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Nonresonant bilinear forms for partially dissipative hyperbolic systems violating the Shizuta-Kawashima condition

Roberta Bianchini ^{*}, Roberto Natalini [†]

Abstract

We consider a simple example of a partially dissipative hyperbolic system violating the Shizuta-Kawashima condition, i.e. such that some eigendirections do not exhibit dissipation at all. In the space-time resonances framework introduced by Germain, Masmoudi and Shatah, we prove that, when the source term has a Nonresonant Bilinear Form, as proposed by Pusateri and Shatah CPAM 2013, the formation of singularities is prevented, despite the lack of dissipation. This allows us to show that smooth solutions to this preliminary case-study model exist globally in time.

Keywords: Partially dissipative hyperbolic systems, Shizuta-Kawashima condition, space-time resonances, dispersion.

1 Introduction

Consider the Cauchy problem for hyperbolic balance laws in several space dimensions

$$\partial_t u + \sum_{i=1}^n \partial_{x_i} F_i(u) = G(u), \quad (1.1)$$

where $u \in \mathbb{R}^k$ and $F = (F_1, \dots, F_n)$ and $G(u)$ are smooth functions and the system is supplemented by the initial condition at $t = 0$

$$u(x, 0) = u_0(x). \quad (1.2)$$

Several mathematical models enter this framework, see for instance [35, 25]. Proving global in time existence of smooth solutions to system (1.1)–(1.2) is

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a difficult problem and, as a matter of fact, in many cases solutions fail to be smooth, because some of their components or their gradients go to infinity pointwise in finite time, see for instance [25, 19]. However, it is possible to characterize some classes of systems such that solutions stay smooth for all times, at least for small initial data, the simplest case being systems with dissipation terms, as for instance the compressible Euler system with damping [27, 31]. In [16, 36, 30] a general framework was proposed to guarantee the global existence in time of smooth solutions. Actually, for systems endowed with a strictly convex entropy function $\eta = \eta(u)$, a preliminary simple assumption is the entropy dissipation, see [8, 16] and references therein. This condition can be stated as follows: given an equilibrium value \bar{u} for system (1.1), i.e. such that $G(\bar{u}) = 0$, for every u in a given domain Ω of \mathbb{R}^k we have:

$$(\eta'(u) - \eta'(\bar{u})) \cdot G(u) \leq 0. \quad (1.3)$$

Unfortunately, even in very simple cases, it is possible to see that condition (1.3) alone is not enough to prevent the appearance of singularities in finite time. An additional assumption, originally introduced in [30], is the so-called Shizuta-Kawashima condition, which can be presented as follows:

[SK] *Shizuta-Kawashima*. Given an equilibrium value \bar{u} for system (1.1), if $z \in \text{Ker}G'(\bar{u})$, then there is no vector $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$ such that z is an eigenvector of the matrix $\sum_{i=1}^n F'_i(\bar{u})\xi_i$.

This condition can be read in many different ways. In terms of stability, it guarantees the necessary coupling between conserved/non conserved quantities in order to have dissipation in each state variable.

In [16, 36] it is proved that, under both entropy dissipation and the Shizuta-Kawashima conditions, and for small initial data, problem (1.1)–(1.2) has smooth solutions for all times. However, there are many physical systems, especially in the multidimensional case, which have dissipative entropies, but do not satisfy [SK]. An interesting example is the model for gas dynamics in rotational and vibrational non-equilibrium in Eulerian coordinates in n -space dimensions, which reads

$$\begin{cases} \partial_t \rho + \text{div}(\rho u) = 0, \\ \partial_t(\rho u) + \text{div}(\rho u \otimes u) + \nabla P(\rho, e) = 0, \\ \partial_t(\rho(\frac{1}{2}|u|^2 + e + q)) + \text{div}((\frac{1}{2}|u|^2 + e + q + P(\rho, e))u) = 0, \\ \partial_t(\rho q) + \text{div}(\rho q u) = \frac{1}{\tau}(\rho(Q(e) - q)), \end{cases} \quad (1.4)$$

where ρ , u , e , and q are respectively the density, the velocity, the internal energy and the internal vibrational energy of the gas, and τ is the relaxation time for the rotational and vibrational modes, see for instance [35]. The meaning of this supplementary equation is that, during rapid changes in the flow, the internal energy e may lag behind the equilibrium value corresponding to the ambient pressure and density. The translational energy reaches the equilibrium faster

than the rotational and vibrational energies. It is easy to show that this system does not satisfy the Shizuta-Kawashima condition in all space dimensions (see for instance [16] for the 1D case). Nevertheless, in the one dimensional case, Y. Zeng in [37] was able to obtain a global existence result using Lagrangian coordinates, while the multidimensional case is still open.

As an additional example of dissipative hyperbolic systems violating [SK], consider the nonlinear Maxwell equations

$$\begin{cases} \partial_t D - \operatorname{rot} H = 0, \\ \partial_t B + \operatorname{rot} E = 0, \\ \operatorname{div} D = \operatorname{div} B = 0, \end{cases} \quad (1.5)$$

where E and H are the electric and magnetic field respectively, and D and B are the related displacements given by

$$B = \nu_0 H, \quad D = \varepsilon_0(1 + \chi)E,$$

where $\varepsilon_0 \chi E$ stands for a polarization term. In the so-called Kerr-Debye model, the term χ is given by a relaxation equation:

$$\partial_t \chi = \frac{1}{\tau} (\varepsilon_r |E|^2 - \chi). \quad (1.6)$$

For more details, see for instance [38]. It is shown in [16], see also [6, 7] and references therein, that when the electric field vanishes, the [SK] condition is no longer satisfied. Nevertheless, in [7] global existence of smooth solutions is proved for all small initial data.

Other examples are given by the hierarchy of equations coming from the kinetic formulation of multi-branch entropy solutions of scalar conservation laws [4, 5], and the very simple example on traffic flow in [21].

Some attempts to go beyond the Shizuta-Kawashima condition are contained in [26], which is restricted to a simple class of one dimensional systems, and in [1], where some requirements weaker than [SK] are assumed on the structure of the linearized system.

In this paper, we try to make a step forward in this direction by taking into account, for the terms which do not satisfy the [SK] condition, some special features of the nonlinearity. Notice that in systems (1.4) and (1.5), and in the class of systems considered in [26], the nonlinear term is linearly degenerate where [SK] is not satisfied: more precisely, the system is linearly degenerate only in the points where the [SK] condition does not hold. This suggests that, beyond the linear structure imposed by the [SK] condition, also the specific type of the nonlinearity plays a role in the development of singularities of the solutions.

In this regard, it is worth considering the case of non dissipative nonlinear systems. The investigation of global existence of smooth solutions to hyperbolic, and more generally dispersive, equations with quadratic source terms and small initial data takes its roots around the Eighties. A deep historical and mathematical survey on the topic can be found in [20]. A general feature is that the

linear dispersive terms of the equation tend to force the solution to spread and to decay, and the contribution of the nonlinear terms is controlled by dispersion. Since dispersion increases with space dimension, a first class of global existence results has been obtained in dimension $n = 4$ by S. Klainerman [23]. As showed by F. John, [18], in lower space dimensions the nonlinearity can lead to blow up in finite time for arbitrarily small data. In this case, a precise structure of the nonlinearity, the so-called *null condition*, introduced by Klainerman [24] and Christodoulou [9], prevents the formation of singularities, at least in the case of systems of nonlinear wave equations and some related examples.

More recently, a crucial contribution to extend the notion of null forms, in the framework of space-time resonances analysis introduced by Germain, Masmoudi and Shatah, was given by Pusateri and Shatah, see [28]. The main idea of space-time resonances analysis is the following: instead of focusing only on time resonances (in other words, instead of dealing with plane waves), one looks also at spatial localization (wave packets) by using space-weighted estimates. A highlighting survey paper is due to Germain, [14]. Normal form methods deal with time resonances, while null forms guarantee spatial localization, so preventing space resonances. Each one tells only a part of the story. A coupling of these two points of view is what is needed to complete the picture.

In the context of long-time behavior for the wave equation with space-time resonances techniques, we refer to a recent paper by Pusateri and Shatah, [28]. Notice that, quadratic source terms satisfying the null condition for the wave equation are actually equivalent to the compatible forms for hyperbolic systems, [15], which are the ones having the weakly sequential continuity described by compensated compactness, [33]. A preliminary result for hyperbolic systems is proved in [11]. Actually, the first result in this direction is due to an unpublished paper by Tartar [34], where he proved global existence of smooth solutions in 1D to a semilinear hyperbolic system with quadratic source satisfying a non-crossing condition for the characteristics (which is actually equivalent to the null condition for this particular case).

With the aim of highlighting the leading idea of this paper, notice that, as already pointed out in [19] (see also [32, 10, 7]), linearly degenerate eigenvalues are strongly connected with the null condition. A clear indication of that is provided for instance by systems (1.4) and (1.5), at the beginning of this introduction. Thus, we think that a more careful study of the interplay between dissipation and linearly degenerate fields (and so the null condition) could give at least some deeper insights on the long time behavior of multidimensional hyperbolic systems.

Beyond these conjectural considerations, the main purpose of this paper is more limited. As a first attempt to take advantage of the structure of the nonlinearity to overcome the Shizuta-Kawashima condition, we consider a very simple model coupling a basic wave equation and the simplest hyperbolic system with dissipation. The wave equation is endowed with a *Nonresonant Bilinear Form* in the wave variable, as in [28], plus any quadratic source depending on the dissipative hyperbolic variables, while the hyperbolic part can handle any kind of quadratic source term. We prove in the following that, under some

restrictions on the form of the nonlinear coupling, the smooth solutions to these systems are well behaved globally in time.

Somehow this means that, at least in the classes of systems considered here, a hyperbolic system with weak dissipation (less than [SK]) can handle a generic quadratic source for long times, *provided that it is nonresonant (see [28]) in terms of the non dissipative variables.*

This shows the potential appeal of our approach in dealing with more concrete and complicated cases, as the examples discussed before. The reason why the coupling between weak dissipation for hyperbolic systems and null forms works is more evident in the high frequency regime and roughly speaking is due to the fact that dissipation prevents time resonances by adding a contribution to the phase. The low frequency regime requires a more careful analysis in quite complicated functional spaces, and this is the main reason for the moment to restrict ourselves to these simple cases.

We restrict our analysis to the three dimensional case, but analogous results (with different decay rates) hold in greater space dimensions.

Plan of the paper The paper is organized as follows. In Section 2 we study a partially dissipative hyperbolic system (of Shizuta-Kawashima type) coupled with the wave equation with nonresonant bilinear source, where the coupling lies in the quadratic source term of the equations for the dissipative variables. Notice that, when restricted only to the part which satisfies the Shizuta-Kawashima condition, we can show that smooth solutions are global for small data with all types of quadratic source terms, just using in a clever way the decay properties of the Green function. Even this result improve on all previous results in [16, 3]. Section 3 concerns the more interesting case where the coupling between the Shizuta-Kawashima system and the wave equation takes place in the quadratic source of the wave equation itself, showing the power of (partial) dissipation in killing resonances and so preventing singularities.

