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Variational formulation for models of shear shallow water flows and ideal turbulence

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Abstract

The shallow water equations without shear effects are similar to the gas dynamics equations with a polytropic equation of state. When the shear effects are taken into account, the equations contain additional evolution equations mathematically analogous to those of the Reynolds stresses in turbulent flows of compressible fluids when the source terms are neglected (*ideal turbulence*). We show that the non-dissipative model of shear shallow water flows and the model of ideal turbulence admit a similar variational formulation where, in the both cases, the equations for the Reynolds stress tensor evolution are considered as non-holonomic constraints.

Keywords: shear shallow water flows, ideal turbulence, Reynolds averaged equations, Hamilton's principle, non-holonomic constraint

1 Introduction

The reason for considering together simplified models of non-dissipative shear shallow water flows and ideal turbulence without source terms is there exists a mathematical similarity between an asymptotic model of shear flows of long waves over flat bottom and the homogeneous Reynolds-averaged equations in turbulence [1, 2, 3, 4, 5, 6].

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The two models can be considered as reversible because they admit a conservative energy balance law. The equation for Reynolds stress tensor evolution and its analogue for shear shallow water flows cannot be integrated: it corresponds to non-holonomic constraints in analytical mechanics. This observation allows us to formulate the Hamilton principle for shear shallow water flows and for ideal turbulence with a non-holonomic constraint governing the Reynolds stress tensor evolution.

The structure of the paper is as follows.

In section 1, we present the both models and establish their equivalence in the dissipationless limit.

In section 2, we derive the momentum equation for both models from Hamilton's principle of stationary action by considering the Reynolds equations as non-holonomic constraints.

A conclusion ends the paper.

1.1 Shear shallow water flows

The governing equations of shear shallow water flows are derived from free surface Euler equations as an exact asymptotic model of weakly sheared flows for long waves over a flat bottom [1] (and [7], where is given a generalization of the model):

$$\left\{ \begin{array}{l} \frac{\partial h}{\partial t} + \operatorname{div}(h \mathbf{U}) = 0, \\ \frac{\partial(h \mathbf{U})}{\partial t} + \left[\operatorname{div} \left(h \mathbf{U} \otimes \mathbf{U} + \frac{gh^2}{2} \mathbf{I} + \mathbf{R} \right) \right]^T = 0, \\ \frac{d\mathbf{R}}{dt} + \mathbf{R} \operatorname{div} \mathbf{U} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{R} + \mathbf{R} \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T = 0. \end{array} \right. \quad (1)$$

The superscript " T " denotes the transposition, \mathbf{I} is the identity tensor, $d/dt = \partial/\partial t + \mathbf{U}^T \nabla$ is the material derivative with respect to the mean motion, $h(t, \mathbf{x})$ is the fluid depth and g is the gravity acceleration. The depth average velocity \mathbf{U} and the Reynolds stress tensor \mathbf{R} are defined as :

$$h \mathbf{U} = \int_0^h \tilde{\mathbf{U}} dz, \quad \mathbf{R} = \int_0^h (\tilde{\mathbf{U}} - \mathbf{U}) \otimes (\tilde{\mathbf{U}} - \mathbf{U}) dz,$$

where $\tilde{\mathbf{U}}(t, \mathbf{x}, z)$ is the instantaneous fluid velocity. Variables $\mathbf{x} = (x, y)^T$ are the horizontal coordinates, z is the vertical coordinate (which is opposed to the direction of gravity acceleration). Equations are written in the limit of weakly sheared flows for three-dimensional long waves.

The reduced Reynolds stress tensor is :

$$\mathbf{P} = \frac{\mathbf{R}}{h}$$

and System (1) admits the energy conservation law :

$$\begin{aligned} & \frac{\partial}{\partial t} \left(h \left(\frac{1}{2} |\mathbf{U}|^2 + \frac{gh}{2} + \frac{\text{tr} \mathbf{P}}{2} \right) \right) \\ & + \text{div} \left(h \mathbf{U} \left(\frac{1}{2} |\mathbf{U}|^2 + \frac{gh}{2} + \frac{\text{tr} \mathbf{P}}{2} \right) + \frac{gh^2}{2} \mathbf{U} + h \mathbf{P} \mathbf{U} \right) = 0. \end{aligned} \quad (2)$$

Taking into account the mass conservation law (1)₁, (equation for h), we obtain the equivalent equation for \mathbf{P} :

$$\frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{P} + \mathbf{P} \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T = 0.$$

.

