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► To cite this version:

| Charaf Ech-Chatbi. A Proof of the Generalized Riemann Hypothesis. 2023. hal-02150289v8

HAL Id: hal-02150289

<https://hal.science/hal-02150289v8>

Preprint submitted on 10 Feb 2023 (v8), last revised 20 Mar 2023 (v9)

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A Simple Proof of the Generalized Riemann's Hypothesis

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Thursday 12 May 2022

Abstract

This article is the continuation of the previous RH proof[1]. We present a simple proof of the Generalized Riemann's Hypothesis (GRH) where we apply the same method we used to prove the RH.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 Dirichlet L -functions

Let's $(z_n)_{n \geq 1}$ be a sequence of complex numbers. A Dirichlet series[3] is a series of the form $\sum_{n=1}^{\infty} \frac{z_n}{n^s}$, where $s = \sigma + i\tau$ is complex. The zeta function is a Dirichlet series. Let's define the function $L(s)$ of the complex s :

$$L(s) = \sum_{n=1}^{\infty} \frac{z_n}{n^s}.$$

- If $(z_n)_{n \geq 1}$ is a bounded, then the corresponding Dirichlet series converges absolutely on the open half-plane where $\Re(s) > 1$.
- If the set of sums $z_n + z_{n+1} + \dots + z_{n+k}$ for each n and $k \geq 0$ is bounded, then the corresponding Dirichlet series converges on the open half-plane where $\Re(s) > 0$.
- In general, if $z_n = O(n^k)$, the corresponding Dirichlet series converges absolutely in the half plane where $\Re(s) > k + 1$.

The function $L(s)$ is analytic on the corresponding open half plane[2-3].

To define Dirichlet L -functions we need to define Dirichlet characters. A function $\chi : \mathbb{Z} \mapsto \mathbb{C}$ is a Dirichlet character modulo q if it satisfies the following criteria:

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- (i) $\chi(n) \neq 0$ if $(n, q) = 1$.
- (ii) $\chi(n) = 0$ if $(n, q) > 1$.
- (iii) χ is periodic with period q : $\chi(n + q) = \chi(n)$ for all n .
- (iv) χ is multiplicative : $\chi(mn) = \chi(m)\chi(n)$ for all integers m and n .

The trivial character is the one with $\chi_0(n) = 1$ whenever $(n, q) = 1$.

Here are some known results for a Dirichlet character modulo q . For any integer n we have $\chi(1) = 1$. Also if $(n, q) = 1$, we have $(\chi(n))^{\phi(q)} = 1$ with ϕ is Euler's totient function. $\chi(n)$ is a $\phi(q)$ -th root of unity. Therefore, $|\chi(n)| = 1$ if $(n, q) = 1$, and $|\chi(n)| = 0$ if $(n, q) > 1$. Also, we recall the cancellation property for Dirichlet characters modulo q : For any n integer

$$\sum_{i=1}^q \chi(i + n) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (1)$$

The Dirichlet L -functions are simply the sum of the Dirichlet series. Let's χ be a Dirichlet character modulo q , The Dirichlet L -function $L(s, \chi)$ is defined for $\Re(s) > 1$ as the following:

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \quad (2)$$

They are a natural generalization of the Riemann zeta-function $\zeta(s)$ to an arithmetic progression and are a powerful tool in analytic number theory. The Dirichlet series, converges absolutely and uniformly in any bounded domain in the complex s -plane for which $\Re(s) \geq 1 + \gamma$, $\gamma > 0$. In the particular case of the trivial character χ_0 , $L(s, \chi_0)$ extends to a meromorphic function[2-5] in $\Re(s) > 0$ with the only pole at $s = 1$.