2 A first model violating the Shizuta-Kawashima condition

We start with the simplest example of partially dissipative hyperbolic system violating the Shizuta-Kawashima condition, hereafter denoted by [SK]. In Fourier variables, the linear part of this system is as follows:

$$\begin{cases} \partial_t \hat{u} + i|\xi| \hat{v} = 0, \\ \partial_t \hat{v} + i|\xi| \hat{u} + \hat{v} = 0, \\ \partial_t \hat{w} + i|\xi| \hat{w} = 0, \end{cases} \quad (2.1)$$

where $(t, \xi) = (t, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^+ \times \mathbb{R}^3$. As we will see in Section 2.1, the variables u, v satisfy [SK], then will be identified as the *dissipative variables*, while w is decoupled and does not dissipate at all. We consider system (2.1)

with a quadratic source term, coupling all the three variables:

$$\begin{cases} \partial_t \hat{u} + i|\xi| \hat{v} = a_u \widehat{u^2} + b_u \widehat{v^2} + c_u \widehat{uv}, \\ \partial_t \hat{v} + i|\xi| \hat{u} + \hat{v} = a_v \widehat{u^2} + b_v \widehat{v^2} + c_v \widehat{uv} + d_v \widehat{uv}, \\ \partial_t \hat{w} + i|\xi| \hat{w} = \widehat{T}_{m(\xi, \eta)}(\hat{w}(\xi - \eta), \hat{w}(\eta)), \end{cases} \quad (2.2)$$

where $a_u, b_u, c_u, a_v, b_v, c_v, d_v$ are constant values,

$$T_{m(\xi, \eta)}(f, g) := \mathcal{F}^{-1} \int m(\xi, \eta) \widehat{f}(\eta) \widehat{g}(\eta - \eta) d\eta \quad (2.3)$$

is the Bilinear Pseudoproduct Operator and $m(\xi, \eta)$ is a Nonresonant Bilinear Form, defined as follows. Considering the profile $\widehat{f}_w(t, \xi) = \exp(i|\xi|t) \hat{w}(t, \xi)$ in Fourier space, we write the Duhamel formula for \hat{w} in (2.2) in terms of $\widehat{f}_w(t, \xi)$:

$$\partial_t \widehat{f}_w(t, \xi) = \int m(\xi, \eta) \exp(i\phi_w(\xi, \eta)t) \widehat{f}_w(\xi - \eta, t) \widehat{f}_w(\eta, t) d\eta, \quad (2.4)$$

where the phase

$$\phi_w(\xi, \eta) = |\xi| - |\xi - \eta| - |\eta|.$$

Definition 2.1 (from [28], Definition 2.1). *A symbol $m(\xi, \eta)$ belongs to the class \mathcal{B}_s if:*

- *it is homogeneous of degree s ;*
- *it is smooth outside $\{\xi = 0\} \cup \{\xi - \eta = 0\} \cup \{\eta = 0\}$;*
- *for any labeling (ξ_1, ξ_2, ξ_3) of the Fourier variables $(\xi, \xi - \eta, \eta)$, the following holds:*

$$\text{for } |\xi_1| \ll |\xi_2|, |\xi_3| \sim 1, \quad m = \mathcal{A}(|\xi_1|, \frac{\xi_1}{|\xi_1|}, \xi_2),$$

for some smooth function \mathcal{A} .

Bilinear pseudoproduct operators T_m associated to symbols $m(\xi, \eta)$ belonging to \mathcal{B}_s are bounded in L^p , i.e.

$$T_m : L^p \times L^q \rightarrow L^r, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 1 < p, q, r < \infty.$$

More precise results will be discussed in the Appendix.

Definition 2.2 (Resonant sets). *The time and space resonant sets are defined respectively as*

$$\mathcal{T} = \{(\xi, \eta) : \phi_w(\xi, \eta) = 0\}, \quad \mathcal{S} = \{(\xi, \eta) : \nabla_\eta \phi_w(\xi, \eta) = 0\}.$$

The time-space resonant set is then

$$\mathcal{R} = \mathcal{T} \cap \mathcal{S}.$$

Intuitively, a bilinear form is *nonresonant* if the time-space resonant set is not “too large”.

Definition 2.3 (Nonresonant Bilinear Form). *Equation (2.4) is called nonresonant if*

$$m(\xi, \eta) = a(\xi, \eta)\phi_w(\xi, \eta) + b(\xi, \eta)\nabla_\eta\phi_w(\xi, \eta),$$

where $a(\xi, \eta) \in \mathcal{B}_1$ and $b(\xi, \eta) \in \mathcal{B}_0$. Two more purely technical requirements on $a(\xi, \eta), b(\xi, \eta)$ are presented in [[28], Definition 2.2].

The main point of Definition 2.3 is that the symbol $m(\xi, \eta)$, then the principal part of the pseudoproduct operator T_m , vanishes on the space-time resonant set \mathcal{R} . As pointed out in [28], it is worth recalling that when $a(\xi, \eta) = 0$, one recovers the Klainerman null condition [24].

The aim of this section is to provide a first example of global in time well-posedness for a partially dissipative hyperbolic system (2.2), whose partial dissipation is weaker than [SK]. We prove that a particular structure of the nonlinearity, i.e. the Nonresonant Bilinear Form, is able to prevent the formation of singularities despite the lack of dissipation for the variable w in (2.2).

2.1 Spectral analysis of the linearized system

Let us set $U = (u, v, w)$. The linear system (2.1) in compact form reads

$$\partial_t \hat{U} - E(i\xi)\hat{U} = 0,$$

where

$$E(i\xi) = \begin{pmatrix} 0 & -i|\xi| & 0 \\ -i|\xi| & -1 & 0 \\ 0 & 0 & -i|\xi| \end{pmatrix} = -i|\xi|A + B, \quad (2.5)$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.6)$$

We write here the eigenvalues associated to $E(i\xi)$,

$$\lambda_1(\xi) = -\frac{1}{2} + \frac{\sqrt{1-4|\xi|^2}}{2}, \quad \lambda_2(\xi) = -\frac{1}{2} - \frac{\sqrt{1-4|\xi|^2}}{2}, \quad \lambda_3 = -i|\xi|,$$

while the related eigenvectors are

$$V_1(\xi) = \begin{pmatrix} 1 \\ -i \left(\frac{1-\sqrt{1-4|\xi|^2}}{2|\xi|} \right) \\ 0 \end{pmatrix}, \quad V_2(\xi) = \begin{pmatrix} 1 \\ -i \left(\frac{1+\sqrt{1-4|\xi|^2}}{2|\xi|} \right) \\ 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We analyse the low frequency regime. Then, fix $0 < a \ll 1$ and consider $|\xi| \leq a$, for a small enough. The eigenvalues expansion yields the following expressions:

$$\lambda_1^a(\xi) = -|\xi|^2 + O(|\xi|^3), \quad \lambda_2^a(\xi) = -1 + |\xi|^2 + O(|\xi|^3).$$

Thus the first two equations in u, v of system (2.1) satisfy [SK]. On the other hand, since the eigenvector V_3 belongs to the kernel of the linear source term B , then the linear part (2.1) of system (2.2) is not [SK].

In the low frequency regime, the eigenprojectors associated to the eigenvalues near 0, i.e. $\lambda_1^a(\xi), \lambda_3(\xi)$, are given by:

$$P_1(i\xi) = \begin{pmatrix} 1 + O(|\xi|^2) & -i|\xi| + O(|\xi|^2) & 0 \\ -i|\xi| + O(|\xi|^2) & -|\xi|^2 + O(|\xi|^3) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.7)$$

The complementing projector reads:

$$P_2(i\xi) = \begin{pmatrix} O(|\xi|^2) & i|\xi| + O(|\xi|^2) & 0 \\ i|\xi| + O(|\xi|^2) & 1 + |\xi|^2 + O(|\xi|^3) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.8)$$

Now, let $\Gamma(t, x)$ be the Green function associated to system (2.1), i.e.

$$\Gamma(t, x) = \mathcal{F}^{-1}(\widehat{\Gamma}(t, \xi)) = \mathcal{F}^{-1}(\exp(E(i\xi)t)).$$

From the previous expansions, we get an expression of the Green function in Fourier space:

$$\begin{aligned} \exp(E(i\xi)t) &= P_1(i\xi) \exp(-c|\xi|^2 t + O(|\xi|^3)t) + P_2(i\xi) \exp(-ct + O(|\xi|^3)) \\ &\quad + P_3 \exp(-i|\xi|t). \end{aligned}$$

Remark 2.4 ([SK] condition and the conservative-dissipative form). *We point out that the dissipative variables u, v of system (2.1) are already in conservative-dissipative form. This important formulation was introduced in [3] and plays a key role in highlighting the effect of dissipation in the context of hyperbolic systems satisfying [SK], see for instance [2] for an application to the Jin-Xin system (see [17]) when the flux of the limit parabolic equation exhibits a linear term. In particular, the spectral analysis of Section 2.1 will prove that the dissipative variable v provides a faster decay in time with respect to u . This point is crucial in order to close the estimates.*

Before starting with the nonlinear analysis, we briefly discuss the particular choice of the quadratic nonlinearity of system (2.2).

Remark 2.5 (The quadratic source of system (2.2)). *Looking at system (2.2), we notice that the coupling between the dissipative variables u, v and the transported variable w lies in the quadratic source term $d_v uw$, in the equation for v . This is obviously a particular choice, which is due to the simplicity of the computations. It is worth pointing out that the argument of this section applies to other similar cases. The first alternative model is the following:*

$$\begin{cases} \partial_t \hat{u} + i|\xi| \hat{v} = a_u \widehat{u^2} + b_u \widehat{v^2} + c_u \widehat{uv}, \\ \partial_t \hat{v} + i|\xi| \hat{u} + \hat{v} = a_v \widehat{u^2} + b_v \widehat{v^2} + c_v \widehat{uv} + d_v \widehat{vw}, \\ \partial_t \hat{w} + i|\xi| \hat{w} = \widehat{T}_{m(\xi, \eta)}(\widehat{w}(\xi - \eta), \widehat{w}(\eta)), \end{cases}$$

where the coupling term uw of system (2.2) has been replaced with a term involving the faster dissipative variable, i.e. vw , and the time decay is actually faster as expected. The other possible system reads:

$$\begin{cases} \partial_t \hat{u} + i|\xi| \hat{v} = a_u \widehat{u^2} + b_u \widehat{v^2} + c_u \widehat{uv} + d_u \widehat{vw}, \\ \partial_t \hat{v} + i|\xi| \hat{u} + \hat{v} = a_v \widehat{u^2} + b_v \widehat{v^2} + c_v \widehat{uv}, \\ \partial_t \hat{w} + i|\xi| \hat{w} = \widehat{T}_{m(\xi, \eta)}(\hat{w}(\xi - \eta), \hat{w}(\eta)), \end{cases}$$

where the coupling term lies in the equation for the slower variable u . Then there is some flexibility on the choice of the coupling in the quadratic source, but there are also some limitations. More general quadratic nonlinearities involving the three variables u, v, w cannot be treated with the method presented in the rest of this section, which is essentially based on decay estimates of the Green function. The reason is that the time decay is not enough to close the estimates in the general case and new ideas are needed. To this end, a “nonlinear Fourier analysis of the profile” (i.e. space-time resonances method, see [12, 13, 14]) will be implemented in the next section on a more relevant partially dissipative hyperbolic system, as a first example in that direction.

