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A Simple Proof of the Generalized Riemann's Hypothesis

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Abstract

This article is the continuation of the previous RH proof[1]. We present a simple proof of the Generalized Riemann's Hypothesis (GRH) where we apply the same method we used to prove the RH.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 Dirichlet L -functions

Let's $(z_n)_{n \geq 1}$ be a sequence of complex numbers. A Dirichlet series[3] is a series of the form $\sum_{n=1}^{\infty} \frac{z_n}{n^s}$, where $s = \sigma + i\tau$ is complex. The zeta function is a Dirichlet series. Let's define the function $L(s)$ of the complex s :

$$L(s) = \sum_{n=1}^{\infty} \frac{z_n}{n^s}.$$

- If $(z_n)_{n \geq 1}$ is a bounded, then the corresponding Dirichlet series converges absolutely on the open half-plane where $\Re(s) > 1$.
- If the set of sums $z_n + z_{n+1} + \dots + z_{n+k}$ for each n and $k \geq 0$ is bounded, then the corresponding Dirichlet series converges on the open half-plane where $\Re(s) > 0$.
- In general, if $z_n = O(n^k)$, the corresponding Dirichlet series converges absolutely in the half plane where $\Re(s) > k + 1$.

The function $L(s)$ is analytic on the corresponding open half plane[2-3].

To define Dirichlet L -functions we need to define Dirichlet characters. A function $\chi : \mathbb{Z} \mapsto \mathbb{C}$ is a Dirichlet character modulo q if it satisfies the following criteria:

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- (i) $\chi(n) \neq 0$ if $(n, q) = 1$.
- (ii) $\chi(n) = 0$ if $(n, q) > 1$.
- (iii) χ is periodic with period q : $\chi(n + q) = \chi(n)$ for all n .
- (iv) χ is multiplicative : $\chi(mn) = \chi(m)\chi(n)$ for all integers m and n .

The trivial character is the one with $\chi_0(n) = 1$ whenever $(n, q) = 1$.

Here are some known results for a Dirichlet character modulo q . For any integer n we have $\chi(1) = 1$. Also if $(n, q) = 1$, we have $(\chi(n))^{\phi(q)} = 1$ with ϕ is Euler's totient function. $\chi(n)$ is a $\phi(q)$ -th root of unity. Therefore, $|\chi(n)| = 1$ if $(n, q) = 1$, and $|\chi(n)| = 0$ if $(n, q) > 1$. Also, we recall the cancellation property for Dirichlet characters modulo q : For any n integer

$$\sum_{i=1}^q \chi(i+n) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (1)$$

The Dirichlet L -functions are simply the sum of the Dirichlet series. Let's χ be a Dirichlet character modulo q , The Dirichlet L -function $L(s, \chi)$ is defined for $\Re(s) > 1$ as the following:

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \quad (2)$$

They are a natural generalization of the Riemann zeta-function $\zeta(s)$ to an arithmetic progression and are a powerful tool in analytic number theory. The Dirichlet series, converges absolutely and uniformly in any bounded domain in the complex s -plane for which $\Re(s) \geq 1 + \gamma$, $\gamma > 0$. In the particular case of the trivial character χ_0 , $L(s, \chi_0)$ extends to a meromorphic function[2-5] in $\Re(s) > 0$ with the only pole at $s = 1$.

If χ is a non-trivial character, we have

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (3)$$

Since the sum in the integrand is bounded, this formula gives an analytic continuation of $L(s, \chi)$ to a regular function in the half-plane $\Re(s) > 0$. Also, like the zeta function, the Dirichlet L -functions have their Euler product[1-2]. For $\Re(s) > 1$:

$$L(s, \chi) = \prod_{p \text{ Prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (4)$$

Therefore, if $\chi = \chi_0$ is a trivial character mod q , we have, for $q = 1$,

$$L(s, \chi_0) = \zeta(s) \quad (5)$$

And for $q > 1$, we have,

$$L(s, \chi_0) = \zeta(s) \prod_{p/q} \left(1 - \frac{1}{p^s}\right) \quad (6)$$

For this reason the properties of $L(s, \chi_0)$ in the entire complex plane are determined by the properties of $\zeta(s)$.

