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ON THE DENOTATIONAL SEMANTICS OF LINEAR LOGIC WITH LEAST AND GREATEST FIXED POINTS OF FORMULAS

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ABSTRACT. We develop a denotational semantics of Linear Logic with least and greatest fixed points in coherence spaces (where both fixed points are interpreted in the same way) and in coherence spaces with totality (where they have different interpretations). These constructions can be carried out in many different denotational models of LL (hypercoherences, Scott semantics, finiteness spaces etc). We also present a natural embedding of Gödel System T in LL with fixed points thus enforcing the expressive power of this system as a programming language featuring both normalization and a huge expressive power in terms of data types.

1. INTRODUCTION

Propositional Linear Logic is a well-established logical system introduced by Girard in [13]. It provides a fine-grain analysis of proofs in intuitionistic and classical logic, and more specifically of their cut-elimination. LL features indeed a well-behaved logical account of the structural rules (weakening, contraction) which are handled implicitly in intuitionistic and classical logic. For this reason, LL has many useful outcomes in the Curry-Howard based approach to the theory of programming: logical understanding of evaluation strategies (CBN and CBV correspond to two different translations from the lambda-calculus into LL, and such translations extend naturally to abstract machines), new presentation of proofs/programs (proof-nets), connections with other branches of mathematics (linear algebra, differential calculus etc), new operational semantics (geometry of interaction)...

However, just as the simply typed lambda-calculus, propositional LL is not a reasonable programming language, by lack of data-types and iteration or recursion principles. This is usually remedied by extending propositional LL to the 2nd order, thus defining a logical system in which Girard's System F [16] can be embedded. As explained in [16] for System F, this extension allows to represent many data-types, for instance the type of natural numbers can be written as $\iota = \forall \zeta !(!\zeta \multimap \zeta) \multimap !\zeta \multimap \zeta$, and then integers are represented as *Church numerals*. The resulting logical system is extremely expressive in terms of representable computable functions: all functions $\mathbb{N} \rightarrow \mathbb{N}$ whose totality can be proven in second order arithmetics are representable. In contrast, its *algorithmic* expressiveness seems quite poor:

as is well-known it is not possible to write a term $! \iota \multimap \iota$ which computes the predecessor function in one (or a uniformly bounded) number of reduction steps.

Another option to turn propositional LL into a programming language is to extend it with least and greatest fixed points of formulas. The kind of extension has early been suggested by Girard in an unpublished note [14], though the first comprehensive proof-theoretic investigation of such an extension of LL is rather recent [1]: Baelde considers an extension of Multiplicative Additive LL sequent calculus with least and greatest fixed points, axiomatized by means of two deduction rules. His main motivation for introducing this extension of LL comes from the need of powerful logical systems for formal verification of programs and the investigations on μ LL were carried out mainly from a proof-search perspectives (cut-elimination, focusing etc). However from a Curry-Howard viewpoint, fixed points allow to define inductive data-types (integers, lists, trees) and coinductive types (streams and other infinite structures). So μ LL can also be considered as a programming language and it is this approach that we develop in this paper. Admittedly the rules associated with fixed points are complex (this is especially true of the ν -introduction rule, the Park’s rule) and lead to subtle cut-elimination proof rewrite rules for which Baelde could prove a restricted form of cut-elimination, sufficient for establishing for instance that a proof of the type of integers $\mu\zeta(1 \oplus \zeta)$ necessarily reduces to an integer. There are also alternative proof-systems for the same logic, involving infinite or cyclic proofs, see [2], whose connections with the aforementioned finitary proof-system are not completely clear.

Since the proof-theory (and hence the “operational semantics”) of μ LL is sophisticated and still under development, it is particularly important to investigate its denotational semantics, whose definition does not rely on the rewrite system μ LL is equipped with. We develop here a semantics of μ LL using coherence spaces [13] equipped with a notion of totality: our model accounts for the termination of computations in μ LL and interprets least and greatest fixed points in different ways. It presents some similarities with the game-theoretic model of Clairambault [4, 5] of intuitionistic logic with least and greatest fixed points (see below).

Girard introduced coherence spaces in [12] in his denotational study of System F. In this model, types (with free variables) are interpreted as continuous (actually, stable) functors on a category of coherence spaces and embedding-retraction pairs. Our basic observation is that such functors admit “least fixed points” in a categorical sense. In the case of μ LL we have to depart from Girard’s setting because formulas must act on proofs and not only on embedding-retraction pairs: given a formula F and a proof π of $A \multimap B$, μ LL uses crucially a proof $F[\pi/\zeta]$ of $F[A/\zeta] \multimap F[B/\zeta]$ defined by induction on F (syntactic functoriality). This means that the functors interpreting formulas must act on all stable morphisms, and not only on embedding-retraction pairs. This is made possible by the fact that, in μ LL, type variables have only positive occurrences in formulas.

μ LL has a construction $\mu\zeta F$ for least fixed points and a dual construction $\nu\zeta F$ for greatest fixed points. The logical rule for introducing $\nu\zeta F$ allows to deduce $\vdash A^\perp, \nu\zeta F$ from $\vdash A^\perp, F[A/\zeta]$ (Park’s Rule). We prefer to consider a slightly generalized version of this rule: deduce $\vdash ?\Gamma, A^\perp, \nu\zeta F$ from $\vdash ?\Gamma, A^\perp, F[A/\zeta]$ (which does not increase expressiveness but makes the embedding of functional formalisms such as Gödel’s System T in μ LL much more natural). We need therefore a version of syntactic functoriality which accommodates these additional contexts $?\Gamma$. The denotational counterpart is that our functors interpreting types are equipped with a *strength* (as in [4, 5], and essentially for the same reason). All together, these requirements lead to the definition of a *variable coherence space* (VCS). Such

a functor has a fixed point which is at the same time an initial algebra and a final coalgebra, meaning that $\mu\zeta F$ and $\nu\zeta F$ have the same interpretation.

This first model, based on coherence spaces, admits morphisms which are not *total* (for instance the interpretation of $\iota \multimap \iota$, where $\iota = \mu\zeta(1 \oplus \zeta)$ is the type of integers, contains non total functions $\mathbb{N} \rightarrow \mathbb{N}$). Following [12] we equip our coherence spaces with a semantic notion of *totality*. We introduce accordingly *variable coherence spaces with totality* which are VCS's equipped with a functorial notion of totality. Using Knaster-Tarski Theorem, we show that these functors have least and greatest fixed points, which now are distinct in general. The benefit of such models is that they give an abstract and compositional account of normalization: the interpretation of any proof π of $\vdash \iota$ is an integer, which is the normal form of π : the model “computes” this integer without reducing the proof.

Contents. In Section 2 we recall basic categorical notions essential in the paper: Seely categories, Eilenberg-Moore and Kleisli categories of the exponential, strong functors and their extension to categories of comodules and, last, initial algebras (least fixed points) and final coalgebras (greatest fixed points) of (strong) functors. Section 3 is devoted to a general presentation of coherence spaces and to a well behaved notion of strong functors on coherence spaces admitting fixed points (Variable Coherence Spaces, VCSs). In this setting, there is no distinction between least fixed points and greatest fixed points, due to the presence of partial computations in the model. Section 4 endows coherence spaces with a notion of totality rejecting such partial morphisms and introduces an adapted notion of variables types (VCSTs), giving rise to a clear distinction between least and greatest fixed points in this model **CohT**. In Section 5 we present μLL and various associated concepts: functorial extension of formulas, cut elimination, interpretation of formulas and proofs in the model **CohT**. The idea that a semantic notion of totality enforces a distinction between least and greatest fixed points is of course not new, it is essential for instance in [4, 5]. In Section 6 we present various examples, stressing in particular that Gödel's System T can be embedded in μLL (following Clairambault [5]) and hence showing that this system has a significant expressive power from the viewpoint of the Curry-Howard correspondence. We also analyze in our model the encoding of exponentials using least and greatest fixed points proposed in [1]; we show in particular that these exponentials do not give rise to a Seely category, thus justifying our choice of considering the whole system μLL , with exponentials. As an outcome of the paper, Section 7 introduces a quite simple and general notion of categorical model of classical μLL , of which the model **CohT** is an instance.

Related work. Most importantly, we want to mention again the work of Pierre Clairambault (see [4] and the long version [5]) who investigates the denotational semantics of an extension of intuitionistic logic with least and greatest fixed points. Our categorical notion of model of classical μLL of Definition 7.1 can probably be seen as a “linearized version” of his notion of μ -closed category (an extension of the notion of cartesian closed categories with least and greatest fixed points) — perhaps through some kind of Kleisli construction — though our setting seems simpler as we do not need contravariant strong functors, only covariant ones. The concrete model considered by Clairambault is based on games and features a notion of totality guaranteeing that all computations terminate and enforcing the distinction between least and greatest fixed points, just as in the present work. Due to technical limitations intrinsically related to game semantics, Clairambault's model is

restricted to the *pseudo-polynomial* fragment where free variables never appear at the right hand side of an implication. No such limitations are necessary in our coherence space interpretation.

Remark 1.1. We use the following notational conventions: \vec{a} stands for a list (a_1, \dots, a_n) . When we write natural transformations, we very often omit the objects where they are taken and prefer to keep these objects implicit for the sake of readability, because they can easily be retrieved from the context. If $\mathcal{F} : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ is a functor and A is an object of \mathcal{A} (notation $A \in \mathcal{A}$) then $\mathcal{F}_A : \mathcal{B} \rightarrow \mathcal{C}$ is the functor defined by $\mathcal{F}_A(B) = \mathcal{F}(A, B)$ and $\mathcal{F}_A(f) = \mathcal{F}(\text{Id}_A, f)$.

2. CATEGORICAL MODELS OF LL

A model of LL consists of the following data (our main reference is the notion of a *Seely category* as presented in [17]. We refer to that survey for all the technical material that we do not record here).

A symmetric monoidal closed category $(\mathcal{L}, \otimes, 1, \lambda, \rho, \alpha, \gamma)$ where $\lambda_X \in \mathcal{L}(1 \otimes X, X)$, $\rho_X \in \mathcal{L}(X \otimes 1, X)$, $\alpha_{X,Y,Z} \in \mathcal{L}((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$ and $\gamma_{X,Y} \in \mathcal{L}(X \otimes Y, Y \otimes X)$ are natural isomorphisms satisfying coherence diagrams that we do not record here. We use $X \multimap Y$ for the object of linear morphisms from X to Y , $\text{ev} \in \mathcal{L}((X \multimap Y) \otimes X, Y)$ for the evaluation morphism and cur for the linear curryfication map $\mathcal{L}(Z \otimes X, Y) \rightarrow \mathcal{L}(Z, X \multimap Y)$. We assume this SMCC to be $*$ -autonomous with dualizing object \perp . We use X^\perp for the object $X \multimap \perp$ of \mathcal{L} (the dual, or linear negation, of X).

\mathcal{L} is cartesian with final object \top , product $(X_1 \& X_2, (\pi_i : X_1 \& X_2 \rightarrow X_i)_{i=1,2})$. By $*$ -autonomy \mathcal{L} is cocartesian with initial object 0 , coproduct \oplus and injections $\bar{\pi}_i$.

We are given a comonad $!_ : \mathcal{L} \rightarrow \mathcal{L}$ with counit $\text{der}_X \in \mathcal{L}(!X, X)$ (*derelection*) and comultiplication $\text{dig}_X \in \mathcal{L}(!X, !!X)$ (*digging*) together with a strong symmetric monoidal structure (Seely natural isos $\text{m}^0 : 1 \rightarrow !\top$ and m^2 with $\text{m}_{X_1, X_2}^2 : !X_1 \otimes !X_2 \rightarrow !(X_1 \& X_2)$, we use n^0 and n_{X_1, X_2}^2 for the inverses of these isos) for the functor $!_$, from the symmetric monoidal category $(\mathcal{L}, \&)$ to the symmetric monoidal category (\mathcal{L}, \otimes) satisfying an additional coherence condition wrt. dig . It is also important to remember that this strong monoidal structure allows to define a weak monoidal structure (μ^0, μ^2) of “!” from (\mathcal{L}, \otimes) to itself. More precisely $\mu^0 \in \mathcal{L}(1, !1)$ and $\mu_{X_1, X_2}^2 \in \mathcal{L}(!X_1 \otimes !X_2, !(X_1 \otimes X_2))$ are defined using m^0 and m^2 . Also, for each object X of \mathcal{L} , there is a canonical structure of commutative \otimes -comonoid on $!X$ given by $w_X \in \mathcal{L}(!X, 1)$ and $\text{contr}_X \in \mathcal{L}(!X, !X \otimes !X)$. The definition of these morphisms involves all the structure of “!” explained above, and in particular the Seely isos. In Section 6.4, we will use the fact that the following equation holds

$$\text{n}_{X_1, X_2}^2 = (!\pi_1 \otimes !\pi_2) \text{contr}_{X_1 \& X_2} \quad (2.1)$$

and also, as a consequence:

$$\begin{aligned} w_{X_1 \& X_2} \text{m}_{X_1, X_2}^2 &= w_{X_1} \otimes w_{X_2} \\ \text{der}_{X_1 \& X_2} \text{m}_{X_1, X_2}^2 &= \langle \text{der}_X \otimes w_Y, w_X \otimes \text{der}_Y \rangle \\ \text{contr}_{X_1 \& X_2} \text{m}_{X_1, X_2}^2 &= (\text{m}_{X_1, X_2}^2 \otimes \text{m}_{X_1, X_2}^2) \gamma_{2,3} (\text{contr}_X \otimes \text{contr}_Y) \end{aligned} \quad (2.2)$$

We use $?_$ for the “De Morgan dual” of $!_$: $?X = (!(X^\perp))^\perp$ and similarly for morphisms. It is a monad on \mathcal{L} .

2.1. Eilenberg-Moore category and free comodules. It is then standard to define the category $\mathcal{L}^!$ of !-coalgebras. An object of this category is a pair $P = (\underline{P}, h_P)$ where $\underline{P} \in \mathcal{L}$ and $h_P \in \mathcal{L}(\underline{P}, !\underline{P})$ is such that $\text{der}_{\underline{P}} h_P = \text{Id}$ and $\text{dig}_{\underline{P}} h_P = !h_P h_P$. Then $f \in \mathcal{L}^!(P, Q)$ iff $f \in \mathcal{L}(\underline{P}, \underline{Q})$ such that $h_Q f = !f h_P$. The functor $!_-$ can be seen as a functor from \mathcal{L} to $\mathcal{L}^!$ mapping X to $(!X, \text{dig}_X)$ and $f \in \mathcal{L}(X, Y)$ to $!f$. It is right adjoint to the forgetful functor $\mathcal{L}^! \rightarrow \mathcal{L}$. Given $f \in \mathcal{L}(\underline{P}, X)$, we use $f^! \in \mathcal{L}^!(P, !X)$ for the morphism associated with f by this adjunction, one has $f^! = !f h_P$. If $g \in \mathcal{L}^!(Q, P)$, we have $f^! g = (f g)^!$.

Then $\mathcal{L}^!$ is cartesian (with product of shape $P \otimes Q = (\underline{P} \otimes \underline{Q}, h_{P \otimes Q})$ and final object $(1, h_1)$, still denoted as 1). Here is the definition of $h_{P \otimes Q}$:

$$\underline{P} \otimes \underline{Q} \xrightarrow{h_P \otimes h_Q} !\underline{P} \otimes !\underline{Q} \xrightarrow{\mu_{\underline{P}, \underline{Q}}^2} !(P \otimes Q).$$

This category is also cocartesian with coproduct of shape $P \oplus Q = (\underline{P} \oplus \underline{Q}, h_{P \oplus Q})$ and initial object $(0, h_0)$ still denoted as 0. Here is the definition of $h_{P \oplus Q}$. One first defines $h_1 : \underline{P} \rightarrow !(P \oplus Q)$ as

$$\underline{P} \xrightarrow{h_P} !\underline{P} \xrightarrow{! \pi_1} !(P \oplus Q)$$

and similarly one defines $h_2 : \underline{Q} \rightarrow !(P \oplus Q)$ and then $h_{P \oplus Q}$ is defined as the unique morphism $\underline{P} \oplus \underline{Q} \rightarrow !(P \oplus Q)$ such that $h_{P \oplus Q} \pi_i = h_i$ for $i = 1, 2$.

More details can be found in [10]. We use $\text{contr}_P \in \mathcal{L}^!(P, P \otimes P)$ (*contraction*) for the diagonal and $w_P \in \mathcal{L}^!(P, 1)$ (*weakening*) for the unique morphism to the final object. These morphisms turn \underline{P} into a commutative \otimes -comonoid, and are defined as

$$\underline{P} \xrightarrow{h_P} !\underline{P} \xrightarrow{!(\text{Id}, \text{Id})} !(P \& P) \xrightarrow{(m^2)^{-1}} !\underline{P} \otimes !\underline{P} \xrightarrow{\text{der}_{\underline{P}} \otimes \text{der}_{\underline{P}}} \underline{P} \otimes \underline{P}$$

and $\underline{P} \xrightarrow{h_P} !\underline{P} \xrightarrow{! \tau_P} !(T) \xrightarrow{(m^0)^{-1}} 1$.

2.1.1. The model of free comodules on a given coalgebra. Given an object P of $\mathcal{L}^!$, we can define a functor $\text{Comod}_P : \mathcal{L} \rightarrow \mathcal{L}$ which maps an object X to $\underline{P} \otimes X$ and a morphism f to $\underline{P} \otimes f$. This functor is clearly a comonad (with structure maps defined using w_P , contr_P and the monoidal structure of \mathcal{L}). A coalgebra for this comonad is a P -comodule¹. It was observed by Girard in [15] that the Kleisli category $\mathcal{L}[P] = \mathcal{L}_{\text{Comod}_P}$ of this comonad (that is, the category of free P -comodules) is in turn a model of LL with operations on objects defined in the same way as in \mathcal{L} , and using the coalgebra structure of P on morphisms. Let us summarize this construction which will be used in the sequel. First let $f_i \in \mathcal{L}[P](X_i, Y_i)$ for $i = 1, 2$. Then we define $f_1 \otimes_P f_2 \in \mathcal{L}[P](X_1 \otimes X_2, Y_1 \otimes Y_2)$ as

$$\underline{P} \otimes X_1 \otimes X_2 \xrightarrow{\text{contr}_{\underline{P}} \otimes \text{Id}} \underline{P} \otimes \underline{P} \otimes X_1 \otimes X_2 \xrightarrow{\sim} \underline{P} \otimes X_1 \otimes \underline{P} \otimes X_2 \xrightarrow{f_1 \otimes f_2} Y_1 \otimes Y_2.$$

The object of linear morphisms from X to Y in $\mathcal{L}[P]$ is $X \multimap Y$, and the evaluation morphism $\text{ev}_P \in \mathcal{L}[P]((X \multimap Y) \otimes X, Y)$ is simply (keeping implicit the \otimes -monoidality isos) $\underline{P} \otimes (X \multimap Y) \otimes X \xrightarrow{w_P \otimes \text{Id}} (X \multimap Y) \otimes X \xrightarrow{\text{ev}} Y$. Then it is easy to check that if $f \in$

¹This is just the dual notion of the standard algebraic notion of an M -module which can be defined as soon as a commutative \otimes -monoid M is given.

$$\begin{array}{ccc}
(!X_1 \otimes !X_2) \otimes \overline{\mathbb{F}}(\vec{Y}) & \xrightarrow{m^2 \otimes \overline{\mathbb{F}}(\vec{Y})} & !(X_1 \& X_2) \otimes \overline{\mathbb{F}}(\vec{Y}) \\
\downarrow !X_1 \otimes \widehat{\mathbb{F}}_{X_2, \vec{Y}} & & \downarrow \widehat{\mathbb{F}}_{X_1 \& X_2, \vec{Y}} \\
!X_1 \otimes \overline{\mathbb{F}}(!X_2 \otimes \vec{Y}) & & 1 \otimes \overline{\mathbb{F}}(\vec{Y}) \xrightarrow{m^0 \otimes \overline{\mathbb{F}}(\vec{Y})} !\top \otimes \overline{\mathbb{F}}(\vec{Y}) \\
\widehat{\mathbb{F}}_{X_1, !X_2 \otimes \vec{Y}} \downarrow & & \wr \downarrow \widehat{\mathbb{F}}_{\top, \vec{Y}} \\
\overline{\mathbb{F}}(!X_1 \otimes !X_2 \otimes \overline{\mathbb{F}}(\vec{Y})) & \xrightarrow{\overline{\mathbb{F}}(m^2 \otimes \vec{Y})} & \overline{\mathbb{F}}(!X_1 \& X_2) \otimes \overline{\mathbb{F}}(\vec{Y}) \\
& & \overline{\mathbb{F}}(1 \otimes \vec{Y}) \xrightarrow{\overline{\mathbb{F}}(m^0 \otimes \vec{Y})} \overline{\mathbb{F}}(!\top \otimes \vec{Y})
\end{array}$$

Figure 1: Monoidality diagrams for strong functors

$\mathcal{L}[P](Z \otimes X, Y)$, that is $f \in \mathcal{L}(\underline{P} \otimes Z \otimes X, Y)$, the morphism $\text{cur } f \in \mathcal{L}[P](Z, X \multimap Y)$ satisfies the required monoidal closeness equations. With these definitions, the category $\mathcal{L}[P]$ is *-autonomous, with \perp as dualizing object. Specifically, given $f \in \mathcal{L}[P](X, Y)$, then $f^{\perp[P]}$

is the following composition of morphisms: $\underline{P} \otimes Y^{\perp} \xrightarrow{\underline{P} \otimes f^{\perp}} \underline{P} \otimes (\underline{P} \multimap X^{\perp}) \xrightarrow{\text{ev}} X^{\perp}$, using implicitly the iso between $(Z \otimes X)^{\perp}$ and $Z \multimap X^{\perp}$, and the *-autonomy of \mathcal{L} allows to prove that indeed $f^{\perp[P]\perp[P]} = f$.

The category $\mathcal{L}[P]$ is easily seen to be cartesian with \top as final object, $X_1 \& X_2$ as cartesian product (and projections defined in the obvious way, using the projections of \mathcal{L} and the counit w_P). Last we define a functor $!_P - : \mathcal{L}[P] \rightarrow \mathcal{L}[P]$ by $!_P X = !X$ and, given $f \in \mathcal{L}[P](X, Y)$, we define $!_P f \in \mathcal{L}[P](!X, !Y)$ as $\underline{P} \otimes !X \xrightarrow{h_P \otimes !X} !\underline{P} \otimes !X \xrightarrow{\mu^2} !(\underline{P} \otimes X) \xrightarrow{!f} !Y$ and this functor is equipped with a comonad structure $(\text{der}[P], \text{dig}[P])$ easily defined using der , dig and w_P (for this crucial construction, we need P to be a !-coalgebra and not simply a commutative \otimes -comonoid; notice however that if ! is the free exponential, as in [15], the latter condition implies the former).

2.2. Strong functors on \mathcal{L} . Given $n \in \mathbb{N}$, an n -ary strong functor on \mathcal{L} is a pair $\mathbb{F} = (\overline{\mathbb{F}}, \widehat{\mathbb{F}})$ where $\overline{\mathbb{F}} : \mathcal{L}^n \rightarrow \mathcal{L}$ is a functor and $\widehat{\mathbb{F}}_{X, \vec{Y}} \in \mathcal{L}(!X \otimes \overline{\mathbb{F}}(\vec{Y}), \overline{\mathbb{F}}(!X \otimes \vec{Y}))$ is a natural transformation, called the *strength* of \mathbb{F} . We use the notation $Z \otimes (Y_1, \dots, Y_n) = (Z \otimes Y_1, \dots, Z \otimes Y_n)$. It is assumed moreover that the diagrams of Figure 1 commute, expressing a monoidality of this strength.

