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| Charaf Ech-Chatbi. A Proof of the Generalized Riemann Hypothesis. 2023. hal-02150289v9

**HAL Id: hal-02150289**

**<https://hal.science/hal-02150289v9>**

Preprint submitted on 20 Mar 2023

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# A Simple Proof of the Generalized Riemann's Hypothesis

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1 September 2022

## Abstract

This article is the continuation of the previous RH proof[1]. We present a simple proof of the Riemann's Hypothesis for Dirichlet  $L$ -functions where we apply the same method we used to prove RH.

**Keywords:** Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

**MSC2020 Classification:** 11Mxx, 11-XX, 26-XX, 30-xx.

## 1 Dirichlet $L$ -functions

Let's  $(z_n)_{n \geq 1}$  be a sequence of complex numbers. A Dirichlet series[3] is a series of the form  $\sum_{n=1}^{\infty} \frac{z_n}{n^s}$ , where  $s = \sigma + i\tau$  is complex. The zeta function is a Dirichlet series. Let's define the function  $L(s)$  of the complex  $s$ :

$$L(s) = \sum_{n=1}^{\infty} \frac{z_n}{n^s}.$$

- If  $(z_n)_{n \geq 1}$  is a bounded, then the corresponding Dirichlet series converges absolutely on the open half-plane where  $\Re(s) > 1$ .
- If the set of sums  $z_n + z_{n+1} + \dots + z_{n+k}$  for each  $n$  and  $k \geq 0$  is bounded, then the corresponding Dirichlet series converges on the open half-plane where  $\Re(s) > 0$ .
- In general, if  $z_n = O(n^k)$ , the corresponding Dirichlet series converges absolutely in the half plane where  $\Re(s) > k + 1$ .

The function  $L(s)$  is analytic on the corresponding open half plane[2-3].

To define Dirichlet  $L$ -functions we need to define Dirichlet characters. A function  $\chi : \mathbb{Z} \mapsto \mathbb{C}$  is a Dirichlet character modulo  $q$  if it satisfies the following criteria:

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- (i)  $\chi(n) \neq 0$  if  $(n, q) = 1$ .
- (ii)  $\chi(n) = 0$  if  $(n, q) > 1$ .
- (iii)  $\chi$  is periodic with period  $q$  :  $\chi(n + q) = \chi(n)$  for all  $n$ .
- (iv)  $\chi$  is multiplicative :  $\chi(mn) = \chi(m)\chi(n)$  for all integers  $m$  and  $n$ .

The trivial character is the one with  $\chi_0(n) = 1$  whenever  $(n, q) = 1$ .

Here are some known results for a Dirichlet character modulo  $q$ . For any integer  $n$  we have  $\chi(1) = 1$ . Also if  $(n, q) = 1$ , we have  $(\chi(n))^{\phi(q)} = 1$  with  $\phi$  is Euler's totient function.  $\chi(n)$  is a  $\phi(q)$ -th root of unity. Therefore,  $|\chi(n)| = 1$  if  $(n, q) = 1$ , and  $|\chi(n)| = 0$  if  $(n, q) > 1$ . Also, we recall the cancellation property for Dirichlet characters modulo  $q$ : For any  $n$  integer

$$\sum_{i=1}^q \chi(i + n) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (1)$$

The Dirichlet  $L$ -functions are simply the sum of the Dirichlet series. Let's  $\chi$  be a Dirichlet character modulo  $q$ , The Dirichlet  $L$ -function  $L(s, \chi)$  is defined for  $\Re(s) > 1$  as the following:

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \quad (2)$$

They are a natural generalization of the Riemann zeta-function  $\zeta(s)$  to an arithmetic progression and are a powerful tool in analytic number theory. The Dirichlet series, converges absolutely and uniformly in any bounded domain in the complex  $s$ -plane for which  $\Re(s) \geq 1 + \gamma$ ,  $\gamma > 0$ . In the particular case of the trivial character  $\chi_0$ ,  $L(s, \chi_0)$  extends to a meromorphic function[2-5] in  $\Re(s) > 0$  with the only pole at  $s = 1$ .

