$ on $\mathscr{O}p_{\overline{M}}\big(\mathrm{Int}(P(\alpha))\big)$;

(iii) $G_n=(e_\alpha{}^{-1}\circ\widetilde{\mathrm{pr}}_\beta)^*(G_\beta)$ on $\mathscr{O}p_{\overline{M}}\big(\mathrm{Int}(P(\beta))\big)$.

We are left to verify that G_n matches Property 4.10(1)–(5):

(1) After Section 3.1.3(e).

(2) After the very definition (i) above, and after the induction, Property 4.10(2), at the order $n-1$.

(3) The only two cells to consider are α and β. The civilization $\mathscr{C}_\alpha$ (resp. $\mathscr{C}_\beta$) is indeed tangential to G_n on $\mathscr{O}p_{\overline{M}}\big(\mathrm{Int}(P(\alpha))\big)$ $\big($resp. $\mathscr{O}p_{\overline{M}}\big(\mathrm{Int}(P(\beta))\big)\big)$ by the very definition (ii) (resp. (iii)), and on $\mathscr{O}p_{\overline{M}}(R(\alpha))$ because G_n coincides with $\mathscr{G}$ there, and because of Lemma 4.8(1).

(4) Let us recollect the construction of the core Σ_n from Σ_{n-1}. Recall or agree that $G_{n-1} = (C_{n-1}, [c_{n-1}], \mathscr{G}_{n-1})$, $G^0 = (C^0, [c^0], \mathscr{G}^0)$, $G^1 = (C^1, [c^1], \mathscr{G}^1)$, $G_\alpha = (C_\alpha, [c_\alpha], \mathscr{G}_\alpha)$ and $G_n = (C_n, [c_n], \mathscr{G}_n)$. Consider the cores $\Sigma_{n-1} := {c_{n-1}}^{-1}(0,0)$, $\Sigma^0 := (c^0)^{-1}(0,0)$, $\Sigma^1 := (c^1)^{-1}(0,0)$, $\Sigma_\alpha := {c_\alpha}^{-1}(0,0)$ and $\Sigma_n := {c_n}^{-1}(0,0)$. Put

$$\widehat{\Sigma} := \Sigma_{n-1} \cap (\widehat{\partial}\alpha \times \mathbb{D}^q).$$

By Claim 4.14(5), Σ^0 is the saturation of ${e_\alpha}^{-1}(\widehat{\Sigma})$ by the flow of V. Hence, it is easy to extend the complex $e_\alpha^*(\Delta_{n-1})$, which is a subdivision of ${e_\alpha}^{-1}(\Sigma^0)$, by a smooth triangulation Δ^0 of Σ^0 collapsing on $e_\alpha^*(\Delta_{n-1})$.

Also (Claim 4.15), Σ^1 is nothing but the disjoint union of Σ^0 with the core $\Sigma^1 \cap H$ of the new fissure in the hole, which is a finite disjoint union of $(i-1)$–spheres disjoint from $\widehat{\partial}\alpha \times \mathbb{D}^q$. Let Δ_H be an arbitrary smooth triangulation of $\Sigma^1 \cap H$.

Also, recalling the definition of G^1, the core $\Sigma_\alpha = {\widetilde{f_1}}^{-1}(\Sigma^1)$ is isotopic with Σ^1 in $\alpha \times \mathbb{D}^q$ rel $\widehat{\partial}\alpha \times \mathbb{D}^q$ (by Claim 4.17 and (3)).

By the above definition (i)–(iii) of G_n, one has

$$\Sigma_n \cap \overline{M}_n = (\Sigma_{n-1} \cap \overline{M}_{n-1}) \cup_{\widehat{\Sigma}} e_\alpha(\Sigma_\alpha). \tag{3}$$

We define a partial smooth triangulation Δ_n of Σ_n as the union of Δ_{n-1} with $e_\alpha({\widetilde{f_1}}^{-1}(\Delta_H))$ and $e_\alpha({\widetilde{f_1}}^{-1}(\Delta^0))$. Property 4.10(4) is verified for G_n.

(5) Will follow as well from the preceding recollection of the successive constructions of the cores. Briefly, by the induction, Property 4.10(5), at the order $n-1$, the projection $\overline{\pi}$ is one-to-one on every connected component of $\Sigma_{n-1} \cap \overline{M}_{n-1}$, and thus, in particular, of $\widehat{\Sigma}$. By (g), the projection $\overline{\pi}$ is one-to-one on every connected component of Σ^0. On the other hand, in view of the construction in the proof of Claim 4.15, each $(i-1)$–sphere composing $\Sigma^1 \cap H$ embeds into $\mathrm{Int}(\alpha)$ through $\overline{\pi}$ (and the image is an embedded $(i-1)$–sphere, close to $\partial\beta$ for the Hausdorff distance). Hence, $\overline{\pi}$ is one-to-one on every connected component of Σ^1. Since the isotopy $\widetilde{f_1}$ is projectable (by (1) of Claim 4.17), the projection $\overline{\pi}$ is one-to-one on every connected component of Σ_α. Finally, after (3), $\overline{\pi}$ is one-to-one on every connected component of $\Sigma_n \cap \overline{M}_n$.

This completes the induction on n, the proof of Proposition 4.3, and the proof of Theorem A′.

4.4 Topology of the cores

We now give a complement to Theorems A and A′ by making explicit the topologies of the cores of the fissures and of the multifold Reeb components resulting from the above proofs (Figure 6).

Recall the notation from Theorem A′.

