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A Proof of the Generalized Riemann Hypothesis

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Abstract

We present a proof of the Generalized Riemann hypothesis (GRH) based on asymptotic expansions and operations on series. The advantage of our method is that it only uses undergraduate mathematics which makes it accessible to a wider audience.

Keywords: Generalized Riemann hypothesis; Zeta; Critical Strip; Prime Number Theorem; Millennium Problems; Dirichlet L-functions.

1 Dirichlet L -functions

Let's $(z_n)_{n \geq 1}$ be a sequence of complex numbers. A Dirichlet series[6] is a series of the form $\sum_{n=1}^{\infty} \frac{z_n}{n^s}$, where $s = \sigma + i\tau$ is complex. The zeta function is a Dirichlet series. Let's define the function $L(s)$ of the complex s :

$$L(s) = \sum_{n=1}^{\infty} \frac{z_n}{n^s}.$$

- If $(z_n)_{n \geq 1}$ is a bounded, then the corresponding Dirichlet series converges absolutely on the open half-plane where $\Re(s) > 1$.
- If the set of sums $z_n + z_{n+1} + \dots + z_{n+k}$ for each n and $k \geq 0$ is bounded, then the corresponding Dirichlet series converges on the open half-plane where $\Re(s) > 0$.
- In general, if $z_n = O(n^k)$, the corresponding Dirichlet series converges absolutely in the half plane where $\Re(s) > k + 1$.

The function $L(s)$ is analytic on the corresponding open half plane[3,6,14].

To define Dirichlet L -functions we need to define Dirichlet characters. A function $\chi : \mathbb{Z} \mapsto \mathbb{C}$ is a Dirichlet character modulo q if it satisfies the following criteria:

- (i) $\chi(n) \neq 0$ if $(n, q) = 1$.
- (ii) $\chi(n) = 0$ if $(n, q) > 1$.

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- (iii) χ is periodic with period q : $\chi(n + q) = \chi(n)$ for all n .
- (iv) χ is multiplicative : $\chi(mn) = \chi(m)\chi(n)$ for all integers m and n .

The trivial character is the one with $\chi_0(n) = 1$ whenever $(n, q) = 1$.

Here are some known results for a Dirichlet character modulo q . For any integer n we have $\chi(1) = 1$. Also if $(n, q) = 1$, we have $(\chi(n))^{\phi(q)} = 1$ with ϕ is Euler's totient function. $\chi(n)$ is a $\phi(q)$ -th root of unity. Therefore, $|\chi(n)| = 1$ if $(n, q) = 1$, and $|\chi(n)| = 0$ if $(n, q) > 1$. Also, we recall the cancellation property for Dirichlet characters modulo q : For any n integer

$$\sum_{i=1}^q \chi(i + n) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (1)$$

The Dirichlet L -functions are simply the sum of the Dirichlet series. Let's χ be a Dirichlet character modulo q , The Dirichlet L -function $L(s, \chi)$ is defined for $\Re(s) > 1$ as the following:

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \quad (2)$$

They are a natural generalization of the Riemann zeta-function $\zeta(s)$ to an arithmetic progression and are a powerful tool in analytic number theory. The Dirichlet series, converges absolutely and uniformly in any bounded domain in the complex s -plane for which $\Re(s) \geq 1 + \gamma$, $\gamma > 0$. In the particular case of the trivial character χ_0 , $L(s, \chi_0)$ extends to a meromorphic function[3,6,7,8] in $\Re(s) > 0$ with the only pole at $s = 1$.

If χ is a non-trivial character, we have

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \quad (3)$$

Since the sum in the integrand is bounded, this formula gives an analytic continuation of $L(s, \chi)$ to a regular function in the half-plane $\Re(s) > 0$. Also, like the zeta function, the Dirichlet L -functions have their Euler product[3,8,10]. For $\Re(s) > 1$:

$$L(s, \chi) = \prod_{p \text{ Prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (4)$$

Therefore, if $\chi = \chi_0$ is a trivial character mod q , we have, for $q = 1$,

$$L(s, \chi_0) = \zeta(s) \quad (5)$$

And for $q > 1$, we have,

$$L(s, \chi_0) = \zeta(s) \prod_{p/q} \left(1 - \frac{1}{p^s}\right) \quad (6)$$

For this reason the properties of $L(s, \chi_0)$ in the entire complex plane are determined by the properties of $\zeta(s)$.

Let's now q' be the smallest divisor (prime) of q . Let's χ' be the Dirichlet character $\chi' \pmod{q'}$. For any integer n such that $(n, q) = 1$, we have also $(n, q') = 1$ and $\chi(n) = \chi'(n)$. χ' is called primitive and $L(s, \chi)$ and $L(s, \chi')$ are related analytically such that if $\chi \neq \chi_0$:

$$L(s, \chi) = L(s, \chi') \prod_{p|q} \left(1 - \frac{\chi'(p)}{p^s}\right) \quad (7)$$

$L(s, \chi)$ and $L(s, \chi')$ have the same zeros in the critical strip $0 < \Re(s) < 1$.

Also, for a primitive character χ , (*i.e.* $\chi = \chi'$) $L(s, \chi)$ has the following functional equation:

$$\tau(\chi) \Gamma\left(\frac{1-s+a}{2}\right) L(1-s, \chi) = \sqrt{\pi} \left(\frac{q}{\pi}\right)^s i^a q^{\frac{1}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \bar{\chi}) \quad (8)$$

Where Γ is the Gamma function and $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$, and $\tau(\chi) = \sum_{k=1}^q \chi(k) \exp\left(\frac{2\pi ki}{q}\right)$.

When $\Re(s) > 1$ there is no zero for $L(s, \chi)$. When $\Re(s) < 0$, for a primitive character χ , we have the trivial zeros of $L(s, \chi)$: $s = a - 2k$, where k is a positive integer and a is defined above. For more details, please refer to the references[7-14].

2 The Generalized Riemann Hypothesis

The Generalized Riemann Hypothesis states that the Dirichlet L -functions have all their non-trivial zeros on the critical line $\Re(s) = \frac{1}{2}$.

It is well known that for any character χ modulo q , all non-trivial zeros of $L(s, \chi)$ lies in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. From the functional equation above we have that if:

- s_0 is a non-trivial zero of $L(s, \chi)$, then $1 - s_0$ is a zero of $L(s, \bar{\chi})$.
- s_0 is a non-trivial zero of $L(s, \bar{\chi})$, then $1 - s_0$ is a zero of $L(s, \chi)$.

Therefore, we just need to prove that for all primitive character χ modulo q , there is no non-trivial zeros of $L(s, \chi)$ in the right hand side of the critical strip $\{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$.

3 Proof of the GRH

Let's take a complex number s such $s = a_0 + ib_0$. Unless we explicitly mention otherwise, let's suppose that $\frac{1}{2} < a_0 < 1$ and $b_0 > 0$. Let's take χ a Dirichlet character.

3.1 Case One: χ non-trivial character

In this case s is a zero of $L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$. Where χ is a non-trivial Dirichlet character χ modulo q . When $q = 2$ there is only one Dirichlet character and it is trivial. So here we are going to assume that $q \geq 3$ is

an odd prime integer.

Let's denote for each $n \geq 1$: $z_n = x_n + iy_n = \chi(n)$.

We are going to develop the sequence $Z_N(s) = \sum_{n=1}^N \frac{z_n}{n^s}$ as follows:
For $N \geq 1$

$$Z_N = \sum_{n=1}^N \frac{z_n}{n^s} = \sum_{n=1}^N \frac{x_n + iy_n}{n^{a_0 + ib_0}} \quad (9)$$

$$= \sum_{n=1}^N \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{n^{a_0}} + i \sum_{n=1}^N \frac{y_n \cos(b_0 \ln(n)) - x_n \sin(b_0 \ln(n))}{n^{a_0}} \quad (10)$$

Let's define the sequences U_n, V_n as follows: For $n \geq 1$

$$U_n = \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{n^{a_0}}, \quad V_n = \frac{y_n \cos(b_0 \ln(n)) - x_n \sin(b_0 \ln(n))}{n^{a_0}} \quad (11)$$

Let's define the series A_n, B_n and Z_n as follows:

$$A_n = \sum_{k=1}^n U_k, \quad B_n = \sum_{k=1}^n V_k, \quad Z_n = A_n + iB_n \quad (12)$$

When we are dealing with complex numbers, it is always insightful to work with the norm. So let's develop further the squared norm of the series Z_N as follows:

$$\|Z_N\|^2 = A_N^2 + B_N^2 = \left(\sum_{n=1}^N U_n \right)^2 + \left(\sum_{n=1}^N V_n \right)^2 \quad (13)$$

So

$$A_N^2 = \sum_{n=1}^N U_n^2 + 2 \sum_{n=1}^N U_n \sum_{k=1}^{n-1} U_k = - \sum_{n=1}^N U_n^2 + 2 \sum_{n=1}^N U_n A_n \quad (14)$$

And the same calculation for B_N

$$B_N^2 = - \sum_{n=1}^N V_n^2 + 2 \sum_{n=1}^N V_n B_n \quad (15)$$

Hence we have the new expression of square norm of Z_N :

$$\|Z_N\|^2 = 2 \sum_{n=1}^N U_n A_n + 2 \sum_{n=1}^N V_n B_n - \sum_{n=1}^N U_n^2 - \sum_{n=1}^N V_n^2 \quad (16)$$

Let's now define F_n and G_n as follows:

$$F_n = U_n A_n, \quad G_n = V_n B_n \quad (17)$$

Therefore

$$A_N^2 = 2 \sum_{n=1}^N F_n - \sum_{n=1}^N U_n^2, \quad B_N^2 = 2 \sum_{n=1}^N G_n - \sum_{n=1}^N V_n^2 \quad (18)$$

Therefore

Conclusion. s is a zero for $L(s, \chi) = 0$, if and only if

$$\lim_{N \rightarrow \infty} A_N = 0 \text{ and } \lim_{N \rightarrow \infty} B_N = 0 \quad (19)$$

Equally, s is a $L(s, \chi)$ zero, $L(s, \chi) = 0$, if and only if

$$\lim_{N \rightarrow \infty} A_N^2 = 0 \text{ and } \lim_{N \rightarrow \infty} B_N^2 = 0 \quad (20)$$

Proof Strategy. The idea is to prove that in the case of a complex s that is in the right hand side of the critical strip $\frac{1}{2} < a_0 \leq 1$ and that is a $L(s, \chi)$ zero, that the limit $\lim_{n \rightarrow \infty} A_n^2 = +/ - \infty$ **OR** the limit $\lim_{n \rightarrow \infty} B_n^2 = +/ - \infty$. This will create a contradiction. Because if s is a $L(s, \chi)$ zero then the $\lim_{n \rightarrow \infty} A_n^2$ should be 0 and the $\lim_{n \rightarrow \infty} B_n^2$ should be 0. And therefore the sequences $(\sum_{n=1}^N F_n)_{N \geq 1}$ and $(\sum_{n=1}^N G_n)_{N \geq 1}$ should converge and their limits should be: $\lim_{n \rightarrow \infty} \sum_{n=1}^N F_n = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^N U_n^2 < +\infty$ and $\lim_{n \rightarrow \infty} \sum_{n=1}^N G_n = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^N V_n^2 < +\infty$.

To prove this result, let's first prove the following lemma:

Lemma 3.1. If the set of the partial sums $z_n + z_{n+1} + \dots + z_{n+k}$ for n and $k \geq 0$ is bounded, then we can write $A_n = \frac{\lambda_n}{n^{a_0}}$ where (λ_n) is a bounded sequence.

Proof. We have $\lim_{N \rightarrow +\infty} A_N = \sum_{n=1}^{+\infty} U_n = 0$. Therefore for each $N \geq 1$:

$$\sum_{n=1}^{+\infty} U_n = \underbrace{\sum_{n=1}^N U_n}_{A_N} + \sum_{n=N+1}^{+\infty} U_n = 0 \quad (21)$$

$$A_N = - \sum_{n=N+1}^{+\infty} U_n \quad (22)$$

We have

$$A_N = - \sum_{n=N+1}^{+\infty} \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{n^{a_0}} \quad (23)$$

Let's denote X_n and Y_n the partial sums of the series x_n and y_n : $X_n = \sum_{k=1}^N x_k$ and $Y_n = \sum_{k=1}^N y_k$. So let's take N and M two integers such that $M \geq N$ and do the Abel summation between $N+1$ and M :

$$\sum_{n=N+1}^M U_n = \sum_{n=N+1}^M \frac{x_n \cos(b_0 \ln(n))}{n^{a_0}} + \sum_{n=N+1}^M \frac{y_n \sin(b_0 \ln(n))}{n^{a_0}} \quad (24)$$

$$= \frac{X_M \cos(b_0 \ln(M))}{M^{a_0}} - \frac{X_N \cos(b_0 \ln(N+1))}{(N+1)^{a_0}} - \sum_{n=N+1}^{M-1} X_n \left(\frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right) \quad (25)$$

$$+ \frac{Y_M \sin(b_0 \ln(M))}{M^{a_0}} - \frac{Y_N \sin(b_0 \ln(N+1))}{(N+1)^{a_0}} - \sum_{n=N+1}^{M-1} Y_n \left(\frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right) \quad (26)$$

Let's define the functions f_n and e_n such that $f_n(t) = \frac{\cos(b_0 \ln(n+t))}{(n+t)^{a_0}}$ and $e_n(t) = \frac{\sin(b_0 \ln(n+t))}{(n+t)^{a_0}}$. For each $n \geq 1$, we can apply the Mean Value Theorem on the interval $[0, 1]$ to find c_1 and c_2 in $(0, 1)$ such that:

$$\left| \frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{(n)^{a_0}} \right| = |e'_n(c_1)(1-0)| \quad (27)$$

$$\left| \frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{(n)^{a_0}} \right| = |f'_n(c_2)(1-0)| \quad (28)$$

We have the derivatives of f_n and e_n such that $f'_n(t) = \frac{-b_0 \sin(b_0 \ln(n+t)) - a_0 \cos(b_0 \ln(n+t))}{(n+t)^{a_0+1}}$ and $e'_n(t) = \frac{b_0 \cos(b_0 \ln(n+t)) - a_0 \sin(b_0 \ln(n+t))}{(n+t)^{a_0+1}}$. Therefore

$$\left| \frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{(n)^{a_0}} \right| \leq \frac{a_0 + b_0}{n^{a_0+1}} \quad (29)$$

$$\left| \frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{(n)^{a_0}} \right| \leq \frac{a_0 + b_0}{n^{a_0+1}} \quad (30)$$

We have the set of the partial sums $z_n + z_{n+1} + \dots + z_{n+k}$ is bounded, then the real part and the imaginary part of the sum partial of $z_n + z_{n+1} + \dots + z_{n+k}$ are also bounded.

Let's $K > 0$ such that for every n and k : $|x_n + x_{n+1} + \dots + x_{n+k}| \leq K$ and $|y_n + y_{n+1} + \dots + y_{n+k}| \leq K$. Therefore for each n : $|X_n| \leq K$ and $|Y_n| \leq K$.

$$\left| \sum_{n=N+1}^M U_n \right| \leq \left| \frac{X_M \cos(b_0 \ln(M))}{M^{a_0}} \right| + \left| \frac{X_N \cos(b_0 \ln(N+1))}{(N+1)^{a_0}} \right| \quad (31)$$

$$+ \sum_{n=N+1}^{M-1} |X_n| \left| \left(\frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right) \right| \quad (32)$$

$$+ \left| \frac{Y_M \sin(b_0 \ln(M))}{M^{a_0}} \right| + \left| \frac{Y_N \sin(b_0 \ln(N+1))}{(N+1)^{a_0}} \right| \quad (33)$$

$$+ \sum_{n=N+1}^{M-1} |Y_n| \left| \left(\frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right) \right| \quad (34)$$

$$\leq \frac{4K}{(N+1)^{a_0}} + 2K(a_0 + b_0) \sum_{n=N+1}^{M-1} \frac{1}{n^{a_0+1}} \quad (35)$$

$$\leq \frac{4K}{(N+1)^{a_0}} + 2K(a_0 + b_0) \int_{N+1}^M \frac{dt}{t^{a_0+1}} \quad (36)$$

$$\leq \frac{4K}{(N+1)^{a_0}} + \frac{2K(a_0 + b_0)}{a_0} \left(\frac{1}{(N+1)^{a_0}} - \frac{1}{M^{a_0}} \right) \quad (37)$$

We tend M to infinity and we get:

$$\left| \sum_{n=N+1}^{+\infty} U_n \right| \leq \frac{4K}{(N+1)^{a_0}} + \frac{2K(a_0 + b_0)}{a_0(N+1)^{a_0}} \quad (38)$$

Therefore

$$|A_N| = \left| - \sum_{n=N+1}^{+\infty} U_n \right| \leq \frac{K_1}{(N+1)^{a_0}} \leq \frac{K_1}{N^{a_0}} \quad (39)$$

where $K_1 = 4K + \frac{2K(a_0+b_0)}{a_0} > 0$.

Let's define the sequence λ_n such that: $A_n = \frac{\lambda_n}{n^{a_0}}$. Therefore for each $n \geq 1$ we have: $|\lambda_n| \leq K_1$. Therefore the sequence (λ_n) is bounded. \blacksquare \square

Lemma 3.2. *The series (F_n) has the following asymptotic expansion:*

$$F_{n+1} - F_n = \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} + \frac{(\lambda_n + 2\gamma_n)\beta_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \quad (40)$$

Where (λ_n) is defined in the previous lemma, and (α_n) , (β_n) , (γ_n) and (ϵ_n) are bounded sequences.