If  $\chi$  is a non-trivial character, we have

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (3)$$

Since the sum in the integrand is bounded, this formula gives an analytic continuation of  $L(s, \chi)$  to a regular function in the half-plane  $\Re(s) > 0$ . Also, like the zeta function, the Dirichlet  $L$ -functions have their Euler product[1-2]. For  $\Re(s) > 1$ :

$$L(s, \chi) = \prod_{p \text{ Prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (4)$$

Therefore, if  $\chi = \chi_0$  is a trivial character mod  $q$ , we have, for  $q = 1$ ,

$$L(s, \chi_0) = \zeta(s) \quad (5)$$

And for  $q > 1$ , we have,

$$L(s, \chi_0) = \zeta(s) \prod_{p/q} \left(1 - \frac{1}{p^s}\right) \quad (6)$$

For this reason the properties of  $L(s, \chi_0)$  in the entire complex plane are determined by the properties of  $\zeta(s)$ .

Let's now  $q'$  be the smallest divisor (prime) of  $q$ . Let's  $\chi'$  be the Dirichlet character  $\chi' \pmod{q'}$ . For any integer  $n$  such that  $(n, q) = 1$ , we have also  $(n, q') = 1$  and  $\chi(n) = \chi'(n)$ .  $\chi'$  is called primitive and  $L(s, \chi)$  and  $L(s, \chi')$  are related analytically such that if  $\chi \neq \chi_0$ :

$$L(s, \chi) = L(s, \chi') \prod_{p|q} \left(1 - \frac{\chi'(p)}{p^s}\right) \quad (7)$$

$L(s, \chi)$  and  $L(s, \chi')$  have the same zeros in the critical strip  $0 < \Re(s) < 1$ .

Also, for a primitive character  $\chi$ , (*i.e.*  $\chi = \chi'$ )  $L(s, \chi)$  has the following functional equation:

$$\tau(\chi) \Gamma\left(\frac{1-s+a}{2}\right) L(1-s, \chi) = \sqrt{\pi} \left(\frac{q}{\pi}\right)^s i^a q^{\frac{1}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \overline{\chi}) \quad (8)$$

Where  $\Gamma$  is the Gamma function and  $a = 0$  if  $\chi(-1) = 1$  and  $a = 1$  if  $\chi(-1) = -1$ , and  $\tau(\chi) = \sum_{k=1}^q \chi(k) \exp\left(\frac{2\pi ki}{q}\right)$ .

When  $\Re(s) > 1$  there is no zero for  $L(s, \chi)$ . When  $\Re(s) < 0$ , for a primitive character  $\chi$ , we have the trivial zeros of  $L(s, \chi)$ :  $s = a - 2k$ , where  $k$  is a positive integer and  $a$  is defined above. For more details, please refer to the references[1-8].

## 2 The Generalized Riemann Hypothesis

The Generalized Riemann Hypothesis states that the Dirichlet  $L$ -functions have all their non-trivial zeros on the critical line  $\Re(s) = \frac{1}{2}$ .

We know that for any character  $\chi$  modulo  $q$ , all non-trivial zeros of  $L(s, \chi)$  lies in the critical strip  $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$ . From the functional equation above we have that if:

- $s_0$  is a non-trivial zero of  $L(s, \chi)$ , then  $1 - s_0$  is a zero of  $L(s, \overline{\chi})$ .
- $s_0$  is a non-trivial zero of  $L(s, \overline{\chi})$ , then  $1 - s_0$  is a zero of  $L(s, \chi)$ .

Therefore, we just need to prove that for any character<sup>1</sup>  $\chi$  modulo  $q$ , there is no non-trivial zeros of  $L(s, \chi)$  in the right hand side of the critical strip  $\{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$ .

## 3 Proof of the Generalized Riemann Hypothesis

Let's take a complex number  $s$  such that  $s = \sigma + i\tau$ . Unless we explicitly mention otherwise, let's suppose that  $0 < \sigma < 1$ ,  $\tau > 0$  and  $L(s, \chi) = 0$  where  $L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$  and  $\chi$  is a non-trivial Dirichlet character  $\chi$  modulo  $q$ . When  $q = 2$  there is only one Dirichlet character and it is

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<sup>1</sup>In fact, we just need to prove it for the primitive characters.

trivial. So here we are going to assume that  $q \geq 3$ .