1.2 Ideal turbulence

The Reynolds averaging of the turbulent flows for barotropic compressible fluids is (for example refer to [8, 9, 10]) :

$$\left\{ \begin{array}{l} \frac{\partial \bar{\rho}}{\partial t} + (\bar{\rho} u_i)_{,i} = 0, \\ \frac{\partial \bar{\rho} u_i}{\partial t} + (\bar{\rho} u_i u_j + \bar{p} \delta_{ij})_{,j} = 0, \\ \frac{\partial \bar{\rho} u_i u_s}{\partial t} + (\bar{\rho} u_i u_k u_s)_{,k} + \bar{p}_{,i} u_s + \bar{p}_{,s} u_i = 0, \end{array} \right. \quad (3)$$

where ρ is the fluid density, $\mathbf{u} = [u_1, u_2, u_3]^T$ is the instantaneous velocity field, *comma* means the derivative with respect to the Eulerian coordinates x_i , $i = 1, 2, 3$, $p = \mathcal{P}(\rho)$ is a given equation of state, δ_{ij} denotes the Kronecker

symbol and repeated indexes mean the summation.

For any function f we have denoted by \overline{f} the Reynolds averaging (for example, time or space averaging). The last equation is obtained by multiplying the momentum equation by u_s and by averaging this new equation.

In the case of compressible fluids, we usually use *the Favre averaging velocity* (mass averaging velocity) [11] :

$$\mathbf{U} = [U_1, U_2, U_3]^T, \quad U_i = \frac{\overline{\rho u_i}}{\overline{\rho}}.$$

In this case, the mass equation yields :

$$\overline{\rho}_t + (\overline{\rho} U_i)_{,i} = 0$$

Introducing the following definitions of fluctuations of density and pressure :

$$\rho = \overline{\rho} + \rho', \quad \overline{\rho'} = 0, \quad p = \overline{p} + p', \quad \overline{p'} = 0,$$

and the mass averaged fluctuations of the velocity,

$$u_i = U_i + u_i'', \quad \overline{\rho u_i''} = 0,$$

The equations (3) can be written as :

$$\left\{ \begin{array}{l} \frac{\partial \overline{\rho}}{\partial t} + (\overline{\rho} U_i)_{,i} = 0, \\ \frac{\partial \overline{\rho} U_i}{\partial t} + (\overline{\rho} U_i U_j + \overline{p} \delta_{ij} + R_{ij})_{,j} = 0, \\ \frac{d R_{is}}{dt} + R_{is} U_{k,k} + R_{ks} U_{i,k} + R_{ki} U_{s,k} = S_{is}, \end{array} \right. \quad (4)$$

where

$$\mathbf{R} = \{R_{is}\}, \quad R_{is} = \overline{\rho u_i'' u_s''}, \quad (i, s = 1, 2, 3),$$

and

$$\mathbf{S} = \{S_{is}\}, \quad S_{is} = -(\overline{\rho u_i'' u_s'' u_k''})_{,k} - \overline{p_{,i} u_s''} - \overline{p_{,s} u_i''}.$$

The expression of S_{is} can also be written as

$$S_{is} = -(\overline{\rho u_i'' u_s'' u_k''})_{,k} - \overline{h'_{,i} \rho u_s''} - \overline{h'_{,s} \rho u_i''},$$

where $h(\rho)$ is the specific gas enthalpy :

$$\frac{dh}{d\rho} = \frac{1}{\rho} \frac{dp}{d\rho}$$

and h' means the enthalpy fluctuation.

