

Design of finite volume schemes minimizing the grid orientation effect in reservoir simulation

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- Dead-oil porous media flow with (a lot of) simplifying assumptions

$$\mathbf{u} + \lambda(s)\nabla p = \mathbf{0}, \quad (1a)$$

$$\operatorname{div} \mathbf{u} = q(t)\delta_{\mathbf{0}}(\mathbf{x}), \quad (1b)$$

$$\phi \partial_t s + \operatorname{div}(f(s)\mathbf{u}) = q(t)\delta_{\mathbf{0}}(\mathbf{x}), \quad (1c)$$

where Dirac source terms mimic a **water injection** at $\mathbf{x} = \mathbf{0}$.

- Fractional flux* formulation using Buckley-Leverett's law

$$f(s) = \frac{\mu_w^{-1}s^2}{\mu_w^{-1}s^2 + \mu_o^{-1}(1-s)^2} = \frac{Ms^2}{Ms^2 + (1-s)^2},$$

$$\lambda(s) = \mu_w^{-1}s^2 + \mu_o^{-1}(1-s)^2 \quad [\text{total mobility}],$$

$$M = \mu_o/\mu_w \quad [\text{viscosity ratio}].$$

- The time stepping is *implicit* w.r.t. pressure, *explicit* w.r.t. saturation

$$\mathbf{u}^{n+1} + \lambda(s^n) \nabla p^{n+1} = \mathbf{0}, \quad (2a)$$

$$\operatorname{div} \mathbf{u}^{n+1} = q^{n+1} \delta_{\mathbf{0}}(\mathbf{x}), \quad (2b)$$

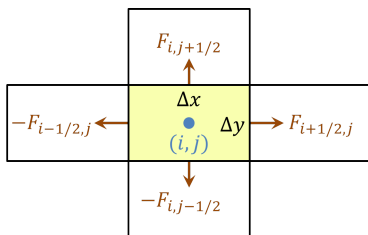
$$\phi \frac{s^{n+1} - s^n}{\Delta t} + \operatorname{div}(f(s^n) \mathbf{u}^{n+1}) = q^{n+1} \delta_{\mathbf{0}}(\mathbf{x}). \quad (2c)$$

- Decoupling into an **elliptic** problem in pressure (2a)–(2b)

$$-\operatorname{div}(\lambda(s^n) \nabla p^{n+1}) = q^{n+1} \delta_{\mathbf{0}}(\mathbf{x}),$$

followed by a **hyperbolic** transport in saturation (2c).

- The domain Ω is divided into rectangles Ω_{ij} of *uniform* size



- The unknowns at the cell centers (i, j) are $s_{i,j}^{n+1}$ and $p_{i,j}^{n+1}$, while those at the edges $\sigma = (i + 1/2, j)$ and $(i, j + 1/2)$ are *fluxes*

$$F_{\sigma} \approx \pm \int_{\sigma} \mathbf{u}^{n+1} \cdot \mathbf{n}_{\sigma}$$

- Equation $\mathbf{u}^{n+1} + \lambda(s^n)\nabla p^{n+1} = \mathbf{0}$ is discretized as

$$F_{i+1/2,j} = \lambda_{i+1/2,j}^n \frac{\Delta y}{\Delta x} (p_{i,j}^{n+1} - p_{i+1,j}^{n+1}), \quad (3a)$$

$$F_{i,j+1/2} = \lambda_{i,j+1/2}^n \frac{\Delta x}{\Delta y} (p_{i,j}^{n+1} - p_{i,j+1}^{n+1}), \quad (3b)$$

where λ_σ^n stems from one's favorite **averaging** procedure.

- Equation $\text{div } \mathbf{u}^{n+1} = q^{n+1}\delta_0$ is discretized as

$$F_{i+1/2,j} - F_{i-1/2,j} + F_{i,j+1/2} - F_{i,j-1/2} = q^{n+1}\kappa_{0,0}(i,j). \quad (4)$$

Plugging (3) into the flux **balances** (4) results in a linear system.

