The epistemic gossip problem

Martin C. Cooper *, Andreas Herzig, Faustine Maffre, Frédéric Maris, Pierre Régnier
IRIT, CNRS, University of Toulouse, 31062 Toulouse Cedex 9, France

A B S T R A C T

In the gossip problem information ('secrets') must be shared among a certain number of agents using the minimum number of calls. We extend the gossip problem to arbitrary epistemic depths. For example, we may require not only that all agents know all secrets but also that all agents know that all agents know all secrets. We give optimal protocols for various versions of this epistemic gossip problem, depending on the graph of com-munication links, in the case of two-way communication, one-way communication and parallel communication. We show, among other things, that increasing epistemic depth from 1 (all agents know all secrets) to 2 (so that all agents know that all agents know all secrets) does not double the required number of calls but increases this number by 3/2 (for a complete graph). We also show that the following counter-intuitive result generalises to the epistemic gossip problem: asymptotically the same number of calls are required whether calls are two-way or one-way.

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1. Introduction

We consider communication problems concerning n agents. We consider that initially, for $i = 1, \ldots, n$, agent i has some information s_i, also known as this agent's secret since, initially, the other agents do not know this information. This corresponds to information that agent i wishes to share with all other agents, such as agent i's signature on a contract or the dates when agent i is available for a meeting. More mundanely, it could simply be some gossip that agent i wants to share. Indeed, the basic (non-epistemic) version of the problem in which all agents want to communicate their secrets to all other agents (using the minimum number of communications) is traditionally known as the gossip problem. A probabilistic version of that problem is relevant in distributed databases [9]. Several variants have been studied in the literature, and a survey of these alternatives and the associated results has been published [16].

The gossip problem and its variants are of great interest in the conception of communication networks [16] and in parallel and distributed computing, but there are other less obvious applications like the management of data on storage devices [18], or the computation of the syntenic distance between two genomes (minimum number of fusions, fissions, and translocations required to transform one into the other) [27].

The original gossip problem is due to A. Boyd in the early 1970s: "There are n ladies, and each of them knows some item of gossip not known to the others. They communicate by telephone, and whenever one lady calls another, they tell each other all that they know at that time. How many calls are required before each gossip knows everything?". The main characteristics of the original gossip problem are: the aim is to find the minimum number of calls so that each person knows all the secrets via two-way communication (full duplex) in a complete graph of possible communication links. According to

* Corresponding author.
E-mail address: cooper@irit.fr (M.C. Cooper).

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these assumptions, $2n-4$ two-way calls are necessary and sufficient to achieve the goal [1,14,29] and any network in which $2n-4$ calls are sufficient contains a 4-cycle [3,19].

The gossip problem can be studied from a centralised or distributed point of view. In the centralised approach, the protocols tell the agents whom they have to call, and when. In the distributed approach, individual agents, on the basis of their own information, decide which other agent to call.

In the centralised approach that we are interested in, numerous variants of the original gossip problem have been studied: restricting the calling process to one-way (half-duplex) communication (such as by e-mail or letter), allowing k-party or conference calls, limiting the amount of redundant information which is sent, restricting the freedom in communication using an incomplete graph (a given person can only call a subset of the other people) or special topologies (such as tree, line or circle) [11,12,24,25].

Another variant is the partial gossip problem which is to determine, in a complete graph, the minimum number of calls needed for each person to know at least k secrets [4] The minimum-time gossip problem considers the amount of time, instead of the number of calls, and the aim is to find an algorithm that minimises the gossip time on a network or to find particular networks which can spread all gossip in minimum time [23,12]. In a similar vein, the minimum-cost gossip problem considers the minimum cost instead of the minimum number of calls (where a cost is associated with each call) [22]. The parallel gossip problem consists in extending the calling process to parallel communication considering that in each time step simultaneous calls can be executed in parallel (although each agent can only make one call in any given time step) [20,10]. The dynamic gossip problem allows dynamic communication networks by permitting agents to exchange not only secrets but also the telephone numbers of other agents they know. For a distributed approach to this version, see [8]. Another variant is perpetual gossipping, in which new secrets may arise at any time, and the objective is to find an efficient call scheme to maintain up-to-date information throughout the network [28].

