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► **To cite this version:**

| Charaf Ech-Chatbi. A Proof of the Generalized Riemann Hypothesis. 2022. hal-02150289v7

HAL Id: hal-02150289

<https://hal.science/hal-02150289v7>

Preprint submitted on 29 Jul 2022 (v7), last revised 20 Mar 2023 (v9)

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A Simple Proof of the Generalized Riemann's Hypothesis

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Thursday 12 May 2022

Abstract

This article is the continuation of the previous RH proof[1]. We present a simple proof of the Generalized Riemann's Hypothesis (GRH) where we apply the same method we used to prove the RH.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 Dirichlet L -functions

Let's $(z_n)_{n \geq 1}$ be a sequence of complex numbers. A Dirichlet series[3] is a series of the form $\sum_{n=1}^{\infty} \frac{z_n}{n^s}$, where $s = \sigma + i\tau$ is complex. The zeta function is a Dirichlet series. Let's define the function $L(s)$ of the complex s :

$$L(s) = \sum_{n=1}^{\infty} \frac{z_n}{n^s}.$$

- If $(z_n)_{n \geq 1}$ is a bounded, then the corresponding Dirichlet series converges absolutely on the open half-plane where $\Re(s) > 1$.
- If the set of sums $z_n + z_{n+1} + \dots + z_{n+k}$ for each n and $k \geq 0$ is bounded, then the corresponding Dirichlet series converges on the open half-plane where $\Re(s) > 0$.
- In general, if $z_n = O(n^k)$, the corresponding Dirichlet series converges absolutely in the half plane where $\Re(s) > k + 1$.

The function $L(s)$ is analytic on the corresponding open half plane[2-3].

To define Dirichlet L -functions we need to define Dirichlet characters. A function $\chi : \mathbb{Z} \mapsto \mathbb{C}$ is a Dirichlet character modulo q if it satisfies the following criteria:

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- (i) $\chi(n) \neq 0$ if $(n, q) = 1$.
- (ii) $\chi(n) = 0$ if $(n, q) > 1$.
- (iii) χ is periodic with period q : $\chi(n + q) = \chi(n)$ for all n .
- (iv) χ is multiplicative : $\chi(mn) = \chi(m)\chi(n)$ for all integers m and n .

The trivial character is the one with $\chi_0(n) = 1$ whenever $(n, q) = 1$.

Here are some known results for a Dirichlet character modulo q . For any integer n we have $\chi(1) = 1$. Also if $(n, q) = 1$, we have $(\chi(n))^{\phi(q)} = 1$ with ϕ is Euler's totient function. $\chi(n)$ is a $\phi(q)$ -th root of unity. Therefore, $|\chi(n)| = 1$ if $(n, q) = 1$, and $|\chi(n)| = 0$ if $(n, q) > 1$. Also, we recall the cancellation property for Dirichlet characters modulo q : For any n integer

$$\sum_{i=1}^q \chi(i+n) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (1)$$

The Dirichlet L -functions are simply the sum of the Dirichlet series. Let's χ be a Dirichlet character modulo q , The Dirichlet L -function $L(s, \chi)$ is defined for $\Re(s) > 1$ as the following:

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \quad (2)$$

They are a natural generalization of the Riemann zeta-function $\zeta(s)$ to an arithmetic progression and are a powerful tool in analytic number theory. The Dirichlet series, converges absolutely and uniformly in any bounded domain in the complex s -plane for which $\Re(s) \geq 1 + \gamma$, $\gamma > 0$. In the particular case of the trivial character χ_0 , $L(s, \chi_0)$ extends to a meromorphic function[2-5] in $\Re(s) > 0$ with the only pole at $s = 1$.