If χ is a non-trivial character, we have

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (3)$$

Since the sum in the integrand is bounded, this formula gives an analytic continuation of $L(s, \chi)$ to a regular function in the half-plane $\Re(s) > 0$. Also, like the zeta function, the Dirichlet L -functions have their Euler product[1-2]. For $\Re(s) > 1$:

$$L(s, \chi) = \prod_{p \text{ Prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (4)$$

Therefore, if $\chi = \chi_0$ is a trivial character mod q , we have, for $q = 1$,

$$L(s, \chi_0) = \zeta(s) \quad (5)$$

And for $q > 1$, we have,

$$L(s, \chi_0) = \zeta(s) \prod_{p/q} \left(1 - \frac{1}{p^s}\right) \quad (6)$$

For this reason the properties of $L(s, \chi_0)$ in the entire complex plane are determined by the properties of $\zeta(s)$.

Let's now q' be the smallest divisor (prime) of q . Let's χ' be the Dirichlet character $\chi' \pmod{q'}$. For any integer n such that $(n, q) = 1$, we have also $(n, q') = 1$ and $\chi(n) = \chi'(n)$. χ' is called primitive and $L(s, \chi)$ and $L(s, \chi')$ are related analytically such that if $\chi \neq \chi_0$:

$$L(s, \chi) = L(s, \chi') \prod_{p|q} \left(1 - \frac{\chi'(p)}{p^s}\right) \quad (7)$$

$L(s, \chi)$ and $L(s, \chi')$ have the same zeros in the critical strip $0 < \Re(s) < 1$.

Also, for a primitive character χ , (*i.e.* $\chi = \chi'$) $L(s, \chi)$ has the following functional equation:

$$\tau(\chi) \Gamma\left(\frac{1-s+a}{2}\right) L(1-s, \chi) = \sqrt{\pi} \left(\frac{q}{\pi}\right)^s i^a q^{\frac{1}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \overline{\chi}) \quad (8)$$

Where Γ is the Gamma function and $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$, and $\tau(\chi) = \sum_{k=1}^q \chi(k) \exp\left(\frac{2\pi ki}{q}\right)$.

When $\Re(s) > 1$ there is no zero for $L(s, \chi)$. When $\Re(s) < 0$, for a primitive character χ , we have the trivial zeros of $L(s, \chi)$: $s = a - 2k$, where k is a positive integer and a is defined above. For more details, please refer to the references[1-8].

2 The Generalized Riemann Hypothesis

The Generalized Riemann Hypothesis states that the Dirichlet L -functions have all their non-trivial zeros on the critical line $\Re(s) = \frac{1}{2}$.

We know that for any character χ modulo q , all non-trivial zeros of $L(s, \chi)$ lies in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. From the functional equation above we have that if:

- s_0 is a non-trivial zero of $L(s, \chi)$, then $1 - s_0$ is a zero of $L(s, \overline{\chi})$.
- s_0 is a non-trivial zero of $L(s, \overline{\chi})$, then $1 - s_0$ is a zero of $L(s, \chi)$.

Therefore, we just need to prove that for any character¹ χ modulo q , there is no non-trivial zeros of $L(s, \chi)$ in the right hand side of the critical strip $\{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$.

3 Proof of the Generalized Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $L(s, \chi) = 0$ where $L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$ and χ is a non-trivial Dirichlet character χ modulo q . When $q = 2$ there is only one Dirichlet character and it is

¹In fact, we just need to prove it for the primitive characters.

trivial. So here we are going to assume that $q \geq 3$.

We have from the integral equation (3) above:

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (9)$$

We have from the Riemann's integral in [1]:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (10)$$

In this article, we leverage on the RH proof in [1] to proof GRH. Also, we will use the same method and notations as in [1].