Proof. Let's $n \geq 1$, we have $A_{n+1} = A_n + U_{n+1}$. Therefore

$$\frac{\lambda_{n+1}}{(n+1)^{a_0}} = \frac{\lambda_n}{n^{a_0}} + \frac{x_{n+1} \cos(b_0 \ln(n+1)) + y_{n+1} \sin(b_0 \ln(n+1))}{(n+1)^{a_0}} \quad (41)$$

And

$$\lambda_{n+1} = \left(1 + \frac{1}{n}\right)^{a_0} \lambda_n + \left(x_{n+1} \cos(b_0 \ln(n+1)) + y_{n+1} \sin(b_0 \ln(n+1))\right) \quad (42)$$

For simplification, let's denote , for $n \geq 1$,

$$u_n = x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n)) \quad (43)$$

$$\lambda_{n+1} = \left(1 + \frac{1}{n}\right)^{a_0} \lambda_n + u_{n+1} \quad (44)$$

First, we apply the previous lemma 3.1 to the case of our Dirichlet series. Thanks to the cancelation property mentioned in (1), we have the partial sums $(\sum_{i=1}^n \chi(i))_{n \geq 1}$ is bounded because our Dirichlet character χ is non-trivial. Therefore we write $A_n = \frac{\lambda_n}{n^{a_0}}$ where the sequence (λ_n) is bounded.

To prove this lemma we proceed with some asymptotic expansions: We have

$$A_{n+1} = A_n + U_{n+1} \quad (45)$$

So F_{n+1} can be written as follows:

$$F_{n+1} = U_{n+1} A_n + U_{n+1}^2 \quad (46)$$

So $F_{n+1} - F_n$ can be written as

$$F_{n+1} - F_n = \left(U_{n+1} - U_n\right) A_n + U_{n+1}^2 \quad (47)$$

Let's do now the asymptotic expansion of $U_{n+1} - U_n$. For this we need the asymptotic expansion of $\cos(b_0 \ln(n+1))$ and $\sin(b_0 \ln(n+1))$.

$$\cos(b_0 \ln(n+1)) = \cos\left(b_0 \ln(n) + b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (48)$$

$$= \cos(b_0 \ln(n)) \cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) - \sin(b_0 \ln(n)) \sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (49)$$

we have the asymptotic expansion of $\ln\left(1 + \frac{1}{n}\right)$ in order two as follow:

$$\ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (50)$$

Using the asymptotic expansion of the functions \sin and \cos that I will speare you the details here, we have

$$\cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) = 1 + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (51)$$

And

$$\sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) = \frac{b_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (52)$$

Hence

$$\cos(b_0 \ln(n+1)) = \cos(b_0 \ln(n)) - \frac{b_0}{n} \sin(b_0 \ln(n)) + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (53)$$

Also the Asymptotic expansion of $\frac{1}{(1+n)^{a_0}}$:

$$\frac{1}{(1+n)^{a_0}} = \frac{1}{n^{a_0}} \left(1 - \frac{a_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \quad (54)$$

Hence

$$\frac{\cos(b_0 \ln(n+1))}{(1+n)^{a_0}} = \frac{\cos(b_0 \ln(n))}{n^{a_0}} - \frac{b_0 \sin(b_0 \ln(n)) + a_0 \cos(b_0 \ln(n))}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (55)$$

$$= \frac{\cos(b_0 \ln(n))}{n^{a_0}} + \frac{c_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (56)$$

And the squared version of the above equation

$$\frac{\cos^2(b_0 \ln(n+1))}{(1+n)^{2a_0}} = \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + 2 \cos(b_0 \ln(n)) \frac{c_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (57)$$

For the asymptotic expansion of $\sin(b_0 \ln(n+1))$.

$$\sin(b_0 \ln(n+1)) = \sin\left(b_0 \ln(n) + b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (58)$$

$$= \sin(b_0 \ln(n)) \cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) + \cos(b_0 \ln(n)) \sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (59)$$

we have the asymptotic expansion of $\ln(1 + \frac{1}{n})$ in order two as follow:

$$\ln(1 + \frac{1}{n}) = \frac{1}{n} + \mathcal{O}(\frac{1}{n^2}) \quad (60)$$

Using the asymptotic expansion of the functions sin and cos that I will speare you the details here, we have

$$\sin\left(b_0 \ln(1 + \frac{1}{n})\right) = \sin\left(\frac{b_0}{n} + \mathcal{O}(\frac{1}{n^2})\right) \quad (61)$$

$$= \frac{b_0}{n} + \mathcal{O}(\frac{1}{n^2}) \quad (62)$$

And

$$\cos\left(b_0 \ln(1 + \frac{1}{n})\right) = 1 + \mathcal{O}(\frac{1}{n^2}) \quad (63)$$

Hence

$$\sin(b_0 \ln(n+1)) = \sin(b_0 \ln(n)) + \frac{b_0}{n} \cos(b_0 \ln(n)) + \mathcal{O}(\frac{1}{n^2}) \quad (64)$$

Also the Asymptotic expansion of $\frac{1}{(1+n)^{a_0}}$:

$$\frac{1}{(1+n)^{a_0}} = \frac{1}{n^{a_0}} \left(1 - \frac{a_0}{n} + \mathcal{O}(\frac{1}{n^2})\right) \quad (65)$$

Hence

$$\frac{\sin(b_0 \ln(n+1))}{(1+n)^{a_0}} = \frac{\sin(b_0 \ln(n))}{n^{a_0}} + \frac{b_0 \cos(b_0 \ln(n)) - a_0 \sin(b_0 \ln(n))}{n^{a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \quad (66)$$

$$= \frac{\sin(b_0 \ln(n))}{n^{a_0}} + \frac{cc_n}{n^{a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \quad (67)$$

And the squared version of the above equation

$$\frac{\sin^2(b_0 \ln(n+1))}{(1+n)^{2a_0}} = \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sin(b_0 \ln(n)) \frac{cc_n}{n^{2a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \quad (68)$$

Where

$$c_n = -\left(b_0 \sin(b_0 \ln(n)) + a_0 \cos(b_0 \ln(n))\right) \quad (69)$$

$$cc_n = b_0 \cos(b_0 \ln(n)) - a_0 \sin(b_0 \ln(n)) \quad (70)$$

Therefore

$$U_{n+1} - U_n = \frac{x_{n+1} \cos(b_0 \ln(n+1)) + y_{n+1} \sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{(n)^{a_0}} \quad (71)$$

$$= x_{n+1} \left(\frac{\cos(b_0 \ln(n))}{n^{a_0}} + \frac{c_n}{n^{a_0+1}} \right) + y_{n+1} \left(\frac{\sin(b_0 \ln(n))}{n^{a_0}} + \frac{cc_n}{n^{a_0+1}} \right) \quad (72)$$

$$- \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{(n)^{a_0}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (73)$$

$$= \frac{(x_{n+1} - x_n) \cos(b_0 \ln(n)) + (y_{n+1} - y_n) \sin(b_0 \ln(n))}{n^{a_0}} + \frac{x_{n+1}c_n + y_{n+1}cc_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (74)$$

$$= \frac{\alpha_n}{n^{a_0}} + \frac{\beta_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (75)$$

And

$$U_{n+1} = \frac{\gamma_n}{n^{a_0}} + \frac{\beta_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (76)$$

Where

$$\alpha_n = (x_{n+1} - x_n) \cos(b_0 \ln(n)) + (y_{n+1} - y_n) \sin(b_0 \ln(n)) \quad (77)$$

$$\beta_n = x_{n+1}c_n + y_{n+1}cc_n \quad (78)$$

$$\gamma_n = x_{n+1} \cos(b_0 \ln(n)) + y_{n+1} \sin(b_0 \ln(n)) \quad (79)$$

Therefore

$$U_{n+1}^2 = \frac{\gamma_n^2}{n^{2a_0}} + 2 \frac{\gamma_n \beta_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (80)$$

So the asymptotic expansion of $F_{n+1} - F_n$ is as follows:

$$F_{n+1} - F_n = \frac{\gamma_n^2}{n^{2a_0}} + \frac{\alpha_n A_n}{n^{a_0}} + \frac{\beta_n A_n}{n^{a_0+1}} + 2 \frac{\gamma_n \beta_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (81)$$

We have $A_n = \frac{\lambda_n}{n^{a_0}}$ where λ_n is bounded. Therefore

$$F_{n+1} - F_n = \frac{\gamma_n^2}{n^{2a_0}} + \frac{\alpha_n \lambda_n}{n^{2a_0}} + \frac{2\gamma_n \beta_n}{n^{2a_0+1}} + \frac{\beta_n \lambda_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (82)$$

Therefore

$$F_{n+1} - F_n = \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} + \frac{(\lambda_n + 2\gamma_n) \beta_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (83)$$

By definition of \mathcal{O} , we know that there is exist a bounded sequence (ϵ_n) and there is exist a number N_0 such that: For each $n \geq N_0$ we have

$$F_{n+1} - F_n = \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} + \frac{(\lambda_n + 2\gamma_n) \beta_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \quad (84)$$

■

□

Let's now study the asymptotic expansion of the dominant term $\frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}}$ in the context of the Dirichlet L -functions.

Lemma 3.3. *Let's q be an odd prime integer. Let's χ be a non-trivial Dirichlet character modulo q . Let's denote that for each k , $z_k = x_k + iy_k = \chi(k)$. Therefore we have the following asymptotic expansion:*

$$\sum_{i=1}^q \frac{\gamma_{qn+i}^2 + \alpha_{qn+i} \lambda_{qn+i}}{(nq+i)^{2a_0}} + \sum_{i=1}^q \frac{(\lambda_{nq+i} + 2\gamma_{nq+i})\beta_{nq+i}}{(nq+i)^{2a_0+1}} = \frac{Ta + Tb \cos(2b_0 \ln(qn)) + Tc \sin(2b_0 \ln(qn))}{(nq)^{2a_0+1}} + O\left(\frac{1}{(qn)^{2a_0+2}}\right) \quad (85)$$

Where $m \in [1, q-2]$ is defined such that $\chi_m(g) = e^{\frac{2\pi mi}{q-1}}$ where g is the generator of the cyclic group $\mathbb{Z}/q\mathbb{Z}^*$.

$$Ta = \begin{cases} -\frac{a_0}{2} \left[\frac{(5q-1)}{2} - \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{2-2\cos(\frac{2\pi m}{q-1})} \right] + \frac{b_0}{2} \left[\frac{(q-1)}{2} + \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{1-\cos(\frac{2\pi m}{q-1})} \right], & \text{if } m \neq \frac{q-1}{2} \\ -\frac{a_0}{4}(5q-1) + \frac{b_0}{4}(q-1), & \text{if } m = \frac{q-1}{2} \end{cases} \quad (86)$$

$$Tb = \begin{cases} \frac{a_0}{2} \left[q+1 - \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{2-2\cos(\frac{2\pi m}{q-1})} \right] - \frac{b_0}{2} \left[\frac{(q-1)}{2} + \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{1-\cos(\frac{2\pi m}{q-1})} \right], & \text{if } m \neq \frac{q-1}{2} \\ \frac{a_0}{4}(4q^2 + 2q - 2) - \frac{b_0}{4}(q-1), & \text{if } m = \frac{q-1}{2} \end{cases} \quad (87)$$

$$Tc = \begin{cases} \frac{a_0}{2} \left[\frac{q-1}{2} + \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{1-\cos(\frac{2\pi m}{q-1})} \right] - \frac{b_0}{2} \left[q+1 + \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{2-2\cos(\frac{2\pi m}{q-1})} \right], & \text{if } m \neq \frac{q-1}{2} \\ \frac{a_0}{4}(q-1) - b_0q, & \text{if } m = \frac{q-1}{2} \end{cases} \quad (88)$$

Proof. In this case we have for each $n \geq 1$:

$$\alpha_n = (x_{n+1} - x_n) \cos(b_0 \ln(n)) + (y_{n+1} - y_n) \sin(b_0 \ln(n)) \quad (89)$$

$$\gamma_n^2 = x_{n+1} \cos(b_0 \ln(n)) + y_{n+1} \sin(b_0 \ln(n)) \quad (90)$$

λ_n is bounded. From the asymptotic expansion of $(1 + \frac{1}{n})^{a_0}$ and the equation (42), we can write:

$$\lambda_{n+1} = \lambda_n + x_{n+1} \cos(b_0 \ln(n+1)) + y_{n+1} \sin(b_0 \ln(n+1)) + \frac{a_0 \lambda_n}{n} + O\left(\frac{1}{n^2}\right) \quad (91)$$

We have $z_1 = 1$, $z_q = 0$ and for each $1 \leq k \leq q-1$, $z_k^{(q-1)} = 1$. Therefore z_k are $q-1$ -th complex root of unity. q is an odd prime number. $\mathbb{Z}/q\mathbb{Z}^*$ is a cyclic group generated by some generator g . We have $g^{q-1} = 1 \pmod{q}$ therefore $\chi(g)^{q-1} = 1$. So $\chi(g) = e^{\frac{2\pi mi}{q-1}}$ with $m \in [1, \dots, q-1]$. We have exactly $q-2$ non-trivial Dirichlet characters defined with their $\chi_m(g) = e^{\frac{2\pi mi}{q-1}}$ where $m \in [1, q-2]$. And $\chi_m(g^k) = e^{\frac{2\pi kmi}{q-1}}$ where $k \in [0, q-2]$.

Without loss of generality, let's take $m \in [1, q-2]$ and put $z_k = \chi_m(g^k) =$

$e^{\frac{2(k-1)\pi im}{q-1}}$ for each $1 \leq k \leq q-1$. Then, by definition we have for each $1 \leq i \leq q$, $x_i = \cos(\frac{2(i-1)\pi m}{q-1})$ and $y_i = \sin(\frac{2(i-1)\pi m}{q-1})$. And for each k , $x_{kq+i} = x_i$ and $y_{kq+i} = y_i$.

We have from above that

$$\cos(b_0 \ln(n+i)) = \cos(b_0 \ln(n)) - \frac{ib_0}{n} \sin(b_0 \ln(n)) + O\left(\frac{1}{n^2}\right) \quad (92)$$

$$\sin(b_0 \ln(n+i)) = \sin(b_0 \ln(n)) + \frac{ib_0}{n} \cos(b_0 \ln(n)) + O\left(\frac{1}{n^2}\right) \quad (93)$$

Therefore for $1 \leq i \leq q$: we have

$$\cos(b_0 \ln(qn+i)) = \cos(b_0 \ln(qn)) - \frac{ib_0}{nq} \sin(b_0 \ln(qn)) + O\left(\frac{1}{(qn)^2}\right) \quad (94)$$

$$\sin(b_0 \ln(qn+i)) = \sin(b_0 \ln(qn)) + \frac{ib_0}{nq} \cos(b_0 \ln(qn)) + O\left(\frac{1}{(qn)^2}\right) \quad (95)$$

$$\lambda_{qn+i+1} = \lambda_{qn+i} + x_{qn+i+1} \cos(b_0 \ln(qn+i+1)) + y_{qn+i+1} \sin(b_0 \ln(qn+i+1)) + \frac{a_0 \lambda_{nq+i}}{nq} + O\left(\frac{1}{(qn)^2}\right) \quad (96)$$

$$= \lambda_{qn+i} + x_{qn+i+1} \left(\cos(b_0 \ln(qn)) - \frac{(i+1)b_0}{nq} \sin(b_0 \ln(qn)) \right) \quad (97)$$

$$+ y_{qn+i+1} \left(\sin(b_0 \ln(qn)) + \frac{(i+1)b_0}{nq} \cos(b_0 \ln(qn)) \right) + \frac{a_0 \lambda_{nq+i}}{nq} + O\left(\frac{1}{(qn)^2}\right) \quad (98)$$

$$= \lambda_{qn+i} + \left(x_{qn+i+1} \cos(b_0 \ln(qn)) + y_{qn+i+1} \sin(b_0 \ln(qn)) \right) \quad (99)$$

$$+ \frac{(i+1)b_0}{nq} \left(-x_{qn+i+1} \sin(b_0 \ln(qn)) + y_{qn+i+1} \cos(b_0 \ln(qn)) \right) + \frac{a_0 \lambda_{nq+i}}{nq} + O\left(\frac{1}{(qn)^2}\right) \quad (100)$$

We need the asymptotic expansion of the order 1 of λ_{nq+i} to replace $\frac{a_0 \lambda_{nq+i}}{nq}$ by its value in function of x_{qn+i} and y_{qn+i} .

$$\lambda_{qn+i+1} = \lambda_{qn+i} + x_{qn+i+1} \cos(b_0 \ln(qn+i+1)) + y_{qn+i+1} \sin(b_0 \ln(qn+i+1)) + O\left(\frac{1}{qn}\right) \quad (101)$$

$$= \lambda_{qn+i} + x_{qn+i+1} \cos(b_0 \ln(qn)) + y_{qn+i+1} \sin(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (102)$$

$$\dots \quad (103)$$

$$\lambda_{qn+1} = \lambda_{qn} + x_{qn+1} \cos(b_0 \ln(qn)) + y_{qn+1} \sin(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (104)$$

Therefore

$$\lambda_{qn+i} = \lambda_{qn} + \cos(b_0 \ln(qn)) \sum_{k=1}^i x_{qn+k} + \sin(b_0 \ln(qn)) \sum_{k=1}^i y_{qn+k} + O\left(\frac{1}{qn}\right) \quad (105)$$