If χ is a non-trivial character, we have

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (3)$$

Since the sum in the integrand is bounded, this formula gives an analytic continuation of $L(s, \chi)$ to a regular function in the half-plane $\Re(s) > 0$. Also, like the zeta function, the Dirichlet L -functions have their Euler product[1-2]. For $\Re(s) > 1$:

$$L(s, \chi) = \prod_{p \text{ Prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (4)$$

Therefore, if $\chi = \chi_0$ is a trivial character mod q , we have, for $q = 1$,

$$L(s, \chi_0) = \zeta(s) \quad (5)$$

And for $q > 1$, we have,

$$L(s, \chi_0) = \zeta(s) \prod_{p/q} \left(1 - \frac{1}{p^s}\right) \quad (6)$$

For this reason the properties of $L(s, \chi_0)$ in the entire complex plane are determined by the properties of $\zeta(s)$.

Let's now q' be the smallest divisor (prime) of q . Let's χ' be the Dirichlet character $\chi' \pmod{q'}$. For any integer n such that $(n, q) = 1$, we have also $(n, q') = 1$ and $\chi(n) = \chi'(n)$. χ' is called primitive and $L(s, \chi)$ and $L(s, \chi')$ are related analytically such that if $\chi \neq \chi_0$:

$$L(s, \chi) = L(s, \chi') \prod_{p|q} \left(1 - \frac{\chi'(p)}{p^s}\right) \quad (7)$$

$L(s, \chi)$ and $L(s, \chi')$ have the same zeros in the critical strip $0 < \Re(s) < 1$.

Also, for a primitive character χ , (i.e. $\chi = \chi'$) $L(s, \chi)$ has the following functional equation:

$$\tau(\chi) \Gamma\left(\frac{1-s+a}{2}\right) L(1-s, \chi) = \sqrt{\pi} \left(\frac{q}{\pi}\right)^s i^a q^{\frac{1}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \bar{\chi}) \quad (8)$$

Where Γ is the Gamma function and $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$, and $\tau(\chi) = \sum_{k=1}^q \chi(k) \exp\left(\frac{2\pi ki}{q}\right)$.

When $\Re(s) > 1$ there is no zero for $L(s, \chi)$. When $\Re(s) < 0$, for a primitive character χ , we have the trivial zeros of $L(s, \chi)$: $s = a - 2k$, where k is a positive integer and a is defined above. For more details, please refer to the references[1-8].

2 The Generalized Riemann Hypothesis

The Generalized Riemann Hypothesis states that the Dirichlet L -functions have all their non-trivial zeros on the critical line $\Re(s) = \frac{1}{2}$.

We know that for any character χ modulo q , all non-trivial zeros of $L(s, \chi)$ lies in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. From the functional equation above we have that if:

- s_0 is a non-trivial zero of $L(s, \chi)$, then $1 - s_0$ is a zero of $L(s, \bar{\chi})$.
- s_0 is a non-trivial zero of $L(s, \bar{\chi})$, then $1 - s_0$ is a zero of $L(s, \chi)$.

Therefore, we just need to prove that for any character¹ χ modulo q , there is no non-trivial zeros of $L(s, \chi)$ in the right hand side of the critical strip $\{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$.

3 Proof of the Generalized Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $L(s, \chi) = 0$ where $L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$ and χ is a non-trivial Dirichlet character χ modulo q . When $q = 2$ there is only one Dirichlet character and it is

¹In fact, we just need to prove it for the primitive characters.

trivial. So here we are going to assume that $q \geq 3$.

We have from the integral equation (3) above:

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (9)$$

We have from the Riemann's integral in [1]:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (10)$$

To continue, we will need to prove the following lemma where we leverage on the RH proof in [1] to proof GRH.