The main purpose of this definition is that one can then define a functor $\mathbb{F}[P] : \mathcal{L}[P]^n \rightarrow \mathcal{L}[P]$ as follows. First one sets $\mathbb{F}[P](\vec{X}) = \overline{\mathbb{F}}(\vec{X})$. Then, given $\vec{f} \in \mathcal{L}[P]^n(\vec{X}, \vec{Y})$ we define $\mathbb{F}[P](\vec{f}) \in \mathcal{L}[P](\mathbb{F}(\vec{X}), \mathbb{F}(\vec{Y}))$ as

$$\underline{P} \otimes \mathbb{F}(\vec{X}) \xrightarrow{h_P \otimes \text{Id}} !\underline{P} \otimes \mathbb{F}(\vec{X}) \xrightarrow{\widehat{\mathbb{F}}} \mathbb{F}(!\underline{P} \otimes \vec{X}) \xrightarrow{\overline{\mathbb{F}}(\text{der}_{\underline{P} \otimes \vec{X}})} \mathbb{F}(\underline{P} \otimes \vec{X}) \xrightarrow{\mathbb{F}(\vec{f})} \mathbb{F}(\vec{Y}).$$

The fact that we have defined a functor results from the two monoidality diagrams of Figure 1 and from the definition of w_P and contr_P based on the Seelye isomorphisms.

Operations on strong functors. There is an obvious unary identity strong functor \mathcal{I} and for each object Y of \mathcal{L} there is an n -ary Y -valued constant strong functor \mathcal{K}^X ; in the first case the strength natural transformation is the identity morphism and in the second case, it is defined using $w_{!X}$. Let \mathbb{F} be an n -ary strong functor and $\mathbb{G}_1, \dots, \mathbb{G}_n$ be k -ary strong functors.

Then one defines a k -ary strong functor $\mathbb{H} = \mathbb{F} \circ (\mathbb{G}_1, \dots, \mathbb{G}_n)$: the functorial component $\overline{\mathbb{H}}$ is defined in the obvious compositional way. The strength is defined as follows

$$!X \otimes \overline{\mathbb{H}}(\vec{Y}) \xrightarrow{\widehat{\mathbb{F}}} \overline{\mathbb{F}}((!X \otimes \overline{\mathbb{G}}_i(\vec{Y}))_{i=1}^n) \xrightarrow{\overline{\mathbb{F}}((\widehat{\mathbb{G}}_i)_{i=1}^k)} \overline{\mathbb{F}}((\overline{\mathbb{G}}_i(!X \otimes \vec{Y}))_{i=1}^n) = \overline{\mathbb{H}}(!X \otimes \vec{Y})$$

and is easily seen to satisfy the required monoidality commutations.

Given an n -ary strong functor, we can define its *De Morgan dual* \mathbb{F}^\perp which is also an n -ary strong functor. On objects, we set $\overline{\mathbb{F}}^\perp(\vec{Y}) = \overline{\mathbb{F}}(\vec{Y}^\perp)^\perp$ and similarly for morphisms. The strength of \mathbb{F}^\perp is defined as the Curry transpose of the following morphism (remember that $!X \multimap \vec{Y}^\perp = (!X \otimes \vec{Y})^\perp$ up to canonical iso):

$$\begin{array}{c} !X \otimes \overline{\mathbb{F}}(\vec{Y}^\perp)^\perp \otimes \overline{\mathbb{F}}(!X \multimap \vec{Y}^\perp) \\ \downarrow \sim \\ !X \otimes \overline{\mathbb{F}}(!X \multimap \vec{Y}^\perp) \otimes \overline{\mathbb{F}}(\vec{Y}^\perp)^\perp \\ \downarrow \widehat{\mathbb{F}} \otimes \text{Id} \\ \overline{\mathbb{F}}(!X \otimes (!X \multimap \vec{Y}^\perp)) \otimes \overline{\mathbb{F}}(\vec{Y}^\perp)^\perp \\ \overline{\mathbb{F}}(\text{ev}) \otimes \text{Id} \downarrow \\ \overline{\mathbb{F}}(\vec{Y}^\perp) \otimes \overline{\mathbb{F}}(\vec{Y}^\perp)^\perp \\ \text{ev } \gamma \downarrow \\ \perp \end{array}$$

Then it is possible to prove, using the $*$ -autonomy of \mathcal{L} , that $\mathbb{F}^{\perp\perp}$ and \mathbb{F} are canonically isomorphic (as strong functors)².

Lemma 2.1. $(\mathbb{F} \circ (\mathbb{G}_1, \dots, \mathbb{G}_n))^\perp = \mathbb{F}^\perp \circ (\mathbb{G}_1^\perp, \dots, \mathbb{G}_n^\perp)$ up to canonical iso.

Proof. Results straightforwardly from the definition of \mathbb{F}^\perp and from the canonical iso between $\mathbb{F}^{\perp\perp}$ and \mathbb{F} . \blacktriangleleft

The bifunctor \otimes can be turned into a strong functor: one defines the strength as

$!X \otimes Y_1 \otimes Y_2 \xrightarrow{\text{contr}_{!X} \otimes \text{Id}} !X \otimes !X \otimes Y_1 \otimes Y_2 \xrightarrow{\sim} !X \otimes Y_1 \otimes !X \otimes Y_2$. By De Morgan duality, this endows \mathfrak{A} with a strength as well. The bifunctor \oplus is also endowed with a strength, simply using the distributivity of \otimes over \oplus (which in turn results from the fact that \mathcal{L} is symmetric monoidal closed). By duality again, $\&$ inherits a strength as well. Last the unary functor “!” can be equipped with a strength as follows $!X \otimes !Y \xrightarrow{\text{dig}_X \otimes !X} !!X \otimes !Y \xrightarrow{\mu^2} !(X \otimes Y)$. The functors on $\mathcal{L}[P]$ that these strong functors induce coincide with the corresponding operations defined in Section 2.1.1.

2.3. Fixed Points of functors. The following definitions and properties are quite standard in the literature on fixed points of functors.

Definition 2.2. Let \mathcal{A} be a category and let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ be a functor. A *coalgebra* of \mathcal{F} is a pair (A, f) where A is an object of \mathcal{A} and $f \in \mathcal{A}(A, \mathcal{F}(A))$. Given two coalgebras (A, f) and (A', f') of \mathcal{F} , a coalgebra morphism from (A, f) to (A', f') is an $h \in \mathcal{A}(A, A')$ such that the following diagram commutes

²In the concrete settings considered in this paper, these canonical isos are actuality identity maps.

$$\begin{array}{ccc}
A & \xrightarrow{h} & A' \\
f \downarrow & & \downarrow f' \\
\mathcal{F}(A) & \xrightarrow{\mathcal{F}(h)} & \mathcal{F}(A')
\end{array}$$

The category of coalgebras of the functor \mathcal{F} will be denoted as $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})$. The notion of algebra of an endofunctor is defined dually (reverse the directions of the arrows f and f') and the corresponding category is denoted as $\mathbf{Alg}_{\mathcal{A}}(\mathcal{F})$.

Lemma 2.3 (Lambek's Theorem). *If (A, f) is a final object in $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})$ then f is an iso.*

In the sequel, we will always assume that this iso is the identity (because this holds in the concrete situations we consider in this paper) so that this final object $(\nu\mathcal{F}, \text{Id})$ satisfies $\mathcal{F}(\nu\mathcal{F}) = \nu\mathcal{F}$. We focus on coalgebras rather than algebras for reasons which will become clear when we shall deal with fixed points of strong functors.

This universal property of $\nu\mathcal{F}$ gives us a powerful tool for proving equalities of morphisms.

Lemma 2.4. *Let A be an object of \mathcal{A} and let $f_1, f_2 \in \mathcal{A}(A, \nu\mathcal{F})$. If there exists $l \in \mathcal{A}(A, \mathcal{F}(A))$ such that $\mathcal{F}(f_i)l = f_i$ for $i = 1, 2$, then $f_1 = f_2$.*

Proof. The assumption means that both f_1 and f_2 are coalgebra morphisms from (A, l) to the final coalgebra, so they must be equal. \blacktriangleleft

Lemma 2.5. *Let $\mathcal{F} : \mathcal{B} \times \mathcal{A} \rightarrow \mathcal{A}$ be a functor such that, for all $B \in \mathcal{B}$, the category $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F}_B)$ has a final object. Then there is a functor $\nu\mathcal{F}$ such that $(\nu\mathcal{F}(B), \text{Id})$ is the final object of $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F}_B)$ (so that $\mathcal{F}(B, \nu\mathcal{F}(B)) = \nu\mathcal{F}(B)$) for each $B \in \mathcal{B}$, and, for each $g \in \mathcal{B}(B, B')$, $\nu\mathcal{F}(g)$ is uniquely characterized by $\mathcal{F}(g, \nu\mathcal{F}(g)) = \nu\mathcal{F}(g)$.*

Proof. We have $\mathcal{F}(g, \nu\mathcal{F}(B)) \in \mathcal{A}(\nu\mathcal{F}(B), \mathcal{F}(B', \nu\mathcal{F}(B)))$ thus defining a $\mathcal{F}_{B'}$ -coalgebra structure on $\nu\mathcal{F}(B)$ and hence there exists a unique morphism $\nu\mathcal{F}(g)$ such that

$$\mathcal{F}(B', \nu\mathcal{F}(g)) \mathcal{F}(g, \nu\mathcal{F}(B)) = \nu\mathcal{F}(g),$$

that is $\mathcal{F}(g, \nu\mathcal{F}(g)) = \nu\mathcal{F}(g)$.

Functoriality follows: consider also $g' \in \mathcal{B}(B', B'')$, then we know that $h = \nu\mathcal{F}(g'g)$ satisfies $\mathcal{F}(g'g, h) = h$ by the definition above. Now $h' = \nu\mathcal{F}(g')\nu\mathcal{F}(g)$ satisfies the same equation by functoriality of \mathcal{F} and because $\mathcal{F}(g, \nu\mathcal{F}(g)) = \nu\mathcal{F}(g)$ and $\mathcal{F}(g', \nu\mathcal{F}(g')) = \nu\mathcal{F}(g')$, and hence $h' = h$ by Lemma 2.5, taking $l = \mathcal{F}(g'g, \nu\mathcal{F}(B))$. In the same way one proves that $\nu\mathcal{F}(\text{Id}) = \text{Id}$. \blacktriangleleft

We consider now the same $\nu\mathcal{F}$ operation applied to strong functors on a model \mathcal{L} of LL. Let \mathbb{F} be an $n + 1$ -ary strong functor on \mathcal{L} (so that $\overline{\mathbb{F}}$ is a functor $\mathcal{L}^{n+1} \rightarrow \mathcal{L}$). Assume that for each $\vec{X} \in \mathcal{L}^n$ the category $\mathbf{Coalg}_{\mathcal{L}}(\overline{\mathbb{F}}_{\vec{X}})$ has a final object. We have defined a functor $\nu\overline{\mathbb{F}} : \mathcal{L}^n \rightarrow \mathcal{L}$ uniquely characterized by $\overline{\mathbb{F}}(\vec{X}, \nu\overline{\mathbb{F}}(\vec{X})) = \nu\overline{\mathbb{F}}(\vec{X})$ and $\overline{\mathbb{F}}(\vec{f}, \nu\overline{\mathbb{F}}(\vec{f})) = \nu\overline{\mathbb{F}}(\vec{f})$ for all $\vec{f} \in \mathcal{L}^n(\vec{X}, \vec{X}')$ (Lemma 2.5). For each $Y, \vec{X} \in \mathcal{L}$, we define $\widehat{\nu\overline{\mathbb{F}}}_{Y, \vec{X}} \in \mathcal{L}(!Y \otimes \nu\overline{\mathbb{F}}(\vec{X}), \nu\overline{\mathbb{F}}(!Y \otimes \vec{X}))$. We have

$$!Y \otimes \nu\overline{\mathbb{F}}(\vec{X}) = !Y \otimes \overline{\mathbb{F}}(\vec{X}, \nu\overline{\mathbb{F}}(\vec{X})) \xrightarrow{\widehat{\mathbb{F}}_{Y, (\vec{X}, \nu\overline{\mathbb{F}}(\vec{X}))}} \overline{\mathbb{F}}(!Y \otimes \vec{X}, !Y \otimes \nu\overline{\mathbb{F}}(\vec{X}))$$

exhibiting a $\widehat{\mathbb{F}}_{!Y \otimes \vec{X}}$ -coalgebra structure on $!Y \otimes \overline{\nu\mathbb{F}}(\vec{X})$. Since $\overline{\nu\mathbb{F}}(!Y \otimes \vec{X})$ is the final coalgebra of the functor $\widehat{\mathbb{F}}_{!Y \otimes \vec{X}}$, we define $\widehat{\nu\mathbb{F}}_{Y, \vec{X}}$ as the unique morphism $!Y \otimes \overline{\nu\mathbb{F}}(\vec{X}) \rightarrow \overline{\nu\mathbb{F}}(!Y \otimes \vec{X})$ such that the following diagram commutes

$$\begin{array}{ccc} !Y \otimes \overline{\nu\mathbb{F}}(\vec{X}) & \xrightarrow{\widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))}} & \overline{\mathbb{F}}(!Y \otimes \vec{X}, !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \\ & \searrow \widehat{\nu\mathbb{F}}_{(Y, \vec{X})} & \downarrow \overline{\mathbb{F}}(!Y \otimes \vec{X}, \widehat{\nu\mathbb{F}}_{Y, \vec{X}}) \\ & & \overline{\mathbb{F}}(!Y \otimes \vec{X}, \overline{\nu\mathbb{F}}(!Y \otimes \vec{X})) = \overline{\nu\mathbb{F}}(!Y \otimes \vec{X}) \end{array} \quad (2.3)$$

Lemma 2.6. *Let \mathbb{F} be an $n + 1$ -ary strong functor on \mathcal{L} such that for each $\vec{X} \in \mathcal{L}^n$, the category $\mathbf{Coalg}_{\mathcal{L}}(\widehat{\mathbb{F}}_{\vec{X}})$ has a final object $\nu\widehat{\mathbb{F}}_{\vec{X}}$. Then there is a unique n -ary strong functor $\nu\mathbb{F}$ such that $\overline{\nu\mathbb{F}}(\vec{X}) = \nu\widehat{\mathbb{F}}_{\vec{X}}$ (and hence $\overline{\mathbb{F}}(\vec{X}, \overline{\nu\mathbb{F}}(\vec{X})) = \overline{\nu\mathbb{F}}(\vec{X})$),*

- $\overline{\mathbb{F}}(\vec{f}, \overline{\nu\mathbb{F}}(\vec{f})) = \overline{\nu\mathbb{F}}(\vec{f})$ for all $\vec{f} \in \mathcal{L}^n(\vec{X}, \vec{X}')$
- and $\overline{\mathbb{F}}(!Y \otimes \vec{X}, \widehat{\nu\mathbb{F}}_{Y, \vec{X}}) \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} = \widehat{\nu\mathbb{F}}_{Y, \vec{X}}$.

Proof. Let us prove the naturality of $\widehat{\nu\mathbb{F}}$ so let $\vec{f} \in \mathcal{L}^n(\vec{X}, \vec{X}')$ and $g \in \mathcal{L}(Y, Y')$, we must prove that the following diagram commutes

$$\begin{array}{ccc} !Y \otimes \overline{\nu\mathbb{F}}(\vec{X}) & \xrightarrow{\widehat{\nu\mathbb{F}}_{Y, \vec{X}}} & \overline{\nu\mathbb{F}}(!Y \otimes \vec{X}) \\ !g \otimes \overline{\nu\mathbb{F}}(\vec{f}) \downarrow & & \downarrow \overline{\nu\mathbb{F}}(!g \otimes \vec{f}) \\ !Y' \otimes \overline{\nu\mathbb{F}}(\vec{X}') & \xrightarrow{\widehat{\nu\mathbb{F}}_{Y', \vec{X}'}} & \overline{\nu\mathbb{F}}(!Y' \otimes \vec{X}') \end{array}$$

Let $h_1 = \widehat{\nu\mathbb{F}}_{Y', \vec{X}'}(!g \otimes \overline{\nu\mathbb{F}}(\vec{f}))$ and $h_2 = \overline{\nu\mathbb{F}}(!g \otimes \vec{f}) \widehat{\nu\mathbb{F}}_{Y, \vec{X}}$ be the two morphisms we must prove equal. We use Lemma 2.4, taking the following morphism l .

$$\begin{array}{c} !Y \otimes \overline{\nu\mathbb{F}}(\vec{X}) = !Y \otimes \overline{\mathbb{F}}(\vec{X}, \overline{\nu\mathbb{F}}(\vec{X})) \\ \downarrow \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\ \overline{\mathbb{F}}(!Y \otimes \vec{X}, !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \\ \downarrow \overline{\mathbb{F}}(!g \otimes \vec{f}, \text{Id}) \\ \overline{\mathbb{F}}(!Y' \otimes \vec{X}', !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \end{array}$$

With these notations we have

$$\begin{aligned} \overline{\mathbb{F}}(!Y' \otimes \vec{X}', h_1) l &= \overline{\mathbb{F}}(!Y' \otimes \vec{X}', \widehat{\nu\mathbb{F}}_{Y', \vec{X}'}) \overline{\mathbb{F}}(!Y' \otimes \vec{X}', !g \otimes \overline{\nu\mathbb{F}}(\vec{f})) \\ &= \overline{\mathbb{F}}(!g \otimes \vec{f}, !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\ &= \overline{\mathbb{F}}(!Y' \otimes \vec{X}', \widehat{\nu\mathbb{F}}_{Y', \vec{X}'}) \overline{\mathbb{F}}(!g \otimes \vec{f}, !g \otimes \overline{\nu\mathbb{F}}(\vec{f})) \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\ &= \overline{\mathbb{F}}(!Y' \otimes \vec{X}', \widehat{\nu\mathbb{F}}_{Y', \vec{X}'}) \widehat{\mathbb{F}}_{Y', (\vec{X}', \overline{\nu\mathbb{F}}(\vec{X}'))} (!g \otimes \overline{\mathbb{F}}(\vec{f}, \overline{\nu\mathbb{F}}(\vec{f}))) \\ &\quad \text{by naturality of } \widehat{\mathbb{F}} \\ &= \widehat{\nu\mathbb{F}}_{Y', \vec{X}'} (!g \otimes \overline{\mathbb{F}}(\vec{f}, \overline{\nu\mathbb{F}}(\vec{f}))) \text{ by (2.3)} \\ &= \widehat{\nu\mathbb{F}}_{Y', \vec{X}'} (!g \otimes \overline{\nu\mathbb{F}}(\vec{f})) \text{ by Lemma 2.5} \end{aligned}$$

so that $\overline{\mathbb{F}}(!Y' \otimes \vec{X}', h_1) l = h_1$ as required. On the other hand we have

$$\begin{aligned}
\overline{\mathbb{F}}(!Y' \otimes \vec{X}', h_2) l &= \overline{\mathbb{F}}(!Y' \otimes \vec{X}', \overline{\nu\mathbb{F}}(!g \otimes \vec{f})) \overline{\mathbb{F}}(!Y' \otimes \vec{X}', \widehat{\nu\mathbb{F}}_{Y, \vec{X}}) \\
&\quad \overline{\mathbb{F}}(!g \otimes \vec{f}, !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\
&= \overline{\mathbb{F}}(!Y' \otimes \vec{X}', \overline{\nu\mathbb{F}}(!g \otimes \vec{f})) \overline{\mathbb{F}}(!g \otimes \vec{f}, !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \\
&\quad \overline{\mathbb{F}}(!Y \otimes \vec{X}, \widehat{\nu\mathbb{F}}_{Y, \vec{X}}) \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\
&= \overline{\mathbb{F}}(!g \otimes \vec{f}, \overline{\nu\mathbb{F}}(!g \otimes \vec{f})) \widehat{\nu\mathbb{F}}_{Y, \vec{X}} \text{ by (2.3)} \\
&= \overline{\nu\mathbb{F}}(!g \otimes \vec{f}) \widehat{\nu\mathbb{F}}_{Y, \vec{X}} \text{ by Lemma 2.5}
\end{aligned}$$

so that $\overline{\mathbb{F}}(!Y' \otimes \vec{X}', h_2) l = h_2$ which proves our contention. The monoidality condition on $\widehat{\nu\mathbb{F}}$ is proved similarly. \blacktriangleleft

Lemma 2.7. *Let \mathbb{F} be an $n + 1$ -ary strong functor on \mathcal{L} such that for each $\vec{X} \in \mathcal{L}^n$, the category $\mathbf{Alg}_{\mathcal{L}}(\overline{\mathbb{F}}_{\vec{X}})$ has an initial object $\mu\overline{\mathbb{F}}_{\vec{X}}$. Then there is a unique n -ary strong functor $\mu\mathbb{F}$ such that $\overline{\mu\mathbb{F}}(\vec{X}) = \mu\overline{\mathbb{F}}_{\vec{X}}$ (and hence $\overline{\mathbb{F}}(\vec{X}, \overline{\mu\mathbb{F}}(\vec{X})) = \overline{\mu\mathbb{F}}(\vec{X})$),*

- $\overline{\mathbb{F}}(\vec{f}, \overline{\mu\mathbb{F}}(\vec{f})) = \overline{\mu\mathbb{F}}(\vec{f})$ for all $\vec{f} \in \mathcal{L}^n(\vec{X}, \vec{X}')$
- and $\overline{\mathbb{F}}(!Y \otimes \vec{X}, \widehat{\mu\mathbb{F}}_{Y, \vec{X}}) \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\mu\mathbb{F}}(\vec{X}))} = \widehat{\mu\mathbb{F}}_{Y, \vec{X}}$.

Moreover $(\mu\mathbb{F})^\perp = \nu(\mathbb{F}^\perp)$

Proof. Apply Lemma 2.6 to the strong functor \mathbb{F}^\perp . \blacktriangleleft

3. COHERENCE SPACES

We consider now the case where \mathcal{L} is the category **Coh** of coherence spaces and linear maps, a well-known model of LL introduced in [12, 13].

A *coherence space* is a structure $E = (|E|, \supseteq_E)$ where $|E|$ is a set (that we always assume to be at most countable since this property is preserved by all the constructions presented in this paper) called the *web* of E and \supseteq_E is a binary reflexive and symmetric relation on $|E|$. A *clique* of E is a subset u of $|E|$ such that $\forall a_1, a_2 \in u \ a_1 \supseteq_E a_2$. We use $\mathbf{Cl}(E)$ for the set of all cliques of E that we consider as a domain, the order relation on $\mathbf{Cl}(E)$ being always inclusion. Observe indeed that $\emptyset \in \mathbf{Cl}(E)$ (that is $\mathbf{Cl}(E)$ has a least element), if $u \subseteq v$ and $v \in \mathbf{Cl}(E)$ then $u \in \mathbf{Cl}(E)$ and last if $D \subseteq \mathbf{Cl}(E)$ is directed then $\cup D \in \mathbf{Cl}(E)$.

3.1. Coherence spaces as a model of LL. Given coherence spaces E and F we define a coherence space $E \multimap F$ whose web is $|E| \times |F|$ and coherence is: $(a_1, b_1) \supseteq_{E \multimap F} (a_2, b_2)$ if $a_1 \supseteq_E a_2 \Rightarrow (b_1 \supseteq_F b_2 \text{ and } b_1 = b_2 \Rightarrow a_1 = a_2)$.