System (4) can be rewritten in tensorial form as :

$$\left\{ \begin{array}{l} \frac{\partial \bar{\rho}}{\partial t} + \operatorname{div}(\bar{\rho} \mathbf{U}) = 0, \\ \frac{\partial(\bar{\rho} \mathbf{U})}{\partial t} + \left[\operatorname{div}(\bar{\rho} \mathbf{U} \otimes \mathbf{U} + \bar{p} \mathbf{I} + \mathbf{R}) \right]^T = 0, \\ \frac{d\mathbf{R}}{dt} + \mathbf{R} \operatorname{div} \mathbf{U} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{R} + \mathbf{R} \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T = \mathbf{S}. \end{array} \right.$$

Using the mass conservation law, the equation of volumic Reynolds stress tensor \mathbf{R} can be rewritten for the specific Reynolds stress tensor \mathbf{P} :

$$\frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{P} + \mathbf{P} \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T = \frac{\mathbf{S}}{\bar{\rho}} \quad \text{where} \quad \mathbf{P} = \frac{\mathbf{R}}{\bar{\rho}}.$$

We focus on the governing equations of mass conservation, momentum equation and specific Reynolds stress evolution without source term \mathbf{S} . Such an ideal system (called sometimes *ideal turbulence system* [12]) appears as a natural step in applying a splitting-up technique in the numerical treatment of the full system of compressible turbulence [6, 5].

$$\left\{ \begin{array}{l} \frac{\partial \bar{\rho}}{\partial t} + \operatorname{div}(\bar{\rho} \mathbf{U}) = 0, \\ \frac{\partial(\bar{\rho} \mathbf{U})}{\partial t} + \left[\operatorname{div}(\bar{\rho} \mathbf{U} \otimes \mathbf{U} + \bar{p} \mathbf{I} + \bar{\rho} \mathbf{P}) \right]^T = 0, \\ \frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{P} + \mathbf{P} \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T = 0. \end{array} \right. \quad (5)$$

A natural hypothesis is that the average pressure is a function of the averaged density (as for example, in case of isothermic ideal gas), *i.e.* $\bar{p} =$

$\mathcal{P}(\bar{\rho})$, and we have a closed system (5).

System (5) admits the energy conservation law :

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\bar{\rho} \left(\frac{1}{2} |\mathbf{U}|^2 + \mathcal{E} + \frac{\text{tr} \mathbf{P}}{2} \right) \right) \\ & + \text{div} \left(\bar{\rho} \mathbf{U} \left(\frac{1}{2} |\mathbf{U}|^2 + \mathcal{E} + \frac{\text{tr} \mathbf{P}}{2} \right) + \bar{\rho} \mathbf{U} + \bar{\rho} \mathbf{P} \mathbf{U} \right) = 0, \end{aligned} \quad (6)$$

where \mathcal{E} is the specific internal energy satisfying the Gibbs identity :

$$d\mathcal{E}(\bar{\rho}) = \frac{\mathcal{P}(\bar{\rho})}{\bar{\rho}^2} d\bar{\rho}.$$

As proven in [2], the last conservation law can be written in the form :

$$\frac{\partial}{\partial t} (\bar{\rho} \Psi) + \text{div} (\bar{\rho} \Psi \mathbf{U}) = 0, \quad \text{with} \quad \Psi = \frac{\det \mathbf{P}}{\bar{\rho}^2}.$$

System (5) coincides with the asymptotic $2D$ -model of weakly sheared shallow water flows (1) when $\bar{\rho}$ is replaced by the fluid depth h , $\mathcal{P}(\bar{\rho})$ by $gh^2/2$, and the specific energy $\mathcal{E}(\bar{\rho})$ by $gh/2$.

1.3 Properties of the two models

In particular case $\text{rot} \mathbf{U} = \mathbf{0}$ (i.e. $(\partial \mathbf{U} / \partial \mathbf{x})^T = \partial \mathbf{U} / \partial \mathbf{x}$), the equation for the specific Reynolds stress tensor can be integrated [13]. Equation (5)₃ corresponds to the evolution of a two-covariant tensor convected by the mean flow. This means that \mathbf{P} has a zero Lie derivative d_L with respect to the average velocity \mathbf{U} . The solution of the equation

$$d_L \mathbf{P} \equiv \frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{P} + \mathbf{P} \frac{\partial \mathbf{U}}{\partial \mathbf{x}} = 0,$$

is

$$\mathbf{P} = (\mathbf{F}^T)^{-1} \mathbf{P}_0(\mathbf{X}) \mathbf{F}^{-1},$$

where $\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X}$ is the deformation gradient of the mean motion and tensor \mathbf{P}_0 is the image of \mathbf{P} in Lagrange coordinates \mathbf{X} [13, 14].

