



HAL
open science

A Proof of the Generalized Riemann Hypothesis

Charaf Ech-Chatbi

► **To cite this version:**

| Charaf Ech-Chatbi. A Proof of the Generalized Riemann Hypothesis. 2019. hal-02150289v1

HAL Id: hal-02150289

<https://hal.science/hal-02150289v1>

Preprint submitted on 7 Jun 2019 (v1), last revised 20 Mar 2023 (v9)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Proof of the Generalized Riemann Hypothesis

Charaf ECH-CHATBI *

Tuesday 28 May 2019

Abstract

We present a proof of the Generalized Riemann hypothesis (GRH) based on asymptotic expansions and operations on series. The advantage of our method is that it only uses undergraduate maths which makes it accessible to a wider audience.

Keywords: Generalized Riemann hypothesis; Zeta; Critical Strip; Prime Number Theorem; Millennium Problems; Dirichlet L-functions.

1 Introduction: Dirichlet L -functions

Let's $(z_n)_{n \geq 1}$ be a sequence of complex numbers. A Dirichlet series[6] is a series of the form $\sum_{n=1}^{\infty} \frac{z_n}{n^s}$, where s is complex. The Riemann zeta function is a Dirichlet series. Let's define the function $L(s)$ of the complex s :

$$L(s) = \sum_{n=1}^{\infty} \frac{z_n}{n^s}.$$

- If $(z_n)_{n \geq 1}$ is a bounded, then the corresponding Dirichlet series converges absolutely on the open half-plane where $\Re(s) > 1$.
- If the set of sums $z_n + z_{n+1} + \dots + z_{n+k}$ for each n and $k \geq 0$ is bounded, then the corresponding Dirichlet series converges on the open half-plane where $\Re(s) > 0$.
- In general, if $z_n = O(n^k)$, the corresponding Dirichlet series converges absolutely in the half plane where $\Re(s) > k + 1$.

The function $L(s)$ is analytic on the corresponding open half plane[3, 6,15].

To define Dirichlet L -functions we need to define Dirichlet characters. A function $\chi : \mathbb{Z} \mapsto \mathbb{C}$ is a Dirichlet character modulo q if it satisfies the following criteria:

*Maybank. One Raffles Quay, North Tower Level 35. 048583 Singapore. Email: charaf@melix.net or charaf.chatbi@gmail.com. Lnikedin/Twitter: @sharafmouzak. The opinions of this article are those of the author and do not reflect in any way the views or business of his employer.

- (i) $\chi(n) \neq 0$ if $(n, q) = 1$.
- (ii) $\chi(n) = 0$ if $(n, q) > 1$.
- (iii) χ is periodic with period q : $\chi(n + q) = \chi(n)$ for all n .
- (iv) χ is multiplicative : $\chi(mn) = \chi(m)\chi(n)$ for all integers m and n .

The trivial character is the one with $\chi_0(n) = 1$ whenever $(n, q) = 1$.

Here are some known results for a Dirichlet character modulo q . For any integer n we have $\chi(1) = 1$. Also if $(n, q) = 1$, we have $(\chi(n))^{\phi(q)} = 1$ with ϕ is Euler's totient function. $\chi(n)$ is a $\phi(q)$ -th root of unity. Therefore, $|\chi(n)| = 1$ if $(n, q) = 1$, and $|\chi(n)| = 0$ if $(n, q) > 1$. Also, we recall the cancellation property for Dirichlet characters modulo q : For any n integer

$$\sum_{i=1}^q \chi(i+n) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (1)$$

The Dirichlet L -functions are simply the sum of the Dirichlet series. Let's χ be a Dirichlet character modulo q , The Dirichlet L -function $L(s, \chi)$ is defined for $\Re(s) > 1$ as the following:

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \quad (2)$$

Where the series $\sum_{n \geq 1} \frac{\chi(n)}{n^s}$ is convergent when $\Re(s) > 0$ and $L(s, \chi)$ is analytic in $\Re(s) > 0$. See([3,6,7,8]). In the particular case of the trivial character χ_0 , $L(s, \chi_0)$ extends to a meromorphic function in $\Re(s) > 0$ with the only pole at $s = 1$ (see [3,6,7,8]).

Also, like the Riemann zeta function, the Dirichlet L -functions have their Euler product[2,3,24]. For $\Re(s) > 1$:

$$L(s, \chi) = \prod_{p \text{ Prime}} \left(1 - \frac{\chi(p)}{p^s}\right) \quad (3)$$

Let's q' be the smallest divisor of q . Let's χ' be the Dirichlet character $\chi' \pmod{q'}$. For any integer n such that $(n, q) = 1$ we have also $(n, q') = 1$ and $\chi(n) = \chi'(n)$. χ' is called primitive and $L(s, \chi)$ and $L(s, \chi')$ are related analytically such that:

$$L(s, \chi) = L(s, \chi') \prod_{p/q} \left(1 - \frac{\chi'(p)}{p^s}\right) \quad (4)$$

$L(s, \chi)$ and $L(s, \chi')$ have the same zeros in the critical strip $0 \leq \Re(s) \leq 1$.

Also, for a primitive character χ , (i.e. $\chi = \chi'$) $L(s, \chi)$ has the following functional equation:

$$\tau(\chi) \Gamma\left(\frac{1-s+a}{2}\right) L(1-s, \chi) = \sqrt{\pi} \left(\frac{q}{\pi}\right)^s i^{-a} q^{\frac{1}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \bar{\chi}) \quad (5)$$

Where Γ is the Gamma function and $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$, and $\tau(\chi) = \sum_{k=1}^q \chi(k) \exp\left(\frac{2\pi ki}{q}\right)$.

When $\Re(s) > 1$ there is no zero for $L(s, \chi)$. When $\Re(s) < 0$, for a primitive character χ , we have the trivial zeros of $L(s, \chi)$: $s = a - 2k$, where k is a positive integer and a is defined above. For more details, please refer to the references[7-14].

2 The Generalized Riemann Hypothesis

The Generalized Riemann Hypothesis states that the Dirichlet L -functions have all their non-trivial zeros on the critical line $\Re(s) = \frac{1}{2}$.

Non trivial in this case means $L(s, \chi) = 0$ for $s \in \mathbb{C}$ and $0 < \Re(s) < 1$. So for any primitive character χ modulo q , all non-trivial zeros of $L(s, \chi)$ lies in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. From the functional equation above we have that if:

- s_0 is a non-trivial zero of $L(s, \chi)$, then $1 - s_0$ is a zero of $L(s, \bar{\chi})$.
- s_0 is a non-trivial zero of $L(s, \bar{\chi})$, then $1 - s_0$ is a zero of $L(s, \chi)$.

Therefore, we just need to prove that for all primitive character χ modulo q , there is no non-trivial zeros of $L(s, \chi)$ in the right hand side of the critical strip $\{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$.

3 Proof of the GRH

Let's take a complex number s such $s = a_0 + ib_0$. Unless we explicitly mention otherwise, let's suppose that $a_0 > 0$ and $b_0 > 0$.

3.1 Case One: $\frac{1}{2} < a_0 \leq 1$

In this case s is a zero of $L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$. Where χ is a Dirichlet character χ modulo q . q is a nonzero positive integer.

Let's denote for each $n \geq 1$: $z_n = x_n + iy_n = \chi(n)$.

We are going to develop the sequence $Z_N(s) = \sum_{n=1}^N \frac{z_n}{n^s}$ as follows: For $N \geq 1$

$$Z_N = \sum_{n=1}^N \frac{z_n}{n^s} \tag{6}$$

$$= \sum_{n=1}^N \frac{x_n + iy_n}{n^{a_0 + ib_0}} \tag{7}$$

$$= \sum_{n=1}^N \frac{(x_n + iy_n)n^{-ib_0}}{n^{a_0}} \tag{8}$$

$$= \sum_{n=1}^N \frac{(x_n + iy_n) \exp(-ib_0 \ln(n))}{n^{a_0}} \tag{9}$$

$$= \sum_{n=1}^N \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n)) + i(y_n \cos(b_0 \ln(n)) - x_n \sin(b_0 \ln(n)))}{n^{a_0}} \tag{10}$$

$$= \sum_{n=1}^N \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{n^{a_0}} + i \sum_{n=1}^{+\infty} \frac{y_n \cos(b_0 \ln(n)) - x_n \sin(b_0 \ln(n))}{n^{a_0}} \tag{11}$$

$$\tag{12}$$

Let's define the sequences U_n, V_n as follows: For $n \geq 1$

$$U_n = \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{n^{a_0}} \quad (13)$$

$$V_n = \frac{y_n \cos(b_0 \ln(n)) - x_n \sin(b_0 \ln(n))}{n^{a_0}} \quad (14)$$

Let's define the series A_n, B_n and Z_n as follows:

$$A_n = \sum_{k=1}^n U_k \quad (15)$$

$$B_n = \sum_{k=1}^n V_k \quad (16)$$

$$Z_n = A_n + iB_n \quad (17)$$

When we are dealing with complex numbers, it is always insightful to work with the norm. So let's develop further the squared norm of the serie Z_N as follows:

$$\|Z_N\|^2 = A_N^2 + B_N^2 \quad (18)$$

$$= \left(\sum_{n=1}^N U_n \right)^2 + \left(\sum_{n=1}^N V_n \right)^2 \quad (19)$$

So

$$A_N^2 = \left(\sum_{n=1}^N U_n \right)^2 \quad (20)$$

$$= \sum_{n=1}^N U_n^2 + 2 \sum_{n=1}^N \sum_{k=1}^{n-1} U_n U_k \quad (21)$$

$$= \sum_{n=1}^N U_n^2 + 2 \sum_{n=1}^N U_n \sum_{k=1}^{n-1} U_k \quad (22)$$

$$= \sum_{n=1}^N U_n^2 + 2 \sum_{n=1}^N U_n (A_n - U_n) \quad (23)$$

$$= - \sum_{n=1}^N U_n^2 + 2 \sum_{n=1}^N U_n A_n \quad (24)$$

And the same calculation for B_N

$$B_N^2 = \left(\sum_{n=1}^N V_n \right)^2 \quad (25)$$

$$= \sum_{n=1}^N V_n^2 + 2 \sum_{n=1}^N \sum_{k=1}^{n-1} V_n V_k \quad (26)$$

$$= \sum_{n=1}^N V_n^2 + 2 \sum_{n=1}^N V_n \sum_{k=1}^{n-1} V_k \quad (27)$$

$$= \sum_{n=1}^N V_n^2 + 2 \sum_{n=1}^N V_n (B_n - V_n) \quad (28)$$

$$= - \sum_{n=1}^N V_n^2 + 2 \sum_{n=1}^N V_n B_n \quad (29)$$

Hence we have the new expression of square norm of Z_N :

$$\|Z_N\|^2 = 2 \sum_{n=1}^N U_n A_n + 2 \sum_{n=1}^N V_n B_n - \sum_{n=1}^N U_n^2 - \sum_{n=1}^N V_n^2 \quad (30)$$

Let's now define F_n and G_n as follows:

$$F_n = U_n A_n \quad (31)$$

$$G_n = V_n B_n \quad (32)$$

Therefore

$$A_N^2 = 2 \sum_{n=1}^N F_n - \sum_{n=1}^N U_n^2 \quad (33)$$

$$B_N^2 = 2 \sum_{n=1}^N G_n - \sum_{n=1}^N V_n^2 \quad (34)$$

Therefore

Conclusion. s is a **zero** for $L(s, \chi) = 0$, if and only if

$$\lim_{N \rightarrow \infty} A_N = 0 \text{ and } \lim_{N \rightarrow \infty} B_N = 0 \quad (35)$$

Equally, s is a $L(s, \chi)$ **zero**, $L(s, \chi) = 0$, if and only if

$$\lim_{N \rightarrow \infty} A_N^2 = 0 \text{ and } \lim_{N \rightarrow \infty} B_N^2 = 0 \quad (36)$$

Proof Strategy. The idea is to prove that in the case of a complex s that is in the right hand side of the critical strip $\frac{1}{2} < a_0 \leq 1$ and that is a $L(s, \chi)$ zero, that the limit $\lim_{n \rightarrow \infty} A_n^2 = +/ - \infty$ **OR** the limit $\lim_{n \rightarrow \infty} B_n^2 = +/ - \infty$. This will create a contradiction. Because if s is a $L(s, \chi)$ zero then the $\lim_{n \rightarrow \infty} A_n^2$ should be 0 and the $\lim_{n \rightarrow \infty} B_n^2$ should be

0. And therefore the sequences $(\sum_{n=1}^N F_n)_{N \geq 1}$ and $(\sum_{n=1}^N G_n)_{N \geq 1}$ should converge and their limits should be: $\lim_{n \rightarrow \infty} \sum_{n=1}^N F_n = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^N U_n^2 < +\infty$ and $\lim_{n \rightarrow \infty} \sum_{n=1}^N G_n = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^N V_n^2 < +\infty$.

