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On more variants of the Majority Problem

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Abstract

The variant of the Majority problem we are considering is the following. A colorblind player is given a set \mathcal{B} of N colored balls. He knows that each ball is colored either red or green, and that there are less green than red balls, but he cannot distinguish the two colors. For any two balls he can ask whether they are colored the same. His goal is to determine the color of each of the balls, asking as few questions as possible. In the case where there are at most p (respectively exactly p) green balls, the minimum number of questions that guarantees the determination of the colors is denoted by $Q(N, p, \leq)$ (respectively $Q(N, p, =)$). We extend results of Aigner on exact values of $Q(N, p, \leq)$, and we provide upper bounds for $Q(N, p, =)$, and even exact values for the first two values of p . Our results lead to several new questions.

Keywords: Combinatorial search; Majority problem; Pairwise comparison.

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1 Introduction

Let $\mathcal{B} = \{b_1, b_2, \dots, b_N\}$ be a set of N balls, each ball being either red or green. The goal is to provide the color of each of the balls by asking questions of the type "Are b_i and b_j colored the same?" ($1 \leq i < j \leq N$). We remark that such comparisons may lead easily to a partition of \mathcal{B} into two subsets \mathcal{B}_1 and \mathcal{B}_2 that are the sets of balls of each color. If we would allow \mathcal{B} to contain an equal number of red and green balls it would be impossible to decide the colors of the balls. Hence we will always assume in the following that \mathcal{B} contains more red balls than green balls and we call such a coloring a *Red-green coloring*. Furthermore we will also consider more restricted kinds of Red-green colorings: given an integer p , a Red-green coloring of N balls will be said *p-majorred* if there are at most p green balls and it will be said *p-equal* if there are exactly p green balls. The difficulty consists in determining a method minimizing the number of comparisons. Given two integers N and $p < \frac{N}{2}$ we call (N, p, \leq) -identification (respectively $(N, p, =)$ -identification) the problem of determining for sure all the colors of N balls colored by a p -majorred Red-green coloring (respectively by a p -equal Red-green coloring) and we denote by $Q(N, p, \leq)$ (respectively $Q(N, p, =)$) the minimum number of comparisons that are necessary to solve any instance of the (N, p, \leq) -identification problem (respectively of the $(N, p, =)$ -identification problem).

In the next section we will present what is known and what are our new results on the values of $Q(N, p, \leq)$ and $Q(N, p, =)$. Then we will see some properties that are similar in both identification problems. In the two last sections we will provide the proofs of the results stated in section 2.

2 Known and new results

2.1 On the (N, p, \leq) -identification problem

The theorem below is due to Aigner (Theorem 4 and end of section 4 in [1]).

Theorem 1. [1] Let N and p be integers such that $0 \leq p < \frac{N}{2}$, we have:

$$\begin{aligned} Q(N, p, \leq) &= N + 1 - \lfloor \frac{N+1}{p+1} \rfloor \text{ or} \\ &= N - \lfloor \frac{N}{p+1} \rfloor. \end{aligned}$$

and $Q(N, p, \leq) = N + 1 - \lfloor \frac{N+1}{p+1} \rfloor$ whenever $N \equiv r [p+1]$ with $r = 0, 1$ or p .

Theorem 1 has an obvious corollary.

Corollary 2. *The following equalities hold :*

- $Q(N, 1, \leq) = N + 1 - \lfloor \frac{N+1}{2} \rfloor$ for every $N \geq 3$
- $Q(N, 2, \leq) = N + 1 - \lfloor \frac{N+1}{3} \rfloor$ for every $N \geq 5$.

Our next theorem extends Theorem 1 and partially answers a question of Wildon (Problem 8.1 in [11]).

Theorem 3. *Let N and p be integers such that $0 \leq p < \frac{N}{2}$, we have:*

$$Q(N, p, \leq) = N + 1 - \lfloor \frac{N + 1}{p + 1} \rfloor$$

whenever $N \equiv r [p + 1]$ with $0 \leq r \leq \lfloor \frac{p}{2} \rfloor$ or $r = p$.

Furthermore we could also show the following.

Proposition 4. $\forall N \geq 7, Q(N, 3, \leq) = N + 1 - \lfloor \frac{N+1}{4} \rfloor$.

A proof of Proposition 4 is provided in the arXiv version of this paper [5].

Let us remark that all these results show only cases where the value of $Q(N, p, \leq)$ is equal to its upper bound provided by Theorem 1. From these it is natural to wonder if the equality of Theorem 3 holds for any N and p such that $N \geq 2p + 1$. This is however not the case: Wildon [11] has checked by a computer search the values of $Q(N, p, \leq)$ for $N \leq 30$. In particular he listed all such $Q(N, p, \leq)$ that are not equal to $N + 1 - \lfloor \frac{N+1}{p+1} \rfloor$. Our (two different) programs confirmed this list. Looking carefully at the results and determining properties of the function $Q(\cdot, \cdot, \leq)$ leads us to ask the following.

Question 1. *Is it true that for any positive integer $p \geq 4$ there exists a smallest integer $N(p) \geq 2p + 1$ such that for any integer $N \geq N(p)$, $N \equiv r [p + 1]$ with $0 \leq r \leq p$, we have:*

If $r \leq \frac{p+1}{2}$ then $Q(N, p, \leq) = N + 1 - \lfloor \frac{N+1}{p+1} \rfloor$ else $Q(N, p, \leq) = N - \lfloor \frac{N}{p+1} \rfloor$?

(Notice that for $r = p$ we have $N - \lfloor \frac{N}{p+1} \rfloor = N + 1 - \lfloor \frac{N+1}{p+1} \rfloor$)

Using the program Main.hs, available from Wildon's website¹, we did generate all couples (N, p) such that $Q(N, p, \leq) < N + 1 - \lfloor \frac{N+1}{p+1} \rfloor$ for $N \leq 51$,

¹www.ma.rhul.ac.uk/~uvah099/Programs/MajorityGame/Main.hs

and among these there is no contradiction to a positive answer to Question 1. Using furthermore Theorem 3 and (2) of Property 16 (see section 4.3) we could completely settle the values of $Q(N, p, \leq)$ in the case of an even $p \leq 12$.

Proposition 5. *For all even $p \leq 12$, $N \geq 2p + 1$, $N \equiv r [p + 1]$ with $0 \leq r \leq p$, we have :*

$$\begin{aligned} \text{if } 0 \leq r \leq \frac{p+1}{2} \text{ then } Q(N, p, \leq) &= N + 1 - \lfloor \frac{N+1}{p+1} \rfloor \\ \text{else } Q(N, p, \leq) &= N - \lfloor \frac{N}{p+1} \rfloor, \end{aligned}$$

with the exception of $Q(18, 6, \leq) = 19 - 2 = 17$, $Q(23, 8, \leq) = 24 - 2 = 22$, $Q(28, 10, \leq) = 29 - 2 = 27$, $Q(33, 12, \leq) = 34 - 2 = 32$.

For odd $p \leq 13$, we could also verify that for N sufficiently large the values of $Q(N, p, \leq)$ are those expected by a positive answer to Question 1, however to completely settle these cases one would need to show that $Q(N, p, \leq) = N + 1 - \lfloor \frac{N+1}{p+1} \rfloor$ whenever $N \equiv \frac{p+1}{2} [p + 1]$.

2.2 On the $(N, p, =)$ -identification problem

To our knowledge the $(N, p, =)$ -identification problem has not been yet studied by other authors. A trivial upper bound for $Q(N, p, =)$ is $Q(N, p, \leq)$. We did obtain a better upper bound.

Theorem 6. *Let N and p be integers such that $1 \leq p < \frac{N}{2}$, and let m be the largest integer such that $2^m \leq \min(N - 2p, 2p)$, we have :*

$$Q(N, p, =) \leq N + 1 - m - \left\lfloor \frac{N + 2 - 2^m}{p + 1} \right\rfloor.$$

For $p = 1$ or 2 we were able to compute the exact value of $Q(N, p, =)$. Proofs for the following two propositions are proposed in the arXiv version of this paper [5].