And thanks to the cancellation property (1), we can write:

$$\lambda_{qn+q} = \lambda_{qn} + O\left(\frac{1}{qn}\right) \quad (106)$$

We inject the equation (105) into the equations (99-100) and get:

$$\lambda_{qn+i+1} = \lambda_{qn+i} + \left(x_{qn+i+1} \cos(b_0 \ln(qn)) + y_{qn+i+1} \sin(b_0 \ln(qn)) \right) \quad (107)$$

$$+ \frac{(i+1)b_0}{nq} \left(-x_{qn+i+1} \sin(b_0 \ln(nq)) + y_{qn+i+1} \cos(b_0 \ln(nq)) \right) \quad (108)$$

$$+ \frac{a_0}{nq} \left(\cos(b_0 \ln(qn)) \sum_{k=1}^i x_{qn+k} + \sin(b_0 \ln(qn)) \sum_{k=1}^i y_{qn+k} \right) \quad (109)$$

$$+ \frac{a_0 \lambda_{nq}}{nq} + O\left(\frac{1}{(qn)^2}\right) \quad (110)$$

$$\dots \quad (111)$$

$$\lambda_{qn+1} = \lambda_{qn} + \left(x_{qn+1} \cos(b_0 \ln(qn)) + y_{qn+1} \sin(b_0 \ln(qn)) \right) \quad (112)$$

$$+ \frac{1b_0}{nq} \left(-x_{qn+1} \sin(b_0 \ln(nq)) + y_{qn+1} \cos(b_0 \ln(nq)) \right) + \frac{a_0 \lambda_{nq}}{nq} + O\left(\frac{1}{(qn)^2}\right) \quad (113)$$

$$(114)$$

Therefore

$$\lambda_{qn+i} = \lambda_{qn} + \left(\cos(b_0 \ln(qn)) \sum_{k=1}^i x_{qn+k} + \sin(b_0 \ln(qn)) \sum_{k=1}^i y_{qn+k} \right) \quad (115)$$

$$+ \frac{b_0}{nq} \left(-\sin(b_0 \ln(nq)) \sum_{k=1}^i k x_{qn+k} + \cos(b_0 \ln(nq)) \sum_{k=1}^i k y_{qn+k} \right) \quad (116)$$

$$+ \frac{a_0}{nq} \left(\cos(b_0 \ln(qn)) \sum_{k=1}^i \sum_{l=1}^{k-1} x_{qn+l} + \sin(b_0 \ln(qn)) \sum_{k=1}^i \sum_{l=1}^{k-1} y_{qn+l} \right) \quad (117)$$

$$+ \frac{a_0 i \lambda_{nq}}{nq} + O\left(\frac{1}{(qn)^2}\right) \quad (118)$$

$$= \lambda_{qn} + A_{i,nq} + \frac{a_0 i \lambda_{nq} + B_{i,nq}}{nq} + O\left(\frac{1}{(qn)^2}\right) \quad (119)$$

In the particular case of $i = q$ we have

$$\lambda_{qn+q} = \lambda_{qn} + A_{q,qn} + \frac{a_0 \lambda_{nq}}{n} + \frac{B_{q,qn}}{nq} + O\left(\frac{1}{(qn)^2}\right) \quad (120)$$

But we have $A_{q,qn} = 0$, thanks to the cancellation property (1) of a non-trivial character. Therefore

$$\lambda_{q(n+1)} = \left(1 + \frac{a_0}{n}\right) \lambda_{qn} + \frac{B_{q,qn}}{qn} + O\left(\frac{1}{(qn)^2}\right) \quad (121)$$

We will see later in the lemma 3.5 that in fact

$$\lambda_{qn} = \frac{1}{q} \left(\alpha \cos(b_0 \ln(qn)) + \beta \sin(b_0 \ln(qn)) \right) + O\left(\frac{1}{qn}\right) \quad (122)$$

Let's now study the term $\sum_{i=1}^q \frac{\alpha_{qn+i} \lambda_{qn+i}}{(nq+i)^{2a_0}}$. For simplicity the notations B_i and A_i will be used instead of $B_{i,nq}$ and $A_{i,nq}$.

$$\sum_{i=1}^q \frac{\alpha_{qn+i} \lambda_{qn+i}}{(nq+i)^{2a_0}} = \sum_{i=1}^q \left(x_{qn+i+1} - x_{qn+i} \right) \lambda_{qn+i} \frac{\cos(b_0 \ln(qn+i))}{(nq+i)^{2a_0}} + \left(y_{qn+i+1} - y_{qn+i} \right) \lambda_{qn+i} \frac{\sin(b_0 \ln(qn+i))}{(nq+i)^{2a_0}} \quad (123)$$

$$= \sum_{i=1}^q \left(x_{qn+i+1} - x_{qn+i} \right) \frac{\lambda_{qn} + A_i + \frac{a_0 i \lambda_{nq} + B_i}{nq}}{(nq+i)^{2a_0}} \left(\cos(b_0 \ln(qn)) - \frac{ib_0}{nq} \sin(b_0 \ln(nq)) \right) \quad (124)$$

$$+ \left(y_{qn+i+1} - y_{qn+i} \right) \frac{\lambda_{qn} + A_i + \frac{a_0 i \lambda_{nq} + B_i}{nq}}{(nq+i)^{2a_0}} \left(\sin(b_0 \ln(qn)) + \frac{ib_0}{nq} \cos(b_0 \ln(nq)) \right) + O\left(\frac{1}{(qn)^{2+2a_0}}\right) \quad (125)$$

$$= O\left(\frac{1}{(qn)^{2+2a_0}}\right) + \frac{1}{(nq)^{2a_0}} \sum_{i=1}^q \left(x_{qn+i+1} - x_{qn+i} \right) \left(\lambda_{nq} + A_i \right) \cos(b_0 \ln(qn)) \quad (126)$$

$$+ \frac{1}{(nq)^{2a_0+1}} \sum_{i=1}^q \left(x_{qn+i+1} - x_{qn+i} \right) \left(\left(a_0 i \lambda_{nq} + B_i - 2a_0 i (\lambda_{qn} + A_i) \right) \cos(b_0 \ln(qn)) \right) \quad (127)$$

$$- ib_0 (\lambda_{qn} + A_i) \sin(b_0 \ln(nq)) \Big) + \frac{1}{(nq)^{2a_0}} \sum_{i=1}^q \left(y_{qn+i+1} - y_{qn+i} \right) \left(\lambda_{nq} + A_i \right) \sin(b_0 \ln(qn)) \quad (128)$$

$$+ \frac{1}{(nq)^{2a_0+1}} \sum_{i=1}^q \left(y_{qn+i+1} - y_{qn+i} \right) \left(\left(a_0 i \lambda_{nq} + B_i - 2a_0 i (\lambda_{qn} + A_i) \right) \sin(b_0 \ln(qn)) \right) \quad (129)$$

$$+ ib_0 (\lambda_{qn} + A_i) \cos(b_0 \ln(nq)) \Big) \quad (130)$$

$$= O\left(\frac{1}{(qn)^{2+2a_0}}\right) + \frac{1}{(nq)^{2a_0}} \left\{ \sum_{i=1}^q \left(x_{i+1} - x_i \right) A_i \cos(b_0 \ln(qn)) + \sum_{i=1}^q \left(y_{i+1} - y_i \right) A_i \sin(b_0 \ln(qn)) \right\} \quad (131)$$

$$+ \frac{1}{(nq)^{2a_0+1}} \left\{ \sum_{i=1}^q \left(x_{i+1} - x_i \right) \left(\left(-a_0 i \lambda_{nq} + B_i - 2a_0 i A_i \right) \cos(b_0 \ln(qn)) \right) \right. \quad (132)$$

$$\left. - ib_0 (\lambda_{qn} + A_i) \sin(b_0 \ln(nq)) \right) + \sum_{i=1}^q \left(y_{i+1} - y_i \right) \left(\left(-a_0 i \lambda_{nq} + B_i - 2a_0 i A_i \right) \sin(b_0 \ln(qn)) \right) \quad (133)$$

$$\left. + ib_0 (\lambda_{qn} + A_i) \cos(b_0 \ln(nq)) \right) \Big\} \quad (134)$$

Therefore

$$\sum_{i=1}^q \frac{\alpha_{qn+i} \lambda_{qn+i}}{(nq+i)^{2a_0}} = O\left(\frac{1}{(qn)^{2+2a_0}}\right) + \frac{1}{(nq)^{2a_0}} \left\{ \sum_{i=1}^q (x_{i+1} - x_i) A_i \cos(b_0 \ln(qn)) + \sum_{i=1}^q (y_{i+1} - y_i) A_i \sin(b_0 \ln(qn)) \right\} \quad (135)$$

$$+ \frac{1}{(nq)^{2a_0+1}} \left\{ \sum_{i=1}^q (x_{i+1} - x_i) \left((B_i - 2a_0 i A_i) \cos(b_0 \ln(qn)) - ib_0 (A_i) \sin(b_0 \ln(nq)) \right) \right\} \quad (136)$$

$$+ \sum_{i=1}^q (y_{i+1} - y_i) \left((B_i - 2a_0 i A_i) \sin(b_0 \ln(qn)) + ib_0 (A_i) \cos(b_0 \ln(nq)) \right) \left. \right\} \quad (137)$$

$$+ \frac{\lambda_{qn}}{(nq)^{2a_0+1}} \left\{ \cos(b_0 \ln(qn)) \left(b_0 \sum_{i=1}^q i (y_{i+1} - y_i) - a_0 \sum_{i=1}^q i (x_{i+1} - x_i) \right) \right. \quad (138)$$

$$\left. - \sin(b_0 \ln(qn)) \left(b_0 \sum_{i=1}^q i (x_{i+1} - x_i) + a_0 \sum_{i=1}^q i (y_{i+1} - y_i) \right) \right\} \quad (139)$$

We have

$$\sum_{i=1}^q i (x_{i+1} - x_i) = q \quad (140)$$

$$\sum_{i=1}^{q-1} i (y_{i+1} - y_i) = 0 \quad (141)$$

And We have from the lemma 3.5 that $\lambda_{qn} = \frac{1}{q} (\alpha \cos(b_0 \ln(qn)) + \beta \sin(b_0 \ln(qn))) + O(\frac{1}{qn})$.

Therefore

$$\sum_{i=1}^q \frac{\alpha_{qn+i} \lambda_{qn+i}}{(nq+i)^{2a_0}} = O\left(\frac{1}{(qn)^{2+2a_0}}\right) + \frac{1}{(nq)^{2a_0}} \left\{ \sum_{i=1}^q (x_{i+1} - x_i) A_i \cos(b_0 \ln(qn)) + \sum_{i=1}^q (y_{i+1} - y_i) A_i \sin(b_0 \ln(qn)) \right\} \quad (142)$$

$$+ \frac{1}{(nq)^{2a_0+1}} \left\{ \sum_{i=1}^q (x_{i+1} - x_i) \left((B_i - 2a_0 i A_i) \cos(b_0 \ln(qn)) - ib_0 (A_i) \sin(b_0 \ln(nq)) \right) \right. \quad (143)$$

$$\left. + \sum_{i=1}^q (y_{i+1} - y_i) \left((B_i - 2a_0 i A_i) \sin(b_0 \ln(qn)) + ib_0 (A_i) \cos(b_0 \ln(nq)) \right) \right\} \quad (144)$$

$$- \frac{1}{(nq)^{2a_0+1}} \left(a_0 \cos(b_0 \ln(qn)) + b_0 \sin(b_0 \ln(qn)) \right) \left(\alpha \cos(b_0 \ln(qn)) + \beta \sin(b_0 \ln(qn)) \right) \quad (145)$$

Now the term $\sum_{i=1}^q \gamma_{qn+i}^2$

$$\gamma_{qn+i} = x_{qn+i+1} \cos(b_0 \ln(qn+i)) + y_{qn+i+1} \sin(b_0 \ln(qn+i)) \quad (146)$$

$$= x_{i+1} \left(\cos(b_0 \ln(nq)) - \frac{ib_0}{nq} \sin(b_0 \ln(nq)) \right) + y_{i+1} \left(\sin(b_0 \ln(nq)) + \frac{ib_0}{nq} \cos(b_0 \ln(nq)) \right) + O\left(\frac{1}{(nq)^2}\right) \quad (147)$$

$$= \left(x_{i+1} \cos(b_0 \ln(nq)) + y_{i+1} \sin(b_0 \ln(nq)) \right) + \frac{ib_0}{nq} \left(-x_{i+1} \sin(b_0 \ln(nq)) + y_{i+1} \cos(b_0 \ln(nq)) \right) \quad (148)$$

$$+ O\left(\frac{1}{(nq)^2}\right)$$

Therefore

$$\begin{aligned} \gamma_{qn+i}^2 &= \left(x_{i+1} \cos(b_0 \ln(nq)) + y_{i+1} \sin(b_0 \ln(nq)) \right)^2 + \frac{2ib_0}{nq} \left(-x_{i+1} \sin(b_0 \ln(nq)) \right. \\ &\quad \left. + y_{i+1} \cos(b_0 \ln(nq)) \right) \left(x_{i+1} \cos(b_0 \ln(nq)) + y_{i+1} \sin(b_0 \ln(nq)) \right) + O\left(\frac{1}{(nq)^2}\right) \end{aligned} \quad (149)$$

And

$$\frac{\gamma_{qn+i}^2}{(nq+i)^{2a_0}} = \frac{\left(x_{i+1} \cos(b_0 \ln(nq)) + y_{i+1} \sin(b_0 \ln(nq)) \right)^2}{(nq)^{2a_0}} \quad (151)$$

$$+ \frac{2i}{(nq)^{2a_0+1}} \left(b_0 \left(-x_{i+1} \sin(b_0 \ln(nq)) + y_{i+1} \cos(b_0 \ln(nq)) \right) \left(x_{i+1} \cos(b_0 \ln(nq)) \right. \right. \quad (152)$$

$$\left. \left. + y_{i+1} \sin(b_0 \ln(nq)) \right) - a_0 \left(x_{i+1} \cos(b_0 \ln(nq)) + y_{i+1} \sin(b_0 \ln(nq)) \right)^2 \right) + O\left(\frac{1}{(nq)^{2a_0+2}}\right) \quad (153)$$

$$\sum_{i=1}^q \frac{\gamma_{qn+i}^2}{(nq+i)^{2a_0}} = \sum_{i=1}^q \frac{\left(x_{i+1} \cos(b_0 \ln(nq)) + y_{i+1} \sin(b_0 \ln(nq)) \right)^2}{(nq)^{2a_0}} \quad (154)$$

$$= \sum_{i=1}^q \frac{2i}{(nq)^{2a_0+1}} \left(b_0 \left(-x_{i+1} \sin(b_0 \ln(nq)) + y_{i+1} \cos(b_0 \ln(nq)) \right) \left(x_{i+1} \cos(b_0 \ln(nq)) \right. \right. \quad (155)$$

$$\left. \left. + y_{i+1} \sin(b_0 \ln(nq)) \right) - a_0 \left(x_{i+1} \cos(b_0 \ln(nq)) + y_{i+1} \sin(b_0 \ln(nq)) \right)^2 \right) + O\left(\frac{1}{(qn)^{2a_0+2}}\right) \quad (156)$$

Now the term $(\lambda_n + 2\gamma_n)\beta_n$. For this term, we need only the first order of the asymptotic expansion. We write from the equations (78-79) and (119):

$$\gamma_{qn+i} = x_{i+1} \cos(b_0 \ln(qn)) + y_{i+1} \sin(b_0 \ln(qn)) + O\left(\frac{1}{nq}\right) \quad (157)$$

$$\lambda_{nq+i} = \lambda_{nq} + A_i + O\left(\frac{1}{nq}\right) \quad (158)$$

$$\beta_{nq+i} = x_{i+1}c_{nq} + y_{i+1}cc_{nq} + O\left(\frac{1}{nq}\right) \quad (159)$$

Therefore

$$(\lambda_{nq+i} + 2\gamma_{qn+i})\beta_{nq+i} = \lambda_{nq} \left(x_{i+1}c_{nq} + y_{i+1}cc_{nq} \right) \quad (160)$$

$$+ \left(A_i + 2x_{i+1} \cos(b_0 \ln(qn)) + 2y_{i+1} \sin(b_0 \ln(qn)) \right) \left(x_{i+1}c_{nq} + y_{i+1}cc_{nq} \right) + O\left(\frac{1}{nq}\right) \quad (161)$$

Therefore, thanks to the cancellation property of a non-trivial character $\left(\sum_{i=1}^q x_{i+1} = \sum_{i=1}^q y_{i+1} = 0 \right)$ and the equations (69-70), we can

write:

$$\sum_{i=1}^q \frac{(\lambda_{nq+i} + 2\gamma_{qn+i})\beta_{nq+i}}{(nq+i)^{2a_0+1}} = \lambda_{nq} \sum_{i=1}^q \frac{(x_{i+1}c_{nq} + y_{i+1}cc_{nq})}{(nq)^{2a_0+1}} \quad (162)$$

$$+ \sum_{i=1}^q \frac{(A_i + 2x_{i+1} \cos(b_0 \ln(qn)) + 2y_{i+1} \sin(b_0 \ln(qn))) (x_{i+1}c_{nq} + y_{i+1}cc_{nq})}{(nq)^{2a_0+1}} + O\left(\frac{1}{(nq)^{2a_0+2}}\right) \quad (163)$$

$$= \sum_{i=1}^q \frac{(A_i + 2x_{i+1} \cos(b_0 \ln(qn)) + 2y_{i+1} \sin(b_0 \ln(qn))) (x_{i+1}c_{nq} + y_{i+1}cc_{nq})}{(nq)^{2a_0+1}} + O\left(\frac{1}{(nq)^{2a_0+2}}\right) \quad (164)$$

Now let's study the term $T_n = \sum_{i=1}^q \frac{\alpha_{qn+i}\lambda_{qn+i}}{(nq+i)^{2a_0}} + \frac{\gamma_{qn+i}^2}{(nq+i)^{2a_0}} + \frac{(\lambda_{nq+i} + 2\gamma_{qn+i})\beta_{nq+i}}{(nq+i)^{2a_0+1}}$.