2.2 Decay estimates on the linearized system

In the low frequency regime, the spectral analysis of Section 2.1 provides the following decomposition:

$$\begin{aligned} \widehat{\Gamma}(t) &= \widehat{K}(t) + \widehat{\mathcal{K}}(t) + \widehat{W}(t) \\ &\approx C \exp(-c|\xi|^2 t) \begin{pmatrix} 1 & |\xi| & 0 \\ |\xi| & |\xi|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + C \exp(-ct) \begin{pmatrix} |\xi|^2 & |\xi| & 0 \\ |\xi| & |\xi|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &+ C \exp(-i|\xi|t) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.9}$$

Therefore the solution $U(t) = (u(t), v(t), w(t)) = \Gamma(t)U_0$ to the linear system (2.1) with initial data $U(t=1) = U_0$ can be decomposed as follows:

$$U(t) = \Gamma(t)U_0 = K(t)U_0 + \mathcal{K}(t)U_0 + W(t)U_0. \quad (2.10)$$

Define the left eigenprojectors onto the null space of the linear source term

$$L_1 = (1, 0, 0), \quad L_3 = (0, 0, 1),$$

and its complementing

$$L_2 = (0, 1, 0).$$

We will use the notation Λ to indicate the operator associated with the symbol $|\xi|$ (but also for $|\xi - \eta|$ and $|\eta|$), while R denotes the Riesz transform.

Let us recall the decay estimates from [3].

Proposition 2.6. *Let $U(t)$ be defined in (2.10). Then, for any multi-index β the following decay estimates hold:*

$$\begin{aligned} i) & \|L_1 D^\beta K(t)U_0\|_{L^2} \leq C \min\{1, t^{-\frac{3}{4} - \frac{|\beta|}{2}}\} \|L_1 U_0\|_{L^1} + C \min\{1, t^{-\frac{5}{4} - \frac{|\beta|}{2}}\} \|L_2 U_0\|_{L^1}, \\ ii) & \|L_2 D^\beta K(t)U_0\|_{L^2} \leq C \min\{1, t^{-\frac{5}{4} - \frac{|\beta|}{2}}\} \|L_1 U_0\|_{L^1} + C \min\{1, t^{-\frac{7}{4} - \frac{|\beta|}{2}}\} \|L_2 U_0\|_{L^1}, \\ iii) & |L_1 D^\beta K(t)U_0|_{L^\infty} \leq C \min\{1, t^{-\frac{3}{2} - \frac{|\beta|}{2}}\} \|L_1 U_0\|_{L^1} + C \min\{1, t^{-2 - \frac{|\beta|}{2}}\} \|L_2 U_0\|_{L^1}, \\ iv) & |L_2 D^\beta K(t)U_0|_{L^\infty} \leq C \min\{1, t^{-2 - \frac{|\beta|}{2}}\} \|L_1 U_0\|_{L^1} + C \min\{1, t^{-\frac{5}{2} - \frac{|\beta|}{2}}\} \|L_2 U_0\|_{L^1}, \\ v) & \|D^\beta \mathcal{K}(t)U_0\|_{L^2} \leq C \exp(-ct) \|D^\beta U_0\|_{L^2}. \end{aligned} \quad (2.11)$$

Some a priori bounds on w are listed in [28].

Proposition 2.7. *Let*

$$w(t) = \exp(i\Lambda t) f_w(t), \quad (2.12)$$

where f_w is the profile for w . Then, for a suitable large N and arbitrarily small fixed constants ε, γ ,

$$\begin{aligned} vi) & \|L_3 W(t)U_0\|_{H^N} = \|w\|_{H^N} \leq t^\varepsilon, \\ vii) & |L_3 W(t)U_0|_{L^\infty} = |w|_{L^\infty} \leq t^{-1}, \quad |L_3 R W(t)U_0|_{L^\infty} = |Rw|_{L^\infty} \leq t^{-1}, \\ viii) & \|x f_w\|_{L^2} \leq t^\gamma, \\ ix) & \|\Lambda x f_w\|_{H^1} \leq 1, \\ x) & \| |x|^2 \Lambda f_w \|_{H^1} \leq t. \end{aligned} \quad (2.13)$$

2.3 Decay estimates on the nonlinear system

In order to deal with the nonlinear system (2.2), we define the following norm:

$$\begin{aligned}
M_0(t) = & \sup_{t \geq 1} \{ t^{\frac{3}{4}} \|u(t)\|_{H^N} + t^{\frac{3}{4}-\varepsilon} \|v(t)\|_{H^N} + t^{\frac{3}{2}-2\varepsilon} |u(t)|_{L^\infty} + t^{\frac{3}{4}-\varepsilon} |v(t)|_{L^\infty} \\
& + t^{-\varepsilon} \|w(t)\|_{H^N} + t |w(t)|_{L^\infty} + t |Rw(t)|_{L^\infty} + t^{-\gamma} \|xf_w(t)\|_{L^2} \\
& + \|\Lambda xf_w(t)\|_{H^1} + t^{-1} \| |x|^2 \Lambda f_w(t) \|_{H^1} \}.
\end{aligned} \tag{2.14}$$

Notice that the equation for w in (2.2) is independent of the dissipative variables u, v and it has already been studied in [28]. Therefore, we will simply implement in our system (2.2) the estimates obtained in [28]. More precisely, we use the following result from [28].

Proposition 2.8. *Let w be the solution to*

$$\partial_t \hat{w} + i|\xi| \hat{w} = \widehat{T}_{m(\xi, \eta)}(\hat{w}(\xi - \eta), \hat{w}(\eta)), \quad \hat{w}(0, \xi) = \hat{w}_0(\xi).$$

Consider the equation for the profile $\widehat{f}_w(t, \xi)$ in (2.12), i.e.

$$\widehat{f}_w(t, \xi) = \hat{w}_0(\xi) + \int_1^t \widehat{T}_{m(\xi, \eta)}(\hat{w}(\xi - \eta, s) \hat{w}(\eta, s)) ds.$$

Then the following estimates hold

$$\begin{aligned}
\|w(t)\|_{H^N} &\leq t^\varepsilon M_0^2(t), \quad |w(t)|_{L^\infty} \leq t^{-1} M_0^2(t), \quad |Rw(t)|_{L^\infty} \leq t^{-1} M_0^2(t), \\
\|xf_w(t)\|_{L^2} &\leq t^\gamma M_0^2(t), \quad \|\Lambda xf_w(t)\|_{H^1} \leq M_0^2(t), \quad \| |x|^2 \Lambda f_w(t) \|_{H^1} \leq t M_0^2(t), \\
\|\partial_t f_w\|_{L^2} &\leq t^{-2+\delta} M_0^2(t), \quad \|x \partial_t f_w\|_{L^2} \leq t^{-1+\delta} M_0^2(t),
\end{aligned}$$

for some δ, γ small enough.

We also need to define the norm of the initial data for $U(t) = (u(t), v(t), w(t))$, (we start at time $t = 1$ in order to simplify the computations):

$$E_N = \max\{\|U(1)\|_{L^1}, \|xU(1)\|_{H^2} + \|\Lambda x^2 U(1)\|_{H^1} + \|U(1)\|_{H^N}\}. \tag{2.15}$$

In the rest of this section, we prove the following result.

Theorem 2.9. *[Global existence and decay estimates for the first weaker [SK] model] Consider the Cauchy problem associated with system (2.2) and initial data at time $t = 1$ belonging to $H^N(\mathbb{R}^3)$, for some large $N > \frac{5}{2}$, with*

$$E_N \leq \varepsilon$$

for some fixed ε small enough. Then there exists a unique global in time solution

$$U \in C([0, \infty); H^N(\mathbb{R}^3)) \cap C^1([0, \infty); H^{N-1}(\mathbb{R}^3)),$$

with time decay estimates provided by the inequality

$$M_0(t) \leq CE_N.$$

Proof. Decay estimates in H^N . We apply the Duhamel formula to system (2.2), recalling that for the slower variable u we expect a decay $\|u(t)\|_{H^N} \approx t^{-\frac{3}{4}}$ as in the linear case in (2.11), while the expected rate for v is quite worse $\|v(t)\|_{H^N} \approx t^{-\frac{3}{4}+\varepsilon}$ with respect to the linear case, because of the mixed term $d_v u v$. Then,

$$\begin{aligned}
\|u(t)\|_{H^N} &\leq CE_N \min\{1, t^{-\frac{3}{4}}\} + CM_0^2(t) \int_1^t \exp(-c(t-s)) \min\{1, s^{-\frac{3}{4}}\} ds \\
&\quad + \int_1^t \min\{1, (t-s)^{-\frac{3}{4}}\} \|u(s)\|_{H^N}^2 ds \\
&\quad + \int_1^t \min\{1, (t-s)^{-\frac{3}{4}}\} \|u(s)\|_{H^N} \|v(s)\|_{H^N} ds \\
&\quad + \int_1^t \min\{1, (t-s)^{-\frac{3}{4}}\} \|v(s)\|_{H^N}^2 ds \\
&\leq C \min\{1, t^{-\frac{3}{4}}\} + CM_0^2(t) \int_0^t \exp(-c(t-s)) \min\{1, s^{-\frac{3}{4}}\} ds \\
&\quad + CM_0^2(t) \int_1^t \min\{1, (t-s)^{-\frac{3}{4}}\} \min\{1, s^{-\frac{3}{2}}\} ds \\
&\quad + CM_0^2(t) \int_1^t \min\{1, (t-s)^{-\frac{3}{4}}\} \min\{1, s^{-\frac{3}{2}+\varepsilon}\} ds \\
&\quad + CM_0^2(t) \int_1^t \min\{1, (t-s)^{-\frac{3}{4}}\} \min\{1, s^{-\frac{3}{2}+2\varepsilon}\} ds.
\end{aligned}$$

We analyse each term separately. Thanks to [[3], Lemma 5.2], one gets:

- $\int_1^t \exp(-c(t-s)) \min\{1, s^{-\frac{3}{4}}\} ds \leq t^{-\frac{3}{4}},$
- $\int_1^t \min\{1, (t-s)^{-\frac{3}{4}}\} \min\{1, s^{-\frac{3}{2}}\} ds \leq t^{-\frac{3}{4}},$
- $\int_1^t \min\{1, (t-s)^{-\frac{3}{4}}\} \min\{1, s^{-\frac{3}{2}+\varepsilon}\} ds \leq t^{-\frac{3}{4}},$
- $\int_1^t \min\{1, (t-s)^{-\frac{3}{4}}\} \min\{1, s^{-\frac{3}{2}+2\varepsilon}\} ds \leq t^{-\frac{3}{4}}.$

This implies that

$$\|u(t)\|_{H^N} \leq CE_N \min\{1, t^{-\frac{3}{4}}\} + Ct^{-\frac{3}{4}} M_0^2(t). \quad (2.16)$$

Next, we consider the dissipative variable.

$$\begin{aligned}
\|v(t)\|_{H^N} &\leq C \min\{1, t^{-\frac{5}{4}}\} E_N + CM_0^2(t) \int_1^t \exp(-c(t-s)) \min\{1, s^{-\frac{3}{4}+\varepsilon}\} ds \\
&+ CM_0^2(t) \int_1^t \min\{1, (t-s)^{-\frac{5}{4}}\} \min\{1, s^{-\frac{3}{2}}\} ds \\
&+ CM_0^2(t) \int_1^t \min\{1, (t-s)^{-\frac{5}{4}}\} \min\{1, s^{-\frac{3}{2}+\varepsilon}\} ds \\
&+ CM_0^2(t) \int_1^t \min\{1, (t-s)^{-\frac{5}{4}}\} \min\{1, s^{-\frac{3}{2}+2\varepsilon}\} ds \\
&+ C \int_1^t \min\{1, (t-s)^{-\frac{5}{4}}\} \|u(s)\|_{H^N} \|w(s)\|_{H^N} ds.
\end{aligned}$$

The first four integrals are bounded by $t^{-\frac{5}{4}}$. We consider the last one.

$$\begin{aligned}
&\int_1^t \min\{1, (t-s)^{-\frac{5}{4}}\} \|u(s)\|_{H^N} \|w(s)\|_{H^N} ds \\
&\leq CM_0^2(t) \int_1^t \min\{1, (t-s)^{-\frac{5}{4}}\} \min\{1, s^{-\frac{3}{4}+\varepsilon}\} ds \\
&\leq Ct^{-\frac{3}{4}+\varepsilon} M_0^2(t).
\end{aligned}$$

This way, for the dissipative variable:

$$\|v(t)\|_{H^N} \leq C \min\{1, t^{-\frac{5}{4}}\} E_N + t^{-\frac{3}{4}+\varepsilon} M_0^2(t). \quad (2.17)$$

Decay estimates in L^∞ . We start from the slower variable u :

$$\begin{aligned}
|u(t)|_{L^\infty} &\leq C \min\{1, t^{-\frac{3}{2}}\} + CM_0^2(t) \int_1^t \exp(-c(t-s)) \min\{1, s^{-\frac{3}{2}+2\varepsilon}\} ds \\
&+ \int_1^t \min\{1, (t-s)^{-\frac{3}{2}}\} \|u(s)\|_{H^N}^2 ds \\
&+ \int_1^t \min\{1, (t-s)^{-\frac{3}{2}}\} \|u(s)\|_{H^N} \|v(s)\|_{H^N} ds \\
&+ \int_1^t \min\{1, (t-s)^{-\frac{3}{2}}\} \|v(s)\|_{H^N}^2 ds
\end{aligned}$$

$$\begin{aligned}
&\leq C \min\{1, t^{-\frac{3}{2}}\} + CM_0^2(t) \int_1^t \exp(-c(t-s)) \min\{1, s^{-\frac{3}{2}+2\varepsilon}\} ds \\
&+ CM_0^2(t) \int_1^t \min\{1, (t-s)^{-\frac{3}{2}}\} \min\{1, s^{-\frac{3}{2}}\} ds \\
&+ CM_0^2(t) \int_1^t \min\{1, (t-s)^{-\frac{3}{2}}\} \min\{1, s^{-\frac{3}{2}+\varepsilon}\} ds \\
&+ CM_0^2(t) \int_1^t \min\{1, (t-s)^{-\frac{3}{2}}\} \min\{1, s^{-\frac{3}{2}+2\varepsilon}\} ds \\
&\leq C \min\{1, t^{-\frac{3}{2}}\} + Ct^{-\frac{3}{2}+2\varepsilon} M_0^2(t).
\end{aligned}$$

This implies that

$$|u(t)|_{L^\infty} \leq CE_N \min\{1, t^{-\frac{3}{2}}\} + Ct^{-\frac{3}{2}+2\varepsilon} M_0^2(t). \quad (2.18)$$

Now consider the dissipative variable.

$$\begin{aligned}
|v(t)|_{L^\infty} &\leq C \min\{1, t^{-2}\} E_N + CM_0^2(t) \int_1^t \exp(-c(t-s)) \min\{1, s^{-\frac{3}{4}+\varepsilon}\} ds \\
&+ CM_0^2(t) \int_1^t \min\{1, (t-s)^{-2}\} \min\{1, s^{-\frac{3}{2}}\} ds \\
&+ CM_0^2(t) \int_1^t \min\{1, (t-s)^{-2}\} \min\{1, s^{-\frac{3}{2}+\varepsilon}\} ds \\
&+ CM_0^2(t) \int_1^t \min\{1, (t-s)^{-2}\} \min\{1, s^{-\frac{3}{2}+2\varepsilon}\} ds \\
&+ C \int_1^t \min\{1, (t-s)^{-2}\} \|uw(s)\|_{L^1} ds.
\end{aligned}$$

The last term reads:

$$\begin{aligned}
&\int_1^t \min\{1, (t-s)^{-2}\} \|uw(s)\|_{L^1} ds \\
&\leq \int_1^t \min\{1, (t-s)^{-2}\} \|u(s)\|_{H^N} \|w(s)\|_{H^N} ds \\
&\leq CM_0^2(t) \int_1^t \min\{1, (t-s)^{-2}\} \min\{1, \{s^{-\frac{3}{4}+\varepsilon}\}\} ds \\
&\leq Ct^{-\frac{3}{4}+\varepsilon} M_0^2(t).
\end{aligned}$$

Thus,

$$|v(t)|_{L^\infty} \leq C \min\{1, t^{-2}\} + Ct^{-\frac{3}{4}+\varepsilon} M_0^2(t). \quad (2.19)$$

□

2.4 Shizuta-Kawashima with any quadratic source

The result of this section can be applied to the following reduced version of system (2.2):

$$\begin{cases} \partial_t \hat{u} + i|\xi| \hat{v} = a_u \widehat{u^2} + b_u \widehat{v^2} + c_u \widehat{uv}, \\ \partial_t \hat{v} + i|\xi| \hat{u} + \hat{v} = a_v \widehat{u^2} + b_v \widehat{v^2} + c_v \widehat{uv}, \end{cases} \quad (2.20)$$

where $(t, \xi) = (t, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^+ \times \mathbb{R}^3$ and $a_u, b_u, c_u, a_v, b_v, c_v$ are constant values. System (2.20) is partially dissipative and satisfies [SK] in the unknown variable $U = (u, v)$. This can be verified by analysing the linearized system

$$\begin{cases} \partial_t \hat{u} + i|\xi| \hat{v} = 0, \\ \partial_t \hat{v} + i|\xi| \hat{u} + \hat{v} = 0, \end{cases}$$

following exactly the procedure of Section 2.1. We point out that this linearized [SK] model is already in the conservative-dissipative form (see Remark 2.4). In this section, we extend the result of [3] in the three dimensional semilinear case, as an application of the method developed for (2.2). In the general form of system (2.20), we are indeed able to handle any quadratic source term in u, v , even in the equation for u .

Starting from the linearized system in Fourier, we have

$$\partial_t \hat{U} - E(i\xi) \hat{U} = 0, \quad E(i\xi) = \begin{pmatrix} 0 & -i|\xi| \\ -i|\xi| & -1 \end{pmatrix}.$$

The method of Section 2.1 applied to this system provides the following eigenvalues expansions in the low frequency regime $|\xi| \ll a$ for a small enough:

$$\lambda^a(\xi) = -|\xi|^2 + O(|\xi|^3), \quad \lambda^-(\xi) = -1 + |\xi|^2 + O(|\xi|^3), \quad (2.21)$$

with related eigenprojectors:

$$P(i\xi) = \begin{pmatrix} 1 + O(|\xi|^2) & -i|\xi| + O(|\xi|^2) \\ -i|\xi| + O(|\xi|^2) & -|\xi|^2 + O(|\xi|^3) \end{pmatrix},$$

$$P_-(i\xi) = \begin{pmatrix} O(|\xi|^2) & i|\xi| + O(|\xi|^2) \\ i|\xi| + O(|\xi|^2) & 1 + |\xi|^2 + O(|\xi|^3) \end{pmatrix},$$

Therefore the Green function in Fourier space is given by:

$$\begin{aligned} \hat{\Gamma}(t, \xi) &= \exp(E(i\xi)t) \\ &= P(i\xi) \exp((-|\xi|^2 + O(|\xi|^3))t) + P_-(i\xi) \exp((-1 + O(|\xi|^2))t) \\ &\approx C \exp(-c|\xi|^2 t) \begin{pmatrix} 1 & |\xi| \\ |\xi| & |\xi|^2 \end{pmatrix} + C \exp(-ct) \begin{pmatrix} |\xi|^2 & |\xi| \\ |\xi| & |\xi|^2 \end{pmatrix} \\ &=: \hat{K}(t) + \hat{\mathcal{K}}(t). \end{aligned}$$

Then the solution $U(t) = (u(t), v(t)) = \Gamma(t)U_0$ to the linear system (2.20) with initial data $U(t = 1) = U_0$ can be decomposed as follows:

$$U(t) = \Gamma(t)U_0 = K(t)U_0 + \mathcal{K}(t)U_0,$$

and the decay estimates i) , ii), iii), iv), v) of Section 2.2 hold. Now define the norm of the initial data for system (2.20):

$$E_N = \max\{\|u(0)\|_{L^1} + \|v(0)\|_{L^1}, \|u(0)\|_{H^N} + \|v(0)\|_{H^N}\},$$

and introduce the functional:

$$M_0(t) = \sup_{0 \leq \tau \leq t} \{\max\{1, \tau^{\frac{3}{4}}\} \|u(\tau)\|_{H^N} + \max\{1, \tau^{\frac{5}{4}}\} \|v(\tau)\|_{H^N}\}.$$

Applying the procedure of Section 2.3, we can prove the following result.

