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► **To cite this version:**

Sergey L Gavriluk, Henri Gouin. Variational formulation of the ideal Reynolds averaged equations. 2019. hal-02148064v1

HAL Id: hal-02148064

<https://hal.science/hal-02148064v1>

Preprint submitted on 5 Jun 2019 (v1), last revised 20 Nov 2019 (v2)

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Variational formulation of the ideal Reynolds averaged equations

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Abstract

The system of equations for turbulent barotropic flows is composed of the mass conservation law, the equation of average momentum and the equation for the Reynolds stress tensor evolution. The momentum evolution equation consists of a non-dissipative part and source terms. In the limit of conservative motions, we neglect the source terms. This non-dissipative model of turbulence admits a variational formulation where the Reynolds stress tensor evolution is considered as a non-holonomic constraint.

Résumé

Le système d'équations des écoulements turbulents barotropes est constitué de la loi de conservation de la masse, de l'équation moyennée de la quantité de mouvement et de l'équation d'évolution du tenseur de Reynolds. L'équation de la quantité de mouvement comprend des termes non dissipatifs et des termes sources. Dans la limite des mouvements conservatifs, nous négligeons les termes sources. Ce modèle non dissipatif de turbulence permet une formulation variationnelle pour laquelle l'équation d'évolution du tenseur de Reynolds peut être considérée comme une liaison non-holonome.

Keywords: Reynolds averaged equation, Hamilton's principle, non-holonomic constraint

1 Introduction

The reason for considering the simplified model of turbulence without source terms is twofold. First, in numerical studies the treatment of the homogeneous part of Reynolds-averaged equations is a natural step in applying the splitting-up technique [1, 2]. Second, there is a mathematical analogy between the homogeneous Reynolds-averaged equations and an exact asymptotic model of shear flows of long waves over flat bottom [3, 4, 5, 6].

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When source terms of the averaged momentum equation are not taken into account, the model can be considered as reversible: it admits the momentum and energy balance laws. Nonetheless, the equation of Reynolds stress tensor evolution cannot be considered in balance form and cannot be integrated in Lagrangian coordinates. Such an equation can be compared with a non-holonomic constraint in analytical mechanics. The comparison allows us to formulate the Hamilton principle for “ideal turbulence” with the non-holonomic constraint governing the Reynolds stress tensor evolution.

2 Ideal Reynolds averaged equations

The governing equations of barotropic turbulent compressible fluids are in the form (see, for example [7, 8, 9]) :

$$\begin{cases} \langle \rho \rangle_t + (\langle \rho \rangle U_i)_{,i} = 0, \\ (\langle \rho \rangle U_i)_t + (\langle \rho \rangle U_i U_j + \langle p \rangle \delta_{ij} + \langle \rho u_i u_j \rangle)_{,j} = 0, \\ \langle \rho u_i u_j \rangle_t + \langle \rho u_i u_j \rangle U_{k,k} + \langle \rho u_k u_j \rangle U_{i,k} + \langle \rho u_i u_k \rangle U_{j,k} = S_{ij}, \end{cases} \quad (1)$$

where the “brackets” mean the averaging, “coma” in subscript means the derivation with respect to the Eulerian coordinates $\mathbf{x} = \{x_i\}$, $i \in \{1, 2, 3\}$, subscript t means the partial derivative with respect to the time, ρ is the fluid density, $\mathbf{U} = \{U_i\}$, $i \in \{1, 2, 3\}$ is the mass average velocity, $\langle p \rangle$ is the averaged pressure which depends only on $\langle \rho \rangle$, $\mathbf{u} = \{u_i\}$, $i \in \{1, 2, 3\}$ is the velocity fluctuation verifying $\langle \rho \mathbf{u} \rangle = 0$, δ_{ij} denotes the Kronecker symbol and repeated indices mean summation. The term $\mathbf{S} = \{S_{ij}\}$, $i, j \in \{1, 2, 3\}$ represents turbulent sources. The Reynolds stress tensor is defined as $\mathbf{R} = \langle \rho \mathbf{u} \otimes \mathbf{u} \rangle$, or $\{R_{ij}\} = \langle \rho u_i u_j \rangle$, $i, j \in \{1, 2, 3\}$.