We have the coefficient of the order of $(nq)^{2a_0}$ in the asymptotic expansion of T_n is zero:

$$0 = \frac{1}{(nq)^{2a_0}} \left\{ \sum_{i=1}^q (x_{i+1} - x_i) A_i \cos(b_0 \ln(qn)) + \sum_{i=1}^q (y_{i+1} - y_i) A_i \sin(b_0 \ln(qn)) \right\} \quad (165)$$

$$+ \sum_{i=1}^q \frac{(x_{i+1} \cos(b_0 \ln(nq)) + y_{i+1} \sin(b_0 \ln(nq)))^2}{(nq)^{2a_0}} \quad (166)$$

We are left with the term of the order $(nq)^{2a_0+1}$. Let's now prove that this term is nonzero. The coefficient of this term is the following: $T_1 + T_2 + T_3 + T_4 + T_5$ where

$$T_1 = \sum_{i=1}^q (x_{i+1} - x_i) \left((B_i - 2a_0 i A_i) \cos(b_0 \ln(qn)) - i b_0 (A_i) \sin(b_0 \ln(nq)) \right) \quad (167)$$

$$T_2 = \sum_{i=1}^q (y_{i+1} - y_i) \left((B_i - 2a_0 i A_i) \sin(b_0 \ln(qn)) + i b_0 (A_i) \cos(b_0 \ln(nq)) \right) \quad (168)$$

$$T_3 = \sum_{i=1}^q 2i \left(b_0 (-x_{i+1} \sin(b_0 \ln(nq)) + y_{i+1} \cos(b_0 \ln(nq))) (x_{i+1} \cos(b_0 \ln(nq)) \right) \quad (169)$$

$$+ y_{i+1} \sin(b_0 \ln(nq)) - a_0 (x_{i+1} \cos(b_0 \ln(nq)) + y_{i+1} \sin(b_0 \ln(nq)))^2 \right) \quad (170)$$

$$T_4 = \sum_{i=1}^q (A_i + 2x_{i+1} \cos(b_0 \ln(qn)) + 2y_{i+1} \sin(b_0 \ln(qn))) (x_{i+1}c_{nq} + y_{i+1}cc_{nq}) \quad (171)$$

$$T_5 = - (a_0 \cos(b_0 \ln(qn)) + b_0 \sin(b_0 \ln(qn))) (\alpha \cos(b_0 \ln(qn)) + \beta \sin(b_0 \ln(qn))) \quad (172)$$

Let's now write the terms T_i in the form of:

$$T_i = T a_i + T b_i \cos(2b_0 \ln(qn)) + T c_i \sin(2b_0 \ln(qn)) \quad (173)$$

$$T_3 = \sum_{i=1}^q \left(ib_0 \left((y_{i+1}^2 - x_{i+1}^2) \sin(2b_0 \ln(nq)) + 2x_{i+1}y_{i+1} \cos(2b_0 \ln(nq)) \right) \right. \quad (174)$$

$$\left. - a_0 i \left((x_{i+1}^2 + y_{i+1}^2) + (x_{i+1}^2 - y_{i+1}^2) \cos(2b_0 \ln(nq)) + 2x_{i+1}y_{i+1} \sin(2b_0 \ln(nq)) \right) \right) \quad (175)$$

$$= -a_0 \sum_{i=1}^q i(x_{i+1}^2 + y_{i+1}^2) + \cos(2b_0 \ln(nq)) \sum_{i=1}^q i \left(a_0(y_{i+1}^2 - x_{i+1}^2) + 2b_0 x_{i+1}y_{i+1} \right) \quad (176)$$

$$+ \sin(2b_0 \ln(nq)) \sum_{i=1}^q i \left(b_0(y_{i+1}^2 - x_{i+1}^2) - 2a_0 x_{i+1}y_{i+1} \right) \quad (177)$$

And

$$Ta_3 = -a_0 \sum_{i=1}^q i(x_{i+1}^2 + y_{i+1}^2) \quad (178)$$

$$Tb_3 = \sum_{i=1}^q i \left(a_0(y_{i+1}^2 - x_{i+1}^2) + 2b_0 x_{i+1}y_{i+1} \right) \quad (179)$$

$$Tc_3 = \sum_{i=1}^q i \left(b_0(y_{i+1}^2 - x_{i+1}^2) - 2a_0 x_{i+1}y_{i+1} \right) \quad (180)$$

We have

$$\sum_{i=1}^q i(x_{i+1}^2 + y_{i+1}^2) = \sum_{i=2}^{q+1} (i-1)(x_i^2 + y_i^2) \quad (181)$$

$$= q + \sum_{i=2}^{q-1} (i-1)(x_i^2 + y_i^2) \quad (x_q = y_q = 0, x_{q+1} = x_1 = 1, y_{q+1} = y_1 = 0) \quad (182)$$

$$= q + \sum_{i=2}^{q-1} (i-1) \quad (183)$$

$$= 1 + \frac{q(q-1)}{2} \quad (184)$$

$$T_4 = \sum_{i=1}^q A_i \left(x_{i+1}c_{nq} + y_{i+1}c_{nq} \right) \quad (185)$$

$$+ 2 \left(x_{i+1} \cos(b_0 \ln(qn)) + y_{i+1} \sin(b_0 \ln(qn)) \right) \left(x_{i+1}c_{nq} + y_{i+1}c_{nq} \right) \quad (186)$$

$$= T_{4,1} + T_{4,2} \quad (187)$$

$$T_{4,2} = 2 \sum_{i=1}^q \left(x_{i+1} \cos(b_0 \ln(qn)) + y_{i+1} \sin(b_0 \ln(qn)) \right) \left(x_{i+1} c_{nq} + y_{i+1} c c_{nq} \right) \quad (188)$$

$$= 2 \sum_{i=1}^q \left(x_{i+1} \cos(b_0 \ln(qn)) + y_{i+1} \sin(b_0 \ln(qn)) \right) \quad (189)$$

$$\left((-a_0 x_{i+1} + b_0 y_{i+1}) \cos(b_0 \ln(qn)) - (b_0 x_{i+1} + a_0 y_{i+1}) \sin(b_0 \ln(qn)) \right) \quad (190)$$

$$= -a_0 \sum_{i=1}^q (x_{i+1}^2 + y_{i+1}^2) + \cos(2b_0 \ln(nq)) \sum_{i=1}^q (a_0(y_{i+1}^2 - x_{i+1}^2) + 2b_0 x_{i+1} y_{i+1}) \quad (191)$$

$$+ \sin(2b_0 \ln(nq)) \sum_{i=1}^q (b_0(y_{i+1}^2 - x_{i+1}^2) - 2a_0 x_{i+1} y_{i+1}) \quad (192)$$

$$(193)$$

And

$$\sum_{i=1}^q (x_{i+1}^2 + y_{i+1}^2) = \sum_{i=2}^{q+1} (x_i^2 + y_i^2) \quad (194)$$

$$= \sum_{i=1}^{q-1} (x_i^2 + y_i^2) \quad (x_q = y_q = 0, x_{q+1} = x_1 = 1, y_{q+1} = y_1 = 0) \quad (195)$$

$$= q - 1 \quad (196)$$

And

$$T_{4,1} = \sum_{i=1}^q A_i (x_{i+1} c_{nq} + y_{i+1} c c_{nq}) \quad (197)$$

$$= \sum_{i=1}^q \left(\cos(b_0 \ln(qn)) \sum_{k=1}^i x_k + \sin(b_0 \ln(qn)) \sum_{k=1}^i y_k \right) \quad (198)$$

$$\left((-a_0 x_{i+1} + b_0 y_{i+1}) \cos(b_0 \ln(qn)) - (b_0 x_{i+1} + a_0 y_{i+1}) \sin(b_0 \ln(qn)) \right) \quad (199)$$

$$= \frac{1}{2} \left(b_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i x_k - b_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i y_k - a_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i x_k - a_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i y_k \right) \quad (200)$$

$$+ \frac{\cos(2b_0 \ln(nq))}{2} \left(b_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i x_k + b_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i y_k - a_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i x_k + a_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i y_k \right) \quad (201)$$

$$+ \frac{\sin(2b_0 \ln(nq))}{2} \left(-a_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i x_k - a_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i y_k - b_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i x_k + b_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i y_k \right) \quad (202)$$

Now T_1

$$T_1 = \sum_{i=1}^q (x_{i+1} - x_i) \left((B_i - 2a_0 i A_i) \cos(b_0 \ln(qn)) - ib_0 A_i \sin(b_0 \ln(qn)) \right) \quad (203)$$

$$B_i = b_0 \left(-\sin(b_0 \ln(nq)) \sum_{k=1}^i kx_k + \cos(b_0 \ln(nq)) \sum_{k=1}^i ky_k \right) \quad (204)$$

$$+ a_0 \left(\cos(b_0 \ln(nq)) \sum_{k=1}^i \sum_{l=1}^{k-1} x_l + \sin(b_0 \ln(nq)) \sum_{k=1}^i \sum_{l=1}^{k-1} y_l \right) \quad (205)$$

$$= b_0 \left(-\sin(b_0 \ln(nq)) \sum_{k=1}^i kx_k + \cos(b_0 \ln(nq)) \sum_{k=1}^i ky_k \right) \quad (206)$$

$$+ a_0 \left(-\cos(b_0 \ln(nq)) \sum_{k=1}^i kx_k - \sin(b_0 \ln(nq)) \sum_{k=1}^i ky_k \right) \quad (207)$$

$$+ a_0 \left(\cos(b_0 \ln(nq)) i \sum_{k=1}^i x_k + \sin(b_0 \ln(nq)) i \sum_{k=1}^i y_k \right) \quad (208)$$

$$= -\sin(b_0 \ln(nq)) \left(b_0 \sum_{k=1}^i kx_k + a_0 \sum_{k=1}^i ky_k - a_0 i \sum_{k=1}^i y_k \right) \quad (209)$$

$$+ \cos(b_0 \ln(nq)) \left(b_0 \sum_{k=1}^i ky_k - a_0 \sum_{k=1}^i kx_k + a_0 i \sum_{k=1}^i x_k \right) \quad (210)$$

And

$$T_{1,1}^i = B_i \cos(b_0 \ln(qn)) \quad (211)$$

$$= \frac{1}{2} \left(b_0 \sum_{k=1}^i ky_k - a_0 \sum_{k=1}^i kx_k + a_0 i \sum_{k=1}^i x_k \right) + \frac{1}{2} \left(b_0 \sum_{k=1}^i ky_k - a_0 \sum_{k=1}^i kx_k + a_0 i \sum_{k=1}^i x_k \right) \cos(2b_0 \ln(qn)) \quad (212)$$

$$- \frac{1}{2} \left(b_0 \sum_{k=1}^i kx_k + a_0 \sum_{k=1}^i ky_k - a_0 i \sum_{k=1}^i y_k \right) \sin(2b_0 \ln(qn)) \quad (213)$$

And

$$T_{1,2}^i = -2a_0 i A_i \cos(b_0 \ln(qn)) - ib_0 A_i \sin(b_0 \ln(nq)) \quad (214)$$

$$= -i A_i \left(2a_0 \cos(b_0 \ln(qn)) + b_0 \sin(b_0 \ln(nq)) \right) \quad (215)$$

$$= -i \left(\cos(b_0 \ln(qn)) \sum_{k=1}^i x_k + \sin(b_0 \ln(nq)) \sum_{k=1}^i y_k \right) \left(2a_0 \cos(b_0 \ln(qn)) + b_0 \sin(b_0 \ln(nq)) \right) \quad (216)$$

$$= -i \left(\frac{1}{2} \left(2a_0 \sum_{k=1}^i x_k + b_0 \sum_{k=1}^i y_k \right) + \frac{1}{2} \left(2a_0 \sum_{k=1}^i x_k - b_0 \sum_{k=1}^i y_k \right) \cos(2b_0 \ln(qn)) \right) \quad (217)$$

$$+ \frac{1}{2} \left(2a_0 \sum_{k=1}^i y_k + b_0 \sum_{k=1}^i x_k \right) \sin(2b_0 \ln(qn)) \quad (218)$$

Therefore

$$T_1 = \sum_{i=1}^q (x_{i+1} - x_i) (T_{1,1}^i + T_{1,2}^i) \quad (219)$$

$$= \sum_{i=1}^q (x_{i+1} - x_i) \left(\frac{b_0 \left(\sum_{k=1}^i ky_k - i \sum_{k=1}^i y_k \right) - a_0 \left(\sum_{k=1}^i kx_k + i \sum_{k=1}^i x_k \right)}{2} \right) \quad (220)$$

$$+ \frac{b_0 \left(\sum_{k=1}^i ky_k + i \sum_{k=1}^i y_k \right) - a_0 \left(\sum_{k=1}^i kx_k + i \sum_{k=1}^i x_k \right)}{2} \cos(2b_0 \ln(qn)) \quad (221)$$

$$+ \frac{-a_0 \left(\sum_{k=1}^i ky_k + i \sum_{k=1}^i y_k \right) - b_0 \left(\sum_{k=1}^i kx_k + i \sum_{k=1}^i x_k \right)}{2} \sin(2b_0 \ln(qn)) \quad (222)$$

And

$$T_{2,1}^i = B_i \sin(b_0 \ln(qn)) \quad (223)$$

$$= -\frac{1}{2} \left(b_0 \sum_{k=1}^i kx_k + a_0 \sum_{k=1}^i ky_k - a_0 i \sum_{k=1}^i y_k \right) + \frac{1}{2} \left(b_0 \sum_{k=1}^i kx_k + a_0 \sum_{k=1}^i ky_k - a_0 i \sum_{k=1}^i y_k \right) \cos(2b_0 \ln(qn)) \quad (224)$$

$$+ \frac{1}{2} \left(b_0 \sum_{k=1}^i ky_k - a_0 \sum_{k=1}^i kx_k + a_0 i \sum_{k=1}^i x_k \right) \sin(2b_0 \ln(qn)) \quad (225)$$

And

$$T_{2,2}^i = -2a_0 i A_i \sin(b_0 \ln(qn)) + i b_0 A_i \cos(b_0 \ln(nq)) \quad (226)$$

$$= i A_i \left(-2a_0 \sin(b_0 \ln(qn)) + b_0 \cos(b_0 \ln(qn)) \right) \quad (227)$$

$$= i \left(\cos(b_0 \ln(qn)) \sum_{k=1}^i x_k + \sin(b_0 \ln(qn)) \sum_{k=1}^i y_k \right) \left(-2a_0 \sin(b_0 \ln(qn)) + b_0 \cos(b_0 \ln(qn)) \right) \quad (228)$$

$$= i \left(\frac{1}{2} \left(b_0 \sum_{k=1}^i x_k - 2a_0 \sum_{k=1}^i y_k \right) + \frac{1}{2} \left(b_0 \sum_{k=1}^i x_k + 2a_0 \sum_{k=1}^i y_k \right) \cos(b_0 \ln(qn)) \right) \quad (229)$$

$$+ \frac{1}{2} \left(b_0 \sum_{k=1}^i y_k - 2a_0 \sum_{k=1}^i x_k \right) \sin(b_0 \ln(qn)) \quad (230)$$

Therefore

$$T_2 = \sum_{i=1}^q (y_{i+1} - y_i) (T_{2,1}^i + T_{2,2}^i) \quad (231)$$

$$= \sum_{i=1}^q (y_{i+1} - y_i) \left(\frac{b_0 \left(i \sum_{k=1}^i x_k - \sum_{k=1}^i kx_k \right) - a_0 \left(i \sum_{k=1}^i y_k + \sum_{k=1}^i ky_k \right)}{2} \right) \quad (232)$$

$$+ \frac{b_0 \left(i \sum_{k=1}^i x_k + \sum_{k=1}^i kx_k \right) + a_0 \left(i \sum_{k=1}^i y_k + \sum_{k=1}^i ky_k \right)}{2} \cos(2b_0 \ln(qn)) \quad (233)$$

$$+ \frac{b_0 \left(i \sum_{k=1}^i y_k + \sum_{k=1}^i ky_k \right) - a_0 \left(i \sum_{k=1}^i x_k + \sum_{k=1}^i kx_k \right)}{2} \sin(2b_0 \ln(qn)) \quad (234)$$

Let's now aggregate the terms T_i in the form of:

$$T = \sum_{i=1}^4 T_i = \sum_{i=1}^4 T a_i + \cos(2b_0 \ln(qn)) \sum_{i=1}^4 T b_i + \sin(2b_0 \ln(qn)) \sum_{i=1}^4 T c_i \quad (235)$$

$$= T a + \cos(2b_0 \ln(qn)) T b + \sin(2b_0 \ln(qn)) T c \quad (236)$$

Let's start with the first term $T a$.

$$T a = \sum_{i=1}^5 T a_i \quad (237)$$

Where

$$T a_1 = \sum_{i=1}^q (x_{i+1} - x_i) \frac{b_0 \left(\sum_{k=1}^i k y_k - i \sum_{k=1}^i y_k \right) - a_0 \left(\sum_{k=1}^i k x_k + i \sum_{k=1}^i x_k \right)}{2} \quad (238)$$

$$T a_2 = \sum_{i=1}^q (y_{i+1} - y_i) \frac{b_0 \left(i \sum_{k=1}^i x_k - \sum_{k=1}^i k x_k \right) - a_0 \left(i \sum_{k=1}^i y_k + \sum_{k=1}^i k y_k \right)}{2} \quad (239)$$

$$T a_3 = -a_0 \sum_{i=1}^q i (x_{i+1}^2 + y_{i+1}^2) \quad (240)$$

$$T a_4 = -a_0 \sum_{i=1}^q (x_{i+1}^2 + y_{i+1}^2) + \frac{1}{2} \left(b_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i x_k - b_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i y_k \right) \quad (241)$$

$$-a_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i x_k - a_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i y_k \quad (242)$$