Lemma 3.1. *Let's consider two variables σ and τ such that $\sigma > 0$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $L(s, \chi)$ be a L -function where χ is a character (not necessarily a Dirichlet character) such that:*

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx \quad (11)$$

With the function $x \rightarrow f(x)$ is bounded and the complex constant $\lambda \neq 0$. If $L(s, \chi) = 0$, then:

$$\sigma = \frac{1}{2} \quad (12)$$

Proof. To prove this lemma, we will use the same method and notations as in the lemmas 1.2 and 1.3 in [1]. Let's denote the following functions:

$$\epsilon(x) = \{x\} \quad (13)$$

$$\phi(x) = \{x\}(1 - \{x\}) \quad (14)$$

$$F(x) = \int_1^x f(t) dt \quad (15)$$

$$\psi_0(x) = \phi(x) \quad (16)$$

$$\psi_{n+1}(x) = \frac{1}{x} \int_0^x \psi_n(t) dt \quad (17)$$

$$\tilde{\psi}_0(x) = \frac{\psi_0(x)}{x^{1-2\sigma}} \quad (18)$$

$$\tilde{\psi}_{n+1}(x) = \frac{1}{x} \int_0^x \tilde{\psi}_n(t) dt \quad (19)$$

Since s is a $L(s, \chi)$ zero, therefore we can write:

$$\frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx = 0 \quad (20)$$

Therefore, since $s \neq 0$ and $\lambda \neq 0$, we have:

$$\int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx = \frac{1}{s-1} - \frac{1}{\lambda} \int_1^{+\infty} \frac{f(x)}{x^{s+1}} dx \quad (21)$$

We use integration by parts to get:

$$\int_1^{+\infty} \frac{\phi(x)}{x^{2+s}} dx = \frac{1}{s(1-s)} + \frac{2}{\lambda} \int_1^{+\infty} \frac{F(x)}{x^{2+s}} dx \quad (22)$$

We use the same technique used in lemma 1.2 lemma 1.3 in [1] to get the following results:

1. For each n :

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+s}} = \frac{1}{s2^n} + \frac{1}{(1-s)3^n} + \frac{2}{\lambda(s+2)^n} \int_1^{+\infty} \frac{F(x)}{x^{2+s}} dx \quad (23)$$

2. For each n :

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-s}} = \frac{1}{(1-s)2^n} + \frac{1}{s3^n} + \frac{2}{\lambda(3-s)^n} \int_1^{+\infty} \frac{F(x)}{x^{3-s}} dx \quad (24)$$

3. For each n :

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \frac{1}{(1-\bar{s})(1+2\sigma)^n} + \frac{1}{\bar{s}(2+2\sigma)^n} + \frac{2}{\lambda(s+2)^n} \int_1^{+\infty} \frac{F(x)}{x^{3-\bar{s}}} dx \quad (25)$$

4. For each n :

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-s}} = \frac{1}{\bar{s}(1+2\sigma)^n} + \frac{1}{(1-\bar{s})(2+2\sigma)^n} + \frac{2}{\lambda(3-s)^n} \int_1^{+\infty} \frac{F(x)}{x^{2+\bar{s}}} dx \quad (26)$$

Where \bar{s} is the complex conjugate of s .

5. For each n :

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{3-\sigma}} = \frac{1}{c_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{2+\sigma}} \quad (27)$$

Where $c_n > 1$ is a real number.

6. For each n :

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} = \frac{1}{d_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma}} \quad (28)$$

Where $d_n > 1$ is a real number.

And

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{3-\sigma}} = \int_1^{+\infty} dx \frac{\psi_n(x) \cos(\tau \ln(x))}{x^{3-\sigma}} + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\sigma}} \quad (29)$$

$$= \frac{1}{2} \left(\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-s}} + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\bar{s}}} \right) + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\sigma}} \quad (30)$$