Let's now q' be the smallest divisor (prime) of q . Let's χ' be the Dirichlet character $\chi' \pmod{q'}$. For any integer n such that $(n, q) = 1$, we have also $(n, q') = 1$ and $\chi(n) = \chi'(n)$. χ' is called primitive and $L(s, \chi)$ and $L(s, \chi')$ are related analytically such that if $\chi \neq \chi_0$:

$$L(s, \chi) = L(s, \chi') \prod_{p|q} \left(1 - \frac{\chi'(p)}{p^s}\right) \quad (7)$$

$L(s, \chi)$ and $L(s, \chi')$ have the same zeros in the critical strip $0 < \Re(s) < 1$.

Also, for a primitive character χ , (i.e. $\chi = \chi'$) $L(s, \chi)$ has the following functional equation:

$$\tau(\chi) \Gamma\left(\frac{1-s+a}{2}\right) L(1-s, \chi) = \sqrt{\pi} \left(\frac{q}{\pi}\right)^s i^a q^{\frac{1}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \bar{\chi}) \quad (8)$$

Where Γ is the Gamma function and $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$, and $\tau(\chi) = \sum_{k=1}^q \chi(k) \exp\left(\frac{2\pi ki}{q}\right)$.

When $\Re(s) > 1$ there is no zero for $L(s, \chi)$. When $\Re(s) < 0$, for a primitive character χ , we have the trivial zeros of $L(s, \chi)$: $s = a - 2k$, where k is a positive integer and a is defined above. For more details, please refer to the references[1-8].

2 The Generalized Riemann Hypothesis

The Generalized Riemann Hypothesis states that the Dirichlet L -functions have all their non-trivial zeros on the critical line $\Re(s) = \frac{1}{2}$.

We know that for any character χ modulo q , all non-trivial zeros of $L(s, \chi)$ lies in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. From the functional equation above we have that if:

- s_0 is a non-trivial zero of $L(s, \chi)$, then $1 - s_0$ is a zero of $L(s, \bar{\chi})$.
- s_0 is a non-trivial zero of $L(s, \bar{\chi})$, then $1 - s_0$ is a zero of $L(s, \chi)$.

Therefore, we just need to prove that for any character¹ χ modulo q , there is no non-trivial zeros of $L(s, \chi)$ in the right hand side of the critical strip $\{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$.

3 Proof of the Generalized Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $L(s, \chi) = 0$ where $L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$ and χ is a non-trivial Dirichlet character χ modulo q . When $q = 2$ there is only one Dirichlet character and it is

¹In fact, we just need to prove it for the primitive characters.

trivial. So here we are going to assume that $q \geq 3$.

We have from the integral equation (3) above:

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (9)$$

We have from the Riemann's integral in [1]:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (10)$$

To continue, we will need to prove the following lemma where we leverage on the RH proof in [1] to proof GRH.

Lemma 3.1. *Let's consider two variables σ and τ such that $\sigma > 0$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $L(s, \chi)$ be a L -function where χ is a character (not necessarily a Dirichlet character) such that:*

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx \quad (11)$$

With the function $x \rightarrow f(x)$ is bounded and the complex constant $\lambda \neq 0$. If $L(s, \chi) = 0$, then:

$$\sigma = \frac{1}{2} \quad (12)$$

Proof. To prove this lemma, we will use the same method and notations as in the lemmas 1.2 and 1.4 in [1]. Let's denote the following functions:

$$\epsilon(x) = \{x\} \quad (13)$$

$$\phi(x) = \{x\}(1 - \{x\}) \quad (14)$$

$$F(x) = \int_1^x f(t) dt \quad (15)$$

$$\psi_0(x) = \phi(x) \quad (16)$$

$$\psi_{n+1}(x) = \frac{1}{x} \int_0^x \psi_n(t) dt \quad (17)$$

$$\tilde{\psi}_0(x) = \frac{\psi_0(x)}{x^{1-2\sigma}} \quad (18)$$

$$\tilde{\psi}_{n+1}(x) = \frac{1}{x} \int_0^x \tilde{\psi}_n(t) dt \quad (19)$$

Since s is a $L(s, \chi)$ zero, therefore we can write:

$$\frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{(\lambda\{x\} + f(x))}{x^{s+1}} dx = 0 \quad (20)$$