The category **Coh** has coherence spaces as objects, and homsets $\mathbf{Coh}(E, F) = \mathbf{Cl}(E \multimap F)$. In this category the identities are the diagonal relations and composition is the ordinary composition of relations.

Remark 3.1. It can be useful to keep in mind that these morphisms can be considered as *linear functions*: a function $f : \mathbf{Cl}(E) \rightarrow \mathbf{Cl}(F)$ is linear if it is stable (that is $\forall u_1, u_2 \in \mathbf{Cl}(E) \ u_1 \cup u_2 \in \mathbf{Cl}(E) \Rightarrow f(u_1 \cap u_2) = f(u_1) \cap f(u_2)$) and commutes with arbitrary well-defined unions of cliques. Such a function f has a *trace* $\text{tr} f = \{(a, b) \in |E| \times |F| \mid b \in f(\{a\})\}$

and this trace operation defines a bijection between $\mathbf{Cl}(E \multimap F)$ and the set of all linear functions from $\mathbf{Cl}(E)$ to $\mathbf{Cl}(F)$. The converse of this operation maps $t \in \mathbf{Cl}(E \multimap F)$ to the function $\text{fun}(t) : \mathbf{Cl}(E) \rightarrow \mathbf{Cl}(F)$ defined by $\text{fun}(t)(u) = \{b \in |F| \mid \exists a \in u (a, b) \in t\}$. We will always write tu instead of $\text{fun}(t)(u)$. In this paper we stick to the relational point of view on morphisms.

This category is monoidal, with tensor product $E_1 \otimes E_2$ having $|E_1| \times |E_2|$ as web and $(a_1, a_2) \circ_{E_1 \otimes E_2} (a'_1, a'_2)$ if $a_i \circ_{E_i} a'_i$ for $i = 1, 2$. Given $t_i \in \mathbf{Coh}(E_i, F_i)$ for $i = 1, 2$, one defines $t_1 \otimes t_2$ as $\{((a_1, a_2), (b_1, b_2)) \mid (a_i, b_i) \in t_i \text{ for } i = 1, 2\} \in \mathbf{Coh}(E_1 \otimes E_2, F_1 \otimes F_2)$ as easily checked. So \otimes is a functor $\mathbf{Coh}^2 \rightarrow \mathbf{Coh}$, which equips \mathbf{Coh} with an obvious symmetric monoidal structure that we will not make explicit here, for a unit object $1 = (\{*\}, =)$. This category is monoidal closed with $E \multimap F$ object of morphisms from E to F (and evaluation morphism $\text{ev} \in \mathbf{Coh}((E \multimap F) \otimes E, F)$ defined by $\text{ev} = \{((a, b), a), b \mid a \in |E| \text{ and } b \in |F|\}$). Taking $\perp = 1$ as dualizing object, \mathbf{Coh} is easily seen to be $*$ -autonomous and the corresponding orthogonality is a functor $\perp : \mathbf{Coh}^{\text{op}} \rightarrow \mathbf{Coh}$ where $E^\perp = (|E|, \succ_E) \simeq (E \multimap \perp)$ (by a trivial iso), the *incoherence* binary relation \succ_E being defined by $a_1 \succ_E a_2$ if $a_1 \circ_E a_2 \Rightarrow a_1 = a_2$. The transpose t^\perp of $t \in \mathbf{Coh}(E, F)$ is simply $\{(b, a) \mid (a, b) \in t\}$. Under this linear negation, the De Morgan dual (par or cotensor) of the tensor product is $E_1 \wp E_2 = (E_1^\perp \otimes E_2^\perp)^\perp$ whose web is $|E_1| \times |E_2|$ and whose coherence relation is characterized by: $(a_1, a_2) \frown_{E_1 \wp E_2} (a'_1, a'_2)$ iff $a_i \frown_{E_i} a'_i$ for $i = 1$ or $i = 2$ (where $a \frown_E b$ means $a \circ_E b$ and $a \neq b$ and is called *strict coherence*; *strict incoherence* \smile_E is defined similarly). Remember that, with these notations, $E \multimap F = E^\perp \wp F$.

\mathbf{Coh} has a final object $\top = (\emptyset, \emptyset)$ and a cartesian product $E_1 \& E_2 = (\{1\} \times |E_1| \cup \{2\} \times |E_2|, \circ_{E_1 \& E_2})$ where the coherence relation is defined by: $(i, a) \circ_{E_1 \& E_2} (j, a')$ if $i = j \Rightarrow a \circ_{E_i} a'$, the associated projections $\pi_i \in \mathbf{Coh}(E_1 \& E_2, E_i)$ being $\pi_i = \{((i, a), a) \mid a \in |E_i|\}$. Dually the initial object is $0 = \top^\perp = \top$ and the coproduct is $E_1 \oplus E_2 = (E_1^\perp \& E_2^\perp)^\perp$ whose web is $\{1\} \times |E_1| \cup \{2\} \times |E_2|$ and whose coherence is characterized by $(i, a) \circ_{E_1 \oplus E_2} (j, a')$ if $i = j$ and $a \circ_{E_i} a'$. There are canonical injections $E_i \rightarrow E_1 \oplus E_2$ which are the transposes of the projections defined above.

We define $!E$ as the coherence space whose web is the set of all finite elements of $\mathbf{Cl}(E)$ and the coherence is: $u_1 \circ_{!E} u_2$ if $u_1 \cup u_2 \in \mathbf{Cl}(E)$ (that is $\forall a_1 \in u_1 \forall a_2 \in u_2 a_1 \circ_E a_2$). This operation is a functor: given $t \in \mathbf{Coh}(E, F)$ one sets $!t = \{(\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}) \in |!E| \times |!F| \mid \forall i (a_i, b_i) \in t\}$. The comonad structure of this functor is given by the natural transformations $\text{der}_E = \{(\{a\}, a) \mid a \in |E|\} \in \mathbf{Coh}(!E, E)$ (dereliction) and $\text{dig}_E = \{(u_1 \cup \dots \cup u_n, \{u_1, \dots, u_n\}) \mid u_1, \dots, u_n \in |!E|\}$ with $u_1 \cup \dots \cup u_n \in \mathbf{Cl}(E)$ (digging). Last there is an obvious isomorphism $\mathfrak{m}^0 \in \mathbf{Coh}(1, !\top)$ and a natural isomorphism $\mathfrak{m}_{E_1, E_2}^2 \in \mathbf{Coh}(!E_1 \otimes !E_2, !(E_1 \& E_2))$ (these isos defining a strong monoidal structure), satisfying an additional technical condition explained in [17] for instance.

3.2. Coherence spaces form a cpo. Let E and F be coherence spaces, we write $E \subseteq F$ if $|E| \subseteq |F|$ and $\forall a, a' \in |E| a \circ_E a' \Leftrightarrow a \circ_F a'$.

Observe that when $E \subseteq F$, one has two linear morphisms $\eta_{E, F}^+ \in \mathbf{Coh}(E, F)$ and $\eta_{E, F}^- \in \mathbf{Coh}(F, E)$ given by $\eta_{E, F}^+ = \eta_{E, F}^- = \{(a, a) \mid a \in |E|\}$.

They satisfy $\eta_{E, F}^- \eta_{E, F}^+ = \text{Id}_E$ and $\eta_{E, F}^+ \eta_{E, F}^- \subseteq \text{Id}_F$, defining an embedding-retraction pair of coherence spaces as considered for instance in [12] (all embedding-retraction pairs are of that shape, up to isomorphism of coherence spaces). One major feature of this order

relation is that it makes linear negation monotonic³, making life quite easy when one needs to compute fixed points of arbitrary LL formulas in this model.

Lemma 3.2. *The relation \subseteq is a partial order relation on coherence spaces, and we have $E \subseteq F \Leftrightarrow E^\perp \subseteq F^\perp$.*

We use \mathbf{Coh}_\subseteq for the class of coherence spaces ordered under the \subseteq partial order relation. This partially ordered class has a least element denoted as \emptyset (the coherence space which has an empty web).

Any countable directed subset \mathcal{E} of \mathbf{Coh}_\subseteq has a lub $\cup\mathcal{E}$, which is the coherence space defined by $|\cup\mathcal{E}| = \cup_{E \in \mathcal{E}} |E|$ and, for all $a, a' \in |\cup\mathcal{E}|$, one has $a \succ_{\cup\mathcal{E}} a'$ iff $a \succ_E a'$ for some $E \in \mathcal{E}$.

The family \mathcal{E} gives rise to two diagrams in the category \mathbf{Coh} :

- the *inductive diagram* \mathcal{E}^+ with morphisms $\eta_{E,F}^+ \in \mathbf{Coh}(E, F)$ when $E, F \in \mathcal{E}$ with $E \subseteq F$
- and the *projective diagram* \mathcal{E}^- with morphisms $\eta_{E,F}^- \in \mathbf{Coh}(F, E)$ when $E, F \in \mathcal{E}$ with $E \subseteq F$.

Lemma 3.3. *Let $G = \cup\mathcal{E}$. Then, in \mathbf{Coh} , the cocone $(E, \eta_{E,G}^+)_{E \in \mathcal{E}}$ is the colimit of the inductive diagram \mathcal{E}^+ and the cone $(E, \eta_{E,G}^-)_{E \in \mathcal{E}}$ is the limit of the projective diagram \mathcal{E}^- .*

Proof. We prove the first statement, the second one following by duality. Let F be a coherence space and, for each $E \in \mathcal{E}$ let $t_E \in \mathbf{Coh}(E, F)$ defining a cocone based on \mathcal{E}^+ , which means $\forall E, E' \in \mathcal{E} \ E \subseteq E' \Rightarrow t_{E'} \eta_{E,E'}^+ = t_E$, that is $\forall E, E' \in \mathcal{E} \ E \subseteq E' \Rightarrow t_E = t_{E'} \cap |E| \times |F|$. Then the unique morphism $t \in \mathbf{Coh}(G, F)$ such that $\forall E \in \mathcal{E} \ t \eta_{E,G}^+ = t_E$ is given by $t = \cup_{E \in \mathcal{E}} t_E$ as easily checked. \blacktriangleleft

Definition 3.4. A functor $\mathcal{F} : \mathbf{Coh}^n \rightarrow \mathbf{Coh}$ is *continuous on objects* if whenever $\vec{E}, \vec{F} \in \mathbf{Coh}^n$ satisfy $E_i \subseteq F_i$ for $i = 1, \dots, n$, one has $\mathcal{F}(\vec{E}) \subseteq \mathcal{F}(\vec{F})$ and $\mathcal{F}(\eta_{E_1, F_1}^+, \dots, \eta_{E_n, F_n}^+) = \eta_{\mathcal{F}(\vec{E}), \mathcal{F}(\vec{F})}^+$ and $\mathcal{F}(\eta_{E_1, F_1}^-, \dots, \eta_{E_n, F_n}^-) = \eta_{\mathcal{F}(\vec{E}), \mathcal{F}(\vec{F})}^-$. Moreover, \mathcal{F} commutes with the lubs of countable directed families of coherence spaces. In other words, for any countable directed families of coherence spaces $\mathcal{E}_1, \dots, \mathcal{E}_n$, one has $\mathcal{F}(\cup\mathcal{E}_1, \dots, \cup\mathcal{E}_n) = \cup\{\mathcal{F}(E_1, \dots, E_n) \mid E_1 \in \mathcal{E}_1, \dots, E_n \in \mathcal{E}_n\}$.

One says that \mathcal{F} is *continuous on morphisms* if when $\vec{f}, \vec{g} \in \mathbf{Coh}^n(\vec{E}, \vec{F})$ satisfy $\vec{f} \subseteq \vec{g}$ (that is $\forall i \ f_i \subseteq g_i$) one has $\mathcal{F}(\vec{f}) \subseteq \mathcal{F}(\vec{g})$ and, if D is a directed subset of $\mathbf{Coh}^n(\vec{E}, \vec{F})$, one has $\mathcal{F}(\cup D) = \cup_{\vec{f} \in D} \mathcal{F}(\vec{f})$ (equivalently $\mathcal{F}(\cup D) \supseteq \cup_{\vec{f} \in D} \mathcal{F}(\vec{f})$).

Last one says that \mathcal{F} is *continuous* if it is both continuous on objects and on morphisms.

Notice that this property is preserved by composition and duality (setting, consistently with Section 2.2, $(\mathcal{F}^\perp)(\vec{E}) = (\mathcal{F}(\vec{E}^\perp))^\perp$ and similarly for morphisms).

Definition 3.5. A (n -ary) variable coherence space (VCS) is a strong functor $\mathbb{E} : \mathbf{Coh}^n \rightarrow \mathbf{Coh}$ such that $\overline{\mathbb{E}}$ is monotonic and continuous.

Proposition 3.6. *The operations $\otimes, \wp, \oplus, \&, !$ and $?$ are VCSs and VCSs are closed under De Morgan duality and composition.*

Proof. This results immediately from the properties of strong functors stated in Section 2.2 and from straightforward computations (for the continuity statement). \blacktriangleleft

³and not antitonic as one might expect having e.g. intersection types in mind

3.3. Least fixed point of a VCS, universal properties wrt. algebras and coalgebras.

Let $\mathcal{F} : \mathbf{Coh} \rightarrow \mathbf{Coh}$ be continuous on objects (in the sense of Definition 3.4). Then we have $\emptyset \subseteq \mathcal{F}(\emptyset) \subseteq \dots \subseteq \mathcal{F}^n(\emptyset) \subseteq \mathcal{F}^{n+1}(\emptyset) \subseteq \dots$ as shown by an easy induction on n . We set $\sigma \mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}^n(\emptyset)$. By Scott continuity of \mathcal{F} , we have $\mathcal{F}(\sigma \mathcal{F}) = \sigma \mathcal{F}$.

Lemma 3.7. *One has $\sigma(\mathcal{F}^\perp) = (\sigma \mathcal{F})^\perp$. If moreover \mathcal{F} is continuous on morphisms then $\sigma \mathcal{F}$ is at the same time the initial object of $\mathbf{Alg}_{\mathbf{Coh}}(\mathcal{F})$ and the final object of $\mathbf{Coalg}_{\mathbf{Coh}}(\mathcal{F})$.*

Proof. The first statement results from the observation that $(\mathcal{F}^\perp)^n = (\mathcal{F}^n)^\perp$. For the second statement, since $\sigma(\mathcal{F}^\perp) = (\sigma \mathcal{F})^\perp$, it suffices to prove that $(\sigma \mathcal{F}, \text{Id})$ is initial in $\mathbf{Alg}_{\mathbf{Coh}}(\mathcal{F})$. This results easily from Lemma 3.3 and from continuity on morphisms. \blacktriangleleft

Let \mathbb{E} be an $n+1$ -ary VCS. Applying Lemma 3.7 to the functors $\overline{\mathbb{E}}_{\vec{E}}$ for all $\vec{E} \in \mathbf{Coh}^n$, Lemma 2.7 shows that there is an n -ary strong functor $\mu\mathbb{E}$ uniquely determined by the following equations

- $\overline{\mu\mathbb{E}}(\vec{E}) = \sigma \overline{\mathbb{E}}_{\vec{E}}$
- $\overline{\mathbb{E}}(\vec{f}, \overline{\mu\mathbb{E}}(\vec{f})) = \overline{\mu\mathbb{E}}(\vec{f})$ for $\vec{f} \in \mathbf{Coh}(\vec{E}, \vec{E}')$
- and $\overline{\mathbb{E}}(!F \otimes \vec{E}, \widehat{\mu\mathbb{E}}_{F, \vec{E}}) \widehat{\mathbb{E}}_{F, (\vec{E}, \overline{\mu\mathbb{E}}(\vec{E}))} = \widehat{\mu\mathbb{E}}_{F, \vec{E}}$.

Proposition 3.8. *The functor $\mu\mathbb{E}$ is a variable coherence space. Defining the dual operation as $\nu\mathbb{E} = (\mu(\mathbb{E}^\perp))^\perp$, one has $\nu\mathbb{E} = \mu\mathbb{E}$. We use $\sigma\mathbb{E}$ for this unique (final and initial) fixed point VCS.*

Proof. The proof that $\overline{\mu\mathbb{E}}$ is monotonic and Scott continuous on $\mathbf{Coh}_{\subseteq}^n$ is a standard domain-theoretic verification. We are left with proving that given \vec{E}, \vec{F} in \mathbf{Coh}^n such that $\vec{E} \subseteq \vec{F}$, one has

$$\overline{\mu\mathbb{E}}(\eta_{\vec{E}, \vec{F}}^+) = \eta_{\overline{\mu\mathbb{E}}(\vec{E}), \overline{\mu\mathbb{E}}(\vec{F})}^+ \quad \overline{\mu\mathbb{E}}(\eta_{\vec{E}, \vec{F}}^-) = \eta_{\overline{\mu\mathbb{E}}(\vec{E}), \overline{\mu\mathbb{E}}(\vec{F})}^-.$$

Let us prove the first equation, the proof of the second one being completely similar. By Lemma 2.4, it suffices to prove $\mathbb{E}(\eta_{\vec{E}, \vec{F}}^+, \eta_{\overline{\mu\mathbb{E}}(\vec{E}), \overline{\mu\mathbb{E}}(\vec{F})}^+) = \eta_{\overline{\mu\mathbb{E}}(\vec{E}), \overline{\mu\mathbb{E}}(\vec{F})}^+$ which in turn results from the assumption that \mathbb{E} is a VCS.

The identity $\mu\mathbb{E} = \nu\mathbb{E}$ results from the uniqueness statements of Lemmas 2.6 and 2.7 and from the fact that $\overline{\nu\mathbb{E}}(\vec{E}) = \sigma \overline{\mathbb{E}}_{\vec{E}} = \overline{\mu\mathbb{E}}(\vec{E})$. \blacktriangleleft

Remark 3.9. The methods developed in this section are by no way specific to coherence spaces and could be used in many other models of LL (relational semantics, Scott semantics, hypercoherence spaces, probabilistic coherence spaces, game models, up to some adaptation since these are not models of classical LL, etc).

4. COHERENCE SPACES WITH TOTALITY

Let E be a coherence space and let $\mathcal{A} \subseteq \mathbf{Cl}(E)$. We set

$$\mathcal{A}^\perp = \{x' \in \mathbf{Cl}(E^\perp) \mid \forall x \in \mathcal{A} \ x \cap x' \neq \emptyset\}.$$

Observe that if $x \cap x' \neq \emptyset$ then this intersection has exactly one element, due to the fact that x and x' are cliques in E and E^\perp respectively.

If $\mathcal{A}, \mathcal{B} \subseteq \mathbf{Cl}(E)$ and $\mathcal{A} \subseteq \mathcal{B}$, we have $\mathcal{B}^\perp \subseteq \mathcal{A}^\perp$, and also $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$. Therefore $\mathcal{A}^\perp = \mathcal{A}^{\perp\perp\perp}$.

A *totality candidate* on E is a set $\mathcal{T} \subseteq \mathbf{Cl}(E)$ such that $\mathcal{T}^{\perp\perp} = \mathcal{T}$, or equivalently $\mathcal{T}^{\perp\perp} \subseteq \mathcal{T}$. This property is equivalent to the existence of a “predual” of \mathcal{T} , that is, of a set $\mathcal{A} \subseteq \mathbf{Cl}(E^\perp)$ such that $\mathcal{T} = \mathcal{A}^\perp$. We use $\mathbf{Tot}(E)$ for the set of all totality candidates of the coherence space E , and we consider this set as a poset, equipped with inclusion.

Lemma 4.1. *The poset $\mathbf{Tot}(E)$ is a complete lattice.*

Proof. Let $\Theta \subseteq \mathbf{Tot}(E)$ and let $\Theta' = \{\mathcal{T}^\perp \mid \mathcal{T} \in \Theta\}$, which is a subset of $\mathbf{Tot}(E^\perp)$. Given $x \in \mathbf{Cl}(E)$, we have $x \in \bigcap \Theta$ iff for all $\mathcal{T}' \in \Theta'$ and all $x' \in \mathcal{T}'$, $x \cap x' \neq \emptyset$. In other words $\bigcap \Theta = (\bigcup_{\mathcal{T}' \in \Theta'} \mathcal{T}'^\perp)^\perp \in \mathbf{Tot}(E)$. \blacktriangleleft

The greatest element of $\mathbf{Tot}(E)$ is $\mathbf{Cl}(E)$ and its least element is $(\mathbf{Cl}(E^\perp))^\perp = \emptyset$ as easily checked. Any subset Θ of $\mathbf{Tot}(E)$ has a least upper bound $\bigvee \Theta$ which is given by $\bigvee \Theta = (\bigcup \Theta)^\perp$ and this biorthogonal closure cannot be disposed of in general ($\bigcup \Theta$ is not necessarily a totality candidate). It is useful to observe that the map $\mathcal{T} \mapsto \mathcal{T}^\perp$ is an isomorphism between the complete lattices $\mathbf{Tot}(E^\perp)$ and $\mathbf{Tot}(E)^{\text{op}}$.

A *coherence space with totality* is a pair $X = (\underline{X}, \top X)$ where \underline{X} is a coherence space (the carrier) and $\top X \in \mathbf{Tot}(\underline{X})$.

4.1. Coherence spaces with totality as a model of LL. Let X and Y be coherence spaces with totality, we define a coherence space with totality $X \multimap Y$ by $\underline{X \multimap Y} = \underline{X} \multimap \underline{Y}$ and $\top(X \multimap Y) = \{t \in \mathbf{Cl}(\underline{X} \multimap \underline{Y}) \mid \forall x \in \top X \ t x \in \top Y\} = \{x \otimes y' \mid x \in \top X \text{ and } y' \in \top Y^\perp\}^\perp$, this latter equation resulting from the equivalence $(tx) \cap y' \neq \emptyset \Leftrightarrow t \cap (x \times y') \neq \emptyset$. It is clear that if $s \in \top(X \multimap Y)$ and $t \in \top(Y \multimap Z)$ then $ts \in \top(X \multimap Z)$, and also that $\text{ld} \in \top(X \multimap X)$, hence we have defined a category that we denote as **CohT**.

The following is a useful tool for proving that a linear morphism (a morphism in **Coh** between the carriers of two coherence spaces with totality) is total.