Lemma 3.1. *If the set of the partial sums $z_n + z_{n+1} + \dots + z_{n+k}$ for n and $k \geq 0$ is bounded, then we can write $A_n = \frac{\lambda_n}{n^{a_0}}$ and $B_n = \frac{\lambda \lambda_n}{n^{a_0}}$ where (λ_n) and $(\lambda \lambda_n)$ are bounded sequences.*

Proof. We have $\lim_{N \rightarrow +\infty} A_N = \sum_{n=1}^{+\infty} U_n = 0$. Therefore for each $N \geq 1$:

$$\sum_{n=1}^{+\infty} U_n = \underbrace{\sum_{n=1}^N U_n}_{A_N} + \sum_{n=N+1}^{+\infty} U_n = 0 \quad (37)$$

$$A_N = - \sum_{n=N+1}^{+\infty} U_n \quad (38)$$

We have

$$A_N = - \sum_{n=N+1}^{+\infty} \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{n^{a_0}} \quad (39)$$

Let's denote X_n and Y_n the partial sums of the series x_n and y_n : $X_n = \sum_{k=1}^N x_k$ and $Y_n = \sum_{k=1}^N y_k$. So let's take N and M two integers such that $M \geq N$ and do the Abel summation between $N+1$ and M :

$$\sum_{n=N+1}^M U_n = \sum_{n=N+1}^M \frac{x_n \cos(b_0 \ln(n))}{n^{a_0}} + \sum_{n=N+1}^M \frac{y_n \sin(b_0 \ln(n))}{n^{a_0}} \quad (40)$$

$$= \frac{X_M \cos(b_0 \ln(M))}{M^{a_0}} - \frac{X_N \cos(b_0 \ln(N+1))}{(N+1)^{a_0}} - \sum_{n=N+1}^{M-1} X_n \left(\frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right) \quad (41)$$

$$+ \frac{Y_M \sin(b_0 \ln(M))}{M^{a_0}} - \frac{Y_N \sin(b_0 \ln(N+1))}{(N+1)^{a_0}} - \sum_{n=N+1}^{M-1} Y_n \left(\frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right) \quad (42)$$

Let's define the functions f_n and e_n such that $f_n(t) = \frac{\cos(b_0 \ln(n+t))}{(n+t)^{a_0}}$ and $e_n(t) = \frac{\sin(b_0 \ln(n+t))}{(n+t)^{a_0}}$. For each $n \geq 1$, we can apply the Mean Value Theorem on the interval $[0, 1]$ to find c_1 and c_2 in $(0, 1)$ such that:

$$\left| \frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right| = |e'_n(c_1)(1-0)| \quad (43)$$

$$\left| \frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right| = |f'_n(c_2)(1-0)| \quad (44)$$

We have the derivatives of f_n and e_n such that $f'_n(t) = \frac{-b_0 \sin(b_0 \ln(n+t)) - a_0 \cos(b_0 \ln(n+t))}{(n+t)^{a_0+1}}$ and $e'_n(t) = \frac{b_0 \cos(b_0 \ln(n+t)) - a_0 \sin(b_0 \ln(n+t))}{(n+t)^{a_0+1}}$. Therefore

$$\left| \frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right| \leq \frac{a_0 + b_0}{n^{a_0+1}} \quad (45)$$

$$\left| \frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right| \leq \frac{a_0 + b_0}{n^{a_0+1}} \quad (46)$$

We have the set of the partial sums $z_n + z_{n+1} + \dots + z_{n+k}$ is bounded, then the real part and the imaginary part of the sum partial of $z_n + z_{n+1} + \dots + z_{n+k}$ are also bounded.

Let's $K > 0$ such that for every n and k : $|x_n + x_{n+1} + \dots + x_{n+k}| \leq K$ and $|y_n + y_{n+1} + \dots + y_{n+k}| \leq K$. Therefore for each n : $|X_n| \leq K$ and $|Y_n| \leq K$.

$$\left| \sum_{n=N+1}^M U_n \right| \leq \left| \frac{X_M \cos(b_0 \ln(M))}{M^{a_0}} \right| + \left| \frac{X_N \cos(b_0 \ln(N+1))}{(N+1)^{a_0}} \right| \quad (47)$$

$$+ \sum_{n=N+1}^{M-1} |X_n| \left| \left(\frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right) \right| \quad (48)$$

$$+ \left| \frac{Y_M \sin(b_0 \ln(M))}{M^{a_0}} \right| + \left| \frac{Y_N \sin(b_0 \ln(N+1))}{(N+1)^{a_0}} \right| \quad (49)$$

$$+ \sum_{n=N+1}^{M-1} |Y_n| \left| \left(\frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right) \right| \quad (50)$$

$$\leq \frac{4K}{(N+1)^{a_0}} + 2K(a_0 + b_0) \sum_{n=N+1}^{M-1} \frac{1}{n^{a_0+1}} \quad (51)$$

$$\leq \frac{4K}{(N+1)^{a_0}} + 2K(a_0 + b_0) \int_{N+1}^M \frac{dt}{t^{a_0+1}} \quad (52)$$

$$\leq \frac{4K}{(N+1)^{a_0}} + \frac{2K(a_0 + b_0)}{a_0} \left(\frac{1}{(N+1)^{a_0}} - \frac{1}{M^{a_0}} \right) \quad (53)$$

We tend M to infinity and we get:

$$\left| \sum_{n=N+1}^{+\infty} U_n \right| \leq \frac{4K}{(N+1)^{a_0}} + \frac{2K(a_0 + b_0)}{a_0(N+1)^{a_0}} \quad (54)$$

Therefore

$$|A_N| = \left| - \sum_{n=N+1}^{+\infty} U_n \right| \leq \frac{K_1}{(N+1)^{a_0}} \leq \frac{K_1}{N^{a_0}} \quad (55)$$

where $K_1 = 4K + \frac{2K(a_0+b_0)}{a_0} > 0$.

Let's define the sequence ϵ_n such that: $A_n = \frac{\lambda_n}{n^{a_0}}$. Therefore for each $n \geq 1$ we have: $|\lambda_n| \leq K_1$. Therefore the sequence (λ_n) is bounded.

The same proof applies for $B_n = \frac{\lambda_n}{n^{a_0}}$ with the sequence (λ_n) bounded. \square

Let's $n \geq 1$, we have $A_{n+1} = A_n + U_{n+1}$. Therefore

$$\frac{\lambda_{n+1}}{(n+1)^{a_0}} = \frac{\lambda_n}{n^{a_0}} + \frac{x_{n+1} \cos(b_0 \ln(n+1)) + y_{n+1} \sin(b_0 \ln(n+1))}{(n+1)^{a_0}} \quad (56)$$

And

$$\lambda_{n+1} = \left(1 + \frac{1}{n}\right)^{a_0} \lambda_n + \left(x_{n+1} \cos(b_0 \ln(n+1)) + y_{n+1} \sin(b_0 \ln(n+1))\right) \quad (57)$$

For the sake of notation simplification, let's denote , for $n \geq 1$,

$$u_n = x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n)) \quad (58)$$

$$\lambda_{n+1} = \left(1 + \frac{1}{n}\right)^{a_0} \lambda_n + u_{n+1} \quad (59)$$

Lemma 3.2. *If $\frac{1}{2} < a_0 \leq 1$, we have either the series $\sum_{n \geq 1} F_n$ **OR** the series $\sum_{n \geq 1} G_n$ diverges.*

Proof. First, we apply the previous lemma 3.1 to the case of our Dirichlet series. Thanks to the cancelation property mentioned in (1), we have the partial sums $(\sum_{i=1}^n \chi(i))_{n \geq 1}$ is bounded. Therefore we write $A_n = \frac{\lambda_n}{n^{a_0}}$ and $B_n = \frac{\lambda \lambda_n}{n^{a_0}}$ where the sequences (λ_n) and $(\lambda \lambda_n)$ are bounded.

To prove this lemma we proceed with some asymptotic expansions: We have

$$A_{n+1} = A_n + U_{n+1} \quad (60)$$

So F_{n+1} can be written as follows:

$$F_{n+1} = U_{n+1} A_n + U_{n+1}^2 \quad (61)$$

So $F_{n+1} - F_n$ can be written as

$$F_{n+1} - F_n = \left(U_{n+1} - U_n\right) A_n + U_{n+1}^2 \quad (62)$$

And the same thing for G_n :

$$G_{n+1} - G_n = \left(V_{n+1} - V_n\right) B_n + V_{n+1}^2 \quad (63)$$

Let's do now the asymptotic expansion of $U_{n+1} - U_n$. For this we need the asymptotic expansion of $\cos(b_0 \ln(n+1))$ and $\sin(b_0 \ln(n+1))$.

$$\cos(b_0 \ln(n+1)) = \cos\left(b_0 \ln(n) + b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (64)$$

$$= \cos(b_0 \ln(n)) \cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) - \sin(b_0 \ln(n)) \sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (65)$$

we have the asymptotic expansion of $\ln\left(1 + \frac{1}{n}\right)$ in order two as follow:

$$\ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (66)$$

Using the asymptotic expansion of the functions \sin and \cos that I will spare you the details here, we have

$$\cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) = 1 + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (67)$$

And

$$\sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) = \frac{b_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (68)$$

Hence

$$\cos(b_0 \ln(n+1)) = \cos(b_0 \ln(n)) - \frac{b_0}{n} \sin(b_0 \ln(n)) + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (69)$$

Also the Asymptotic expansion of $\frac{1}{(1+n)^{a_0}}$:

$$\frac{1}{(1+n)^{a_0}} = \frac{1}{n^{a_0}} \left(1 - \frac{a_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \quad (70)$$

Hence

$$\frac{\cos(b_0 \ln(n+1))}{(1+n)^{a_0}} = \frac{\cos(b_0 \ln(n))}{n^{a_0}} - \frac{b_0 \sin(b_0 \ln(n)) + a_0 \cos(b_0 \ln(n))}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (71)$$

$$= \frac{\cos(b_0 \ln(n))}{n^{a_0}} + \frac{c_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (72)$$

And the squared version of the above equation

$$\frac{\cos^2(b_0 \ln(n+1))}{(1+n)^{2a_0}} = \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + 2 \cos(b_0 \ln(n)) \frac{c_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (73)$$

For the asymptotic expansion of $\sin(b_0 \ln(n+1))$.

$$\sin(b_0 \ln(n+1)) = \sin\left(b_0 \ln(n) + b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (74)$$

$$= \sin(b_0 \ln(n)) \cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) + \cos(b_0 \ln(n)) \sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (75)$$

we have the asymptotic expansion of $\ln\left(1 + \frac{1}{n}\right)$ in order two as follow:

$$\ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (76)$$

Using the asymptotic expansion of the functions \sin and \cos that I will speare you the details here, we have

$$\sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) = \sin\left(\frac{b_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \quad (77)$$

$$= \frac{b_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (78)$$

And

$$\cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) = 1 + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (79)$$

Hence

$$\sin(b_0 \ln(n+1)) = \sin(b_0 \ln(n)) + \frac{b_0}{n} \cos(b_0 \ln(n)) + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (80)$$

Also the Asymptotic expansion of $\frac{1}{(1+n)^{a_0}}$:

$$\frac{1}{(1+n)^{a_0}} = \frac{1}{n^{a_0}} \left(1 - \frac{a_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \quad (81)$$

Hence

$$\frac{\sin(b_0 \ln(n+1))}{(1+n)^{a_0}} = \frac{\sin(b_0 \ln(n))}{n^{a_0}} + \frac{b_0 \cos(b_0 \ln(n)) - a_0 \sin(b_0 \ln(n))}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (82)$$

$$= \frac{\sin(b_0 \ln(n))}{n^{a_0}} + \frac{cc_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (83)$$