Therefore, since $s \neq 0$ and $\lambda \neq 0$, we have:

$$\int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx = \frac{1}{s-1} - \frac{1}{\lambda} \int_1^{+\infty} \frac{f(x)}{x^{s+1}} dx \quad (21)$$

We use integration by parts to get:

$$\int_1^{+\infty} \frac{\phi(x)}{x^{2+s}} dx = \frac{1}{s(1-s)} + \frac{2}{\lambda} \int_1^{+\infty} \frac{F(x)}{x^{2+s}} dx \quad (22)$$

We use the same technique used in lemma 1.2 lemma 1.4 in [1] to get the following results:

1. For each n :

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+s}} = \frac{1}{s2^n} + \frac{1}{(1-s)3^n} + \frac{2}{\lambda(s+2)^n} \int_1^{+\infty} \frac{F(x)}{x^{2+s}} dx \quad (23)$$

2. For each n :

$$\frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+s}} = \frac{1}{s(1-s)(2+s)^n} - \frac{2}{\lambda(s+2)^n} \int_1^{+\infty} \frac{F(x)}{x^{2+s}} dx \quad (24)$$

3. For each n :

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \frac{1}{(1-\bar{s})(1+2\sigma)^n} + \frac{1}{\bar{s}(2+2\sigma)^n} + \frac{2}{\lambda(s+2)^n} \int_1^{+\infty} \frac{F(x)}{x^{3-\bar{s}}} dx \quad (25)$$

4. For each n :

$$\frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_1^x dt \tilde{\psi}_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+s}} = \frac{1}{\bar{s}(1-\bar{s})(2+s)^n} - \frac{2}{\lambda(s+2)^n} \int_1^{+\infty} \frac{F(x)}{x^{3-\bar{s}}} dx \quad (26)$$

Where \bar{s} is the complex conjugate of s .

5. For each n :

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x)) \int_1^x dt \tilde{\psi}_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} = \frac{1}{c_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2\left(\frac{\tau}{2} \ln(x)\right) \int_1^x dt \psi_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (27)$$

$$- \frac{1}{c^{1-2\sigma}} \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (28)$$

Where $c_n > 1$ and $c > 1$ are real numbers.

Let's write the following:

$$\frac{1}{s(1-s)} = \frac{e^{-i\alpha}}{\|s(1-s)\|} \quad (29)$$

$$\frac{1}{2+s} = \frac{e^{-i\beta}}{\|2+s\|} \quad (30)$$

$$\frac{2}{\lambda} \int_1^{+\infty} \frac{F(x)}{x^{2+s}} dx = \left\| \frac{2}{\lambda} \int_1^{+\infty} \frac{F(x)}{x^{2+s}} dx \right\| e^{i\gamma_1} = \lambda_1 e^{i\gamma_1} \quad (31)$$

$$\frac{2}{\lambda} \int_1^{+\infty} \frac{F(x)}{x^{3-\bar{s}}} dx = \left\| \frac{2}{\lambda} \int_1^{+\infty} \frac{F(x)}{x^{3-\bar{s}}} dx \right\| e^{i\gamma_2} = \lambda_2 e^{i\gamma_2} \quad (32)$$

Where $\alpha \in (0, \pi)$ and $\beta \in (0, \pi)$. Without loss of generality, we can assume α and β are in $(0, \frac{\pi}{2})$ since $0 < \sigma < \frac{1}{2}$ and $\tau > 0$.