Let's denote the following functions:

$$\epsilon(x) = \{x\} \quad (11)$$

$$F(x) = \int_1^x f(x) dx \quad (12)$$

$$\epsilon_0(x) = \epsilon(x) \quad (13)$$

$$\epsilon_{n+1}(x) = \frac{1}{x} \int_0^x \epsilon_n(t) \quad (14)$$

$$(15)$$

Lemma 3.1. *Let's consider two variables σ and τ such that $\sigma > 0$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $L(s, \chi)$ be a L -function where χ is a character (not necessarily a Dirichlet character) such that:*

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx \quad (16)$$

With the function $x \rightarrow f(x)$ is bounded and null over the interval $[0, 1]$ and the complex constant $\lambda \neq 0$. Let's define the sequence of functions f_n and \bar{f}_n such that $f_0(x) = \bar{f}_0(x) = \frac{f(x)}{\lambda}$ and for each $n \geq 1$:

$$f_{n+1}(x) = \frac{1}{x} \int_0^x f_n(t) \quad (17)$$

$$\bar{f}_{n+1}(x) = \frac{1}{x} \int_1^x \bar{f}_n(t) \quad (18)$$

If $L(s, \chi) = 0$, then:

1. *For each n :*

$$\int_1^{+\infty} dx \frac{\epsilon_n(x) + f_n(x)}{x^{1+s}} = -\frac{1}{(1-s)2^n} \quad (19)$$

2. *For each n :*

$$\int_1^{+\infty} dx \frac{\epsilon_n(x) + f_n(x)}{x^{2-s}} = -\frac{1}{(s)2^n} \quad (20)$$

3. *For each $x \geq 1$:*

$$f_n(x) = \bar{f}_n(x) \quad (21)$$

4. For each n and $x \geq 1$:

$$f_n(x) = \frac{1}{(n-1)!} \frac{1}{x} \int_1^x dt f_0(t) \left(\ln\left(\frac{x}{t}\right) \right)^{n-1} \quad (22)$$

5. For each $x \geq 1$:

$$\lim_{n \rightarrow +\infty} 2^n f_n(x) = 0 \quad (23)$$

Proof. Let's take s such that $L(s, \chi) = 0$, therefore we can write:

$$\frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx = 0 \quad (24)$$

Therefore, since $s \neq 0$ and $\lambda \neq 0$, we have:

$$\int_1^{+\infty} \frac{\epsilon_0(x) + f_0(x)}{x^{s+1}} dx = \frac{1}{s-1} \quad (25)$$

We use the same method used in the lemma 1.2 in [1] to prove this lemma. \square

Lemma 3.2. *Let's consider two variables σ and τ such that $0 < \sigma \leq \frac{1}{2}$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $L(s, \chi)$ be a L -function where χ is a character (not necessarily a Dirichlet character) such that:*

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx \quad (26)$$

With the function $x \rightarrow f(x)$ is bounded and null over the interval $[0, 1]$ and the complex constant $\lambda \neq 0$.

Let's define the sequence of functions $E_{n,\sigma}(x)$, $F_{n,\sigma}(x)$, $G_{n,\sigma}(x)$ and $H_{n,\sigma}(x)$ over $[1, +\infty)$ for each $n \geq 1$ as follows:

$$E_{n,\sigma}(x) = \int_1^x dt \frac{\sin(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) \quad (27)$$

$$F_{n,\sigma}(x) = \int_1^x dt \frac{\cos(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) \quad (28)$$

$$G_{n,\sigma}(x) = (E_{n,\sigma}(x))^2 + (F_{n,\sigma}(x))^2 \quad (29)$$

$$H_{n,\sigma}(x) = G_{n,1-\sigma}(x) - G_{n,\sigma}(x) \quad (30)$$

Let's define the sequence of functions $I_{n,\sigma}(x)$ over $[0, +\infty)$ for each $n \geq 1$ as follows:

$$I_{n,\sigma}(x) = H_{n,\sigma}(x + e^{\frac{3}{2\tau}}) \quad (31)$$

We denote $a_0 = e^{\frac{3}{2\tau}}$ and $b_0 = e^{\frac{\pi}{2\tau}}$. If $L(s, \chi) = 0$, then:

1. For a large enough integer n there exists $x_n > b_0$ such as:

$$H_{n,\sigma}(x_n) = 0 \quad (32)$$

2. For a large enough integer n there exists $x_n > b_0 - a_0 > 0$ such as:

$$I_{n,\sigma}(x_n) = 0 \quad (33)$$

In other terms:

$$G_{n,1-\sigma}(x_n + a_0) = G_{n,\sigma}(x_n + a_0) \quad (34)$$

Proof. Let's prove the first point. Let's take $n \geq 1$ an integer. Since $\sigma \leq \frac{1}{2}$. We have for each $x > 1$: $\frac{1}{x^{1+\sigma}} > \frac{1}{x^{2-\sigma}}$. Therefore

$$E_{n,\sigma}(b_0) = \int_1^{e^{\frac{\pi}{2\tau}}} dt \frac{\sin(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) > 0 \quad (35)$$

$$F_{n,\sigma}(b_0) = \int_1^{e^{\frac{\pi}{2\tau}}} dt \frac{\cos(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) > 0 \quad (36)$$

$$E_{n,\sigma}(b_0) > E_{n,1-\sigma}(b_0) > 0 \quad (37)$$

$$F_{n,\sigma}(b_0) > F_{n,1-\sigma}(b_0) > 0 \quad (38)$$

Therefore

$$(E_{n,\sigma}(b_0))^2 > (E_{n,1-\sigma}(b_0))^2 \quad (39)$$

$$(F_{n,\sigma}(b_0))^2 > (F_{n,1-\sigma}(b_0))^2 \quad (40)$$

And therefore

$$G_{n,\sigma}(b_0) > G_{n,1-\sigma}(b_0) \quad (41)$$

$$H_{n,\sigma}(b_0) < 0 \quad (42)$$

Thanks to lemma 3.1, we have for each $x \geq 1$:

$$\lim_{n \rightarrow +\infty} 2^n f_n(x) = 0 \quad (43)$$

Therefore, for each $x \geq 1$:

$$\lim_{n \rightarrow +\infty} 2^n \frac{f_n(x)}{x^{1+s}} = 0 \quad (44)$$

Thanks to the dominated convergence theorem, we can conclude that:

$$\lim_{n \rightarrow +\infty} 2^n \int_1^{+\infty} dx \frac{f_n(x)}{x^{1+s}} = 0 \quad (45)$$

Therefore, we can write:

$$\lim_{n \rightarrow +\infty} 2^n \int_1^{+\infty} dx \frac{\epsilon_n(x)}{x^{1+s}} = -\frac{1}{(1-s)} \quad (46)$$

$$\lim_{n \rightarrow +\infty} 2^n \int_1^{+\infty} dx \frac{\epsilon_n(x)}{x^{2-s}} = -\frac{1}{(s)} \quad (47)$$

Therefore:

$$\begin{aligned} \lim_{x \rightarrow +\infty} G_{n,\sigma}(x) &= \left(\int_1^{+\infty} dt \frac{\sin(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) \right)^2 + \left(\int_1^{+\infty} dt \frac{\cos(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) \right)^2 \quad (48) \\ &\sim_{n \rightarrow +\infty} \frac{1}{2^{2n} \|1-s\|^2} \quad (49) \end{aligned}$$

And the same for the case with $1-\sigma$, we can write:

$$\begin{aligned} \lim_{x \rightarrow +\infty} G_{n,1-\sigma}(x) &= \left(\int_1^{+\infty} dt \frac{\sin(\tau \ln(t))}{x^{2-\sigma}} \epsilon_n(x) \right)^2 + \left(\int_1^{+\infty} dt \frac{\cos(\tau \ln(t))}{x^{2-\sigma}} \epsilon_n(x) \right)^2 \quad (50) \\ &\sim_{n \rightarrow +\infty} \frac{1}{2^{2n} \|s\|^2} \quad (51) \end{aligned}$$

Therefore:

$$\lim_{x \rightarrow +\infty} H_{n,\sigma}(x) \sim_{n \rightarrow +\infty} \frac{1}{2^{2n}} \left(\frac{1}{\|s\|^2} - \frac{1}{\|1-s\|^2} \right) > 0 \quad (52)$$