Proposition 7. $\forall N \geq 3$, $Q(N, 1, =) = N - \lfloor \frac{N}{2} \rfloor$.

Proposition 8. $\forall N \geq 6$, $Q(N, 2, =) = N - \lfloor \frac{N+1}{3} \rfloor$.

Propositions 7 and 8 show that, for $p = 1$ or 2 , the upper bound $Q_+(N, p, =)$ of $Q(N, p, =)$ given by Theorem 6, is in fact the right value of $Q(N, p, =)$. This is however not true for any value of p as shown by our programs. For $N \leq 30$ and $p < \frac{N}{2}$ it happens several times that $Q(N, p, =) = Q_+(N, p, =) - 1$ (see Table 1), once that $Q(N, p, =) = Q_+(N, p, =) - 2$ (for $(N, p) = (30, 4)$) and for all other couples (N, p) the equality $Q(N, p, =) = Q_+(N, p, =)$ holds.

From these results one could expect that $Q(N, p, =)$ is never very far from the upper bound given by Theorem 6. As we will see in Section 5, this is however not the case as soon as p is at least 3. We will explain there that better bounds than the one of Theorem 6 may be obtained for N very large compare to p by a method that we call "Towers Method".

N	17	20	21	22	23	24	25	26	27	28	29	30
p	3	3, 4	3	5	5	3, 4	3, 4, 6	6	3, 4, 5	3, 4, 5, 7	3, 4, 5, 6, 7	3, 6, 8

Table 1: The values of N and p such that $Q(N, p, =) = Q_+(N, p, =) - 1$.

3 A tool and some useful Lemmas

In order to describe algorithms solving the (N, p, \leq) -identification problem, or the $(N, p, =)$ -identification problem, we use methods putting balls in *boxes divided into two sides*; in the rest of the paper we will consider only non empty boxes of this kind and will, all along, manage to have in each box all the balls with a same color put in a same side.

Given a non empty box B we will call *big side of B* the side of B with the most balls. In case there is the same number of balls in each side, the big side of B will be the one containing the ball b_j where j is the highest index of a ball in B . The side of B which is not the big side will be called *the small side*.

A non empty box will be said of *Type (x, y)* if there are x balls in its small side and y in its big side (hence x and y are two integers such that $\max(x, 1) \leq y$). A box whose small side is empty will be said *monocolored*, and a box of Type (x, x) will be said *balanced*.

Of course it has no sense to compare balls that are in a same box since the answer is already known. Given two distinct boxes B and B' , the balls

in their big sides either have the same color, or not, and this can be decided by comparing any ball of B with any ball of B' . For that reason we may consider that we do "boxes comparisons" rather than "balls comparisons". After the comparison of B and B' we may place all their balls into one box, in a way compatible with their colors, and throw away the empty boxes. If the Types of B and B' were (x, y) and (x', y') then the Type of the new box is either $(x + x', y + y')$ or $(\min(x + y', x' + y), \max(x + y', x' + y))$, according to whether the balls in the big sides of B and B' have the same color or not. From a partition \mathcal{P} of the set \mathcal{B} of balls into boxes, we can deduce the possible colorings of the balls : a p -majored (resp. p -equal) Red-green coloring is *compatible* with \mathcal{P} when any two balls in a same box are colored the same if and only if they are in the same side of the box. We will denote by $\mathcal{C}(\mathcal{P})$ the set of p -majored (resp. p -equal) Red-green colorings that are compatible with a partition \mathcal{P} . Our goal is to reach, within a minimum number of boxes comparisons, a partition \mathcal{P} such that $|\mathcal{C}(\mathcal{P})| = 1$. Notice that $|\mathcal{C}(\mathcal{P})|$ depends only on the number of balls in the sides of the boxes : a partition of the balls into boxes will be said *of Type* $(x_1, y_1)^{N_1}(x_2, y_2)^{N_2} \dots (x_k, y_k)^{N_k}$ if it consists in exactly N_i boxes of Type (x_i, y_i) , for $i = 1$ to k .

From our remarks above it is natural to represent a method M solving the (N, p, \leq) or $(N, p, =)$ -identification problem by a *labeled binary tree* T_M as follows :

- Each vertex u is labeled with a partition $\mathcal{P}(u)$ of \mathcal{B} into boxes,
- The partition $\mathcal{P}(R)$ of the root R of T_M is made of N boxes B_1, B_2, \dots, B_N of Type $(0, 1)$ where each B_i contains only the ball b_i .
- Each non-leaf vertex u is furthermore labeled with a couple $(B(u), B'(u))$ of boxes belonging to $\mathcal{P}(u)$, and it has
 - one child connected to u by an edge labeled " $=$ ". This child is labeled by the partition obtained from $\mathcal{P}(u)$ by replacing $B(u)$ and $B'(u)$ by the new box obtained in case the comparison of $B(u)$ and $B'(u)$ provides that the balls in the big sides of $B(u)$ and $B'(u)$ have the same color.
 - one child connected to u by an edge labeled " \neq ". This child is labeled by the partition obtained from $\mathcal{P}(u)$ by replacing $B(u)$ and $B'(u)$ by the new box obtained in case the comparison of $B(u)$ and $B'(u)$ provides that the balls in the big sides of $B(u)$ and $B'(u)$ have different colors.

- Each leaf-vertex ℓ is such that $|\mathcal{C}(\mathcal{P}(\ell))| = 1$.

The binary tree T_M is in a sense a "user manual" of the method M as it gives at any step which boxes to compare. However, as T_M should contain all possible cases we will use more synthetic ways to describe a method. Furthermore, the partition associated to a vertex of T_M is the exact expression of the knowledge on the balls colors provided by the comparisons done when reaching this vertex. We will see in the following that T_M may be useful to compute bounds on $Q(N, p, \leq)$ and $Q(N, p, =)$.

If v and w are the two children of a vertex u of T then $\mathcal{C}(\mathcal{P}(u))$ is equal to the disjoint union of the two non empty sets $\mathcal{C}(\mathcal{P}(v))$ and $\mathcal{C}(\mathcal{P}(w))$.

A vertex of T_M will be said at *level* k if it is at distance k from the root. The *height* of T_M is the maximum level of a vertex of T_M .

Let v be a vertex of T_M at level $k \geq 1$. We call *parent of* v , denoted by $p(v)$, the unique neighbor of v which is at level $k - 1$. The vertex $v' \neq v$ such that $p(v') = p(v)$ will be said the *sibling of* v . Remark that in a binary tree, any path P issued from the root contains $p(v)$ for each $v \in P$ which is not the root.

From its definition and some preceding remarks we get the following properties of T_M .

Property 9. *Any binary tree T_M associated to a method M solving the (N, p, \leq) -identification problem satisfies the following properties:*

- (i) *each p -majored Red-green coloring of \mathcal{B} labels exactly one leaf,*
- (ii) *the partition of a vertex at level k consists in $N - k$ boxes obtained after k comparisons,*
- (iii) *the maximum number of comparisons used by an execution of M is equal to the height of T_M .*

Notice that similar remarks and properties are valid in the case of the $(N, p, =)$ -identification problem. In particular the **Property 9'** obtained from Property 9 by replacing " (N, p, \leq) " by " $(N, p, =)$ " and " p -majored Red-green coloring" by " p -equal Red-green coloring" is also valid.

Remark 10. *Trivially $N - 1$ comparisons are enough to solve both identification problems. The partition labeling a vertex at level $N - 1$ (if any) consists in one single box, no more comparisons are possible, and only one side of the box contains less than $p + 1$ balls and those are exactly the green*

balls of \mathcal{B} . Hence such a vertex should be a leaf of T_M and the height of T_M is at most $N - 1$.

From the (easy and well known) fact that a binary tree of height k has at most 2^k leaves, Property 9'(i) and Property 9'(iii), we get the following lower bound.

Lemma 11. *For every two integers N and $p < \frac{N}{2}$: $Q(N, p, =) \geq \log_2(\binom{N}{p})$.*

Let \perp denotes either " $=$ " or " \leq ". The two following lemmas are also useful.