- Equation $\phi \frac{s^{n+1} - s^n}{\Delta t} + \text{div}(f(s^n)\mathbf{u}^{n+1}) = q^{n+1}\delta_0$ is discretized as

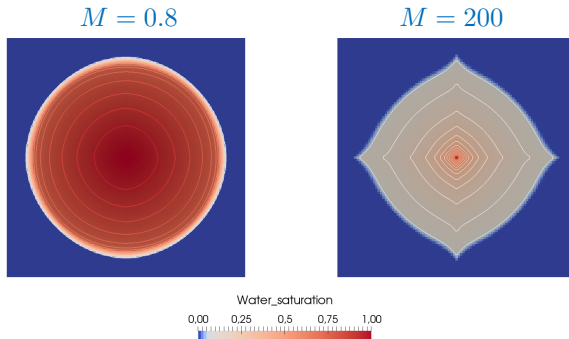
$$\phi \frac{\Delta x \Delta y}{\Delta t} (s_{i,j}^{n+1} - s_{i,j}^n) + (f(s)F)_{i+1/2,j} - (f(s)F)_{i-1/2,j} + (f(s)F)_{i,j+1/2} - (f(s)F)_{i,j-1/2} = q^{n+1}\kappa_{0,0}(i,j) \quad (5)$$

- The fractional fluxes are **upwinded** by

$$(f(s)F)_{i+1/2,j} = f(s_{i,j}^n) [F_{i+1/2,j}]^+ + f(s_{i+1,j}^n) [F_{i+1/2,j}]^-$$
$$(f(s)F)_{i,j+1/2} = f(s_{i,j}^n) [F_{i,j+1/2}]^+ + f(s_{i,j+1}^n) [F_{i,j+1/2}]^-$$

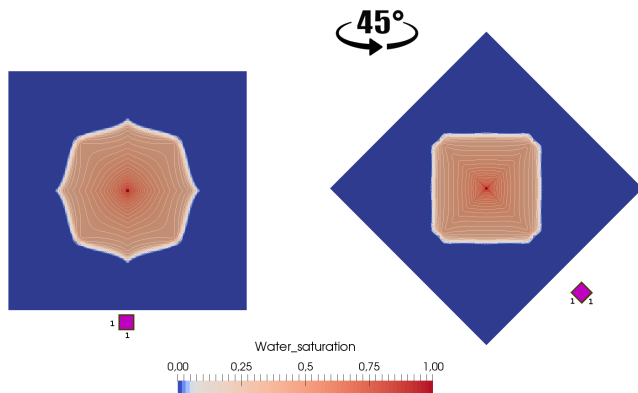
where $[\cdot]^+ = \max(\cdot, 0)$ and $[\cdot]^- = \min(\cdot, 0)$.

- The computed solution propagates faster along the axes of the grid.



- For $M \gg 1$, mathematical instabilities intrinsic to the model will amplify this numerical error.

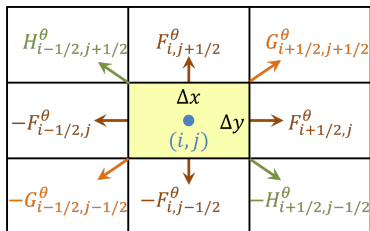
- The GOE issue is annoying for predicting production at wells.



- Alleviate the GOE. *Make the front round again* in an “cheap” way.

The 9-point scheme: Principle of construction (1)

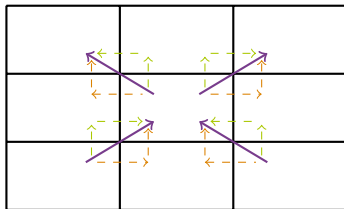
- YANOSIK-McKRACKEN (1979), COATS (1983), POTEPA (1985), DING (1990) studied a class of schemes involving 4 diagonal cells



- The two diagonal fluxes $G_{i+1/2, j+1/2}^{\theta}$ and $H_{i+1/2, j+1/2}^{\theta}$ at vertex $(i + 1/2, j + 1/2)$ do not correspond to any actual outgoing flux $\pm \int_{\sigma} \mathbf{u} \cdot \mathbf{n}$ for any physical "edge" σ .

The 9-point scheme: Principle of construction (2)

- To further decrease anisotropy, use **two** parameters instead of one. This is possible thanks to a “smarter” definition of the diagonal fluxes.



- Every diagonal flux is a linear **combination** of horizontal and vertical fluxes arising from the 5-point scheme.