And the squared version of the above equation

$$\frac{\sin^2(b_0 \ln(n+1))}{(1+n)^{2a_0}} = \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sin(b_0 \ln(n)) \frac{cc_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (84)$$

Therefore

$$U_{n+1} - U_n = \frac{x_{n+1} \cos(b_0 \ln(n+1)) + y_{n+1} \sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{(n)^{a_0}} \quad (85)$$

$$= x_{n+1} \left(\frac{\cos(b_0 \ln(n))}{n^{a_0}} + \frac{c_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \right) \quad (86)$$

$$+ y_{n+1} \left(\frac{\sin(b_0 \ln(n))}{n^{a_0}} + \frac{cc_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \right) \quad (87)$$

$$- \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{(n)^{a_0}} \quad (88)$$

$$= \frac{(x_{n+1} - x_n) \cos(b_0 \ln(n)) + (y_{n+1} - y_n) \sin(b_0 \ln(n))}{n^{a_0}} + \frac{x_{n+1}c_n + y_{n+1}cc_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (89)$$

$$= \frac{\alpha_n}{n^{a_0}} + \frac{\beta_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (90)$$

And

$$U_{n+1} = \frac{\gamma_n}{n^{a_0}} + \frac{\beta_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (91)$$

Where

$$\alpha_n = (x_{n+1} - x_n) \cos(b_0 \ln(n)) + (y_{n+1} - y_n) \sin(b_0 \ln(n)) \quad (92)$$

$$\beta_n = x_{n+1}c_n + y_{n+1}cc_n \quad (93)$$

$$\gamma_n = x_{n+1} \cos(b_0 \ln(n)) + y_{n+1} \sin(b_0 \ln(n)) \quad (94)$$

$$c_n = -\left(b_0 \sin(b_0 \ln(n)) + a_0 \cos(b_0 \ln(n))\right) \quad (95)$$

$$cc_n = b_0 \cos(b_0 \ln(n)) - a_0 \sin(b_0 \ln(n)) \quad (96)$$

Therefore

$$U_{n+1}^2 = \frac{\gamma_n^2}{n^{2a_0}} + 2\frac{\gamma_n\beta_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (97)$$

So the asymptotic expansion of $F_{n+1} - F_n$ is as follows:

$$F_{n+1} - F_n = \frac{\gamma_n^2}{n^{2a_0}} + \frac{\alpha_n A_n}{n^{a_0}} + \frac{\beta_n A_n}{n^{a_0+1}} + 2\frac{\gamma_n\beta_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (98)$$

We have $A_n = \frac{\lambda_n}{n^{a_0}}$ where λ_n is bounded. Therefore

$$F_{n+1} - F_n = \frac{\gamma_n^2}{n^{2a_0}} + \frac{\alpha_n \lambda_n}{n^{2a_0}} + \frac{2\gamma_n\beta_n}{n^{2a_0+1}} + \frac{\beta_n \lambda_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (99)$$

Therefore

$$F_{n+1} - F_n = \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} + \frac{(\lambda_n + 2\gamma_n)\beta_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (100)$$

And the same thing for $V_{n+1} - V_n$:

$$V_{n+1} - V_n = \frac{y_{n+1} \cos(b_0 \ln(n+1)) - x_{n+1} \sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{y_n \cos(b_0 \ln(n)) - x_n \sin(b_0 \ln(n))}{(n)^{a_0}} \quad (101)$$

$$= y_{n+1} \left(\frac{\cos(b_0 \ln(n))}{n^{a_0}} + \frac{c_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \right) \quad (102)$$

$$- x_{n+1} \left(\frac{\sin(b_0 \ln(n))}{n^{a_0}} + \frac{cc_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \right) \quad (103)$$

$$- \frac{y_n \cos(b_0 \ln(n)) - x_n \sin(b_0 \ln(n))}{(n)^{a_0}} \quad (104)$$

$$= \frac{(y_{n+1} - y_n) \cos(b_0 \ln(n)) - (x_{n+1} - x_n) \sin(b_0 \ln(n))}{n^{a_0}} + \frac{y_{n+1}c_n - x_{n+1}cc_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (105)$$

$$= \frac{\alpha\alpha_n}{n^{a_0}} + \frac{\beta\beta_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (106)$$

And

$$V_{n+1} = \frac{\gamma\gamma_n}{n^{a_0}} + \frac{\beta\beta_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (107)$$

Where

$$\alpha\alpha_n = (y_{n+1} - y_n) \cos(b_0 \ln(n)) - (x_{n+1} - x_n) \sin(b_0 \ln(n)) \quad (108)$$

$$\beta\beta_n = y_{n+1}c_n - x_{n+1}cc_n \quad (109)$$

$$\gamma\gamma_n = y_{n+1} \cos(b_0 \ln(n)) - x_{n+1} \sin(b_0 \ln(n)) \quad (110)$$

Therefore

$$V_{n+1}^2 = \frac{\gamma\gamma_n^2}{n^{2a_0}} + 2\frac{\gamma\gamma_n\beta\beta_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (111)$$

So the asymptotic expansion of $G_{n+1} - G_n$ is as follows:

$$G_{n+1} - G_n = \frac{\gamma\gamma_n^2}{n^{2a_0}} + \frac{\alpha\alpha_n A_n}{n^{a_0}} + \frac{\beta\beta_n A_n}{n^{a_0+1}} + 2\frac{\gamma\gamma_n\beta_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (112)$$

We have $B_n = \frac{\lambda\lambda_n}{n^{a_0}}$ where $\lambda\lambda_n$ is bounded. Therefore

$$G_{n+1} - G_n = \frac{\gamma\gamma_n^2 + \alpha\alpha_n\lambda\lambda_n}{n^{2a_0}} + \frac{2\gamma\gamma_n\beta\beta_n}{n^{2a_0+1}} + \frac{\beta\beta_n\lambda\lambda_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (113)$$

Therefore

$$G_{n+1} - G_n = \frac{\gamma\gamma_n^2 + \alpha\alpha_n\lambda\lambda_n}{n^{2a_0}} + \frac{(\lambda\lambda_n + 2\gamma\gamma_n)\beta\beta_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (114)$$

By definition of \mathcal{O} , we know that there is exist a bounded sequence (ϵ_n) and there is exist a number N_0 such that: For each $n \geq N_0$ we have
Therefore

$$F_{n+1} - F_n = \frac{\gamma_n^2 + \alpha_n\lambda_n}{n^{2a_0}} + \frac{(\lambda_n + 2\gamma_n)\beta_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \quad (115)$$

$$G_{n+1} - G_n = \frac{\gamma\gamma_n^2 + \alpha\alpha_n\lambda\lambda_n}{n^{2a_0}} + \frac{(\lambda\lambda_n + 2\gamma\gamma_n)\beta\beta_n}{n^{2a_0+1}} + \frac{\epsilon\epsilon_n}{n^{a_0+2}} \quad (116)$$

Let's now study the asymptotic expansion of the dominant term $\frac{\gamma_n^2 + \alpha_n\lambda_n}{n^{2a_0}}$ and $\frac{\gamma\gamma_n^2 + \alpha\alpha_n\lambda\lambda_n}{n^{2a_0}}$ in the context of the Dirichlet L -functions.

Lemma 3.3. *Let's q be a nonzero integer. Let's suppose that for each k , $x_{kq+1} = x_{kq+1}^2 = 1$ and $y_{kq+1} = y_{kq+1}^2 = 0$. Therefore we have the following asymptotic expansions:*

$$\sum_{i=1}^q \left(\gamma_{qn+i}^2 + \alpha_{qn+i}\lambda_{qn+i} \right) = \sum_{i=1}^q \frac{x_{qn+i} + y_{qn+i}}{2} + \cos(2b_0 \ln(qn)) \sum_{i=1}^q \frac{x_{qn+i} - y_{qn+i}}{2} \quad (117)$$

$$+ \sin(2b_0 \ln(qn)) \sum_{i=2}^q \frac{x_{qn+i} + y_{qn+i}}{2} + \mathcal{O}\left(\frac{1}{qn}\right) \quad (118)$$

And

$$\sum_{i=1}^q \left(\gamma_{qn+i}^2 + \alpha_{qn+i}\lambda_{qn+i} \right) = \sum_{i=1}^q \frac{y_{qn+i} - x_{qn+i} + 2x_{qn+i}^2}{2} \quad (119)$$

$$+ \cos(2b_0 \ln(qn)) \sum_{i=1}^q \frac{y_{qn+i} + x_{qn+i} - 2x_{qn+i}^2}{2} \quad (120)$$

$$+ \sin(2b_0 \ln(qn)) \sum_{i=2}^q \frac{y_{qn+i} - x_{qn+i} - 2x_{qn+i}y_{qn+i}}{2} + \mathcal{O}\left(\frac{1}{qn}\right) \quad (121)$$

Proof. In this case we have for each $n \geq 1$:

$$\alpha_n = (x_{n+1} - x_n) \cos(b_0 \ln(n)) + (y_{n+1} - y_n) \sin(b_0 \ln(n)) \quad (122)$$

$$\gamma_n^2 = x_{n+1} \cos(b_0 \ln(n)) + y_{n+1} \sin(b_0 \ln(n)) \quad (123)$$

λ_n is bounded. From the asymptotic expansion of $\cos(b_0 \ln(n+1))$ and $\sin(b_0 \ln(n+1))$:

$$\lambda_{n+1} = \lambda_n + x_{n+1} \cos(b_0 \ln(n+1)) + y_{n+1} \sin(b_0 \ln(n+1)) + O\left(\frac{1}{n}\right) \quad (124)$$

Let q be a nonzero integer. And let's χ be the Dirichlet character modulo q associated to our Dirichlet series $\sum_{n \geq 1} \frac{z_n}{n^s}$. $\chi(n) = z_n = x_n + iy_n$ for each $n \geq 1$. Then, by definition we have for each k , $x_{kq+1} = x_{kq+1}^2 = 1$ and $y_{kq+1} = y_{kq+1}^2 = 0$. And for each $1 \leq i \leq q$, $x_{kq+i} = x_i$ and $y_{kq+i} = y_i$.

We have from above that

$$\cos(b_0 \ln(n+1)) = \cos(b_0 \ln(n)) + O\left(\frac{1}{n}\right) \quad (125)$$

$$\sin(b_0 \ln(n+1)) = \sin(b_0 \ln(n)) + O\left(\frac{1}{n}\right) \quad (126)$$

Therefore for $1 \leq i \leq q$: we have

$$\cos(b_0 \ln(qn+i)) = \cos(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (127)$$

$$\sin(b_0 \ln(qn+i)) = \sin(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (128)$$

$$\lambda_{qn+i+1} = \lambda_{qn+i} + x_{qn+i+1} \cos(b_0 \ln(qn)) + y_{qn+i+1} \sin(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (129)$$

Therefore

$$\sum_{i=1}^q \alpha_{qn+i} \lambda_{qn+i} = \sum_{i=1}^q (x_{qn+i+1} - x_{qn+i}) \lambda_{qn+i} \cos(b_0 \ln(qn+i)) \quad (130)$$

$$+ (y_{qn+i+1} - y_{qn+i}) \lambda_{qn+i} \sin(b_0 \ln(qn+i)) \quad (131)$$

$$= \cos(b_0 \ln(qn)) \sum_{i=1}^q (x_{qn+i+1} - x_{qn+i}) \lambda_{qn+i} \quad (132)$$

$$+ \sin(b_0 \ln(qn)) \sum_{i=1}^q (y_{qn+i+1} - y_{qn+i}) \lambda_{qn+i} + O\left(\frac{1}{qn}\right) \quad (133)$$