Lemma 12. *For every two integers N and $1 \leq p < \frac{N}{2}$ we have :*

$$Q(N, p, \perp) \geq 2 + Q(N - 2, p - 1, \perp).$$

Proof. Let T_M be the tree associated to a method solving the (N, p, \perp) -identification problem within $Q(N, p, \perp)$ comparisons. Since $p \geq 1$ the height of T_M , which is equal to $Q(N, p, \perp)$ by Property 9(iii), is at least 2 and there is in T_M an edge labeled " \neq " between the root R and a vertex v . In $\mathcal{P}(v)$, the first two balls b, b' that are compared by M are in a box B^* of Type (1, 1) and all other boxes are of Type (0, 1) and contain the balls of $\mathcal{B} \setminus \{b, b'\}$. Notice that the set $\mathcal{B} \setminus \{b, b'\}$ contains $N - 2$ balls and exactly one less green ball than \mathcal{B} . On another hand, for any leaf ℓ of the subtree T_v of T_M of root v we have $|\mathcal{C}(\mathcal{P}(\ell))| = 1$ and hence the balls b and b' should be in an unbalanced box of $\mathcal{P}(\ell)$. This means that, on any path of T_M from v to a leaf, there is a vertex where one of the two boxes that are compared is B^* containing only b and b' .

We claim that T_v gives a method to find the $(p - 1)$ -equal, or $(p - 1)$ -majorized (depending on the value of \perp), Red-green coloring of $N - 2$ balls: use the $N - 2$ boxes of cardinality 1 of $\mathcal{P}(v) \setminus B^*$, ignore b and b' in the partitions, do the box-comparisons and follow the edges as indicated by the labelings, except when you reach a vertex of T_v labeled with the "empty" box B^* : then don't do any comparison and go to any child. Then, for any leaf ℓ of T_v the resulting partition $\mathcal{P}_v(\ell)$ is the same as the one obtained from $\mathcal{P}(\ell)$ by withdrawing b and b' that are in different sides of one box. Hence $|\mathcal{C}(\mathcal{P}_v(\ell))| = 1$, else we would contradict the fact that $|\mathcal{C}(\mathcal{P}(\ell))| = 1$. So the height of T_v minus 1 (we skipped exactly one comparison) is at least $Q(N - 2, p - 1, \perp)$. Since the height of T_v is at most the height of T_M minus 1 we get that $Q(N, p, \perp) - 2 \geq Q(N - 2, p - 1, \perp)$. \square

Lemma 13. For every two integers N and $p < \frac{N-1}{3}$ we have :

$$Q(N, p, \perp) \leq p + Q(N - (p + 1), p, \perp).$$

Proof. In order to prove the upper-bound on $Q(N, p, \perp)$ it is sufficient to exhibit a method solving the (N, p, \perp) -identification problem within at most $p + Q(N - p - 1, p, \perp)$ comparisons. So let us assume that we have a set of N balls b_1, b_2, \dots, b_N colored by a p -equal or p -majored (depending on the value of \perp) Red-green coloring. We may first compare b_1 to b_2, b_3, \dots, b_i for some $2 \leq i \leq N$, thus obtaining a partition of Type $(x, y)^1(0, 1)^{N-i}$ for some integers $0 \leq x \leq y$ such that $x + y = i$. The rule will be to do so until $y = x + p + 1$, which will happen since we must have one side containing at most p balls and $N > 3p + 1$. At this stage, we have done $2x + p$ comparisons, all balls in the side of cardinality $y = x + p + 1 > p$ of the only bicolored box should be red and those $x \leq p$ in the other side are green : it remains $N - (p + 2x + 1)$ other balls whose colors have to be determined and the only information we have about them is that there are at most $p - x$, or exactly $p - x$ in case \perp means "=", green balls among them. Since $x \leq p$ and we assumed $N > 3p + 1$ then $N - (p + 2x + 1) > 2(p - x)$ and we can determine the colors of the remaining balls using at most $Q(N - (p + 2x + 1), p - x, \perp)$ comparisons. So $Q(N, p, \perp) \leq \text{Max}_{0 \leq x \leq p}(p + 2x + Q(N - (p + 2x + 1), p - x, \perp))$. This maximum is attained for $x = 0$ since by Lemma 12 we have for any $1 \leq x \leq p$:

$$\begin{aligned} p + Q(N - (p + 1), p, \perp) &\geq p + 2 + Q(N - (p + 3), p - 1, \perp) \\ &\geq \dots \\ &\geq p + 2x + Q(N - (p + 2x + 1), p - x, \perp). \end{aligned}$$

□

4 Proofs concerning the (N, p, \leq) -identification problem

In order to have a self-contained paper we begin this section with a proof of the first statement in Aigner's Theorem 1 (the second statement is contained in Theorem 3 which is proved in the next subsection).

4.1 Proof of $Q(N, p, \leq) = N + 1 - \left\lfloor \frac{N+1}{p+1} \right\rfloor$ or $N - \left\lfloor \frac{N}{p+1} \right\rfloor$

We first show that $N + 1 - \left\lfloor \frac{N+1}{p+1} \right\rfloor$ is an upper bound of $Q(N, p, \leq)$.

Let N and p be integers such that $0 \leq p < \frac{N}{2}$.

For $p = 0$: $Q(N, 0, \leq) = 0 = N + 1 - \left\lfloor \frac{N+1}{1} \right\rfloor$, and the bound is correct. Let us consider the case where $p \geq 1$.

Assume first that $N < 3p + 2$. Then $\left\lfloor \frac{N+1}{p+1} \right\rfloor = 2$ and $N + 1 - \left\lfloor \frac{N+1}{p+1} \right\rfloor = N - 1$ which is always an upper bound of $Q(N, p, \leq)$, as already noticed in Remark 10.

Let us consider now the case where $N \geq 3p + 2$. We may use consecutively Lemma 13 as long as we have a number of balls which is at least $3p + 2$:

$$Q(N, p, \leq) \leq p + Q(N - (p + 1), p, \leq) \leq \dots \leq \ell p + Q(N - \ell(p + 1), p, \leq).$$

Then $\ell \geq 1$ is such that

$$\begin{aligned} 2p + 1 \leq N - \ell(p + 1) < 3p + 2 &\iff 2p + 2 \leq N + 1 - \ell(p + 1) < 3p + 3 \\ &\iff N + 1 - 2(p + 1) \geq \ell(p + 1) > N + 1 - 3(p + 1) \\ &\iff \ell = \left\lfloor \frac{N + 1}{p + 1} \right\rfloor - 2. \end{aligned}$$

Then we have

$$\begin{aligned} Q(N, p, \leq) \leq \ell p + Q(N - \ell(p + 1), p, \leq) &\leq \ell p + N - \ell(p + 1) - 1 \\ &= N - \ell - 1 = N + 1 - \left\lfloor \frac{N + 1}{p + 1} \right\rfloor. \end{aligned}$$

We now show that $N - \left\lfloor \frac{N}{p+1} \right\rfloor$ is a lower bound of $Q(N, p, \leq)$.

Let N and p be integers such that $0 \leq p < \frac{N}{2}$.

For $p = 0$: $Q(N, 0, \leq) = 0 = N - \left\lfloor \frac{N}{1} \right\rfloor$, and the bound is correct. Let us consider the case where $p \geq 1$.

Let M be any method that solves the (N, p, \leq) -identification problem. The binary tree T_M has a leaf ℓ which is connected to the root by a path whose all edges are labeled with "=". The partition $\mathcal{P}(\ell)$ contains then only monocolored boxes.

In case one box B of $\mathcal{P}(\ell)$ is of cardinality at most p , there will be at least two p -majored Red-green colorings compatible with $\mathcal{P}(\ell)$: one where all

balls are red, and one where all balls in B are green and all other balls are red. This is in contradiction with the fact that ℓ is a leaf of T_M .