The system (1) can be rewritten in tensorial form :

$$\begin{cases} \frac{\partial \langle \rho \rangle}{\partial t} + \text{div}(\langle \rho \rangle \mathbf{U}) = 0, \\ \frac{\partial (\langle \rho \rangle \mathbf{U})}{\partial t} + \left[\text{div}(\langle \rho \rangle \mathbf{U} \otimes \mathbf{U} + \langle p \rangle \mathbf{I} + \mathbf{R}) \right]^T = 0, \\ \frac{d\mathbf{R}}{dt} + \mathbf{R} \text{div} \mathbf{U} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{R} + \mathbf{R} \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T = \mathbf{S}, \end{cases} \quad (2)$$

where d/dt is the material derivative with respect to the mean motion :

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{U}^T \nabla,$$

superscript “ T ” means the transposition, and \mathbf{I} denotes the identity tensor. Using the mass conservation law, the equation for the volumic Reynolds stress tensor \mathbf{R} can be rewritten for the specific (or per unit mass) Reynolds stress tensor \mathbf{P} :

$$\frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{P} + \mathbf{P} \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T = \frac{\mathbf{S}}{\langle \rho \rangle} \quad \text{where} \quad \mathbf{P} = \frac{\mathbf{R}}{\langle \rho \rangle}.$$

We concentrate on the governing equations of mass conservation, momentum equation and specific Reynolds stress evolution without the source terms :

$$\left\{ \begin{array}{l} \frac{\partial \langle \rho \rangle}{\partial t} + \operatorname{div} (\langle \rho \rangle \mathbf{U}) = 0, \\ \frac{\partial (\langle \rho \rangle \mathbf{U})}{\partial t} + \left[\operatorname{div} \left(\langle \rho \rangle \mathbf{U} \otimes \mathbf{U} + \langle p \rangle \mathbf{I} + \langle \rho \rangle \mathbf{P} \right) \right]^T = 0, \\ \frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{P} + \mathbf{P} \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T = 0. \end{array} \right. \quad (3)$$

Equations (3) admit the energy conservation law :

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\langle \rho \rangle \left(\frac{1}{2} |\mathbf{U}|^2 + \varepsilon(\langle \rho \rangle) + e_T \right) \right) \\ & + \operatorname{div} \left(\langle \rho \rangle \mathbf{U} \left(\frac{1}{2} |\mathbf{U}|^2 + \varepsilon(\langle \rho \rangle) + e_T \right) + \langle p \rangle \mathbf{U} + \langle \rho \rangle \mathbf{P} \mathbf{U} \right) = 0, \end{aligned} \quad (4)$$

where

$$e_T = \frac{\operatorname{tr} \mathbf{P}}{2},$$

and $\varepsilon(\langle \rho \rangle)$ is the specific internal energy satisfying the Gibbs identity :

$$d\varepsilon = \frac{\langle p \rangle}{\langle \rho \rangle^2} d\langle \rho \rangle.$$

The Gibbs identity yields the definition of pressure $\langle p \rangle$ as a function of $\langle \rho \rangle$. As proven in [4], the last conservation law can be written in the form :

$$\frac{\partial}{\partial t} (\langle \rho \rangle \Psi) + \operatorname{div} (\langle \rho \rangle \Psi \mathbf{U}) = 0, \quad \text{with} \quad \Psi = \frac{\det \mathbf{P}}{\langle \rho \rangle^2}.$$

System (3), coming from system (2), can be seen as the Reynolds averaged system for barotropic turbulent flows where the third order correlations of the velocity fluctuations and the correlations of the velocity fluctuations with the pressure gradient are omitted [1, 2, 9]. In addition, system (3) is an asymptotic $2D$ model of weakly sheared shallow water flows where $\langle \rho \rangle$ should be replaced by the fluid depth h , and the specific energy can be taken as $\varepsilon(h) = gh/2$, [3, 4, 5, 6].