Lemma 4.2. *Let $t \in \mathbf{Coh}(\underline{X}, \underline{Y})$ and let $\mathcal{A} \subseteq \mathbf{Cl}(\underline{X})$ be such that $\top X = \mathcal{A}^{\perp\perp}$. If $\forall x \in \mathcal{A} \ t x \in \top Y$ then $t \in \mathbf{CohT}(X, Y)$.*

Proof. Let $x \in \top X$, we have to prove that $tx \in \top Y$ so let $y' \in \top Y^\perp$, we must prove that $(tx) \cap y' \neq \emptyset$. This statement is equivalent to $t \cap (x \otimes y') \neq \emptyset \Leftrightarrow (t^\perp y') \cap x \neq \emptyset$. So we must prove $\forall x \in \top X \ \forall y' \in \top Y^\perp \ (t^\perp y') \cap x \neq \emptyset$, that is $t^\perp \in \mathbf{CohT}(Y^\perp, X^\perp)$. So let $y' \in \top Y^\perp$, we must prove that $t^\perp y' \in \top X^\perp = \mathcal{A}^\perp$ which results from our assumption by the same reasoning. \blacktriangleleft

We equip 1 and \perp with the same totality, namely $\{\{\ast\}\}$. We define $X_1 \otimes X_2$ by $\underline{X_1 \otimes X_2} = \underline{X_1} \otimes \underline{X_2}$ and $\top(X_1 \otimes X_2) = \{x_1 \otimes x_2 \mid x_i \in \top X_i \text{ for } i = 1, 2\}^{\perp\perp}$, so that $\underline{X \otimes Y} = (\underline{X} \multimap \underline{Y}^\perp)^\perp$. Then it is easy to check that **CohT** is \ast -autonomous, *with the same operations on morphisms as in Coh* (for instance one checks that if $t_i \in \mathbf{CohT}(X_i, Y_i)$ then $t_1 \otimes t_2 \in \mathbf{CohT}(X_1 \otimes X_2, Y_1 \otimes Y_2)$ which is easy using Lemma 4.2; in the same way one proves easily that $\text{ev} \in \mathbf{CohT}((X \multimap Y) \otimes X, Y)$ etc). Similarly one shows that the cartesian structure on **Coh** gives rise to a cartesian structure on **CohT**: $\underline{X_1 \& X_2} = \underline{X_1} \& \underline{X_2}$ and $\{1\} \times x_1 \cup \{2\} \times x_2 \in \top(X_1 \& X_2)$ if $x_i \in \top X_i$ for $i = 1, 2$. The total cliques of $\underline{X_1} \oplus \underline{X_2}$ are the $\{i\} \times z$ for $i = 1, 2$ and $z \in \top X_i$. Notice that \top and 0 are different coherence spaces with totality: $\top \top = \{\emptyset\}$ and $\top 0 = \emptyset$.

Last $!X$ is given by $\underline{!X} = \underline{!X}$ and $\top(!X) = \{x^\dagger \mid x \in \top X\}^{\perp\perp}$ (where $x^\dagger = \mathcal{P}_{\text{fin}}(x)$). Then one proves easily that $t \in \mathbf{CohT}(X, Y) \Rightarrow !t \in \mathbf{CohT}(!X, !Y)$ again using Lemma 4.2. It is also easy to check that $\text{der}_{\underline{X}} \in \mathbf{CohT}(!X, X)$ and that $\text{dig}_{\underline{X}} \in \mathbf{CohT}(!X, !!X)$ so we

denote these morphisms as der_X and dig_X turning “!” into a comonad on \mathbf{CohT} . The same holds for the monoidal structure (Seely isomorphisms): $m^0 \in \mathbf{CohT}(1, !\top)$ and $m_{X_1, X_2}^2 \in \mathbf{CohT}(!X_1 \otimes !X_2, !(X_1 \& X_2))$.

4.2. Variable coherence spaces with totality (VCST). We first recall the well-known Knaster-Tarski’s Theorem.

Theorem 4.3. *Let S be a complete lattice and let $f : S \rightarrow S$ be a monotonic function. Then f has a least fixed point in S . Let (s_α) be the ordinal-indexed family of elements of S defined by $s_{\alpha+1} = f(s_\alpha)$ and $s_\lambda = \bigvee_{\alpha < \lambda} s_\alpha$ for λ limit ordinal. Then this sequence is monotonic and there is an ordinal θ such that $s_\theta = s_{\theta+1}$. Moreover s_θ is the least fixed point of f in S .*

We denote s_α as $f^\alpha(0)$ where 0 is the least element of S ; this notation coincides with a finite iterations of f when α is a finite ordinal (we have $s_0 = 0$ considering $0 \in \mathbb{N}$ as a limit ordinal).

4.2.1. General definition of a VCST. To make the notations more readable, when \mathbb{E} is a VCS (see Definition 3.5), we use \mathbb{E} (instead of $\overline{\mathbb{E}}$) to denote its functorial part. We keep denoting as $\widehat{\mathbb{E}}$ the associated strength natural transformation.

Definition 4.4. An n -ary variable coherence space with totality (VCST) is a pair $\mathbb{X} = (\underline{\mathbb{X}}, \text{T}\mathbb{X})$ where

- $\underline{\mathbb{X}} : \mathbf{Coh}^n \rightarrow \mathbf{Coh}$ is an n -ary VCS called the *carrier* of \mathbb{X}
- and $\text{T}\mathbb{X}$ is an operation, called the *totality* of \mathbb{X} , which, with each n -tuple \vec{X} of coherence spaces with totality, associates $\text{T}\mathbb{X}(\vec{X}) \in \text{Tot}(\underline{\mathbb{X}}(\vec{X}))$ — and we use the notation $\mathbb{X}(\vec{X})$ for the coherence space with totality $(\underline{\mathbb{X}}(\vec{X}), \text{T}\mathbb{X}(\vec{X}))$.

Moreover the two following properties must hold.

- If \vec{X} and \vec{Y} are objects of \mathbf{CohT}^n and $f \in \mathbf{CohT}^n(\vec{X}, \vec{Y})$, then the \mathbf{Coh} morphism $\underline{\mathbb{X}}(f)$ belongs actually to $\mathbf{CohT}(\mathbb{X}(\vec{X}), \mathbb{X}(\vec{Y}))$, so that \mathbb{X} defines a functor $\mathbf{CohT}^n \rightarrow \mathbf{CohT}$ (denoted simply as \mathbb{X}).
- If \vec{X} is an object of \mathbf{CohT}^n and Y is an object of \mathbf{CohT} then the \mathbf{Coh} morphism $\widehat{\underline{\mathbb{X}}}_{Y, \vec{X}}$ belongs actually to $\mathbf{CohT}(!Y \otimes \mathbb{X}(\vec{X}), \mathbb{X}(!Y \otimes \vec{X}))$. We denote this total morphism as $\widehat{\mathbb{X}}_{Y, \vec{X}}$.

So we can consider \mathbb{X} as a strong functor $\mathbf{CohT}^n \rightarrow \mathbf{CohT}$ (the monoidality diagram commutations of Figure 1 hold because the LL operations on morphisms are interpreted in the same way in \mathbf{CohT} and in \mathbf{Coh}).

Remark 4.5. To fully understand this definition, it is essential to keep in mind that, if $f \in \mathbf{CohT}(\vec{X}, \vec{Y})$ then actually $f \in \mathbf{Coh}(\underline{\mathbb{X}}(\vec{X}), \underline{\mathbb{X}}(\vec{Y}))$ so that the morphism $\underline{\mathbb{X}}(f) \in \mathbf{Coh}(\underline{\mathbb{X}}(\vec{X}), \underline{\mathbb{X}}(\vec{Y}))$ is defined, *independently of the notions of totality on \vec{X} and \vec{Y}* and similarly for $\widehat{\underline{\mathbb{X}}}_{Y, \vec{X}}$. This decoupling of the totality-free part of the notions involved from the totality dependent ones makes life much simpler. This situation can certainly be axiomatized categorically, around the obvious forgetful functor $\mathbf{CohT} \rightarrow \mathbf{Coh}$ which commutes with all LL constructs; this abstract categorical analysis is postponed to further work.

Remark 4.6. Strictly speaking, an n -ary VCST \mathbb{X} is not a strong functor $\mathbf{CohT}^n \rightarrow \mathbf{CohT}$ but a structure which induces — as explained above — such a strong functor \mathcal{F} , that we have denoted simply as \mathbb{X} . This choice of notation is motivated by the fact that \mathbb{X} can very simply be recovered from \mathcal{F} . We have indeed a forgetful functor $\mathcal{U} : \mathbf{CohT} \rightarrow \mathbf{Coh}$ which maps X to \underline{X} and acts as the identity on morphisms. This functor has a left adjoint $\mathcal{Z} : \mathbf{Coh} \rightarrow \mathbf{CohT}$ which maps a coherence space E to (E, \emptyset) (no cliques of E are total) and acts as the identity on morphisms. Then we have $\underline{\mathbb{X}} = \mathcal{U} \circ \mathcal{F} \circ \mathcal{Z} = \mathcal{U} \circ \mathcal{F} \circ \mathcal{Z}^\perp$ (for the functorial part of $\underline{\mathbb{X}}$) and for the strength $\widehat{\underline{\mathbb{X}}}_{F, \vec{E}} = \widehat{\mathcal{F}}_{\mathcal{Z}(F), \mathcal{Z}(\vec{E})}$, and $\mathcal{T} = \mathbb{T}\mathbb{X}(\vec{X})$ is defined by the fact that the coherence space with totality $\mathcal{F}(\vec{X})$ is of shape (F, \mathcal{T}) . In these definitions, the choice of \mathcal{Z} as “inverse” of \mathcal{U} is arbitrary. By the definition of VCSTs we could have used the right adjoint \mathcal{Z}^\perp (it maps E to $(E, \mathbf{Cl}(E))$ where all cliques are total) or any other functor $\mathcal{Y} : \mathbf{Coh} \rightarrow \mathbf{CohT}$ such that $\mathcal{U} \circ \mathcal{Y} = \text{Id}$ instead: the resulting \mathbb{X} would have been the same. For these reasons, it is meaningful to consider VCSTs as strong functors $\mathbf{CohT}^n \rightarrow \mathbf{CohT}$, what we do now.

This observation also motivates our general notion of model presented in Definition 7.1.

Proposition 4.7. *The operations $\otimes, \wp, \oplus, \&, !$ and $?$ are VCSTs and VCSTs are closed under De Morgan duality and composition.*

This is a consequence of Proposition 3.6.

4.2.2. Fixed Points of VCST's. We deal first with least fixed points of unary VCST's, so let \mathbb{X} be a unary VCST (whose strength is not used in this first step). We define a coherence space with totality $\underline{\mu\mathbb{X}}$. First, we set $\underline{\mu\mathbb{X}} = \sigma \underline{\mathbb{X}}$.

We define a map $\Theta(\mathbb{X}) : \text{Tot}(\underline{\mu\mathbb{X}}) \rightarrow \text{Tot}(\underline{\mu\mathbb{X}})$ as follows: if $\mathcal{T} \in \text{Tot}(\underline{\mu\mathbb{X}})$, then $\mathbb{T}\mathbb{X}(\underline{\mu\mathbb{X}}, \mathcal{T}) \in \text{Tot}(\underline{\mathbb{X}}(\underline{\mu\mathbb{X}})) = \text{Tot}(\underline{\mu\mathbb{X}})$ and we set $\Theta(\mathbb{X})(\mathcal{T}) = \mathbb{T}\mathbb{X}(\underline{\mu\mathbb{X}}, \mathcal{T})$. We contend that this mapping is monotonic on the lattice $\text{Tot}(\underline{\mu\mathbb{X}})$. Assume that $\mathcal{T}, \mathcal{T}' \in \text{Tot}(\underline{\mu\mathbb{X}})$ with $\mathcal{T} \subseteq \mathcal{T}'$. Then $\text{Id} \in \mathbf{CohT}((\underline{\mu\mathbb{X}}, \mathcal{T}), (\underline{\mu\mathbb{X}}, \mathcal{T}'))$ (see Section 4) and hence $\text{Id} = \underline{\mathbb{X}}(\text{Id}) \in \mathbf{CohT}((\underline{\mu\mathbb{X}}, \Theta(\mathbb{X})(\mathcal{T})), (\underline{\mu\mathbb{X}}, \Theta(\mathbb{X})(\mathcal{T}')))$ by Definition 4.4, from which it follows that $\Theta(\mathbb{X})(\mathcal{T}) \subseteq \Theta(\mathbb{X})(\mathcal{T}')$.

Let \mathcal{U} be the least fixed point of $\Theta(\mathbb{X})$ (applying Theorem 4.3), we set $\mathcal{T}(\underline{\mu\mathbb{X}}) = \mathcal{U}$ and this ends the definition of the coherence space with totality $\underline{\mu\mathbb{X}}$, which satisfies $\mathbb{X}(\underline{\mu\mathbb{X}}) = \underline{\mu\mathbb{X}}$. Now we prove that it is initial in $\mathbf{Alg}_{\mathbf{CohT}}(\underline{\mathbb{X}})$.

For this we shall use the following sequence of candidates of totality for $\underline{\mu\mathbb{X}}$, indexed by ordinals: $\mathcal{U}_{\alpha+1} = \Theta(\mathbb{X})(\mathcal{U}_\alpha)$ and $\mathcal{U}_\lambda = (\bigcup_{\alpha < \lambda} \mathcal{U}_\alpha)^{\perp\perp}$ when λ is a limit ordinal. Remember that there is an ordinal θ such that $\mathcal{U}_{\theta+1} = \mathcal{U}_\theta$, and that we have $\mathcal{U} = \mathcal{U}_\theta$ (see Theorem 4.3).

Proposition 4.8. *$\underline{\mu\mathbb{X}}$ is initial in the category $\mathbf{Alg}_{\mathbf{CohT}}(\underline{\mathbb{X}})$.*

Proof. Let (X, g) be an object in $\mathbf{Alg}_{\mathbf{CohT}}(\underline{\mathbb{X}})$, that is $g \in \mathbf{CohT}(\underline{\mathbb{X}}(X), X)$. This means in particular that $g \in \mathbf{Coh}(\underline{\mathbb{X}}(\underline{X}), \underline{X})$ so that, by Proposition 3.8, we know that there is exactly one morphism $\hat{g} \in \mathbf{Coh}(\underline{\mu\mathbb{X}}, \underline{X})$ such that

$$g \underline{\mathbb{X}}(\hat{g}) = \hat{g}.$$

We have to prove that $\hat{g} \in \mathbf{CohT}(\underline{\mu\mathbb{X}}, X)$. By induction on the ordinal α , we prove that

$$\hat{g} \in \mathbf{CohT}((\underline{\mu\mathbb{X}}, \mathcal{U}_\alpha), X)$$

for all ordinal α . Assume first that the property holds for α and let us prove it for $\alpha + 1$. By Definition 4.4 we get $\underline{\mathbb{X}}(\hat{g}) \in \mathbf{CohT}((\underline{\mu\mathbb{X}}, \mathcal{U}_{\alpha+1}), \underline{\mathbb{X}}(X))$ and hence $\hat{g} = g \underline{\mathbb{X}}(\hat{g}) \in$

$\mathbf{CohT}((\underline{\mu}\mathbb{X}, \mathcal{U}_{\alpha+1}), X)$. Let now λ be a limit ordinal and assume that $\hat{g} \in \mathbf{CohT}((\underline{\mu}\mathbb{X}, \mathcal{U}_\alpha), X)$ for all $\alpha < \lambda$. It will be sufficient to prove that $\hat{g}^\perp \in \mathbf{CohT}(X^\perp, (\bigcup_{\alpha < \lambda} \mathcal{U}_\alpha)^\perp)$ so let $x' \in \mathbb{T}X^\perp$, we must prove that $\hat{g}^\perp x' \in (\bigcup_{\alpha < \lambda} \mathcal{U}_\alpha)^\perp$ so let $y \in \mathcal{U}_\alpha$ for some $\alpha < \lambda$, we must prove that $(\hat{g}^\perp x') \cap y \neq \emptyset$, that is $x' \cap gy \neq \emptyset$ which results from our inductive hypothesis applied to ordinal α .

So we have proven the existence of $\hat{g} \in \mathbf{CohT}(\underline{\mu}\mathbb{X}, X)$ such that $g\underline{\mathbb{X}}(\hat{g}) = \hat{g}$. Uniqueness follows from the uniqueness property for $\underline{\mu}\mathbb{X}$. \blacktriangleleft

We consider now the case of several variables, so let \mathbb{X} be an $n + 1$ -ary VCST. Given $\vec{X} \in \mathbf{CohT}^n$ consider the unary VCST $\mathbb{X}_{\vec{X}}$ defined as follows: $\underline{\mathbb{X}}_{\vec{X}} = \underline{\mathbb{X}}_{\vec{X}}$ and $\mathbb{T}(\mathbb{X}_{\vec{X}})(X) = \mathbb{T}\mathbb{X}(\vec{X}, X)$ (the strength can be defined in a similar way though this is not needed actually because the proof of Proposition 4.8 does not involve the strength). Then by Proposition 4.8 applied to $\mathbb{X}_{\vec{X}}$ and Lemma 2.7 we have an n -ary strong functor $\Phi = (\overline{\Phi}, \hat{\Phi})$ on \mathbf{CohT} such that $\overline{\Phi}(\vec{X}) = \underline{\mu}(\underline{\mathbb{X}}_{\vec{X}})$ and whose action on morphisms and strength are uniquely characterized by

- $\overline{\mathbb{X}}(\vec{f}, \overline{\Phi}(\vec{f})) = \overline{\Phi}(\vec{f})$ for all $\vec{f} \in \mathbf{CohT}^n(\vec{X}, \vec{X}')$
- and $\overline{\mathbb{X}}(!Y \otimes \vec{X}, \hat{\Phi}_{Y, \vec{X}}) \widehat{\mathbb{X}}_{Y, (\vec{X}, \overline{\Phi}(\vec{X}))} = \hat{\Phi}_{Y, \vec{X}}$.

By Proposition 3.8, the first equation implies that $\overline{\Phi}(\vec{f}) = \overline{\sigma}\overline{\mathbb{X}}(\vec{f})$ (remember that actually $\vec{f} \in \mathbf{Coh}^n(\vec{X}, \vec{X}')$ and that $\sigma\mathbb{X}$ is an n -ary VCS characterized by that proposition) and the second equation shows that $\hat{\Phi}_{Y, \vec{X}} = \overline{\sigma}\widehat{\mathbb{X}}_{Y, \vec{X}}$. This proves that $\overline{\sigma}\overline{\mathbb{X}}(\vec{f}) \in \mathbf{CohT}(\underline{\mu}(\underline{\mathbb{X}}_{\vec{X}}), \underline{\mu}(\underline{\mathbb{X}}_{\vec{X}}'))$ and that $\overline{\sigma}\widehat{\mathbb{X}}_{Y, \vec{X}} \in \mathbf{CohT}(Y \otimes \underline{\mu}(\underline{\mathbb{X}}_{\vec{X}}), \underline{\mu}(\underline{\mathbb{X}}_{Y \otimes \vec{X}}))$. Therefore we have defined a VCST $\underline{\mu}\mathbb{X}$ whose carrier $\underline{\mu}\underline{\mathbb{X}}$ is the VCS $\overline{\sigma}\underline{\mathbb{X}}$ and whose totality $\mathbb{T}(\underline{\mu}\mathbb{X})$ is such that $(\overline{\sigma}\underline{\mathbb{X}}(\vec{X}), \mathbb{T}(\underline{\mu}\mathbb{X})(\vec{X})) = \underline{\mu}(\underline{\mathbb{X}}_{\vec{X}})$ for all $\vec{X} \in \mathbf{CohT}^n$. We can summarize our constructions as follows.

Theorem 4.9. *Let \mathbb{X} be an $n + 1$ -ary VCST. There is a unique VCST $\underline{\mu}\mathbb{X}$ whose carrier is $\overline{\sigma}\underline{\mathbb{X}}$ and whose totality is such that $\mathbb{X}(\vec{X}, \underline{\mu}\mathbb{X}(\vec{X})) = \underline{\mu}\mathbb{X}(\vec{X})$ and $(\underline{\mu}\mathbb{X}(\vec{X}), \text{ld})$ is initial in the category $\mathbf{Alg}_{\mathbf{CohT}}(\underline{\mathbb{X}}_{\vec{X}})$.*

Moreover, we have provided a “concrete” way for defining this operation (which involves an ordinal iteration).

Now we can define “greatest fixed points” by De Morgan duality. So let \mathbb{X} be an $n + 1$ -ary VCST. Given an n -tuple of coherence spaces with totality \vec{X} , we set $\nu\mathbb{X}(\vec{X}) = (\underline{\mu}((\underline{\mathbb{X}})^\perp)(\vec{X}^\perp))^\perp$. More precisely, this means that the carrier of $\nu\mathbb{X}$ is the VCS $\overline{\sigma}\underline{\mathbb{X}}$ (the *very same* as for $\underline{\mu}\mathbb{X}$), and that $\mathbb{T}(\nu\mathbb{X})(\vec{X}) \in \mathbf{Tot}(\overline{\sigma}\underline{\mathbb{X}}(\vec{X}))$ is given by $\mathbb{T}(\nu\mathbb{X})(\vec{X}) = (\mathbb{T}(\underline{\mu}(\underline{\mathbb{X}}^\perp))(\vec{X}^\perp))^\perp$ which indeed makes sense because $\mathbb{T}\underline{\mu}(\underline{\mathbb{X}}^\perp)(\vec{X}^\perp) \in \mathbf{Tot}(\overline{\sigma}\underline{\mathbb{X}}^\perp(\vec{X}^\perp))$ and $\overline{\sigma}\underline{\mathbb{X}}^\perp(\vec{X}^\perp) = (\overline{\sigma}\underline{\mathbb{X}}(\vec{X}))^\perp$ by definition of the De Morgan dual of a VCS.

More concretely, this means that $\mathbb{T}(\nu\mathbb{X})(\vec{X}) = (\overline{\sigma}\underline{\mathbb{X}}(\vec{X}), \mathcal{V})$ where \mathcal{V} is the greatest totality candidate of $\overline{\sigma}\underline{\mathbb{X}}(\vec{X})$ such that $F(\mathcal{V}) = \mathcal{V}$ where $F(\mathcal{T}) = \mathbb{T}\mathbb{X}(\vec{X}, (\overline{\sigma}\underline{\mathbb{X}}(\vec{X}), \mathcal{T}))$. In other words, $\mathcal{V} = \bigcap_\alpha F^\alpha(\mathbf{Cl}(\overline{\sigma}\underline{\mathbb{X}}(\vec{X})))$.

5. $\underline{\mu}\mathbf{LL}$ AND ITS INTERPRETATION

We assume to be given an infinite set of propositional variables \mathcal{V} (ranged over by Greek letters $\zeta, \xi \dots$). We introduce a language of propositional LL formulas with least and greatest

fixed points.

$$\begin{aligned}
A, B, \dots := & 1 \mid \perp \mid A \otimes B \mid A \wp B \\
& \mid 0 \mid \top \mid A \oplus B \mid A \& B \\
& \mid !A \mid ?A \\
& \mid \zeta \mid \mu\zeta A \mid \nu\zeta A.
\end{aligned}$$

The notion of closed types is defined as usual, the two last constructions being the only binders.

Remark 5.1. In contrast with second-order linear logic or dependent type systems where open formulas or types play a crucial role — and it is necessary there to provide an interpretation of proofs or programs having non-closed types —, in the case of fixed points, all formulas appearing in sequents and other syntactical objects allowing to give types to programs will be closed. In our setting, open types appear only locally, for allowing the expression of (least and greatest) fixed points. This is made possible by the fact that the deduction rules we consider for these operations preserve closeness of formulas in both directions (upwards and downwards) unlike the rule for quantifiers (think of \forall -intro).

We can define two basic operations on formulas.

- *Substitution*: $A[B/\zeta]$, taking care of not binding free variables (uses α -conversion).
- *Negation* or *dualization*: defined by induction on formulas $1^\perp = \perp$, $\perp^\perp = 1$, $(A \wp B)^\perp = A^\perp \otimes B^\perp$, $(A \otimes B)^\perp = A^\perp \wp B^\perp$, $0^\perp = \top$, $\top^\perp = 0$, $(A \& B)^\perp = A^\perp \oplus B^\perp$, $(A \oplus B)^\perp = A^\perp \& B^\perp$, $(!A)^\perp = ?A^\perp$, $(?A)^\perp = !A^\perp$, $\zeta^\perp = \zeta$, $(\mu\zeta A)^\perp = \nu\zeta A^\perp$ and $(\nu\zeta A)^\perp = \mu\zeta A^\perp$.