We have here Abel's sums so:

$$\sum_{i=1}^q (x_{qn+i+1} - x_{qn+i}) \lambda_{qn+i} = \sum_{i=1}^q x_{qn+i+1} \lambda_{qn+i} - \sum_{i=1}^q x_{qn+i} \lambda_{qn+i} \quad (134)$$

$$= \sum_{i=2}^{q+1} x_{qn+i} \lambda_{qn+i-1} - \sum_{i=1}^q x_{qn+i} \lambda_{qn+i} \quad (135)$$

$$= x_{qn+q+1} \lambda_{qn+q} - x_{qn+1} \lambda_{qn+1} + \sum_{i=2}^q x_{qn+i} (\lambda_{qn+i-1} - \lambda_{qn+i}) \quad (136)$$

$$= \lambda_{qn+q} - \lambda_{qn+1} \quad (137)$$

$$- \sum_{i=2}^q x_{qn+i} (x_{qn+i} \cos(b_0 \ln(qn)) + y_{qn+i} \sin(b_0 \ln(qn))) + O\left(\frac{1}{qn}\right) \quad (138)$$

$$= \sum_{i=2}^q \lambda_{qn+i} - \lambda_{qn+i-1} \quad (139)$$

$$- \sum_{i=2}^q x_{qn+i} (x_{qn+i} \cos(b_0 \ln(qn)) + y_{qn+i} \sin(b_0 \ln(qn))) + O\left(\frac{1}{qn}\right) \quad (140)$$

$$= \sum_{i=2}^q (x_{qn+i} \cos(b_0 \ln(qn)) + y_{qn+i} \sin(b_0 \ln(qn))) + O\left(\frac{1}{qn}\right) \quad (141)$$

$$- \sum_{i=2}^q x_{qn+i} (x_{qn+i} \cos(b_0 \ln(qn)) + y_{qn+i} \sin(b_0 \ln(qn))) + O\left(\frac{1}{qn}\right) \quad (142)$$

$$= \sum_{i=2}^q (x_{qn+i} - x_{qn+i}^2) \cos(b_0 \ln(qn)) \quad (143)$$

$$+ (y_{qn+i} - x_{qn+i} y_{qn+i}) \sin(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (144)$$

And

$$\sum_{i=1}^q (y_{qn+i+1} - y_{qn+i}) \lambda_{qn+i} = \sum_{i=2}^q (y_{qn+i} - y_{qn+i}^2) \sin(b_0 \ln(qn)) \quad (145)$$

$$+ (x_{qn+i} - y_{qn+i} x_{qn+i}) \cos(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (146)$$

Therefore:

$$\sum_{i=1}^q \alpha_{qn+i} \lambda_{qn+i} = \sum_{i=2}^q (x_{qn+i} - x_{qn+i}^2) \cos^2(b_0 \ln(qn)) \quad (147)$$

$$+ (y_{qn+i} - x_{qn+i} y_{qn+i}) \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \quad (148)$$

$$+ \sum_{i=2}^q (y_{qn+i} - y_{qn+i}^2) \sin^2(b_0 \ln(qn)) \quad (149)$$

$$+ (x_{qn+i} - y_{qn+i} x_{qn+i}) \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (150)$$

Now the term $\sum_{i=1}^q \gamma_{qn+i}^2$

$$\sum_{i=1}^q \gamma_{qn+i}^2 = \sum_{i=1}^q \left(x_{qn+i+1} \cos(b_0 \ln(qn+i)) + y_{qn+i+1} \sin(b_0 \ln(qn+i)) \right)^2 \quad (151)$$

$$= \sum_{i=1}^q \left\{ x_{qn+i+1}^2 \cos^2(b_0 \ln(qn)) + y_{qn+i+1}^2 \sin^2(b_0 \ln(qn)) \right. \quad (152)$$

$$\left. + 2x_{qn+i+1}y_{qn+i+1} \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \right\} + O\left(\frac{1}{qn}\right) \quad (153)$$

$$= \cos^2(b_0 \ln(qn)) \sum_{i=2}^{q+1} x_{qn+i}^2 + \sin^2(b_0 \ln(qn)) \sum_{i=2}^{q+1} y_{qn+i}^2 \quad (154)$$

$$+ 2 \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^{q+1} x_{qn+i}y_{qn+i} + O\left(\frac{1}{qn}\right) \quad (155)$$

We have for each n , $x_{kq+1} = x_{kq+1}^2 = 1$ and $y_{kq+1} = y_{kq+1}^2 = 0$.
Therefore

$$\sum_{i=1}^q \gamma_{qn+i}^2 + \sum_{i=1}^q \alpha_{qn+i} \lambda_{qn+i} = \cos^2(b_0 \ln(qn)) \sum_{i=2}^{q+1} x_{qn+i}^2 + \sin^2(b_0 \ln(qn)) \sum_{i=2}^{q+1} y_{qn+i}^2 \quad (156)$$

$$+ 2 \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^{q+1} x_{qn+i}y_{qn+i} + O\left(\frac{1}{qn}\right) \quad (157)$$

$$+ \cos^2(b_0 \ln(qn)) \sum_{i=2}^q \left(x_{qn+i} - x_{qn+i}^2 \right) \quad (158)$$

$$+ \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^q \left(y_{qn+i} - x_{qn+i}y_{qn+i} \right) \quad (159)$$

$$+ \sin^2(b_0 \ln(qn)) \sum_{i=2}^q \left(y_{qn+i} - y_{qn+i}^2 \right) \quad (160)$$

$$+ \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^q \left(x_{qn+i} - y_{qn+i}x_{qn+i} \right) + O\left(\frac{1}{qn}\right) \quad (161)$$

$$= \cos^2(b_0 \ln(qn))x_{qn+1}^2 + \sin^2(b_0 \ln(qn))y_{qn+1}^2 \quad (162)$$

$$+ 2 \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn))x_{qn+1}y_{qn+1} \quad (163)$$

$$+ \cos^2(b_0 \ln(qn)) \sum_{i=2}^q x_{qn+i} + \sin^2(b_0 \ln(qn)) \sum_{i=2}^q y_{qn+i} \quad (164)$$

$$+ \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^q \left(x_{qn+i} + y_{qn+i} \right) + O\left(\frac{1}{qn}\right) \quad (165)$$

$$= \cos^2(b_0 \ln(qn)) \sum_{i=1}^q x_{qn+i} + \sin^2(b_0 \ln(qn)) \sum_{i=1}^q y_{qn+i} \quad (166)$$

$$+ \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^q \left(x_{qn+i} + y_{qn+i} \right) + O\left(\frac{1}{qn}\right) \quad (167)$$

$$(168)$$

Therefore

$$\sum_{i=1}^q \left(\gamma_{qn+i}^2 + \alpha_{qn+i} \lambda_{qn+i} \right) = \sum_{i=1}^q \frac{x_{qn+i} + y_{qn+i}}{2} + \cos(2b_0 \ln(qn)) \sum_{i=1}^q \frac{x_{qn+i} - y_{qn+i}}{2} \quad (169)$$

$$+ \sin(2b_0 \ln(qn)) \sum_{i=2}^q \frac{x_{qn+i} + y_{qn+i}}{2} + O\left(\frac{1}{qn}\right) \quad (170)$$

Let's now study the case of $\sum_{i=1}^q \alpha_{qn+i} \lambda \lambda_{qn+i}$.

$$\sum_{i=1}^q \alpha_{qn+i} \lambda \lambda_{qn+i} = \sum_{i=1}^q \left(y_{qn+i+1} - y_{qn+i} \right) \lambda_{qn+i} \cos(b_0 \ln(qn+i)) \quad (171)$$

$$- \left(x_{qn+i+1} - x_{qn+i} \right) \lambda \lambda_{qn+i} \sin(b_0 \ln(qn+i)) \quad (172)$$

$$= \cos(b_0 \ln(qn)) \sum_{i=1}^q \left(y_{qn+i+1} - y_{qn+i} \right) \lambda \lambda_{qn+i} \quad (173)$$

$$- \sin(b_0 \ln(qn)) \sum_{i=1}^q \left(x_{qn+i+1} - x_{qn+i} \right) \lambda \lambda_{qn+i} + O\left(\frac{1}{qn}\right) \quad (174)$$

We have here Abel's sums so:

$$\sum_{i=1}^q \left(y_{qn+i+1} - y_{qn+i} \right) \lambda \lambda_{qn+i} = \sum_{i=1}^q y_{qn+i+1} \lambda \lambda_{qn+i} - \sum_{i=1}^q y_{qn+i} \lambda \lambda_{qn+i} \quad (175)$$

$$= \sum_{i=2}^{q+1} y_{qn+i} \lambda \lambda_{qn+i-1} - \sum_{i=1}^q y_{qn+i} \lambda \lambda_{qn+i} \quad (176)$$

$$= y_{qn+q+1} \lambda \lambda_{qn+q} - y_{qn+1} \lambda \lambda_{qn+1} + \sum_{i=2}^q y_{qn+i} \left(\lambda \lambda_{qn+i-1} - \lambda \lambda_{qn+i} \right) \quad (177)$$

$$= \lambda \lambda_{qn+q} - \lambda \lambda_{qn+1} \quad (178)$$

$$- \sum_{i=2}^q y_{qn+i} \left(y_{qn+i} \cos(b_0 \ln(qn)) - x_{qn+i} \sin(b_0 \ln(qn)) \right) + O\left(\frac{1}{qn}\right) \quad (179)$$

$$= \sum_{i=2}^q \lambda \lambda_{qn+i} - \lambda \lambda_{qn+i-1} \quad (180)$$

$$- \sum_{i=2}^q y_{qn+i} \left(y_{qn+i} \cos(b_0 \ln(qn)) - x_{qn+i} \sin(b_0 \ln(qn)) \right) + O\left(\frac{1}{qn}\right) \quad (181)$$

$$= \sum_{i=2}^q \left(y_{qn+i} \cos(b_0 \ln(qn)) - x_{qn+i} \sin(b_0 \ln(qn)) \right) + O\left(\frac{1}{qn}\right) \quad (182)$$

$$- \sum_{i=2}^q y_{qn+i} \left(y_{qn+i} \cos(b_0 \ln(qn)) - x_{qn+i} \sin(b_0 \ln(qn)) \right) + O\left(\frac{1}{qn}\right) \quad (183)$$

$$= \sum_{i=2}^q \left(y_{qn+i} - y_{qn+i}^2 \right) \cos(b_0 \ln(qn)) \quad (184)$$

$$- \sum_{i=2}^q \left(x_{qn+i} - x_{qn+i} y_{qn+i} \right) \sin(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (185)$$

And

$$\sum_{i=1}^q (x_{qn+i+1} - x_{qn+i}) \lambda \lambda_{qn+i} = \sum_{i=2}^q (x_{qn+i} - x_{qn+i}^2) \sin(b_0 \ln(qn)) \quad (186)$$

$$- (y_{qn+i} - x_{qn+i} y_{qn+i}) \cos(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (187)$$

Therefore:

$$\sum_{i=1}^q \alpha \alpha_{qn+i} \lambda \lambda_{qn+i} = \sum_{i=2}^q (y_{qn+i} - y_{qn+i}^2) \cos^2(b_0 \ln(qn)) \quad (188)$$

$$- (x_{qn+i} - x_{qn+i} y_{qn+i}) \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \quad (189)$$

$$- \sum_{i=2}^q (x_{qn+i} - x_{qn+i}^2) \sin^2(b_0 \ln(qn)) \quad (190)$$

$$+ (y_{qn+i} - x_{qn+i} y_{qn+i}) \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) + O\left(\frac{1}{qn}\right) \quad (191)$$

Now the term $\sum_{i=1}^q \gamma \gamma_{qn+i}^2$

$$\sum_{i=1}^q \gamma \gamma_{qn+i}^2 = \sum_{i=1}^q (y_{qn+i+1} \cos(b_0 \ln(qn+i)) - x_{qn+i+1} \sin(b_0 \ln(qn+i)))^2 \quad (192)$$

$$= \sum_{i=1}^q \left\{ y_{qn+i+1}^2 \cos^2(b_0 \ln(qn)) + x_{qn+i+1}^2 \sin^2(b_0 \ln(qn)) \quad (193) \right.$$

$$\left. - 2x_{qn+i+1} y_{qn+i+1} \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \right\} + O\left(\frac{1}{qn}\right) \quad (194)$$

$$= \cos^2(b_0 \ln(qn)) \sum_{i=2}^{q+1} y_{qn+i}^2 + \sin^2(b_0 \ln(qn)) \sum_{i=2}^{q+1} x_{qn+i}^2 \quad (195)$$

$$- 2 \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^{q+1} x_{qn+i} y_{qn+i} + O\left(\frac{1}{qn}\right) \quad (196)$$

We have for each n , $x_{kq+1} = x_{kq+1}^2 = 1$ and $y_{kq+1} = y_{kq+1}^2 = 0$.