However, hypothesis $\text{rot} \mathbf{U} = 0$ is not compatible with systems (1) and (5). Indeed, if initially $\text{rot} \mathbf{U} = 0$, this zero value is not conserved along the motion (the Kelvin theorem is not valid for system (1) and system (5)),

and the equation for \mathbf{P} cannot be integrated in Lagrange coordinates. The general case ($\text{rot } \mathbf{U} \neq 0$) was studied in [2].

Now we ask the following question :

Since systems (1) and (5) are conservative, are we able to derive the governing equations from the Hamilton principle of stationary action as in case of classical non-dissipative models?

Both, energy equations (2) and (6) suggest to formulate Hamilton's action in the form :

$$a = \int_{t_1}^{t_2} \mathcal{L} dt$$

where for (2), the Lagrangian is

$$\mathcal{L} = \int_{\Omega(t)} h \left(\frac{1}{2} |\mathbf{U}|^2 - \frac{gh}{2} - \frac{\text{tr } \mathbf{P}}{2} \right) d\Omega,$$

and for (6),

$$\mathcal{L} = \int_{\Omega(t)} \bar{\rho} \left(\frac{1}{2} |\mathbf{U}|^2 - \mathcal{E}(\bar{\rho}) - \frac{\text{tr } \mathbf{P}}{2} \right) d\Omega,$$

where t_1 and t_2 are two fixed times, $\Omega(t)$ is the material volume associated with average velocity \mathbf{U} and $d\Omega$ denotes the convected volume element in $\Omega(t)$. We have to set which equations can be considered as imposed constraints and which equations are derived from Hamilton's principle.

2 Variational formulation for shear shallow water flows and ideal turbulence

2.1 Virtual motion

We recall the notion of *virtual motion* and of *virtual displacement*. Let a one-parameter family of *virtual motions* :

$$\mathbf{x} = \Phi(t, \mathbf{X}, \lambda)$$

where \mathbf{x} denotes the Euler coordinates, \mathbf{X} denotes the Lagrange coordinates, t is the time, and $\lambda \in \mathcal{O}$ is a real number (\mathcal{O} is an open interval containing

0). When $\lambda = 0$,

$$\Phi(t, \mathbf{X}, 0) = \phi(t, \mathbf{X}),$$

where $\phi(t, \mathbf{X})$ denotes the real motion associated with the averaged velocity field \mathbf{U} .

As usually, we assume at the boundary of $[t_1, t_2] \times \Omega(t)$,

$$\Phi(t, \mathbf{X}, \lambda) = \phi(t, \mathbf{X}).$$

The virtual displacement of the particle is denoted $\tilde{\delta}\mathbf{x}$ and is defined as [14, 15, 16] :

$$\tilde{\delta}\mathbf{x}(t, \mathbf{X}) = \left. \frac{\partial \Phi(t, \mathbf{X}, \lambda)}{\partial \lambda} \right|_{\lambda=0}.$$

In the following, symbol $\tilde{\delta}$ means the derivative with respect to λ , at fixed Lagrange coordinates \mathbf{X} , when $\lambda = 0$. We denote by $\zeta(t, \mathbf{x})$ the virtual displacement expressed as a function of Euler coordinates :

$$\zeta(t, \mathbf{x}) = \zeta(t, \phi(t, \mathbf{X})) = \tilde{\delta}\mathbf{x}(t, \mathbf{X}).$$

As for ζ , for the sake of simplicity, we use for all quantities the same notation in both Euler and Lagrange coordinates.

2.2 Lagrangian

The equations for shear shallow water and ideal turbulence are identical when we identify the quantities $\bar{\rho}$ and h , $\mathcal{E}(\bar{\rho})$ and $\frac{gh}{2}$.

Consequently, let us consider the Lagrangian in the general form :

$$\mathcal{L} = \int_{\Omega(t)} \bar{\rho} \left(\frac{1}{2} |\mathbf{U}|^2 - \mathcal{E}(\bar{\rho}) - \frac{\text{tr} \mathbf{P}}{2} \right) d\Omega.$$

We consider two constraints :

- The first one corresponds to the mass conservation law,

$$\frac{\partial \bar{\rho}}{\partial t} + \text{div}(\bar{\rho} \mathbf{U}) = 0,$$

which can be integrated in the form :

$$\bar{\rho} \det F = \rho_0(\mathbf{X}).$$

It corresponds to a holonomic constraint.