Remark 5.2. The only subtle point of this definition is negation of propositional variables: $\zeta^\perp = \zeta$. The purpose of this is that we have $(B[A/\zeta])^\perp = B^\perp[A^\perp/\zeta]$ as easily proven by induction on B . If we consider B as a compound connective with placeholders labeled by variables then B^\perp is its De Morgan dual. This is also a very natural way of preventing the introduction of fixed points wrt. variables with negative occurrences. For instance $D = \mu\zeta (1 \& (!\zeta \multimap \zeta))$ is *not* a formula of μLL . Indeed D should be written $\mu\zeta (1 \& (?(\zeta^\perp) \wp \zeta))$ which is not a formula of μLL (if we do not remove the linear negation) or has not the intended meaning if we apply the equation $\zeta^\perp = \zeta$. The rejection of such general recursive types is coherent with the fact that D allows, for instance, to type all pure lambda-terms and hence also non-normalizing ones.

We give now the deduction rules, in a standard unilateral Linear Logic sequent calculus as in [1].

The identity fragment:

$$\frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \text{ (cut)}$$

The multiplicative fragment:

$$\frac{}{\vdash 1} \text{ (1)} \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \text{ (\otimes)} \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp} \text{ (\perp)} \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \text{ (\wp)}$$

The additive fragment:

$$\frac{}{\vdash \Gamma, \top} (\top) \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} (\oplus_1) \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} (\oplus_2) \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} (\&)$$

The exponential fragment:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A} (w) \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} (c) \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} (d) \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} (p)$$

The fixed point fragment:

$$\frac{\vdash \Gamma, F [\mu\zeta F/\zeta]}{\vdash \Gamma, \mu\zeta F} (\mu - \text{fold}) \quad \frac{\vdash \Gamma, F [\nu\zeta F/\zeta]}{\vdash \Gamma, \nu\zeta F} (\nu - \text{fold})$$

$$\frac{\vdash \Delta, A \quad \vdash ?\Gamma, A^\perp, F[A/\zeta]}{\vdash \Delta, ?\Gamma, \nu\zeta F} (\nu - \text{rec})$$

By taking, in the last rule, $\Delta = A^\perp$ and proving the left premise by an axiom, we obtain the following derived rule

$$\frac{\vdash ?\Gamma, A^\perp, F[A/\zeta]}{\vdash ?\Gamma, A^\perp, \nu\zeta F} (\nu - \text{rec}')$$

The only cut-elimination rule that we give is $(\mu - \text{fold})/(\nu - \text{rec})$, in Section 5.1; for the other ones, see for instance [13] or any other presentation of the classical LL Sequent Calculus. We refer to [1] for a proof that this system admits cut-elimination⁴. Observe that a cut-free proof has not the sub-formula property in general because of rule $(\nu - \text{rec})$. But Baelde's theorem makes sure that a proof of a sequent *which does not contain any ν -formula* has a cut-free proof with the sub-formula property. This is the main motivation for this apparently weird formulation of the ν -rule.

5.1. Functoriality of formulas. Let ζ be a variable, F a formula and τ be a proof of $\vdash ?\Gamma, A^\perp, B$, let $\vec{\xi} = (\xi_1, \dots, \xi_n)$ and $\vec{C} = (C_1, \dots, C_n)$ be closed formulas. Then one can define a proof $F \left[\tau/\zeta, \vec{C}/\vec{\xi} \right]$ of $\vdash ?\Gamma, F \left[A/\zeta, \vec{C}/\vec{\xi} \right]^\perp, F \left[B/\zeta, \vec{C}/\vec{\xi} \right]$ by induction on F , see [1]. As an example, assume that $F = \mu\xi G$. The proof $F \left[\tau/\zeta, \vec{C}/\vec{\xi} \right]$ is defined by (setting $G' = G \left[\vec{C}/\vec{\xi} \right]$)

$$G \left[\tau/\zeta, (\mu\xi G') [B/\zeta]/\xi, \vec{C}/\vec{\xi} \right]$$

$$\frac{\vdash ?\Gamma, G' [A/\zeta, (\mu\xi G') [B/\zeta]/\xi]^\perp, G' [B/\zeta, (\mu\xi G') [B/\zeta]/\xi]}{\vdash ?\Gamma, G' [A/\zeta, (\mu\xi G') [B/\zeta]/\xi]^\perp, (\mu\xi G') [B/\zeta]} (\mu - \text{fold})$$

$$\frac{\vdash ?\Gamma, G' [A/\zeta, (\mu\xi G') [B/\zeta]/\xi]^\perp, (\mu\xi G') [B/\zeta]}{\vdash ?\Gamma, (\mu\xi G') [A/\zeta]^\perp, (\mu\xi G') [B/\zeta]} (\nu - \text{rec}')$$

Let us also deal with the case $F = !G$. Then, with the same conventions as above, $F \left[\tau/\zeta, \vec{C}/\vec{\xi} \right]$ is defined as

⁴The system considered by Baelde is slightly different: no exponentials, no context in the $(\nu - \text{rec})$ rule. Though it seems quite clear that his proof can be adapted to the system presented here which has the same as Baelde's, in terms of provability. Observe however that, denotationally, our extension μLL is quite meaningful as explained in Section 6.4

$$\begin{array}{c}
G [\tau/\zeta, \vec{G}/\vec{\xi}] \\
\frac{\vdash ?\Gamma, G' [A/\zeta]^\perp, G' [B/\zeta]}{\vdash ?\Gamma, (!G' [A/\zeta])^\perp, G' [B/\zeta]} \text{ (d)} \\
\frac{\vdash ?\Gamma, (!G' [A/\zeta])^\perp, G' [B/\zeta]}{\vdash ?\Gamma, (!G' [A/\zeta])^\perp, !G' [B/\zeta]} \text{ (p)}
\end{array}$$

Where we crucially use the fact that the context is made of ?-formulas. This feature is also essential in the case where $F = G_1 \otimes G_2$, for instance.

5.2. Cut elimination. The only two reductions that we will mention here are the $(\mu - \text{fold})/(\nu - \text{rec})$ and $(\mu - \text{fold})/(\nu - \text{fold})$. Consider first a proof θ of shape

$$\frac{\frac{\frac{\pi}{\vdash \Lambda, F [\mu\zeta F/\zeta]} (\mu - \text{fold})}{\vdash \Lambda, \mu\zeta F} \quad \frac{\frac{\lambda}{\vdash \Delta, A^\perp} \quad \frac{\rho}{\vdash ?\Gamma, A, (F [A/\zeta])^\perp} (\nu - \text{rec})}{\vdash \Delta, ?\Gamma, (\mu\zeta F)^\perp}}{\vdash \Lambda, \Delta, ?\Gamma}$$

and let ρ' be the proof $\frac{\vdash A, A^\perp \quad \vdash ?\Gamma, A, (F [A/\zeta])^\perp}{\vdash ?\Gamma, A, (\mu\zeta F)^\perp}$. Then θ reduces to

$$\frac{\frac{\frac{F [\rho'/\zeta]}{\vdash ?\Gamma, F [A/\zeta], F [\mu\zeta F/\zeta]^\perp} \quad \frac{\pi}{\vdash \Lambda, F [\mu\zeta F/\zeta]}}{\vdash \Lambda, ?\Gamma, F [A/\zeta]} \quad \frac{\rho}{\vdash ?\Gamma, A, (F [A/\zeta])^\perp}}{\frac{\frac{\vdash \Lambda, ?\Gamma, ?\Gamma, A}{\vdash \Lambda, ?\Gamma, A} \text{ (c)} \quad \lambda}{\vdash \Delta, \Lambda, ?\Gamma}} \quad \frac{\lambda}{\vdash \Delta, A^\perp}$$

This reduction rule uses the functoriality of formulas as well as the ?-contexts in the $(\nu - \text{rec})$ rule.

Next, a proof of shape

$$\frac{\frac{\frac{\lambda}{\vdash \Gamma, A [\mu\zeta A/\zeta]} (\mu - \text{fold})}{\vdash \Gamma, \mu\zeta A} \quad \frac{\frac{\rho}{\vdash \Delta, (A [\mu\zeta A/\zeta])^\perp} (\nu - \text{fold})}{\vdash \Delta, (\mu\zeta A)^\perp}}{\vdash \Gamma, \Delta}$$

reduces to

$$\frac{\frac{\lambda}{\vdash \Gamma, A [\mu\zeta A/\zeta]} \quad \frac{\rho}{\vdash \Delta, (A [\mu\zeta A/\zeta])^\perp}}{\vdash \Gamma, \Delta}$$

Remark 5.3. In terms of *provability*, the rule $(\nu - \text{fold})$ is redundant since it can be derived as follows

$$\frac{\frac{A [\pi/\zeta]}{\vdash \Gamma, A [\nu\zeta A/\zeta]} \quad \frac{\vdash (A [\nu\zeta A/\zeta])^\perp, A [A [\nu\zeta A/\zeta]/\zeta]}{\vdash \Gamma, \nu\zeta A} (\nu - \text{rec})$$

where π is the following proof

$$\frac{\vdash (A [\nu\zeta A/\zeta])^\perp, A [\nu\zeta A/\zeta]}{\vdash (\nu\zeta A)^\perp, A [\nu\zeta A/\zeta]} (\mu - \text{fold})$$

Though, in terms of *algorithmic expressiveness*, the (ν -fold) rule is essential since it corresponds to pattern matching. For instance, with the type $\iota = \mu\zeta(1 \oplus \zeta)$ of integers (see 6.1), the (ν -fold) allows to define a function of type $\iota \multimap \iota$ which computes the predecessor of an integer n in a fixed number of reduction steps whereas the predecessor function defined using the above (ν -rec)-based definition of (ν -fold) requires a number of steps proportional to n .

5.3. Interpreting formulas and proofs. With any formula F and any repetition-free sequence $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$ of variables containing all variables free in F , one can associate an n -ary VCST $\llbracket F \rrbracket_{\vec{\zeta}}$ by induction on F , using straightforwardly the constructions of Section 4.2. If F is closed (again, this holds for any formula occurring in a sequent) then $\llbracket F \rrbracket$ is simply an object of **CohT**. If $\Gamma = (C_1, \dots, C_n)$ is a sequence of closed formulas then $\llbracket \Gamma \rrbracket = \llbracket C_1 \rrbracket \wp \dots \wp \llbracket C_n \rrbracket$.

Lemma 5.4. *Let F be a formula and $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$ be a repetition-free list of variables containing all free variables of F . Let G_1, \dots, G_n be a list of formulas and let $\vec{\xi} = (\xi_1, \dots, \xi_k)$ be a repetition-free list of variables containing all free variables of G_1, \dots, G_n . Then*

$$\llbracket F [G_1/\zeta_1, \dots, G_n/\zeta_n] \rrbracket_{\vec{\xi}} = \llbracket F \rrbracket_{\vec{\zeta}} \circ (\llbracket G_1 \rrbracket_{\vec{\xi}}, \dots, \llbracket G_n \rrbracket_{\vec{\xi}}).$$

The proof is a straightforward induction on F .

Next, with any proof π of a sequent $\vdash \Gamma$, we can associate $\llbracket \pi \rrbracket \in \mathbb{T}(\llbracket \Gamma \rrbracket)$. More precisely, $\llbracket \pi \rrbracket \in \mathbf{Cl}(\llbracket \Gamma \rrbracket)$ and it turns out that this clique is total in $\llbracket \Gamma \rrbracket$. We describe now this interpretation. For simplifying the notations, we drop the “ $_$ ” notation and write simply “ $\llbracket \pi \rrbracket \in \mathbf{Cl}(\llbracket \Gamma \rrbracket)$ ” instead of “ $\llbracket \pi \rrbracket \in \mathbf{Cl}(\llbracket \Gamma \rrbracket)$ ”.

5.3.1. *Reminder: interpreting the rules of LL.* If π is

$$\frac{}{\vdash A^+, A}$$

then $\llbracket \pi \rrbracket = \text{Id}_{\llbracket A \rrbracket} = \{(a, a) \mid a \in \llbracket A \rrbracket\}$. If π is

$$\frac{\lambda \quad \rho}{\frac{\vdash \Gamma, A \quad \vdash A^+, \Delta}{\vdash \Gamma, \Delta}}$$

then, considering that $\llbracket \lambda \rrbracket \in \mathbf{Coh}(\llbracket \Gamma \rrbracket^\perp, \llbracket A \rrbracket)$ and $\llbracket \rho \rrbracket \in \mathbf{Coh}(\llbracket A \rrbracket, \llbracket \Delta \rrbracket)$ then $\llbracket \pi \rrbracket = \llbracket \rho \rrbracket \llbracket \lambda \rrbracket$ that is $\llbracket \pi \rrbracket = \{(\gamma, \delta) \mid \exists a \in \llbracket A \rrbracket (\gamma, a) \in \llbracket \lambda \rrbracket \text{ and } (a, \delta) \in \llbracket \rho \rrbracket\}$. If π is

$$\frac{}{\vdash 1}$$

then $\llbracket \pi \rrbracket = \{*\}$. If π is

$$\frac{\lambda \quad \rho}{\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}}$$

then $\llbracket \pi \rrbracket = \llbracket \lambda \rrbracket \otimes \llbracket \rho \rrbracket$ considering that $\llbracket \lambda \rrbracket \in \mathbf{Coh}(\llbracket \Gamma \rrbracket^\perp, \llbracket A \rrbracket)$ and $\llbracket \rho \rrbracket \in \mathbf{Coh}(\llbracket \Delta \rrbracket^\perp, \llbracket B \rrbracket)$, that is $\llbracket \pi \rrbracket = \{(\gamma, \delta, (a, b)) \mid (\gamma, a) \in \llbracket \lambda \rrbracket \text{ and } (\delta, b) \in \llbracket \rho \rrbracket\}$. If π is

$$\frac{\lambda}{\frac{\vdash \Gamma}{\vdash \Gamma, \perp}}$$

then $\llbracket \pi \rrbracket = \{(\gamma, *) \mid \gamma \in \llbracket \lambda \rrbracket\}$. If π is

$$\frac{\lambda}{\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}}$$

then $\llbracket \pi \rrbracket = \llbracket \lambda \rrbracket$ if we consider that $\llbracket \lambda \rrbracket \in \mathbf{Coh}(\llbracket \Gamma \rrbracket^\perp, \llbracket A \wp B \rrbracket)$ that is $\llbracket \pi \rrbracket = \{(\gamma, (a, b)) \mid (\gamma, a, b) \in \llbracket \lambda \rrbracket\}$. If π is

$$\frac{}{\vdash \Gamma, \top}$$

then $\llbracket \pi \rrbracket = \emptyset$. If π is

$$\frac{\lambda \quad \rho}{\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}}$$

then $\llbracket \pi \rrbracket = \langle \llbracket \lambda \rrbracket, \llbracket \rho \rrbracket \rangle$ considering that $\llbracket \lambda \rrbracket \in \mathbf{Coh}(\llbracket \Gamma \rrbracket^\perp, A)$ and $\llbracket \rho \rrbracket \in \mathbf{Coh}(\llbracket \Gamma \rrbracket^\perp, B)$, that is $\llbracket \pi \rrbracket = \{(\gamma, (1, a)) \mid (\gamma, a) \in \llbracket \lambda \rrbracket\} \cup \{(\gamma, (2, b)) \mid (\gamma, b) \in \llbracket \rho \rrbracket\}$. If π is

$$\frac{\lambda}{\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}}$$

then $\llbracket \pi \rrbracket = \bar{\pi}_1 \llbracket \lambda \rrbracket$ where $\bar{\pi}_1 \in \mathbf{Coh}(\llbracket A \rrbracket, \llbracket A \oplus B \rrbracket)$, that is $\llbracket \pi \rrbracket = \{(\gamma, (1, a)) \mid (\gamma, a) \in \llbracket \lambda \rrbracket\}$, considering that $\llbracket \lambda \rrbracket \in \mathbf{Coh}(\llbracket \Gamma \rrbracket^\perp, \llbracket A \rrbracket)$. If π ends with a right \oplus -rule, the interpretation is similar. If π ends with

$$\frac{\lambda}{\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}}$$

then $\llbracket \pi \rrbracket = \mathbf{der}_{\llbracket A \rrbracket^\perp}^\perp \llbracket \lambda \rrbracket$, that is $\llbracket \pi \rrbracket = \{(\gamma, \{a\}) \mid (\gamma, a) \in \llbracket \lambda \rrbracket\}$. If π is

$$\frac{\lambda}{\frac{\vdash \Gamma}{\vdash \Gamma, ?A}}$$

then $\llbracket \pi \rrbracket = \mathbf{w}_{\llbracket A \rrbracket^\perp}^\perp \llbracket \lambda \rrbracket$, that is $\llbracket \pi \rrbracket = \{(\gamma, \emptyset) \mid (\gamma, a) \in \llbracket \lambda \rrbracket\}$. If π is

$$\frac{\lambda}{\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}}$$

then $\llbracket \pi \rrbracket = \mathbf{contr}_{\llbracket A \rrbracket^\perp}^\perp \llbracket \lambda \rrbracket$, that is $\llbracket \pi \rrbracket = \{(\gamma, x_1 \cup x_2) \mid (\gamma, x_i) \in \llbracket \lambda \rrbracket \text{ for } i = 1, 2 \text{ and } x_1 \cup x_2 \in \llbracket ?A \rrbracket\}$, considering that $\llbracket \lambda \rrbracket \in \mathbf{Coh}(\llbracket \Gamma \rrbracket^\perp, \llbracket ?A \rrbracket \wp \llbracket ?A \rrbracket)$. If π is

$$\frac{\lambda}{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}}$$

then, considering that $\llbracket \lambda \rrbracket \in \mathbf{Coh}(!E_1 \otimes \dots \otimes !E_k, \llbracket A \rrbracket)$ where $\Gamma = (C_1, \dots, C_k)$ and $E_j = \llbracket C_j \rrbracket^\perp$ for $j = 1, \dots, k$, we set $\llbracket \pi \rrbracket = \llbracket \lambda \rrbracket^!$ that is $\llbracket \pi \rrbracket = \{(\gamma^1 \cup \dots \cup \gamma^p, \{a_1, \dots, a_p\}) \mid (\gamma^l, a_l) \in \llbracket \lambda \rrbracket \text{ for } l = 1, \dots, p \text{ and } \gamma_j^1 \cup \dots \cup \gamma_j^p \in \mathbf{Cl}(E_j) \text{ for } j = 1, \dots, k\}$.

5.3.2. *Interpreting the fixed point rules.* Assume that π is

$$\frac{\lambda}{\frac{\vdash \Gamma, A[\mu\zeta A/\zeta]}{\vdash \Gamma, \mu\zeta A} (\mu - \text{fold})}$$

We have $\llbracket \lambda \rrbracket \in \mathbf{CohT}(\llbracket \Gamma \rrbracket^\perp, \llbracket A \rrbracket_\zeta(\mu \llbracket A \rrbracket_\zeta))$ by inductive hypothesis and Lemma 5.4, that is $\llbracket \lambda \rrbracket \in \mathbf{CohT}(\llbracket \Gamma \rrbracket^\perp, \mu \llbracket A \rrbracket_\zeta)$, so that we simply set $\llbracket \pi \rrbracket = \llbracket \lambda \rrbracket$. The interpretation is similar if π ends with $(\nu - \text{fold})$. Last, assume that π is

$$\frac{\lambda}{\frac{\vdash ?\Gamma, B^\perp, A[B/\zeta]}{\vdash ?\Gamma, B^\perp, \nu\zeta A} (\nu - \text{rec}')}$$

Let $(C_1, \dots, C_k) = \Gamma$, let $X_i = \llbracket C_i \rrbracket^\perp$ for $i = 1, \dots, k$, and let $Y = X_1 \& \dots \& X_k$ so that setting $P = !X_1 \otimes \dots \otimes !X_k$ we have a (generalized) Seelye iso $m_{\bar{X}} \in \mathbf{CohT}(P, !Y)$. Then we have $\llbracket \lambda \rrbracket \in \mathbf{CohT}(P \otimes \llbracket B \rrbracket, \llbracket A \rrbracket_\zeta(\llbracket B \rrbracket))$ by inductive hypothesis, using also Lemma 5.4. Then we consider the object $(P \otimes \llbracket B \rrbracket, f)$ of $\mathbf{Alg}_{\mathbf{CohT}}(\llbracket A \rrbracket_\zeta)$ where f is defined as the following composition of morphisms

$$\begin{array}{c} P \otimes \llbracket B \rrbracket \\ \downarrow \text{contr} \otimes \llbracket B \rrbracket \\ P \otimes P \otimes \llbracket B \rrbracket \\ \downarrow m_{\bar{X}} \otimes \llbracket \lambda \rrbracket \\ !Y \otimes \llbracket A \rrbracket_\zeta(\llbracket B \rrbracket) \\ \downarrow \widehat{\llbracket A \rrbracket}_{\zeta Y, \llbracket B \rrbracket}} \\ \llbracket A \rrbracket_\zeta(!Y \otimes \llbracket B \rrbracket) \\ \downarrow \llbracket A \rrbracket_\zeta(m_{\bar{X}}^{-1} \otimes \llbracket B \rrbracket) \\ \llbracket A \rrbracket_\zeta(P \otimes \llbracket B \rrbracket) \end{array}$$

Then $\llbracket \pi \rrbracket$ is the unique morphism of $\mathbf{Alg}_{\mathbf{CohT}}(\llbracket A \rrbracket_\zeta)$ from the coalgebra $(P \otimes \llbracket B \rrbracket, f)$ to the final coalgebra $(\llbracket \nu\zeta A \rrbracket, \text{Id})$. So $\llbracket \pi \rrbracket = \bigcup_{n=0}^{\infty} g_n$ where the morphisms $g_n \in \mathbf{Coh}(\underline{P} \otimes \underline{\llbracket B \rrbracket}, \sigma \llbracket A \rrbracket_\zeta)$ are defined by: $g_0 = \emptyset$ and g_{n+1} is

$$\begin{array}{c} \underline{P} \otimes \underline{\llbracket B \rrbracket} \\ \downarrow \text{contr} \otimes \underline{\llbracket B \rrbracket}} \\ \underline{P} \otimes \underline{P} \otimes \underline{\llbracket B \rrbracket} \\ \downarrow m_{\bar{X}} \otimes \llbracket \lambda \rrbracket \\ \underline{!Y} \otimes \underline{\llbracket A \rrbracket}_\zeta(\underline{\llbracket B \rrbracket}) \\ \downarrow \widehat{\underline{\llbracket A \rrbracket}}_{\zeta Y, \underline{\llbracket B \rrbracket}}} \\ \underline{\llbracket A \rrbracket}_\zeta(\underline{!Y} \otimes \underline{\llbracket B \rrbracket}) \\ \downarrow \underline{\llbracket A \rrbracket}_\zeta(m_{\bar{X}}^{-1} \otimes \underline{\llbracket B \rrbracket}) \\ \underline{\llbracket A \rrbracket}_\zeta(\underline{P} \otimes \underline{\llbracket B \rrbracket}) \\ \downarrow \underline{\llbracket A \rrbracket}_\zeta(g_n) \\ \underline{\llbracket A \rrbracket}_\zeta(\sigma \underline{\llbracket A \rrbracket}_\zeta) = \sigma \underline{\llbracket A \rrbracket}_\zeta \end{array}$$

Indeed $(g_n)_{n \in \mathbb{N}}$ is a monotonic sequence of cliques and so $g = \bigcup_{n \in \mathbb{N}} g_n \in \mathbf{Coh}(P \otimes \llbracket B \rrbracket, \sigma \llbracket A \rrbracket_\zeta)$. To prove our contention that $\llbracket \pi \rrbracket = g$, it suffices to prove that $g \in \mathbf{Alg}_{\mathbf{Coh}}(\llbracket A \rrbracket_\zeta)((\underline{P} \otimes \llbracket B \rrbracket, f), (\sigma \llbracket A \rrbracket_\zeta, \mathbf{Id}))$, that is $g = \llbracket A \rrbracket_\zeta(g) f$ which follows readily from the observation that $\llbracket A \rrbracket_\zeta(g_n) f = g_{n+1}$ by continuity of $\llbracket A \rrbracket_\zeta$ (remember that the functor $\llbracket A \rrbracket_\zeta$ is continuous in the sense of Definition 3.4).