Even if equations (3) of “ideal turbulence” [10] admit the mass, momentum and energy conservation laws, they cannot be written in conservative form: the number of unknowns is larger than the number of conservation laws (for a proof in the case of shear shallow water flows, see [11]). This means that the mathematical methods developed for quasi-linear systems of conservation laws cannot be applied here : non-classical Rankine-Hugoniot relations are needed to describe the shocks in system (3) (see [11] where such additional relations are formulated for the case of shear shallow water flows).

The specific Reynolds stress tensor can be integrated in the particular case $\operatorname{rot} \mathbf{U} = \mathbf{0}$

[12]. In such a case $(\partial\mathbf{U}/\partial\mathbf{x})^T = \partial\mathbf{U}/\partial\mathbf{x}$ and Eq. (3)₃ corresponds to a two-covariant tensor convected by the mean flow. This means that \mathbf{P} has a zero Lie derivative d_L with respect to the velocity field \mathbf{U} and the tensor \mathbf{P}_0 , image of \mathbf{P} in Lagrange coordinates (t, \mathbf{X}) , only depends on $\mathbf{X} = \{X_i\}$, $i \in \{1, 2, 3\}$

$$d_L\mathbf{P} \equiv \frac{d\mathbf{P}}{dt} + \frac{\partial\mathbf{U}}{\partial\mathbf{x}}\mathbf{P} + \mathbf{P}\frac{\partial\mathbf{U}}{\partial\mathbf{x}} = 0, \quad \text{which implies } \mathbf{P} = (F^T)^{-1}\mathbf{P}_0(\mathbf{X})F^{-1},$$

where $F = \partial\mathbf{x}/\partial\mathbf{X}$ is the deformation gradient of the mean motion [13].

Nonetheless, even if condition $\text{rot}\mathbf{U} = \mathbf{0}$ is satisfied at a particular time, the property is not conserved along the mean motion (one can verify that the Kelvin conservation theorem is not satisfied). So, the equation for \mathbf{P} cannot be integrated in the Lagrangian coordinates.

In this paper, we ask for the following question: since the system is conservative, are we able to derive the governing equations from the Hamilton principle of stationary action as in cases of classical non-dissipative models? Indeed, the energy equation (4) allows us to formulate the Hamilton action in the form :

$$a = \int_{t_0}^{t_1} \mathcal{L} dt \quad \text{where} \quad \mathcal{L} = \int_{\Omega(t)} \langle \rho \rangle \left(\frac{1}{2}|\mathbf{U}|^2 - \varepsilon(\langle \rho \rangle) - \frac{\text{tr}\mathbf{P}}{2} \right) d\omega.$$

Here t_0 and t_1 are two fixed times, $\Omega(t)$ is the material volume associated with the average velocity \mathbf{U} and $d\omega$ denotes the volume element in $\Omega(t)$. We have to set which equations can be considered as imposed constraints and which equations are derived from Hamilton's principle.

3 Variational formulation of the governing equations

3.1 Virtual motion

We introduce the notion of *virtual motion* and *virtual displacement*. Let a one-parameter family of *virtual motions* :

$$\mathbf{x} = \Phi(t, \mathbf{X}, \lambda)$$

where \mathbf{x} denotes the Eulerian coordinates, \mathbf{X} denotes the Lagrangian coordinates, t is the time, and $\lambda \in \mathcal{O}$ is a real number (\mathcal{O} is an open interval containing 0). When $\lambda = 0$,

$$\Phi(t, \mathbf{X}, 0) = \phi(t, \mathbf{X}),$$

where $\phi(t, \mathbf{X})$ denotes the real motion associated with the averaged velocity field \mathbf{U} . As usually, we assume that,

$$\Phi(t_0, \mathbf{X}, \lambda) = \phi(t_0, \mathbf{X}), \quad \Phi(t_1, \mathbf{X}, \lambda) = \phi(t_1, \mathbf{X}),$$

and at the boundary of $[t_0, t_1] \times \Omega(t)$, one has $\Phi(t, \mathbf{X}, \lambda) = \phi(t, \mathbf{X})$. The virtual displacement of the particle denoted $\tilde{\delta}\mathbf{x}$ is defined as [13, 14, 15] :

$$\tilde{\delta}\mathbf{x}(t, \mathbf{X}) = \left. \frac{\partial\Phi(t, \mathbf{X}, \lambda)}{\partial\lambda} \right|_{\lambda=0}.$$