We have from the integral equation (3) above:

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (9)$$

We have from the Riemann's integral in [1]:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (10)$$

In this article, we leverage on the RH proof in [1] to proof GRH. Also, we will use the same method and notations as in [1]. Let's recall from [1] the following notations:

$$\epsilon_0(x) = \{x\} \quad (11)$$

$$\epsilon_{n+1}(x) = \frac{1}{x} \int_0^x dt \epsilon_n(t) \quad (12)$$

$$\epsilon(x) = \sum_{k=0}^{+\infty} \epsilon_k(x) \quad (13)$$

**Lemma 3.1.** *Let's consider two variables  $\sigma$  and  $\tau$  such that  $\sigma > 0$ ,  $\tau > 0$  and  $s = \sigma + i\tau$ . Let's  $L(s, \chi)$  be a  $L$ -function where  $\chi$  is a character (not necessarily a Dirichlet character) such that:*

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda \{x\} + f_\lambda(x))}{x^{s+1}} dx \quad (14)$$

*With the function  $x \rightarrow f_\lambda(x)$  is bounded over  $\mathbb{R}^+$  and null over the interval  $[0, 1]$  and the complex constant  $\lambda \neq 0$ . Let's define the sequence of functions  $f_n$  and  $\bar{f}_n$  such that  $f_0(x) = \bar{f}_0(x) = \frac{f_\lambda(x)}{\lambda}$  and for each  $n \geq 1$ :*

$$f_{n+1}(x) = \frac{1}{x} \int_0^x dt f_n(t) \quad (15)$$

$$\bar{f}_{n+1}(x) = \frac{1}{x} \int_1^x dt \bar{f}_n(t) \quad (16)$$

$$f(x) = \sum_{k=0}^{+\infty} f_k(x) \quad (17)$$

*If  $L(s, \chi) = 0$ , then:*

1. *For each  $n$ :*

$$\int_1^{+\infty} dx \frac{\epsilon_n(x) + f_n(x)}{x^{1+s}} = -\frac{1}{(1-s) 2^n} \quad (18)$$

2. *For each  $n$ :*

$$\int_1^{+\infty} dx \frac{\epsilon_n(x) + f_n(x)}{x^{2-s}} = -\frac{1}{(s) 2^n} \quad (19)$$

3. For each  $x \geq 1$ :

$$f_n(x) = \overline{f}_n(x) \quad (20)$$

4. For each  $n$  and  $x \geq 1$ :

$$f_n(x) = \frac{1}{(n-1)!} \frac{1}{x} \int_1^x dt f_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1} \quad (21)$$

5. For each  $x \geq 1$ :

$$\lim_{n \rightarrow +\infty} 2^n f_n(x) = 0 \quad (22)$$

6. For each  $x \geq 0$ :

$$f(x) = f_0(x) + \int_1^x dt \frac{f_0(t)}{t} \quad (23)$$

7. There exists a constant  $M > 0$  such that for each  $x \geq 0$ :

$$|\epsilon(x) + f(x)| \leq 2 + M + (M+1) \ln(x) \quad (24)$$

*Proof.* Let's take  $s$  such that  $L(s, \chi) = 0$ , therefore we can write:

$$\frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx = 0 \quad (25)$$

Therefore, since  $s \neq 0$  and  $\lambda \neq 0$ , we have:

$$\int_1^{+\infty} \frac{\epsilon_0(x) + f_0(x)}{x^{s+1}} dx = \frac{1}{s-1} \quad (26)$$

We use the same method used in the lemma 1.2 in [1] to prove this lemma. □

**Lemma 3.2.** *Let's consider two variables  $\sigma$  and  $\tau$  such that  $0 < \sigma \leq \frac{1}{2}$ ,  $\tau > 0$  and  $s = \sigma + i\tau$ . Let's  $L(s, \chi)$  be a  $L$ -function where  $\chi$  is a character (not necessarily a Dirichlet character) such that:*

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx \quad (27)$$

*With the function  $x \rightarrow f(x)$  is bounded over  $\mathbb{R}^+$  and null over the interval  $[0, 1]$  and the complex constant  $\lambda \neq 0$ .*