Therefore, we can write:

$$\frac{1}{(2+s)^n} = \frac{e^{-i\beta n}}{\|2+s\|^n} \quad (33)$$

Therefore, when we combine the equations (23-28) and get the following:

$$\begin{aligned} X_n &= \frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{2 \cos^2\left(\frac{\tau}{2} \ln(x)\right) \int_1^x dt \psi_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (34) \\ &= \frac{1}{\|s(1-s)\| \|2+s\|^n} \frac{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c^{1-2\sigma}} - \|s(1-1)\| \left(\lambda_2 \cos(\gamma_2 - \beta n) - \frac{\lambda_1 \cos(\gamma_1 - \beta n)}{c^{1-2\sigma}} \right)}{\frac{1}{c_n^{1-2\sigma}} - \frac{1}{c^{1-2\sigma}}} \quad (35) \end{aligned}$$

And

$$\begin{aligned} Y_n &= \frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (36) \\ &= \frac{1}{\|s(1-s)\| \|2+s\|^n} \frac{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c_n^{1-2\sigma}} - \|s(1-1)\| \left(\lambda_2 \cos(\gamma_2 - \beta n) - \frac{\lambda_1 \cos(\gamma_1 - \beta n)}{c_n^{1-2\sigma}} \right)}{\frac{1}{c_n^{1-2\sigma}} - \frac{1}{c^{1-2\sigma}}} \quad (37) \end{aligned}$$

We can also calculate X_n and Y_n in another way as follows:

$$X_n = \frac{U_0}{(2+\sigma)^n} + \frac{\cos(\alpha + \beta n)}{\|s(1-s)\| \|2+s\|^n} - \frac{\lambda_1 \cos(\gamma_1 - \beta n)}{\|2+s\|^n} \quad (38)$$

And

$$Y_n = \frac{U_0}{(2+\sigma)^n} \quad (39)$$

Where

$$U_0 = \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{2+\sigma}} \quad (40)$$

We have from the equations (34-37):

$$\frac{X_n}{Y_n} = 1 + \frac{(2+\sigma)^n \cos(\alpha + \beta n)}{U_0 \|s(1-s)\| \|2+s\|^n} - \frac{\lambda_1 \cos(\gamma_1 - \beta n) (2+\sigma)^n}{U_0 \|2+s\|^n} \quad (41)$$

$$\begin{aligned} &= \frac{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c^{1-2\sigma}} - \|s(1-1)\| \left(\lambda_2 \cos(\gamma_2 - \beta n) - \frac{\lambda_1 \cos(\gamma_1 - \beta n)}{c^{1-2\sigma}} \right)}{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c_n^{1-2\sigma}} - \|s(1-1)\| \left(\lambda_2 \cos(\gamma_2 - \beta n) - \frac{\lambda_1 \cos(\gamma_1 - \beta n)}{c_n^{1-2\sigma}} \right)} \quad (42) \end{aligned}$$

$$= \frac{A \cos(\beta n) + B \sin(\beta n)}{A_n \cos(\beta n) + B_n \sin(\beta n)} \quad (43)$$

Where

$$A_n = \cos(\alpha) \left(1 - \frac{1}{c_n^{1-2\sigma}}\right) - \|s(1-s)\| \left(\lambda_2 \cos(\gamma_2) - \frac{\lambda_1 \cos(\gamma_1)}{c_n^{1-2\sigma}}\right) \quad (44)$$

$$B_n = \sin(\alpha) \left(1 + \frac{1}{c_n^{1-2\sigma}}\right) - \|s(1-s)\| \left(\lambda_2 \sin(\gamma_2) - \frac{\lambda_1 \sin(\gamma_1)}{c_n^{1-2\sigma}}\right) \quad (45)$$

Where the sequences (A_n) and (B_n) converge respectively to A and B defined as follows:

$$A = \cos(\alpha) \left(1 - \frac{1}{c^{1-2\sigma}}\right) - \|s(1-s)\| \left(\lambda_2 \cos(\gamma_2) - \frac{\lambda_1 \cos(\gamma_1)}{c^{1-2\sigma}}\right) \quad (46)$$

$$B = \sin(\alpha) \left(1 + \frac{1}{c^{1-2\sigma}}\right) - \|s(1-s)\| \left(\lambda_2 \sin(\gamma_2) - \frac{\lambda_1 \sin(\gamma_1)}{c^{1-2\sigma}}\right) \quad (47)$$

The sequence $(\frac{1}{c_n})_{n \geq 1}$ is bounded. From the equations (35) and (38), we deduce that the limit of $(\frac{1}{c_n})_{n \geq 1}$ is $\frac{1}{c} < 1$. We will distinguish between two cases.