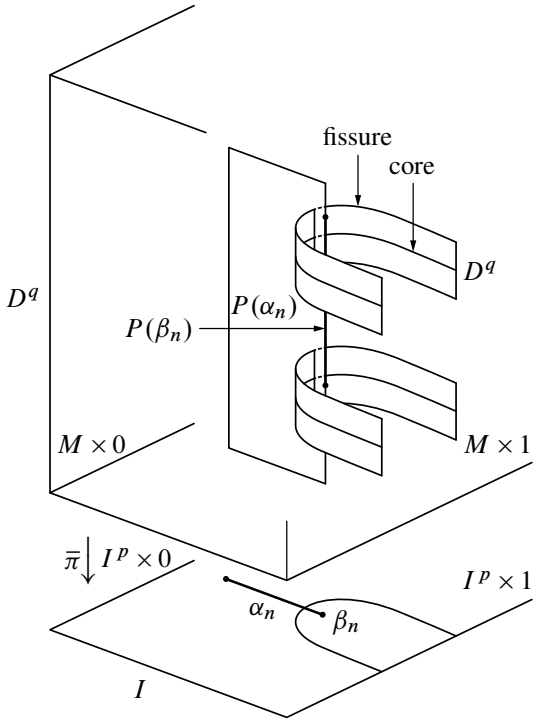

Figure 6: Schematic view on a fissure component which appears while foliating $P(\alpha_n)$ (Claim 4.15), on its propagation during the rest of the induction, and on its projection into the base $\mathbb{I}^{p+1}$. Beware that this schematic low-dimensional drawing can be misleading in several ways: actually, there appears a fissure while foliating $P(\alpha_n)$ only for $i := \dim(\beta_n) \geq 1$; in the topology of the core, here the factor $\mathbb{S}^{i-1}$ has been assimilated to two points; in the base $\mathbb{I}^{p+1}$, the projection of the core is actually of codimension 2, not 1.

Proposition 4.21 *The core $\Sigma := c^{-1}(0,0)$ of the cleft Γ_q–structure Γ built in the above proof of Theorem A′ is diffeomorphic with a disjoint union of products $\mathbb{S}^{i-1} \times \mathbb{D}^{p-i}$, where $p := \dim(M) - q$ and where $1 \leq i \leq p$.*

In particular, the core $\Sigma \cap (M \times 1) = \partial\Sigma$ of the cleft foliation $\Gamma|_{M\times 1}$ is diffeomorphic with a disjoint union of products $\mathbb{S}^{i-1} \times \mathbb{S}^{p-i-1}$, where $1 \leq i \leq p-1$.

Proof By Property 4.10(4) for $n = N$, the smooth compact manifold $\Sigma = \Sigma_N$ admits a smooth (global) triangulation Δ_N collapsing on a system of spheres (Vocabulary 4.9) interior to Σ. After excising an open tubular neighbourhood of these spheres, and after an obvious restoration of the triangulation close to the new boundary components, one is reduced to the well-known fact that every collapsing cobordism is trivial (see Proposition A.1). □

Now, recall the notation from Theorem A.

Corollary 4.22 *The core of the multifold Reeb component of the foliation γ' built in the above proof of Theorem A is diffeomorphic with a disjoint union of products $\mathbb{S}^{i-1} \times \mathbb{S}^{p-i-1}$, where $1 \leq i \leq p-1$.*

5 Proof of the Mather–Thurston theorem as a corollary of Theorem A′

Let $q \geq 1$. Recalling Notation 3.1, fix an element $\varphi \in \mathrm{Diff}(\mathbb{D}^q)^{\mathbb{I}}$ which is a product of commutators,

$$\varphi = [\alpha_1, \beta_1] \dots [\alpha_g, \beta_g]$$

with $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g \in \mathrm{Diff}(\mathbb{D}^q)^{\mathbb{I}}$, and such that the image $\varphi_1 \in \mathrm{Diff}(\mathbb{D}^q)$ is not the identity.

Remark 5.1 In fact, every $\varphi \in \mathrm{Diff}(\mathbb{D}^q)^{\mathbb{I}}$ is of this form, but we don't need this difficult perfectness result, for which one can find references and a discussion in [30]. Moreover, in view of Remark 1.4, recall that it remains unknown whether perfectness holds in the differentiability class C^{q+1}.

The interest for us of such a commutators decomposition is that it allows an (obvious) suspension construction over a surface. Namely, let S_g be the compact orientable

surface bounded by $\mathbb{S}^1$ and of genus g, with a basepoint on the boundary. Consider the representation

$$\rho\colon \pi_1(S_g) \to \mathrm{Diff}(\mathbb{D}^q)^{\mathbb{I}}$$

mapping the canonical free basis to $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$; hence $\rho(\partial S_g) = \varphi$: the suspension $\mathscr{S}_\varphi$ of φ over the circle ∂S_g bounds the codimension-q foliation $\mathscr{S}_\rho$ on $S_g \times \mathbb{D}^q$ that is the suspension of ρ.

Proof of Theorem 1.8 Let X, V, $\mathscr{F}_V$ and γ be as in the hypotheses of Theorem 1.8; to fix ideas, assume moreover that X is closed. Recall that $\overline{V} := V \times \mathbb{I}$, that $\widehat{V} \subset \overline{V}$ is the union of $V \times 0$ with $\partial V \times \mathbb{I}$, that $\widehat{\mathrm{pr}}$ is the restriction to $\widehat{V} \times X$ of the projection $V \times \mathbb{I} \times X \to V \times X$, and let $\pi_V\colon V \times X \to V$ denote the first projection.

The hypothesis that γ restricts to a foliated product over ∂V amounts to say that at every point $a \in \partial V \times X$, the differential $d_a\gamma$ is a linear retraction of $\tau_a(V \times X)$ onto $\nu_a\gamma = \tau_a\mathscr{F}_V$. Hence, Theorem A′ provides a *cleft* Γ_q–structure on $\overline{V} \times X$ which coincides with $\widehat{\mathrm{pr}}^*(\gamma)$ over $\mathscr{O}p_{\overline{V}}(\widehat{V})$ and which is, over $V \times 1 \cong V$, complementary to $\mathscr{F}_V$ *on the complement of the fissure.* Our proof of the Mather–Thurston theorem will consist in enlarging the fissure into a hole and filling the hole by a foliation complementary to the fibres, at the price of changing the base $\overline{V}$ into a more general cobordism. To this end, the main tool will be the pullback construction that has been described in Section 3.1.3. A difficulty is that in $\overline{V}$, the projections of the connected components of the fissure will in general intersect each other. To solve this difficulty, we need an inductive process, filling at each step only a union of components whose projections are pairwise disjoint, and performing successive surgeries on the base. During the process, the cores of the remaining components will get surgered too.