*Let's define the sequence of functions  $E_{n,\sigma}(x)$ ,  $F_{n,\sigma}(x)$ ,  $G_{n,\sigma}(x)$  and  $H_{n,\sigma}(x)$  over  $[1, +\infty)$  for each  $n \geq 1$  as follows:*

$$E_{n,\sigma}(x) = \Im \left[ \int_1^x dt \frac{(f_n(t) + \epsilon_n(t))}{t^{1+s}} \right] \quad (28)$$

$$F_{n,\sigma}(x) = \Re \left[ \int_1^x dt \frac{(f_n(t) + \epsilon_n(t))}{t^{1+s}} \right] \quad (29)$$

$$G_{n,\sigma}(x) = (E_{n,\sigma}(x))^2 + (F_{n,\sigma}(x))^2 \quad (30)$$

$$H_{n,\sigma}(x) = G_{n,1-\sigma}(x) - G_{n,\sigma}(x) \quad (31)$$

Where  $\Im(z)$  and  $\Re(z)$  are the imaginary and the real parts of the complex  $z$ . Let's denote  $a_0 = e^{\frac{3}{2\tau}}$  and  $b_0 = e^{\frac{\pi}{2\tau}}$ . If  $L(s, \chi) = 0$ , then:

1. There is exist a large enough integer  $N_0$  such that for each  $n \geq N_0$ , there exists  $x_n > b_0$  such as:

$$H_{n,\sigma}(x_n) = 0 \quad (32)$$

2. There is exist a large enough integer  $N_0$  such that for each  $n \geq N_0$ , there exists  $x_n > b_0$  such as:

$$\sum_{k=n+1}^{+\infty} H_{k,\sigma}(x_n) = 0 \quad (33)$$

In other terms:

$$\sum_{k=n+1}^{+\infty} G_{k,1-\sigma}(x_n) = \sum_{k=n+1}^{+\infty} G_{k,\sigma}(x_n) \quad (34)$$

*Proof.* Let's prove the first point. We have  $\sigma \leq \frac{1}{2}$  and  $a_0 = e^{\frac{3}{2\tau}}$ ,  $b_0 = e^{\frac{\pi}{2\tau}}$ . From lemma 1.2 in [1], we have:

$$\lim_{n \rightarrow +\infty} 2^n \epsilon_n(x) = x \quad (35)$$

And from lemma 3.1, we have:

$$\lim_{n \rightarrow +\infty} 2^n f_n(x) = 0 \quad (36)$$

Therefore for each  $x \geq 1$ :

$$\lim_{n \rightarrow +\infty} \Im \left[ \frac{2^n (\epsilon_n(x) + f_n(x))}{x^{1+s}} \right] = -\frac{\sin(\tau \ln(x))}{x^\sigma} \quad (37)$$

$$\lim_{n \rightarrow +\infty} \Re \left[ \frac{2^n (\epsilon_n(x) + f_n(x))}{x^{1+s}} \right] = \frac{\cos(\tau \ln(x))}{x^\sigma} \quad (38)$$

$$\lim_{n \rightarrow +\infty} \Im \left[ \frac{2^n (\epsilon_n(x) + f_n(x))}{x^{2-s}} \right] = \frac{\sin(\tau \ln(x))}{x^{1-\sigma}} \quad (39)$$

$$\lim_{n \rightarrow +\infty} \Re \left[ \frac{2^n (\epsilon_n(x) + f_n(x))}{x^{2-s}} \right] = \frac{\cos(\tau \ln(x))}{x^{1-\sigma}} \quad (40)$$

Thanks to the Dominated Convergence Theorem applied on the sequence of functions in (37 – 40) over the bounded interval  $[1, a_0]$ , we can write:

$$\lim_{n \rightarrow +\infty} 2^{2n} G_{n,\sigma}(a_0) = \left( \int_1^{a_0} dt \frac{\sin(\tau \ln(t))}{t^\sigma} \right)^2 + \left( \int_1^{a_0} dt \frac{\cos(\tau \ln(t))}{t^\sigma} \right)^2 \quad (41)$$

$$\lim_{n \rightarrow +\infty} 2^{2n} G_{n,1-\sigma}(a_0) = \left( \int_1^{a_0} dt \frac{\sin(\tau \ln(t))}{t^{1-\sigma}} \right)^2 + \left( \int_1^{a_0} dt \frac{\cos(\tau \ln(t))}{t^{1-\sigma}} \right)^2 \quad (42)$$