Since \bar{s} is a zero of $L(s, \bar{\chi})$, therefore we can use the equation(24) to write:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{3-\sigma}} = \frac{1}{2} \left(\frac{1}{(1-s)2^n} + \frac{1}{s3^n} + \frac{1}{(1-\bar{s})2^n} + \frac{1}{\bar{s}3^n} + \frac{2}{\lambda(3-s)^n} \int_1^{+\infty} \frac{F(x)}{x^{3-s}} dx \right) \quad (31)$$

$$+ \frac{2}{\lambda(3-\bar{s})^n} \int_1^{+\infty} \frac{\bar{F}(x)}{x^{3-\bar{s}}} dx \Big) + \frac{1}{(1-\sigma)2^n} + \frac{1}{\sigma 3^n} + \frac{1}{(3-\sigma)^n} \left[U_0(1-\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (32)$$

$$= \frac{(1-\sigma)}{\|1-s\|^2 2^n} + \frac{1}{(1-\sigma)2^n} + \frac{\sigma}{\|s\|^2 3^n} + \frac{1}{\sigma 3^n} \quad (33)$$

$$+ \frac{A_n(1-s)}{\|3-s\|^n} + \frac{1}{(3-\sigma)^n} \left[U_0(1-\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (34)$$

Where $U_0(1-\sigma)$ is defined as follows:

$$U_0(1-\sigma) = \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{3-\sigma}} \quad (35)$$

$$\frac{A_n(1-s)}{\|3-s\|^n} = \frac{2}{\lambda(3-s)^n} \int_1^{+\infty} \frac{F(x)}{x^{3-s}} dx + \frac{2}{\lambda(3-\bar{s})^n} \int_1^{+\infty} \frac{\bar{F}(x)}{x^{3-\bar{s}}} dx \quad (36)$$

And the same

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{2+\sigma}} = \int_1^{+\infty} dx \frac{\psi_n(x) \cos(\tau \ln(x))}{x^{2+\sigma}} + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+\sigma}} \quad (37)$$

$$= \frac{1}{2} \left(\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+s}} + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+\bar{s}}} \right) + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+\sigma}} \quad (38)$$

Since \bar{s} is a zero of $L(s, \bar{\chi})$, therefore we can use the equation (23) to write:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{2+\sigma}} = \frac{1}{2} \left(\frac{1}{s2^n} + \frac{1}{(1-s)3^n} + \frac{1}{\bar{s}2^n} + \frac{1}{(1-\bar{s})3^n} + \frac{2}{\lambda(2+s)^n} \int_1^{+\infty} \frac{F(x)}{x^{2+s}} dx \right) \quad (39)$$

$$+ \frac{2}{\lambda(2+\bar{s})^n} \int_1^{+\infty} \frac{\bar{F}(x)}{x^{2+\bar{s}}} dx \Big) + \frac{1}{\sigma 2^n} + \frac{1}{(1-\sigma)3^n} + \frac{1}{(2+\sigma)^n} \left[U_0(\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (40)$$

$$= \frac{\sigma}{\|s\|^2 2^n} + \frac{1}{\sigma 2^n} + \frac{1-\sigma}{\|1-s\|^2 3^n} + \frac{1}{\sigma 3^n} \quad (41)$$

$$+ \frac{A_n(s)}{\|2+s\|^n} + \frac{1}{(2+\sigma)^n} \left[U_0(\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (42)$$

Therefore, from the equations (27), (33-34) and (41-42), we conclude:

$$\frac{1}{c_n^{1-2\sigma}} \sim_{n \rightarrow +\infty} \frac{\frac{(1-\sigma)}{\|1-s\|^2} + \frac{1}{(1-\sigma)}}{\frac{\sigma}{\|s\|^2} + \frac{1}{\sigma}} \quad (43)$$

Since $c_n > 1$ for each n , we conclude that:

$$\frac{\frac{(1-\sigma)}{\|1-s\|^2} + \frac{1}{(1-\sigma)}}{\frac{\sigma}{\|s\|^2} + \frac{1}{\sigma}} \leq 1 \quad (44)$$

And we do the same for $\tilde{\psi}_n$. From the equations (25), (26) and (28), we can write the following for $n \geq 1$

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} = \frac{1}{d_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma}} \quad (45)$$

Where $d_n > 1$ is a real number.