More precisely, one can paraphrase the conclusions of Theorem A′ by the following properties (1)–(3) and (I)–(IV), where $V^* := V$, where $W := V \times \mathbb{I}$, and where $\Sigma_1, \dots, \Sigma_\ell$ are the connected components of Σ. One has

(1) an oriented cobordism (V, W, V^*) rel ∂V;

(2) on $W \times X$, a cleft Γ_q–structure $\Gamma = (C, [c], \overline{\gamma})$ whose monodromy is φ, and whose normal bundle is $\tau\mathscr{F}_W$;

(3) a partition of $\Sigma := c^{-1}(0,0)$ as a finite disjoint sum of compact components

$$\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_\ell$$

(each Σ_i is thus a union of connected components of Σ)

such that

(I) C is disjoint from $\widehat{V}\times X$, and $\overline{\gamma}$ coincides in restriction to $\widehat{V}\times X$ with $\widehat{\mathrm{pr}}^*(\gamma)$;

(II) C is in standard position with respect to $\mathscr{F}_W$ (Definition 3.7);

(III) $\Gamma|_{V^*\times X}$ is on $V^*\times X$ a cleft foliation quasicomplementary to $\mathscr{F}_{V^*}$ (recall Definitions 3.4 and 3.8);

(IV) the projection π_W is one-to-one in restriction to each component $\Sigma_1,\dots,\Sigma_\ell$.

(Recall that, after the remarks following Definitions 3.2 and 3.3, in the case $\dim(W)=1$, a "cleft Γ_q–structure" has no cleft at all: C and Σ are empty, and the proof of Theorem 1.8 is complete. From now on, we assume that $\dim(W)\geq 2$.)

Let us describe how such data can be modified so that the length ℓ of the partition drops by one.

Fix a smooth map $\alpha\colon S_g\to\mathbb{D}^2$ inducing the identity between the boundaries. Let $H=h(\Sigma\times\mathbb{D}^2\times\mathbb{D}^q)$ be a thin neighbourhood of C in $W\times X$ as in Lemma 3.10. By (III), provided that we choose H thin enough, the equidimensional immersion

$$j:\Sigma_\ell\times\mathbb{D}^2\to W,\quad (\sigma,z)\mapsto\pi_W(h(\sigma,z,0)),$$

is an embedding. Let W' (resp. $(V^*)'$) be the $(p+1)$–fold (resp. p–fold) obtained from W (resp. V^*) by cutting $j(\Sigma_\ell\times\mathbb{D}^2)$ (resp. $j(\partial\Sigma_\ell\times\mathbb{D}^2)$) and pasting $\Sigma_\ell\times S_g$ (resp. $\partial\Sigma_\ell\times S_g$). The triple $(V,W',(V^*)')$ is a new oriented cobordism rel ∂V. Let $a\colon W'\to W$ be the smooth map defined on $\Sigma_\ell\times S_g\subset W'$ as

$$(\sigma,x)\mapsto j(\sigma,\alpha(x))$$

and as the identity on the complement. Consider the product map

$$A:=a\times\mathrm{id}_X\colon W'\times X\to W\times X.$$

The map A being leafwise étale (Section 3.1.3(a)) with respect to $\mathscr{F}_{W'}$ and $\mathscr{F}_W$, according to Lemma 3.12, after a generic small horizontal perturbation of the fissure C, one can arrange that C is pullable through A. The above properties (I)–(III) still hold (Section 3.1.2); we choose the perturbation small enough that (IV) also still holds.

On the one hand, on $A^{-1}(H)\cong\Sigma_\ell\times S_g\times\mathbb{D}^q$, define $\overline{\gamma}'$ as $(\mathrm{pr}_2\times\mathrm{pr}_3)^*(\mathscr{S}_\rho)$: a genuine foliation complementary to $\mathscr{F}_{W'}$ there. On the other hand, consider the complement $R:=(W\times X)\setminus\mathrm{Int}(H)$ and, on $A^{-1}(R)$, the cleft Γ_q–structure $A^{-1}(\Gamma|_R)$.

The two constructions obviously coincide on $\mathcal{O}p_{W'\times X}(A^{-1}(\partial H))$, thus defining on $W'\times X$ a global cleft Γ_q–structure Γ'. The above properties (I)–(III) hold for W', $(V^*)'$, Γ' instead of W, (V^*), Γ (after Section 3.1.3(c), (e)). As for (IV), the core of the fissure of Γ' is the disjoint union of $\Sigma'_i := A^{-1}(\Sigma_i)$ for $1\leq i\leq \ell-1$. Since for each i the projection π_W is one-to-one on the component Σ_i, clearly $\pi_{W'}$ is one-to-one on Σ'_i.

After the last step of the inductive process, $\ell=0$; in other words, there is no fissure any more, and Theorem 1.8 is proved. □

Appendix The "collapsing cobordism theorem"

Proposition A.1 *Let (V_0, W, V_1) be a cobordism between two closed manifolds V_0 and V_1. Assume that W admits a smooth triangulation K collapsing onto V_0. Then W is diffeomorphic with $V_0\times\mathbb{I}$.*

This fact, which Thurston used [37, last lines of paragraph 7] for the same purpose as we do in the present paper, is of course covered by the classical theorems of PL topology (I thank Larry Siebenmann for pointing this out to me). Indeed, by the normal neighbourhood theorem [17], K is PL–isomorphic with the product $(K|_{V_0})\times\mathbb{I}$. Then, by the product smoothing theorem (eg [16]), W is diffeomorphic with $V_0\times\mathbb{I}$.

If $\dim(W)\geq 6$, alternatively, a collapsing cobordism being a s–cobordism, one can apply the s–cobordism theorem. This second argument, which does not use the smoothness of the triangulation, is restricted to the large dimensions.

Both these arguments involving some rather elaborate results, we give a direct, comparatively short and pedestrian proof. (We also believe that it is worth giving because the product smoothing theorem is a particular case of Proposition A.1: the case where K is a Whitney product triangulation).

Our proof consists in building a smooth Morse function on W whose critical points are exactly the barycenters of the cells of K, and such that for each (or most) of the elementary collapses composing the given global collapse, the corresponding pair of critical points is cancellable. After the cancellations, one obtains a noncritical smooth function, and thus a diffeomorphism of W with $V_0\times\mathbb{I}$. The only subtleties in the construction lie of course in some smoothness topics.