We have $T a_1$ and $T a_2$ contain many terms that cancel each other.

$$T a_1 + T a_2 = -\frac{a_0}{2} \left(\sum_{i=1}^q (x_{i+1} - x_i) \left(\sum_{k=1}^i k x_k + i \sum_{k=1}^i x_k \right) + (y_{i+1} - y_i) \left(i \sum_{k=1}^i y_k + \sum_{k=1}^i k y_k \right) \right) \quad (243)$$

$$+ \frac{b_0}{2} \left(\sum_{i=1}^q (x_{i+1} - x_i) \left(\sum_{k=1}^i k y_k - i \sum_{k=1}^i y_k \right) + \sum_{i=1}^{q-1} (y_{i+1} - y_i) \left(i \sum_{k=1}^i x_k - \sum_{k=1}^i k x_k \right) \right) \quad (244)$$

$$= -\frac{a_0}{2} \left(\sum_{k=1}^q k x_k + \sum_{k=1}^q k y_k - 2 \sum_{i=1}^q k x_k^2 + \sum_{k=1}^{q-1} x_k \sum_{i=1}^k x_i - 2 \sum_{i=1}^q k y_k^2 + \sum_{k=1}^{q-1} y_k \sum_{i=1}^k y_i \right) \quad (245)$$

$$+ \frac{b_0}{2} \left(\sum_{k=1}^q k y_k - \sum_{k=1}^q k x_k + 2 \sum_{i=1}^{q-1} y_i \sum_{k=1}^i x_k - \sum_{k=1}^q x_k y_k \right) \quad (246)$$

And

$$Ta_4 = -a_0 \sum_{i=1}^q (x_{i+1}^2 + y_{i+1}^2) + \frac{1}{2} \left(b_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i x_k - b_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i y_k \right) \quad (247)$$

$$-a_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i x_k - a_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i y_k \quad (248)$$

$$= -a_0 \sum_{i=1}^q (x_{i+1}^2 + y_{i+1}^2) + \frac{1}{2} \left(-2b_0 \sum_{i=1}^{q-1} x_i \sum_{k=1}^i y_k - a_0 \sum_{i=1}^{q-1} x_i \sum_{k=1}^i x_k \right) \quad (249)$$

$$-a_0 \sum_{i=1}^{q-1} y_i \sum_{k=1}^i y_k - a_0 \sum_{i=1}^{q-1} (x_i^2 + y_i^2) \quad (250)$$

$$= -\frac{2a_0 + 1}{2} \sum_{i=1}^{q-1} (x_i^2 + y_i^2) + \frac{1}{2} \left(2b_0 \sum_{i=1}^{q-1} y_i \sum_{k=1}^i x_k - 2b_0 \sum_{i=1}^{q-1} x_i y_i - a_0 \sum_{i=1}^{q-1} x_i \sum_{k=1}^i x_k \right) \quad (251)$$

$$-a_0 \sum_{i=1}^{q-1} y_i \sum_{k=1}^i y_k \quad (252)$$

And

$$Ta_3 = -a_0 \sum_{i=1}^q i(x_{i+1}^2 + y_{i+1}^2) \quad (253)$$

$$= -a_0 \sum_{i=2}^{q+1} (i-1)(x_i^2 + y_i^2) \quad (254)$$

$$= -a_0 q - a_0 \sum_{i=1}^{q-1} (i-1)(x_i^2 + y_i^2) \quad (255)$$

And

$$Ta_5 = -\frac{a_0 \alpha + b_0 \beta}{2} \quad (256)$$

And finally

$$Ta = -a_0 q - \frac{2a_0 + 1}{2} \sum_{i=1}^{q-1} (x_i^2 + y_i^2) + a_0 \sum_{i=1}^{q-1} (i - (i-1))(x_i^2 + y_i^2) - \frac{a_0}{2} \left(\sum_{k=1}^{q-1} kx_k + \sum_{k=1}^{q-1} ky_k \right) \quad (257)$$

$$- \frac{2a_0}{2} \left(\sum_{i=1}^{q-1} x_i \sum_{k=1}^i x_k + \sum_{i=1}^{q-1} y_i \sum_{k=1}^i y_k \right) + \frac{b_0}{2} \left(\sum_{k=1}^{q-1} ky_k - \sum_{k=1}^q kx_k + 4 \sum_{k=1}^{q-1} y_k \sum_{i=1}^k x_i - 3 \sum_{k=1}^{q-1} x_k y_k \right) \quad (258)$$

$$- \frac{a_0 \alpha + b_0 \beta}{2} \quad (259)$$

$$= -a_0 q - \frac{a_0}{2} \sum_{i=1}^{q-1} (x_i^2 + y_i^2) - \frac{a_0}{2} \left(\sum_{k=1}^{q-1} kx_k + \sum_{k=1}^{q-1} ky_k \right) - \frac{2a_0}{2} \left(\sum_{i=1}^{q-1} x_i \sum_{k=1}^i x_k + \sum_{i=1}^{q-1} y_i \sum_{k=1}^i y_k \right) \quad (260)$$

$$+ \frac{b_0}{2} \left(\sum_{k=1}^{q-1} ky_k - \sum_{k=1}^q kx_k + 4 \sum_{k=1}^{q-1} y_k \sum_{i=1}^k x_i - 3 \sum_{k=1}^{q-1} x_k y_k \right) - \frac{a_0 \alpha + b_0 \beta}{2} \quad (261)$$

Let's now deal with the second term Tb .

$$Tb = \sum_{i=1}^5 Tb_i \quad (262)$$

Where

$$Tb_1 = \sum_{i=1}^q (x_{i+1} - x_i) \frac{b_0 \left(\sum_{k=1}^i ky_k + i \sum_{k=1}^i y_k \right) - a_0 \left(\sum_{k=1}^i kx_k + i \sum_{k=1}^i x_k \right)}{2} \quad (263)$$

$$Tb_2 = \sum_{i=1}^q (y_{i+1} - y_i) \frac{b_0 \left(i \sum_{k=1}^i x_k + \sum_{k=1}^i kx_k \right) + a_0 \left(i \sum_{k=1}^i y_k + \sum_{k=1}^i ky_k \right)}{2} \quad (264)$$

$$Tb_3 = 2b_0 \sum_{i=1}^q ix_{i+1}y_{i+1} + a_0 \sum_{i=1}^{q-1} i(x_{i+1}^2 - y_{i+1}^2) \quad (265)$$

$$Tb_4 = 2b_0 \sum_{i=1}^q x_{i+1}y_{i+1} + a_0 \sum_{i=1}^{q-1} (x_{i+1}^2 - y_{i+1}^2) \quad (266)$$

$$+ \frac{1}{2} \left(b_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i x_k + b_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i y_k - a_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i x_k + a_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i y_k \right) \quad (267)$$

We have Tb_1 and Tb_2 contain many terms that cancel each other.

$$Tb_1 + Tb_2 = \frac{a_0}{2} \left(\sum_{i=1}^q - (x_{i+1} - x_i) \left(\sum_{k=1}^i kx_k + i \sum_{k=1}^i x_k \right) + (y_{i+1} - y_i) \left(i \sum_{k=1}^i y_k + \sum_{k=1}^i ky_k \right) \right) \quad (268)$$

$$+ \frac{b_0}{2} \left(\sum_{i=1}^q (x_{i+1} - x_i) \left(\sum_{k=1}^i ky_k + i \sum_{k=1}^i y_k \right) + \sum_{i=1}^q (y_{i+1} - y_i) \left(i \sum_{k=1}^i x_k + \sum_{k=1}^i kx_k \right) \right) \quad (269)$$

$$= \frac{a_0}{2} \left(- \left[\sum_{i=1}^{q-1} kx_k - 2 \sum_{i=1}^{q-1} kx_k^2 + \sum_{k=1}^{q-1} x_k \sum_{i=1}^k x_i \right] + \sum_{i=1}^{q-1} ky_k - 2 \sum_{i=1}^{q-1} ky_k^2 + \sum_{k=1}^{q-1} y_k \sum_{i=1}^k y_i \right) \quad (270)$$

$$+ \frac{b_0}{2} \left(\sum_{k=1}^{q-1} ky_k + \sum_{i=1}^{q-1} y_i \sum_{k=1}^i x_k - 2 \sum_{k=1}^{q-1} kx_k y_k + \sum_{k=1}^{q-1} kx_k - 2 \sum_{k=1}^{q-1} kx_k y_k - \sum_{k=1}^{q-1} y_k \sum_{i=1}^k x_i + \sum_{k=1}^{q-1} x_k y_k \right) \quad (271)$$

And

$$Tb_4 = 2b_0 \sum_{i=1}^q x_{i+1}y_{i+1} + a_0 \sum_{i=1}^q (x_{i+1}^2 - y_{i+1}^2) + \frac{1}{2} \left(b_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i x_k \right. \quad (272)$$

$$\left. + b_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i y_k - a_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i x_k + a_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i y_k \right) \quad (273)$$

$$= 2b_0 \sum_{i=1}^{q-1} x_i y_i + a_0 \sum_{i=1}^{q-1} (x_i^2 - y_i^2) + \frac{1}{2} \left(-b_0 \sum_{i=1}^{q-1} x_i \sum_{k=1}^i y_k + b_0 \sum_{i=1}^{q-1} x_i \sum_{k=1}^i y_k \right. \quad (274)$$

$$\left. - b_0 \sum_{i=1}^{q-1} x_i y_i - a_0 \sum_{i=1}^{q-1} x_i \sum_{k=1}^i x_k + a_0 \sum_{i=1}^{q-1} y_i \sum_{k=1}^i y_k + a_0 \sum_{i=1}^{q-1} (x_i^2 - y_i^2) \right) \quad (275)$$

$$= \frac{3b_0}{2} \sum_{i=1}^{q-1} x_i y_i + \frac{2a_0 + 1}{2} \sum_{i=1}^{q-1} (x_i^2 - y_i^2) + \frac{a_0}{2} \left(\sum_{i=1}^{q-1} y_i \sum_{k=1}^i y_k - \sum_{i=1}^{q-1} x_i \sum_{k=1}^i x_k \right) \quad (276)$$

And

$$Tb_3 = 2b_0 \sum_{i=1}^q i x_{i+1} y_{i+1} + a_0 \sum_{i=1}^q i (x_{i+1}^2 - y_{i+1}^2) \quad (277)$$

$$= 2b_0 \sum_{i=2}^{q+1} (i-1) x_i y_i + a_0 \sum_{i=2}^{q+1} (i-1) (x_i^2 - y_i^2) \quad (278)$$

And

$$Tb_5 = -\frac{a_0 \alpha - b_0 \beta}{2} \quad (279)$$

And finally

$$Tb = \frac{a_0}{2} \left(- \left[\sum_{k=1}^{q-1} k x_k - 2 \sum_{i=1}^{q-1} k x_k^2 + \sum_{k=1}^{q-1} x_k \sum_{i=1}^k x_i \right] + \sum_{k=1}^{q-1} k y_k - 2 \sum_{i=1}^{q-1} k y_k^2 + \sum_{k=1}^{q-1} y_k \sum_{i=1}^k y_i \right) \quad (280)$$

$$+ \frac{b_0}{2} \left(\sum_{k=1}^{q-1} k y_k - 4 \sum_{k=1}^{q-1} k x_k y_k + \sum_{k=1}^{q-1} k x_k + \sum_{k=1}^{q-1} x_k y_k \right) \quad (281)$$

$$+ \frac{3b_0}{2} \sum_{i=1}^{q-1} x_i y_i + \frac{2a_0 + 1}{2} \sum_{i=1}^{q-1} (x_i^2 - y_i^2) + \frac{a_0}{2} \left(\sum_{i=1}^{q-1} y_i \sum_{k=1}^i y_k - \sum_{i=1}^{q-1} x_i \sum_{k=1}^i x_k \right) \quad (282)$$

$$+ 2b_0 \sum_{i=1}^{q+1} (i-1) x_i y_i + a_0 \sum_{i=1}^{q+1} (i-1) (x_i^2 - y_i^2) - \frac{a_0 \alpha - b_0 \beta}{2} \quad (283)$$

Let's now deal with the third term Tc .

$$Tc = \sum_{i=1}^5 Tc_i \quad (284)$$

Where

$$Tc_1 = - \sum_{i=1}^q (x_{i+1} - x_i) \frac{a_0 \left(\sum_{k=1}^i ky_k + i \sum_{k=1}^i y_k \right) + b_0 \left(\sum_{k=1}^i kx_k + i \sum_{k=1}^i x_k \right)}{2} \quad (285)$$

$$Tc_2 = \sum_{i=1}^q (y_{i+1} - y_i) \frac{b_0 \left(i \sum_{k=1}^i y_k + \sum_{k=1}^i ky_k \right) - a_0 \left(i \sum_{k=1}^i x_k + \sum_{k=1}^i kx_k \right)}{2} \quad (286)$$

$$Tc_3 = 2b_0 \sum_{i=1}^q ix_{i+1}y_{i+1} + a_0 \sum_{i=1}^{q-1} i(x_{i+1}^2 - y_{i+1}^2) \quad (287)$$

$$Tc_4 = 2b_0 \sum_{i=1}^q x_{i+1}y_{i+1} + a_0 \sum_{i=1}^{q-1} (x_{i+1}^2 - y_{i+1}^2) \quad (288)$$

$$+ \frac{1}{2} \left(b_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i x_k + b_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i y_k - a_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i x_k + a_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i y_k \right) \quad (289)$$

We have Tc_1 and Tc_2 contain many terms that cancel each other.

$$Tc_1 + Tc_2 = \frac{b_0}{2} \left(\sum_{i=1}^q (y_{i+1} - y_i) \left(\sum_{k=1}^i ky_k + i \sum_{k=1}^i y_k \right) - (x_{i+1} - x_i) \left(i \sum_{k=1}^i x_k + \sum_{k=1}^i kx_k \right) \right) \quad (290)$$

$$- \frac{a_0}{2} \left(\sum_{i=1}^{q-1} (x_{i+1} - x_i) \left(\sum_{k=1}^i ky_k + i \sum_{k=1}^i y_k \right) + \sum_{i=1}^{q-1} (y_{i+1} - y_i) \left(i \sum_{k=1}^i x_k + \sum_{k=1}^i kx_k \right) \right) \quad (291)$$

$$= \frac{b_0}{2} \left(\left[\sum_{i=1}^{q-1} ky_k - 2 \sum_{i=1}^{q-1} ky_k^2 + \sum_{k=1}^{q-1} y_k \sum_{i=1}^k y_i \right] - \left[\sum_{i=1}^{q-1} kx_k - 2 \sum_{i=1}^{q-1} kx_k^2 + \sum_{k=1}^{q-1} x_k \sum_{i=1}^k x_i \right] \right) \quad (292)$$

$$- \frac{a_0}{2} \left(\sum_{i=1}^{q-1} ky_k + \sum_{i=1}^{q-1} y_i \sum_{k=1}^i x_k - 2 \sum_{k=1}^{q-1} kx_k y_k + \sum_{i=1}^{q-1} kx_k - \sum_{k=1}^{q-1} y_k \sum_{i=1}^k x_i - 2 \sum_{k=1}^{q-1} kx_k y_k + \sum_{k=1}^{q-1} x_k y_k \right) \quad (293)$$

$$= \frac{b_0}{2} \left(\left[\sum_{i=1}^{q-1} ky_k - 2 \sum_{i=1}^{q-1} ky_k^2 + \sum_{k=1}^{q-1} y_k \sum_{i=1}^k y_i \right] - \left[\sum_{i=1}^{q-1} kx_k - 2 \sum_{i=1}^{q-1} kx_k^2 + \sum_{k=1}^{q-1} x_k \sum_{i=1}^k x_i \right] \right) \quad (294)$$

$$- \frac{a_0}{2} \left(\sum_{i=1}^{q-1} kx_k + \sum_{i=1}^{q-1} ky_k - 4 \sum_{k=1}^{q-1} kx_k y_k + \sum_{k=1}^{q-1} x_k y_k \right) \quad (295)$$

And

$$Tc_4 = b_0 \sum_{i=1}^q (y_{i+1}^2 - x_{i+1}^2) - 2a_0 \sum_{i=1}^q x_{i+1} y_{i+1} + \frac{1}{2} \left(-a_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i x_k \right. \quad (296)$$

$$\left. -a_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i y_k - b_0 \sum_{i=1}^{q-1} x_{i+1} \sum_{k=1}^i x_k + b_0 \sum_{i=1}^{q-1} y_{i+1} \sum_{k=1}^i y_k \right) \quad (297)$$

$$= -2a_0 \sum_{i=1}^{q-1} x_i y_i + b_0 \sum_{i=1}^{q-1} (y_i^2 - x_i^2) + \frac{1}{2} \left(a_0 \sum_{i=1}^{q-1} x_i \sum_{k=1}^i y_k - a_0 \sum_{i=1}^{q-1} x_i \sum_{k=1}^i y_k \right. \quad (298)$$

$$\left. + a_0 \sum_{i=1}^{q-1} x_i y_i - b_0 \sum_{i=1}^{q-1} x_i \sum_{k=1}^i x_k + b_0 \sum_{i=1}^{q-1} y_i \sum_{k=1}^i y_k + b_0 \sum_{i=1}^{q-1} (x_i^2 - y_i^2) \right) \quad (299)$$

$$= -\frac{2a_0 + 1}{2} \sum_{i=1}^{q-1} x_i y_i - \frac{b_0}{2} \sum_{i=1}^{q-1} (x_i^2 - y_i^2) + \frac{b_0}{2} \left(\sum_{i=1}^{q-1} y_i \sum_{k=1}^i y_k - \sum_{i=1}^{q-1} x_i \sum_{k=1}^i x_k \right) \quad (300)$$