Theorem 2.10. *[Global existence and decay estimates for the [SK] system] Consider the Cauchy problem associated with system (2.20) and initial data belonging to $H^N(\mathbb{R}^3)$, with $N > \frac{5}{2}$ and $E_N \leq \varepsilon$ for some fixed ε small enough. Then there exists a unique global in time solution*

$$U = (u, v) \in C([0, \infty); H^N(\mathbb{R}^3)) \cap C^1([0, \infty); H^{N-1}(\mathbb{R}^3)),$$

with time decay estimates:

$$\|u(t)\|_s \leq t^{-\frac{3}{4}} E_N, \quad \|v(t)\|_s \leq t^{-\frac{5}{4}} E_N.$$

Remark 2.11. *We conclude this section with some comments on the theorems.*

- *Theorem 2.10 extends the result of [3] about asymptotic behavior and decay rates in the case of semilinear symmetrizable hyperbolic [SK] systems in three space dimensions with quadratic nonlinearity, since the source term of (2.20) is the most general one.*
- *The decay estimates allow to prove also the global in time existence of smooth solutions to systems (2.2)-(2.20), using a Nishida-type functional, see [16, 3]. Differently from [3], the global in time existence is not an assumption of the method.*
- *The linear part of system (2.20) is quite general in the context of semilinear partially dissipative hyperbolic systems satisfying [SK]. This is due to the general structure of symmetrizable [SK] models, as they can be put in the conservative-dissipative form, see [3] and Remark 2.4. Then the same result holds for any semilinear symmetrizable hyperbolic [SK] model in three space dimensions.*
- *For quasilinear systems, we have a partial result. The decay rates can be obtained as in Theorem 2.10, but we need to start from the global in time existence of smooth solutions, as in [3]. In other words, the Nishida approach does not provide global solutions in the quasilinear case, because of a loss of derivatives.*

3 A step towards models of physical interest: when dissipation prevents time resonances

In this section, we consider a hyperbolic system with quadratic source, whose linear part is exactly the linear operator (2.1), while the coupling between the dissipative variables u, v and w lies in the quadratic source term of the equation for w . The system is the following:

$$\begin{cases} \partial_t \hat{u} + i|\xi| \hat{v} = \widehat{v^2}, \\ \partial_t \hat{v} + i|\xi| \hat{u} + \hat{v} = \widehat{v^2}, \\ \partial_t \hat{w} + i|\xi| \hat{w} = \widehat{vw} + \widehat{T}_{m(\xi, \eta)}(\hat{w}(\xi - \eta), \hat{w}(\eta)). \end{cases} \quad (3.1)$$

The source term of the equation for w is the sum of a Nonresonant Bilinear Form for the wave operator (as in Definition 2.3, where $T_{m(\xi, \eta)}$ is the bilinear pseudoproduct in (2.3)) and a coupling term with the dissipative variables. We will show that system (3.1) is well behaved for long times, despite the facts that:

- the last equation (the wave propagator) does not dissipate at all;
- the whole source of w , i.e. $vw + T_m$ is not a Nonresonant Bilinear Form for the wave propagator (see Definition 2.3).

In the rest of this section we prove the global in time existence and decay of the smooth solutions to system (3.1). In spite of the lack of dissipation of the linear part (system (3.1) is weaker than [SK], see Section 2.1), the coupling term vw is able to transfer enough dissipation to the equation for w to prevent the formation of singularities.

Remark 3.1 (A comment on the choice of the model). *System (3.1) is one of the simplest model to show that a cooperation between weak dissipation and Nonresonant Bilinear Forms is indeed possible. The space-time resonances method works also for a more general system with linear part (2.1) and any quadratic source depending on the three variables u, v, w . However, the procedure in this general case would require several further steps in terms of normal form transformations and weighted/localized estimates, because of the different decay rates of u, v . In this first investigative paper, we limit ourselves to the simple case of (3.1), which however shows that a particular structure of the nonlinearity (Nonresonant Bilinear Form) in terms of the non dissipative variable as an addend of the source of the equation for the non dissipative variable itself is what is needed to prevent singularities. More general systems including some relevant physical models fitting this framework will be studied in the future, as anticipated in the introduction.*

3.1 Decay estimates on the nonlinear system in the low frequency regime

The spectral analysis and the decay estimates on the linearized system are provided in Section 2.2. We define the following norm:

$$\begin{aligned}
M_0(t) = \sup_{t \geq 1} \{ & t^{\frac{3}{4}} \|u(t)\|_{H^N} + t^{\frac{5}{4}} \|v(t)\|_{H^N} + t^{\frac{3}{2}} |u(t)|_{L^\infty} + t^{\frac{5}{2}} |v(t)|_{L^\infty} + \\
& + t^{-\varepsilon} \|w(t)\|_{H^N} + t |w(t)|_{L^\infty} + t |Rw(t)|_{L^\infty} + t^{-\gamma} \|xf_w(t)\|_{L^2} \\
& + \|\Lambda xf_w(t)\|_{H^1} + t^{-1} \| |x|^2 \Lambda f_w(t) \|_{H^1} \}. \quad (3.2)
\end{aligned}$$

Concerning the initial data, we recall the definition of E_N in (2.15). The H^N estimates for dissipative variables u, v are provided in Section 2.4:

$$\|u(t)\|_{H^N} \approx t^{-\frac{3}{4}}, \quad \|v(t)\|_{H^N} \approx t^{-\frac{5}{4}}.$$

Similar computations lead to the L^∞ estimates, namely:

$$|u(t)|_{L^\infty} \approx t^{-\frac{3}{2}}, \quad |v(t)|_{L^\infty} \approx t^{-\frac{5}{2}}.$$

Then it remains to deal with the estimates for w . We treat each term of the norm (3.2) separately. As before, we use the notation Λ to indicate the operator associated with the symbol $|\xi|$ (but also for $|\xi - \eta|$ and $|\eta|$). The main result of this section is stated here.

Theorem 3.2. *[Global existence and decay estimates for the 2nd weaker [SK] model] Consider the Cauchy problem associated with system (3.1) and initial data at initial time $t = 1$ belonging to $H^N(\mathbb{R}^3)$, for some large $N > \frac{5}{2}$, with*

$$E_N \leq \varepsilon$$

for some fixed ε small enough. Then there exists a unique global in time solution

$$U = (u, v) \in C([0, \infty); H^N(\mathbb{R}^3)) \cap C^1([0, \infty); H^{N-1}(\mathbb{R}^3)),$$

with time decay estimates provided by the inequality

$$M_0(t) \leq CE_N.$$

In the rest of this section, we prove Theorem 3.2. Let us start from the Duhamel formula for $w(t)$,

$$w(t) = w_0 + \int_1^t \exp(i\Lambda(t-s)) T_{m(\xi, \eta)}(w, w) ds + \int_1^t \exp(i\Lambda(t-s)) vw ds. \quad (3.3)$$

We will use also the *profile* for the three variables in Fourier space $\hat{u}, \hat{v}, \hat{w}$:

$$\hat{f}(\xi, t) = \begin{pmatrix} \hat{f}_u \\ \hat{f}_v \\ \hat{f}_w \end{pmatrix} = \exp(-E(i\xi)t) \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix},$$

with $E(i\xi)$ in (2.5). In the low frequency regime, we recall the following expansions from Section 2.2:

$$\begin{aligned}\widehat{U} &= \begin{pmatrix} \widehat{u} \\ \widehat{v} \\ \widehat{w} \end{pmatrix} = \exp(E(i\xi)t) \begin{pmatrix} \widehat{f}_u \\ \widehat{f}_v \\ \widehat{f}_w \end{pmatrix} \\ &= \begin{pmatrix} \left(\begin{array}{cc} 1 + O(|\xi|^2) & -i|\xi| + O(|\xi|^2) \\ -i|\xi| + O(|\xi|^2) & -|\xi|^2 + O(|\xi|^3) \end{array} \right) \exp(-|\xi|^2 t) & 0 \\ 0 & \exp(-i|\xi|t) \end{pmatrix} \begin{pmatrix} \widehat{f}_u \\ \widehat{f}_v \\ \widehat{f}_w \end{pmatrix}.\end{aligned}$$

Then at main order (the remainders decay faster in L^2 and L^∞) one gets:

$$\begin{aligned}\widehat{u}(\xi, t) &= \exp(-|\xi|^2 t) \widehat{f}_u, \\ \widehat{v}(\xi, t) &= |\xi| \exp(-|\xi|^2 t) \widehat{f}_u, \\ \widehat{w}(\xi, t) &= \exp(-i|\xi|t) \widehat{f}_w.\end{aligned}\tag{3.4}$$

Now we write the Duhamel formula for the profile \widehat{f}_w :

$$\begin{aligned}\widehat{f}_w(\xi, t) &= \widehat{w}_0(\xi) + \int_1^t \exp(i|\xi|s) \widehat{T}_{m(\xi, \eta)}(\exp(-i|\xi - \eta|s) \widehat{f}_w, \exp(-i|\eta|s) \widehat{f}_w) ds \\ &\quad + \int_1^t \int \exp(i\phi(\xi, \eta)s) \widehat{f}_w(\xi - \eta, s) |\eta| \widehat{f}_u(\eta, s) d\eta ds,\end{aligned}\tag{3.5}$$

where we used (3.4) and

$$\phi(\xi, \eta) = |\xi| - |\xi - \eta| + \frac{2i|\eta|^2}{1 + \sqrt{1 - 4|\eta|^2}} \approx |\xi| - |\xi - \eta| + i|\eta|^2.\tag{3.6}$$

The first two terms of (3.3) and (3.5) have already been estimated in every functional space involved in $M_0(t)$ in (3.2). These results, due to [28], are listed in Proposition 2.8. Therefore, we will only focus on the last integral,

$$\int_1^t \exp(i\Lambda(t-s)) v w ds.\tag{3.7}$$

Similarly, to estimate the profile f_w it remains to deal with

$$\int_1^t \int \exp(i\phi(\xi, \eta)s) \widehat{f}_w(\xi - \eta, s) |\eta| \widehat{f}_u(\eta, s) d\eta ds.\tag{3.8}$$

Estimate of $\|w(t)\|_{H^N}$.

$$\begin{aligned}\int_1^t \|\exp(i\Lambda(t-s)) u w\|_{H^N} ds &\leq \int_1^t \|v(s)\|_{H^N} \|w(s)\|_{H^N} ds \\ &\leq M_0^2(t) \int_1^t s^{-\frac{5}{4} + \varepsilon} ds \leq t^{-\frac{1}{4} + \varepsilon} M_0^2(t).\end{aligned}$$

From Proposition 2.8 and formula (3.3), it follows that $\|w(t)\|_{H^N} \leq t^\varepsilon M_0^2(t)$.