We begin with some elementary tools for handling smooth triangulations. A simplex of dimension at least 2 being a smooth manifold with *cornered* boundary, the (short) proofs are given.

For $0 \leq p \leq n$, endow $\mathbb{R}^n$ with the Euclidean coordinates $x_1, \ldots, x_n$; define the sector $\mathbb{R}^n_p \subset \mathbb{R}^n$ by $x_i \geq 0$ for $n-p+1 \leq i \leq n$; and consider for each $n-p+1 \leq i \leq n$ the hyperface $\eta_i \subset \mathbb{R}^n_p$ defined by $x_i = 0$.

Lemma A.2 *Every real function f on $\partial\mathbb{R}^n_p$ whose restriction to each hyperface η_i is smooth extends to a smooth real function on $\mathbb{R}^n_p$. Moreover, this extension process can be made continuous with respect to the smooth topologies on spaces of smooth functions.*

Proof There is indeed a simple formula, playing with the projections to the faces. Let I be the finite set $n-p+1, \ldots, n$; for every subset $J \subset I$, let $\eta_J \subset \mathbb{R}^n_p$ be the face defined by $x_i = 0$ for every $i \in J$, and let

$$\mathrm{pr}_J : \mathbb{R}^n_p \to \eta_J$$

be the orthogonal projection. It is enough to verify that, on $\partial\mathbb{R}^n_p$,

$$\sum_{J \subset I} (-1)^{|J|} f \circ \mathrm{pr}_J = 0 \tag{4}$$

(indeed, this equation amounts to express the term f, corresponding to $J = \varnothing$, as minus the sum of the other terms, which obviously extend smoothly to $\mathbb{R}^n_p$).

Fixing $i \in I$, one verifies (4) on η_i by splitting the set of subsets $\mathscr{P}(I)$ into pairs J and J', where $i \notin J$ and $J' = J \cup \{i\}$. On η_i, one has $\mathrm{pr}_J = \mathrm{pr}_{J'}$, hence the terms of (4) cancel by pairs. □

Lemma A.3 *Given a simplicial complex K and a subcomplex $L \subset K$, every real simplexwise smooth function f on L extends to a real simplexwise smooth function F on K. Moreover, the support of F can be contained in any neighbourhood of the support of f; and the extension process can be made continuous with respect to the smooth topologies on spaces of smooth functions.*

Proof By induction on the number of cells of K not in L, one is reduced to the case where K is a single cell α, and $L = \partial\alpha$. Every point $x \in \partial\alpha$ admits in α an open neighbourhood U_x diffeomorphic to a sector. By Lemma A.2, a smooth extension F_x of $f|_{U_x \cap \partial\alpha}$ exists on U_x. Pick any mathematical object $* \notin \partial\alpha$; put $X := \partial\alpha \cup \{*\}$

and $U_* := \mathrm{Int}(\alpha)$; let (u_x) $(x \in X)$ be a smooth partition of the unity on α subordinate to the open cover (U_x) $(x \in X)$; define

$$F := \sum_{x \in \partial\alpha} u_x F_x. \qquad \square$$

Definition A.4 (i) A *smooth partial triangulation* K of a manifold with boundary W consists of a finite linear geometric simplicial complex $|K|$ and a topological embedding $h\colon |K| \to W$ such that

- h embeds smoothly each cell of $|K|$ into W;
- $h^{-1}(\partial W)$ is a subcomplex of $|K|$.

(ii) If moreover $h(|K|) = W$, then K is a *smooth triangulation* of W.

(iii) An *isotopy* of the smooth partial triangulation K is a continuous family (h_t) $(t \in \mathbb{I})$ of topological embeddings $h_t\colon |K| \to W$ such that

- $h_0 = h$;
- for each cell α of $|K|$, the restriction $(h_t|_\alpha)$ is a smooth isotopy of smooth embeddings of this cell;
- ${h_t}^{-1}(\partial W) = h^{-1}(\partial W)$ for every $t \in \mathbb{I}$.

(iv) Given an isotopy as in (iii), consider the simplexwise smooth, time-dependent vector field $v_t := \partial h_t/\partial t$. Fix an auxiliary smooth (resp. simplexwise linear) embedding of W (resp. $|K|$) in a large-dimensional Euclidean space. All Euclidean norms are denoted by $|\cdot|$. We define the *C^1–norm* of the isotopy as

$$\sup_{x \in \alpha} \big(|v_t(x)| + |\partial_\alpha v_t(x)|\big),$$

where $\partial_\alpha v_t(x)$ is the spatial differential of $v_t|_\alpha$ at x.

Lemma A.5 *Given a smooth partial triangulation K of a manifold W and a subcomplex $L \subset K$, every isotopy (h_t) of L which is C^1–small enough, extends to some isotopy (H_t) of K which we can make arbitrarily C^1–small. Moreover, the support of (H_t) can be contained in any neighbourhood of the support of (h_t).*

Proof By induction on the number of cells of K not in L, one is reduced to the case where $K = L \cup \alpha$ for a cell α such that $\partial\alpha = \alpha \cap L$. Since everything takes place in W in a small neighbourhood of the embedded simplex α, we can assume that W is the half-space $\mathbb{R}^n_1$.

The time-dependent vector field $\partial h_t/\partial t$ amounts over $\partial\alpha$ to a simplexwise smooth 1–parameter family of maps $v_t\colon \partial\alpha \to \mathbb{R}^n$. By Lemma A.3, this family extends to a

simplexwise smooth 1–parameter family of maps $V_t : \alpha \to \mathbb{R}^n$. If $h(\alpha) \subset \partial W = \mathbb{R}^{n-1}$, one has $v_t(\partial\alpha) \subset \mathbb{R}^{n-1}$, hence one can choose V_t to take values in $\mathbb{R}^{n-1}$. We can choose the values $V_t(x)$ and the spacial differential $\partial_\alpha V_t$ to be arbitrarily close to 0 on α, provided that those of v_t are close enough to 0 on each proper face of α. We can also choose V_t to be supported in an arbitrary neighbourhood of the support of v_t. Integrating the time-dependent vector field V_t, one obtains an isotopy of immersions $h_t^\alpha : \alpha \to W$ such that each h_t^α is close to $h|_\alpha$ *in the* C^1 *topology.* If close enough, then the global map

$$H_t := h_t \cup h_t^\alpha : K \to W$$

is one-to-one, hence a simplexwise smooth global embedding. □

Recall that a *local coordinate chart* for W at a point y is a smooth diffeomorphism ϕ between an open neighbourhood of y in W and an open neighbourhood of $\phi(y)$ in the half-space $\mathbb{R}^n_1$.