- Horizontal and vertical fluxes

$$\tilde{F}_{i+1/2,j}^\theta = (1 - 4\theta_x)F_{i+1/2,j} \quad (6a)$$

$$\tilde{F}_{i,j+1/2}^\theta = (1 - 4\theta_y)F_{i,j+1/2} \quad (6b)$$

- Diagonal fluxes

$$\begin{aligned} \tilde{G}_{i+1/2,j+1/2}^\theta &= \theta_y F_{i,j+1/2} + \theta_x F_{i+1/2,j+1} + \theta_x F_{i+1/2,j} + \theta_y F_{i+1,j+1/2} \\ \tilde{H}_{i-1/2,j+1/2}^\theta &= \theta_y F_{i,j+1/2} - \theta_x F_{i-1/2,j+1} - \theta_x F_{i-1/2,j} + \theta_y F_{i-1,j+1/2} \end{aligned} \quad (6c)$$

- The free parameters $\theta = (\theta_x, \theta_y) \in [0, 1/4]^2$ are to be tuned later.

- Equation $-\operatorname{div}(\lambda(s^n)\nabla p^{n+1}) = q^{n+1}\delta_0$ is discretized as

$$\begin{aligned} \tilde{F}_{i+1/2,j}^\theta - \tilde{F}_{i-1/2,j}^\theta + \tilde{F}_{i,j+1/2}^\theta - \tilde{F}_{i,j-1/2}^\theta \\ + \tilde{G}_{i+1/2,j+1/2}^\theta - \tilde{G}_{i-1/2,j-1/2}^\theta \\ + \tilde{H}_{i-1/2,j+1/2}^\theta - \tilde{H}_{i+1/2,j-1/2}^\theta = q^{n+1}\kappa_{0,0}(i,j). \end{aligned} \quad (7)$$

- Plugging (6) into the flux balances (7) yields a linear system.

- Equation $\phi \frac{s^{n+1} - s^n}{\Delta t} + \text{div}(f(s^n)\mathbf{u}^{n+1}) = q^{n+1}\delta_0$ is discretized as

$$\begin{aligned} \phi \frac{\Delta x \Delta y}{\Delta t} (s_{i,j}^{n+1} - s_{i,j}^n) &+ (f(s)\tilde{F})_{i+1/2,j}^\theta - (f(s)\tilde{F})_{i-1/2,j}^\theta \\ &+ (f(s)\tilde{F})_{i,j+1/2}^\theta - (f(s)\tilde{F})_{i,j-1/2}^\theta \\ &+ (f(s)\tilde{G})_{i+1/2,j+1/2}^\theta - (f(s)\tilde{G})_{i-1/2,j-1/2}^\theta \\ &+ (f(s)\tilde{H})_{i-1/2,j+1/2}^\theta - (f(s)\tilde{H})_{i+1/2,j-1/2}^\theta = q^{n+1}\kappa_{0,0}(i,j) \end{aligned} \quad (8)$$

- The fractional fluxes are upwinded by

$$\begin{aligned} (f(s)\tilde{G})_{i+1/2,j+1/2}^\theta &= f(s_{i,j}^n) [\tilde{G}_{i+1/2,j+1/2}^\theta]^+ + f(s_{i+1,j+1}^n) [\tilde{G}_{i+1/2,j+1/2}^\theta]^- \\ (f(s)\tilde{H})_{i-1/2,j+1/2}^\theta &= f(s_{i,j}^n) [\tilde{H}_{i-1/2,j+1/2}^\theta]^+ + f(s_{i-1,j+1}^n) [\tilde{H}_{i-1/2,j+1/2}^\theta]^- \end{aligned}$$

1. Quantify the **anisotropy** of the numerical error along each direction, by means of Fourier analysis or equivalent modified equation
2. Introduce an **ideal** behavior of this angular error, declared to be the “least anisotropic” one.
3. Minimize the total discrepancy over all direction between the angular error of the scheme and that of the ideal one.
 - Do not seek to achieve higher-order accuracy, but simply to alter the distribution of angular error.

- Neglect the effect of averaging λ_σ^n , focus on the limit $\Delta t \rightarrow 0$, carry out Taylor expansions at $(i\Delta x, j\Delta y)$, switch to **polar** coordinates (r, φ) .
- The saturation transport scheme is a second-order approximation of

$$\phi \partial_t s + \operatorname{div}(f(s)\mathbf{u}) = (r^{-1}\partial_r(r\cdot), r^{-1}\partial_\varphi) \left\{ f'(s) \tilde{\mathbf{D}}_{\Delta x, \Delta y}^\theta \left(\begin{pmatrix} \partial_r \\ r^{-1}\partial_\varphi \end{pmatrix} s \right) \right\},$$

where the diffusion matrix in polar coordinates reads

$$\tilde{\mathbf{D}}_{\Delta x, \Delta y}^\theta = \begin{bmatrix} \tilde{A}_{\Delta x, \Delta y}^\theta(\varphi) & \tilde{B}_{\Delta x, \Delta y}^\theta(\varphi) \\ \tilde{B}_{\Delta x, \Delta y}^\theta(\varphi) & \tilde{C}_{\Delta x, \Delta y}^\theta(\varphi) \end{bmatrix}.$$

- The first diagonal entry $\tilde{A}_{\Delta x, \Delta y}^\theta(\varphi)$ reflects the **longitudinal** diffusion in the direction of the current location.