Therefore

$$\sum_{i=1}^q \gamma \gamma_{qn+i}^2 + \sum_{i=1}^q \alpha \alpha_{qn+i} \lambda \lambda_{qn+i} = \cos^2(b_0 \ln(qn)) \sum_{i=2}^{q+1} y_{qn+i}^2 + \sin^2(b_0 \ln(qn)) \sum_{i=2}^{q+1} x_{qn+i}^2 \quad (197)$$

$$- 2 \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^{q+1} x_{qn+i} y_{qn+i} + O\left(\frac{1}{qn}\right) \quad (198)$$

$$+ \cos^2(b_0 \ln(qn)) \sum_{i=2}^q (y_{qn+i} - y_{qn+i}^2) \quad (199)$$

$$+ \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^q (y_{qn+i} - x_{qn+i} y_{qn+i}) \quad (200)$$

$$- \sin^2(b_0 \ln(qn)) \sum_{i=2}^q (x_{qn+i} - x_{qn+i}^2) \quad (201)$$

$$- \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^q (x_{qn+i} - y_{qn+i} x_{qn+i}) + O\left(\frac{1}{qn}\right) \quad (202)$$

$$= \cos^2(b_0 \ln(qn)) y_{qn+1}^2 + \sin^2(b_0 \ln(qn)) x_{qn+1}^2 \quad (203)$$

$$- 2 \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) x_{qn+1} y_{qn+1} \quad (204)$$

$$+ \cos^2(b_0 \ln(qn)) \sum_{i=2}^q y_{qn+i} + \sin^2(b_0 \ln(qn)) \sum_{i=2}^q (2x_{qn+i}^2 - x_{qn+i}) \quad (205)$$

$$+ \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^q (y_{qn+i} - x_{qn+i} - 2x_{qn+i} y_{qn+i}) \quad (206)$$

$$+ O\left(\frac{1}{qn}\right) \quad (207)$$

$$= \cos^2(b_0 \ln(qn)) \sum_{i=1}^q y_{qn+i} + \sin^2(b_0 \ln(qn)) \sum_{i=1}^q (2x_{qn+i}^2 - x_{qn+i}) \quad (208)$$

$$+ \sin(b_0 \ln(qn)) \cos(b_0 \ln(qn)) \sum_{i=2}^q (y_{qn+i} - x_{qn+i} - 2x_{qn+i} y_{qn+i}) \quad (209)$$

$$+ O\left(\frac{1}{qn}\right) \quad (210)$$

Therefore

$$\sum_{i=1}^q (\gamma \gamma_{qn+i}^2 + \alpha \alpha_{qn+i} \lambda \lambda_{qn+i}) = \sum_{i=1}^q \frac{y_{qn+i} - x_{qn+i} + 2x_{qn+i}^2}{2} \quad (211)$$

$$+ \cos(2b_0 \ln(qn)) \sum_{i=1}^q \frac{y_{qn+i} + x_{qn+i} - 2x_{qn+i}^2}{2} \quad (212)$$

$$+ \sin(2b_0 \ln(qn)) \sum_{i=2}^q \frac{y_{qn+i} - x_{qn+i} - 2x_{qn+i} y_{qn+i}}{2} + O\left(\frac{1}{qn}\right) \quad (213)$$

□

Proof Strategy. *The idea here is to prove that we can choose N_0 such*

that $R_{N_0} + F_{N_0} > 0$. In such case, we will have the $\lim_{N \rightarrow +\infty} \sum_{n=1}^N F_{n+1} = +\infty$.

Divergence of $\sum_{n \geq 1} F_n$

For large $N \geq N_0$ we have the following expression of F_{N+1} :

$$F_{N+1} = F_{N_0} + \sum_{n=N_0}^N \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} + \frac{(\lambda_n + 2\gamma_n)\beta_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \quad (214)$$

$$= F_{N_0} + \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \quad (215)$$

Where

$$C_n = \gamma_n^2 + \alpha_n \lambda_n \quad (216)$$

$$D_n = (\lambda_n + 2\gamma_n)\beta_n \quad (217)$$

We do another summation and we have the following expression:

So for each $N \geq N_0$:

$$\sum_{n=N_0}^N F_{n+1} = (N - N_0 + 1)F_{N_0} \quad (218)$$

$$+ \sum_{n=N_0}^N \frac{C_n}{n^{a_0+1}}(N - n + 1) \quad (219)$$

$$+ \sum_{n=N_0}^N \frac{D_n}{n^{2a_0+1}}(N - n + 1) \quad (220)$$

$$+ \sum_{n=N_0}^N \frac{\epsilon_n}{n^{a_0+2}}(N - n + 1) \quad (221)$$

$$\sum_{n=N_0}^N \frac{C_n}{n^{a_0+1}}(N - n + 1) = (N + 1) \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} - \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} n \quad (222)$$

$$= (N + 1) \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} - \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} \quad (223)$$

$$\sum_{n=N_0}^N \frac{D_n}{n^{2a_0+1}}(N - n + 1) = (N + 1) \sum_{n=N_0}^N \frac{D_n}{n^{2a_0+1}} - \sum_{n=N_0}^N \frac{D_n}{n^{2a_0+1}} n \quad (224)$$

$$= (N + 1) \sum_{n=N_0}^N \frac{D_n}{n^{2a_0+1}} - \sum_{n=N_0}^N \frac{D_n}{n^{2a_0}} \quad (225)$$

And

$$\sum_{n=N_0}^N \frac{\varepsilon_n}{n^{2a_0+1}} (N-n+1) = (N+1) \sum_{n=N_0}^N \frac{\varepsilon_n}{n^{a_0+2}} - \sum_{n=N_0}^N \frac{\varepsilon_n}{n^{a_0+2}} n \quad (226)$$

$$= (N+1) \sum_{n=N_0}^N \frac{\varepsilon_n}{n^{a_0+2}} - \sum_{n=N_0}^N \frac{\varepsilon_n}{n^{a_0+1}} \quad (227)$$

So for each $N \geq N_0$:

$$\sum_{n=N_0}^N F_{n+1} = (N-N_0+1)F_{N_0} \quad (228)$$

$$+ (N+1) \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (229)$$

$$- \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} \quad (230)$$

We factorise by $N+1$ then we have this expression

$$\sum_{n=N_0}^N F_{n+1} = (N+1) \left(\frac{(N-N_0+1)}{N+1} F_{N_0} \right) \quad (231)$$

$$+ \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (232)$$

$$- \frac{1}{N+1} \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} \quad (233)$$

We have the sequences (C_n) , (D_n) and (ε_n) are bounded and $2a_0 > 1$, so the serie $\sum_{n \geq 1}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}}$ is converging absolutely. Let's prove now that the sequence in the equation (233) is converging to zero when N goes to infinity.

As we have $2a_0 > 1$, So

$$\lim_{n \rightarrow \infty} \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} = 0 \quad (234)$$

Therefore following the Césaro theorem we have:

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} = 0 \quad (235)$$

We denote $FF(N, N_0)$ as:

$$FF(N, N_0) = \left(\frac{(N - N_0 + 1)}{N + 1} F_{N_0} \right. \quad (236)$$

$$+ \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (237)$$

$$\left. - \frac{1}{N + 1} \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} \right) \quad (238)$$

Therefore:

$$\sum_{n=N_0}^N F_{n+1} = (N + 1)FF(N, N_0) \quad (239)$$

The serie $\sum_{n \geq 1} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}}$ is converging absolutely. Let's denote

$$R_{N_0} = \lim_{N \rightarrow +\infty} \sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (240)$$

Lemma 3.4. *We can choose N_0 such that $R_{N_0} + F_{N_0} > 0$.*

Proof. Let's now study the term $\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}}$.

We have from above lemmas that:

$$\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} = \sum_{n=qN_0+1}^{Nq} \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} \quad (241)$$

$$= \sum_{n=N_0}^N \sum_{i=1}^q \frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn+i)^{2a_0}} \quad (242)$$

We have for each

$1 \leq i \leq q$:

$$\frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn+i)^{2a_0}} = \frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn)^{2a_0}} + O\left(\frac{1}{(qn)^{2a_0+1}}\right) \quad (243)$$

Therefore

$$\sum_{i=1}^q \frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn+i)^{2a_0}} = \sum_{i=1}^q \frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn)^{2a_0}} + O\left(\frac{1}{(qn)^{2a_0+1}}\right) \quad (244)$$

$$= \frac{1}{(qn)^{2a_0}} \left\{ \sum_{i=1}^q \frac{x_{qn+i} + y_{qn+i}}{2} + \cos(2b_0 \ln(qn)) \sum_{i=1}^q \frac{x_{qn+i} - y_{qn+i}}{2} \right. \quad (245)$$

$$\left. + \sin(2b_0 \ln(qn)) \sum_{i=2}^q \frac{x_{qn+i} + y_{qn+i}}{2} \right\} + O\left(\frac{1}{(qn)^{2a_0+1}}\right) \quad (246)$$

$$= \frac{1}{(qn)^{2a_0}} \left\{ \sum_{i=1}^q \frac{x_i + y_i}{2} + \cos(2b_0 \ln(qn)) \sum_{i=1}^q \frac{x_i - y_i}{2} \right. \quad (247)$$

$$\left. + \sin(2b_0 \ln(qn)) \sum_{i=2}^q \frac{x_i + y_i}{2} \right\} + O\left(\frac{1}{(qn)^{2a_0+1}}\right) \quad (248)$$

$$= \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \alpha \sin(2b_0 \ln(qn))}{(qn)^{2a_0}} + O\left(\frac{1}{(qn)^{2a_0+1}}\right) \quad (249)$$

Where

$$\alpha = \sum_{i=1}^q \frac{x_i + y_i}{2} \quad (250)$$

$$\beta = \sum_{i=1}^q \frac{x_i - y_i}{2} \quad (251)$$

From the cancelation property of the Dirichlet characters, we can calculate the exact values of α and β .

$$\alpha = \frac{\Re\left(\sum_{i=1}^q \chi(i)\right) + \Im\left(\sum_{i=1}^q \chi(i)\right)}{2} = \begin{cases} \frac{\phi(q)}{2}, & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (252)$$

$$\beta = \frac{\Re\left(\sum_{i=1}^q \chi(i)\right) - \Im\left(\sum_{i=1}^q \chi(i)\right)}{2} = \begin{cases} \frac{\phi(q)}{2}, & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (253)$$

Therefore we can write the following for a large n :

$$\sum_{i=1}^q \frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn+i)^{2a_0}} = \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \alpha \sin(2b_0 \ln(qn))}{(qn)^{2a_0}} + \frac{\xi_n}{(qn)^{2a_0+1}} \quad (254)$$

Where the sequence ξ_n is bounded. Therefore

$$\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} = \sum_{n=N_0}^N \sum_{i=1}^q \frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn+i)^{2a_0}} \quad (255)$$

$$= \sum_{n=N_0}^N \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \alpha \sin(2b_0 \ln(qn))}{(qn)^{2a_0}} + \frac{\xi_n}{(qn)^{2a_0+1}} \quad (256)$$

$$= \sum_{n=N_0}^N \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \alpha \sin(2b_0 \ln(qn))}{(qn)^{2a_0}} + \sum_{n=N_0}^N \frac{\xi_n}{(qn)^{2a_0+1}} \quad (257)$$

And

$$\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} = \sum_{n=N_0}^N \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \alpha \sin(2b_0 \ln(qn))}{(qn)^{2a_0}} \quad (258)$$

$$+ \sum_{n=N_0}^N \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_0+1}^{Nq} \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (259)$$

The serie $\sum_{n \geq 1} \frac{\alpha + \beta \cos(2b_0 \ln(qn)) + \alpha \sin(2b_0 \ln(qn))}{(qn)^{2a_0}}$ was studied in the lemma 3.1 in [1]. Let's denote the function g as the following:

$$g(x) = \frac{\alpha + \beta \cos(2b_0 \ln(qx)) + \alpha \sin(2b_0 \ln(qx))}{(qx)^{2a_0}} \quad (260)$$

We have the following:

$$\left| \sum_{n=N_0+1}^{+\infty} g(n) - \int_{N_0+1}^{+\infty} g(x) \right| \leq \frac{K}{(N_0)^{2a_0}} \quad (261)$$

And the primitive function G of the function g (see lemma 3.1 in [1]):

$$G(x) = \frac{1}{2q} \left(-\frac{\alpha}{2a_0 - 1} + \frac{(2b_0\beta + \alpha(1 - 2a_0)) \sin(2b_0 \ln(x)) + (\beta(1 - 2a_0) - 2\alpha b_0) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1 - 2a_0)_0^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (262)$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} g(n) + G(N_0 + 1) \right| \leq \frac{K}{(N_0)^{2a_0}} \quad (263)$$

Case of Trivial Character In this case we have $\alpha = \beta = \frac{\phi(q)}{2}$. As an example, the case of the trivial character mod 3 where $\chi(1) = 1$, $\chi(2) = 1$ and $\chi(3) = 0$. Here we have $\alpha = 1$ and $\beta = 1$. $L(s, \chi) = \sum_{n=0}^{+\infty} \left(\frac{1}{(3n+1)^s} + \frac{1}{(3n+2)^s} \right)$.