Definition A.6 The smooth partial triangulation K of the n–manifold W is *linear* at the point $y = h(x)$ with respect to the local coordinate chart ϕ if the germ of $\phi \circ h$ at x is linear in every cell of $|K|$ through x. We call K *linearizable* at y if there exists such a local coordinate chart.

Note A.7 Beware that the linearizability property *generically fails* provided that the combinatorial link of x in K is complicated enough — here, "generically" refers to the simplexwise smooth embedding h. Indeed, the linear representations of quivers come into play.

For a simple example, consider a smooth triangulation K of a manifold of dimension $n \geq 3$, and a codimension-2 cell α of K lying in the boundary of at least four codimension-1 cells, η_i of K for $1 \leq i \leq 4$. At every point $y \in \alpha$, the four lines $\tau_y\eta_i/\tau_y\alpha$ have in the 2–plane $\tau_y W/\tau_y\alpha$ a cross-ratio $c(y) \in \mathbb{R} \setminus \{0, 1\}$; and the function c on α is in general not constant on any neighbourhood of a given point $y_0 \in \alpha$. Then K is not linearizable at y_0.

In the same spirit, note that for any vector field ∇ of class C^1 which is tangential to each cell of K, the derivative $\nabla \cdot c$ must vanish identically on α. For a generic position of $\eta_1, \ldots, \eta_4$, the function c is Morse on α. Then ∇ must be tangential to its level sets, and vanish at the critical points. If α is contained in at least $n+1$ codimension-1 cells, then one gets a number $\dim(\alpha)$ of such cross-ratio functions on α; and for a generic position of these cells, ∇ must vanish identically on α.

Lemma A.8 *Given a smooth partial triangulation $K = (|K|, h)$ of a manifold W, a point $y = h(x)$, and a local coordinate chart $\phi\colon \mathcal{O}p_W(y) \to \mathbb{R}^n_1$, there is an isotopy (h_t) of smooth partial triangulations such that $h_0 = h$ and that h_1 is linear at y with respect to ϕ.*

Moreover, one can choose the isotopy to be supported in an arbitrarily small neighbourhood of x.

Given moreover a subcomplex $L \subset K$ containing x which is already linear at y with respect to ϕ, there is such an isotopy (h_t) which is stationary on L.

Proof By induction on the number of cells of $|K|$ not in $|L|$ and containing x. Consider a minimal such cell α; in particular,

$$\partial\alpha \cap \mathrm{St}(x) = \alpha \cap |L| \cap \mathrm{St}(x),$$

where $\mathrm{St}(x)$ is the open star of x (the interior of the union of the cells of $|K|$ containing x). The 1–jet at x of the smooth embedding $\phi \circ h|_\alpha$ extends to a linear embedding $\ell\colon \alpha \hookrightarrow \mathbb{R}^n$ coinciding with $\phi \circ h$ on $\partial\alpha \cap \mathcal{O}p_\alpha(x)$. The obvious local isotopy of embeddings

$$h_t(z) := \phi^{-1}\big((1-t)\phi(h(z)) + t\ell(z)\big) \quad (z \in \mathcal{O}p_\alpha(x),\, t \in \mathbb{I})$$

is arbitrarily C^1–small (Definition A.4(iv)) on a small enough neighbourhood of x, and stationary on $\mathcal{O}p_{\partial\alpha}(x)$. By the ordinary isotopy extension property for embeddings, the germ of (h_t) at x extends to a global isotopy of embeddings $h^\alpha_t\colon \alpha \hookrightarrow W$, stationary on $\partial\alpha$. Clearly, h^α_t can be chosen to be supported in an arbitrarily small neighbourhood of x, arbitrarily C^1–small, and stationary on $|L|$. One thus gets a C^1–small isotopy of $L \cup \alpha$ stationary on L and coinciding with (h^α_t) on α, which extends to the rest of K with support in an arbitrarily small neighbourhood of x (Lemma A.5). □

We now begin to prove Proposition A.1. Let W, V_0, V_1 and K be as in this proposition. First, we somehow normalize K close to the upper boundary V_1.

Lemma A.9 *After changing K, one can arrange that, moreover, there is a smooth function $w\colon W \to [0,1]$ such that*

(1) *$V_i = w^{-1}(i)$ for $i = 0, 1$;*

(2) *every critical value of w lies in the open interval $\big(0, \frac{1}{2}\big)$;*

(3) *the smooth hypersurface $w^{-1}\big(\frac{1}{2}\big)$ is a subcomplex of K;*

(4) *the subcomplex $K_{1/2} := K|_{w^{-1}([0,1/2])}$ collapses onto V_0.*

Proof Consider the triangulation $K|_{V_1}$ of the manifold V_1; put an arbitrary total order on the set of its vertices; then the Whitney subdivision L of the cartesian product $(K|_{V_1})\times\mathbb{I}$ (see [39, page 365]) is on $V_1\times\mathbb{I}$ a smooth triangulation, and $L|_{V_1\times 0}=K|_{V_1}$, and L collapses onto $V_1\times 0$. The manifold $\widehat{W}:=W\cup_{V_1}(V_1\times\mathbb{I})$ is diffeomorphic with W, and $\widehat{W}$ admits the smooth triangulation $\widehat{K}:=K\cup_{K|_{V_1}}L$. Composing the collapses $L\searrow V_1\times 0$ and $K\searrow V_0$, one gets a global collapse $\widehat{K}\searrow V_0$. The function $\widehat{w}$ on $\widehat{W}$ is defined as $\widehat{w}(x,t):=\frac{1}{2}(1+t)$ in $V_1\times\mathbb{I}$, and its extension to the rest of $\widehat{W}$ is not less obvious. □

Lemma A.10 *After changing K again, one can arrange that, moreover, K is linearizable at the barycenter of each of its cells.*