Estimate of $|w(t)|_{L^\infty}$. We use the dispersive properties of the wave propagator $\exp(i\Lambda t)$ in Lemma A.1.

$$\begin{aligned}
\int_1^t |\exp(-i\Lambda(s-t))vw(s)|_{L_x^\infty} ds &\leq \int_1^t \frac{1}{s} (\|vw(s)\|_{\dot{W}^{2,1}} + \|\Lambda vw(s)\|_{\dot{W}^{1,1}}) ds \\
&\leq \int_1^t s^{-1} \|v(s)\|_{H^N} \|w(s)\|_{H^N} ds \\
&\leq M_0^2(t) \int_1^t s^{-1} s^{-\frac{5}{4}+\varepsilon} ds \\
&\leq t^{-\frac{5}{4}+\varepsilon} M_0^2(t),
\end{aligned}$$

Thus, from (3.3) and Proposition 2.8, $|w(t)|_{L^\infty} \leq t^{-1} M_0^2(t)$.

Estimate of $|Rw(t)|_{L^\infty}$. The Riesz transform R does not play any role on the estimates for the dissipative variables u, v . Then we get the same decay of the previous paragraph. From Proposition 2.8, $|Rw(t)|_{L^\infty} \leq t^{-1} M_0^2(t)$.

Estimate of $\|xf_w(t)\|_{L^2}$. Differentiating (3.8) with respect to ξ we have:

$$\begin{aligned}
&\int_1^t \int is \partial_\xi \phi \exp(i\phi s) \widehat{f}_w(\xi - \eta, s) |\eta| \widehat{f}_u(\eta, s) d\eta ds \quad (i) \\
&+ \int_1^t \int \exp(i\phi s) \partial_\xi \widehat{f}_w(\xi - \eta, s) |\eta| \widehat{f}_u(\eta, s) d\eta ds \quad (ii).
\end{aligned}$$

Let us start with (ii).

$$\begin{aligned}
\|(ii)\|_{L^2} &\leq \int_1^t \|\exp(i\phi s) \partial_\xi \widehat{f}_w(\xi - \eta, s) |\eta| \widehat{f}_u(\eta, s)\|_{L^2} ds \\
&= \int_1^t \|\exp(-i\Lambda s) x f_w \Lambda \exp(\Delta s) f_u\|_{L^2} ds \\
&\leq \int_1^t \|\exp(-i\Lambda s) x f_w\|_{L^6} \|\Lambda \exp(\Delta s) f_u\|_{L^3} ds \\
&\leq \int_1^t \|x f_w(s)\|_{H^1} \|v(s)\|_{L^3} ds \\
&\leq \int_1^t \|x f_w(s)\|_{H^1} \|v(s)\|_{H^{\frac{1}{2}}} ds \\
&\leq \int_1^t \|x f_w(s)\|_{H^1} \|v(s)\|_{H^N} ds \\
&\leq M_0^2(t) \int_1^t s^{\gamma-\frac{5}{4}} ds \leq t^{-\frac{1}{4}+\gamma} M_0^2(t),
\end{aligned}$$

where we used standard Sobolev embeddings and $v \approx \Lambda \exp(\Delta s) f_u$ from (3.4).

Now let us integrate in time (i). We use the following identity:

$$\exp(i\phi s) = \frac{(\frac{1}{s} + \partial_s) \exp(i\phi s)}{i\phi + 1/s}.$$

Then

$$\begin{aligned} (i) &= \left[is \int \exp(i\phi s) \frac{\partial_\xi \phi |\eta|}{i\phi + 1/s} \widehat{f}_w \widehat{f}_u d\eta \right]_1^t - \int_1^t \int is \exp(i\phi s) \frac{\partial_\xi \phi |\eta|}{i\phi + 1/s} \partial_s \widehat{f}_w \widehat{f}_u d\eta ds \\ &- \int_1^t \int is \exp(i\phi s) \frac{\partial_\xi \phi |\eta|}{i\phi + 1/s} \widehat{f}_w \partial_s \widehat{f}_u d\eta ds + \int_1^t \int \frac{i}{s} \exp(i\phi s) \frac{\partial_\xi \phi |\eta|}{(i\phi + 1/s)^2} \widehat{f}_w \widehat{f}_u d\eta ds \\ &=: (i)_1 + (i)_2 + (i)_3 + (i)_4. \end{aligned}$$

Before going further, we need the following lemmas. Hereafter we use the notation μ_0, μ_1 to denote any symbol in $\mathcal{B}_0, \mathcal{B}_1$ respectively.

Lemma 3.3. *The operator associated to the symbol*

$$\mu_0(\xi, \eta) = \frac{\partial_\xi \phi |\xi - \eta|}{i\phi + 1/s} \quad (3.9)$$

belongs to \mathcal{B}_0 , as in Definition 2.1.

Proof. For $|\xi| \ll 1$ and $|\xi - \eta|, |\eta| \approx 1$,

$$\begin{aligned} \mu_0(\xi, \eta) &\approx \frac{\frac{\xi |\xi - \eta|}{|\xi|} - (\xi - \eta)}{i(|\xi| - |\xi - \eta|) + (1/s - \frac{2|\xi - \eta|^2}{1 + \sqrt{1 - 4|\xi - \eta|^2}} + O(|\xi|))} \\ &= \mathcal{A}(|\xi|, \frac{\xi}{|\xi|}, \xi - \eta), \end{aligned}$$

where \mathcal{A} is a smooth function. □

Lemma 3.4. *We have the following estimates:*

$$\begin{aligned} (a) \quad &\|\partial_t f_w\|_{L^2} \approx t^{-\frac{2}{3} - \frac{5}{4} + \gamma + O(\delta)} M_0^2(t) = t^{-\frac{23}{12} + O(\delta)} M_0^2(t), \\ (b) \quad &\|x \partial_t f_w\|_{L^2} \approx t^{-\frac{1}{4} - \frac{3}{5} + \gamma + O(\delta)} M_0^2(t) = t^{-\frac{17}{20} + O(\delta)} M_0^2(t). \\ (c) \quad &\|x \partial_t f_w\|_{H^2} \approx t^{-\frac{1}{4} - \frac{3}{5} + \gamma + O(\delta)} M_0^2(t) = t^{-\frac{17}{20} + O(\delta)} M_0^2(t). \end{aligned} \quad (3.10)$$

where δ is small enough.

Proof. Let us start from estimate (a).

$$\begin{aligned} \partial_t \widehat{f}_w(\xi, t) &= \exp(i\Lambda t) \widehat{T}_{m(\xi, \eta)}(\widehat{w}, \widehat{w}) \\ &+ \int \exp(i|\xi|t) \exp(-i|\xi - \eta|t) \widehat{f}_w(\xi - \eta, t) |\eta| \exp(-|\eta|^2 t) \widehat{f}_u(\eta, t) d\eta. \end{aligned}$$

The first term has been handled in [28] and, according to Propostion 2.8, provides a decay $t^{-2+\delta}$ for some δ small. Consider the next term:

$$\begin{aligned}
\|\exp(i\Lambda t)\exp(-i\Lambda t)f_w\exp(\Delta t)\Lambda f_u\|_{L^2} &\leq \|\exp(-i\Lambda t)f_w\|_{L^{6-\delta}}\|\exp(\Delta t)\Lambda f_u\|_{L^{\frac{12-2\delta}{4-\delta}}} \\
&\leq \frac{1}{t^{\frac{4-\delta}{6-\delta}}}\|\Lambda^{\frac{8-2\delta}{6-\delta}}f_w\|_{L^{\frac{6-\delta}{5-\delta}}}\|v\|_{L^{\frac{12-2\delta}{4-\delta}}} \\
&\leq t^{-\frac{2}{3}+O(\delta)}\|\Lambda^{\frac{8-2\delta}{6-\delta}}f_w\|_{L^{\frac{6-\delta}{5-\delta}}}\|v\|_{H^{\frac{1}{2}+O(\delta)}} \\
&\leq t^{-\frac{2}{3}+O(\delta)}(\|xf_w\|_{H^1}+\|\Lambda xf_w\|_{H^1})\|v\|_{H^N} \\
&\leq t^{-\frac{2}{3}+\gamma-\frac{5}{4}+O(\delta)}M_0^2(t),
\end{aligned}$$

where we used Lemma A.1 and the embedding $L^2(\langle x \rangle) \subset L^{\frac{6-\delta}{5-\delta}}$ in \mathbb{R}^3 . Now we deal with the weighted estimate (b). Differentiating with respect to ξ :

$$\begin{aligned}
\partial_\xi \partial_t \widehat{f}_w(t, \xi) &= \partial_\xi (\exp(i\Lambda t) \widehat{T}_m (\exp(-i\Lambda t) \widehat{w}, \exp(-i\Lambda t) \widehat{w})) \\
&\quad + it \int \partial_\xi \phi \exp(i\phi t) \widehat{f}_w(\xi - \eta, t) |\eta| \widehat{f}_u(\eta, t) d\eta \\
&\quad + \int \exp(i\phi t) \partial_\xi \widehat{f}_w(\xi - \eta, t) |\eta| \widehat{f}_u(\eta, t) d\eta,
\end{aligned}$$

where the first line on the right hand side has already been treated in [28] and, as recalled in Proposition 2.8, decays like $t^{-1+\delta}M_0^2(t)$. Moreover, from Definition 2.1 it is easy to verify that

$$\mu_1(\xi, \eta) = \partial_\xi \phi |\xi - \eta| \in \mathcal{B}_1,$$

where we recall that

$$\partial_\xi \phi = \frac{\xi}{|\xi|} - \frac{(\xi - \eta)}{|\xi - \eta|}.$$

Thus we have:

$$\begin{aligned}
&t\|\exp(i\Lambda t)T_{\mu_1}(\exp(-i\Lambda t)\Lambda^{-1}f_w, \exp(\Delta t)\Lambda f_u)\|_{L^2} \\
&\quad + \|\exp(i\Lambda t)(\exp(-i\Lambda t)xf_w, \exp(\Delta t)\Lambda f_u)\|_{L^2} \\
&\leq t\|\Lambda^{-1}\exp(-i\Lambda t)f_w\|_{W^{1,5}}\|\exp(\Delta t)\Lambda f_u\|_{L^{\frac{10}{3}}} \\
&\quad + t\|\Lambda^{-1}\exp(-i\Lambda t)f_w\|_{L^5}\|\exp(\Delta t)\Lambda f_u\|_{W^{1, \frac{10}{3}}} \\
&\quad + \|\exp(-i\Lambda t)xf_w\|_{L^6}\|\exp(\Delta t)\Lambda f_u\|_{L^3} \\
&\leq t^{1-\frac{3}{5}+\gamma}M_0(t)t^{-\frac{5}{4}}M_0(t) + \|xf_w\|_{H^1}\|v(t)\|_{H^{\frac{1}{2}}} \\
&\leq t^{1+\gamma-\frac{5}{4}-\frac{3}{5}}M_0^2(t) + t^{\gamma-\frac{5}{4}}M_0^2(t) \\
&\leq t^{-\frac{1}{4}-\frac{3}{5}+\gamma}M_0^2(t),
\end{aligned}$$

where first we used Lemma A.3, then estimate (A.4) of Lemma A.1 in the fifth line and Sobolev embeddings.