Back to the general case. Concerning the term F_{N_0} . We have

$$F_{N_0} = \frac{u_{N_0}}{(N_0)^{a_0}} A_{N_0} = \frac{u_{N_0} \lambda_{N_0}}{(N_0)^{2a_0}} \quad (264)$$

Where the sequence $(u_n \lambda_n)$ is bounded.

Therefore we have the asymptotic expansion of $F_{N_0} + \sum_{n=N_0+1}^{+\infty} g(k)$ as follows:

$$F_{N_0} + \sum_{n=N_0+1}^{+\infty} g(k) = -G(N_0 + 1) + \mathcal{O}\left(\frac{1}{(N_0)^{2a_0}}\right) \quad (265)$$

Remark. The remaining terms in the expression of R_{N_0} are all of the order of $\frac{1}{(N_0)^{2a_0}}$ and above. So the dominant term in the expression R_{N_0} is the term $-G(N_0+1)$. As we will see later that the function G is nonzero function. Therefore this term is the dominant term in the expression of $\frac{1}{N} \sum_{n=N_0}^{N-1} F_{n+1}$. Hence based on the sign of $F(N_0 + 1)$ we can show that the limit of $\sum_{n=1}^{N-1} F_{n+1}$ can be both $+\infty$ and $-\infty$.

Let's now study the function f :

$$f(x) = -\frac{\alpha}{2a_0 - 1} + \frac{(2b_0\beta + \alpha(1 - 2a_0)) \sin(2b_0x) + (\beta(1 - 2a_0) - 2\alpha b_0) \cos(2b_0x)}{(2b_0)^2 + (1 - 2a_0)_0^2} \quad (266)$$

$$= \alpha_1 + \beta_1 \sin(2b_0x) + \gamma_1 \cos(2b_0x) \quad (267)$$

The function f is also nonzero function. Otherwise we will have $\alpha = 0$ and $\beta = 0$. The function f is a linear combination of the functions \sin and \cos . So the function f is differentiable and bounded. The function f is periodic of period $\frac{\pi}{b_0}$.

Let's calculate the function f values at the following points:

$$f(0) = \alpha_1 + \gamma_1 \quad (268)$$

$$f\left(\frac{\pi}{b_0}\right) = \alpha_1 - \gamma_1 \quad (269)$$

We have either $f(0) \neq 0$ or $f\left(\frac{\pi}{b_0}\right) \neq 0$, otherwise we will have $\alpha = 0$ and $\beta = 0$. Let's assume $f(0) \neq 0$. From the values above we have the following:

- If $(\alpha_1 + \gamma_1) > 0$ then $f(0) > 0$.
- If $(\alpha_1 + \gamma_1) < 0$ then $f(0) < 0$.

In case of $f(0) > 0$ we can prove that the limit of $\sum_{n=1}^{N-1} F_{n+1}$ goes to $-\infty$ and in the case of $f(0) < 0$ we can prove that the limit of $\sum_{n=1}^{N-1} F_{n+1}$ goes to $+\infty$. Therefore, in all case we have the series $\sum_{n \geq 1} F_{n+1}$ diverges. So let's assume from now on that $f(0) < 0$ as the same proof can be done in both cases.

From the lemma 3.3 in [1], there exist $N_1 \geq N_0$ such that $|\cos(2b_0 \ln(N_1 + 1)) - 1| \leq \epsilon$ and $|\sin(2b_0 \ln(N_1 + 1))| \leq \epsilon$ for any $1 > \epsilon > 0$.

Let's define $\beta_0 = -f(0)$. $\beta_0 > 0$.

Let's fix $\epsilon > 0$ to be very small such that $0 < \epsilon < \min\left(1, \frac{\beta_0}{10000}\right)$.

Let's define $\gamma_0 = |\beta_1| + |\gamma_1|$

Let's N_1 be such that $|\cos(2b_0 \ln(N_1 + 1)) - 1| \leq \epsilon/\gamma_0$ and also $|\sin(2b_0 \ln(N_1 + 1)) - 0| \leq \epsilon/\gamma_0$. Therefore:

$$\left| G(N_1 + 1) - \frac{f(0)}{(N_1 + 1)^{2a_0 - 1}} \right| \leq \left\{ \frac{\left((\beta_1) \sin(2b_0 \ln(N_1 + 1)) + (\gamma_1) \cos(2b_0 \ln(N_1 + 1)) \right)}{(N_1 + 1)^{2a_0 - 1}} \right. \quad (270)$$

$$\left. - \frac{\left((\beta_1) \sin(2b_0 0) + (\gamma_1) \cos(2b_0 0) \right)}{(N_1 + 1)^{2a_0 - 1}} \right\} \quad (271)$$

$$\leq \frac{\left(|\beta_1| |\sin(2b_0 \ln(N_1 + 1))| + |\gamma_1| |\cos(2b_0 \ln(N_1 + 1)) - 1| \right)}{(N_1 + 1)^{2a_0 - 1}} \quad (272)$$

$$\leq \frac{\left(|\beta_1| \epsilon / \gamma_0 + |\gamma_1| \epsilon / \gamma_0 \right)}{(N_1 + 1)^{2a_0 - 1}} \quad (273)$$

$$\leq \frac{\epsilon}{(N_1 + 1)^{2a_0 - 1}} \quad (274)$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} g(n) - \frac{\beta_0}{(N_0 + 1)^{2a_0 - 1}} \right| \leq \left\{ \left| \sum_{n=N_0+1}^{+\infty} g(n) + G(N_1 + 1) \right| \quad (275)$$

$$+ \left| -G(N_1 + 1) - \frac{-f(0)}{(N_1 + 1)^{2a_0 - 1}} \right| \right\} \leq \frac{\epsilon}{(N_1 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_1)^{2a_0}} \quad (276)$$

Hence

$$\left| \sum_{n=N_1+1}^{+\infty} g(n) - \frac{\beta_0}{(N_1 + 1)^{2a_0 - 1}} \right| \leq \frac{\epsilon}{(N_1 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_1)^{2a_0}} \quad (277)$$

Therefore

$$\sum_{n=N_1+1}^{+\infty} g(n) \geq \frac{\beta_0 - \epsilon}{(N_1 + 1)^{2a_0 - 1}} - \frac{K}{2a_0(N_1)^{2a_0}} \quad (278)$$

Therefore

$$R_{N_1} = \sum_{k=qN_1+1}^{+\infty} g(k) + F_{N_0} + \lim_{N \rightarrow +\infty} \left[\sum_{n=N_1}^N \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_1+1}^{Nq} \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \right] \quad (279)$$

$$\geq \frac{\beta_0 - \epsilon}{(N_0 + 1)^{2a_0 - 1}} - \frac{K}{2a_0(N_1)^{2a_0}} + F_{N_1} + \lim_{N \rightarrow +\infty} \left[\sum_{n=N_1}^N \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_1+1}^{Nq} \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \right] \quad (280)$$

We have the sequences (ξ_k) , (D_k) and (ε_k) are bounded. Plus we have $a_0 + 2 \geq 2a_0 + 1 > a_0 + 1 \geq 2a_0 > 1$, therefore the series $\sum_{k \geq 1} \frac{D_k}{k^{2a_0+1}}$, $\sum_{k \geq 1} \frac{\varepsilon_k}{k^{a_0+2}}$ are converging absolutely.

Let the positive constant M such that:
For each $k \geq N_1$ and $N \geq N_1$:

$$|c_k| \leq M \quad (281)$$

$$|D_k| \leq M \quad (282)$$

$$|\varepsilon_k| \leq M \quad (283)$$

Therefore, we have $a_0 > 0$:

$$\left| \sum_{n=N_1}^{+\infty} \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_1+1}^{+\infty} \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \right| \quad (284)$$

$$\leq \frac{M}{2a_0(qN_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0+1)(N_1)^{a_0+1}} \quad (285)$$

Therefore

$$R_{N_1} \geq \frac{\beta_0 - \epsilon}{(N_1 + 1)^{2a_0-1}} - \frac{K}{2a_0(N_1)^{2a_0}} - \frac{M}{2a_0(qN_1)^{2a_0}} - \frac{M}{2a_0(N_1)^{2a_0}} - \frac{M}{(a_0+1)(N_1)^{a_0+1}} \quad (286)$$

Let's define the sequence (δ_n) :

$$\delta_n = \frac{u_n \lambda_n (n+1)^{2a_0-1}}{n^{2a_0}} - \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} - \frac{M(n+1)^{2a_0-1}}{2a_0(qn)^{2a_0}} - \frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} - \frac{M(n+1)^{2a_0-1}}{(a_0+1)(n)^{a_0+1}} \quad (287)$$

As we have $\frac{1}{2} < a_0 < 1$, therefore $2a_0 - 1 < a_0 < 2a_0 < a_0 + 1$
Therefore

$$\lim_{n \rightarrow +\infty} -\frac{u_n \lambda_n (n+1)^{2a_0-1}}{n^{2a_0}} + \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{2a_0(qn)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(a_0+1)(n)^{a_0+1}} = 0 \quad (288)$$

Therefore the $\lim_{n \rightarrow +\infty} \delta_n = 0$.

So we can choose N_1 such that $|\delta_{N_1}| < \epsilon$, i.e $\delta_{N_1} \leq \epsilon$.

Therefore

$$R_{N_1} \geq \frac{1}{(N_1 + 1)^{2a_0-1}} (\beta_0 + 2\epsilon) > 0 \quad (289)$$

Therefore $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^N F_n = +\infty$.

Case of Non-Trivial Character

Remark. *In this particular case we have the function $G = 0$. Therefore*

$$F_{N_0} + \sum_{n=N_0+1}^{+\infty} g(k) = \mathcal{O}\left(\frac{1}{(N_0)^{2a_0}}\right) \quad (290)$$

Thus, we don't have the term of order $\frac{1}{(N_0)^{2a_0-1}}$. So the method we applied in the case above cannot work here. However, as we will see in the paragraph below, we will still be able to prove in this case that $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^N G_n = +\infty$.

□

□

Divergence of $\sum_{n \geq 1} G_n$

We are going to follow the same the steps as in the case of F_n . So We will skip the calculation steps except when it is necessary and different from the case of F_n above.