So all boxes of $\mathcal{P}(\ell)$ are of cardinality at least $p + 1$; hence $\mathcal{P}(\ell)$ contains at most $\lfloor \frac{N}{p+1} \rfloor$ boxes. These are obtain after at least $N - \lfloor \frac{N}{p+1} \rfloor$ comparisons. Since M was chosen as any method solving the (N, p, \leq) -identification problem, this implies that indeed $Q(N, p, \leq) \geq N - \lfloor \frac{N}{p+1} \rfloor$.

Noticing that the upper bound and lower bounds obtained above differ by at most 1 the proof is done. \square

We now prove Theorem 3 which contains the second statement of Theorem 1.

4.2 Proof of Theorem 3

We have to show that $Q(N, p, \leq) = N + 1 - \lfloor \frac{N+1}{p+1} \rfloor$ whenever $N \equiv r [p + 1]$ with $0 \leq r \leq \lfloor \frac{p}{2} \rfloor$ or $r = p$.

Consider first the case where $N \equiv p [p + 1]$. Then $N + 1 - \lfloor \frac{N+1}{p+1} \rfloor = N - \lfloor \frac{N}{p+1} \rfloor$, hence by Theorem 1 we may conclude that $Q(N, p, \leq)$ is indeed equal to $N - \lfloor \frac{N}{p+1} \rfloor$.

Let us assume now that $N \equiv r [p + 1]$ for some $0 \leq r \leq \lfloor \frac{p}{2} \rfloor$ and let then $q := \lfloor \frac{N+1}{p+1} \rfloor = \lfloor \frac{N}{p+1} \rfloor$.

By the first statement of Theorem 1 we have $N - q \leq Q(N, p, \leq) \leq N + 1 - q$. Assume, by contradiction, that there exists a method M^* that solves the (N, p, \leq) -identification problem using only $N - q$ comparisons. By Property 9(iii), the height of T_{M^*} is equal to $N - q$ and the partition associated to a leaf of T_{M^*} should then contain at least q boxes. As in the proof of the lower bound in section 4.1 we know that there exists a leaf ℓ^* of T_{M^*} whose associated partition $\mathcal{P}^*(\ell^*)$ is made of at most q , and hence exactly q , monocolored boxes.

Let $(R, u_1, u_2, \dots, u_{N-q} = \ell^*)$ be the path from the root R to ℓ^* in T_{M^*} and let $i = \min\{1 \leq j \leq N - q - 1 \mid \mathcal{P}^*(u_{j+1}) \text{ has } q \text{ boxes of cardinality } \geq p + 1\}$ (i does exist since $N - q - 1$ satisfies the requirement).

Consider the vertex v_{i+1} connected to u_i by an edge labeled " \neq " (that is v_{i+1} is the sibling of u_{i+1}). From our assumptions, the partition $\mathcal{P}^*(v_{i+1})$ consists into $q - 1$ monocolored boxes of cardinality at least $p + 1$, called *big monocolored boxes of $\mathcal{P}^*(v_{i+1})$* , one bicolored box B of cardinality at least $p + 1$ containing x balls in one side and y in the other where $1 \leq x \leq y \leq p$, and $N - (i + 1) - q$ monocolored boxes of cardinality at most p , called *small*

monocolored boxes of $\mathcal{P}^*(v_{i+1})$.

Considering $\mathcal{P}^*(v_{i+1})$, we denote by \mathcal{A} the set of balls belonging to big monocolored boxes, \mathcal{X} the set of the x balls in the small side of B , \mathcal{Y} the set of the y balls in the big side of B , and \mathcal{C} the set of balls belonging to small monocolored boxes. Notice that $\mathcal{A}, \mathcal{X}, \mathcal{Y}, \mathcal{C}$ is a partition of the whole set \mathcal{B} of balls.

From v_{i+1} we will follow a path in T_{M^*} using the following rule that will all along ensure partitions with exactly one bicolored box, containing \mathcal{X} in one side and \mathcal{Y} in the other, and monocolored boxes containing balls from $\mathcal{A} \cup \mathcal{C}$.

Rule : We start from the vertex v_{i+1} which by definition satisfies the requirement. As long as we are on a non-leaf vertex v we go down in the tree by repeating the following :

- if $B(v)$ and $B'(v)$ (the two boxes of $\mathcal{P}^*(v)$ that will be compared at this step) are both monocolored we follow the edge labeled " = " and get to the next vertex v ,
- else one of them, say $B(v)$, is the bicolored box of $\mathcal{P}^*(v)$. In that case, if in $\mathcal{P}^*(v)$ the balls in $B'(v)$ are all in \mathcal{C} then we follow the edge labeled with the answer that merges the balls in $B'(v)$ with those of \mathcal{X} and else we follow the edge with the answer that merges the balls in $B'(v)$ with those of \mathcal{Y} ; and get to the next vertex v .

The process stops when we reach a leaf-vertex l . By the rule, $\mathcal{P}^*(l)$ has monocolored boxes whose contents are included in $\mathcal{A} \cup \mathcal{C}$ and exactly one bicolored box B_l . Let \mathcal{X}' be the set of balls of B_l in the side containing \mathcal{X} and let \mathcal{Y}' be the set of balls of B_l in the side containing \mathcal{Y} . By definition the balls of \mathcal{X}' that are not in \mathcal{X} are in \mathcal{C} and those of \mathcal{Y}' that are not in \mathcal{Y} are in $\mathcal{A} \cup \mathcal{C}$.

Claim 14. $|\mathcal{X}'| + |\mathcal{C}| \leq p$.

Proof. There are at least $(q-1)(p+1)$ balls in \mathcal{A} and exactly $N = q(p+1)+r$ balls in \mathcal{B} , so we have at most $p+1+r$ balls in $\mathcal{X}' \cup \mathcal{Y}' \cup \mathcal{C}$. Since $|\mathcal{X}' \cup \mathcal{Y}'| \geq p+1$, one has $|\mathcal{X}' \cup \mathcal{Y}'| = p+1+s$, for some $0 \leq s \leq r \leq \lfloor \frac{p}{2} \rfloor$. Then $0 \leq |\mathcal{C}| \leq N - (q(p+1) + s) = r - s$, and $|\mathcal{X}'| \leq \lfloor \frac{p+1+s}{2} \rfloor \leq \lceil \frac{p+1+s}{2} \rceil \leq |\mathcal{Y}'| \leq p$, so that

$$|\mathcal{X}'| + |\mathcal{C}| \leq \left\lfloor \frac{p+1+s}{2} \right\rfloor + r - s = \left\lfloor \frac{p+1-s}{2} \right\rfloor + r \leq \left\lfloor \frac{p+1}{2} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor = p.$$

The claim is proved.

Claim 15. *No monocolored box of $\mathcal{P}^*(l)$ is included in \mathcal{C} .*

Proof. Let us assume the contrary and let $\mathcal{C}_1 \subseteq \mathcal{C}$ be the set of balls of a monocolored box of $\mathcal{P}^*(l)$. By definition $\mathcal{X}' \cup \mathcal{C}_1 \subseteq \mathcal{X} \cup \mathcal{C}$, so by Claim 14 we have $|\mathcal{X}' \cup \mathcal{C}_1| \leq p$. Then there are at least two p -majored Red-green colorings compatible with $\mathcal{P}^*(l)$: one colors in green only the vertices of \mathcal{X}' and the other colors in green the vertices of $\mathcal{X}' \cup \mathcal{C}_1$. This contradicts the fact that l is a leaf of T_M^* and ends the proof of the claim.