1. Case 1: There exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$: $c_n = c$. Therefore, from the equation (35), we can deduce that for $n \geq n_0$:

$$A \cos(\beta n) + B \sin(\beta n) = 0 \quad (48)$$

Therefore

$$A = B = 0 \quad (49)$$

Therefore

$$\cos(\alpha) \left(1 - \frac{1}{c^{1-2\sigma}}\right) = \|s(1-s)\| \left(\lambda_2 \cos(\gamma_2) - \frac{\lambda_1 \cos(\gamma_1)}{c^{1-2\sigma}}\right) \quad (50)$$

$$\sin(\alpha) \left(1 + \frac{1}{c^{1-2\sigma}}\right) = \|s(1-s)\| \left(\lambda_2 \sin(\gamma_2) - \frac{\lambda_1 \sin(\gamma_1)}{c^{1-2\sigma}}\right) \quad (51)$$

From the equations (41) and (43) we can write:

$$\cos(\alpha + \beta n) = \lambda_1 \|s(1-s)\| \cos(\gamma_1 - \beta n) \quad (52)$$

Therefore

$$\cos(\alpha) = \lambda_1 \|s(1-s)\| \cos(\gamma_1) \quad (53)$$

$$\sin(\alpha) = -\lambda_1 \|s(1-s)\| \sin(\gamma_1) \quad (54)$$

Therefore

$$\cos^2(\alpha) + \sin^2(\alpha) = \lambda_1^2 \|s(1-s)\|^2 \cos^2(\gamma_1) + \lambda_1^2 \|s(1-s)\|^2 \sin^2(\gamma_1) = 1 \quad (55)$$

Therefore

$$\lambda_1 = \frac{1}{\|s(1-s)\|} \quad (56)$$

Therefore by taking into account the equations (50-51)

$$\cos(\alpha) = \lambda_2 \|s(1-s)\| \cos(\gamma_2) \quad (57)$$

$$\sin(\alpha) = \lambda_2 \|s(1-s)\| \sin(\gamma_2) \quad (58)$$

And the same

$$\lambda_2 = \frac{1}{\|s(1-s)\|} \quad (59)$$

Therefore

$$\cos(\alpha) = \cos(\gamma_1) = \cos(\gamma_2) \quad (60)$$

$$\sin(\alpha) = -\sin(\gamma_1) = \sin(\gamma_2) \quad (61)$$

Therefore

$$\gamma_1 = (k + k')\pi \quad (62)$$

$$\gamma_2 = (k - k')\pi \quad (63)$$

Where $k, k' \in \mathbb{Z}$. Therefore

$$\sin(\alpha) = 0 \quad (64)$$

Which is a contradiction since $\alpha \in (0, \frac{\pi}{2})$.

2. Case 2: For each $n_0 \in \mathbb{N}$ there exists $n \geq n_0$: $c_n \neq c$.

We have from the equations (41-43):

$$\frac{X_n}{Y_n} = \frac{A \cos(\beta n) + B \sin(\beta n)}{A_n \cos(\beta n) + B_n \sin(\beta n)} \quad (65)$$

The case where $A = B = 0$ is similar to the previous case and it is not possible since $X_n > 0$. So, let assume that $AB \neq 0$. And since the sequence (c_n) terms and c are different, therefore the sequence (A_n) terms are different from A and the sequence (B_n) terms are different from B , similarly like in the lemma 1.4 in [1], we can find n_0 such that $A_{n_0} \cos(\beta n_0) + B_{n_0} \sin(\beta n_0) > 0$ and $A \cos(\beta n_0) + B \sin(\beta n_0) < 0$ or $A_{n_0} \cos(\beta n_0) + B_{n_0} \sin(\beta n_0) < 0$ and $A \cos(\beta n_0) + B \sin(\beta n_0) > 0$.

Therefore, in such case, we will have $\frac{X_{n_0}}{Y_{n_0}} < 0$. Which means either $X_{n_0} < 0$ or $Y_{n_0} < 0$. Which is a contradiction.