For large $N \geq N_0$ we have the following expression of G_{N+1} :

$$G_{N+1} = G_{N_0} + \sum_{n=N_0}^N \frac{\gamma\gamma_n^2 + \alpha_n\lambda\lambda_n}{n^{2a_0}} + \frac{(\lambda\lambda_n + 2\gamma\gamma_n)\beta_n}{n^{2a_0+1}} + \frac{\epsilon\epsilon_n}{n^{a_0+2}} \quad (291)$$

$$= G_{N_0} + \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\epsilon\epsilon_n}{n^{a_0+2}} \quad (292)$$

Where

$$C_n = \gamma\gamma_n^2 + \alpha_n\lambda\lambda_n \quad (293)$$

$$D_n = (\lambda\lambda_n + 2\gamma\gamma_n)\beta_n \quad (294)$$

We do another summation and we have the following expression:
So for each $N \geq N_0$:

$$\sum_{n=N_0}^N G_{n+1} = (N - N_0 + 1)G_{N_0} \quad (295)$$

$$+ (N + 1) \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\epsilon\epsilon_n}{n^{a_0+2}} \quad (296)$$

$$- \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\epsilon\epsilon_n}{n^{a_0+1}} \quad (297)$$

We factorise by $N + 1$ then we have this expression

$$\sum_{n=N_0}^N G_{n+1} = (N + 1) \left(\frac{(N - N_0 + 1)}{N + 1} G_{N_0} \right) \quad (298)$$

$$+ \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\epsilon\epsilon_n}{n^{a_0+2}} \quad (299)$$

$$- \frac{1}{N + 1} \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\epsilon\epsilon_n}{n^{a_0+1}} \quad (300)$$

We have the sequences (C_n) , (D_n) and $(\epsilon\epsilon_n)$ are bounded and $2a_0 > 1$, so the serie $\sum_{n \geq 1} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\epsilon\epsilon_n}{n^{a_0+2}}$ is converging absolutely. Let's prove now that the sequence in the equation (300) is converging to zero when N goes to infinity.

As we have $2a_0 > 1$, So

$$\lim_{n \rightarrow \infty} \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon \varepsilon_n}{n^{a_0+1}} = 0 \quad (301)$$

Therefore following the Césaro theorem we have:

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon \varepsilon_n}{n^{a_0+1}} = 0 \quad (302)$$

We denote $GG(N, N_0)$ as:

$$GG(N, N_0) = \left(\frac{(N - N_0 + 1)}{N + 1} G_{N_0} \right. \quad (303)$$

$$+ \sum_{n=N_0}^N \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon \varepsilon_n}{n^{a_0+2}} \quad (304)$$

$$\left. - \frac{1}{N+1} \sum_{n=N_0}^N \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon \varepsilon_n}{n^{a_0+1}} \right) \quad (305)$$

And

$$\sum_{n=N_0}^N G_{n+1} = (N+1)GG(N, N_0) \quad (306)$$

The serie $\sum_{n \geq 1} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon \varepsilon_n}{n^{a_0+2}}$ is converging absolutely. Let's denote

$$R_{N_0} = \lim_{N \rightarrow +\infty} \sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon \varepsilon_n}{n^{a_0+2}} \quad (307)$$

Lemma 3.5. *We can choose N_0 such that $R_{N_0} + G_{N_0} > 0$.*

Proof. Concerning the term $\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}}$.

We have from above lemmas that:

$$\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} = \sum_{n=qN_0+1}^{Nq} \frac{\gamma \gamma_n^2 + \alpha \alpha_n \lambda \lambda_n}{n^{2a_0}} \quad (308)$$

$$= \sum_{n=N_0}^N \sum_{i=1}^q \frac{\gamma \gamma_{qn+i}^2 + \alpha \alpha_n \lambda \lambda_{qn+i}}{(qn+i)^{2a_0}} \quad (309)$$

We have for each $1 \leq i \leq q$:

$$\frac{\gamma \gamma_{qn+i}^2 + \alpha \alpha_n \lambda \lambda_{qn+i}}{(qn+i)^{2a_0}} = \frac{\gamma \gamma_{qn+i}^2 + \alpha \alpha_n \lambda \lambda_{qn+i}}{(qn)^{2a_0}} + O\left(\frac{1}{(qn)^{2a_0+1}}\right) \quad (310)$$

Therefore

$$\sum_{i=1}^q \frac{\gamma\gamma_{qn+i}^2 + \alpha_n\lambda\lambda_{qn+i}}{(qn+i)^{2a_0}} = \sum_{i=1}^q \frac{\gamma\gamma_{qn+i}^2 + \alpha\alpha_n\lambda\lambda_{qn+i}}{(qn)^{2a_0}} + O\left(\frac{1}{(qn)^{2a_0+1}}\right) \quad (311)$$

$$= \frac{1}{(qn)^{2a_0}} \left\{ \sum_{i=1}^q \frac{y_i - x_i + 2x_i^2}{2} \right. \quad (312)$$

$$+ \cos(2b_0 \ln(qn)) \sum_{i=1}^q \frac{y_i + x_i - 2x_i^2}{2} \quad (313)$$

$$\left. + \sin(2b_0 \ln(qn)) \sum_{i=2}^q \frac{y_i - x_i - 2x_i y_i}{2} \right\} + O\left(\frac{1}{(qn)^{2a_0+1}}\right) \quad (314)$$

$$= \frac{\alpha\alpha + \beta\beta \cos(2b_0 \ln(qn)) + \gamma\gamma \sin(2b_0 \ln(qn))}{(qn)^{2a_0}} + O\left(\frac{1}{(qn)^{2a_0+1}}\right) \quad (315)$$

Where

$$\alpha\alpha = \sum_{i=1}^q \frac{y_i - x_i + 2x_i^2}{2} \quad (316)$$

$$\beta\beta = \sum_{i=1}^q \frac{y_i + x_i - 2x_i^2}{2} \quad (317)$$

$$\gamma\gamma = \sum_{i=2}^q \frac{y_i - x_i - 2x_i y_i}{2} \quad (318)$$

From the cancelation property of the Dirichlet characters, we can calculate the values of $\alpha\alpha$, $\beta\beta$ and $\gamma\gamma$.

$$\sum_{i=1}^q x_i = \Re\left(\sum_{i=1}^q \chi(i)\right) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (319)$$

And

$$\sum_{i=1}^q y_i = \Im\left(\sum_{i=1}^q \chi(i)\right) = \begin{cases} 0, & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (320)$$

Therefore

$$\alpha\alpha = \begin{cases} \frac{\phi(q)}{2}, & \text{if } \chi = \chi_0 \text{ the trivial character} \\ \sum_{i=1}^q x_i^2, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (321)$$

$$\beta\beta = \begin{cases} -\frac{\phi(q)}{2}, & \text{if } \chi = \chi_0 \text{ the trivial character} \\ -\sum_{i=1}^q x_i^2, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (322)$$

$$\gamma\gamma = \begin{cases} -\frac{\phi(q)}{2}, & \text{if } \chi = \chi_0 \text{ the trivial character} \\ -\sum_{i=2}^q x_i y_i, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \quad (323)$$

Therefore we can write the following for a large n :

$$\sum_{i=1}^q \frac{\gamma\gamma_{qn+i}^2 + \alpha\alpha_n\lambda\lambda_{qn+i}}{(qn+i)^{2a_0}} = \frac{\alpha\alpha + \beta\beta \cos(2b_0 \ln(qn)) + \gamma\gamma \sin(2b_0 \ln(qn))}{(qn)^{2a_0}} + \frac{\xi_n}{(qn)^{2a_0+1}} \quad (324)$$

Where the sequence ξ_n is bounded.

Therefore

$$\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} = \sum_{n=N_0}^N \sum_{i=1}^q \frac{\gamma\gamma_{qn+i}^2 + \alpha\alpha_n\lambda\lambda_{qn+i}}{(qn+i)^{2a_0}} \quad (325)$$

$$= \sum_{n=N_0}^N \frac{\alpha\alpha + \beta\beta \cos(2b_0 \ln(qn)) + \gamma\gamma \sin(2b_0 \ln(qn))}{(qn)^{2a_0}} + \sum_{n=N_0}^N \frac{\xi_n}{(qn)^{2a_0+1}} \quad (326)$$

And

$$\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} = \sum_{n=N_0}^N \frac{\alpha\alpha + \beta\beta \cos(2b_0 \ln(qn)) + \gamma\gamma \sin(2b_0 \ln(qn))}{(qn)^{2a_0}} \quad (327)$$

$$+ \sum_{n=N_0}^N \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_0+1}^{Nq} \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (328)$$

The serie $\sum_{n \geq 1} \frac{\alpha\alpha + \beta\beta \cos(2b_0 \ln(qn)) + \gamma\gamma \sin(2b_0 \ln(qn))}{(qn)^{2a_0}}$ was studied in the lemma 3.1 in [1]. Let's denote the function g as the following:

$$g(x) = \frac{\alpha\alpha + \beta\beta \cos(2b_0 \ln(qx)) + \gamma\gamma \sin(2b_0 \ln(qx))}{(qx)^{2a_0}} \quad (329)$$

We have the following:

$$\left| \sum_{n=N_0+1}^{+\infty} g(n) - \int_{N_0+1}^{+\infty} g(x) \right| \leq \frac{K}{(N_0)^{2a_0}} \quad (330)$$

And the primitive function G of the function g (see lemma 3.1 in [1]):

$$G(x) = \frac{1}{2q} \left\{ -\frac{\alpha\alpha}{(2a_0-1)(x)^{2a_0-1}} \right. \quad (331)$$

$$\left. + \frac{(2b_0\beta\beta + \gamma\gamma(1-2a_0)) \sin(2b_0 \ln(x)) + (\beta\beta(1-2a_0) - 2\gamma\gamma b_0) \cos(2b_0 \ln(x))}{\left((2b_0)^2 + (1-2a_0)_0^2 \right) (x)^{2a_0-1}} \right\} \quad (332)$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} g(n) + G(N_0+1) \right| \leq \frac{K}{(N_0)^{2a_0}} \quad (333)$$

Case of Non-Trivial Character In this case we have:

$$\alpha\alpha = \sum_{i=1}^q x_i^2 \quad (334)$$

$$\beta\beta = -\sum_{i=1}^q x_i^2 \quad (335)$$

$$\gamma\gamma = -\sum_{i=2}^q x_i y_i \quad (336)$$

As an example, the case where $q = 3$ and $\chi(1) = 1$, $\chi(2) = -1$ and $\chi(3) = 0$. we have $\alpha\alpha = 2$, $\beta\beta = -2$ and $\gamma\gamma = 0$. $L(s, \chi) = \sum_{n=0}^{+\infty} \left(\frac{1}{(3n+1)^s} - \frac{1}{(3n+2)^s} \right)$. Also, the Dirichlet η function falls in this case: $q = 2$ and $\chi(1) = 1$, $\chi(2) = -1$. we have $\alpha\alpha = 2$, $\beta\beta = -2$ and $\gamma\gamma = 0$. $\eta(s) = \sum_{n=0}^{+\infty} \left(\frac{1}{(2n+1)^s} - \frac{1}{(2n+2)^s} \right)$.

Back to the general case, we have $x_1 = 1$ as we are in the case of a Dirichlet L -function where the character $\chi(1) = 1$. Therefore $\alpha\alpha > 1$ and $\beta\beta = -\alpha\alpha$. Therefore $G \neq 0$.

Concerning the term G_{N_0} . We have

$$G_{N_0} = \frac{v_{N_0}}{(N_0)^{a_0}} B_{N_0} = \frac{v_{N_0} \lambda \lambda_{N_0}}{(N_0)^{2a_0}} \quad (337)$$

Where the sequence $(v_n \lambda_n)$ is bounded.

Therefore we have the asymptotic expansion of $G_{N_0} + \sum_{n=N_0+1}^{+\infty} g(k)$ as follows:

$$G_{N_0} + \sum_{n=N_0+1}^{+\infty} g(k) = -G(N_0 + 1) + \mathcal{O}\left(\frac{1}{(N_0)^{2a_0}}\right) \quad (338)$$

Let's now study the function f :

$$f(x) = -\frac{\alpha\alpha}{2a_0 - 1} + \frac{(2b_0\beta\beta + \gamma\gamma(1 - 2a_0)) \sin(2b_0x) + (\beta\beta(1 - 2a_0) - 2\gamma\gamma b_0) \cos(2b_0x)}{(2b_0)^2 + (1 - 2a_0)^2} \quad (339)$$

$$= \alpha_2 + \beta_2 \sin(2b_0x) + \gamma_2 \cos(2b_0x) \quad (340)$$

The function f is also nonzero function. Otherwise we will have $\alpha\alpha = 0$ which is in contradiction with the fact that $\alpha\alpha > 1$.

The function f is a linear combination of the functions sin and cos. So the function f is differentiable and bounded. The function f is periodic of period $\frac{\pi}{b_0}$.