And

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} = \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x) \cos(\tau \ln(x))}{x^{3-\sigma}} + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-\sigma}} \quad (46)$$

$$= \frac{1}{2} \left(\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-s}} + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-\bar{s}}} \right) + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-\sigma}} \quad (47)$$

Since \bar{s} is a zero of $L(s, \bar{\chi})$, therefore we can use the equation (26) to write:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} = \frac{1}{2} \left(\frac{1}{\bar{s}(1+2\sigma)^n} + \frac{1}{(1-\bar{s})(2+2\sigma)^n} + \frac{1}{s(1+2\sigma)^n} + \frac{1}{(1-s)(2+2\sigma)^n} \right) \quad (48)$$

$$+ \frac{2}{\lambda(3-s)^n} \int_1^{+\infty} \frac{F(x)}{x^{2+\bar{s}}} dx + \frac{2}{\lambda(3-\bar{s})^n} \int_1^{+\infty} \frac{\bar{F}(x)}{x^{2+s}} dx \quad (49)$$

$$+ \frac{1}{\sigma(1+2\sigma)^n} + \frac{1}{(1-\sigma)(2+2\sigma)^n} + \frac{1}{(3-\sigma)^n} \left[\tilde{U}_0(1-\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (50)$$

$$= \frac{\sigma}{\|s\|^2(1+2\sigma)^n} + \frac{1}{\sigma(1+2\sigma)^n} + \frac{1-\sigma}{\|1-s\|^2(2+2\sigma)^n} + \frac{1}{(1-\sigma)(2+2\sigma)^n} \quad (51)$$

$$+ \frac{\tilde{A}_n(1-s)}{\|3-s\|^n} + \frac{1}{(3-\sigma)^n} \left[\tilde{U}_0(1-\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (52)$$

Where $\tilde{U}_0(1-\sigma)$ is defined as follows:

$$\tilde{U}_0(1-\sigma) = \int_1^{+\infty} dt \frac{\tilde{\psi}_0(t)}{t^{3-\sigma}} \quad (53)$$

$$\frac{\tilde{A}_n(1-s)}{\|3-s\|^n} = \frac{2}{\lambda(3-s)^n} \int_1^{+\infty} \frac{F(x)}{x^{2+\bar{s}}} dx + \frac{2}{\lambda(3-\bar{s})^n} \int_1^{+\infty} \frac{\bar{F}(x)}{x^{2+s}} dx \quad (54)$$

And the same

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma}} = \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x) \cos(\tau \ln(x))}{x^{2+\sigma}} + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+\sigma}} \quad (55)$$

$$= \frac{1}{2} \left(\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+\bar{s}}} \right) + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+\sigma}} \quad (56)$$

Since \bar{s} is a zero of $L(s, \bar{\chi})$, therefore we can use the equation (25) to write:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\pi}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma}} = \frac{1}{2} \left(\frac{1}{(1-\bar{s})(1+2\sigma)^n} + \frac{1}{\bar{s}(2+2\sigma)^n} + \frac{1}{(1-s)(1+2\sigma)^n} + \frac{1}{s(2+2\sigma)^n} \right) \quad (57)$$

$$+ \frac{2}{\lambda(2+s)^n} \int_1^{+\infty} \frac{F(x)}{x^{3-\bar{s}}} dx + \frac{2}{\lambda(2+\bar{s})^n} \int_1^{+\infty} \frac{\bar{F}(x)}{x^{3-s}} dx \quad (58)$$

$$+ \frac{1}{(1-\sigma)(1+2\sigma)^n} + \frac{1}{\sigma(2+2\sigma)^n} + \frac{1}{(2+\sigma)^n} \left[\tilde{U}_0(\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (59)$$

$$= \frac{1-\sigma}{\|1-s\|^2(1+2\sigma)^n} + \frac{1}{(1-\sigma)(1+2\sigma)^n} + \frac{\sigma}{\|s\|^2(2+2\sigma)^n} + \frac{1}{\sigma(2+2\sigma)^n} \quad (60)$$

$$+ \frac{\tilde{A}_n(s)}{\|2+s\|^n} + \frac{1}{(2+\sigma)^n} \left[\tilde{U}_0(\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (61)$$