Estimate (c) is analogous to (b). \square

Now we come back to the estimates of $(i)_1, (i)_2, (i)_3, (i)_4$. Recall that $\mu_0(\xi, \eta) = \frac{\partial_\xi \phi |\xi - \eta|}{i\phi + 1/s} \in \mathcal{B}_0$ thanks to Lemma 3.3. Then

$$\begin{aligned}
\|(i)_1\|_{L^2} &\leq t \|\exp(i\Lambda t) T_{\mu_0}(\Lambda^{-1} \exp(-i\Lambda t) f_w, \exp(\Delta t) \Lambda f_u)\|_{L^2} \\
&\leq t \|\Lambda^{-1} \exp(-i\Lambda t) f_w\|_{L^6} \|\exp(\Delta t) \Lambda f_u\|_{L^3} \\
&\leq t \|f_w\|_{L^2} \|\exp(\Delta t) \Lambda f_u\|_{H^{\frac{1}{2}}} \\
&\leq t \|w\|_{H^N} \|u\|_{H^N} \\
&\leq t^{1+\varepsilon - \frac{5}{4}} M_0^2(t) \approx t^{-\frac{1}{4} + \varepsilon} M_0^2(t),
\end{aligned}$$

where we used Lemma A.3 and Lemma A.2. Next, we deal with $(i)_2$.

$$\begin{aligned}
\|(i)_2\|_{L^2} &\leq \int_1^t s \|\exp(i\Lambda s) T_{\mu_0}(\exp(-i\Lambda s) \Lambda^{-1} \partial_s f_w, \exp(\Delta s) \Lambda f_u)\|_{L^2} ds \\
&\leq \int_1^t s \|\Lambda^{-1} \exp(-i\Lambda s) \partial_s f_w\|_{L^6} \|\exp(\Delta s) \Lambda f_u\|_{L^3} ds \\
&\leq \int_1^t s \|\partial_s f_w\|_{L^2} \|v(s)\|_{H^2} ds \\
&\leq M_0^2(t) \int_1^t s^{1 - \frac{2}{3} - \frac{5}{4} + \gamma - \frac{5}{4} + O(\delta)} ds \approx t^{-\frac{7}{6} + \gamma + O(\delta)} M_0^2(t).
\end{aligned}$$

The estimate for $(i)_3$ is similar and provides a better decay. Now, as a consequence of Lemma 3.3, it is easy to see that

$$\mu_0(\xi, \eta) = \frac{\partial_\xi \phi |\xi - \eta| |\eta|}{(i\phi + 1/s)^2} \in \mathcal{B}_0.$$

Then

$$\begin{aligned}
\|(i)_4\|_{L^2} &\leq \int_1^t s^{-1} \|\exp(i\Lambda s) T_{\mu_0(\xi, \eta)}(\exp(-i\Lambda s) \Lambda^{-1} f_w, \exp(\Delta s) f_u)\|_{L^2} ds \\
&\leq \int_1^t s^{-1} \|\exp(-i\Lambda s) \Lambda^{-1} f_w\|_{L^6} \|\exp(\Delta s) f_u\|_{L^2} ds \\
&\leq \int_1^t s^{-1} \|f_w\|_{L^2} \|u\|_{L^2} ds \\
&\leq M_0^2(t) \int_1^t s^{-1 + \varepsilon - \frac{3}{4}} ds \approx t^{\varepsilon - \frac{3}{4}} M_0^2(t).
\end{aligned}$$

Estimate of $\|\Lambda x f_w(t)\|_{H^1}$. It is enough to do again the estimates of the previous paragraph in H^2 and check that the mixed term vw provides a better decay.

Estimate of $\| |x|^2 \Lambda f_w(t) \|_{H^1}$. We start by recalling that

$$\begin{aligned} |\xi| \widehat{f}_w &= |\xi| \widehat{w}_0(\xi) + \int_1^t \exp(i|\xi|s) |\xi| \widehat{T}_{m(\xi, \eta)}(\exp(-i|\xi - \eta|s) \widehat{f}_w, \exp(-i|\eta|s) \widehat{f}_w) ds \\ &\quad + \int_1^t \int |\xi| \exp(i\phi s) \widehat{f}_w |\eta| \widehat{f}_u d\eta ds. \end{aligned}$$

Differentiating with respect to ξ one has:

$$\begin{aligned} \partial_\xi (|\xi| \widehat{f}_w) &= \frac{\xi}{|\xi|} \widehat{w}_0 + |\xi| \partial_\xi \widehat{w}_0 \\ &\quad + \int_1^t \partial_\xi (\exp(i\Lambda s) |\xi| \widehat{T}_{m(\xi, \eta)}(\exp(-i\Lambda s) \widehat{f}_w, \exp(-i\Lambda s) \widehat{f}_w)) ds \\ &\quad + \int_1^t \int \frac{\xi}{|\xi|} \exp(i\phi s) \widehat{f}_w |\eta| \widehat{f}_u d\eta ds \\ &\quad + \int_1^t \int |\xi| i s \partial_\xi \phi \exp(i\phi s) |\eta| \widehat{f}_w \widehat{f}_u d\eta ds \\ &\quad + \int_1^t \int |\xi| \exp(i\phi s) |\eta| \partial_\xi \widehat{f}_w \widehat{f}_u d\eta ds. \end{aligned}$$

Differentiating again in ξ ,

$$\begin{aligned} \partial_\xi^2 (|\xi| \widehat{f}_w) &= \frac{\xi^2}{|\xi|^3} \widehat{w}_0 + 2 \frac{\xi}{|\xi|} \partial_\xi \widehat{w}_0 + |\xi| \partial_\xi^2 \widehat{w}_0 \\ &\quad + \int_1^t \partial_\xi^2 (\exp(i\Lambda s) |\xi| \widehat{T}_{m(\xi, \eta)}(\exp(-i\Lambda s) \widehat{f}_w, \exp(-i\Lambda s) \widehat{f}_w)) ds \\ &\quad + \int_1^t \int s^2 \exp(i\phi s) |\partial_\xi \phi|^2 |\eta| |\xi| \widehat{f}_w \widehat{f}_u d\eta ds \quad (a) \\ &\quad + \int_1^t \int s \exp(i\phi s) \partial_\xi^2 \phi |\xi| |\eta| \widehat{f}_w \widehat{f}_u d\eta ds \quad (b) \\ &\quad + \int_1^t \int s \exp(i\phi s) \frac{\xi}{|\xi|} \partial_\xi \widehat{f}_w |\eta| \widehat{f}_u d\eta ds \quad (c) \\ &\quad + \int_1^t \int s \exp(i\phi s) |\xi| |\eta| \partial_\xi \phi \partial_\xi \widehat{f}_w \widehat{f}_u d\eta ds \quad (d) \\ &\quad + \int_1^t \int i s \exp(i\phi s) \partial_\xi \phi |\eta| \frac{\xi}{|\xi|} \widehat{f}_w \widehat{f}_u d\eta ds \quad (e) \\ &\quad + \int_1^t \int \exp(i\phi s) \frac{\xi^2}{|\xi|^3} \widehat{f}_w |\eta| \widehat{f}_u d\eta ds \quad (f) \end{aligned}$$

$$\begin{aligned}
& + \int_1^t \int \exp(i\phi s) |\eta| \frac{\xi}{|\xi|} \partial_\xi \widehat{f}_w \widehat{f}_u \, d\eta \, ds \quad (g) \\
& + \int_1^t \int \exp(i\phi s) |\xi| |\eta| \partial_\xi^2 \widehat{f}_w \widehat{f}_u \, d\eta \, ds \quad (h).
\end{aligned}$$

The first integral is studied in [28]. As recalled in Proposition 2.8, it grows like t . Let us start from (a). Integrating by parts in time,

$$\begin{aligned}
(a) & = \left[\int s^2 \exp(i\phi s) \frac{(\partial_\xi \phi)^2}{(i\phi + 1/s)} |\eta| |\xi| \widehat{f}_w \widehat{f}_u \, d\eta \right]_1^t \quad (a)_1 \\
& - \int_1^t s^2 \exp(i\phi s) \frac{(\partial_\xi \phi)^2}{(i\phi + 1/s)} |\eta| |\xi| \partial_s \widehat{f}_w \widehat{f}_u \, d\eta \, ds \quad (a)_2 \\
& - \int_1^t s^2 \exp(i\phi s) \frac{(\partial_\xi \phi)^2}{(i\phi + 1/s)} |\eta| |\xi| \widehat{f}_w \partial_s \widehat{f}_u \, d\eta \, ds \quad (a)_3 \\
& \quad \quad \quad + \text{similar terms.}
\end{aligned}$$

Notice that, as a consequence of Lemma 3.3,

$$\mu_1(\xi, \eta) = \frac{(\partial_\xi \phi)^2 |\xi - \eta|^2}{(i\phi + 1/s)} \in \mathcal{B}_1.$$

Then:

$$\begin{aligned}
\|(a)_1\|_{L^2} & \leq t^2 \|\Lambda \exp(i\Lambda t) T_{\mu_1} (\Lambda^{-2} \exp(i\Lambda t) f_w, \Lambda \exp(\Delta s) f_u)\|_{L^2} \\
& \leq t^2 \|\Lambda^{-2} \exp(i\Lambda t) f_w\|_{W^{2,6+\delta}} \|\exp(\Delta s) \Lambda f_u\|_{W^{2, \frac{12+2\delta}{4+\delta}}} \\
& \leq t^2 \|f_w\|_{W^{2, \frac{18+3\delta}{15+2\delta}}} \|\Lambda \exp(\Delta s) f_u\|_{H^{2+\frac{1}{3}+O(\delta)}} \\
& \leq t^2 \|x f_w\|_{H^2} \|v\|_{H^N} \approx t^{\frac{3}{4}+\gamma} M_0^2(t).
\end{aligned}$$

Next we look at $(a)_2$.