And

$$Tc_3 = -b_0 \sum_{i=1}^q i(x_{i+1}^2 - y_{i+1}^2) - 2a_0 \sum_{i=1}^q i x_{i+1} y_{i+1} \quad (301)$$

$$= -2a_0 \sum_{i=1}^{q+1} (i-1) x_i y_i - b_0 \sum_{i=1}^{q+1} (i-1) (x_i^2 - y_i^2) \quad (302)$$

And

$$Tc_5 = -\frac{a_0 \beta + b_0 \alpha}{2} \quad (303)$$

And finally

$$Tc = \frac{b_0}{2} \left(\left[\sum_{i=1}^{q-1} k y_k - 2 \sum_{i=1}^{q-1} k y_k^2 + \sum_{k=1}^{q-1} y_k \sum_{i=1}^k y_i \right] - \left[\sum_{i=1}^{q-1} k x_k - 2 \sum_{i=1}^{q-1} k x_k^2 + \sum_{k=1}^{q-1} x_k \sum_{i=1}^k x_i \right] \right) \quad (304)$$

$$- \frac{a_0}{2} \left(\sum_{i=1}^{q-1} k x_k + \sum_{i=1}^{q-1} k y_k - 4 \sum_{k=1}^{q-1} k x_k y_k + \sum_{k=1}^{q-1} x_k y_k \right) \quad (305)$$

$$- \frac{2a_0 + 1}{2} \sum_{i=1}^{q-1} x_i y_i - \frac{b_0}{2} \sum_{i=1}^{q-1} (x_i^2 - y_i^2) + \frac{b_0}{2} \left(\sum_{i=1}^{q-1} y_i \sum_{k=1}^i y_k - \sum_{i=1}^{q-1} x_i \sum_{k=1}^i x_k \right) \quad (306)$$

$$- 2a_0 \sum_{i=1}^{q+1} (i-1) x_i y_i - b_0 \sum_{i=1}^{q+1} (i-1) (x_i^2 - y_i^2) - \frac{a_0 \beta + b_0 \alpha}{2} \quad (307)$$

To calculate Ta , Tb and Tc we need to calculate the following terms: A , B , A_1 , B_1 , C_1 , A_2 , B_2 , C_2 , A_3 , B_3 , C_3 . For $1 \leq m \leq q-2$ and $m \neq \frac{q-1}{2}$ we have:

$$A = \sum_{i=1}^q kx_k = -\frac{q-1}{2}, B = \sum_{i=1}^q ky_k = -\frac{(q-1)\sin(\frac{2\pi m}{q-1})}{2-2\cos(\frac{2\pi m}{q-1})} \quad (m \neq \frac{q-1}{2}) \quad (308)$$

$$A_1 = \sum_{i=1}^q x_k^2 = \frac{q-1}{2}, B_1 = \sum_{i=1}^q y_k^2 = \frac{q-1}{2}, C_1 = \sum_{i=1}^q x_k y_k = 0 \quad (m \neq \frac{q-1}{2}) \quad (309)$$

$$A_2 = \sum_{i=1}^q kx_k^2 = \frac{(q-1)^2}{4}, B_2 = \sum_{i=1}^q ky_k^2 = \frac{(q-1)(q+1)}{4}, \quad (m \neq \frac{q-1}{2}) \quad (310)$$

$$C_2 = \sum_{i=1}^q kx_k y_k = -\frac{(q-1)\sin(\frac{4\pi m}{q-1})}{4-4\cos(\frac{4\pi m}{q-1})} \quad (m \neq \frac{q-1}{2}) \quad (311)$$

$$A_3 = \sum_{k=1}^{q-1} x_k \sum_{i=1}^k x_i = \frac{q-1}{4}, B_3 = \sum_{k=1}^{q-1} y_k \sum_{i=1}^k y_i = \frac{q-1}{4}, \quad (312)$$

$$C_3 = \sum_{k=1}^{q-1} y_k \sum_{i=1}^k x_i = \frac{(q-1)\sin(\frac{2\pi m}{q-1})}{4-4\cos(\frac{2\pi m}{q-1})} \quad (m \neq \frac{q-1}{2}) \quad (313)$$

And when $m = \frac{q-1}{2}$:

$$A = \sum_{i=1}^q kx_k = -\frac{q-1}{2}, B = \sum_{i=1}^q ky_k = 0 \quad (m = \frac{q-1}{2}) \quad (314)$$

$$A_1 = \sum_{i=1}^q x_k^2 = q-1, B_1 = \sum_{i=1}^q y_k^2 = 0, C_1 = \sum_{i=1}^q x_k y_k = 0 \quad (m = \frac{q-1}{2}) \quad (315)$$

$$A_2 = \sum_{i=1}^q kx_k^2 = \frac{q(q-1)}{2}, B_2 = \sum_{i=1}^q ky_k^2 = 0, C_2 = \sum_{i=1}^q kx_k y_k = 0 \quad (m = \frac{q-1}{2}) \quad (316)$$

$$A_3 = \sum_{k=1}^{q-1} x_k \sum_{i=1}^k x_i = \frac{q-1}{2}, B_3 = \sum_{k=1}^{q-1} y_k \sum_{i=1}^k y_i = 0, C_3 = \sum_{k=1}^{q-1} y_k \sum_{i=1}^k x_i = 0 \quad (m = \frac{q-1}{2}) \quad (317)$$

And

$$\alpha = -\frac{a_0 a + b_0 b}{b_0^2 + a_0^2}, \beta = \frac{b_0 a - a_0 b}{b_0^2 + a_0^2}, b = -(b_0 A + a_0 B), a = b_0 B - a_0 A \quad (318)$$

Therefore

$$Ta = \begin{cases} -\frac{a_0}{2} \left[\frac{(5q-1)}{2} - \frac{(q-1)\sin(\frac{2\pi m}{q-1})}{2-2\cos(\frac{2\pi m}{q-1})} \right] + \frac{b_0}{2} \left[\frac{(q-1)}{2} + \frac{(q-1)\sin(\frac{2\pi m}{q-1})}{1-\cos(\frac{2\pi m}{q-1})} \right], & \text{if } m \neq \frac{q-1}{2} \\ -\frac{a_0}{4}(5q-1) + \frac{b_0}{4}(q-1), & \text{if } m = \frac{q-1}{2} \end{cases} \quad (319)$$

$$Tb = \begin{cases} \frac{a_0}{2} \left[q+1 - \frac{(q-1)\sin(\frac{2\pi m}{q-1})}{2-2\cos(\frac{2\pi m}{q-1})} \right] - \frac{b_0}{2} \left[\frac{(q-1)}{2} + \frac{(q-1)\sin(\frac{2\pi m}{q-1})}{1-\cos(\frac{2\pi m}{q-1})} \right], & \text{if } m \neq \frac{q-1}{2} \\ \frac{a_0}{4}(4q^2+2q-2) - \frac{b_0}{4}(q-1), & \text{if } m = \frac{q-1}{2} \end{cases} \quad (320)$$

$$Tc = \begin{cases} \frac{a_0}{2} \left[\frac{q-1}{2} + \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{1 - \cos(\frac{2\pi m}{q-1})} \right] - \frac{b_0}{2} \left[q + 1 + \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{2 - 2 \cos(\frac{2\pi m}{q-1})} \right], & \text{if } m \neq \frac{q-1}{2} \\ \frac{a_0}{4} (q-1) - b_0 q, & \text{if } m = \frac{q-1}{2} \end{cases} \quad (321)$$

■

□

Lemma 3.4. *The case of the Riemann Zeta. The Riemann zeta function is extended to the part of the complex plane where $\Re(s) > 0$ by the Dirichlet η function: $\eta(s) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$. Where $q = 2$ and $\chi(kq+1) = 1$, $\chi(kq+2) = -1$. In this case, the function χ is not a Dirichlet character but it is not a stop to apply our method as our method is not Dirichlet character specific. In this case, the function χ also has the cancellation property like a Dirichlet character. And hence we can use the lemmas 3.1 and 3.2. We keep the same notation as the lemma 3.3. Therefore we have the same asymptotic expansion as in the previous lemma:*

$$\sum_{i=1}^q \frac{\gamma_{qn+i}^2 + \alpha_{qn+i} \lambda_{qn+i}}{(nq+i)^{2a_0}} + \sum_{i=1}^q \frac{(\lambda_{nq+i} + 2\gamma_{qn+i})\beta_{nq+i}}{(nq+i)^{2a_0+1}} = \frac{Ta + Tb \cos(2b_0 \ln(qn)) + Tc \sin(2b_0 \ln(qn))}{(nq)^{2a_0+1}} + O\left(\frac{1}{(qn)^{2a_0+2}}\right) \quad (322)$$

with

$$Ta = -a_0, Tb = -a_0, c = -b_0 \quad (323)$$

Proof. In this case we have $q = 2$ and for each $k \geq 1$:

$$x_{kq+1} = (-1)^{2k+1+1} = 1, x_{kq+2} = (-1)^{2k+2+1} = -1 \quad (324)$$

$$y_{kq+1} = 0, y_{kq+2} = 0 \quad (325)$$

and we have the same equations like in (140-141):

$$\begin{aligned} \sum_{i=1}^2 i(x_{i+1} - x_i) &= 2 \\ \sum_{i=1}^{2-1} i(y_{i+1} - y_i) &= 0 \end{aligned}$$

We apply the same calculation as in the lemma 3.3 and we replace by the new values of x_k and y_k . ■ □

Lemma 3.5. *From the lemma 3.3, equation(121), we have the asymptotic expansion of λ_{qn} :*

$$\lambda_{q(n+1)} = \lambda_{qn} + \frac{a_0 \lambda_{nq}}{n} + \frac{B_q}{qn} + O\left(\frac{1}{(qn)^2}\right) \quad (326)$$

Therefore

$$\lambda_{qn} = \frac{1}{q} \left(\alpha \cos(b_0 \ln(nq)) + \beta \sin(b_0 \ln(nq)) \right) + O\left(\frac{1}{qn}\right) \quad (327)$$

Where

$$\alpha = -\frac{a_0 a + b_0 b}{b_0^2 + a_0^2}, \beta = \frac{b_0 a - a_0 b}{b_0^2 + a_0^2}, b = -(b_0 A + a_0 B), a = b_0 B - a_0 A \quad (328)$$

And A and B are defined in the proof of the lemma 3.3.

Proof. Let's denote $v_n = \lambda_{qn}$ for $n \geq 1$.

$$B_{q,nq} = -\sin(b_0 \ln(nq)) \left(b_0 \sum_{k=1}^q kx_k + a_0 \sum_{k=1}^q ky_k - a_0q \sum_{k=1}^q y_k \right) \quad (329)$$

$$+ \cos(b_0 \ln(nq)) \left(b_0 \sum_{k=1}^q ky_k - a_0 \sum_{k=1}^q kx_k + a_0q \sum_{k=1}^q x_k \right) \quad (330)$$

$$= -\sin(b_0 \ln(nq)) (b_0A + a_0B) + \cos(b_0 \ln(nq)) (b_0B - a_0A) \quad (331)$$

Where A and B are defined in the the proof of the lemma 3.3. Let's denote $b = -(b_0A + a_0B)$ and $a = b_0B - a_0A$. Let's denote B_{nq} such that

$$B_{q,nq} = B_{qn} = a \cos(b_0 \ln(nq)) + b \sin(b_0 \ln(nq)) \quad (332)$$

We can write

$$v_{n+1} = v_n \left(1 + \frac{a_0}{n} \right) + \frac{B_{qn}}{qn} + \frac{\epsilon_n}{n^2} \quad (333)$$

with ϵ_n is a bounded sequence.

Let's denote ξ_n such that $\frac{\xi_n}{n} = \frac{B_{qn}}{qn} + \frac{\epsilon_n}{n^2}$. The sequence ξ_n is also bounded.

Therefore for each $n \geq 1$:

$$v_{n+1} = v_n \left(1 + \frac{a_0}{n} \right) + \frac{\xi_n}{n} \quad (334)$$

Therefore for $n \geq m \geq 1$. We have:

$$v_{n+1} = v_m \left(1 + \frac{a_0}{m} \right) \left(1 + \frac{a_0}{m+1} \right) \dots \left(1 + \frac{a_0}{n} \right) + \frac{\xi_m}{m} \left(1 + \frac{a_0}{m+1} \right) \left(1 + \frac{a_0}{m+2} \right) \dots \left(1 + \frac{a_0}{n} \right) \quad (335)$$

$$+ \frac{\xi_{m+1}}{m+1} \left(1 + \frac{a_0}{m+2} \right) \left(1 + \frac{a_0}{m+3} \right) \dots \left(1 + \frac{a_0}{n} \right) + \dots + \frac{\xi_{n-1}}{n-1} \left(1 + \frac{a_0}{n} \right) + \frac{\xi_n}{n} \quad (336)$$

For a large m and n we have the product $\left(1 + \frac{a_0}{m+1} \right) \left(1 + \frac{a_0}{m+2} \right) \dots \left(1 + \frac{a_0}{n} \right)$ equivalent to $\left(\frac{n}{m} \right)^{a_0}$. Because:

We have the Euler-Mclaurin formula that gives us the asymptotic expansion of the harmonic series:

$$\sum_{k=1}^n \frac{1}{k} = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \frac{1}{240n^8} - \frac{1}{132n^{10}} + O\left(\frac{1}{n^{12}}\right) \quad (337)$$

Where $\gamma \approx 0,577215$ is the famous Euler-Mascheroni constant.

Therefore there is exist a bounded sequence θ_n such that:

$$\sum_{k=1}^n \frac{1}{k} = \ln n + \gamma + \frac{1}{2n} + \frac{\theta_n}{n^2} \quad (338)$$

Let's put $Y_{m,n} = \left(1 + \frac{a_0}{m+1} \right) \left(1 + \frac{a_0}{m+2} \right) \dots \left(1 + \frac{a_0}{n} \right)$.

Therefore

$$\ln(Y_{m,n}) = \sum_{i=m+1}^n \ln \left(1 + \frac{a_0}{i} \right) \quad (339)$$

We use the asymptotic expansion of

$$\ln\left(1 + \frac{a_0}{n}\right) = \frac{a_0}{n} + O\left(\frac{1}{n^2}\right) \quad (340)$$

And we write

$$\ln\left(1 + \frac{a_0}{n}\right) = \frac{a_0}{n} + \frac{\alpha_n}{n^2} \quad (341)$$

where α_n is a bounded sequence.

$$\ln(Y_{m,n}) = \sum_{i=m+1}^n \ln\left(1 + \frac{a_0}{i}\right) = a_0 \sum_{i=m+1}^n \frac{1}{i} + \frac{\alpha_i}{i^2} \quad (342)$$

$$= a_0 \ln\left(\frac{n}{m}\right) + \frac{a_0}{2} \left(\frac{1}{n} - \frac{1}{m}\right) + \left(\frac{\theta_n}{n^2} - \frac{\theta_m}{m^2}\right) + a_0 \sum_{i=m+1}^n \frac{\alpha_i}{i^2} \quad (343)$$

Therefore

$$Y_{m,n} = \left(\frac{n}{m}\right)^{a_0} e^{\frac{a_0}{2} \left(\frac{1}{n} - \frac{1}{m}\right) + \left(\frac{\theta_n}{n^2} - \frac{\theta_m}{m^2}\right) + a_0 \sum_{i=m+1}^n \frac{\alpha_i}{i^2}} \quad (344)$$

Therefore we can write $Y_{m,n}$ as the following:

$$Y_{m,n} = \left(\frac{n}{m}\right)^{a_0} \left(1 + \frac{\beta_m}{m}\right) \quad (345)$$

where β_m is a bounded sequence. Therefore

$$Y_{m,n} = \left(\frac{n}{m}\right)^{a_0} + \frac{\beta_m n^{a_0}}{m^{a_0+1}} \quad (346)$$

Let's now write V_{n+1} in function of $Y_{m,n}$:

$$v_{n+1} = v_m Y_{m-1,n} + \frac{\xi_m}{m} Y_{m,n} + \frac{\xi_{m+1}}{m+1} Y_{m+1,n} \quad (347)$$

$$+ \dots + \frac{\xi_{n-1}}{n-1} Y_{n-1,n} + \frac{\xi_n}{n} Y_{n,n} \quad (348)$$

Therefore V_{n+1} can be written as:

$$v_{n+1} = v_m \left(\frac{n}{m}\right)^{a_0} + \frac{\xi_m}{m} \left(\frac{n}{m}\right)^{a_0} + \frac{\xi_{m+1}}{m+1} \left(\frac{n}{m+1}\right)^{a_0} \quad (349)$$

$$+ \dots + \frac{\xi_{n-1}}{n-1} \left(\frac{n}{n-1}\right)^{a_0} + \frac{\xi_n}{n} \left(\frac{n}{n}\right)^{a_0} + Z_{m,n} \quad (350)$$

And the remaining term is

$$Z_{m,n} = \frac{\xi_m \beta_m}{m^2} \left(\frac{n}{m}\right)^{a_0} + \frac{\xi_{m+1} \beta_{m+1}}{(m+1)^2} \left(\frac{n}{m+1}\right)^{a_0} \quad (351)$$

$$+ \dots + \frac{\xi_{n-1} \beta_{n-1}}{(n-1)^2} \left(\frac{n}{n-1}\right)^{a_0} + \frac{\xi_n \beta_n}{n^2} \left(\frac{n}{n}\right)^{a_0} \quad (352)$$

Therefore

$$v_{n+1} = v_m \left(\frac{n}{m}\right)^{a_0} + n^{a_0} \left(\frac{\xi_m}{m^{a_0+1}} + \frac{\xi_{m+1}}{(m+1)^{a_0+1}} + \dots + \frac{\xi_n}{n^{a_0+1}}\right) \quad (353)$$

$$= v_m \left(\frac{n}{m}\right)^{a_0} + n^{a_0} \left(X_{n,m}\right) + Z_{m,n} \quad (354)$$

Now since the sequences β_n and ξ_n are bounded we have:

$$|Z_{m,n}| \leq C n^{a_0} \int_{m-1}^n \frac{dt}{t^{a_0+2}} \quad (355)$$

$$\leq \frac{C n^{a_0}}{a_0 + 1} \left(\frac{1}{(m-1)^{a_0+1}} - \frac{1}{n^{a_0+1}} \right) \quad (356)$$

Where C is a positive constant.