Where $\tilde{U}_0(\sigma)$ is defined as follows:

$$\tilde{U}_0(\sigma) = \int_1^{+\infty} dt \frac{\tilde{\psi}_0(t)}{t^{2+\sigma}} \quad (62)$$

Therefore, from the equations (45), (51-52) and (60-61), we conclude:

$$\frac{1}{d_n^{1-2\sigma}} \sim_{n \rightarrow +\infty} \frac{\frac{\sigma}{\|s\|^2} + \frac{1}{\sigma}}{\frac{(1-\sigma)}{\|1-s\|^2} + \frac{1}{(1-\sigma)}} \quad (63)$$

Since $d_n > 1$ for each n , we conclude that:

$$\frac{\frac{\sigma}{\|s\|^2} + \frac{1}{\sigma}}{\frac{(1-\sigma)}{\|1-s\|^2} + \frac{1}{(1-\sigma)}} \leq 1 \quad (64)$$

From the equations (44) and (64), we conclude that:

$$\frac{\sigma}{\|s\|^2} + \frac{1}{\sigma} = \frac{(1-\sigma)}{\|1-s\|^2} + \frac{1}{(1-\sigma)} \quad (65)$$

Since the function $g(x) \rightarrow \frac{x}{x^2+\tau^2} + \frac{1}{x}$ is strictly decreasing since its first derivative is strictly negative: for each $x > 0$:

$$g'(x) = \frac{-2x^4 - \tau^4 - x^2\tau^2}{x^2(x^2 + \tau^2)^2} < 0 \quad (66)$$

$$\sigma = 1 - \sigma = \frac{1}{2} \quad (67)$$

This ends the proof of our lemma. ■

□

From the lemma 3.1 above we can conclude the following corollary.

Corollary 3.1.1. *Let's consider two variables σ and τ such that $\sigma > 0$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $q > 1$ be an positive integer. Let's $L(s, \chi)$ be a L -function where χ is a Dirichlet character mod q with no cancellation property such that:*

$$\sum_{n=1}^q \chi(n) \neq 0 \quad (68)$$

If $L(s, \chi) = 0$, then:

$$\sigma = \frac{1}{2} \quad (69)$$

Proof. Let's take $x > 1$ a real number. Let's denote $n_x = [x]$ be the integer part of x . We have $n_x = x - \{x\}$. We denote q_x and r_x the quotient and the rest of the Euclidean division of n_x by q as follows: $n_x = qq_x + r_x$. Therefore we can write:

$$\sum_{n=1}^x \chi(n) = \sum_{n=0}^{q_x-1} \sum_{k=1}^q \chi(nq+k) + \sum_{k=1}^{r_x} \chi(nq+k) \quad (70)$$

$$= q_x \sum_{n=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (71)$$

$$= \frac{x - \{x\} - r_x}{q} \sum_{k=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (72)$$

$$= \frac{x}{q} \sum_{k=1}^q \chi(n) - \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (73)$$

Let's define the constant λ and the function g as follows:

$$\lambda = \frac{\sum_{k=1}^q \chi(n)}{q} \quad (74)$$

$$g(x) = \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi(n) - \sum_{k=1}^{r_x} \chi(k) \quad (75)$$

$$= \lambda\{x\} + \underbrace{\lambda r_x - \sum_{k=1}^{r_x} \chi(k)}_{f(x)} \quad (76)$$

It is clear that the function $f(x)$ is bounded. And by assumption we have $\lambda \neq 0$. Therefore, we can calculate $L(s, \chi)$ as follows:

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (77)$$

$$= s \int_1^{+\infty} \frac{\lambda x - g(x)}{x^{s+1}} dx \quad (78)$$

$$= s \int_1^{+\infty} \frac{\lambda}{x^s} dx - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (79)$$

$$(80)$$

If $\Re(s) > 1$, we can conclude:

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (81)$$

We can extend the equation (80) over the region $\Re(s) > 0$. $L(s, \chi)$ has a holomorphic continuation as a function of s to the whole complex plane (see [3, 6, 7, 8]). Therefore, we apply the lemma 3.1 to prove that $\sigma = \frac{1}{2}$. \blacksquare \square

Remark. As a direct result of the above corollary, we can see that GRH is true for the series $(\sum_{n \geq 1} \frac{1}{(qn+p)^\sigma})$ with $p > 0$ and $q > 1$ are integers.

Lemma 3.2. Let's consider two variables σ and τ such that $\sigma > 0$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $q > 1$ be an positive integer. Let's $L(s, \chi)$ be a L -function where χ is a Dirichlet character mod q with cancellation property such that:

$$\sum_{n=1}^q \chi(n) = 0 \quad (82)$$

Therefore:

$$L(s, \chi) = \left(\frac{1-q^{1-s}}{q}\right) \frac{s}{s-1} - s \int_1^{+\infty} \frac{\left(\frac{1-q^{1-s}}{q}\right)\{x\} + f(x)}{x^{s+1}} dx \quad (83)$$

Where f is a bounded function.

Proof. We will use the zeta function $\zeta(s)$ to prove our case.

Let's now write the following for $s > 1$:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (84)$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^q \frac{1}{(qn+i)^s} \quad (85)$$

$$= \sum_{i=1}^q \sum_{n=0}^{\infty} \frac{1}{(qn+i)^s} \quad (86)$$

Therefore

$$\left(1 - \frac{1}{q^s}\right)\zeta(s) = \sum_{i=1}^{q-1} \sum_{n=0}^{\infty} \frac{1}{(qn+i)^s} \quad (87)$$

Also

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \quad (88)$$

$$= \sum_{n=0}^{+\infty} \sum_{i=1}^q \frac{\chi(i)}{(qn+i)^s} \quad (89)$$

$$= \sum_{i=1}^q \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (90)$$

Therefore since $\chi(q) = 0$

$$L(s, \chi) = \sum_{i=1}^{q-1} \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (91)$$

Since $\sum_{n=1}^q \chi(n) = 0$ and $\chi(1) = 1$, we can write the following:

$$\sum_{n=0}^{+\infty} \frac{1}{(qn+1)^s} = L(s, \chi) - \sum_{i=2}^{q-1} \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (92)$$

$$\sum_{n=0}^{+\infty} \frac{1}{(qn+1)^s} = \left(1 - \frac{1}{q^s}\right) \zeta(s) - \sum_{i=2}^{q-1} \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (93)$$

Therefore

$$\left(1 - \frac{1}{q^s}\right) \zeta(s) - L(s, \chi) = \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (94)$$

Let's denote $L(s, i)_{2 \leq i \leq q-1}$ as follows:

$$L(s, i) = \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (95)$$

We can write also:

$$L(s, i) = s \int_1^{+\infty} \frac{\sum_{n=1}^{\lfloor x \rfloor} \chi_i(n)}{x^{s+1}} dx \quad (96)$$

Where χ_i is the character of $L(s, i)$:

- (i) $\chi_i(qn+i) = 1$
- (ii) $\chi_i(qn+j) = 0$ if $j \neq i$

We use the equation (90-94) to write:

$$\lambda_i = \frac{\sum_{k=1}^q \chi_i(k)}{q} = \frac{1}{q} \quad (97)$$

$$f_i(x) = \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi_i(k) - \sum_{k=1}^{r_x} \chi_i(k) \quad (98)$$

$$= \frac{(\{x\} + r_x)}{q} - \sum_{k=1}^{r_x} \chi_i(k) \quad (99)$$

Therefore from the equation (96):

$$L(s, i) = \frac{s}{q(s-1)} - s \int_1^{+\infty} \frac{f_i(x)}{x^{s+1}} dx \quad (100)$$