Let's calculate the function f values at the following points:

$$f(0) = \alpha_2 + \gamma_2 \quad (341)$$

$$f\left(\frac{\pi}{b_0}\right) = \alpha_2 - \gamma_2 \quad (342)$$

We have either $f(0) \neq 0$ or $f\left(\frac{\pi}{b_0}\right) \neq 0$, otherwise we will have $\alpha\alpha = 0$ which is again in contradiction with the fact that $\alpha\alpha > 1$. Let's assume $f(0) \neq 0$. From the values above we have the following:

- If $(\alpha_2 + \gamma_2) > 0$ then $f(0) > 0$.
- If $(\alpha_2 + \gamma_2) < 0$ then $f(0) < 0$.

In case of $f(0) > 0$ we can prove that the limit of $\sum_{n=1}^{N-1} G_{n+1}$ goes to $-\infty$ and in the case of $f(0) < 0$ we can prove that the limit of $\sum_{n=1}^{N-1} G_{n+1}$ goes to $+\infty$. Therefore, in all case we have the series $\sum_{n \geq 1} G_{n+1}$ diverges. So let's assume from now on that $f(0) < 0$ as the same proof can be done in both cases.

From the lemma 3.3 in [1], there exist $N_1 \geq N_0$ such that $|\cos(2b_0 \ln(N_1 + 1)) - 1| \leq \epsilon$ and $|\sin(2b_0 \ln(N_1 + 1))| \leq \epsilon$ for any $1 > \epsilon > 0$.

Let's define $\beta_0 = -f(0)$. $\beta_0 > 0$.

Let's fix $\epsilon > 0$ to be very small such that $0 < \epsilon < \min\left(1, \frac{\beta_0}{10000}\right)$.

Let's define $\gamma_0 = |\beta_2| + |\gamma_2|$. We have the same proof steps as in the case of F_n .

Let's N_1 be such that $|\cos(2b_0 \ln(N_1 + 1)) - 1| \leq \epsilon/\gamma_0$ and also $|\sin(2b_0 \ln(N_1 + 1)) - 0| \leq \epsilon/\gamma_0$. Therefore:

$$\left| G(N_1 + 1) - \frac{f(0)}{(N_1 + 1)^{2a_0 - 1}} \right| \leq \left| \left\{ \frac{\left((\beta_2) \sin(2b_0 \ln(N_1 + 1)) + (\gamma_2) \cos(2b_0 \ln(N_1 + 1)) \right)}{(N_1 + 1)^{2a_0 - 1}} \right. \right. \quad (343)$$

$$\left. \left. - \frac{\left((\beta_2) \sin(2b_0 0) + (\gamma_2) \cos(2b_0 0) \right)}{(N_1 + 1)^{2a_0 - 1}} \right\} \right| \quad (344)$$

$$\leq \frac{\left(|\beta_2| |\sin(2b_0 \ln(N_1 + 1))| + |\gamma_2| |\cos(2b_0 \ln(N_1 + 1)) - 1| \right)}{(N_1 + 1)^{2a_0 - 1}} \quad (345)$$

$$\leq \frac{\left(|\beta_2| \epsilon/\gamma_0 + |\gamma_2| \epsilon/\gamma_0 \right)}{(N_1 + 1)^{2a_0 - 1}} \quad (346)$$

$$\leq \frac{\epsilon}{(N_1 + 1)^{2a_0 - 1}} \quad (347)$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} g(n) - \frac{\beta_0}{(N_0 + 1)^{2a_0 - 1}} \right| \leq \left| \sum_{n=N_0+1}^{+\infty} g(n) + G(N_1 + 1) \right| \quad (348)$$

$$+ \left| -G(N_1 + 1) - \frac{-f(0)}{(N_1 + 1)^{2a_0 - 1}} \right| \leq \frac{\epsilon}{(N_1 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_1)^{2a_0}} \quad (349)$$

Hence

$$\left| \sum_{n=N_1+1}^{+\infty} g(n) - \frac{\beta_0}{(N_1 + 1)^{2a_0 - 1}} \right| \leq \frac{\epsilon}{(N_1 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_1)^{2a_0}} \quad (350)$$

Therefore

$$\sum_{n=N_1+1}^{+\infty} g(n) \geq \frac{\beta_0 - \epsilon}{(N_1 + 1)^{2a_0-1}} - \frac{K}{2a_0(N_1)^{2a_0}} \quad (351)$$

Therefore

$$R_{N_1} = \sum_{k=qN_1+1}^{+\infty} g(k) + F_{N_0} + \lim_{N \rightarrow +\infty} \left[\sum_{n=N_1}^N \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_1+1}^{Nq} \frac{D_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \right] \quad (352)$$

$$\geq \frac{\beta_0 - \epsilon}{(N_0 + 1)^{2a_0-1}} - \frac{K}{2a_0(N_1)^{2a_0}} + F_{N_1} + \lim_{N \rightarrow +\infty} \left[\sum_{n=N_1}^N \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_1+1}^{Nq} \frac{D_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \right] \quad (353)$$

We have the sequences (ξ_k) , (D_k) and (ϵ_k) are bounded. Plus we have $a_0 + 2 \geq 2a_0 + 1 > a_0 + 1 \geq 2a_0 > 1$, therefore the series $\sum_{k \geq 1} \frac{D_k}{k^{2a_0+1}}$, $\sum_{k \geq 1} \frac{\epsilon_k}{k^{a_0+2}}$ are converging absolutely.

Let the positive constant M such that:

For each $k \geq N_1$ and $N \geq N_1$:

$$|c_k| \leq M \quad (354)$$

$$|D_k| \leq M \quad (355)$$

$$|\epsilon_k| \leq M \quad (356)$$

Therefore, we have $a_0 > 0$:

$$\left| \sum_{n=N_1}^{+\infty} \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_1+1}^{+\infty} \frac{D_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \right| \quad (357)$$

$$\leq \frac{M}{2a_0(qN_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0 + 1)(N_1)^{a_0+1}} \quad (358)$$

Therefore

$$R_{N_1} \geq \frac{\beta_0 - \epsilon}{(N_1 + 1)^{2a_0-1}} - \frac{K}{2a_0(N_1)^{2a_0}} - \frac{M}{2a_0(qN_1)^{2a_0}} - \frac{M}{2a_0(N_1)^{2a_0}} - \frac{M}{(a_0 + 1)(N_1)^{a_0+1}} \quad (359)$$

Let's define the sequence (δ_n) :

$$\delta_n = \frac{v_n \lambda_n (n+1)^{2a_0-1}}{n^{2a_0}} - \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} - \frac{M(n+1)^{2a_0-1}}{2a_0(qn)^{2a_0}} - \frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} - \frac{M(n+1)^{2a_0-1}}{(a_0 + 1)(n)^{a_0+1}} \quad (360)$$

As we have $\frac{1}{2} < a_0 < 1$, therefore $2a_0 - 1 < a_0 < 2a_0 < a_0 + 1$

Therefore

$$\lim_{n \rightarrow +\infty} -\frac{v_n \lambda_n (n+1)^{2a_0-1}}{n^{2a_0}} + \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{2a_0(qn)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(a_0 + 1)(n)^{a_0+1}} = 0 \quad (361)$$

Therefore the $\lim_{n \rightarrow +\infty} \delta_n = 0$.

So we can choose N_1 such that $|\delta_{N_1}| < \epsilon$, i.e $\delta_{N_1} \leq \epsilon$.

Therefore

$$R_{N_1} \geq \frac{1}{(N_1 + 1)^{2a_0-1}} (\beta_0 + 2\epsilon) > 0 \quad (362)$$

Therefore $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^N G_n = +\infty$.

Case of Trivial Character In this case we have:

$$\alpha\alpha = \frac{\phi(q)}{2} \quad (363)$$

$$\beta\beta = -\frac{\phi(q)}{2} \quad (364)$$

$$\gamma\gamma = -\frac{\phi(q)}{2} \quad (365)$$

We note that we are in a similar situation like the case of Trivial Character of the series $\sum_{n \geq 1} F_n$. Therefore we apply the same method to prove that $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^N G_n = +\infty$. □

Remark. Note that for the purpose of proving GRH: we need only the divergence of either the series $\sum_{n \geq 1} F_{n+1}$ or $\sum_{n \geq 1} G_{n+1}$ and not the divergence of the both of them.

Conclusion. We saw above that the serie $\sum_{n \geq 1} (U_n^2 + V_n^2)$ is converging absolutely. We have from the lemmas 3.2 that either:

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N F_n = +\infty \text{ OR } \lim_{N \rightarrow \infty} \sum_{n=1}^N G_n = +\infty \quad (366)$$

Therefore

$$\lim_{N \rightarrow \infty} A_N^2 = +\infty \text{ OR } \lim_{N \rightarrow \infty} B_N^2 = +\infty \quad (367)$$

thanks to $2a_0 > 1$ that makes the series $\sum_{n \geq 1} U_n^2$ and $\sum_{n \geq 1} V_n^2$ converging absolutely. This result is in contradiction with the fact that s is a $L(s, \chi)$ zero therefore the limit $\lim_{N \rightarrow \infty} A_N = 0$ and the limit $\lim_{N \rightarrow \infty} B_N = 0$. Therefore s with $\frac{1}{2} < a_0 = \Re(s) < 1$ cannot be a zero for $L(s, \chi)$.

3.2 Conclusion

We saw that if s is a $L(s, \chi)$ zero, then real part $\Re(s)$ can only be $\frac{1}{2}$ as all other possibilities can be discarded using the functional equation like in [1]. Therefore the Generalized Riemann hypothesis is true: *The non-trivial zeros of $L(s, \chi)$ have real part equal to $\frac{1}{2}$.*

References

- [1] Charaf Ech-Chatbi. *A Proof of the Riemann's Hypothesis*
<https://arxiv.org/abs/1905.09385>
- [2] Léo Agélas *Generalized Riemann Hypothesis*
<https://hal.archives-ouvertes.fr/hal-00747680v2/document>
- [3] Thai Pham. *Dirichlet's Theorem on Arithmetic Progressions*
https://web.stanford.edu/~thaipham/papers/MIT_18.104_Review_Paper.pdf

- [4] Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*
http://wayback.cecm.sfu.ca/pborwein/TEMP_PROTECTED/book.pdf
- [5] G. H. Hardy. *Sur les zeros de la fonction $\zeta(s)$ de Riemann*, *C. R. Acad. Sci. Paris* 158 (1914), 1012-1014.
- [6] G. H. Hardy. *The general theory of dirichlet series*.
<https://archive.org/details/generaltheoryofd029816mbp/page/n9>
- [7] Garrett, Paul. *Primes in arithmetic progressions*, 2011.
http://www.math.umn.edu/garrett/m/mfms/notes_c/dirichlet.pdf
- [8] A. Wintner. *The fundamental lemma in Dirichlet's theory of the arithmetical progressions*. *Amer. J. Math.*, 68:285-292, 1946.
- [9] E. C. Titchmarsh. *The Theory of the Riemann Zeta-function*, 2nd edition (revised by D. R. Heath-Brown), Oxford University Press, Oxford, 1986.
- [10] G. Bhowmik, D. Essouabri and B. Lichtin. *Meromorphic continuation of multivariable Euler products*, *Forum Math.* p. 1-29, DOI 10.1515, 2006.
- [11] E. Bombieri and D. A. Hejhal. *On the distribution of zeros of linear combinations of Euler products*. *Duke Math. J.*, 80(3):821-862, 1995.
- [12] H. Cohen. *Number theory. Vol. II. Analytic and modern tools, volume 240 of Graduate Texts in Mathematics*. Springer, New York, 2007.
- [13] H. Davenport. *Multiplicative Number Theory, 2nd edition, revised by H. Montgomery, Graduate Texts in Mathematics, Vol. 74, Springer-Verlag, New York, 1980*.
- [14] H. L. Montgomery and R. C. Vaughan *Multiplicative Number Theory I. Classical Theory*, Cambridge University Press 2006.
- [15] Bernhard Riemann. *On the Number of Prime Numbers less than a Given Quantity*
<https://www.claymath.org/sites/default/files/ezeta.pdf>
- [16] Enrico Bombieri. *Problems of the Millennium: the Riemann Hypothesis*
https://www.claymath.org/sites/default/files/official_problem_description.pdf
- [17] Peter Sarnak. *Problems of the Millennium: The Riemann Hypothesis (2004)*
https://www.claymath.org/sites/default/files/sarnak_rh.0.pdf