Proof Consider the hypersurface $H:=w^{-1}\left(\frac{1}{2}\right)$ and its smooth triangulation $K|_H$. For every cell α of $K|_H$, apply Lemma A.8 at the barycenter of α. One gets a smooth triangulation of H isotopic to $K|_H$ and linearizable at each barycenter. Then, extend this isotopy into an isotopy of K supported in a small neighbourhood of H (Lemma A.5). Finally, applying Lemma A.8 at the barycenter of each cell of K with $L=K|_H$, one obtains a triangulation linearizable at the barycenter of each cell. □

Each triangulation bears a canonical, piecewise Euclidean, Riemannian metric. Precisely, for each $d\geq 0$, the Euclidean metric on $\mathbb{R}^{d+1}$, restricted to the standard d–simplex $\Delta^d\subset\mathbb{R}^{d+1}$ (the convex hull of the canonical basis), is invariant by the symmetric group $\mathfrak{S}_{d+1}$. One thus obtains a natural Riemannian metric on each linear simplex, and in particular on each cell of $|K|$. Its image under h is a Riemannian metric g_α on each cell α of K. Define on α the quadratic function

$$q_\alpha : x\mapsto \tfrac{1}{2}|x-\flat(\alpha)|^2,$$

where $|-|$ denotes of course the g_α–distance. On every face $\beta\subset\alpha$, one has $g_\alpha|_\beta=g_\beta$ and (after Pythagoras)

$$q_\alpha|_\beta - q_\beta = \text{constant}. \tag{5}$$

Still consider any cell α of K. On a small open neighbourhood U_α of $\flat(\alpha)$ in W, let $\phi=(x_1,\dots,x_n)$ be a local coordinate chart linearizing K (Definition A.6 and Lemma A.10). Put $d:=\dim(\alpha)$ and $a:=\phi(\flat(\alpha))\in\mathbb{R}^n_1$. Composing ϕ with an affine automorphism of $\mathbb{R}^n$ preserving $\mathbb{R}^n_1$, we arrange that moreover

- $\alpha\cap U_\alpha$ is defined in U_α by the equations $x_i=a_i$ for $d+1\leq i\leq n$;
- $g_\alpha|_{\alpha\cap U_\alpha}=\sum_{i=1}^d dx_i^2$.

Endow U_α with the quadratic function

$$f_\alpha := d - \frac{1}{2}\sum_{i=1}^{d}(x_i - a_i)^2 + \frac{1}{2}\sum_{i=d+1}^{n}(x_i - a_i)^2$$

(hence, $f_\alpha = q_\alpha + d$ on $\alpha \cap U_\alpha$) and with the linear vector field

$$\nabla_\alpha := \sum_{i=1}^{d}(x_i - a_i)\frac{\partial}{\partial x_i} - \sum_{i=d+1}^{n}(x_i - a_i)\frac{\partial}{\partial x_i}.$$

Lemma A.11 (1) *∇_α is tangential in U_α to every cell of K containing α.*

(2) *For every face $\beta \subset \alpha$, one has $\nabla_\beta \cdot q_\alpha > 0$ on $\mathcal{O}p_\alpha(\flat(\beta)) \setminus \flat(\beta)$.*

The verifications are immediate.

Lemma A.12 *There are a global*, continuous *vector field ∇ on W, and real values $0 < \tau_0 < \frac{1}{2} < \tau_1 < 1$, such that for every cell $\alpha \in K$:*

(1) *∇ coincides with ∇_α on some neighbourhood of $\flat(\alpha)$ in W.*

(2) *∇ is tangential to α and smooth on α.*

(3) *$\nabla \cdot q_\alpha > 0$ on $\mathrm{Int}(\alpha) \setminus \flat(\alpha)$.*

(4) *$w(\flat(\alpha)) \notin (0, \tau_0] \cup \left(\frac{1}{2}, \tau_1\right]$.*

(5) *Every critical value of w is $> \tau_0$.*

(6) *$\nabla \cdot w < 0$ on $w^{-1}(\tau_i)$ for $i = 0, 1$.*

Proof On every cell α of K, let the vector field r_α be the (ascending) g_α–gradient of q_α.

Let us first fix τ_0 and τ_1. Recall that $w^{-1}(0) = V_0$ and $w^{-1}\left(\frac{1}{2}\right)$ are subcomplexes of K.

Claim *For $i = 0, 1$, let α be a cell of K contained in $w^{-1}\left[\frac{i}{2}, 1\right]$ but not in $w^{-1}\left(\frac{i}{2}\right)$. Then $r_\alpha \cdot w < 0$ on $\alpha \cap w^{-1}\left(\frac{i}{2}\right)$.*

Indeed, the intersection $\alpha \cap w^{-1}\left(\frac{i}{2}\right)$, if not empty, is a proper face of α. At any point x on this face, the vector field r_α exits α, transversely to every hyperface of α containing x. On the other hand, in the tangent vector space $\tau_x W$, the convex cone

tangent to α admits $\ker(d_x w)$ as a supporting hyperplane at point 0. Hence, $r_\alpha \cdot w < 0$ at x. The claim is proved.

For $i = 0, 1$, we choose $\tau_i > \frac{i}{2}$ so close to $\frac{i}{2}$ that (5) is satisfied and that for every cell α of K, (4) is satisfied, and (thanks to the claim) $r_\alpha \cdot w < 0$ on $(w|_\alpha)^{-1}(\tau_i)$.

One builds ∇ cell after cell, by induction on the dimensions of the cells. Let α be a cell of K; assume by induction that a continuous vector field ∇ is already defined on $\partial\alpha$; that (1)–(3) are satisfied for every proper face of α instead of α; and that (6) holds at every point of $\partial\alpha \cap w^{-1}(\tau_i)$ for $i = 0, 1$.

The extension of ∇ to α will be a local construction: we shall first define, for every point $x \in \alpha$, a vector field X_x on an open neighbourhood N_x of x in α. The construction of X_x depends on x being the barycenter of α, or the barycenter of a proper face, or not a face barycenter.

On $N_{\flat(\alpha)} := \mathrm{Int}(\alpha)$, let $X_{\flat(\alpha)} := r_\alpha$. In particular, $X_{\flat(\alpha)} \cdot q_\alpha > 0$ on $N_{\flat(\alpha)} \setminus \flat(\alpha)$.