Thanks to lemma 1.4 in [1], we can calculate the integrals in the equations (41 – 42), we conclude:

$$2^{2n} G_{n,\sigma}(a_0) \sim_{n \rightarrow +\infty} f_{a_0}(1 - \sigma) = \frac{a_0^{2(1-\sigma)} - 2a_0^{(1-\sigma)} \cos(\tau \ln(a_0)) + 1}{(1 - \sigma)^2 + \tau^2} \quad (43)$$

$$2^{2n} G_{n,1-\sigma}(a_0) \sim_{n \rightarrow +\infty} f_{a_0}(\sigma) = \frac{a_0^{2\sigma} - 2a_0^\sigma \cos(\tau \ln(a_0)) + 1}{\sigma^2 + \tau^2} \quad (44)$$

Where  $f_a$  is strictly increasing as we saw in lemma 1.4 in [1]. Therefore

$$\lim_{n \rightarrow +\infty} 2^{2n} H_{n,\sigma}(a_0) = f_{a_0}(\sigma) - f_{a_0}(1 - \sigma) < 0 \quad (45)$$

Therefore, there is exist a large enough integer  $N_0$  such that for each  $n \geq N_0$ :

$$2^{2n} H_{n,\sigma}(a_0) < 0 \quad (46)$$

Let's take now  $n \geq N_0$ . From lemma 3.1, have:

$$2^n \int_1^{+\infty} dt \frac{(\epsilon_n(t) + f_n(t))}{t^{1+s}} = -\frac{1}{(1-s)} \quad (47)$$

$$2^n \int_1^{+\infty} dt \frac{(\epsilon_n(t) + f_n(t))}{t^{2-s}} = -\frac{1}{(s)} \quad (48)$$

Therefore:

$$\begin{aligned} \lim_{x \rightarrow +\infty} G_{n,\sigma}(x) &= \Im \left( \int_1^{+\infty} dt \frac{(\epsilon_n(t) + f_n(t))}{t^{1+s}} \right)^2 + \Re \left( \int_1^{+\infty} dt \frac{(\epsilon_n(t) + f_n(t))}{t^{1+s}} \right)^2 \quad (49) \\ &= \frac{1}{2^{2n} \|1-s\|^2} \quad (50) \end{aligned}$$

And the same for the case with  $1-s$ , we can write:

$$\lim_{x \rightarrow +\infty} G_{n,1-\sigma}(x) = \Im \left( \int_1^{+\infty} dt \frac{(\epsilon_n(t) + f_n(t))}{t^{2-s}} \right)^2 + \Re \left( \int_1^{+\infty} dt \frac{(\epsilon_n(t) + f_n(t))}{t^{2-s}} \right)^2 \quad (51)$$

$$= \frac{1}{2^{2n} \|s\|^2} \quad (52)$$

Therefore:

$$\lim_{x \rightarrow +\infty} H_{n,\sigma}(x) = \frac{1}{2^{2n}} \left( \frac{1}{\|s\|^2} - \frac{1}{\|1-s\|^2} \right) > 0 \quad (53)$$

Since the function  $H_{n,\sigma}$  is continuous over  $[1, +\infty)$ . From (46) and (53) and thanks to the Mean value theorem, we can conclude that there exists an  $x_n > b_0$  such that:

$$H_{n,\sigma}(x_n) = 0 \quad (54)$$

■



To prove the second point, we use the same method as in the second point of lemma 1.3 in [1]. Let's  $N_0$  be the same as defined during the first point. Let's define now the sequence of functions  $s(n, \sigma, x)$  such that for  $n \geq N_0$  and  $x \geq 1$ :

$$s(n, \sigma, x) = \sum_{k=N_0}^n G_{k, \sigma}(x) \quad (55)$$

From the equations (45-46), we can deduce that:

$$\sum_{k=N_0}^{+\infty} H_{k, \sigma}(a_0) \leq 0 \quad (56)$$

In order to use the Mean Value theorem like in the point 1) of this lemma, 1) we will prove that the function  $s(+\infty, \sigma, x)$  (resp.  $s(+\infty, 1 - \sigma, x)$ ) is continuous over  $[1, +\infty)$ . and 2) we will prove that:

$$\lim_{x \rightarrow +\infty} \sum_{k=N_0}^{+\infty} H_{k, \sigma}(x) = \frac{1}{3 \cdot 4^{N_0-1}} \left( \frac{1}{\|s\|^2} - \frac{1}{\|1-s\|^2} \right) > 0 \quad (57)$$