In the following, symbol $\tilde{\delta}$ means the derivative with respect to λ , when $\lambda = 0$, at fixed Lagrangian coordinates \mathbf{X} . We denote by $\zeta(t, \mathbf{x})$ the virtual displacement expressed as a function of the Eulerian coordinates :

$$\zeta(t, \mathbf{x}) = \zeta(t, \phi(t, \mathbf{X})) = \tilde{\delta}\mathbf{x}(t, \mathbf{X}).$$

For the sake of simplicity, as for ζ , we use the same notation in both Eulerian and Lagrangian coordinates for all the quantities as $\langle \rho \rangle$, \mathbf{U} etc.

3.2 Lagrangian

Let us consider the Lagrangian in the form :

$$\mathcal{L} = \int_{\Omega(t)} \langle \rho \rangle \left(\frac{1}{2} |\mathbf{U}|^2 - \varepsilon(\langle \rho \rangle) - \frac{\text{tr} \mathbf{P}}{2} \right) d\omega.$$

We consider two constraints :

- The first one corresponds to the mass conservation law ,

$$\frac{\partial \langle \rho \rangle}{\partial t} + \text{div}(\langle \rho \rangle \mathbf{U}) = 0,$$

which can be integrated in the form :

$$\langle \rho \rangle \det F = \rho_0(\mathbf{X})$$

and is a holonomic constraint for the motion.

- The second one is :

$$\frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{P} + \mathbf{P} \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T = 0,$$

which is not integrable along the motion [15].

Two types of variations for unknowns ρ , \mathbf{U} and \mathbf{P} can be used [14, 15] :

- The previous one, at fixed Lagrangian coordinates (denoted by $\tilde{\delta}$),
- Another equivalent variation at fixed Eulerian coordinates (denoted by $\hat{\delta}$).

These variations are related : for any variable f , the connection between the two variations writes,

$$\hat{\delta}f = \tilde{\delta}f - \nabla f \cdot \zeta. \quad (5)$$

We consider that the gradient operator, as all space operators, is taken in Eulerian coordinates. The mass constraint allows us to obtain the variation of ρ at fixed Lagrangian and Eulerian coordinates in the form [14, 15] :

$$\tilde{\delta} \langle \rho \rangle = - \langle \rho \rangle \text{div}(\zeta) \quad \text{and} \quad \hat{\delta} \langle \rho \rangle = -\text{div}(\langle \rho \rangle \zeta). \quad (6)$$

The variations of the velocity \mathbf{U} at fixed Lagrangian (or Eulerian) coordinates are given respectively as [14, 15] :

$$\tilde{\delta}\mathbf{U} = \frac{\partial\tilde{\delta}\mathbf{x}}{\partial t} = \frac{d\boldsymbol{\zeta}}{dt} \quad \text{and} \quad \hat{\delta}\mathbf{U} = \frac{d\boldsymbol{\zeta}}{dt} - \frac{\partial\mathbf{U}}{\partial\mathbf{x}}\boldsymbol{\zeta}. \quad (7)$$

However, equation (3)₃ for \mathbf{P} is not integrable in Lagrangian coordinates. It can be considered as a non-holonomic constraint. Let us recall m non-holonomic constraints in analytical mechanics for a system with n degrees of freedom $\mathbf{q} = (q_1, q_2, \dots, q_n)^T$, $m < n$, are in the form :

$$\mathbf{A}(\mathbf{q}, t) \frac{d\mathbf{q}}{dt} + \mathbf{b}(t) = 0.$$

Matrix \mathbf{A} is a matrix with n columns and m lines and \mathbf{b} is a time dependent vector in \mathbb{R}^n . Even if the system of constraints cannot be reduced to pure holonomic constraints, the variations of \mathbf{q} corresponding to these non-holonomic constraints are expressed as [16] :

$$\mathbf{A}(\mathbf{q}, t) \delta\mathbf{q} = 0.$$

Similarly, the equation for \mathbf{P} can be seen as a non-holonomic constraint, and consequently the Lagrangian variation of \mathbf{P} can be written in the form :

$$\tilde{\delta}\mathbf{P} = -\frac{\partial\boldsymbol{\zeta}}{\partial\mathbf{x}}\mathbf{P} - \mathbf{P} \left(\frac{\partial\boldsymbol{\zeta}}{\partial\mathbf{x}} \right)^T.$$