For each proper face $\beta \subsetneq \alpha$, let $N_{\flat(\beta)}$ be a small open neighbourhood of $\flat(\beta)$ in α (not containing any other face barycenter, and disjoint from $w^{-1}(\tau_0)$ and $w^{-1}(\tau_1)$) such that the linear vector field $X_{\flat(\beta)} := \nabla_\beta|_\alpha$, which is tangential to α there (Lemma A.11(1)), satisfies $X_{\flat(\beta)} \cdot q_\alpha > 0$ on $N_{\flat(\beta)} \setminus \flat(\beta)$ (Lemma A.11(2)).

Now, consider a boundary point $x \in \partial\alpha$ which is not the barycenter of any face. Call $\beta \subset \partial\alpha$ the smallest face of α through x. By induction, ∇ is already defined on each hyperface η of α through x, and tangential to η. Let N_x be in α a small open neighbourhood of x diffeomorphic to a sector. By Lemma A.2, there is a smooth vector field X_x on N_x which coincides with ∇ on every hyperface of α through x. By the induction hypothesis (3) applied to β and by (5), one has $\nabla \cdot q_\alpha > 0$ at x. After shrinking N_x, it does not contain any cell barycenter; and $X_x \cdot q_\alpha > 0$ on N_x. Moreover, if $w(x) = 0$ or $\frac{1}{2}$, by induction, (6) holds at x; hence, after shrinking N_x, one has $X_x \cdot w < 0$ on N_x.

Finally, one defines ∇ on α as

$$\nabla := \sum_{x \in \partial\alpha \cup \flat(\alpha)} u_x X_x,$$

where (u_x) is a partition of the unity on α subordinate to the open cover (N_x). The properties (1)–(3) hold on α, and (6) holds on $\alpha \cap w^{-1}(\tau_i)$ for $i = 0, 1$. □

For every cell α of K, the stable manifold $W^s(\flat(\alpha))$ and the unstable manifold $W^u(\flat(\alpha))$ of the barycenter $\flat(\alpha)$ with respect to ∇ are well defined, since ∇, being simplexwise smooth, is Lipschitz.

Corollary A.13 (1) *$W^s(\flat(\alpha))$ is contained in the open star of α (the interior of the union of the cells of K containing α).*

(2) $W^u(\flat(\alpha)) = \mathrm{Int}(\alpha)$.

This follows immediately from Lemma A.12(1)–(3).

Lemma A.14 *There is on W a smooth Morse function f such that*

(1) *f coincides with f_α on a neighbourhood of $\flat(\alpha)$ in W for every $\alpha \in K$;*

(2) *$\nabla \cdot f < 0$ except at the barycenters of the cells of K.*

Proof Let α be a cell of K. By induction on $d := \dim(\alpha)$, assume that a smooth function f_U is already defined on some small open neighbourhood U of $\partial\alpha$ in W; that for every proper face $\beta \subsetneq \alpha$, one has $f_U = f_\beta$ on a neighbourhood of $\flat(\beta)$ in U; and that $\nabla \cdot f_U < 0$ on U, except at the barycenters of the proper faces of α.

First, let us extend f_U to α itself. To this aim, note that, by Corollary A.13(2), every local maximum of the continuous function $f_U|_{\partial\alpha}$ is the barycenter $\flat(\eta)$ of a hyperface $\eta \subset \alpha$, while

$$f_U(\flat(\eta)) = f_\eta(\flat(\eta)) = d - 1.$$

Hence, shrinking U if necessary, one has $f_U < d - \frac{1}{2}$ on U.

Again by Corollary A.13(2), there is on $\mathrm{Int}(\alpha)$ a smooth function g such that

- $g = f_\alpha$ on a neighbourhood of $\flat(\alpha)$;
- $\nabla \cdot g < 0$ on $\mathrm{Int}(\alpha) \setminus \flat(\alpha)$;
- $g > d - \frac{1}{2}$.

Again by Corollary A.13(2), one has on α a smooth plateau function u such that $u = 1$ on a neighbourhood in α of $\alpha \setminus (U \cap \alpha)$, and $\mathrm{spt}(u) \subset \mathrm{Int}(\alpha)$, and $\nabla \cdot u \le 0$. On α, define

$$F := (1-u) f_U + u g.$$

Then, on $\mathrm{Int}(\alpha) \setminus \flat(\alpha)$,

$$\nabla \cdot F = (1-u)\nabla \cdot f_U + u \nabla \cdot g + (g - f_U)\nabla \cdot u < 0.$$

Hence, any smooth extension of F to W coinciding with f_α close to $\flat(\alpha)$ and with f_U close to $\partial\alpha$ will also satisfy (2) on a neighbourhood of α. □

Note that f is self-indexing.

Informally, at this step, the vector field ∇ is in some sense gradient-like for f; and every pair of critical points $\flat(\alpha_j)$ and $\flat(\beta_j)$ for $1 \leq j \leq J$ is in some sense in cancellation position with respect to ∇. The problem is that ∇ is in general not smooth, and in many cases *cannot* be smooth (Note A.7). In particular, for every cell α of K, the stable manifold $W^s(\flat(\alpha))$ is in general not smooth. However, the *germ* of $W^s(\flat(\alpha))$ at $\flat(\alpha)$ is smooth (Lemma A.12(1)).

Recall (Lemma A.9(4)) that the subcomplex $K_{1/2} := K|_{w^{-1}([0,1/2])}$ collapses to V_0; such a collapse means a filtration of the simplicial complex $K_{1/2}$ by subcomplexes (K_j) for $0 \leq j \leq J$ such that $K_0 = K|_{V_0}$, and $K_J = K_{1/2}$, and that, for each $1 \leq j \leq J$,

- $K_{j-1} \subset K_j$;
- K_j has exactly two cells α_j and β_j not in K_{j-1};
- $\beta_j \subset \alpha_j$ is a hyperface.