Therefore, we can write the equation (94) as follows:

$$\left(1 - \frac{1}{q^s}\right) \zeta(s) - L(s, \chi) = \sum_{i=2}^{q-1} (1 - \chi(i)) L(s, i) \quad (101)$$

$$= \frac{s}{q(s-1)} \sum_{i=2}^{q-1} (1 - \chi(i)) - s \sum_{i=2}^{q-1} (1 - \chi(i)) \int_1^{+\infty} \frac{f_i(x)}{x^{s+1}} dx \quad (102)$$

Thanks to the cancellation property, we can conclude that:

$$\left(1 - \frac{1}{q^s}\right)\zeta(s) - L(s, \chi) = \frac{s(q-1)}{q(s-1)} - s \int_1^{+\infty} \frac{\sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x)}{x^{s+1}} dx \quad (103)$$

From the equation (10) we replace the $\zeta(s)$ by its L - function:

$$L(s, \chi) = \frac{1 - q^{1-s}}{q} \frac{s}{(s-1)} - s \int_1^{+\infty} \frac{(1 - q^{-s})\{x\} - \sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x)}{x^{s+1}} dx \quad (104)$$

Let's define the function $g(x)$ as follows:

$$g(x) = (1 - q^{-s})\{x\} - \sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x) \quad (105)$$

$$= \frac{1 - q^{1-s}}{q} \{x\} - \left(1 - \frac{1}{q}\right) r_x + \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{k=1}^{r_x} \chi_i(k) \quad (106)$$

$$= \frac{1 - q^{1-s}}{q} \{x\} + f(x) \quad (107)$$

Where f is unrelated to s , bounded and defined as follows:

$$f(x) = -\left(1 - \frac{1}{q}\right) r_x + \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{k=1}^{r_x} \chi_i(k) \quad (108)$$

$$= \frac{r_x}{q} - \sum_{k=1}^{r_x} \chi(k) \quad (109)$$

And

$$L(s, \chi) = \left(\frac{1 - q^{1-s}}{q}\right) \frac{s}{s-1} - s \int_1^{+\infty} \frac{\left(\frac{1 - q^{1-s}}{q}\right)\{x\} + f(x)}{x^{s+1}} dx \quad (110)$$

□

Lemma 3.3. *Let's consider two variables σ and τ such that $0 < \sigma < 1$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $q > 1$ be an positive integer. Let's $L(s, \chi)$ be a L -function where χ is a Dirichlet character mod q with cancellation property such that:*

$$\sum_{n=1}^q \chi(n) = 0 \quad (111)$$

Let's suppose that $s = \sigma + i\tau$ is a zero for $L(s, \chi)$. i.e $L(s, \chi) = 0$. Therefore

$$\sigma = \frac{1}{2} \quad (112)$$

And the Generalized Riemann's Hypothesis is true.

Proof. The proof is straightforward as we can use the lemmas 3.1 and 3.2. From lemma 3.2 we can write the following:

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (113)$$

$$\lambda = \frac{1 - q^{1-s}}{q} \quad (114)$$

Where f is a bounded function. Then, we apply the lemma 3.1 to conclude that

$$\sigma = \frac{1}{2} \quad (115)$$

And the Generalized Riemann's Hypothesis is true.

■

□

3.1 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore, the Generalized Riemann Hypothesis (GRH) is true: *The non-trivial zeros of a $L - Dirichlet$ function have real part equal to $\frac{1}{2}$.* With this, we can also conclude that $P = NP$.

Acknowledgments

I would like to thank Farhat Latrach, Giampiero Esposito, Jacques Gélinas, Michael Milgram, Léo Agélas, Ronald F. Fox, Kim Y.G, Masumi Nakajima, Maksym Radziwill and Shekhar Suman for thoughtful comments and discussions on my paper versions on the RH. All errors are mine.

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