A few remarks are now enough to end the proof of Theorem 3. Let us remind that $\mathcal{P}^*(v_{i+1})$ contains $N-(i+1)$ boxes and among them $N-(i+1)-q$ are monocolored boxes included in \mathcal{C} . By Claim 15 all these monocolored boxes disappear after doing all the comparisons indicated by the path from v_{i+1} to l obtained by applying the Rule. As we already know, each of these comparisons decreases the number of boxes by 1 and it is straightforward to verify that, furthermore, the number of monocolored boxes included in \mathcal{C} either stays the same or decreases by 1 (only in case at least one of the two boxes that are compared is a monocolored box included in \mathcal{C}). So the number of boxes in $\mathcal{P}^*(l)$ is at most q . Moreover, by our assumption on M^* , the height of T_{M^*} and hence the level of l is at most $N - q$, so that the number of boxes in $\mathcal{P}^*(l)$ is at least q . From all these facts we deduce that each comparison done to get $\mathcal{P}^*(l)$ from $\mathcal{P}^*(v_{i+1})$ concerns at least one monocolored box included in \mathcal{C} . So, as we followed the Rule, the bicolored box B_l of $\mathcal{P}^*(l)$ is such that $\mathcal{X}' \subseteq \mathcal{X} \cup \mathcal{C}$ and $\mathcal{Y}' = \mathcal{Y}$.

Then, by Claim 14 and the fact that $|\mathcal{Y}'| = y \leq p$, each side of B_l contains at most p balls and we have at least two p -majored Red-green colorings compatible with $\mathcal{P}(l)$: everything colored red except the balls in one of the two sides of B_l . This contradicts the fact that l is a leaf of T_M . \square

4.3 The motivation of Question 1

From the preceding results we get the following properties of $Q(N, p, \leq)$.

Property 16. (1) *Let p be a nonnegative integer, the function $: N \mapsto Q(N, p, \leq)$ is nondecreasing and 1-lipschitz, that is for $N \geq 2p + 1$:*

$$Q(N, p, \leq) \leq Q(N + 1, p, \leq) \leq 1 + Q(N, p, \leq)$$

(2) *If N and p are such that $N \geq 2p + 1$ and $Q(N, p, \leq) < N + 1 - \lfloor \frac{N+1}{p+1} \rfloor$*

then for any $N' = N + k(p + 1)$ ($k \geq 0$) we have:

$$Q(N', p, \leq) < N' + 1 - \lfloor \frac{N' + 1}{p + 1} \rfloor$$

Proof. (1) From the first statement of Theorem 1 one get the lower bound of $Q(N + 1, p, \leq)$:

$$Q(N, p, \leq) \leq N + 1 - \left\lfloor \frac{N + 1}{p + 1} \right\rfloor \leq Q(N + 1, p, \leq).$$

Let now M be a method determining a p -majored Red-green coloring of N balls within at most $Q(N, p, \leq)$ comparisons. It is easy to derive from M a method determining a p -majored Red-green coloring of $N + 1$ balls using at most one more comparison : put aside one of the $N + 1$ balls, you have then a p -majored Red-green coloring of N balls, determine their colors using M , if you get less that p green balls one more comparison will be necessary to know the color of the ball that was put aside. Thus the second inequality of Property 16 (1) is proved.

(2) Let us consider $N' = N + k(p + 1)$ for some $k \geq 0$. By Lemma 13 we know that $Q(N', p, \leq) \leq kp + Q(N, p, \leq)$ and then by the assumption that $Q(N, p, \leq) < N + 1 - \lfloor \frac{N+1}{p+1} \rfloor$ we get

$$\begin{aligned} Q(N', p, \leq) &< kp + N + 1 - \left\lfloor \frac{N + 1}{p + 1} \right\rfloor = N' + 1 - k - \left\lfloor \frac{N+1}{p+1} \right\rfloor \\ &= N' + 1 - \left\lfloor \frac{N'+1}{p+1} \right\rfloor. \end{aligned}$$

□

For a given $p \geq 1$ and any $N \geq 2p + 1$, let us say that N is of *character* p^- in case $Q(N, p, \leq) < N + 1 - \left\lfloor \frac{N+1}{p+1} \right\rfloor$ and else N is said of *character* p^+ .

Theorem 3 shows that N is of character p^+ as soon as $N \equiv r \pmod{p + 1}$ with $0 \leq r < \frac{p+1}{2}$. From Property 16 and Theorem 1 it is easy to derive other properties of the character of N depending on the remainder of the Euclidean division of N by $p + 1$:

Property 17. *Given $p \geq 1$ and $N \geq 2p + 1$, the following holds :*

1. If N is of character p^- then $N + 1$ is p^- except if $N \equiv p - 1 \ [p + 1]$,
2. If N is of character p^- then N' is of character p^- for any $N' \geq N$ such that $N' \equiv N \ [p + 1]$.

From this last property we derive that the sequence of characters of N becomes periodic starting from a big enough N .

Property 18. For every nonnegative integer p exactly one of the two following statements is satisfied :

1. The character of every $N \geq 2p + 1$ is p^+ , or
2. There exist two integers $\frac{p+1}{2} \leq r_p \leq p - 1$ and $N_p \geq 2p + 1$ such that for every $N \geq N_p$:

$$N \text{ is of character } p^- \iff N \equiv r \ [p + 1] \text{ with } r_p \leq r \leq p - 1.$$

Proof. Let us assume that the first statement is not satisfied. Then the set $\mathcal{S} = \{N \mid N \geq 2p + 1 \text{ and } N \text{ is of character } p^-\}$ is not empty. So we may choose r_p as the minimum r for which there exists $N \in \mathcal{S}$ such that $N \equiv r \ [p + 1]$. Let now N_p be the minimum $N \in \mathcal{S}$ such that $N \equiv r_p \ [p + 1]$.

By Theorem 3 (page 3) and the definition of r_p we know that $\frac{p+1}{2} \leq r_p \leq p - 1$ and that every $N \geq 2p + 1$ such that $N \equiv r \ [p + 1]$ with $0 \leq r \leq r_p - 1$ or $r = p$ is of character p^+ .

Furthermore, by Property 17, every $N \geq N_p$ such that $N \equiv r \ [p + 1]$ with $r_p \leq r \leq p - 1$ is of character p^- .

□

As mentioned at the end of section 2.1, we looked at all couples (N, p) such that $Q(N, p, \leq) < N + 1 - \lfloor \frac{N+1}{p+1} \rfloor$ for $N \leq 51$. We did not find any contradiction to the hypothesis that r_p is always equal to $\lceil \frac{p}{2} \rceil + 1$. This motivated our Question 1 (page 3).

5 Proofs concerning the $(N, p, =)$ -identification problem

5.1 A Lemma

The next Lemma will allow us to prove Theorem 6.

Lemma 19. *Let N and p be integers such that $1 \leq p < \frac{N}{2}$, and let $\nu \geq 0$ be an integer such that $2^\nu \leq N - 2p$, then*

$$Q(N, p, =) \leq 2^\nu - 1 - \nu + Q(N + 1 - 2^\nu, p, \leq).$$

Proof. Assume first that $\nu = 0$. It is always true that $Q(N, p, =) \leq Q(N, p, \leq)$, so Lemma 19 is verified in the case $\nu = 0$.

From now on we will assume that $\nu \geq 1$. As for the preceding proofs, we will use a method which compares boxes, but this time we will have several locations for the boxes : the *laboratory* which will initially contain all the balls of \mathcal{B} into boxes of Type $(0, 1)$, the *reserve* that all along contains only balanced bicolored boxes and a *podium* which contains steps numbered starting from 0: the i th step of the podium is provided to receive a monocolored box of size 2^i .

Any situation where the laboratory contains only boxes of Type $(0, 1)$ and the reserve only balanced boxes will be said *a correct situation*. The *overage of a correct situation* is equal to the overage of red over green among balls that are in the laboratory or in the reserve. Notice that it is the same as the overage among balls in the laboratory since the boxes in the reserve are balanced.

We now describe a "subroutine" $\mathcal{S}(i)$, for $i \geq 0$, which starting from any correct situation with an overage $O_i > 2^i$, builds one monocolored box of size 2^i , and ends with a correct situation with a new overage of at least $O_i - 2^i > 0$.