Lemma 5.5. *Let π be a proof of $\vdash (!\Gamma)^\perp, A^\perp, B$ and consider $\llbracket \pi \rrbracket$ as an element of $\mathbf{CohT}(!\llbracket \Gamma \rrbracket)(\llbracket A \rrbracket, \llbracket B \rrbracket)$ (see the definition of this category of free comodules in Section 2.1; we use the canonical structure of $!$ -coalgebra of $!\llbracket \Gamma \rrbracket$). Let F be a formula and $\vec{C} = (C_1, \dots, C_k)$ be closed formulas and $\zeta, \zeta_1, \dots, \zeta_k$ be pairwise distinct variables. Then $\llbracket F \left[\pi / \zeta, \vec{C} / \vec{\zeta} \right] \rrbracket = \llbracket F \rrbracket_{\zeta, \vec{\zeta}}(\llbracket \pi \rrbracket, \llbracket \vec{C} \rrbracket / \vec{\zeta})$ (see Section 2.2 for the action of a strong functor on a category of free comodules).*

This means that our definition of the functorial action of formulas on proofs in Section 5.1 is compatible with the definition of the category of free comodules and of the extension of a strong functor to this category explained in Section 2.2. The proof is a simple induction on F .

As usual a main feature of this interpretation is the following.

Theorem 5.6. *If π reduces to π' , then $\llbracket \pi \rrbracket = \llbracket \pi' \rrbracket$.*

The proof is a lengthy and boring verification which uses crucially Lemma 5.5 in the most interesting case, which is the $(\mu - \text{fold}) / (\nu - \text{rec})$ cut reduction.

6. EXAMPLES

6.1. Some data and non-data types.

Strict integers. The type of strict integers is $\iota = \mu\zeta(1 \oplus \zeta)$. The following deduction rules are derivable in μLL :

$$\frac{}{\vdash \iota} (0) \quad \frac{\vdash \Delta, \iota}{\vdash \Delta, \iota} (\text{succ}) \quad \frac{\vdash ?\Gamma, A \quad \vdash ?\Gamma, C^\perp, C}{\vdash ?\Gamma, \iota^\perp, C} (\text{it}_\iota)$$

The first rule corresponds to the constant 0, the second one to the successor function and the third one should be understood as an iteration principle: the first premise is the base case and the second one is the “inductive step”.

More precisely these proofs are defined as follows.

$$\frac{\frac{}{\vdash 1}}{\vdash 1 \oplus \iota} (\mu - \text{fold}) \quad \frac{\vdash \Delta, \iota}{\vdash \Delta, 1 \oplus \iota} (\mu - \text{fold}) \quad \frac{\frac{\vdash ?\Gamma, C}{\vdash ?\Gamma, \perp, C} \quad \vdash ?\Gamma, C^\perp, C}{\vdash ?\Gamma, \perp \& C^\perp, C} (\nu - \text{rec}')$$

The coherence space $\llbracket \iota \rrbracket$ is the least fixed point of the VCS $E \mapsto 1 \oplus E$ so that, up to trivial iso, $\llbracket \iota \rrbracket$ is the coherence space $\mathbf{N} = (\mathbb{N}, =)$. For computing $\mathbf{T}[\llbracket \iota \rrbracket]$, the method explained in Section 4.2.2 boils down to computing the least fixed point of the map $\Theta : \mathbf{Tot}(\mathbf{N}) \rightarrow \mathbf{Tot}(\mathbf{N})$ such that $\Theta(T) = \{\{0\}\} \cup \{x+1 \mid x \in T\}$ (where $x+1 = \{n+1 \mid n \in x\}$). We have $\Theta^n(\emptyset) = \{\{0\}, \{1\}, \dots, \{n-1\}\}$ so $T = \bigcup_{n \in \mathbb{N}} \Theta^n(\emptyset) = \{\{n\} \mid n \in \mathbb{N}\}$. Hence $T^\perp = \{\mathbf{N}\}$

and it follows that $T = T^{\perp\perp}$, hence T is the least fixed point of Θ in $\text{Tot}(\mathbf{N})$. Finally $\mathbb{T}[\iota] = \{\{n\} \mid n \in \mathbf{N}\}$.

We specialize the definition of Section 5.3.2 taking (with the notations of that section) $A = \perp \& \zeta$ and $B = C^{\perp}$. If π is

$$\frac{}{\vdash \iota} \text{ (0)}$$

then $\llbracket \pi \rrbracket = \{0\}$. If π is

$$\frac{\lambda}{\frac{\vdash \Delta, \iota}{\vdash \Delta, \iota} \text{ (succ)}}$$

then $\pi = \{(\delta, n+1) \mid (\delta, n) \in \llbracket \lambda \rrbracket\}$. For the (it_ι) rule, assume that π is

$$\frac{\frac{\lambda}{\vdash ?\Gamma, C} \quad \frac{\rho}{\vdash ?\Gamma, C^{\perp}, C}}{\vdash ?\Gamma, \iota^{\perp}, C} \text{ (it}_\iota\text{)}$$

One has first to compute $\llbracket \perp \& \zeta \rrbracket_{\zeta}$ in this special case applying the general definition of formula interpretation as explained in Section 5.3. Then given coherence spaces E and F , the morphism $(\llbracket \perp \& \zeta \rrbracket_{\zeta})_{F,E} \in \mathbf{Coh}(!F \otimes (\perp \& E), \perp \& (!F \otimes E))$ is $\{((\emptyset, (1, *)), (1, *))\} \cup \{((y_0, (2, a)), (2, (y_0, a))) \mid y_0 \in !F \text{ and } a \in |E|\}$. Next applying the recipe at the end of Section 5.3.2 we obtain that $\llbracket \pi \rrbracket$ is the least $g \in \mathbf{Coh}(!\llbracket \Gamma \rrbracket^{\perp} \otimes \mathbf{N}, \llbracket C \rrbracket)$ such that

$$\begin{aligned} g = & \{(\gamma, 0, c) \mid (\gamma, c) \in \llbracket \lambda \rrbracket\} \\ & \cup \{(\gamma^1 \cup \gamma^2, n+1, c) \mid \exists c' \in \llbracket C \rrbracket \text{ } (\gamma^1, n, c') \in g, \\ & \quad (\gamma^2, c', c) \in \llbracket \rho \rrbracket \text{ and } \gamma^1 \cup \gamma^2 \in !\llbracket \Gamma \rrbracket^{\perp}\}. \end{aligned}$$

Setting $(A_1, \dots, A_k) = \Gamma^{\perp}$ we can consider $\llbracket \lambda \rrbracket$ as a stable function $h : \mathbf{Cl}(\llbracket A_1 \rrbracket) \times \dots \times \mathbf{Cl}(\llbracket A_1 \rrbracket) \rightarrow \mathbf{Cl}(\llbracket C \rrbracket)$ and $\llbracket \rho \rrbracket$ as a stable function $k : \mathbf{Cl}(\llbracket A_1 \rrbracket) \times \dots \times \mathbf{Cl}(\llbracket A_1 \rrbracket) \times \mathbf{Cl}(\llbracket C \rrbracket) \rightarrow \mathbf{Cl}(\llbracket C \rrbracket)$ which is *linear in its last argument*. Then g can be seen as the stable function $g : \mathbf{Cl}(\llbracket A_1 \rrbracket) \times \dots \times \mathbf{Cl}(\llbracket A_1 \rrbracket) \times \mathbf{Cl}(\llbracket \mathbf{N} \rrbracket) \rightarrow \mathbf{Cl}(\llbracket C \rrbracket)$, linear in its last argument, such that $g(\vec{x}, \{0\}) = h(\vec{x})$ and $g(\vec{x}, \{n+1\}) = k(\vec{x}, g(\vec{x}, \{n\}))$.

Remark 6.1. If π is a proof of $\vdash \iota^{\perp}, \iota$ then $\llbracket \pi \rrbracket$ is a clique of $\mathbf{N} \multimap \mathbf{N}$ such that $\forall n \exists p (n, p) \in \llbracket \pi \rrbracket$, that is, a total function $\mathbf{N} \rightarrow \mathbf{N}$. Using the derived rules above, we shall see in Section 6.3 that a version of Gödel's System T can be represented in μLL , which shows that all functions $\mathbf{N} \rightarrow \mathbf{N}$ representable in System T appear as interpretations of such proofs π .

Lazy integers. One can define a type of lazy integers as $\iota_l = \mu\zeta (1 \oplus !\zeta)$. The following deduction rules are derivable in μLL :

$$\frac{}{\vdash \iota_l} \quad \frac{\vdash \Delta, !\iota_l}{\vdash \Delta, \iota_l} \quad \frac{\vdash ?\Gamma, A \quad \vdash ?\Gamma, !A \multimap A}{\vdash ?\Gamma, \iota_l \multimap A}$$

Then $\llbracket \iota_l \rrbracket$ is an object of \mathbf{CohT} , that is, a coherence space L equipped with a totality T which can be described as follows. One defines a sequence of coherence spaces L_n by: L_0 is the coherence space which has $\{(1, *)\}$ as web (where $*$ is the unique element of $|1|$) and, if L_n is known then an element of $|L_{n+1}|$ is either $(1, *)$ or a pair $(2, x_0)$ where x_0 is a finite clique of L_n . The coherence relation given by $(i, a) \frown (j, b)$ iff $i = j = 2$ and $a \frown_{!L_n} b$. Last $|L|$ is the union of all the $|L_n|$ (which form a monotone sequence of sets)

and $a \supset_L b$ if $a \supset_{L^n} b$ for some n . With each integer k is associated a total clique $x(k)$ of L : $x(0) = \{(1, *)\}$ and $x(k+1) = \{(2, x_0) \mid x_0 \subseteq x(k) \text{ and } x_0 \text{ finite}\}$ so that $x(1) = \{(2, \emptyset), (2, \{(1, *)\})\}$, $x(2) = \{(2, \emptyset), (2, \{(2, \emptyset)\}), (2, \{(2, \{(1, *)\})\}), (2, \{(2, \emptyset), (2, \{(1, *)\})\})\}$ etc. The clique $\{(2, \emptyset)\}$, which is not total, represents a “partial integer” that one could denote as $\text{succ}(\Omega)$ and $\{((1, *), (1, *)), ((2, \emptyset), (2, *))\}$ is the interpretation of a proof of $\iota \multimap 1 \oplus 1$ which tests whether an integer is zero or not without evaluating it completely:

$$\frac{\frac{\overline{\vdash 1}}{\vdash 1 \oplus 1} (\oplus_1) \quad \frac{\frac{\overline{\vdash 1}}{\vdash 1 \oplus 1} (\oplus_2) \quad \frac{\overline{\vdash 1}}{\vdash 1 \oplus 1} (\oplus_2)}{\vdash ?(\perp \& \perp), 1 \oplus 1} (w)}{\vdash !(1 \oplus 1) \multimap 1 \oplus 1}}{\vdash \iota \multimap 1 \oplus 1}$$

Streams of booleans. Let $\gamma = \nu\zeta(1 \& (\zeta \oplus \zeta))$, let C be the carrier of $\llbracket \gamma \rrbracket$ and T be the totality. Then $|C|$ can be described as the set $\{0, 1\}^{<\omega}$ of finite sequences of booleans, and $s \supset_C t$ if s is a prefix of t or conversely. So a clique of C is set x of finite sequences which is totally ordered by the prefix order, and one checks easily that T is the set of all maximal such sets: up to iso, $T = \{0, 1\}^\omega$. Dually a total clique of $\llbracket \gamma^\perp \rrbracket$ is a set x' of pairwise incomparable sequences such that each element of $\{0, 1\}^\omega$ has a prefix in x' . Such an x' is finite by compactness of the Cantor Space.

Another definition is considered in the literature for the type of streams of booleans, namely $\gamma' = \nu\zeta(\zeta \oplus \zeta)$. A simple computation shows that $\llbracket \gamma' \rrbracket = \top$ which may seem weird since the formula γ' has many different proofs in μLL . This is due to the fact that there is no way to write a proof π of $\gamma' \multimap 1 \oplus 1$ which would allow to extract a finite information from a stream of type γ' (for instance, its first element). Actually, this model provides a proof of this fact: if such a proof would exist, it would map \emptyset (total in $\llbracket \gamma' \rrbracket = \top$) to \emptyset (not total in $1 \oplus 1$).

6.2. Polarization in μLL . Among all formulas of μLL , we define two subsets: *positive* and *negative* formulas. We use letters P, Q, \dots to denote positive formulas and letters M, N, \dots for negative formulas and as before A, B, \dots for general formulas.

$$\begin{aligned} P, Q, \dots &:= 0 \mid P \oplus Q \mid 1 \mid P \otimes Q \mid !A \mid \mu\zeta P \mid \zeta \\ M, N, \dots &:= \top \mid M \& N \mid \perp \mid M \wp N \mid ?A \mid \nu\zeta M \mid \zeta \end{aligned}$$

The only formulas which are at the same time positive and negative are the variables ζ 's. So there is no closed formula which is both negative and positive, and M is negative iff M^\perp is positive. There are of course μLL formulas, like $0 \otimes \perp$, which are neither positive nor negative.

From our Curry-Howard point of view, the main property of positive formulas is the following, which expresses that positive formulas have “structural rules”, meaning that they are types of storable values.

Theorem 6.2. *For each positive formula P , each sequence of pairwise distinct variables $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$ and each sequence of closed formulas $\vec{A} = (A_1, \dots, A_n)$ there is a proof $\chi(P, !\vec{A}/\vec{\zeta})$ in μLL of $\vdash P^\perp, !P'$ where $P' = P \left[\vec{A}/\vec{\zeta} \right]$.*

Proof. By induction on P . If A is a formula, we use A' for $A \left[\vec{A}/\vec{\zeta} \right]$. The proof $\chi(0, !\vec{A}/\vec{\zeta})$ is an instance of the (\top) axiom. The proof $\chi(\zeta_i, !\vec{A}/\vec{\zeta})$ is

$$\frac{\overline{\vdash (!A_i)^\perp, !A_i}}{\vdash (!A_i)^\perp, !!A_i}$$

The proof $\chi(P_1 \oplus P_2, !\vec{A}/\vec{\zeta})$ is

$$\frac{\frac{\frac{\overline{\vdash P_1^\perp, P_1'}}{\vdash P_1^\perp, P_1 \oplus P_2'} \quad \chi(P_1, !\vec{A}/\vec{\zeta})}{\vdash P_1^\perp, !P_1} \quad \frac{\overline{\vdash P_2^\perp, P_2'}}{\vdash P_2^\perp, P_1 \oplus P_2'} \quad \chi(P_2, !\vec{A}/\vec{\zeta})}{\vdash P_1^\perp, !P_1 \quad \vdash P_2^\perp, !P_2} \text{ (cut)} \quad \frac{\overline{\vdash P_2^\perp, P_2'}}{\vdash P_2^\perp, !P_2} \quad \frac{\overline{\vdash P_1^\perp, P_1'}}{\vdash P_1^\perp, P_1 \oplus P_2'} \quad \chi(P_1, !\vec{A}/\vec{\zeta})}{\vdash P_2^\perp, !P_2 \quad \vdash P_1^\perp, !P_1} \text{ (cut)} \quad \frac{\vdash P_1^\perp, !(P_1 \oplus P_2)}{\vdash P_1^\perp, !(P_1 \oplus P_2)} \text{ (&)} \quad \frac{\vdash P_2^\perp, !(P_1 \oplus P_2)}{\vdash P_2^\perp, !(P_1 \oplus P_2)} \text{ (&)}}{\vdash P_1^\perp \& P_2^\perp, !(P_1 \oplus P_2)} \text{ (&)}$$

The proof $\chi(1, !\vec{A}/\vec{\zeta})$ is

$$\frac{\overline{\vdash 1}}{\vdash !1} \quad \frac{\vdash !1}{\vdash \perp, !1}$$

The proof $\chi(P_1' \otimes P_2', !\vec{A}/\vec{\zeta})$ is

$$\frac{\frac{\frac{\overline{\vdash P_1'^\perp, P_1'}}{\vdash P_1'^\perp, !P_1'} \quad \chi(P_1', !\vec{A}/\vec{\zeta})}{\vdash P_1'^\perp, !P_1'} \quad \frac{\overline{\vdash P_2'^\perp, P_2'}}{\vdash P_2'^\perp, !P_2'} \quad \chi(P_2', !\vec{A}/\vec{\zeta})}{\vdash P_2'^\perp, !P_2'} \quad \frac{\overline{\vdash P_1'^\perp, P_1'}}{\vdash P_1'^\perp, P_1 \otimes P_2'} \quad \frac{\overline{\vdash P_2'^\perp, P_2'}}{\vdash P_2'^\perp, P_1 \otimes P_2'}}{\vdash P_1'^\perp, !P_1' \quad \vdash P_2'^\perp, !P_2'} \text{ (cut)} \quad \frac{\overline{\vdash P_1'^\perp, P_1'}}{\vdash P_1'^\perp, P_1 \otimes P_2'} \quad \frac{\overline{\vdash P_2'^\perp, P_2'}}{\vdash P_2'^\perp, P_1 \otimes P_2'}}{\vdash P_1'^\perp, P_2'^\perp, !(P_1' \otimes P_2')} \text{ (cut)} \quad \frac{\vdash P_1'^\perp, P_2'^\perp, !(P_1' \otimes P_2')}{\vdash P_1'^\perp \wp P_2'^\perp, !(P_1' \otimes P_2')}}{\vdash P_1'^\perp \wp P_2'^\perp, !(P_1' \otimes P_2')}$$

The proof $\chi(!A, !\vec{A}/\vec{\zeta})$ is

$$\frac{\overline{\vdash ?A'^\perp, !A'}}{\vdash ?A'^\perp, !!A'}$$

The proof $\chi(\mu\zeta P, !\vec{A}/\vec{\zeta})$ is

$$\frac{\frac{\frac{\overline{\vdash P[\delta/\zeta]}}{\vdash P[!\mu\zeta P/\zeta]^\perp, P[\mu\zeta P/\zeta]} \quad \chi(P, !(\vec{A}, \mu\zeta P)/\vec{\zeta}, \zeta)}{\vdash P[!\mu\zeta P/\zeta]^\perp, !P[!\mu\zeta P/\zeta]} \quad \frac{\overline{\vdash P[!\mu\zeta P/\zeta]^\perp, \mu\zeta P}}{\vdash (!P[!\mu\zeta P/\zeta])^\perp, \mu\zeta P} \quad \frac{\overline{\vdash P[!\mu\zeta P/\zeta]^\perp, \mu\zeta P}}{\vdash (!P[!\mu\zeta P/\zeta])^\perp, !(\mu\zeta P)}}{\vdash P[!\mu\zeta P/\zeta]^\perp, !(\mu\zeta P)} \text{ (}\nu\text{-rec')}$$

where δ is the proof

$$\frac{\overline{\vdash \mu\zeta P^\perp, \mu\zeta P}}{\vdash (!\mu\zeta P)^\perp, \mu\zeta P}$$

As an example (up to some cut-eliminations) the proof $\chi(\iota)$ is

$$\frac{\frac{\overline{\vdash \iota}}{\vdash !\iota} (0) \quad \frac{\overline{\vdash \iota^\perp, \iota}}{\vdash \iota^\perp, \iota} (\text{succ})}{\vdash \perp, !\iota} \quad \frac{\overline{\vdash (!\iota)^\perp, \iota}}{\vdash (!\iota)^\perp, !\iota} (\&)}{\vdash \perp \& (!\iota)^\perp, !\iota} (\nu - \text{rec}') \quad \frac{}{\vdash \iota^\perp, !\iota}$$

Theorem 6.3. *Let N_1, \dots, N_k be closed negative formulas and let A be a closed formula. Then the generalized promotion rule and the generalized structural rules*

$$\frac{\vdash N_1, \dots, N_k, A}{\vdash N_1, \dots, N_k, !A} (\text{gp}) \quad \frac{\vdash \Gamma}{\vdash N, \Gamma} (\text{gw}) \quad \frac{\vdash N, N, \Gamma}{\vdash N, \Gamma} (\text{gc})$$

are derivable in μLL .

Proof. Cut the proof

$$\frac{\vdash ?N_1, \dots, ?N_k, A}{\vdash ?N_1, \dots, ?N_k, !A}$$

against the proofs $\chi(N_i^\perp)$ (for $i = 1, \dots, k$) and similarly for the two other proofs. \blacktriangleleft

6.3. Representing a System T in μLL . We define a version of Gödel's System T where integers can be handled in a strict way thanks to an additional let construct⁵ (in contrast with the system presented in [16]) that we call T. We first define the syntax of our language:

$$\sigma, \tau \dots := \text{nat} \mid \sigma \Rightarrow \tau \quad (\text{types})$$

$$s, t, u \dots := \underline{n} \mid x \mid (s)t \mid \lambda x^\sigma s \mid \text{succ}(s) \mid \text{rec}(s, t, u) \mid \text{let}(x, s, t) \quad (\text{terms})$$

The typing rules are

$$\frac{n \in \mathbb{N}}{\Phi \vdash \underline{n} : \text{nat}} \quad \frac{}{\Phi, x : \sigma \vdash x : \sigma} \quad \frac{\Phi \vdash s : \sigma \Rightarrow \tau \quad \Phi \vdash t : \sigma}{\Phi \vdash (s)t : \tau}$$

$$\frac{\Phi, x : \sigma \vdash s : \tau}{\Phi \vdash \lambda x^\sigma s : \sigma \Rightarrow \tau} \quad \frac{\Phi \vdash s : \text{nat}}{\Phi \vdash \text{succ}(s) : \text{nat}} \quad \frac{\Phi \vdash s : \text{nat} \quad \Phi, x : \text{nat} \vdash t : \sigma}{\Phi \vdash \text{let}(x, s, t) : \sigma}$$

$$\frac{\Phi \vdash s : \text{nat} \quad \Phi \vdash t : \sigma \quad \Phi \vdash u : \text{nat} \Rightarrow \sigma \Rightarrow \sigma}{\Phi \vdash \text{rec}(s, t, u) : \sigma}$$

In these rules Φ denotes a typing context $\Phi = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$. We define an operational semantics by means of a weak-head reduction relation specified by the following deduction rules.

$$\frac{}{(\lambda x^\sigma s)t \beta_{wh} s [t/x]} \quad \frac{}{\text{let}(x, \underline{n}, s) \beta_{wh} s [\underline{n}/x]} \quad \frac{}{\text{succ}(\underline{n}) \beta_{wh} \underline{n} + 1}$$

⁵One motivation for focusing on such a language is explained in [11], in a context of probabilistic computing. Indeed the setting presented here can easily and meaningfully be applied to such computations whereas a purely CBN version of system T wouldn't really make sense in a probabilistic setting from an algorithmic expressiveness viewpoint.

$$\begin{array}{c}
\frac{}{\text{rec}(\underline{0}, t, u) \beta_{wh} t} \quad \frac{}{\text{rec}(\underline{n+1}, t, u) \beta_{wh} (u) \underline{n} \text{rec}(\underline{n}, t, u)} \quad \frac{s \beta_{wh} s'}{(s) t \beta_{wh} (s') t} \\
\frac{s \beta_{wh} s'}{\text{let}(x, s, t) \beta_{wh} \text{let}(x, s', t)} \quad \frac{s \beta_{wh} s'}{\text{succ}(s) \beta_{wh} \text{succ}(s')} \quad \frac{s \beta_{wh} s'}{\text{rec}(s, t, u) \beta_{wh} \text{rec}(s', t, u)}
\end{array}$$

Of course β_{wh} enjoys Subject Reduction as easily checked. The first five rules describe the three forms of redexes in our System \mathbb{T} , the let construct offering the ability to handle integers in a CBV manner. One can also define a general reduction relation allowing to reduce these redexes anywhere in a term and it can be proved that this general reduction is Church-Rosser, but here we focus on β_{wh} which is a deterministic reduction strategy turning \mathbb{T} into a programming language.