Since the function $H_{n,\sigma}$ is continuous over $[1, +\infty)$. From (42) and (52) and thanks to the Mean value theorem, we can conclude that there exists an $x_n > b_0$ such that:

$$H_{n,\sigma}(x_n) = 0 \quad (53)$$

■

To prove the second point, we proceed by the same approach as in the first point. We remark that since $b_0 - a_0 > 0$:

$$I_{n,\sigma}(b_0 - a_0) = H_{n,\sigma}(a_0 + b_0 - a_0) = H_{n,\sigma}(b_0) < 0 \quad (54)$$

And the same like above, for a large enough n :

$$\lim_{x \rightarrow +\infty} I_{n,\sigma}(x) = \lim_{x \rightarrow +\infty} H_{n,\sigma}(a_0 + x) \quad (55)$$

$$\sim_{n \rightarrow +\infty} \frac{1}{2^{2n}} \left(\frac{1}{\|s\|^2} - \frac{1}{\|1-s\|^2} \right) > 0 \quad (56)$$

And the same we conclude thanks to the Mean value theorem.

■

□

Lemma 3.3. *Let's consider two variables σ and τ such that $0 < \sigma \leq \frac{1}{2}$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $L(s, \chi)$ be a L -function where χ is a character (not necessarily a Dirichlet character) such that:*

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx \quad (57)$$

With the function $x \rightarrow f(x)$ is bounded and null over the interval $[0, 1]$ and the complex constant $\lambda \neq 0$.

If $L(s, \chi) = 0$, then:

$$\sigma = \frac{1}{2} \quad (58)$$

Proof. With the result of the lemma 3.2, we can use the same methods used in lemma 1.4 and lemma 1.6 in [1] to prove this lemma. \square

From the lemma 3.3 above we can conclude the following corollary.

Corollary 3.3.1. *Let's consider two variables σ and τ such that $\sigma > 0$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $q > 1$ be an positive integer. Let's $L(s, \chi)$ be a L -function where χ is a Dirichlet character mod q with no cancellation property such that:*

$$\sum_{n=1}^q \chi(n) \neq 0 \quad (59)$$

If $L(s, \chi) = 0$, then:

$$\sigma = \frac{1}{2} \quad (60)$$

Proof. Let's take $x > 1$ a real number. Let's denote $n_x = \lfloor x \rfloor$ be the integer part of x . We have $n_x = x - \{x\}$. We denote q_x and r_x the quotient and the rest of the Euclidean division of n_x by q as follows: $n_x = qq_x + r_x$. Therefore we can write:

$$\sum_{n=1}^x \chi(n) = \sum_{n=0}^{q_x-1} \sum_{k=1}^q \chi(nq + k) + \sum_{k=1}^{r_x} \chi(nq + k) \quad (61)$$

$$= q_x \sum_{n=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (62)$$

$$= \frac{x - \{x\} - r_x}{q} \sum_{k=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (63)$$

$$= \frac{x}{q} \sum_{k=1}^q \chi(n) - \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (64)$$

Let's define the constant λ and the function g as follows:

$$\lambda = \frac{\sum_{k=1}^q \chi(n)}{q} \quad (65)$$

$$g(x) = \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi(n) - \sum_{k=1}^{r_x} \chi(k) \quad (66)$$

$$= \lambda \{x\} + \underbrace{\lambda r_x - \sum_{k=1}^{r_x} \chi(k)}_{f(x)} \quad (67)$$

It is clear that the function $f(x)$ is bounded and null over $[0, 1]$. And by assumption we have $\lambda \neq 0$. Therefore, we can calculate $L(s, \chi)$ as follows:

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (68)$$

$$= s \int_1^{+\infty} \frac{\lambda x - g(x)}{x^{s+1}} dx \quad (69)$$

$$= s \int_1^{+\infty} \frac{\lambda}{x^s} dx - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (70)$$

$$(71)$$

If $\Re(s) > 1$, we can conclude:

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (72)$$

We can extend the equation (72) over the region $\Re(s) > 0$. $L(s, \chi)$ has a holomorphic continuation as a function of s to the whole complex plane (see [3, 6, 7, 8]). Therefore, we apply the lemma 3.3 to prove that $\sigma = \frac{1}{2}$. \blacksquare

Remark. As a direct result of the above corollary, we can see that GRH is true for the series $(\sum_{n \geq 1} \frac{1}{(qn+p)^s})$ with $p > 0$ and $q > 1$ are integers.