- The second one corresponds to the specific Reynolds stress tensor evolution,

$$\frac{d\mathbf{P}}{dt} + \frac{\partial\mathbf{U}}{\partial\mathbf{x}} \mathbf{P} + \mathbf{P} \left(\frac{\partial\mathbf{U}}{\partial\mathbf{x}} \right)^T = 0,$$

which is not integrable along the motion [16]. So, it corresponds to a non-holonomic constraint.

Two types of variations for unknowns $\bar{\rho}$, \mathbf{U} and \mathbf{P} can be used [15, 16] :

- The previous one, at fixed Lagrangian coordinates (denoted by $\tilde{\delta}$),
- Another equivalent variation at fixed Eulerian coordinates (denoted by $\hat{\delta}$).

These variations are related : for any variable f , the connection between the two variations is :

$$\hat{\delta}f = \tilde{\delta}f - \nabla f \cdot \boldsymbol{\zeta}. \quad (7)$$

We consider that the gradient operator, as all space operators, is taken in Euler coordinates. The mass constraint allows us to obtain the variation of $\bar{\rho}$ at fixed Lagrange and Euler coordinates in the form [15, 16] :

$$\tilde{\delta}\bar{\rho} = -\bar{\rho} \operatorname{div}(\boldsymbol{\zeta}) \quad \text{and} \quad \hat{\delta}\bar{\rho} = -\operatorname{div}(\bar{\rho} \boldsymbol{\zeta}). \quad (8)$$

The variations of velocity \mathbf{U} at fixed Lagrange (or Euler) coordinates are given respectively as [15, 16] :

$$\tilde{\delta}\mathbf{U} = \frac{\partial\tilde{\delta}\mathbf{x}}{\partial t} = \frac{d\boldsymbol{\zeta}}{dt} \quad \text{and} \quad \hat{\delta}\mathbf{U} = \frac{d\boldsymbol{\zeta}}{dt} - \frac{\partial\mathbf{U}}{\partial\mathbf{x}} \boldsymbol{\zeta}. \quad (9)$$

However, equation (5)₃ for \mathbf{P} is not integrable in Lagrange coordinates. Let us recall that m non-holonomic constraints in analytical mechanics for a system with n degrees of freedom $\mathbf{q} = (q_1, q_2, \dots, q_n)^T$, where $m < n$, are in the form :

$$\mathbf{A}(\mathbf{q}, t) \frac{d\mathbf{q}}{dt} + \mathbf{b}(t) = 0.$$

Matrix \mathbf{A} is a matrix with n columns and m lines and \mathbf{b} is a time dependent vector in \mathbb{R}^m . Even if the system of constraints cannot be reduced to

pure holonomic constraints, the variations of \mathbf{q} corresponding to these non-holonomic constraints are expressed as [17] :

$$\mathbf{A}(\mathbf{q}, t) \delta \mathbf{q} = 0.$$

Similarly, equation for \mathbf{P} can be seen as a non-holonomic constraint, and consequently the Lagrangian variation of \mathbf{P} can be written in the form :

$$\tilde{\delta} \mathbf{P} = -\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{x}} \mathbf{P} - \mathbf{P} \left(\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{x}} \right)^T.$$

It implies :

$$\tilde{\delta}(\bar{\rho} \mathbf{P}) = -\bar{\rho} \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{x}} \mathbf{P} - \bar{\rho} \mathbf{P} \left(\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{x}} \right)^T - \bar{\rho} \mathbf{P} \operatorname{div} \boldsymbol{\zeta}.$$

Since the operator tr and variation $\tilde{\delta}$ commute, we obtain :

$$\tilde{\delta} [\operatorname{tr}(\bar{\rho} \mathbf{P})] = -2 \operatorname{tr} \left(\bar{\rho} \mathbf{P} \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{x}} \right) - \operatorname{tr}(\bar{\rho} \mathbf{P}) \operatorname{div} \boldsymbol{\zeta}.$$