Now let's study the term $X_{m,n}$. We have

$$X_{n,m} = \frac{\xi_m}{m^{a_0+1}} + \frac{\xi_{m+1}}{(m+1)^{a_0+1}} + \dots + \frac{\xi_n}{n^{a_0+1}} \quad (357)$$

$$= \underbrace{\frac{B_{qm}}{qm^{a_0+1}} + \frac{B_{q(m+1)}}{q(m+1)^{a_0+1}} + \dots + \frac{B_{qn}}{qn^{a_0+1}}}_{W_{m,n}} + \underbrace{\frac{\epsilon_m}{m^{a_0+2}} + \frac{\epsilon_{m+1}}{(m+1)^{a_0+2}} + \dots + \frac{\epsilon_n}{n^{a_0+2}}}_{ZZ_{m,n}} \quad (358)$$

Similarly to $Z_{m,n}$, we have the same inequality for $ZZ_{m,n}$:

$$|ZZ_{m,n}| \leq C_1 \int_{m-1}^n \frac{dt}{t^{a_0+2}} \quad (359)$$

$$\leq \frac{C_1}{a_0 + 1} \left(\frac{1}{(m-1)^{a_0+1}} - \frac{1}{n^{a_0+1}} \right) \quad (360)$$

Where C_1 is a positive constant.

Let's denote the function g as the following:

$$g(x) = \frac{a \cos(b_0 \ln(x)) + b \sin(b_0 \ln(x))}{x^{a_0+1}} \quad (361)$$

We can write the sequence $W_{m,n}$ as the following:

$$W_{m,n} = q^{a_0} \sum_{i=m+1}^n g(qi) \quad (362)$$

Using the same technique we used in the lemma 3.1 and the same technique used in lemma 3.1 and 3.2 in [1], we have the following:

$$\left| q^{a_0} \sum_{i=m+1}^n g(qi) - q^{a_0} \int_{q(m+1)}^{qn} g(x) \right| \leq \frac{K}{m^{a_0+1}} \quad (363)$$

Therefore

$$|W_{m,n} - q^{a_0-1}(G(qn) - G(qm))| \leq \frac{K}{m^{a_0+1}} \quad (364)$$

And

$$|W_{m,n} + q^{a_0-1}G(qm)| \leq q^{a_0-1} |G(qn)| + \frac{K}{m^{a_0+1}} \quad (365)$$

And the primitive function G of the function g (see lemma 3.1 in [1]):

$$G(x) = \frac{(b_0 a - a_0 b) \sin(b_0 \ln(x)) - (a_0 a + b_0 b) \cos(b_0 \ln(x))}{(b_0^2 + a_0^2) x^{a_0}} \quad (366)$$

We have

$$v_m \left(\frac{n}{m}\right)^{a_0} + n^{a_0} W_{n,m} = v_{n+1} - n^{a_0} Z Z_{m,n} - Z_{m,n} \quad (367)$$

The sequence (v_n) is bounded since the sequence (λ_n) is bounded thanks to lemma 3.1. So, let's M be the nonzero positive constant such that for each $n \geq 1$:

$$|v_n| \leq M \quad (368)$$

Therefore

$$\left| v_m \left(\frac{n}{m}\right)^{a_0} + n^{a_0} W_{n,m} \right| \leq |v_{n+1}| + n^{a_0} |Z Z_{m,n}| + |Z_{m,n}| \quad (369)$$

$$\leq M + \frac{(C + C_1) n^{a_0}}{a_0 + 1} \left(\frac{1}{(m-1)^{a_0+1}} - \frac{1}{n^{a_0+1}} \right) \quad (370)$$

Therefore

$$\left| v_m \left(\frac{n}{m}\right)^{a_0} - \frac{(nq)^{a_0}}{q} G(qm) \right| \leq \left| \frac{(nq)^{a_0}}{q} G(qm) + n^{a_0} W_{n,m} \right| + |v_{n+1}| + n^{a_0} |Z Z_{m,n}| + |Z_{m,n}| \quad (371)$$

$$\leq \frac{(nq)^{a_0}}{q} |G(qn)| + \frac{K n^{a_0}}{m^{a_0+1}} + M + \frac{(C + C_1) n^{a_0}}{a_0 + 1} \left(\frac{1}{(m-1)^{a_0+1}} - \frac{1}{n^{a_0+1}} \right) \quad (372)$$

We have

$$(nq)^{a_0} G(qn) = \left(\frac{b_0 a - a_0 b}{b_0^2 + a_0^2} \sin(b_0 \ln(qn)) - \frac{a_0 a + b_0 b}{b_0^2 + a_0^2} \cos(b_0 \ln(qn)) \right) \quad (373)$$

$$= \alpha \cos(b_0 \ln(qn)) + \beta \sin(b_0 \ln(qn)) \quad (374)$$

Therefore the sequence $(nq)^{a_0} G(qn)$ is bounded.

Therefore we can write:

$$\left| v_m \left(\frac{n}{m}\right)^{a_0} - \frac{(nq)^{a_0}}{q} G(qm) \right| \leq (nq)^{a_0} |G(qn)| + \frac{K n^{a_0}}{m^{a_0+1}} + M + \frac{(C + C_1) n^{a_0}}{a_0 + 1} \left(\frac{1}{(m-1)^{a_0+1}} - \frac{1}{n^{a_0+1}} \right) \quad (375)$$

$$\leq C_2 + \frac{K n^{a_0}}{m^{a_0+1}} + \frac{(C + C_1) n^{a_0}}{a_0 + 1} \frac{1}{(m-1)^{a_0+1}} \quad (376)$$

Now we choose $n = m^{1+\frac{1}{a_0}}$. Then $n^{a_0} = m^{1+a_0}$. Therefore $\left(\frac{n}{m}\right)^{a_0} = m$. Therefore

$$\left| m v_m - \frac{m}{q} \left(\alpha \cos(b_0 \ln(qm)) + \beta \sin(b_0 \ln(qm)) \right) \right| \leq C_2 + \frac{K m^{a_0+1}}{m^{a_0+1}} + \frac{(C + C_1) m^{a_0+1}}{a_0 + 1} \frac{1}{(m-1)^{a_0+1}} \quad (377)$$

$$\leq C_2 + K + \frac{C + C_1}{a_0 + 1} \left(\frac{m}{m-1} \right)^{1+a_0} \quad (378)$$

$$\leq C_2 + K + \frac{C + C_1}{a_0 + 1} 2^{1+a_0} = C' \quad (379)$$

Therefore for any large m we have.

$$\left| v_m - \frac{1}{q} \left(\alpha \cos(b_0 \ln(qm)) + \beta \sin(b_0 \ln(qm)) \right) \right| \leq \frac{C'}{m} \quad (380)$$

where C' is a positive constant.

Therefore

$$v_n = \frac{1}{q} \left(\alpha \cos(b_0 \ln(qn)) + \beta \sin(b_0 \ln(qn)) \right) + O\left(\frac{1}{(qn)}\right) \quad (381)$$

Where

$$\beta = \frac{b_0 a - a_0 b}{b_0^2 + a_0^2}, \quad \alpha = -\frac{a_0 a + b_0 b}{b_0^2 + a_0^2} \quad (382)$$

■

□

Remark. The results obtained in the lemmas 3.3-3.5 concern the sequence (F_n) . Similar results can be obtained for the sequence (G_n) by replacing $(x_n, y_n)_{n \geq 1}$ in the formulas above by $(x'_n, y'_n)_{n \geq 1}$ where for each $n \geq 1$, $x'_n = y_n$ and $y'_n = -x_n$.

Divergence of $\sum_{n \geq 1} F_n$

Lemma 3.6. If $\frac{1}{2} < a_0 \leq 1$, the series $\sum_{n \geq 1} F_n$ diverges.

Proof. For large $N \geq N_0$ we have the following expression of F_{N+1} from the lemma 3.2:

$$F_{N+1} = F_{N_0} + \sum_{n=N_0}^N \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} + \frac{(\lambda_n + 2\gamma_n)\beta_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \quad (383)$$

$$= F_{N_0} + \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \quad (384)$$

Where

$$C_n = \gamma_n^2 + \alpha_n \lambda_n \quad (385)$$

$$D_n = (\lambda_n + 2\gamma_n)\beta_n \quad (386)$$

We do another summation and we have the following expression:

So for each $N \geq N_0$:

$$\sum_{n=N_0}^N F_{n+1} = (N - N_0 + 1)F_{N_0} \quad (387)$$

$$+ \sum_{n=N_0}^N \frac{C_n}{n^{a_0+1}} (N - n + 1) \quad (388)$$

$$+ \sum_{n=N_0}^N \frac{D_n}{n^{2a_0+1}} (N - n + 1) \quad (389)$$

$$+ \sum_{n=N_0}^N \frac{\epsilon_n}{n^{a_0+2}} (N - n + 1) \quad (390)$$

$$\sum_{n=N_0}^N \frac{C_n}{n^{a_0+1}}(N-n+1) = (N+1) \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} - \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}}n \quad (391)$$

$$= (N+1) \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} - \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} \quad (392)$$

$$\sum_{n=N_0}^N \frac{D_n}{n^{2a_0+1}}(N-n+1) = (N+1) \sum_{n=N_0}^N \frac{D_n}{n^{2a_0+1}} - \sum_{n=N_0}^N \frac{D_n}{n^{2a_0+1}}n \quad (393)$$

$$= (N+1) \sum_{n=N_0}^N \frac{D_n}{n^{2a_0+1}} - \sum_{n=N_0}^N \frac{D_n}{n^{2a_0}} \quad (394)$$

And

$$\sum_{n=N_0}^N \frac{\varepsilon_n}{n^{2a_0+1}}(N-n+1) = (N+1) \sum_{n=N_0}^N \frac{\varepsilon_n}{n^{a_0+2}} - \sum_{n=N_0}^N \frac{\varepsilon_n}{n^{a_0+2}}n \quad (395)$$

$$= (N+1) \sum_{n=N_0}^N \frac{\varepsilon_n}{n^{a_0+2}} - \sum_{n=N_0}^N \frac{\varepsilon_n}{n^{a_0+1}} \quad (396)$$

So for each $N \geq N_0$:

$$\sum_{n=N_0}^N F_{n+1} = (N-N_0+1)F_{N_0} \quad (397)$$

$$+ (N+1) \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (398)$$

$$- \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} \quad (399)$$

We factorise by $N+1$ then we have this expression

$$\sum_{n=N_0}^N F_{n+1} = (N+1) \left(\frac{(N-N_0+1)}{N+1} F_{N_0} \right) \quad (400)$$

$$+ \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (401)$$

$$- \frac{1}{N+1} \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} \quad (402)$$

We have the sequences (C_n) , (D_n) and (ε_n) are bounded and $2a_0 > 1$, so the series $\sum_{n \geq 1}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}}$ is converging absolutely. Let's prove now that the sequence in the equation (398) is converging to

zero when N goes to infinity.

As we have $2a_0 > 1$, So

$$\lim_{n \rightarrow \infty} \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} = 0 \quad (403)$$

Therefore following the Césaro theorem we have:

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} = 0 \quad (404)$$

We denote $FF(N, N_0)$ as:

$$FF(N, N_0) = \left(\frac{(N - N_0 + 1)}{N + 1} F_{N_0} \right) \quad (405)$$

$$+ \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (406)$$

$$- \frac{1}{N+1} \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} \quad (407)$$

Therefore:

$$\sum_{n=N_0}^N F_{n+1} = (N+1)FF(N, N_0) \quad (408)$$

The series $\sum_{n \geq 1} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}}$ is converging absolutely. Let's denote

$$R_{N_0} = \lim_{N \rightarrow +\infty} \sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (409)$$

Proof Strategy. *The idea here is to prove that we can choose N_0 such that $R_{N_0} + F_{N_0} > 0$. In such case, we will have the $\lim_{N \rightarrow +\infty} \sum_{n=1}^N F_{n+1} = +\infty$.*

Lemma 3.7. *We can choose N_0 such that $R_{N_0} + F_{N_0} > 0$.*

Proof. Let's now study the term $\sum_{n=qN_0+1}^{Nq} \left(\frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} \right)$.

We have from above lemmas that:

$$\sum_{n=qN_0+1}^{Nq} \left(\frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} \right) = \sum_{n=N_0}^{N-1} \left(\sum_{i=1}^q \frac{\gamma_{qn+i}^2 + \alpha_{qn+i} \lambda_{qn+i}}{(nq+i)^{2a_0}} + \sum_{i=1}^q \frac{(\lambda_{nq+i} + 2\gamma_{qn+i}) \beta_{nq+i}}{(nq+i)^{2a_0+1}} \right) \quad (410)$$

We have from the lemma 3.3 and 3.4

$$\sum_{i=1}^q \frac{\gamma_{qn+i}^2 + \alpha_{qn+i} \lambda_{qn+i}}{(nq+i)^{2a_0}} + \sum_{i=1}^q \frac{(\lambda_{nq+i} + 2\gamma_{qn+i}) \beta_{nq+i}}{(nq+i)^{2a_0+1}} = \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \gamma \sin(2b_0 \ln(qn))}{(nq)^{2a_0+1}} + O\left(\frac{1}{(qn)^{2a_0+2}}\right) \quad (411)$$

Therefore we can write the following for a large n :

$$\sum_{i=1}^q \frac{\gamma_{qn+i}^2 + \alpha_{qn+i} \lambda_{qn+i}}{(nq+i)^{2a_0}} + \sum_{i=1}^q \frac{(\lambda_{nq+i} + 2\gamma_{qn+i})\beta_{nq+i}}{(nq+i)^{2a_0+1}} = \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \gamma \sin(2b_0 \ln(qn))}{(nq)^{2a_0+1}} + \frac{\xi_n}{(qn)^{2a_0+2}} \quad (412)$$

Where the sequence ξ_n is bounded.

Therefore

$$\sum_{n=qN_0+1}^{Nq} \left(\frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} \right) = \sum_{n=N_0}^{N-1} \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \gamma \sin(2b_0 \ln(qn))}{(qn)^{2a_0+1}} + \sum_{n=N_0}^{N-1} \frac{\xi_n}{(qn)^{2a_0+2}} \quad (413)$$

And

$$\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} = \sum_{n=N_0}^{N-1} \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \gamma \sin(2b_0 \ln(qn))}{(qn)^{2a_0+1}} \quad (414)$$

$$+ \sum_{n=N_0}^{N-1} \frac{\xi_n}{(qn)^{2a_0+2}} + \sum_{n=qN_0+1}^{Nq} \frac{\varepsilon_n}{n^{a_0+2}} \quad (415)$$

The series $\sum_{n \geq 1} \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \gamma \sin(2b_0 \ln(qn))}{(qn)^{2a_0+1}}$ was studied in the lemma 3.1 in [1]. Let's denote the function g as the following:

$$g(x) = \frac{\alpha + \beta \cos(2b_0 \ln(qx)) + \gamma \sin(2b_0 \ln(qx))}{(qx)^{2a_0+1}} \quad (416)$$

We have the following:

$$\left| \sum_{n=N_0+1}^{+\infty} g(n) - \int_{N_0+1}^{+\infty} g(x) \right| \leq \frac{K}{(N_0)^{2a_0+1}} \quad (417)$$

And the primitive function G of the function $x \mapsto \frac{1}{q}g(\frac{x}{q})$ (see lemma 3.1 in [1]):

$$G(x) = \frac{1}{q} \left\{ -\frac{\alpha}{(2a_0)x^{2a_0}} + \frac{(2b_0\beta - (2a_0)\gamma) \sin(2b_0 \ln(x)) + (-2a_0\beta - 2b_0\gamma) \cos(2b_0 \ln(x))}{((2b_0)^2 + (2a_0)^2)x^{2a_0}} \right\} \quad (418)$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} g(n) + G(q(N_0+1)) \right| \leq \frac{K}{(N_0)^{2a_0+1}} \quad (419)$$

Case of Primitive and Non-Trivial Character From the lemma 3.3, if we keep the same notation as the lemma 3.3, we have:

$$\alpha = \begin{cases} -\frac{a_0}{2} \left[\frac{(5q-1)}{2} - \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{2-2 \cos(\frac{2\pi m}{q-1})} \right] + \frac{b_0}{2} \left[\frac{(q-1)}{2} + \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{1-\cos(\frac{2\pi m}{q-1})} \right], & \text{if } m \neq \frac{q-1}{2} \\ -\frac{a_0}{4}(5q-1) + \frac{b_0}{4}(q-1), & \text{if } m = \frac{q-1}{2} \end{cases} \quad (420)$$

$$\beta = \begin{cases} \frac{a_0}{2} \left[q + 1 - \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{2-2\cos(\frac{2\pi m}{q-1})} \right] - \frac{b_0}{2} \left[\frac{(q-1)}{2} + \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{1-\cos(\frac{2\pi m}{q-1})} \right], & \text{if } m \neq \frac{q-1}{2} \\ \frac{a_0}{4} (4q^2 + 2q - 2) - \frac{b_0}{4} (q - 1), & \text{if } m = \frac{q-1}{2} \end{cases} \quad (421)$$

$$\gamma = \begin{cases} \frac{a_0}{2} \left[\frac{q-1}{2} + \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{1-\cos(\frac{2\pi m}{q-1})} \right] - \frac{b_0}{2} \left[q + 1 + \frac{(q-1) \sin(\frac{2\pi m}{q-1})}{2-2\cos(\frac{2\pi m}{q-1})} \right], & \text{if } m \neq \frac{q-1}{2} \\ \frac{a_0}{4} (q - 1) - b_0 q, & \text{if } m = \frac{q-1}{2} \end{cases} \quad (422)$$

for some $m \in [1, q - 2]$.