It implies :

$$\tilde{\delta}(\langle\rho\rangle\mathbf{P}) = -\langle\rho\rangle\frac{\partial\boldsymbol{\zeta}}{\partial\mathbf{x}}\mathbf{P} - \langle\rho\rangle\mathbf{P} \left(\frac{\partial\boldsymbol{\zeta}}{\partial\mathbf{x}} \right)^T - \langle\rho\rangle\mathbf{P} \operatorname{div}\boldsymbol{\zeta}.$$

Since the operator tr and variation $\tilde{\delta}$ commute, we obtain:

$$\tilde{\delta}[\operatorname{tr}(\langle\rho\rangle\mathbf{P})] = -2\operatorname{tr}\left(\langle\rho\rangle\mathbf{P}\frac{\partial\boldsymbol{\zeta}}{\partial\mathbf{x}}\right) - \operatorname{tr}(\langle\rho\rangle\mathbf{P})\operatorname{div}\boldsymbol{\zeta}.$$

Its Eulerian variation $\hat{\delta}$ (considered at fixed Eulerian coordinates) is obtained according to relation (5) :

$$\begin{aligned} \hat{\delta}[\operatorname{tr}(\langle\rho\rangle\mathbf{P})] &= -2\operatorname{tr}\left(\langle\rho\rangle\mathbf{P}\frac{\partial\boldsymbol{\zeta}}{\partial\mathbf{x}}\right) - \operatorname{tr}(\langle\rho\rangle\mathbf{P})\operatorname{div}\boldsymbol{\zeta} - \{\nabla[\operatorname{tr}(\langle\rho\rangle\mathbf{P})]\}^T\boldsymbol{\zeta} \\ &= -2\operatorname{tr}\left(\langle\rho\rangle\mathbf{P}\frac{\partial\boldsymbol{\zeta}}{\partial\mathbf{x}}\right) - \operatorname{div}[\operatorname{tr}(\langle\rho\rangle\mathbf{P})\boldsymbol{\zeta}]. \end{aligned} \quad (8)$$

The Hamilton action is :

$$a = \int_{t_1}^{t_2} \mathcal{L} dt.$$

The variation of Hamilton's action in Eulerian coordinates is :

$$\hat{\delta}a = \int_{t_1}^{t_2} \int_{\Omega(t)} \left(\frac{\hat{\delta}\langle\rho\rangle}{2} |\mathbf{U}|^2 + \langle\rho\rangle \mathbf{U}^T \hat{\delta}\mathbf{U} - \frac{\partial(\langle\rho\rangle\varepsilon)}{\partial\langle\rho\rangle} \hat{\delta}\langle\rho\rangle - \frac{\hat{\delta}\operatorname{tr}(\langle\rho\rangle\mathbf{P})}{2} \right) d\omega dt.$$

Using the formula for Eulerian variations (6), (7) and (8), we obtain :

$$\begin{aligned} \hat{\delta}a &= \int_{t_1}^{t_2} \int_{\Omega(t)} \left[-\frac{\operatorname{div}(\langle \rho \rangle \boldsymbol{\zeta})}{2} |\mathbf{U}|^2 + \langle \rho \rangle \mathbf{U}^T \left(\frac{d\boldsymbol{\zeta}}{dt} - \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \boldsymbol{\zeta} \right) + \frac{\partial(\langle \rho \rangle \varepsilon)}{\partial \langle \rho \rangle} \operatorname{div}(\langle \rho \rangle \boldsymbol{\zeta}) \right. \\ &\quad \left. + \operatorname{tr} \left(\langle \rho \rangle \mathbf{P} \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{x}} \right) + \frac{1}{2} \operatorname{div} [\operatorname{tr}(\langle \rho \rangle \mathbf{P}) \boldsymbol{\zeta}] \right] d\omega dt. \end{aligned}$$