Its Eulerian variation $\hat{\delta}$ (considered at fixed Euler coordinates) is obtained according to relation (7) :

$$\begin{aligned} \hat{\delta} [\operatorname{tr}(\bar{\rho} \mathbf{P})] &= -2 \operatorname{tr} \left(\bar{\rho} \mathbf{P} \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{x}} \right) - \operatorname{tr}(\bar{\rho} \mathbf{P}) \operatorname{div} \boldsymbol{\zeta} - \{ \boldsymbol{\nabla} [\operatorname{tr}(\bar{\rho} \mathbf{P})] \}^T \boldsymbol{\zeta} \\ &= -2 \operatorname{tr} \left(\bar{\rho} \mathbf{P} \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{x}} \right) - \operatorname{div} [\operatorname{tr}(\bar{\rho} \mathbf{P}) \boldsymbol{\zeta}]. \end{aligned} \quad (10)$$

The variation of Hamilton's action in Euler coordinates is :

$$\hat{\delta} a = \int_{t_1}^{t_2} \int_{\Omega(t)} \left(\frac{\hat{\delta} \bar{\rho}}{2} |\mathbf{U}|^2 + \bar{\rho} \mathbf{U}^T \hat{\delta} \mathbf{U} - \frac{\partial(\bar{\rho} \mathcal{E})}{\partial \bar{\rho}} \hat{\delta} \bar{\rho} - \frac{\hat{\delta} \operatorname{tr}(\bar{\rho} \mathbf{P})}{2} \right) d\Omega dt.$$

By using formula (8), (9) and (10) for Euler variations, we obtain :

$$\begin{aligned} \hat{\delta} a &= \int_{t_1}^{t_2} \int_{\Omega(t)} \left[-\frac{\operatorname{div}(\bar{\rho} \boldsymbol{\zeta})}{2} |\mathbf{U}|^2 + \bar{\rho} \mathbf{U}^T \left(\frac{d\boldsymbol{\zeta}}{dt} - \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \boldsymbol{\zeta} \right) \right. \\ &\quad \left. + \frac{\partial(\bar{\rho} \mathcal{E})}{\partial \bar{\rho}} \operatorname{div}(\bar{\rho} \boldsymbol{\zeta}) + \operatorname{tr} \left(\bar{\rho} \mathbf{P} \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{x}} \right) + \frac{1}{2} \operatorname{div} [\operatorname{tr}(\bar{\rho} \mathbf{P}) \boldsymbol{\zeta}] \right] d\Omega dt. \end{aligned}$$

The Gauss-Ostrogradsky formula and the fact that the variations vanish at the boundary of the domain $[t_1, t_2] \times \Omega_t$ imply :

$$\hat{\delta}a = - \int_{t_1}^{t_2} \int_{\Omega(t)} \left[\frac{\partial(\bar{\rho} \mathbf{U})^T}{\partial t} + \operatorname{div}(\bar{\rho} \mathbf{U} \otimes \mathbf{U} + \mathcal{P}(\bar{\rho}) \mathbf{I} + \bar{\rho} \mathbf{P}) \right] \boldsymbol{\zeta} \, d\Omega \, dt.$$

For all vector field $\boldsymbol{\zeta}$, the variation of Hamilton's action in Euler coordinates vanishes, the fundamental lemma of variation calculus yields momentum equation (5)₂.

The case of non-isentropic compressible turbulent flows can be treated in the same way.

3 Conclusion

We have established that the momentum equation of the non-dissipative model for shear shallow water flows and for ideal turbulence can be obtained by the Hamilton principle of stationary action. As usually, the mass conservation law corresponds to a holonomic (or integrable) constraint, but the evolution equation for the Reynolds stress tensor which is not integrable, corresponds to a non-holonomic constraint.

Systems (1) and (5) belong to the class of physical models subject to Hamilton's principle of stationary action, as is generally the case for conservative systems with holonomic constraints.

Systems (1) and (5) are hyperbolic and shock waves can be formed. It can be proved, that systems (1) and (5) cannot be written in conservative form [19]. So, a definition of weak solutions is questionable. The fact that the equations admit the variational formulation could allow us to formulate the corresponding Rankine-Hugoniot relations (shock relations) for shear shallow water flows as it was done, for example, for two-velocity flows in [18]. They could confirm empirical Rankine-Hugoniot relations proposed for systems (1) and (5) in [19, 20].

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