- Let $\varphi_{\bullet}^{\theta} = \arctan(\theta_x \Delta y / \theta_y \Delta x)$ be the **transition angle**, where the stencil for saturation changes due to upwinding.
- If $0 \leq \varphi \leq \varphi_{\bullet}^{\theta}$,

$$\begin{aligned}\tilde{A}_{\Delta x, \Delta y}^{\theta}(\varphi) &= \frac{\Delta x}{2} \cos^3 \varphi + \frac{\Delta y}{2} (1 - 4\theta_y) \sin^3 \varphi \\ &\quad + 2\Delta y \left(2\theta_y \frac{\Delta x}{\Delta y} + \theta_x \frac{\Delta y}{\Delta x} \right) \cos \varphi \sin^2 \varphi\end{aligned}$$

If $\varphi_{\bullet}^{\theta} \leq \varphi \leq \pi/2$,

$$\begin{aligned}\tilde{A}_{\Delta x, \Delta y}^{\theta}(\varphi) &= \frac{\Delta y}{2} \sin^3 \varphi + \frac{\Delta x}{2} (1 - 4\theta_x) \cos^3 \varphi \\ &\quad + 2\Delta x \left(2\theta_x \frac{\Delta y}{\Delta x} + \theta_y \frac{\Delta x}{\Delta y} \right) \sin \varphi \cos^2 \varphi\end{aligned}$$

- For all $\theta \in [0, 1/4]^2$,
 - $\tilde{A}_{\Delta x, \Delta y}^\theta$ is a continuous, piecewise differentiable function of $\varphi \in [0, \pi/2]$.
 - $\tilde{A}_{\Delta x, \Delta y}^\theta(\varphi)$ is of **first-order** in Δx and Δy .
 - $\tilde{A}_{\Delta x, \Delta y}^\theta(0) = \Delta x/2$ and $\tilde{A}_{\Delta x, \Delta y}^\theta(\pi/2) = \Delta y/2$ do *not* depend on θ .
- The **ideal** behavior is declared to be **affine** in $\sin^2\varphi$, i.e.,

$$A_{\Delta x, \Delta y}^*(\varphi) = \frac{\Delta x}{2} \cos^2\varphi + \frac{\Delta y}{2} \sin^2\varphi$$

- Exact minimization: find

$$\theta^* = \arg \min_{\theta \in [0, 1/4]^2} \int_0^{\pi/2} |\tilde{A}_{\Delta x, \Delta y}^{\theta}(\varphi) - A_{\Delta x, \Delta y}^*(\varphi)|^2 2 \sin \varphi \cos \varphi \, d\varphi.$$

Optimal but slightly tedious to compute.

- Approximate minimization: find $\theta^b = (\theta_x^b, \theta_y^b) \in [0, 1/4]^2$ such that

$$\begin{aligned} \varphi_{\bullet}^{\theta^b} &\text{ is as close to } \pi/4 \text{ as possible,} \\ \tilde{A}_{\Delta x, \Delta y}^{\theta^b}(\varphi_{\bullet}^{\theta^b}) &= A_{\Delta x, \Delta y}^*(\varphi_{\bullet}^{\theta^b}). \end{aligned}$$

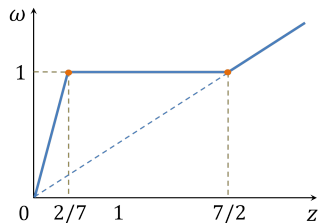
The closer $\varphi_{\bullet}^{\theta^b}$ is to $\pi/4$, the better.

- For $2/7 \leq \Delta y / \Delta x \leq 7/2$, it is possible to prescribe $\varphi_{\bullet}^{\theta^b} = \pi/4$ without violating the constraint $\theta^b \in [0, 1/4]^2$.