Let's take a sequence  $(y_n \geq 1)$  such that it has a limit  $y$ .  $y$  can be finite or  $+\infty$ . The proof is similar for both cases. So, let's consider the double sequence  $s(n, \sigma, m)$  such that:

$$s(n, \sigma, m) = s(n, \sigma, y_m) \quad (58)$$

We will prove that the double sequence  $(s(n, \sigma, m))$  is a Cauchy sequence using the same method as in lemma 1.3 in [1]. We will use the points (6) and (7) of the lemma 3.1 to support the same steps as in [1].  $\blacksquare$

**Lemma 3.3.** *Let's consider two variables  $\sigma$  and  $\tau$  such that  $0 < \sigma \leq \frac{1}{2}$ ,  $\tau > 0$  and  $s = \sigma + i\tau$ . Let's  $L(s, \chi)$  be a  $L$ -function where  $\chi$  is a character (not necessarily a Dirichlet character) such that:*

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx \quad (59)$$

*With the function  $x \rightarrow f(x)$  is bounded over  $\mathbb{R}^+$  and null over the interval  $[0, 1]$  and the complex constant  $\lambda \neq 0$ .*

*If  $L(s, \chi) = 0$ , then:*

$$\sigma = \frac{1}{2} \quad (60)$$

*Proof.* With the result of lemmas 3.1 and 3.2, we use the same methods used in lemmas 1.4, 1.5 and 1.6 in [1] to prove this lemma.  $\square$

From the lemma 3.3 above we can conclude the following corollary.

**Corollary 3.3.1.** *Let's consider two variables  $\sigma$  and  $\tau$  such that  $\sigma > 0$ ,  $\tau > 0$  and  $s = \sigma + i\tau$ . Let's  $q > 1$  be an positive integer. Let's  $L(s, \chi)$  be a  $L$ -function where  $\chi$  is a Dirichlet character mod  $q$  with no cancellation property such that:*

$$\sum_{n=1}^q \chi(n) \neq 0 \quad (61)$$

If  $L(s, \chi) = 0$ , then:

$$\sigma = \frac{1}{2} \quad (62)$$

*Proof.* Let's take  $x > 1$  a real number. Let's denote  $n_x = [x]$  be the integer part of  $x$ . We have  $n_x = x - \{x\}$ . We denote  $q_x$  and  $r_x$  the quotient and the rest of the Euclidean division of  $n_x$  by  $q$  as follows:  $n_x = qq_x + r_x$ . Therefore we can write:

$$\sum_{n=1}^x \chi(n) = \sum_{n=0}^{q_x-1} \sum_{k=1}^q \chi(nq + k) + \sum_{k=1}^{r_x} \chi(nq + k) \quad (63)$$

$$= q_x \sum_{n=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (64)$$

$$= \frac{x - \{x\} - r_x}{q} \sum_{k=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (65)$$

$$= \frac{x}{q} \sum_{k=1}^q \chi(n) - \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (66)$$

Let's define the constant  $\lambda$  and the function  $g$  as follows:

$$\lambda = \frac{\sum_{k=1}^q \chi(n)}{q} \quad (67)$$

$$g(x) = \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi(n) - \sum_{k=1}^{r_x} \chi(k) \quad (68)$$

$$= \lambda\{x\} + \underbrace{\lambda r_x - \sum_{k=1}^{r_x} \chi(k)}_{f(x)} \quad (69)$$

It is clear that the function  $f(x)$  is bounded over  $\mathbb{R}^+$  and null over  $[0, 1]$ . And by assumption we have  $\lambda \neq 0$ . Therefore, we can calculate  $L(s, \chi)$  as follows:

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (70)$$

$$= s \int_1^{+\infty} \frac{\lambda x - g(x)}{x^{s+1}} dx \quad (71)$$

$$= s \int_1^{+\infty} \frac{\lambda}{x^s} dx - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (72)$$

$$(73)$$

If  $\Re(s) > 1$ , we can conclude:

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (74)$$

We can extend the equation (74) over the region  $\Re(s) > 0$ .  $L(s, \chi)$  has a holomorphic continuation as a function of  $s$  to the whole complex plane (see [3, 6, 7, 8]). Therefore, we apply the lemma 3.3 to prove that  $\sigma = \frac{1}{2}$ . ■  $\square$

**Remark.** As a direct result of the above corollary, we can see that GRH is true for the series  $(\sum_{n \geq 1} \frac{1}{(qn+p)^s})$  with  $p > 0$  and  $q > 1$  are integers.