Lemma 3.4. Let's consider two variables σ and τ such that $\sigma > 0$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $q > 1$ be an positive integer. Let's $L(s, \chi)$ be a L -function where χ is a Dirichlet character mod q with cancellation property such that:

$$\sum_{n=1}^q \chi(n) = 0 \quad (73)$$

Therefore:

$$L(s, \chi) = \left(\frac{1-q^{1-s}}{q}\right) \frac{s}{s-1} - s \int_1^{+\infty} \frac{\left(\frac{1-q^{1-s}}{q}\right)\{x\} + f(x)}{x^{s+1}} dx \quad (74)$$

Where f is a bounded function and null over the interval $[0, 1]$.

Proof. We will use the zeta function $\zeta(s)$ to prove our case.

Let's now write the following for $s > 1$:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (75)$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^q \frac{1}{(qn+i)^s} \quad (76)$$

$$= \sum_{i=1}^q \sum_{n=0}^{\infty} \frac{1}{(qn+i)^s} \quad (77)$$

Therefore

$$\left(1 - \frac{1}{q^s}\right) \zeta(s) = \sum_{i=1}^{q-1} \sum_{n=0}^{\infty} \frac{1}{(qn+i)^s} \quad (78)$$

Also

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \quad (79)$$

$$= \sum_{n=0}^{+\infty} \sum_{i=1}^q \frac{\chi(i)}{(qn+i)^s} \quad (80)$$

$$= \sum_{i=1}^q \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (81)$$

Therefore since $\chi(q) = 0$

$$L(s, \chi) = \sum_{i=1}^{q-1} \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (82)$$

Since $\sum_{n=1}^q \chi(n) = 0$ and $\chi(1) = 1$, we can write the following:

$$\sum_{n=0}^{+\infty} \frac{1}{(qn+1)^s} = L(s, \chi) - \sum_{i=2}^{q-1} \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (83)$$

$$\sum_{n=0}^{+\infty} \frac{1}{(qn+1)^s} = \left(1 - \frac{1}{q^s}\right) \zeta(s) - \sum_{i=2}^{q-1} \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (84)$$

Therefore

$$\left(1 - \frac{1}{q^s}\right) \zeta(s) - L(s, \chi) = \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (85)$$

Let's denote $L(s, i)_{2 \leq i \leq q-1}$ as follows:

$$L(s, i) = \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (86)$$

We can write also:

$$L(s, i) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi_i(n)}{x^{s+1}} dx \quad (87)$$

Where χ_i is the character of $L(s, i)$:

- (i) $\chi_i(qn+i) = 1$
- (ii) $\chi_i(qn+j) = 0$ if $j \neq i$

Let's take $x > 1$ a real number. Let's denote $n_x = \lfloor x \rfloor$ be the integer part of x . We have $n_x = x - \{x\}$. We denote q_x and r_x the quotient and the rest of the Euclidean division of n_x by q as follows: $n_x = qq_x + r_x$. Let's denote the following:

$$\lambda_i = \frac{\sum_{k=1}^q \chi_i(k)}{q} = \frac{1}{q} \quad (88)$$

$$f_i(x) = \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi_i(k) - \sum_{k=1}^{r_x} \chi_i(k) \quad (89)$$

$$= \frac{(\{x\} + r_x)}{q} - \sum_{k=1}^{r_x} \chi_i(k) \quad (90)$$

Therefore from the equation (72):

$$L(s, i) = \frac{s}{q(s-1)} - s \int_1^{+\infty} \frac{f_i(x)}{x^{s+1}} dx \quad (91)$$