- For $i = 0$, every box of the laboratory has cardinality 2^i . After putting one of it aside, the situation is still correct and the overage decreases by at most one (in case the ball in the box we have chosen happened to be red).
- For $i = 1$, let us compare, one by one, couples of balls in the laboratory: as long as we obtain a balanced box we put it in the reserve. Since it is assumed that $O_1 > 2$, there are at least three more red balls than green balls and thus we are sure to get once a monocolored couple. Putting it aside, we obtain a correct situation whose overage has decreased by at most 2.
- The same way, we may define inductively $\mathcal{S}(i + 1)$ from $\mathcal{S}(i)$. Assume $\mathcal{S}(i)$ exists, we will show that then $\mathcal{S}(i + 1)$ exists. Suppose that we are in a correct situation with an overage $O_{i+1} > 2^{i+1}$.

We may apply $\mathcal{S}(i)$ in order to get a monocolored box B of size 2^i . We put B aside. By the induction hypothesis, the situation is then correct and the remaining balls in the laboratory have an overage $O > 2^{i+1} - 2^i = 2^i$. We may apply $\mathcal{S}(i)$ again in order to obtain a second monocolored box B' of size 2^i . We then compare B and B' . If this comparison results in a bicolored box, this box is balanced and we put it into the reserve : the overage in the laboratory is still more than 2^{i+1} and we may repeat the procedure. Each time we get a bicolored box the number of balls in the laboratory decreases, but not the overage, so that we will finally obtain a monocolored box of size 2^{i+1} . After putting this box aside, by the induction hypothesis, the situation is correct. If the withdrawn balls are green, the overage of the final situation increases, and else it decreases by at most 2^{i+1} . So $\mathcal{S}(i+1)$ does exist.

Let μ be the largest integer such that $2^\mu \leq N - 2p$; since we assumed that $\nu \geq 1$, we have $\mu \geq 1$. We will define a method $M(\nu)$ that solves the $(N, p, =)$ -identification problem within at most $2^\nu - 1 - \nu + Q(N+1-2^\nu, p, \leq)$ comparisons, thus proving the lemma. We describe now $M(\nu)$.

At Step 0 of $M(\nu)$ we start with the initial state, that is an empty podium, an empty reserve, and a laboratory containing all the balls in boxes of Type $(0, 1)$. So, at this stage, the situation is correct and the overage $\mathcal{O}(0) = N - 2p$ is by definition at least $2^\mu > \sum_{j=0}^{\mu-1} 2^j \geq 2^0$. We apply $\mathcal{S}(0)$ in order to obtain a monocolored box of cardinality 2^0 which is put on the 0th step of the podium. Then we are in a correct situation with a positive overage $\mathcal{O}(1)$. If $\mu > 1$ then $\mathcal{O}(1) > \sum_{j=1}^{\mu-1} 2^j \geq 2^1$. We may continue: for each $i \leq \nu - 1 \leq \mu - 1$, Step i starts with a correct situation with an overage $\mathcal{O}(i) > \sum_{j=i}^{\mu-1} 2^j \geq 2^i$ and we apply $\mathcal{S}(i)$ in order to obtain a monocolored box of cardinality 2^i which is put on the i th step of the podium. We are then in a correct situation with a positive overage $\mathcal{O}(i+1)$. If $\mu > i+1$ then $\mathcal{O}(i+1) > \sum_{j=i+1}^{\mu-1} 2^j \geq 2^{i+1}$.

At the end of Step $\nu - 1$ of $M(\nu)$ we have filled ν steps of the podium with monocolored boxes of cardinality $1, 2, \dots, 2^{\nu-1}$. Let \mathcal{B}_ν be the set of balls that are not on the podium and $N_\nu = |\mathcal{B}_\nu| = N - (2^\nu - 1)$.

The reserve contains only balanced boxes and hence an even number, let us say $2k$ ($k \geq 0$), of balls. The laboratory contains $N_\nu - 2k = N + 1 - 2^\nu - 2k$ balls being each contained in one box of Type $(0, 1)$ and at most $p - k$ of these balls are green. The overage of red balls in the laboratory is the same as the overage in \mathcal{B}_ν and we know that it is positive.

So we may determine the colors of all balls in the laboratory within at most $Q(N_\nu - 2k, p - k, \leq)$ comparisons and these will lead to a partition \mathcal{P} of the balls of the laboratory into at least $N_\nu - 2k - Q(N_\nu - 2k, p - k, \leq)$ boxes. By Lemma 12 (page 8) we know that $Q(N_\nu - 2k, p - k, \leq) \leq Q(N_\nu, p, \leq) - 2k$; from this we deduce that \mathcal{P} has at least $N_\nu - Q(N_\nu, p, \leq)$ boxes. Comparing one of these boxes with all boxes in the reserve will then provide a partition \mathcal{P}_ν of \mathcal{B}_ν which determines completely the colors of the balls in \mathcal{B}_ν and has the same number of boxes than \mathcal{P} .

Then we know exactly the number $p' \leq p$ of green balls in \mathcal{B}_ν and there should be $p - p'$ green balls on the podium. The binary representation of $p - p'$ gives us the boxes of the podium containing the green balls without any further comparisons. So we have a partition of the whole set of balls to which corresponds a unique Red-green coloring and it consists into ν boxes for balls on the podium and at least $N_\nu - Q(N_\nu, p, \leq)$ boxes for balls in \mathcal{B}_ν . The number of comparisons that are done to obtain such a partition is then at most

$$N - (\nu + N_\nu - Q(N_\nu, p, \leq)) = 2^\nu - 1 - \nu + Q(N + 1 - 2^\nu, p, \leq). \quad \square$$

5.2 The proof of Theorem 6

In this subsection we provide the proof of the theorem stated page 4 which gives an upper bound of $Q(N, p, =)$.

Combining Lemma 19 and Theorem 3 we get that

$$Q(N, p, =) \leq \min_{0 \leq \nu \leq \mu} \left(N + 1 - \nu - \left\lfloor \frac{N + 2 - 2^\nu}{p + 1} \right\rfloor \right),$$

for μ being the largest integer such that $2^\mu \leq N - 2p$.

It remains to show which value of ν provides the minimum of $f(x) = N + 1 - x - \lfloor \frac{N+2-2^x}{p+1} \rfloor$.

We will do it by comparing the value of f for two consecutive values ν and $\nu + 1$ such that $2^{\nu+1} \leq N - 2p$. We may write $N = 2p + 2^{\nu+1} + \ell = 2p + 2^\nu + 2^\nu + \ell$ where $\ell \geq 0$:

- $f(\nu) = N + 1 - \nu - \lfloor \frac{N+2-2^\nu}{p+1} \rfloor = N + 1 - \nu - \lfloor \frac{2p+2^\nu+\ell+2}{p+1} \rfloor = N - \nu - 1 - \lfloor \frac{2^\nu+\ell}{p+1} \rfloor$
- $f(\nu+1) = N - \nu - \lfloor \frac{N+2-2^{\nu+1}}{p+1} \rfloor = N - \nu - \lfloor \frac{2p+\ell+2}{p+1} \rfloor = N - \nu - 2 - \lfloor \frac{\ell}{p+1} \rfloor$

So, if $2^\nu \leq p$ then $f(\nu + 1) \leq f(\nu)$ and else $f(\nu + 1) \geq f(\nu)$.

In case $N - 2p \leq 2p$, then any $\nu + 1 \leq \mu$ verifies that $2^{\nu+1} \leq 2^\mu \leq N - 2p \leq 2p$ and then $2^\nu \leq p$. So, then $\min_{0 \leq \nu \leq \mu} (N + 1 - \nu - \lfloor \frac{N+2-2^\nu}{p+1} \rfloor)$ is attained for $\nu = \mu = m$.

If $2p < N - 2p$, then the minimum will be attained for m equal to the largest ν such that $2^{\nu-1} \leq p$, or equivalently $2^\nu \leq 2p$. \square

However as mentioned in subsection 2.2 the bound in Theorem 6 is not always equal to $Q(N, p, =)$.

5.3 The Towers Method

In this part it will be convenient to use the notation $B(N, p, =)$ for the value $N - Q(N, p, =)$. This value represents the minimum number of boxes we may have at the end of any optimal method solving the $(N, p, =)$ -problem (i.e. using at most $Q(N, p, =)$ comparisons). Similarly we define $B_+(N, p, =) = N - Q_+(N, p, =)$. (Notice that $B_+(N, p, =)$ is then a lower bound of $B(N, p, =)$). By Theorem 6 we know that for a "sufficiently big N ", $B_+(N, p, =)$ is approximately $\frac{N}{p+1}$. We will show in the following that a better ratio may be obtained for any $p \geq 3$. Let us first consider the case where $p = 3$.