We can also prove it in an another way: from the equations (41-43), we deduce that the sequence $(\frac{X_n}{Y_n})$ converges to 1. And therefore the sequence $(\frac{A \cos(\beta n) + B \sin(\beta n)}{A_n \cos(\beta n) + B_n \sin(\beta n)})$ converges to 1. But $AB \neq 0$, so, the sequence $(\frac{A \cos(\beta n) + B \sin(\beta n)}{A_n \cos(\beta n) + B_n \sin(\beta n)})$ cannot converge. Hence the contradiction. Therefore we must have $A = B = 0$. And therefore $\sin(\alpha)$ be zero.

Therefore

$$\sigma = \frac{1}{2} \quad (66)$$

In the case of $\sigma > \frac{1}{2}$. We work with $1 - s$ that is also another zero and here we have $1 - \sigma < \frac{1}{2}$. And we will reach to the same contradiction. Therefore, σ can only be $\frac{1}{2}$. This ends the proof of our lemma. \blacksquare

□

From the lemma above we can conclude the following corollary.

Corollary 3.1.1. *Let's consider two variables σ and τ such that $\sigma > 0$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $q > 1$ be an positive integer. Let's $L(s, \chi)$ be a L -function where χ is a Dirichlet character mod q with no cancellation property such that:*

$$\sum_{n=1}^q \chi(n) \neq 0 \quad (67)$$

If $L(s, \chi) = 0$, then:

$$\sigma = \frac{1}{2} \quad (68)$$

Proof. Let's take $x > 1$ a real number. Let's denote $n_x = [x]$ be the integer part of x . We have $n_x = x - \{x\}$. We denote q_x and r_x the quotient and the rest of the Euclidean division of n_x by q as follows: $n_x = qq_x + r_x$. Therefore we can write:

$$\sum_{n=1}^x \chi(n) = \sum_{n=0}^{q_x-1} \sum_{k=1}^q \chi(nq + k) + \sum_{k=1}^{r_x} \chi(nq + k) \quad (69)$$

$$= q_x \sum_{n=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (70)$$

$$= \frac{x - \{x\} - r_x}{q} \sum_{k=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (71)$$

$$= \frac{x}{q} \sum_{k=1}^q \chi(n) - \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi(n) + \sum_{k=1}^{r_x} \chi(k) \quad (72)$$

Let's define the constant λ and the function g as follows:

$$\lambda = \frac{\sum_{k=1}^q \chi(n)}{q} \quad (73)$$

$$g(x) = \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi(n) - \sum_{k=1}^{r_x} \chi(k) \quad (74)$$

$$= \lambda\{x\} + \underbrace{\lambda r_x - \sum_{k=1}^{r_x} \chi(k)}_{f(x)} \quad (75)$$

It is clear that the function $f(x)$ is bounded. And by assumption we have $\lambda \neq 0$. Therefore, we can calculate $L(s, \chi)$ as follows:

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (76)$$

$$= s \int_1^{+\infty} \frac{\lambda x - g(x)}{x^{s+1}} dx \quad (77)$$

$$= s \int_1^{+\infty} \frac{\lambda}{x^s} dx - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (78)$$

$$(79)$$

If $\Re(s) > 1$, we can conclude:

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (80)$$

We can extend the equation (80) over the region $\Re(s) > 0$. $L(s, \chi)$ has a holomorphic continuation as a function of s to the whole complex plane (see [3, 6, 7, 8]). Therefore, we apply the lemma 3.1 to prove that $\sigma = \frac{1}{2}$. \blacksquare \square

Remark. As a direct result of the above corollary, we can see that GRH is true for the series $(\sum_{n \geq 1} \frac{1}{(qn+p)^s})$ with $p > 0$ and $q > 1$ are integers.

Lemma 3.2. Let's consider two variables σ and τ such that $\sigma > 0$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $q > 1$ be an positive integer. Let's $L(s, \chi)$ be a L -function where χ is a Dirichlet character mod q with cancellation property such that:

$$\sum_{n=1}^q \chi(n) = 0 \quad (81)$$

Therefore:

$$L(s, \chi) = \left(\frac{1-q^{1-s}}{q}\right) \frac{s}{s-1} - s \int_1^{+\infty} \frac{\left(\frac{1-q^{1-s}}{q}\right)\{x\} + f(x)}{x^{s+1}} dx \quad (82)$$

Where f is a bounded function.