The Gauss-Ostrogradsky formula and the fact that the variations vanish at the boundary of the domain $[t_1, t_2] \times \Omega_t$ imply :

$$\hat{\delta}a = - \int_{t_1}^{t_2} \int_{\Omega(t)} \left[\frac{\partial(\langle \rho \rangle \mathbf{U})^T}{\partial t} + \operatorname{div} \left(\langle \rho \rangle \mathbf{U} \otimes \mathbf{U} + \langle p \rangle \mathbf{I} + \langle \rho \rangle \mathbf{P} \right) \right] \boldsymbol{\zeta} d\omega dt.$$

Using the fact that, for all vector field $\boldsymbol{\zeta}$, the variation of Hamilton's action in Eulerian coordinates vanishes, the fundamental lemma of variation calculus yields momentum equation (3)₂.

The case of compressible non-isentropic turbulent flows can be treated in the same way.

4 Conclusion

We have seen that the momentum equation of the non-dissipative model of turbulence can be obtained through the Hamilton principle of stationary action. As usually, the mass conservation law corresponds to a holonomic (or integrable) constraint, but the evolution equation of the Reynolds stress tensor which is not integrable, can be considered as a non-holonomic constraint. This equation implies an expression for the variation of the Reynolds stress tensor which makes possible to obtain the momentum equation of the "ideal" turbulence.

References

- [1] C. Berthon, F. Coquel, J.-M. Hérard, M. Uhlmann, An approximate solution of the Riemann problem for a realisable second-moment turbulent closure, *Shock Waves* **11** (2002) 245–269.
- [2] B. Audebert, F. Coquel, Structural stability of shock solutions of hyperbolic systems in nonconservation form via kinetic relations. In S. Benzoni-Gavage, D. Serre (Eds.) *Hyperbolic Problems: Theory, Numerics, Applications*, Springer, Berlin, 2008.
- [3] V. M. Teshukov, Gas dynamic analogy for vortex free-boundary flows, *J. Appl. Mech. Tech. Phys.* **48** (2007) 303–309.
- [4] S. L. Gavriluk, H. Gouin, Geometric evolution of the Reynolds stress tensor, *Int. J. Eng. Sci.* **59** (2012) 65–73.

- [5] G. L. Richard, S. L. Gavriluk, A new model of roll waves: comparison with Brocks experiments, *J. Fluid Mech.* **698** (2012) 374–405.
- [6] G. L. Richard, S. L. Gavriluk, The classical hydraulic jump in a model of shear shallow–water flows, *J. Fluid Mech.* **725** (2013) 492–521.
- [7] S. B. Pope, *Turbulent Flows*, Cambridge Univ. Press, Cambridge, 2005.
- [8] D. Wilcox, *Turbulence Modeling for CFD*, DCW Industries, 1998.
- [9] B. Mohammadi, O. Pironneau, *Analysis of the K-epsilon Turbulence Model*, John Wiley & Sons, New York, 1994.
- [10] O. V. Troshkin, On wave properties of an incompressible turbulent fluid, *Physica A* **168** (1990) 881–899.
- [11] S. L. Gavriluk, K. Ivanova, N. Favrie, Multidimensional shear shallow water flows: problems and solutions, *J. Comput. Phys.* **366** (2018) 252–280.
- [12] J.-F. Debiève, H. Gouin, J. Gaviglio, Evolution of the Reynolds stress tensor in a shock wave-turbulence interaction, *Indian Journal of Technology* **20** (1982) 90-97.
- [13] J. Serrin, *Mathematical principles of classical fluid mechanics*, in Encyclopedia of Physics VIII/1, S. Flügge (ed.), p.p. 125-263, Springer, Berlin, 1960.
- [14] V. L. Berdichevsky, *Variational Principles of Continuum Mechanics: I. Fundamentals*. Springer, New York, 2009.
- [15] S. Gavriluk, Multiphase flow modelling via Hamilton’s principle, In F. dell’Isola, S. L. Gavriluk (Eds.), *Variational Models and Methods in Solid and Fluid Mechanics*, Springer, Wien, 2011.
- [16] V. F. Zhuravlev, *Basic principles of theoretical mechanics*, Fizmatlit, Moscow, 2001 (in Russian).