$$\begin{aligned}
\|(a)_2\|_{L^2} & \leq \int_1^t s^2 \|\exp(i\Lambda s) T_{\mu_1} (\Lambda^{-2} \exp(i\Lambda s) \partial_s f_w, \exp(\Delta s) \Lambda f_u)\|_{L^2} \, ds \\
& \leq \int_1^t s^2 \|\Lambda^{-2} \exp(i\Lambda s) \partial_s f_w\|_{W^{2,6+\delta}} \|\exp(\Delta s) \Lambda f_u\|_{W^{1, \frac{12+2\delta}{4+\delta}}} \, ds \\
& \leq \int_1^t s^2 \|\partial_s f_w\|_{W^{2, \frac{18+3\delta}{15+2\delta}}} s^{-\frac{5}{4}} \, ds \\
& \leq \int_1^t s^2 \|x \partial_s f_w\|_{H^2} s^{-\frac{5}{4}} \, ds \\
& \leq M_0^2(t) \int_1^t s^{2-\frac{1}{4}-\frac{3}{5}-\frac{5}{4}+\gamma} \, ds \\
& \approx t^{\frac{9}{10}+\gamma} M_0^2(t),
\end{aligned}$$

where we used Lemma A.3, Lemma A.2 and Sobolev embeddings. The estimate of $(a)_3$ is similar. Now consider (b). Notice that

$$\partial_\xi^2 \phi = \frac{\xi^2}{|\xi|^3} - \frac{(\xi - \eta)^2}{|\xi - \eta|^3},$$

then

$$|\partial_\xi^2 \phi| \leq \frac{1}{|\xi|} + \frac{1}{|\xi - \eta|}.$$

Therefore,

$$|\eta||\xi||\partial_\xi^2 \phi| \leq |\eta| + \frac{|\xi||\eta|}{|\xi - \eta|}.$$

Using this observation in the estimate for (b), we have:

$$\begin{aligned} \|(b)\|_{L^2} &\leq \int_1^t s \|\exp(i\Lambda s) \exp(-i\Lambda s) f_w \exp(\Delta s) \Lambda f_u\|_{L^2} ds \\ &\quad + \int_1^t s \|\Lambda \exp(i\Lambda s) \Lambda^{-1} \exp(-i\Lambda s) f_w \exp(\Delta s) \Lambda f_u\|_{L^2} ds \\ &\leq \int_1^t s \|\exp(-i\Lambda s) f_w\|_{L^3} \|\exp(\Delta s) \Lambda f_u\|_{L^6} ds \\ &\quad + \int_1^t s \|\Lambda^{-1} \exp(-i\Lambda s) f_w\|_{W^{1,6}} \|\exp(\Delta s) \Lambda f_u\|_{W^{1,3}} ds \\ &\leq CM_0^2(t) \int_1^t s^{1+\varepsilon-\frac{5}{4}} ds \\ &\approx t^{\frac{3}{4}+\varepsilon} M_0^2(t). \end{aligned}$$

Note that (c) and (e) are similar. Let us consider (d). The symbol

$$\mu_1(\xi, \eta) = |\xi| \partial_\xi \phi \in \mathcal{B}_1.$$

$$\begin{aligned} \|(d)\|_{L^2} &\leq \int_1^t s \|\exp(i\Lambda s) T_{\mu_1}(\exp(-i\Lambda s) x f_w, \exp(\Delta s) \Lambda f_u)\|_{L^2} ds \\ &\leq \int_1^t s \|\exp(-i\Lambda s) x f_w\|_{W^{1,6}} \|\exp(\Delta s) \Lambda f_u\|_{W^{1,3}} ds \\ &\leq M_0^2(t) \int_1^t s^{1+\gamma-\frac{5}{4}} ds \approx s^{\frac{3}{4}+\gamma} M_0^2(t). \end{aligned}$$

We estimate (f).

$$\begin{aligned}
\|(f)\|_{L^2} &\leq \int_1^t s \|\Lambda^{-1} \exp(i\Lambda s) T_{\mu_0}(\exp(-i\Lambda s) f_w, \exp(\Delta s) \Lambda f_u)\|_{L^2} ds \\
&\leq \int_1^t s \|\exp(i\Lambda s) T_{\mu_0}(\exp(-i\Lambda s) f_w, \exp(\Delta s) \Lambda f_u)\|_{L^{\frac{6}{5}}} ds \\
&\leq \int_1^t s \|\exp(-i\Lambda s) f_w\|_{L^3} \|\exp(\Delta s) \Lambda f_u\|_{L^2} ds \\
&\leq M_0(t) \int_1^t s^{1-\frac{1}{3}} \|x f_w\|_{L^2} s^{-\frac{5}{4}} ds \\
&\leq M_0^2(t) \int_1^t s^{1-\frac{1}{3}+\gamma-\frac{5}{4}} ds \\
&\approx t^{\frac{5}{12}+\gamma} M_0^2(t),
\end{aligned}$$

where we used Lemma A.2, Lemma A.3 and Lemma A.1. Notice that (g) is even simpler, then the last one is (h).

$$\begin{aligned}
\|(h)\|_{L^2} &\leq \int_1^t \|\exp(i\Lambda s) \Lambda \exp(-i\Lambda s) x^2 f_w \exp(\Delta s) \Lambda f_u\|_{L^2} ds \\
&\leq \int_1^t \|x^2 \Lambda f_w\|_{L^6} \|\exp(\Delta s) \Lambda f_u\|_{L^3} ds \\
&\leq M_0(t) \int_1^t \|\Lambda x^2 f_w\|_{H^1} s^{-\frac{5}{4}} ds \\
&\leq M_0^2(t) \int_1^t s^{1-\frac{5}{4}} ds \approx t^{\frac{3}{4}} M_0^2(t).
\end{aligned}$$

3.2 The high frequency regime

Denoting the quadratic source term by $Q(U, U)$, where $U = (u, v, w)$, we recall that system (3.1) can be written in compact form in Fourier variables as follows:

$$\begin{aligned}
\widehat{U}(\xi, t) &= \exp(E(i\xi)t)\widehat{U}_0 + \int_1^t \int \exp(E(i\xi)(t-s))Q(\widehat{U}(\xi-\eta), \widehat{U}(\eta)) d\eta ds \\
&= \exp(E(i\xi)t)\chi_{\{|\xi|\leq a\}}\widehat{U}_0 \\
&\quad + \int_1^t \int \exp(E(i\xi)(t-s))_{|\xi|\leq a} Q(\widehat{U}(\xi-\eta), \widehat{U}(\eta))\chi_{\{|\xi|\leq a, |\eta|\leq a\}} d\eta ds \quad (a) \\
&\quad + \exp(E(i\xi)t)\chi_{\{|\xi|\leq a\}}\widehat{U}_0 \\
&\quad + \int_1^t \int \exp(E(i\xi)(t-s))_{|\xi|\leq a} Q(\widehat{U}(\xi-\eta), \widehat{U}(\eta))\chi_{\{|\xi|\leq a, |\eta|>a\}} d\eta ds \quad (b) \\
&\quad + \exp(E(i\xi)t)\chi_{\{|\xi|>a\}}\widehat{U}_0 \\
&\quad + \int_1^t \int \exp(E(i\xi)(t-s))_{|\xi|>a} Q(\widehat{U}(\xi-\eta), \widehat{U}(\eta))\chi_{\{|\xi|>a, |\eta|\leq a\}} d\eta ds \quad (c) \\
&\quad + \exp(E(i\xi)t)\chi_{\{|\xi|>a\}}\widehat{U}_0 \\
&\quad + \int_1^t \int \exp(E(i\xi)(t-s))_{|\xi|>a} Q(\widehat{U}(\xi-\eta), \widehat{U}(\eta))\chi_{\{|\xi|>a, |\eta|>a\}} d\eta ds \quad (d).
\end{aligned}$$

Now (a) is exactly what we considered until now. Let us look at the other terms. From [3] and Section 2.2, we know the behavior of the dissipative variables in the high frequency regime

$$u, v \approx \exp(-ct).$$

We look at \widehat{w} ,

$$\widehat{w}(\xi, t) = \exp(-i|\xi|t)\widehat{w}_0 + \int_1^t \int \exp(-i|\xi|(t-s))\widehat{w}(\xi-\eta)\widehat{v}(\eta) d\eta ds,$$

where $\widehat{v} \approx \exp(-ct)$. Considering the profile for w , at main order we have

$$\begin{aligned}
\widehat{f}_w^{(b)}(\xi, t) &= \widehat{w}_0(\xi) \\
&\quad + \int_1^t \int \exp(i|\xi|s) \exp(-i|\xi-\eta|s) \widehat{f}_w(\xi-\eta, s) \exp(-cs) (\widehat{f}_v + i|\eta|\widehat{f}_u) d\eta ds,
\end{aligned}$$

which is exponentially small. Moreover, the time frequency

$$\phi(\xi, \eta) = |\xi| - |\xi-\eta| + ic$$

does not vanish, then the time resonant set $\mathcal{T} = \emptyset$, according to Definition 2.2. This is analogous to the case of the Klein-Gordon equation, see [29]. The remaining terms are similar. Then the high frequency regime does not affect the result at all and the long time behavior is completely determined by the low frequencies.

A An appendix section

First, we collect two estimates on the wave propagator $\exp(i\Lambda t)$.

Lemma A.1 ([28]). *The following estimates on the wave propagator $\exp(i\Lambda t)$ hold:*

$$|\exp(i\Lambda t)f|_{L^\infty} \leq t^{-1}[\|f\|_{\dot{W}^{2,1}} + \|\Lambda f\|_{\dot{W}^{1,1}}]. \quad (\text{A.1})$$

For $2 \leq p < \infty$,

$$\|\exp(i\Lambda t)f\|_{L^p} \leq t^{-1+\frac{2}{p}}\|\Lambda^{2-\frac{4}{p}}f\|_{L^{p'}}. \quad (\text{A.2})$$

They also provide the following bounds:

$$\|w\|_{W^{1,p}} \leq t^{-1+\frac{2}{p}}M_0(t), \quad \text{for } 2 \leq p \leq 4; \quad (\text{A.3})$$

$$\|\Lambda^{-1}w\|_{W^{1,p}} \leq t^{\gamma-1+\frac{2}{p}}M_0(t), \quad \text{for } 4 \leq p < 6. \quad (\text{A.4})$$

Then we recall two estimates on fractional integration.

Lemma A.2 ([28]). *For $0 < \alpha < \frac{3}{p}$,*

$$\left\| \frac{1}{\Lambda^\alpha} f \right\|_{L^q} \leq \|f\|_{L^p} \quad \text{for } 1 < p, q < \infty \text{ and } \alpha = \frac{3}{p} - \frac{3}{q}. \quad (\text{A.5})$$

$$\left\| \frac{1}{\Lambda^\alpha} \exp(i\Lambda t)f \right\|_{L^q} \leq \|f\|_{L^p} \quad \text{for } 1 < p \leq 2 \leq q < \infty \text{ and } \alpha = \frac{3}{p} - \frac{3}{q}. \quad (\text{A.6})$$

We conclude this appendix section with the generalized version of the Hölder inequality for symbols in \mathcal{B}_s .

Lemma A.3 ([28]). *Let p, q, r be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $1 < p, q, r < \infty$. If $m(\xi, \eta) \in \mathcal{B}_s$, $s > 0$ as in Definition 2.1, then:*

$$\|\Lambda^k T_m(f, g)\|_{L^r} \leq \|f\|_{W^{s+k,p}} \|g\|_{L^q} + \|f\|_{L^p} \|g\|_{W^{s+k,q}}.$$

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