As an example, the case where $q = 3$ and $\chi(1) = 1$, $\chi(2) = -1$ and $\chi(3) = 0$. $L(s, \chi) = \sum_{n=0}^{+\infty} \left(\frac{1}{(3n+1)^s} - \frac{1}{(3n+2)^s} \right)$.

From the formulas (417) and (419-421) above we have the function $G \neq 0$. Otherwise $\alpha = \beta = \gamma = 0$, which implies $a_0(q - 1) = 0$ in the case of $m \neq \frac{q-1}{2}$ and $a_0(4q^2 - 3q - 1) = a_0 \underbrace{(q - 1)(4q + 1)}_{\geq 9} = 0$ in the case of

$m = \frac{q-1}{2}$. As $q \geq 3$, therefore $a_0 = 0$. This will also imply that $b_0 = 0$ which is in contradiction with our assumption of $a_0 > \frac{1}{2}$ and $b_0 > 0$.

Concerning the term F_{N_0} . We have

$$F_{N_0} = \frac{u_{N_0}}{(N_0)^{a_0}} A_{N_0} = \frac{u_{N_0} \lambda_{N_0}}{(N_0)^{2a_0}} \quad (423)$$

Where the sequence (λ_n) is bounded (lemma 3.1). If we choose $N_0 = n_0 q$ as a multiple of q we have $u_{n_0 q} = 0$ (because $\chi(n_0 q) = 0$ in the case of a non-trivial character). Therefore, here, we simply need to choose N_0 such that:

$$F_{N_0} = F_{n_0 q} = 0 \quad (424)$$

Case of Riemann Zeta Hypothesis In this case from the lemma 3.4: $q = 2$, $\alpha = -a_0$, $\beta = -a_0$ and $\gamma = -b_0$.

Here, we don't have $\chi(n_0 q) = 0$. So, we will choose $N_0 = n_0 q$. We have thanks to lemma 3.5:

$$F_{N_0} = \frac{u_{q n_0} \lambda_{q n_0}}{(N_0)^{2a_0}} \quad (425)$$

$$= \frac{\cos(b_0 \ln(qn_0))}{q} \left\{ \frac{\alpha \cos(b_0 \ln(qn_0)) + \beta \sin(b_0 \ln(qn_0))}{qn_0^{2a_0}} \right\} + O\left(\frac{1}{(qn_0)^{2a_0+1}}\right) \quad (426)$$

$$= \frac{1}{2q} \left\{ \frac{\alpha}{(qn_0)^{2a_0}} + \frac{\alpha \cos(2b_0 \ln(qn_0)) + \beta \sin(2b_0 \ln(qn_0))}{(qn_0)^{2a_0}} \right\} + O\left(\frac{1}{(qn_0)^{2a_0+1}}\right) \quad (427)$$

Let's consider the function $G(x)$ defined as follows:

$$G(x) = \frac{1}{2q} \left\{ \frac{\alpha}{(x)^{2a_0}} + \frac{\alpha \cos(2b_0 \ln(x)) + \beta \sin(2b_0 \ln(x))}{(x)^{2a_0}} \right\} \quad (428)$$

So in this case, instead of working with the function G like in the case of non-trivial character above, we work with the function $G - G'$ and we replace the inequation (418) by the following one:

$$\left| F_{N_0} + \sum_{n=N_0+1}^{+\infty} g(n) + G(q(N_0 + 1)) - G'(q(N_0 + 1)) \right| \leq \frac{Kt}{(N_0)^{2a_0+1}} \quad (429)$$

Also, the function $G - G' \neq 0$. Otherwise $\alpha = \beta = \gamma = 0$, which implies $a_0 = b_0 = 0$ which is in contradiction with our assumption of $a_0 > \frac{1}{2}$ and $b_0 > 0$. The reader can also refer to [1] for another application of this method to the particular case of the Riemann Zeta.

Conclusion. *In both cases, the functions G and $G - G'$ are nonzero.*

Remark. *The remaining terms in the expression of $F_{N_0} + R_{N_0}$ are all of the order of $\frac{1}{(N_0)^{2a_0+1}}$ and above. So the dominant term in the expression $F_{N_0} + R_{N_0}$ is the term $-G(q(N_0 + 1))$ (resp. $(G - G')(q(N_0 + 1))$). Therefore this term is the dominant term in the expression of $\frac{1}{N} \sum_{n=N_0}^{N-1} F_{n+1}$. We will show that based on the sign of $G(N_0 + 1)$ the limit of $\sum_{n=1}^{N-1} F_{n+1}$ can be both $+\infty$ and $-\infty$.*

Let's now study the function f :

$$f(x) = -\frac{\alpha}{2a_0} + \frac{(2b_0\beta - (2a_0)\gamma) \sin(2b_0x) + (-2a_0\beta - 2b_0\gamma) \cos(2b_0x)}{(2b_0)^2 + (2a_0)^2} \quad (430)$$

$$= \alpha_2 + \beta_2 \cos(2b_0x) + \gamma_2 \sin(2b_0x) \quad (431)$$

The function f is also nonzero function. Otherwise we will have $\alpha = \beta = \gamma = 0$ which is in contradiction with the fact that $a_0 > \frac{1}{2}$. The function f is a linear combination of the functions \sin and \cos . So the function f is differentiable and bounded. The function f is periodic of period $\frac{\pi}{b_0}$.

Let's calculate the function f values at the following points:

$$f(0) = \alpha_2 + \beta_2 \quad (432)$$

$$f\left(\frac{\pi}{b_0}\right) = \alpha_2 - \beta_2 \quad (433)$$

We have either $f(0) \neq 0$ or $f\left(\frac{\pi}{b_0}\right) \neq 0$, otherwise we will have $a_0 = b_0 = 0$ which is again in contradiction with the fact that $a_0 > \frac{1}{2}$ and $b_0 > 0$. Let's assume $f(0) \neq 0$. From the values above we have the following:

- If $(\alpha_2 + \beta_2) > 0$ then $f(0) > 0$.
- If $(\alpha_2 + \beta_2) < 0$ then $f(0) < 0$.

In case of $f(0) > 0$ we can prove that the limit of $\sum_{n=1}^{N-1} F_{n+1}$ goes to $-\infty$ and in the case of $f(0) < 0$ we can prove that the limit of $\sum_{n=1}^{N-1} F_{n+1}$ goes to $+\infty$. Therefore, in all cases we have the series $\sum_{n \geq 1} F_{n+1}$ diverges. So

let's assume from now on that $f(0) < 0$ as the same proof can be done in both cases.

From the lemma 3.3 in [1], we know that there exist $N_1 \geq N_0$ such that $|\cos(2b_0 \ln(N_1 + 1)) - 1| \leq \epsilon$ and $|\sin(2b_0 \ln(N_1 + 1))| \leq \epsilon$ for any $1 > \epsilon > 0$.

Let's define $\beta_0 = -\frac{f(0)}{q^{2a_0+1}}$. $\beta_0 > 0$.

Let's fix $\epsilon > 0$ to be very small such that $0 < \epsilon < \min\left(1, \frac{\beta_0}{10000}\right)$.

Let's define $\gamma_0 = \frac{|\beta_2| + |\gamma_2|}{q^{2a_0+1}}$.

Without loss of generality, let's N_1 be such that $|\cos(2b_0 \ln q(N_1 + 1)) - 1| \leq \epsilon/\gamma_0$ and also $|\sin(2b_0 \ln q(N_1 + 1)) - 0| \leq \epsilon/\gamma_0$. Therefore:

$$\left| G(q(N_1 + 1)) - \frac{f(0)}{q(q(N_1 + 1))^{2a_0}} \right| \leq \left| \left\{ \frac{(\gamma_2 \sin(2b_0 \ln q(N_1 + 1)) + \beta_2) \cos(2b_0 \ln q(N_1 + 1))}{q(q(N_1 + 1))^{2a_0}} \right. \right. \quad (434)$$

$$\left. - \frac{(\gamma_2 \sin(2b_0) + \beta_2) \cos(2b_0)}{q(q(N_1 + 1))^{2a_0}} \right\} \quad (435)$$

$$\leq \frac{(|\gamma_2| |\sin(2b_0 \ln q(N_1 + 1))| + |\beta_2| |\cos(2b_0 \ln q(N_1 + 1)) - 1|)}{q(q(N_1 + 1))^{2a_0}} \quad (436)$$

$$\leq \frac{(|\beta_2| \epsilon/\gamma_0 + |\gamma_2| \epsilon/\gamma_0)}{q(q(N_1 + 1))^{2a_0}} \quad (437)$$

$$\leq \frac{\epsilon}{(N_1 + 1)^{2a_0}} \quad (438)$$

Therefore

$$\left| \sum_{n=N_1+1}^{+\infty} g(n) - \frac{\beta_0}{(N_1 + 1)^{2a_0}} \right| \leq \left| \sum_{n=N_1+1}^{+\infty} g(n) + G(q(N_1 + 1)) \right| + \left| -G(q(N_1 + 1)) - \frac{-f(0)}{q(q(N_1 + 1))^{2a_0}} \right| \quad (439)$$

$$\leq \frac{\epsilon}{(N_1 + 1)^{2a_0}} + \frac{K}{(N_1)^{2a_0+1}} \quad \left(\text{from (418) and (437), or (428) and (437)} \right) \quad (440)$$

Hence

$$\left| \sum_{n=N_1+1}^{+\infty} g(n) - \frac{\beta_0}{(N_1 + 1)^{2a_0}} \right| \leq \frac{\epsilon}{(N_1 + 1)^{2a_0}} + \frac{K}{(N_1)^{2a_0+1}} \quad (441)$$

Therefore

$$\sum_{n=N_1+1}^{+\infty} g(n) \geq \frac{\beta_0 - \epsilon}{(N_1 + 1)^{2a_0}} - \frac{K}{(N_1)^{2a_0+1}} \quad (442)$$

Therefore, without loss of generality let's suppose $N_1 = qN_2$.

$$R_{N_1} = \sum_{k=qN_2+1}^{+\infty} g(k) + \underbrace{F_{qN_2}}_{=0} + \lim_{N \rightarrow +\infty} \left[\sum_{n=N_2}^{N-1} \frac{\xi_n}{(qn)^{2a_0+2}} + \sum_{n=qN_2+1}^{Nq} \frac{\varepsilon_n}{n^{a_0+2}} \right] \quad \left(\text{from (413) and (414)} \right) \quad (443)$$

$$\geq \frac{\beta_0 - \epsilon}{(qN_2 + 1)^{2a_0}} - \frac{K}{(qN_2)^{2a_0+1}} + \lim_{N \rightarrow +\infty} \left[\sum_{n=N_2}^{N-1} \frac{\xi_n}{(qn)^{2a_0+2}} + \sum_{n=qN_2+1}^{Nq} \frac{\varepsilon_n}{n^{a_0+2}} \right] \quad (444)$$

We have the sequences (ξ_k) and (ε_k) are bounded. Plus we have $a_0 + 2 \geq 2a_0 + 1 > 3a_0 > 2a_0 > 1$, therefore the series $\sum_{k \geq 1} \frac{\xi_k}{k^{2a_0+2}}$ and $\sum_{k \geq 1} \frac{\varepsilon_k}{k^{a_0+2}}$ are converging absolutely.

Let the positive constant M such that:
For each $k \geq N_2$ and $N \geq N_2$:

$$|\xi_k| \leq M \quad (445)$$

$$|\varepsilon_k| \leq M \quad (446)$$

Therefore, we have $a_0 > 0$:

$$\left| \sum_{n=N_2}^{+\infty} \frac{\xi_n}{(qn)^{2a_0+2}} + \sum_{n=qN_2+1}^{+\infty} \frac{\varepsilon_n}{n^{a_0+2}} \right| \leq \frac{M}{(2a_0 + 1)q^{2a_0+2}(N_2 - 1)^{2a_0+1}} + \frac{M}{(a_0 + 1)(qN_2)^{a_0+1}} \quad (447)$$

Therefore

$$R_{N_1} \geq \frac{\beta_0 - \epsilon}{(qN_2 + 1)^{2a_0}} - \frac{K}{(qN_2)^{2a_0+1}} - \frac{M}{(2a_0 + 1)q^{2a_0+2}(N_2 - 1)^{2a_0+1}} - \frac{M}{(a_0 + 1)(qN_2)^{a_0+1}} \quad (448)$$

$$\geq \frac{1}{(qN_2 + 1)^{2a_0}} (\beta_0 - \epsilon + \delta_{qN_2}) \quad (449)$$

Where the sequence (δ_n) is defined as the following:

$$\delta_{qn} = -\frac{K(qn + 1)^{2a_0}}{(qn)^{2a_0+1}} - \frac{M(qn + 1)^{2a_0}}{(2a_0 + 1)q^{2a_0+2}(n - 1)^{2a_0+1}} - \frac{M(qn + 1)^{2a_0}}{(a_0 + 1)(qn)^{a_0+1}} \quad (450)$$

As we have $\frac{1}{2} < a_0 < 1$, therefore $0 < 1 - a_0 < \frac{1}{2} < a_0 < 2a_0 < a_0 + 1 < 3a_0 < 2a_0 + 1$.

Therefore

$$\lim_{n \rightarrow +\infty} \frac{K(qn + 1)^{2a_0}}{(qn)^{2a_0+1}} + \frac{M(qn + 1)^{2a_0}}{(2a_0 + 1)q^{2a_0+2}(n - 1)^{2a_0+1}} + \frac{M(qn + 1)^{2a_0}}{(a_0 + 1)(qn)^{a_0+1}} = 0 \quad (451)$$

Therefore the $\lim_{n \rightarrow +\infty} \delta_{qn} = 0$.

So we can choose N_1 such that $|\delta_{N_1}| < \epsilon$, i.e $\delta_{N_1} \geq -\epsilon$.

Therefore

$$R_{N_1} \geq \frac{1}{(N_1 + 1)^{2a_0}} (\beta_0 - 2\epsilon) > 0 \quad (452)$$

Therefore $\lim_{N \rightarrow +\infty} \sum_{n=N_1}^N F_n = +\infty$. ■ □

□

Conclusion. We have the series $\sum_{n \geq 1} U_n^2$ is converging absolutely thanks to $2a_0 > 1$. We have from the lemmas 3.1 that:

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N F_n = +\infty \quad (453)$$

Therefore

$$\lim_{N \rightarrow \infty} A_N^2 = +\infty \quad (454)$$

This result is in contradiction with the fact that s is a $L(s, \chi)$ zero and therefore with the fact that the limit $\lim_{N \rightarrow \infty} A_N = 0$ should be zero. Therefore s with $\frac{1}{2} < a_0 = \Re(s) < 1$ cannot be a zero for $L(s, \chi)$.

Remark. Note that when $a_0 = \frac{1}{2}$, from the equations (397-399), we cannot apply the same method to determine the limit $\lim_{N \rightarrow \infty} \sum_{n=1}^N F_n$, because the limit of the term in the equation (399) becomes unknown since we don't have here the $\lim_{N \rightarrow +\infty} (\frac{1}{N+1} \sum_{k=1}^N \frac{C_k}{k^{2a_0-1}}) = 0$ like in the case where $a_0 > \frac{1}{2}$.

3.2 Case Two: χ trivial character

For the trivial character χ_0 modulus q we have:

$$\zeta(s) = L(s, \chi_0) \prod_{p \text{ Prime}, p/q} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (455)$$

Where ζ is the Riemann Zeta function.

The product $\prod_{p \text{ Prime}, p/q} \left(1 - \frac{1}{p^s}\right)^{-1}$ is finite, bounded and nonzero as $\frac{1}{2} < \Re(s) = a_0 \leq 1$. From the equation above we have: if s is a zero for $L(s, \chi_0)$, then it is also a zero for the Riemann $\zeta(s)$. We saw in the previous case that if $\frac{1}{2} < \Re(s) = a_0 \leq 1$, it is not possible for the Riemann zeta function to have such a zero. Therefore $L(s, \chi_0)$ cannot have a zero where $\frac{1}{2} < \Re(s) = a_0 \leq 1$.

3.3 Conclusion

We saw that if s is a non-trivial zero of $L(s, \chi)$, then real part $\Re(s)$ can only be $\frac{1}{2}$ as all other possibilities can be discarded using the functional equation like in [1]. Therefore the Generalized Riemann hypothesis is true: *The non-trivial zeros of $L(s, \chi)$ have real part equal to $\frac{1}{2}$.*

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