Proof. We will use the zeta function $\zeta(s)$ to prove our case.

Let's now write the following for $s > 1$:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (83)$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^q \frac{1}{(qn+i)^s} \quad (84)$$

$$= \sum_{i=1}^q \sum_{n=0}^{\infty} \frac{1}{(qn+i)^s} \quad (85)$$

Therefore

$$\left(1 - \frac{1}{q^s}\right)\zeta(s) = \sum_{i=1}^{q-1} \sum_{n=0}^{\infty} \frac{1}{(qn+i)^s} \quad (86)$$

Also

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \quad (87)$$

$$= \sum_{n=0}^{+\infty} \sum_{i=1}^q \frac{\chi(i)}{(qn+i)^s} \quad (88)$$

$$= \sum_{i=1}^q \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (89)$$

Therefore since $\chi(q) = 0$

$$L(s, \chi) = \sum_{i=1}^{q-1} \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (90)$$

Since $\sum_{n=1}^q \chi(n) = 0$ and $\chi(1) = 1$, we can write the following:

$$\sum_{n=0}^{+\infty} \frac{1}{(qn+1)^s} = L(s, \chi) - \sum_{i=2}^{q-1} \chi(i) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (91)$$

$$\sum_{n=0}^{+\infty} \frac{1}{(qn+1)^s} = \left(1 - \frac{1}{q^s}\right)\zeta(s) - \sum_{i=2}^{q-1} \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (92)$$

Therefore

$$\left(1 - \frac{1}{q^s}\right)\zeta(s) - L(s, \chi) = \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (93)$$

Let's denote $L(s, i)_{2 \leq i \leq q-1}$ as follows:

$$L(s, i) = \sum_{n=0}^{+\infty} \frac{1}{(qn+i)^s} \quad (94)$$

We can write also:

$$L(s, i) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi_i(n)}{x^{s+1}} dx \quad (95)$$

Where χ_i is the character of $L(s, i)$:

- (i) $\chi_i(qn+i) = 1$
- (ii) $\chi_i(qn+j) = 0$ if $j \neq i$

We use the equation (90-94) to write:

$$\lambda_i = \frac{\sum_{k=1}^q \chi_i(n)}{q} = \frac{1}{q} \quad (96)$$

$$f_i(x) = \frac{(\{x\} + r_x)}{q} \sum_{k=1}^q \chi_i(n) - \sum_{k=1}^{r_x} \chi_i(k) \quad (97)$$

$$= \frac{(\{x\} + r_x)}{q} - \sum_{k=1}^{r_x} \chi_i(k) \quad (98)$$

Therefore from the equation (96):

$$L(s, i) = \frac{s}{q(s-1)} - s \int_1^{+\infty} \frac{f_i(x)}{x^{s+1}} dx \quad (99)$$

Therefore, we can write the equation (94) as follows:

$$\left(1 - \frac{1}{q^s}\right) \zeta(s) - L(s, \chi) = \sum_{i=2}^{q-1} (1 - \chi(i)) L(s, i) \quad (100)$$

$$= \frac{s}{q(s-1)} \sum_{i=2}^{q-1} (1 - \chi(i)) - s \sum_{i=2}^{q-1} (1 - \chi(i)) \int_1^{+\infty} \frac{f_i(x)}{x^{s+1}} dx \quad (101)$$

Thanks to the cancellation property, we can conclude that:

$$\left(1 - \frac{1}{q^s}\right) \zeta(s) - L(s, \chi) = \frac{s(q-1)}{q(s-1)} - s \int_1^{+\infty} \frac{\sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x)}{x^{s+1}} dx \quad (102)$$

From the equation (10) we replace the $\zeta(s)$ by its L - function:

$$L(s, \chi) = \frac{1 - q^{1-s}}{q} \frac{s}{(s-1)} - s \int_1^{+\infty} \frac{(1 - q^{-s})\{x\} - \sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x)}{x^{s+1}} dx \quad (103)$$