Normalization of \mathbb{T} . It is well known that System \mathbb{T} is strongly normalizing, but, since our presentation of this system is slightly different from the usual one (by our CBV handling of integers) it is meaningful to give a direct proof of this fact. We actually don't prove strong normalization, but normalization of the β_{wh} strategy, and only for closed terms of type nat , which will be enough for our purpose and can be done in a few lines.

Lemma 6.4. *If $\vdash s : \text{nat}$ and s is β_{wh} -normal then there is an integer n such that $s = \underline{n}$.*

Proof. Simple inductive analysis of the structure of s . ◀

Theorem 6.5. *The reduction β_{wh} is normalizing for closed terms of type ι .*

Proof. We adapt the standard reducibility proof. For each type σ we define a set Red_σ of reducible closed terms s such that $\vdash s : \sigma$. The definition, by induction on σ , is: Red_{nat} is the set of all normalizing closed terms of type nat , and $\text{Red}_{\sigma \Rightarrow \tau}$ is the set of all s such that $\vdash s : \sigma \Rightarrow \tau$ and, for all $t \in \text{Red}_\sigma$, one has $(s) t \in \text{Red}_\tau$.

One proves first that if $\vdash s : \sigma$ and $s \beta_{wh} s' \in \text{Red}_\sigma$, then $s \in \text{Red}_\sigma$. This is done by induction on σ . The case $\sigma = \text{nat}$ results immediately from the definition of Red_{nat} so assume that $\sigma = (\tau \Rightarrow \varphi)$. Let $t \in \text{Red}_\tau$, we have $(s) t \beta_{wh} (s') t \in \text{Red}_\tau$ by definition of β_{wh} and hypothesis on s' , and hence $(s) t \in \text{Red}_\tau$ by inductive hypothesis.

Last we prove by induction on s that if $x_1 : \sigma_1, \dots, x_k : \sigma_k \vdash s : \tau$, then for all $s_1 \in \text{Red}_{\sigma_1}, \dots, s_k \in \text{Red}_{\sigma_k}$, one has $s[s_1/x_1, \dots, s_k/x_k] \in \text{Red}_\tau$. We set $\Phi = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$ and $\bar{w} = w[s_1/x_1, \dots, s_k/x_k]$ for all term w .

- If $s = x_i$ or $s = \underline{n}$ the reasoning is straightforward.
- Assume $s = \text{succ}(t)$ with $\Phi \vdash t : \text{nat}$. By inductive hypothesis $\bar{t} \in \text{Red}_{\text{nat}}$ and hence $\bar{t} \beta_{wh}^* \underline{n}$ for some $n \in \mathbb{N}$, and hence $\bar{s} \beta_{wh}^* \underline{n+1}$ by definition of β_{wh} .
- Assume $s = (t) u$ with $\Phi \vdash t : \sigma \Rightarrow \tau$ and $\Phi \vdash u : \sigma$ so that, by inductive hypothesis $\bar{t} \in \text{Red}_{\sigma \Rightarrow \tau}$ and $\bar{u} \in \text{Red}_\sigma$ and hence $\bar{s} = (\bar{t}) \bar{u} \in \text{Red}_\tau$ by definition of $\text{Red}_{\sigma \Rightarrow \tau}$.
- Assume $s = \lambda x^\sigma t$ with $\Phi, x : \sigma \vdash t : \tau$. We must prove that $\bar{s} = \lambda x^\sigma \bar{t} \in \text{Red}_{\sigma \Rightarrow \tau}$, so let $u \in \text{Red}_\sigma$ and let us prove that $(\bar{s}) u = (\lambda x^\sigma \bar{t}) u \in \text{Red}_\tau$. We have $(\lambda x^\sigma \bar{t}) u \beta_{wh} \bar{t}[u/x]$ and we know that $\bar{t}[u/x] \in \text{Red}_\tau$ by inductive hypothesis applied to t . We conclude that $\bar{s} \in \text{Red}_{\sigma \Rightarrow \tau}$ by Lemma 6.4.
- Assume $s = \text{rec}(t, u, v)$ with $\Phi \vdash t : \text{nat}$, $\Phi \vdash u : \sigma$ and $\Phi \vdash v : \text{nat} \Rightarrow \sigma \Rightarrow \sigma$ so that $\bar{t} \in \text{Red}_{\text{nat}}$, $\bar{u} \in \text{Red}_\sigma$ and $\bar{v} \in \text{Red}_{\text{nat} \Rightarrow \sigma \Rightarrow \sigma}$ by inductive hypothesis. By definition of Red_{nat} there is $n \in \mathbb{N}$ such that $\bar{t} \beta_{wh}^* \underline{n}$ and therefore $\bar{s} \beta_{wh}^* \text{rec}(\underline{n}, \bar{u}, \bar{v})$ by definition of β_{wh} . An easy induction on n (using Lemma 6.4) shows that $\text{rec}(\underline{n}, \bar{u}, \bar{v}) \in \text{Red}_\sigma$ and hence $\bar{s} \in \text{Red}_\sigma$ by Lemma 6.4 again.

- If $s = \text{let}(x, t, u)$ with $\Phi \vdash t : \text{nat}$ and $\Phi, x : \text{nat} \vdash u : \sigma$ we have $\bar{t} \in \text{Red}_{\text{nat}}$ and hence $\bar{t} \beta_{wh}^* \underline{n}$ for some $n \in \mathbb{N}$. Therefore $\text{let}(x, \bar{t}, \bar{u}) \beta_{wh}^* \text{let}(x, \underline{n}, \bar{u}) \beta_{wh} \bar{u} [\underline{n}/x]$ by definition of β_{wh} and we have $\bar{u} [\underline{n}/x] \in \text{Red}_\tau$ by inductive hypothesis on u . It follows that $\bar{s} \in \text{Red}_\sigma$ by Lemma 6.4 again. \blacktriangleleft

Therefore, given s such that $\vdash s : \text{nat} \Rightarrow \text{nat}$, there is a uniquely defined total function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall n \in \mathbb{N} (s) \underline{n} \beta_{wh}^* f(n)$. Following [16], one can show that these functions which are definable in \mathbb{T} are exactly those whose totality can be proved in Peano's Arithmetics.

Denotational semantics of \mathbb{T} in \mathbf{CohT} . With each type σ one associates an object $\llbracket \sigma \rrbracket$ of \mathbf{CohT} using the structure of LL model of this category and with each s such that $\Phi \vdash s : \sigma$ (with $\Phi = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$) one associates an element of $\mathbf{CohT}(\llbracket \Phi \rrbracket, \llbracket \sigma \rrbracket)$ where $\llbracket \Phi \rrbracket = \llbracket \sigma_1 \rrbracket \otimes \dots \otimes \llbracket \sigma_k \rrbracket$. For $\llbracket \text{nat} \rrbracket$ we take the coherence space with totality \mathbb{N} such that $\underline{\mathbb{N}} = (\mathbb{N}, =)$ and $\mathbb{T}\mathbb{N} = \{\{n\} \mid n \in \mathbb{N}\}$. Then we set $\llbracket \sigma \Rightarrow \tau \rrbracket = (\llbracket \sigma \rrbracket \multimap \llbracket \tau \rrbracket)$. There is a morphism $h_{\mathbb{N}} \in \mathbf{CohT}(\mathbb{N}, \mathbb{T}\mathbb{N})$ which turns \mathbb{N} into a $!$ -coalgebra, namely $h_{\mathbb{N}} = \{(n, x) \in \mathbb{N} \times \mathcal{P}(\mathbb{N}) \mid x = \emptyset \text{ or } x = \{n\}\}$. Given $n \in \mathbb{N}$, we set $\bar{n} = \{n\} \in \mathbf{Cl}(\mathbb{N})$, that we also consider as an element of $\mathbf{CohT}(1, \mathbb{N})$.

If $\Phi \vdash s : \sigma$ then $\llbracket s \rrbracket_\Phi \in \mathbf{CohT}(\llbracket \Phi \rrbracket, \llbracket \sigma \rrbracket)$ is defined by induction on s using standard algebraic constructs. We give here a direct concrete description of this interpretation.

- $\llbracket x_i \rrbracket_\Phi = \{(x_1, \dots, x_k, n) \mid n \in \mathbb{N}, x_i = \{n\} \text{ and } x_j = \emptyset \text{ if } j \neq i\}$.
- $\llbracket \underline{n} \rrbracket_\Phi = \{(\emptyset, \dots, \emptyset, n)\}$.
- $\llbracket \text{succ}(s) \rrbracket_\Phi = \{(\vec{x}, n+1) \mid (\vec{x}, n) \in \llbracket s \rrbracket_\Phi\}$.
- $\llbracket \lambda x^\sigma s \rrbracket_\Phi = \{(\vec{x}, (x_0, a)) \mid (\vec{x}, x_0, a) \in \llbracket s \rrbracket_{\Phi, x: \sigma}\}$.
- Assume $\Phi \vdash s : \sigma \Rightarrow \tau$ and $\Phi \vdash t : \sigma$ so that $\Phi \vdash (s)t : \tau$. Then $\llbracket (s)t \rrbracket_\Phi = \{(\vec{x} \cup \vec{y}^1 \cup \dots \cup \vec{y}^p, a) \in \llbracket \llbracket \Phi \rrbracket \multimap \llbracket \tau \rrbracket \rrbracket \mid \exists a_1, \dots, a_p \in \llbracket \llbracket \sigma \rrbracket \rrbracket (\vec{x}, (\{a_1, \dots, a_p\}, a)) \in \llbracket s \rrbracket_\Phi \text{ and } (\vec{y}^j, a_j) \in \llbracket t \rrbracket_\Phi \text{ for } j = 1, \dots, p\}$.
- Let X be a coherence space with totality. We define by induction on $n \in \mathbb{N}$ a family of morphisms $r_n \in \mathbf{CohT}(!X \otimes !(\mathbb{N} \multimap !X \multimap X), X)$ as follows (keeping monoidal isos implicit). The morphism r_0 is

$$!X \otimes !(\mathbb{N} \multimap !X \multimap X) \xrightarrow{!X \otimes w_X} !X \xrightarrow{\text{der}_X} X$$

and the morphism r_{n+1} is

$$\begin{aligned} & !X \otimes !(\mathbb{N} \multimap !X \multimap X) \\ & \quad \downarrow !X \otimes \text{contr} \otimes \bar{n}^! \\ & !X \otimes !(\mathbb{N} \multimap !X \multimap X) \otimes !(\mathbb{N} \multimap !X \multimap X) \otimes !\mathbb{N} \\ & \quad \downarrow r_n^! \otimes \text{der}_{\mathbb{N} \multimap !X \multimap X} \otimes !\mathbb{N} \\ & !X \otimes !(\mathbb{N} \multimap !X \multimap X) \otimes !\mathbb{N} \\ & \quad \downarrow e \\ & X \end{aligned}$$

where e uses twice the evaluation morphism ev . Stated otherwise, if we consider r_n as a binary stable function $\mathbf{Cl}(X) \times \mathbf{Cl}(\mathbb{N} \multimap !X \multimap X) \rightarrow \mathbf{Cl}(X)$, we have $r_0(x, f) = x$ and $r_{n+1}(x, f) = f(\{n\}, r_n(x, f))$ (considering $f \in \mathbf{Cl}(\mathbb{N} \multimap !X \multimap X)$ as a binary stable function $\mathbf{Cl}(\mathbb{N}) \times \mathbf{Cl}(X) \rightarrow \mathbf{Cl}(X)$). Then one defines $r \in \mathbf{CohT}(\mathbb{N} \otimes !X \otimes !(\mathbb{N} \multimap !X \multimap X), X)$

by

$$r = \{((n, x_0, f_0), a) \mid n \in \mathbb{N} \text{ and } ((x_0, f_0), a) \in r_n\}.$$

Indeed, the elements of $\mathbb{T}\mathbb{N}$ are the $\{n\}$ for $n \in \mathbb{N}$ and we have $r'(\{n\}) = r_n \in \mathcal{T}(!X \otimes !(N \multimap !X \multimap X) \multimap X)$ where r' is the Curry transpose of r ; it follows that $r \in \mathbf{CohT}(\mathbb{N} \otimes !X \otimes !(N \multimap !X \multimap X), X)$.

Assume that $\Phi \vdash s : \mathbf{nat}$, $\Phi \vdash t : \sigma$ and $\Phi \vdash u : \mathbf{nat} \Rightarrow \sigma \Rightarrow \sigma$ and take $X = \llbracket \sigma \rrbracket$, then we have $\llbracket s \rrbracket_\Phi \in \mathbf{CohT}(\llbracket \Phi \rrbracket, \mathbb{N})$, $\llbracket t \rrbracket_\Phi \in \mathbf{CohT}(\llbracket \Phi \rrbracket, X)$ and $\llbracket u \rrbracket_\Phi \in \mathbf{CohT}(\llbracket \Phi \rrbracket, !N \multimap !X \multimap X)$ and we define $\llbracket \mathbf{rec}(s, t, u) \rrbracket_\Phi$ as the following composition of morphisms is \mathbf{CohT}

$$\begin{array}{c} \llbracket \Phi \rrbracket \\ \downarrow c \\ \llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \\ \downarrow \llbracket s \rrbracket_\Phi \otimes \llbracket t \rrbracket_\Phi \otimes \llbracket u \rrbracket_\Phi \\ \mathbb{N} \otimes !X \otimes !(N \multimap !X \multimap X) \\ \downarrow r \\ X \end{array}$$

where c is obtained by combining two copies of $\mathbf{contr}_{\llbracket \Phi \rrbracket}$.

- Assume last that $s = \mathbf{let}(x, t, u)$ where $\Phi \vdash t : \mathbf{nat}$ and $\Phi, x : \mathbf{nat} \vdash u : \sigma$. By inductive hypothesis we have $\llbracket t \rrbracket_\Phi \in \mathbf{CohT}(\llbracket \Phi \rrbracket, \mathbb{N})$ and $\llbracket s \rrbracket_{\Phi, x : \mathbf{nat}} \in \mathbf{CohT}(\llbracket \Phi \rrbracket \otimes !\mathbb{N}, \llbracket \sigma \rrbracket)$. Then $\llbracket \mathbf{let}(x, t, u) \rrbracket_\Phi$ is

$$\llbracket \Phi \rrbracket \xrightarrow{c} \llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \xrightarrow{\llbracket \Phi \rrbracket \otimes \llbracket t \rrbracket_\Phi} \llbracket \Phi \rrbracket \otimes \mathbb{N} \xrightarrow{\llbracket \Phi \rrbracket \otimes h_{\mathbb{N}}} \llbracket \Phi \rrbracket \otimes !\mathbb{N} \xrightarrow{\llbracket u \rrbracket_{\Phi, x : \mathbf{nat}}} \llbracket \sigma \rrbracket$$

Proposition 6.6. *Assume that $\Phi \vdash s : \sigma$ and that $s \beta_{wh} s'$ (so that $\Phi \vdash s' : \sigma$). Then $\llbracket s' \rrbracket_\Phi = \llbracket s \rrbracket_\Phi$.*

Proof. We only sketch the proof which is quite standard and relies on a preliminary Substitution Lemma: if $\Phi, x : \sigma \vdash s : \tau$ and $\Phi \vdash t : \sigma$, the morphism $\llbracket s[t/x] \rrbracket_\Phi$ is equal to

$$\llbracket \Phi \rrbracket \xrightarrow{\mathbf{contr}_{\llbracket \Phi \rrbracket}} \llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \xrightarrow{\llbracket \Phi \rrbracket \otimes \llbracket t \rrbracket_\Phi} \llbracket \Phi \rrbracket \otimes !\llbracket \sigma \rrbracket \xrightarrow{\llbracket s \rrbracket_{\Phi, x : \sigma}} \llbracket \tau \rrbracket$$

which is proven by a simple induction on s . In the proof of the proposition itself the only delicate case is when $s = \mathbf{let}(x, \underline{n}, t)$ where $\Phi, x : \iota \vdash t : \sigma$. To deal with it it suffices to observe that $h_{\mathbb{N}} \llbracket \underline{n} \rrbracket = \llbracket \underline{n} \rrbracket^!$, in other words, $\llbracket \underline{n} \rrbracket$ is a $!$ -coalgebra morphism from 1 to \mathbb{N} (see [10] to read more on this point of view on values). ◀

Notice that the elements of $\mathbb{T}(\mathbb{N} \multimap \mathbb{N})$ are exactly the total functions $\mathbb{N} \rightarrow \mathbb{N}$. Those which are of shape $\llbracket s \rrbracket h_{\mathbb{N}}$ for some s such that $\vdash s : \mathbf{nat} \Rightarrow \mathbf{nat}$ are exactly the functions which are provably total in Peano Arithmetics. We prove now that these functions are definable in $\mu\mathbf{LL}$.

Translating \mathbb{T} into $\mu\mathbf{LL}$. First we translate types: given a type σ we define a $\mu\mathbf{LL}$ formula σ^* by: $\mathbf{nat}^* = \iota$ and $(\sigma \Rightarrow \tau)^* = !\sigma^* \multimap \tau^*$. Observe that $\llbracket \sigma \rrbracket$ (in the sense of the semantics of \mathbb{T}) coincides with $\llbracket \sigma^* \rrbracket$ (in the sense of the semantics of $\mu\mathbf{LL}$).

Now we define a translation of terms: given s , $\Phi = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$ and τ such that $\Phi \vdash s : \tau$ (our typing system is such that when τ exists, it is unique), a proof s_Φ^* of the sequent $\vdash (!\sigma_1^*)^\perp, \dots, (!\sigma_k^*)^\perp, \tau^*$. We use the notation Φ^* for the sequence $(!\sigma_1^*)^\perp, \dots, (!\sigma_k^*)^\perp$. This translation is defined by induction on s as follows. In the course of this inductive definition, we argue that $\llbracket s_\Phi^* \rrbracket = \llbracket s \rrbracket_\Phi$. Observe indeed that $\llbracket s_\Phi^* \rrbracket \in \mathbf{Cl}(\llbracket !\sigma_1^* \rrbracket^\perp \wp \dots \wp \llbracket !\sigma_k^* \rrbracket^\perp \wp \llbracket \tau^* \rrbracket)$,

$\llbracket s \rrbracket_{\Phi} \in \mathbf{Coh}(\llbracket \sigma_1 \rrbracket \otimes \cdots \otimes \llbracket \sigma_k \rrbracket, \llbracket \tau \rrbracket)$ and that $\mathbf{Cl}(\llbracket !\sigma_1^* \rrbracket^{\perp} \wp \cdots \wp \llbracket !\sigma_k^* \rrbracket^{\perp} \wp \llbracket \tau^* \rrbracket) = \mathbf{Coh}(\llbracket \sigma_1 \rrbracket \otimes \cdots \otimes \llbracket \sigma_k \rrbracket, \llbracket \tau \rrbracket)$ up to trivial iso (using the fact that $\llbracket \sigma^* \rrbracket = \llbracket \sigma \rrbracket$ for all type σ of \mathbb{T}).

If $s = x_i$ then s_{Φ}^* is

$$\frac{\frac{\vdash (\sigma_i^*)^{\perp}, \sigma_i^*}{\vdash (!\sigma_i^*)^{\perp}, \sigma_i^*}}{\vdash \Phi^*, \sigma_i^*} (w)$$

Then $(\gamma, a) \in \llbracket s_{\Phi}^* \rrbracket$ iff $\gamma_i = \{a\}$ and $\gamma_j = \emptyset$ if $j \neq i$, $\llbracket s_{\Phi}^* \rrbracket = \llbracket s \rrbracket_{\Phi}$ follows.

If $s = \underline{0}$ then s_{Φ}^* is

$$\frac{\frac{}{\vdash \iota} (0)}{\vdash \Phi^*, \iota} (w)$$

Then $(\gamma, k) \in \llbracket s_{\Phi}^* \rrbracket$ iff $k = 0$ and $\gamma_i = \emptyset$ for all i , $\llbracket s_{\Phi}^* \rrbracket = \llbracket s \rrbracket_{\Phi}$ follows.

If $s = \underline{n+1}$ then s_{Φ}^* is

$$\frac{\frac{\mathcal{U}_{\Phi}^*}{\vdash \Phi^*, \iota} (\text{succ})}{\vdash \Phi^*, \iota} (\text{succ})$$

Then $(\gamma, k) \in \llbracket s_{\Phi}^* \rrbracket$ iff $k = n+1$ and $\gamma_i = \emptyset$ for all i , $\llbracket s_{\Phi}^* \rrbracket = \llbracket s \rrbracket_{\Phi}$ follows.

If $s = \text{succ}(t)$ then s_{Φ}^* is

$$\frac{\frac{t_{\Phi}^*}{\vdash \Phi^*, \iota} (\text{succ})}{\vdash \Phi^*, \iota} (\text{succ})$$

Then $(\gamma, k) \in \llbracket s_{\Phi}^* \rrbracket$ iff $k > 0$ and $(\gamma, k-1) \in \llbracket t_{\Phi}^* \rrbracket$, and $\llbracket s_{\Phi}^* \rrbracket = \llbracket s \rrbracket_{\Phi}$ follows since $\llbracket t_{\Phi}^* \rrbracket = \llbracket t \rrbracket_{\Phi}$ by inductive hypothesis.

If $s = (t)u$ with $\Phi \vdash t : \sigma \Rightarrow \tau$ and $\Phi \vdash u : \sigma$ then s_{Φ}^* is

$$\frac{\frac{\frac{t_{\Phi}^*}{\vdash \Phi^*, (!\sigma^*)^{\perp} \wp \tau^*} \quad \frac{\frac{\frac{\vdash (!\sigma^*)^{\perp}, !\sigma^*}{\vdash (!\sigma^*)^{\perp}, \tau^*, !\sigma^* \otimes (\tau^*)^{\perp}} \quad \frac{\vdash \tau^*, (\tau^*)^{\perp}}{\vdash \tau^*, (\tau^*)^{\perp}}}{\vdash \Phi^*, !\sigma^{*\perp}, \tau^*}}{\vdash \Phi^*, \Phi^*, \tau^*} (c)}{\vdash \Phi^*, !\sigma^*} (p)}{\vdash \Phi^*, \sigma^*} (p)$$

Then $(\gamma, b) \in \llbracket s_{\Phi}^* \rrbracket$ iff there are $a_1, \dots, a_k \in \llbracket \llbracket \sigma^* \rrbracket \rrbracket$ and $\gamma^0, \gamma_1, \dots, \gamma_k$ such that $\gamma = \gamma^0 \cup \gamma^1 \cup \cdots \cup \gamma^k \in \llbracket \llbracket \Phi^* \rrbracket \rrbracket$ with $(\gamma^0, \{a_1, \dots, a_k\}, b) \in \llbracket t_{\Phi}^* \rrbracket$ and $(\gamma^j, a_j) \in \llbracket u_{\Phi}^* \rrbracket$ for $j = 1, \dots, k$ and the required equation follows by inductive hypothesis.