Choice of suboptimal parameters

Let $z = \frac{\Delta y}{\Delta x}$. Set

$$\omega(z) = \begin{cases} \frac{7}{2}z & \text{if } z \leq \frac{2}{7}, \\ 1 & \text{if } \frac{2}{7} \leq z \leq \frac{7}{2}, \\ \frac{2}{7}z & \text{if } z \geq \frac{7}{2} \end{cases}$$



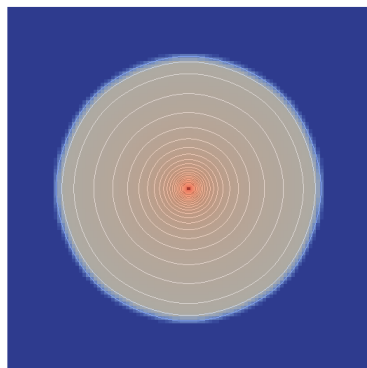
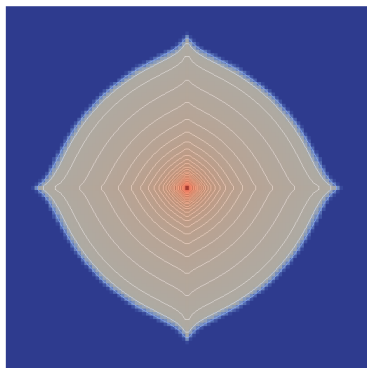
Then, the recommended values for θ^b are

$$\theta_x^b = \frac{(1 + \omega^2(z)z)\sqrt{1 + \omega^2(z)} - (1 + \omega^3(z)z)}{8\omega(z)z}$$

$$\theta_y^b = \frac{(1 + \omega^2(z)z)\sqrt{1 + \omega^2(z)} - (1 + \omega^3(z)z)}{8\omega^2(z)}$$

5-point vs. 9-point (1)

$$\Delta y / \Delta x = 1, M = 200$$



1
1

1
1

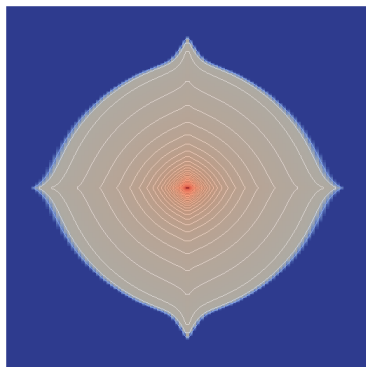
Water_saturation

0,00 0,25 0,5 0,75 1,00

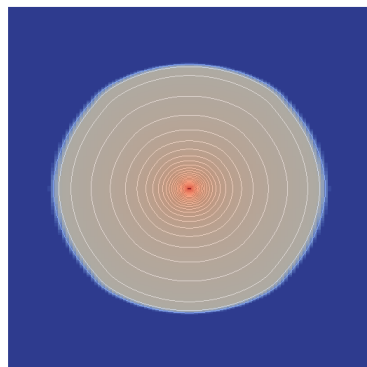


5-point vs. 9-point (2)

$$\Delta y / \Delta x = 1/3, M = 200$$



1 
3



1 
3

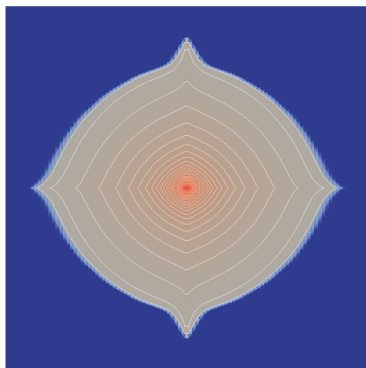
Water_saturation

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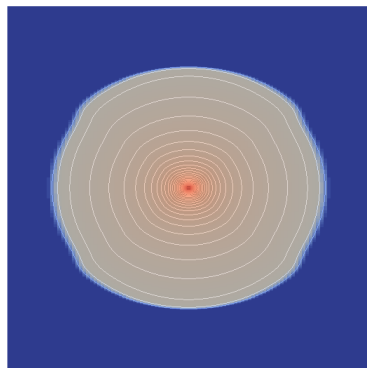


5-point vs. 9-point (3)

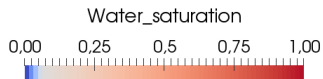
$$\Delta y / \Delta x = 1/5, M = 200$$



1
5



1
5



- Revisited an old scheme with new insights and proposed a **quantitative approach** by clarifying the notion of least anisotropic behavior.
- Incorporated more sophisticated **physical effects** such as heterogeneous permeabilities, capillary pressure and gravity.
- Extended this optimization strategy to other transport schemes, in particular that of KEILEGAVLEN-KOZDON-MALLISON (2012) where the **tuning degree of freedom is a function**.