Proposition 20. *There exists a positive constant D such that $\forall N \geq 7$, $B(N, 3, =) \geq \lfloor \frac{3N}{10} \rfloor - D$.*

Proof. To prove the bound we exhibit a method giving the colors of all balls that ends with at least $\lfloor \frac{3N}{10} \rfloor - D$ boxes, for a constant $D > 0$.

We first compare b_1 to b_2, b_3, \dots until to obtain 4 balls of the same color (and hence red). Thus we get a partition containing exactly one bicolored box and, more precisely, of Type $(u, 4)^1(0, 1)^{N-(u+4)}$ for some integer $u \leq 3$.

Then we build monocolored boxes of cardinality alternately 2 and 3 : during this process, each time we get a bicolored box we compare it to the previous lonely bicolored box, this will happen at most $3-u$ times, and doing so we keep the property of having exactly one bicolored box. We stop this process when there is no more cardinality 1 boxes. At this stage, if we don't have the same number of boxes of cardinality 2 and 3, we compare the boxes obtained during the last trial with the unique bicolored box, increasing its cardinality by at most 4. So at the end of this process, we have a partition of the balls of Type $(x, y)^1(0, 2)^a(0, 3)^a$ where $y \geq 4$, $x + y \leq 17$ and $a \geq \frac{N-17}{5}$. It is not possible that $x = 2$, since else the remaining green ball cannot be in a box of Type $(0, 2)$ or $(0, 3)$. So either $x = 3$ and we know the colors of

all balls, or $x = 0$ and the three green balls are in a monocolored box of size 3 or $x = 1$ and the two remaining green balls are in a monocolored box of size 2. In the last two cases, we have to solve a problem equivalent to the one of finding one green ball among a balls, which by Proposition 7 can be done leaving at least $\frac{a}{2}$ boxes. So in total we have at least $1 + a + \frac{a}{2}$ boxes which is at least $1 + \frac{3a}{2} \geq \frac{3N}{10} - 4.1$, so more than $\frac{3N}{10} - 5$. \square

We observe that the ratio $\frac{3}{10}$ is slightly better than the ratio $\frac{1}{4}$ obtained by Theorem 6. Hence, for a big enough N the method described in the proof of Proposition 20 is more efficient than the one in the proof of Theorem 6.

The method used in the proof of Proposition 20 can be generalized for any value of p by the *Towers Method* that we describe now : this name comes from the fact that we may consider that we build, from the boxes obtained by the chosen comparisons, towers each containing all monocolored boxes of a given size, the *height of a tower* corresponding to the number of boxes it is made of. For a given p the method depends on the following parameters :

- a non empty set $E = \{u_1, \dots, u_k\} \subseteq \{2, \dots, p\}$ of *sizes of the monocolored boxes* that will be built. This subset should satisfy the following "**unicity property**" : For every $0 \leq l \leq p$, there exists at most one k -tuple of non-negative integers (q_1, \dots, q_k) such that $l = q_1 u_1 + \dots + q_k u_k$. We denote by $\mathcal{U}(p)$ the set of subsets of $\{2, \dots, p\}$ that have the unicity property and by L_E , E being in $\mathcal{U}(p)$, the set of values $0 \leq l \leq p$, for which there exists a k -tuple associated with the above expression of l as a sum of values in E .

The cardinality $k \leq p - 1$ of E will be the number of towers that will be erected. In the proof of Proposition 20 we had $E = \{u_1, u_2\}$ with $u_1 = 2$ and $u_2 = 3$ and indeed 1 cannot be expressed as a sum of 2's and 3's, and 0, 2 or 3 can, but by only one way ; so $L_E = \{0, 2, 3\}$.

- a set of k integers h_1, \dots, h_k , expressing each the number of monocolored boxes of size u_i that are added to the u_i -tower at each step. In the proof of Proposition 20 we had $h_1 = h_2 = 1$.

Now we may describe the Towers method.

As usual all balls are initially in boxes of Type $(0, 1)$. At the first step we compare b_1 to b_2, b_3, \dots until to obtain $p + 1$ balls of the same color (and hence red). Thus we get a partition containing exactly one bicolored box of Type $(u, p + 1)$ for some integer $u \leq p$, with all other boxes of cardinality 1.

At the second step we use a procedure "add floors to each tower" by consecutively building h_1 monocolored boxes of cardinality u_1 added to the

u_1 -tower, h_2 monocolored boxes of cardinality u_2 added to the u_2 -tower, \dots , h_k monocolored boxes of cardinality u_k added to the u_k -tower. During this process, each time it happens that we create a new bicolored box we compare it to the previous lonely bicolored box in order to keep a partition with exactly one bicolored box. We continue to "add floors" as long as we can. If we have to stop because of lack of cardinality 1 boxes before finishing the "add floors" procedure, then we compare each box created during the last and partial "add floors" procedure with the unique bicolored box. At that point, the partition is of Type $(x, y)^1(0, u_1)^{h_1c} \dots (0, u_k)^{h_kc}$ where $x \leq p$, $y \geq p + 1$, $c = \frac{N-(x+y)}{h_1u_1+\dots+h_ku_k}$ (so c corresponds to the number of times we could perform entirely the procedure "add floors").

At this stage we know for sure x green balls and we have to detect the remaining $l = p - x$ ($0 \leq l \leq p$). Since these balls are in the towers, by the "unicity property", there exists a unique $(q_1(l), \dots, q_k(l))$ such that $l = \sum_{1 \leq i \leq k} q_i(l)u_i$ and $q_i(l) \geq 0$ for each i ; that is $l \in L_E$. It remains now to solve the $(h_i c, q_i(l), =)$ -problem for each $1 \leq i \leq k$ in order to find all green balls, since one monocolored box may be considered as just one ball of the same color. Any method solving these problems may be used. In case we know for each $(N, q_i(l), =)$ -problem a constant $C(q_i(l))$ such that the $(N, q_i(l), =)$ -problem may be solved leaving at least $C(q_i(l))N - D_i$ boxes, for some constant D_i , since exactly as in the case of the proof of Proposition 20 the value of $x + y$ is bounded by a constant, we would then get at least $\frac{\sum_{1 \leq i \leq k} C(q_i(l))h_i}{h_1u_1+\dots+h_ku_k}N - D_l$ remaining boxes, for some constant D_l . Notice that we know from Theorem 6 that coefficients $C(q_i(l)) \geq \frac{1}{q_i(l)+1}$ do exist.

As we cannot fix the value $l \in L_E$ of balls remaining to be detected we obtain $B(N, p, =) \geq \min_{l \in L_E} \left\{ \frac{\sum_{1 \leq i \leq k} C(q_i(l))h_i}{h_1u_1+\dots+h_ku_k} \right\} N - D$, for some constant D .

Notice that since $1 \notin E$ we have $q_i(l) < p$. Hence we may show now how, based on best known values $C^T(i)$ for $i < p$ we may compute recursively, for each $p \geq 3$, the Towers Method's parameters providing the maximum constant $C^T(p)$, by solving a linear programming problem. We will need values of $C^T(i)$ for $i < 3$: the $(N, 0, =)$ -problem can obviously be solved leaving N boxes. By Propositions 7 and 8, $B(N, 1, =) \geq \frac{1}{2}N - \frac{1}{2}$ and $B(N, 2, =) \geq \frac{1}{3}N - \frac{1}{3}$. So we set $C^T(0) = 1$, $C^T(1) = \frac{1}{2}$ and $C^T(2) = \frac{1}{3}$.