**Lemma 3.4.** Let's consider two variables  $\sigma$  and  $\tau$  such that  $\sigma > 0$ ,  $\tau > 0$  and  $s = \sigma + i\tau$ . Let's  $q > 1$  be an positive integer. Let's  $L(s, \chi)$  be a  $L$ -function where  $\chi$  is a Dirichlet character mod  $q$  with cancellation property such that:

$$\sum_{n=1}^q \chi(n) = 0 \quad (75)$$

Therefore:

$$L(s, \chi) = \left(\frac{1-q^{1-s}}{q}\right) \frac{s}{s-1} - s \int_1^{+\infty} \frac{\left(\frac{1-q^{1-s}}{q}\right)\{x\} + f(x)}{x^{s+1}} dx \quad (76)$$

Where  $f$  is a bounded function over  $\mathbb{R}^+$  and null over the interval  $[0, 1]$ .

*Proof.* We will use the zeta function  $\zeta(s)$  to prove our case.

Let's now write the following for  $s > 1$ :

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (77)$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^q \frac{1}{(qn+i)^s} \quad (78)$$

$$= \sum_{i=1}^q \sum_{n=0}^{\infty} \frac{1}{(qn+i)^s} \quad (79)$$

Therefore

$$\left(1 - \frac{1}{q^s}\right) \zeta(s) = \sum_{i=1}^{q-1} \sum_{n=0}^{\infty} \frac{1}{(qn+i)^s} \quad (80)$$

Also

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \quad (81)$$

$$= \sum_{n=0}^{+\infty} \sum_{i=1}^q \frac{\chi(i)}{(qn+i)^s} \quad (82)$$

$$= \sum_{i=1}^q \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (83)$$

Therefore since  $\chi(q) = 0$

$$L(s, \chi) = \sum_{i=1}^{q-1} \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (84)$$

Since  $\sum_{n=1}^q \chi(n) = 0$  and  $\chi(1) = 1$ , we can write the following:

$$\sum_{n=0}^{+\infty} \frac{1}{(qn+1)^s} = L(s, \chi) - \sum_{i=2}^{q-1} \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (85)$$

$$\sum_{n=0}^{+\infty} \frac{1}{(qn+1)^s} = \left(1 - \frac{1}{q^s}\right) \zeta(s) - \sum_{i=2}^{q-1} \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (86)$$

Therefore

$$\left(1 - \frac{1}{q^s}\right) \zeta(s) - L(s, \chi) = \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (87)$$

Let's denote  $L(s, i)_{2 \leq i \leq q-1}$  as follows:

$$L(s, i) = \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (88)$$

We can write also:

$$L(s, i) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi_i(n)}{x^{s+1}} dx \quad (89)$$

Where  $\chi_i$  is the character of  $L(s, i)$ :

- (i)  $\chi_i(qn+i) = 1$
- (ii)  $\chi_i(qn+j) = 0$  if  $j \neq i$

Let's take  $x > 1$  a real number. Let's denote  $n_x = \lfloor x \rfloor$  be the integer part of  $x$ . We have  $n_x = x - \{x\}$ . We denote  $q_x$  and  $r_x$  the quotient and the rest of the Euclidean division of  $n_x$  by  $q$  as follows:  $n_x = qq_x + r_x$ . Let's denote the following:

$$\lambda_i = \frac{\sum_{k=1}^q \chi_i(k)}{q} = \frac{1}{q} \quad (90)$$

$$f_i(x) = \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi_i(k) - \sum_{k=1}^{r_x} \chi_i(k) \quad (91)$$

$$= \frac{(\{x\} + r_x)}{q} - \sum_{k=1}^{r_x} \chi_i(k) \quad (92)$$

Therefore from the equation (74):

$$L(s, i) = \frac{s}{q(s-1)} - s \int_1^{+\infty} \frac{f_i(x)}{x^{s+1}} dx \quad (93)$$