Lemma A.15 *There are real values (v_j) for $1 \leq j \leq J$ such that*

(1) $\dim(\beta_j) < v_j < \dim(\alpha_j)$

(2) *the compact disk*

$$D_j^s := W^s(\flat(\beta_j)) \cap f^{-1}(-\infty, v_j]$$

is contained in U_{β_j}, and thus smooth;

(3) $w < \tau_1$ *on* D_j^s;

(4) *if $1 \leq j < j' \leq k$ and if $\beta_j \subset \alpha_{j'}$, then $v_j < v_{j'}$.*

Proof One defines v_j by descending induction on j. At each step, any value close enough to $\dim(\beta_j)$ from above works. □

Also consider, for $1 \leq j \leq J$, the smooth compact disk

$$D_j^u := \alpha_j \cap f^{-1}[v_j, +\infty).$$

In view of Corollary A.13(2), of Lemma A.12(1)–(2), and of Lemma A.15(1), in the regular level set $f^{-1}(v_j)$, the two attachment spheres ∂D_j^s and ∂D_j^u obviously meet transversely in a single point.

Now, apply the elementary Weierstrass approximation theorem. The continuous vector field ∇ on W being already smooth on a neighbourhood of the barycenter of every cell, and on every disk D_j^u for $1 \le j \le J$, and on a neighbourhood of every disk D_j^s for $1 \le j \le J$, we can approximate ∇ on W, arbitrarily closely in the C^0 topology, by a smooth vector field $\widetilde{\nabla}$ which coincides with ∇ on a neighbourhood of the barycenter of every cell, and on every disk D_j^u for $1 \le j \le J$, and on a neighbourhood of every disk D_j^s for $1 \le j \le J$. We can choose $\widetilde{\nabla}$ to be, like ∇, tangential to V_0 and V_1; hence the flow $\widetilde{\nabla}^t$ is well defined on W for $t \in \mathbb{R}$. We choose $\widetilde{\nabla}$ so close to ∇, in the C^0 topology, that $\widetilde{\nabla} \cdot f < 0$ except at the barycenters of the cells, and that $\widetilde{\nabla} \cdot w < 0$ on $w^{-1}(\tau_i)$ for $i = 0, 1$.

Thus, $\widetilde{\nabla}$ is a genuine smooth, gradient-like vector field for f. Consider, for every $1 \le j \le J$, the unstable manifold $\widetilde{W}^u(\flat(\alpha_j))$ of the critical point $\flat(\alpha_j)$ for $\widetilde{\nabla}$, and the stable manifold $\widetilde{W}^s(\flat(\beta_j))$ of the critical point $\flat(\beta_j)$ for $\widetilde{\nabla}$. By the above definition of $\widetilde{\nabla}$, one has

$$D_j^u \subset \widetilde{W}^u(\flat(\alpha_j)), \tag{6}$$

$$D_j^s \subset \widetilde{W}^s(\flat(\beta_j)). \tag{7}$$

We can arrange moreover that

$$w|_{D_j^u} > \tau_0 \quad (1 \le j \le J). \tag{8}$$

Indeed, after the inclusion (6) and Lemma A.12(4), for a large enough time $T > 0$, one has $w \circ \widetilde{\nabla}^{-T} > \tau_0$ on each D_j^u. We change w on W to $w \circ \widetilde{\nabla}^{\theta \circ w}$, where θ is any smooth function on $\mathbb{I}$ such that $\theta(\tau_0) = -T$ and $\theta(\tau_1) = 0$.

Then consider $W' := w^{-1}[\tau_0, \tau_1]$, a cobordism between $V_0' := w^{-1}(\tau_0)$ and $V_1' := w^{-1}(\tau_1)$, diffeomorphic with W since the critical values of w lie between τ_0 and τ_1. We shall prove that W' is diffeomorphic with $V_0' \times \mathbb{I}$.

The critical points of f in W' are exactly the pairs $\flat(\alpha_j)$ and $\flat(\beta_j)$ for $1 \le j \le J$. For each j, one has $D_j^u \subset W'$ (by the inequality (8), and since $\widetilde{\nabla} \cdot w < 0$ on V_1') and $D_j^s \subset W'$ (by Lemma A.15(3), and since $\widetilde{\nabla} \cdot w < 0$ on V_0').

The major consequence of the inclusions (6)–(7) is of course that $\widetilde{W}^u(\flat(\alpha_j))$ and $\widetilde{W}^s(\flat(\beta_j))$ meet transversely on a single gradient line. It follows that the pair of critical points $\flat(\alpha_j)$ and $\flat(\beta_j)$ can be cancelled in Morse's way by changing f *in an arbitrarily small neighbourhood of the bouquet* $D_j^u \cup D_j^s$. For this localization, see [27, Lemma 2.3] — or alternatively choose $\epsilon > 0$ so small that no critical value of f

is ϵ–close to v_j; then apply, in an arbitrarily small neighbourhood of the bouquet, the standard modification of f that brings both critical values to $v_j \pm \frac{1}{2}\epsilon$ without changing the pseudogradient; and finally verify in the proof of Theorem 5.4 in [28] that the modification of f in the cobordism $f^{-1}[v_j - \epsilon, v_j + \epsilon]$ takes place in an arbitrary small neighbourhood of the union of the stable and unstable manifolds of the two critical points.

After Corollary A.13(1)–(2) and Lemma A.15(4), the bouquets $D^u_j \cup D^s_j$ for $1 \leq j \leq J$ are pairwise disjoint. Hence, one can perform simultaneously the J cancellations, yielding on W' a function f' without critical points which coincides with f on neighbourhoods of V'_0 and V'_1.

The end of the proof of Proposition A.1 is much classical: By means of a partition of the unity, one makes on W' a smooth vector field ∇' such that $\nabla' = \widetilde{\nabla}$ on neighbourhoods of V'_0 and V'_1, and that $\nabla' \cdot f' < 0$ on W'. Consequently, every orbit of ∇' must descend from V'_1 to V'_0. After a rescaling of ∇', the time $t = -1$ of the flow $(\nabla')^t$ carries V'_0 onto V'_1. The mapping $(x, t) \mapsto (\nabla')^{-t}(x)$ thus yields a diffeomorphism of $V'_0 \times \mathbb{I}$ with W'.

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Laboratoire de Mathématiques de Bretagne Atlantique, Université de Bretagne Sud (LMBA UBS, UMR CNRS 6205)
Vannes, France

gael.meigniez@univ-ubs.fr

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