First we have to compute all subsets $E = \{u_1, \dots, u_k\} \in \mathcal{U}(p)$. For each such E , we will show how to compute heights h_1, h_2, \dots, h_k providing the best proportion $C^E(p)$ of boxes we are sure to obtain by a Towers Method based on E . For that purpose, we need to determine what is in such an optimal method, the proportion x_r of balls placed in monocolored boxes of size u_r

among the set of balls placed in monocolored boxes, for each $1 \leq r \leq k$. So x_r represents $\frac{h_r u_r}{h_1 u_1 + \dots + h_k u_k}$. We will need one more variable y_E , and the linear programming problem P_E will be the following :

$$\text{Maximize } y_E \text{ subject to } \begin{cases} \sum_{i=1}^k x_i = 1, \\ y_E - \sum_{1 \leq i \leq k} C(q_i(l)) u_i^{-1} x_i \leq 0 \quad \text{for each } l \in L_E, \\ x_i \geq 0 \end{cases}$$

It is clear that P_E is feasible. Let $(x_1, x_2, \dots, x_k, y_E)$ be an optimal vertex of P_E . Since all coefficients of P_E are rational we have that all x_i 's are rational numbers, and then there exist integers a_1, \dots, a_k, b such that $x_r = \frac{a_r}{b} = \frac{a_r \prod_{1 \leq i \leq k} u_i}{b \prod_{1 \leq i \leq k} u_i}$ for each $1 \leq r \leq k$. A Towers Method with parameters E and $h_r = a_r \prod_{1 \leq i \leq k, i \neq r} u_i$ for each $1 \leq r \leq k$, will provide, for a big enough number of balls, a proportion x_r of balls placed in monocolored boxes of size u_r among balls placed in monocolored boxes.

We can solve P_E for each $E \in \mathcal{U}(p)$ and the best ratio $C^T(p) = \max_{E \in \mathcal{U}(p)} y_E$. Notice that it is enough to consider only maximal subsets E in $\mathcal{U}(p)$.

In particular one can show the following :

Proposition 21. *For $3 \leq p \leq 6$, the best ratio obtained by the Towers Method is*

- $C^T(3) = \frac{3}{10}$, with $E = \{2, 3\}$, $u_1 = 2$, $u_2 = 3$, $h_1 = 1$ and $h_2 = 1$,
- $C^T(4) = \frac{5}{18}$, with $E = \{2, 3\}$, $u_1 = 2$, $u_2 = 3$, $h_1 = 3$ and $h_2 = 4$,
- $C^T(5) = \frac{2}{9}$, with $E = \{2, 3\}$, $u_1 = 2$, $u_2 = 3$, $h_1 = 3$ and $h_2 = 1$,
- $C^T(6) = \frac{1}{5}$, with $E = \{3, 4\}$, $u_1 = 3$, $u_2 = 4$, $h_1 = 3$ and $h_2 = 4$.

However one may also wonder if there are other methods that could give better bounds.

For any positive integer p , let us define I_p as the set of values $C \in [0, 1]$ for which there exists a constant $D \geq 0$ such that, for each $N > 2p$, we have

$$B(N, p, =) \geq CN - D.$$

Let C_p be the supremum of I_p . It is easy to see that I_p is equal either to $[0, C_p]$ or to $[0, C_p[$, and that, by Theorem 6, $C_p \geq \frac{1}{p+1}$ for every $p \geq 0$.

Proposition 22. *The sequence (C_p) is non-increasing and $\lim_{p \rightarrow \infty} pC_p = \infty$.*

Proof. Let us consider an integer $p \geq 1$ and an integer $N \geq 2p + 1$. By Lemma 12 one has

$B(N-2, p-1, =) = N-2-Q(N-2, p-1, =) \geq N-Q(N, p, =) = B(N, p, =)$. Assume that for some $C > 0$ there exists $D \geq 0$ such that $B(N, p, =) \geq CN - D$. Then by the previous inequality we get that $B(N-2, p-1, =) \geq C(N-2) - D$ and hence $C_{p-1} \geq C_p$.

To state $\lim_{p \rightarrow \infty} pC_p = \infty$, we will use a coefficient provided by the Towers Method.

Given an integer $q \geq 2$, let p be any integer such that $p \geq q^q$ and set $k_p = \lfloor \frac{p}{q} \rfloor$.

Claim 23. *The set $E = \{u_i | u_i = k_p + q^{i-1}, i \in \{1, \dots, q-1\}\}$ belongs to $\mathcal{U}(p)$.*

In order to prove this Claim we have to show that for any $0 \leq l \leq p$ there exists at most one $(q-1)$ -tuple (x_1, \dots, x_{q-1}) of non-negative integers such that $l = \sum_{1 \leq i \leq q-1} x_i u_i$. So, let us consider $0 \leq l \leq p$ and let $s = \lfloor \frac{l}{k_p+1} \rfloor$. Since $l \leq p$, by the definition of k_p , we have $0 \leq s \leq q-1$. Then, since each $u_i \geq k_p + 1$, we have $\sum_{1 \leq i \leq q-1} x_i \leq s$.

On another hand we have $k_p \geq q^{q-1} > (q-2)q^{q-2} \geq (s-1)q^{q-2} - s$, and then by definition of s we have $l \geq s(k_p + 1) = (s-1)k_p + k_p + s > (s-1)(k_p + q^{q-2})$. Then we have $\sum_{1 \leq i \leq q-1} x_i \geq s$. By the inequality above we conclude that $\sum_{1 \leq i \leq q-1} x_i = s$. There is a unique way to write $l - sk_p$ in basis q and hence at most one way to obtain l balls as the union of boxes of cardinalities $k_p + q^0, \dots, k_p + q^{q-2}$. The Claim is proved.

Let us now assume that for some "big" N we use the Towers method with E as defined in Claim 23 and heights $h_i = 1$ for each $1 \leq i \leq q-1$. Let l be the number of green balls that should be detected at the end of the erection of the towers. Notice that the towers all have the same number a of monocolored boxes. By the proof of Claim 23 there are $s = \lfloor \frac{l}{k_p+1} \rfloor$ monocolored boxes to discover and $0 \leq s \leq q-1$. We claim that the case leaving the smallest number of boxes is when there is one green box in each of the $q-1$ towers, in which case we end with at least $(q-1)\frac{a}{2}$ boxes. Indeed if we have less than $q-1$ green boxes to find, then obviously we won't end with less boxes, and in case we have to find $q-1$ green boxes among at most $q-2$ towers we gain at least $\frac{a}{2}$ boxes and loose at most $\frac{a}{2} - C_{q-1}a \leq \frac{a}{2}$. It remains to observe that N is approximately equal to

$$a \sum_{1 \leq l \leq q-1} (k_p + q^{l-1}) = a \left((q-1)k_p + \frac{q^{q-1}-1}{q-1} \right) \leq aqk_p \leq ap.$$

Hence $B(N, p, =) \geq (q-1)\frac{N}{2^p} - R$ for some constant R which means that $C_p \geq \frac{q-1}{2^p}$ and then $pC_p \geq \frac{q-1}{2}$, thus $\lim_{p \rightarrow \infty} pC_p = \infty$. \square

6 Conclusion

The problems we considered here on a set of bicolored balls are also known with other equivalent formulations or generalizations: detection of fake coins with a two-pan balance, minimization of the number of questions allowing to find a truth-teller in a population composed by "knights" who always tell the truth, and "knaves", who always lie, chips testing... ([8], [2], [4], [6]...). They can also be formulated as two-players games ([7], [9], [6], [1], [10]...). See [5] for more details.

In Theorem 3 of this paper, for a given integer p , we provide the exact value of $Q(N, p, \leq)$ for half of the integers N and for the other half we guessed in Question 1 a formula which is verified for any $N \leq 51$.

We also proposed some bounds on $Q(N, p, =)$, but the exact value is proved only for $p = 1$ or 2 . In fact the $(N, p, =)$ -identification problem seems to be much more difficult to handle than the (N, p, \leq) -identification problem.

Remark that it is easy to decide if a given partition of balls into boxes is compatible with a (unique) p -majority Red-green coloring. We conclude with a last question : *What is the complexity of deciding if a given partition of balls into boxes is compatible with a (unique) p -equal Red-green coloring ?*

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