Therefore, we can write the equation (87) as follows:

$$(1 - \frac{1}{q^s})\zeta(s) - L(s, \chi) = \sum_{i=2}^{q-1} (1 - \chi(i))L(s, i) \quad (94)$$

$$= \frac{s}{q(s-1)} \sum_{i=2}^{q-1} (1 - \chi(i)) - s \sum_{i=2}^{q-1} (1 - \chi(i)) \int_1^{+\infty} \frac{f_i(x)}{x^{s+1}} dx \quad (95)$$

Thanks to the cancellation property, we can conclude that:

$$(1 - \frac{1}{q^s})\zeta(s) - L(s, \chi) = \frac{s(q-1)}{q(s-1)} - s \int_1^{+\infty} \frac{\sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x)}{x^{s+1}} dx \quad (96)$$

From the equation (10) we replace the  $\zeta(s)$  by its  $L$ -function:

$$L(s, \chi) = \frac{1 - q^{1-s}}{q} \frac{s}{(s-1)} - s \int_1^{+\infty} \frac{(1 - q^{-s})\{x\} - \sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x)}{x^{s+1}} dx \quad (97)$$

Let's define the function  $g(x)$  as follows:

$$g(x) = (1 - q^{-s})\{x\} - \sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x) \quad (98)$$

$$= \frac{1 - q^{1-s}}{q} \{x\} - (1 - \frac{1}{q}) r_x + \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{k=1}^{r_x} \chi_i(k) \quad (99)$$

$$= \frac{1 - q^{1-s}}{q} \{x\} + f(x) \quad (100)$$

Where  $f$  is unrelated to  $s$ , bounded over  $\mathbb{R}^+$  and null over  $[0, 1]$ . We can also write:

$$f(x) = -(1 - \frac{1}{q}) r_x + \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{k=1}^{r_x} \chi_i(k) \quad (101)$$

$$= \frac{r_x}{q} - \sum_{k=1}^{r_x} \chi(k) \quad (102)$$

And

$$L(s, \chi) = (\frac{1 - q^{1-s}}{q}) \frac{s}{s-1} - s \int_1^{+\infty} \frac{(\frac{1 - q^{1-s}}{q})\{x\} + f(x)}{x^{s+1}} dx \quad (103)$$

□

**Lemma 3.5.** *Let's consider two variables  $\sigma$  and  $\tau$  such that  $0 < \sigma < 1$ ,  $\tau > 0$  and  $s = \sigma + i\tau$ . Let's  $q > 1$  be an positive integer. Let's  $L(s, \chi)$  be a  $L$ -function where  $\chi$  is a Dirichlet character mod  $q$  with cancellation property such that:*

$$\sum_{n=1}^q \chi(n) = 0 \quad (104)$$

Let's suppose that  $s = \sigma + i\tau$  is a zero for  $L(s, \chi)$ . i.e  $L(s, \chi) = 0$ .  
Therefore

$$\sigma = \frac{1}{2} \quad (105)$$

And the Generalized Riemann's Hypothesis is true.

*Proof.* The proof is straightforward as we can use the lemmas 3.3 and 3.4.  
From lemma 3.4 we can write the following:

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (106)$$

$$\lambda = \frac{1 - q^{1-s}}{q} \quad (107)$$

Where  $f$  is a bounded function over  $\mathbb{R}^+$ . Then, we apply the lemma 3.3 to conclude that

$$\sigma = \frac{1}{2} \quad (108)$$

And the Generalized Riemann's Hypothesis is true.

■

□

### 3.1 Conclusion

We saw that if  $s$  is a zeta zero, then real part  $\Re(s)$  can only be  $\frac{1}{2}$ . Therefore, the Generalized Riemann Hypothesis (GRH) is true: *The non-trivial zeros of a  $L$ -Dirichlet function have real part equal to  $\frac{1}{2}$ .* With this, we conclude the Factoring problem is in P and therefore  $P = NP$ .

### Acknowledgments

I would like to thank Farhat Latrach, Giampiero Esposito, Jacques G  linas, Michael Milgram, L  o Ag  las, Ronald F. Fox, Kim Y.G, Masumi Nakajima, Maksym Radziwill and Shekhar Suman for thoughtful comments and discussions on my paper versions on the RH. All errors are mine.

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