If $s = \lambda x^{\sigma} t$ with $\Phi, x : \sigma \vdash s : \tau$ then s^* is

$$\frac{\frac{t_{\Phi, x : \sigma}^*}{\vdash \Phi^*, (!\sigma^*)^{\perp}, \tau^*}}{\vdash \Phi^*, (!\sigma^*)^{\perp} \wp \tau^*}$$

and the required equation results straightforwardly from the inductive hypothesis.

If $s = \text{let}(x, t, u)$ with $\Phi \vdash t : \text{nat}$ and $\Phi, x : \text{nat} \vdash u : \sigma$. Then s_{Φ}^* is

$$\frac{\frac{\frac{t_{\Phi}^*}{\vdash \Phi^*, \iota} \quad \chi(\iota)}{\vdash \Phi^*, !\iota} \quad \frac{u_{\Phi, x: \text{nat}}^*}{\vdash \Phi^*, (!\iota)^\perp, \sigma^*}}{\frac{\vdash \Phi^*, \Phi^*, \sigma^*}{\vdash \Phi^*, \sigma^*}}$$

and the required equation follows directly from the fact that $\llbracket \chi(\iota) \rrbracket = h_{\mathbb{N}}$.

If $s = \text{rec}(t, u, v)$ with $\Phi \vdash t : \text{nat}$, $\Phi \vdash u : \sigma$, $\Phi \vdash v : \text{nat} \Rightarrow \sigma \Rightarrow \sigma$ then s_{Φ}^* is

$$\frac{\frac{\frac{u_{\Phi}^*}{\vdash \Phi^*, \sigma^*} \quad \frac{\vdash \iota}{\vdash !\iota}}{\varphi : \vdash \Phi^*, !\iota \otimes !\sigma^*} \quad (0) \quad \frac{\frac{\frac{v_{\Phi}^*}{\vdash \Phi^*, (!\iota)^\perp \wp (!\sigma^*)^\perp \wp \sigma^*} \quad \frac{\frac{\frac{\vdash \iota^\perp, \iota}{\vdash \iota^\perp, \iota} \text{ (succ)}}{\vdash (!\iota)^\perp, \iota}}{\vdash \Phi^*, (!\iota)^\perp, !\sigma^*}}{\vdash \Phi^*, (!\iota)^\perp, (!\sigma^*)^\perp, !\sigma^*}}{\psi : \vdash \Phi^*, (!\iota)^\perp, (!\sigma^*)^\perp, !\iota \otimes !\sigma^*}}{\theta : \vdash \Phi^*, \iota^\perp, !\iota \otimes !\sigma^*} \quad (\text{it}_\iota) \quad \frac{\frac{\frac{\vdash (\sigma^*)^\perp, \sigma^*}{\vdash (!\sigma^*)^\perp, \sigma^*}}{\vdash (!\iota)^\perp, (!\sigma^*)^\perp, \sigma^*}}{\vdash (!\iota)^\perp \wp (!\sigma^*)^\perp, \sigma^*}}{\vdash \Phi^*, \iota^\perp, \sigma^*} \quad \frac{t_{\Phi}^*}{\vdash \Phi^*, \iota}}{\frac{\vdash \Phi^*, \Phi^*, \sigma^*}{\vdash \Phi^*, \sigma^*}}$$

where ρ consists of two cuts against tensor rules. The fact that $\llbracket s_{\Phi}^* \rrbracket = \llbracket s \rrbracket_{\Phi}$ can be seen as follows (we actually apply an old trick for encoding primitive recursion using iteration).

First observe that $\llbracket v_{\Phi}^* \rrbracket$ can be seen as a stable function $\mathbf{Cl}(\llbracket \sigma_1^* \rrbracket) \times \cdots \times \mathbf{Cl}(\llbracket \sigma_k^* \rrbracket) \times \mathbf{Cl}(\mathbb{N}) \times \mathbf{Cl}(\llbracket \sigma^* \rrbracket) \rightarrow \llbracket \sigma^* \rrbracket$, that $\llbracket u_{\Phi}^* \rrbracket$ can be seen as a stable function $\mathbf{Cl}(\llbracket \sigma_1^* \rrbracket) \times \cdots \times \mathbf{Cl}(\llbracket \sigma_k^* \rrbracket) \rightarrow \llbracket \sigma^* \rrbracket$ and that $\llbracket t_{\Phi}^* \rrbracket$ can be seen as a stable function $\mathbf{Cl}(\llbracket \sigma_1^* \rrbracket) \times \cdots \times \mathbf{Cl}(\llbracket \sigma_k^* \rrbracket) \rightarrow \llbracket \mathbb{N} \rrbracket$.

Then the proof φ above the sequent $\vdash \Phi^*, !\iota \otimes !\sigma^*$ satisfies $\llbracket \varphi \rrbracket(\vec{x}) = \llbracket u_{\Phi}^* \rrbracket(\vec{x})^\perp \otimes \{0\}^\perp$. The proof ψ can be seen as a stable function $\mathbf{Cl}(\llbracket \sigma_1^* \rrbracket) \times \cdots \times \mathbf{Cl}(\llbracket \sigma_k^* \rrbracket) \times \mathbf{Cl}(\mathbb{N}) \times \mathbf{Cl}(\llbracket \sigma^* \rrbracket) \rightarrow !\mathbb{N} \otimes !\llbracket \sigma^* \rrbracket$ such that $\llbracket \psi \rrbracket(\vec{x}, y, x) = (y^\perp)^\perp \otimes \llbracket v_{\Phi}^* \rrbracket(\vec{x}, y, x)^\perp$ where $y^\perp = \{n + 1 \mid n \in y\}$. Applying the interpretation of the derived rule (it_ι) as described in Section 6.1, an easy induction shows that

$$\begin{aligned} \llbracket \theta \rrbracket(\vec{x}, \{0\}) &= \{0\}^\perp \otimes \llbracket u_{\Phi}^* \rrbracket(\vec{x})^\perp \\ \llbracket \theta \rrbracket(\vec{x}, \{n + 1\}) &= \{n + 1\}^\perp \otimes \llbracket v_{\Phi}^* \rrbracket(\vec{x}, \{n\}, \llbracket \theta \rrbracket(\vec{x}, \{n\}))^\perp \end{aligned}$$

from which the required equation follows. So we have proved the following result.

Theorem 6.7. *If s is a term of \mathbb{T} such that $\Phi \vdash s : \tau$ then $\llbracket s_{\Phi}^* \rrbracket = \llbracket s \rrbracket_{\Phi}$.*

Theorem 6.8. *For any function $f : \mathbb{N} \rightarrow \mathbb{N}$, if f can be represented by a term of \mathbb{T} then there is a proof π if μLL of $\vdash \iota^\perp, \iota$ such that for all $n \in \mathbb{N}$ the proof*

$$\frac{\frac{\pi}{\vdash \iota^\perp, \iota} \quad \frac{n}{\vdash \iota}}{\vdash \iota}$$

reduces to $f(n)$.

Proof. Let s be a closed term of \mathbb{T} such that $\vdash s : \text{nat} \Rightarrow \text{nat}$ and s represents f (that is, for all $n \in \mathbb{N}$, $(s) \underline{n} \beta_{wh}^* f(n)$). Let $\pi = s^*$ which is a proof of $\vdash (!\iota)^\perp \wp \iota$, that is of $\vdash (!\iota)^\perp, \iota$

(up to a cut with a trivial proof). Baelde’s normalization theorem tells us that for all $n \in \mathbb{N}$ the following proof⁶ π_n

$$\frac{\frac{\pi}{\vdash (!\iota)^\perp, \iota} \quad \frac{\underline{n}^*}{\vdash !\iota} \text{ (p)}}{\vdash \iota}$$

reduces to a proof \underline{k}^* for some $k \in \mathbb{N}$ so that $\llbracket \pi_n \rrbracket = \{k\}$ by Theorem 5.6. By Theorem 6.7 we have $\{f(n)\} = \llbracket (s) \underline{n} \rrbracket = \llbracket \pi_n \rrbracket = \{k\}$ and hence $k = f(n)$ which proves our contention. \blacktriangleleft

6.4. Encoding of the exponentials. In [1] it is noticed that the exponential connectives of LL can be encoded using least and greatest fixed points. It is also mentioned that this encoding is shallow in the sense that it allows, on formulas involving exponentials, proofs which do not make sense in LL. We propose here a short denotational analysis of this encoding showing that it gives rise to a resource comonad which however does not validate the Seely isomorphism. This is our motivation for including the exponentials in μ LL from the beginning. So let us set

$$!A = \nu\zeta (1 \& A \& (\zeta \otimes \zeta)) \quad \text{and} \quad ?A = (!A^\perp)^\perp = \mu\zeta (\perp \oplus A \oplus (\zeta \wp \zeta)).$$

6.4.1. Functoriality and comonadic structure. We consider the binary VCST \mathbb{X} given by $\mathbb{X}(X, U) = 1 \& X \& (U \otimes U)$ and we set $!X = \nu \mathbb{X}_X$ (remember that this notation \mathbb{X}_X is introduced at the end of the Introduction for arbitrary functors) so that $\llbracket !A \rrbracket = ![A]$. We set $\text{der}'_X = \pi_1 \in \mathbf{CohT}(!X, X)$ since $\mathbb{X}(X, !X) = !X$ and similarly $w'_X = \pi_0 \in \mathbf{CohT}(!X, 1)$ and $\text{contr}'_X = \pi_2 \in \mathbf{CohT}(!X, !X \otimes !X)$, these maps exhibit on $!X$ a structure of “free co-magma” generated by X .

For $g \in \mathbf{CohT}(Y, \mathbb{X}_X(Y))$, there is exactly one $\text{ext}'(g) \in \mathbf{CohT}(Y, !X)$ such that $\text{ext}'(g) = \mathbb{X}_X(\text{ext}'(g))g$, that is

$$w'_Y \text{ext}'(g) = \pi_0 g \quad \text{der}'_Y \text{ext}'(g) = \pi_1 g \quad \text{contr}'_Y \text{ext}'(g) = (\text{ext}'(g) \otimes \text{ext}'(g)) \pi_2 g. \quad (6.1)$$

Let $f \in \mathbf{CohT}(!X, Y)$, we have $\langle w'_X, f, \text{contr}'_X \rangle \in \mathbf{CohT}(!X, \mathbb{X}_Y(!X))$ and so we can set $f' = \text{ext}'(\langle w'_X, f, \text{contr}'_X \rangle) \in \mathbf{CohT}(!X, !Y)$ so that this morphism f' is fully characterized by the following equations

$$w'_Y f' = w'_X \quad \text{der}'_Y f' = f \quad \text{contr}'_Y f' = (f' \otimes f') \text{contr}'_X. \quad (6.2)$$

Then the action of the exponential on morphisms is defined as $!f = (f \text{der}'_X)'$ for $f \in \mathbf{CohT}(X, Y)$ and is easily seen to be functorial using Equations (6.2). Observe that $!f$ is characterized by the following equations.

$$w'_Y !f = w'_X \quad \text{der}'_Y !f = f \text{der}'_Y \quad \text{contr}'_Y !f = (!f \otimes !f) \text{contr}'_X. \quad (6.3)$$

Last the comultiplication of the exponential is defined as

$$\text{dig}'_X = (\text{Id}_{!X})' \in \mathbf{CohT}(!X, !!X)$$

⁶Observe that \underline{n}^* is the canonical representation of n as a proof of μ LL consisting of a stack of n (succ) rules topped by a (0) axiom.

and is easily checked to satisfy the required commutations for the comonad structure, using again Equations (6.2).

Now we will argue that this exponential lacks the required Seely isomorphisms that we mentioned in Section 2. Indeed, if such isomorphisms $\mathbf{m}_{X,Y}^{2'} \in \mathbf{CohT}(!X \otimes !Y, !(X \& Y))$ and $\mathbf{n}_{X,Y}^{2'} \in \mathbf{CohT}(!(X \& Y), !X \otimes !Y)$ existed they would satisfy the following equations

$$\begin{aligned} w'_{X\&Y} \mathbf{m}_{X,Y}^{2'} &= w'_X \otimes w'_Y \\ \mathbf{der}'_{X\&Y} \mathbf{m}_{X,Y}^{2'} &= \langle \mathbf{der}'_X \otimes w'_Y, w'_X \otimes \mathbf{der}'_Y \rangle \\ \mathbf{contr}'_{X\&Y} \mathbf{m}_{X,Y}^{2'} &= \left(\mathbf{m}_{X,Y}^{2'} \otimes \mathbf{m}_{X,Y}^{2'} \right) \gamma_{2,3} (\mathbf{contr}'_X \otimes \mathbf{contr}'_Y) \end{aligned} \quad (6.4)$$

and

$$\mathbf{n}_{X,Y}^{2'} = (!\pi_1 \otimes !\pi_2) \mathbf{contr}'_{X\&Y} . \quad (6.5)$$

Equation (6.5) can be understood as a mere definition of $\mathbf{n}_{X,Y}^{2'}$ since we know how to compute its right hand member. And by the characterization (6.1), the equations (6.4) determine $\mathbf{m}_{X,Y}^{2'}$ uniquely. So it suffices to compute these two morphisms in our model \mathbf{CohT} to convince ourselves that they do not define an isomorphism.

First of all, given a coherence space with totality X , observe that the elements α, α_1, \dots of the web of the coherence space $!X$ can be described by the following grammar

$$\alpha, \alpha_1, \dots := w \mid d(a) \mid c(\alpha_1, \alpha_2)$$

where a stands for the elements of $|X|$. In other words, an element of $!X$ is a finite binary tree with two kinds of leaves: “weakening leaves” w and “dereliction leaves” $d(a)$. Coherence in $!X$ is given by: $\alpha \supset_{!X} \alpha'$ if $\alpha = d(a)$, $\alpha' = d(a')$ $\Rightarrow a \supset_X a'$ and $\alpha = c(\alpha_1, \alpha_2)$, $\alpha' = c(\alpha'_1, \alpha'_2) \Rightarrow \alpha_i \supset_{!X} \alpha'_i$ for $i = 1, 2$. Totality $\mathbb{T}!X$ is defined as a greatest fixed point as explained in Section 4.2, and we don’t know yet any more explicit presentation of this totality. We call *paths* the elements of $\{1, 2\}^{<\omega}$ and we define a partial function $\mathbf{st} : !X \times \{1, 2\}^{<\omega} \rightarrow !X$ by

$$\begin{aligned} \mathbf{st}(\alpha, \langle \rangle) &= \alpha \\ \mathbf{st}(c(\alpha_1, \alpha_2), i\theta) &= \mathbf{st}(\alpha_i, \theta) \end{aligned}$$

and $\mathbf{paths}(\alpha)$ is the set of paths θ such that $\mathbf{st}(\alpha, \theta)$ is defined (it is a finite set).

Let $f \in \mathbf{CohT}(X, Y)$. By Equations (6.3) and by the fact that $\mathbf{ext}'(g)$ is defined as a least fixed point (see Section 4.2) we get that $!f$ is the least set such that $(\alpha, \beta) \in !f$ iff $\alpha = \beta = w$ or $\alpha = d(a)$, $\beta = d(b)$ and $(a, b) \in f$ or $\alpha = c(\alpha_1, \alpha_2)$, $\beta = c(\beta_1, \beta_2)$ and $(\alpha_i, \beta_i) \in !f$ for $i = 1, 2$. In other words $(\alpha, \beta) \in !f$ iff $\mathbf{paths}(\alpha) = \mathbf{paths}(\beta) = D$ and for all $\theta \in D$ which is maximal for the prefix order:

- either $\mathbf{st}(\alpha, \theta) = \mathbf{st}(\beta, \theta) = w$
- or $\mathbf{st}(\alpha, \theta) = d(a)$ and $\mathbf{st}(\beta, \theta) = d(b)$ with $(a, b) \in f$.

That is, α and β have exactly the same shape and corresponding $d(_)$ -leaves of α and β are related by f .

Now Equation (6.5) gives us an explicit definition of $\mathbf{n}_{X,Y}^{2'}$, namely: $(\gamma, (\alpha, \beta)) \in \mathbf{n}_{X,Y}^{2'}$ iff $\gamma = c(1 \cdot \alpha, 2 \cdot \beta)$ where $1 \cdot w = w$, $1 \cdot d(a) = d(1, a)$ and $1 \cdot c(\alpha_1, \alpha_2) = c(1 \cdot \alpha_1, 1 \cdot \alpha_2)$

and similarly for $2 \cdot \beta$. The map $(\alpha, \beta) \mapsto c(1 \cdot \alpha, 2 \cdot \beta)$ is injective but not surjective and so $n_{X,Y}^{2'}$ is not an iso⁷.

This suffices to show that $!_-$ does not endow **CohT** with a structure of Seely category (and hence is not a categorical model of LL in the usual sense), but for the sake of completeness we also give $m_{X,Y}^{2'}$. It is the least set such that $((\alpha, \beta), \gamma) \in m_{X,Y}^{2'}$ iff

- $\alpha = \beta = \gamma = w$
- or $\gamma = d(1, a)$, $\alpha = d(a)$ and $\beta = w$
- or $\gamma = d(2, b)$, $\alpha = w$ and $\beta = d(b)$
- or $\gamma = c(\gamma_1, \gamma_2)$, $\alpha = c(\alpha_1, \alpha_2)$, $\beta = c(\beta_1, \beta_2)$ and $((\alpha_i, \beta_i), \gamma_i) \in m_{X,Y}^{2'}$ for $i = 1, 2$.

In other words, $((\alpha, \beta), \gamma) \in m_{X,Y}^{2'}$ iff $\alpha = p_1(\gamma)$ and $\beta = p_2(\gamma)$ where $\text{paths}(p_1(\gamma)) = \text{paths}(\gamma) = D$, and for all $\theta \in D$, if θ is maximal (for the prefix order), $\text{st}(\gamma, \theta) = d(1, a) \Rightarrow \text{st}(p_1(\gamma), \theta) = d(a)$ and $\text{st}(\gamma, \theta) = d(2, b) \Rightarrow \text{st}(p_1(\gamma), \theta) = w$. That is, $p_1(\gamma)$ is γ where the b 's are replaced with w . The definition of $p_2(\gamma)$ is symmetrical. Again, the map $\gamma \mapsto (p_1(\gamma), p_2(\gamma))$ is injective but not surjective since $p_1(\gamma)$ and $p_2(\gamma)$ have the same ‘‘internal’’ tree structure (namely, that of γ).

Remark 6.9. Such non-commutative exponentials in coherence spaces have been introduced by Myriam Quatrini in her PhD thesis [18] as early as in 1995 and it would be interesting to compare them with this $!_-$ construction. One major feature of this exponential is that, unlike the usual set-based and multiset-based exponentials in coherence spaces, the $!_-$ is non-uniform, meaning that the elements of $|E|$ occurring in an element α of $!|E|$ are not required to be pairwise coherent. There is a price to pay for this feature: the Seely isos are not available⁸ and so the Kleisli category **Coh_v** is not cartesian closed. In [3] we exhibited another way of accommodating non-uniform exponentials with coherence spaces, in a true model of LL and there, the price to pay was the loss of coherence’s reflexivity (on the web). It would certainly be interesting to explore the connection between these two kinds of non-uniform exponentials in coherence spaces.

7. OTHER MODELS AND GENERALIZATIONS

The main feature of our model construction is that it is based on a two-level structure:

- The LL model **Coh** of coherence spaces, where a notion of variable object can be defined for which least and greatest fixed-points coincide
- and the LL model of **CohT** of coherence spaces with totality where we can define a notion of variable types by equipping the variable types of the first level with an additional structure (here, a totality structure). In this second level, least and greatest fixed points are interpreted in different ways.

The same two-level structure can certainly be developed on the model of hypercoherence spaces [7], on the Scott semantics of LL [9] or on the model of probabilistic coherence space [6]. In all these cases a notion of totality similar to that of **CohT** can be defined.

A slightly different structure can be obtained by using the category **Rel** of sets and relations as base category and the category of finiteness spaces [8] (which are sets equipped with a finiteness structure) as level-2 structure. In this case, an interesting challenge will

⁷Remember that $f \in \mathbf{Coh}(E, F)$ is an iso iff, as a relation, f is the graph of a bijection $\varphi : |E| \rightarrow |F|$ such that $a \succ_E a' \Leftrightarrow \varphi(a) \succ_F \varphi(a')$

⁸They are replaced by injections, which are not so far from being bijections after all. . .

be to understand how the fixed point constructs interact with the *linearization* typical of finiteness spaces: given a field, any finiteness space can canonically be turned into a topological vector space over this field (considered as a discrete space). In this case there might even be a third level since these linearizations can be equipped with a totality which is simply a closed affine subspace as explained in [19].

7.1. A categorical axiomatization models of μLL . In Section 2 we tried to give as much general categorical definitions as possible, to serve as a background for the concrete developments in further sections, and also for a general categorical presentation of μLL models extending naturally the concept of Seely category. Here is a tentative such definition, compatible with the example we have presented in the paper and the other examples that we mentioned.

Definition 7.1. A *categorical model of μLL* is a family $\vec{\mathcal{L}} = (\mathcal{L}_n)_{n \in \mathbb{N}}$ where

- \mathcal{L}_0 is a Seely category
- \mathcal{L}_n is a family of strong functors $\mathcal{L}_0^n \rightarrow \mathcal{L}_0$ and our choice of notations for \mathcal{L}_0 means that all constant functors are in \mathcal{L}_0 (see Section 2.2 for basic definitions on strong functors in our LL categorical setting)
- if $\mathbb{X} \in \mathcal{L}_n$ and $\mathbb{X}_i \in \mathcal{L}_k$ (for $i = 1, \dots, n$) then $\mathbb{X} \circ \vec{\mathbb{X}} \in \mathcal{L}_k$
- the strong functors \otimes and $\&$ belong to \mathcal{L}_2 , the strong functor $!_-$ belongs to \mathcal{L}_1 and, if $\mathbb{X} \in \mathcal{L}_n$, then $\mathbb{X}^\perp \in \mathcal{L}_n$
- and last, for all $\mathbb{X} \in \mathcal{L}_1$ the category $\mathbf{Coalg}_{\mathcal{L}_0}(\mathbb{X})$ (see Section 2.3) has a final object or, equivalently, the category $\mathbf{Alg}_{\mathcal{L}_0}(\mathbb{X})$ has an initial object.

Of course we should interpret, in such an abstract model, all sequents and proofs of μLL and we are confident that this can be done following the pattern we developed for the interpretation in **CohT** (where \mathcal{L}_n is of course the class of n -ary VCSTs, considered as strong functors as explained in Section 4.2); such a formal verification is postponed to further work.

8. CONCLUSION

One of the main goals of this work was to develop syntax-independent tools to study new proof-systems for μLL and more specifically infinite proof-systems as in [2], and denotational semantics is clearly a natural framework for such tools. A first step in this study will be to prove that these infinite proofs can be interpreted in the model of VCSTs. It would be also quite interesting to understand how the notion of totality is related with that of *productiveness*, crucial in the study of coinductive types. Our models should also suggest natural notions of proof-nets for μLL .

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