Let's define the function $g(x)$ as follows:

$$g(x) = (1 - q^{-s})\{x\} - \sum_{i=2}^{q-1} (1 - \chi(i)) f_i(x) \quad (104)$$

$$= \frac{1 - q^{1-s}}{q} \{x\} - \left(1 - \frac{1}{q}\right) r_x + \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{k=1}^{r_x} \chi_i(k) \quad (105)$$

$$= \frac{1 - q^{1-s}}{q} \{x\} + f(x) \quad (106)$$

Where f is unrelated to s , bounded and defined as follows:

$$f(x) = -\left(1 - \frac{1}{q}\right) r_x + \sum_{i=2}^{q-1} (1 - \chi(i)) \sum_{k=1}^{r_x} \chi_i(k) \quad (107)$$

$$= \frac{r_x}{q} - \sum_{k=1}^{r_x} \chi(k) \quad (108)$$

And

$$L(s, \chi) = \left(\frac{1 - q^{1-s}}{q}\right) \frac{s}{s-1} - s \int_1^{+\infty} \frac{\left(\frac{1 - q^{1-s}}{q}\right)\{x\} + f(x)}{x^{s+1}} dx \quad (109)$$

□

Lemma 3.3. *Let's consider two variables σ and τ such that $0 < \sigma < 1$, $\tau > 0$ and $s = \sigma + i\tau$. Let's $q > 1$ be an positive integer. Let's $L(s, \chi)$ be a L -function where χ is a Dirichlet character mod q with cancellation property such that:*

$$\sum_{n=1}^q \chi(n) = 0 \quad (110)$$

Let's suppose that $s = \sigma + i\tau$ is a zero for $L(s, \chi)$. i.e $L(s, \chi) = 0$. Therefore

$$\sigma = \frac{1}{2} \quad (111)$$

And the Generalized Riemann's Hypothesis is true.

Proof. The proof is straightforward as we can use the lemmas 3.1 and 3.2. From lemma 3.2 we can write the following:

$$L(s, \chi) = \frac{\lambda s}{s-1} - s \int_1^{+\infty} \frac{\lambda\{x\} + f(x)}{x^{s+1}} dx \quad (112)$$

$$\lambda = \frac{1 - q^{1-s}}{q} \quad (113)$$

Where f is a bounded function. Then, we apply the lemma 3.1 to conclude that

$$\sigma = \frac{1}{2} \quad (114)$$

And the Generalized Riemann's Hypothesis is true.

■

□

3.1 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore, the Generalized Riemann Hypothesis (GRH) is true: *The non-trivial zeros of a L -Dirichlet function have real part equal to $\frac{1}{2}$.* With this, we conclude that $P = NP$.

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References

- [1] Charaf E. Chatbi *A Simple Proof of the Riemann's Hypothesis*
<https://arxiv.org/abs/1905.09385>
- [2] Bernhard Riemann. *On the Number of Prime Numbers less than a Given Quantity*
<https://www.claymath.org/sites/default/files/ezeta.pdf>
- [3] Aleksandar Ivic. *The Riemann Zeta-Function: Theory and Applications*
- [4] Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*
http://wayback.cecm.sfu.ca/~pborwein/TEMP_PROTECTED/book.pdf
- [5] Jørgen Veisdal. *The Riemann Hypothesis, explained*
<https://medium.com/cantors-paradise/the-riemann-hypothesis-explained-fa01c1f75d3f>
- [6] Thai Pham. *Dirichlet's Theorem on Arithmetic Progressions*
https://web.stanford.edu/~thaipham/papers/MIT_18.104_Review_Paper.pdf
- [7] G. H. Hardy. *The general theory of dirichlet series.*
<https://archive.org/details/generaltheoryofd029816mbp/page/n9>
- [8] Garrett, Paul. *Primes in arithmetic progressions, 2011.*
[http : //www.math.umn.edu/~garrett/m/mfms/notes_c/dirichlet.pdf](http://www.math.umn.edu/~garrett/m/mfms/notes_c/dirichlet.pdf)