Therefore, we can write the equation (85) as follows:

$$(1 - \frac{1}{q^s})\zeta(s) - L(s, \chi) = \sum_{i=2}^{q-1} (1 - \chi(i)) L(s, i) \quad (92)$$

$$= \frac{s}{q(s-1)} \sum_{i=2}^{q-1} (1 - \chi(i)) - s \sum_{i=2}^{q-1} (1 - \chi(i)) \int_1^{+\infty} \frac{f_i(x)}{x^{s+1}} dx \quad (93)$$

Thanks to the cancellation property, we can conclude that:

$$(1 - \frac{1}{q^s})\zeta(s) - L(s, \chi) = \frac{s(q-1)}{q(s-1)} - s \int_1^{+\infty} \frac{\sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x)}{x^{s+1}} dx \quad (94)$$

From the equation (10) we replace the $\zeta(s)$ by its L - function:

$$L(s, \chi) = \frac{1 - q^{1-s}}{q} \frac{s}{(s-1)} - s \int_1^{+\infty} \frac{(1 - q^{-s})\{x\} - \sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x)}{x^{s+1}} dx \quad (95)$$

Let's define the function $g(x)$ as follows:

$$g(x) = (1 - q^{-s})\{x\} - \sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x) \quad (96)$$

$$= \frac{1 - q^{1-s}}{q} \{x\} - (1 - \frac{1}{q}) r_x + \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{k=1}^{r_x} \chi_i(k) \quad (97)$$

$$= \frac{1 - q^{1-s}}{q} \{x\} + f(x) \quad (98)$$

Where f is unrelated to s , bounded and null over $[0, 1]$. We can also write:

$$f(x) = -(1 - \frac{1}{q}) r_x + \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{k=1}^{r_x} \chi_i(k) \quad (99)$$

$$= \frac{r_x}{q} - \sum_{k=1}^{r_x} \chi(k) \quad (100)$$

And

$$L(s, \chi) = (\frac{1 - q^{1-s}}{q}) \frac{s}{s-1} - s \int_1^{+\infty} \frac{(\frac{1 - q^{1-s}}{q})\{x\} + f(x)}{x^{s+1}} dx \quad (101)$$

□

Lemma 3.5. *Let's consider two variables σ and τ such that $0 < \sigma < 1$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $q > 1$ be an positive integer. Let's $L(s, \chi)$*

be a L -function where χ is a Dirichlet character mod q with cancellation property such that:

$$\sum_{n=1}^q \chi(n) = 0 \quad (102)$$

Let's suppose that $s = \sigma + i\tau$ is a zero for $L(s, \chi)$. i.e $L(s, \chi) = 0$. Therefore

$$\sigma = \frac{1}{2} \quad (103)$$

And the Generalized Riemann's Hypothesis is true.

Proof. The proof is straightforward as we can use the lemmas 3.3 and 3.4. From lemma 3.4 we can write the following:

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (104)$$

$$\lambda = \frac{1 - q^{1-s}}{q} \quad (105)$$

Where f is a bounded function. Then, we apply the lemma 3.3 to conclude that

$$\sigma = \frac{1}{2} \quad (106)$$

And the Generalized Riemann's Hypothesis is true.

■

□

3.1 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore, the Generalized Riemann Hypothesis (GRH) is true: *The non-trivial zeros of a L - Dirichlet function have real part equal to $\frac{1}{2}$.* With this, we conclude that $P = NP$.

Acknowledgments

I would like to thank Farhat Latrach, Giampiero Esposito, Jacques G  linas, Michael Milgram, L  o Ag  las, Ronald F. Fox, Kim Y.G, Masumi Nakajima, Maksym Radziwill and Shekhar Suman for thoughtful comments and discussions on my paper versions on the RH. All errors are mine.

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