Our contribution is to study the gossip problem at different epistemic depths. In the classic gossip problem, the goal is for all agents to know all secrets (which corresponds to epistemic depth 1). The equivalent goal at epistemic depth 2 is that all agents know that all agents know all the secrets; at depth 3, all agents must know that all agents know that all agents know all the secrets. For example, in a commercial setting, if the secrets are the agents' agreement to the terms of a joint contract, then an agent may not authorise expenditure on the project before knowing that all other agents know that all agents agree to the terms of the contract. We provide algorithms for these variants and establish their optimality in most of the cases.

Even in the case of cooperating agents, knowledge of the communication protocol does not imply blind trust that the other agents respect the protocol nor a 100% confidence in the reliability of communication links. Thus, as with registered letters, extra communication is required for agents to be sure what other agents know.

The paper is organised as follows. In Section 2 we formally introduce the epistemic version of the classic gossip problem $\mathrm{Gossip}_G(d)$, where G is the graph of direct communication links and d is the epistemic depth. In Section 3 we study $\mathrm{Gossip}_G(d)$ for different kinds of graphs G. We show, among other things, that increasing epistemic depth from 1 (all agents know all secrets) to 2 (so that all agents know that all agents know all secrets) does not double the required number of calls but increases this number by 3/2 (for a complete graph). In Section 4 we turn our attention to the version of this problem in which all communications are one-way (such as e-mails rather than telephone calls). We show that the following counter-intuitive result generalises to the epistemic gossip problem: asymptotically the same number of calls are required whether calls are two-way or one-way. In Section 5 we study a parallel version in which calls between different agents can take place simultaneously. In each of these three cases, we give a protocol which is optimal (given certain conditions on the graph G). In Section 6 we generalise an asymptotically optimal protocol for the one-way parallel version of the classical gossip problem to the epistemic case. We conclude with a discussion in Section 7.

2. The epistemic gossip problem

We use the notation $K_i s_j$ to represent the fact that agent i knows the secret s_j of j, the notation $K_i K_j s_k$ to represent the fact that agent i knows that agent j knows the secret of k, etc. We consider the secrets s_i as propositional constants and that agents never forget; an epistemic proposition of the form $K_{i_1} \dots K_{i_r} s_j$, once true, can never become false.

An instance of a planning problem consists of a set of actions, an initial state and a goal. A solution plan (or protocol) is a sequence of actions which when applied in this order to the initial state produces a state in which the goal formula is true. The *epistemic gossip problem* on n agents and an undirected graph $G = \langle \{1, \dots, n\}, E_G \rangle$ is the planning problem in which the actions are $\mathrm{C}(i, j)$ for $\{i, j\} \in E_G$ (i.e. there is an edge between i and j in G if and only if they can call each other). The effect of the call action $\mathrm{C}(i, j)$ is that agents i and j share all their knowledge. We go further and assume that the two agents know that they have shared all their knowledge, so that, if we had $K_i f$ or $K_j f$ before the execution of $\mathrm{C}(i, j)$, for any proposition f, then we have $K_{i_1} \dots K_{i_r} f$ just afterwards, for any r and for any sequence $i_1, \dots, i_r \in \{i, j\}$. Observe that $\mathrm{C}(i, j)$ and $\mathrm{C}(j, i)$ have the same effect. The initial state contains $K_i s_i$ for $i = 1, \dots, n$ (and implicitly all propositions of the form $K_{i_1} \dots K_{i_r} s_j$ with $i_r = j$). We use $\mathrm{Gossip}_G(d)$ to denote the epistemic gossip problem on a graph G in which the goal is the conjunction of all positive propositions of the form $K_{i_1} \dots K_{i_d} s_j$. Thus, the parameter d specifies the epistemic depth of the goal formula. Observe that $K_{i_1} \dots K_{i_d} s_j$ implies $K_{i_1} \dots K_{i_r} s_j$ $(1 \leq r < d)$ and hence the goal formula at epistemic depth d subsumes the goal formula for all epistemic depths r such that $1 \leq r < d$. Versions with one-way and parallel communication will be defined in Sections 4 and 5.

The following abbreviation will be useful: for $r \geq 1$, let T_r be the conjunction of $K_{i_1} \dots K_{i_{r-1}} s_{i_r}$ for all $i_1, \dots, i_r \in \{1, \dots, n\}$). The goal of $\mathrm{Gossip}_G(d)$ is to establish T_{d+1}. We consider that $K_i T_r$ abbreviates the conjunction $K_i K_{i_1} \dots K_{i_{r-1}} s_{i_r}$ for all $i_1, \dots, i_r \in \{1, \dots, n\}$. So T_{r+1} is the same as $K_1 T_r \wedge \dots \wedge K_n T_r$.

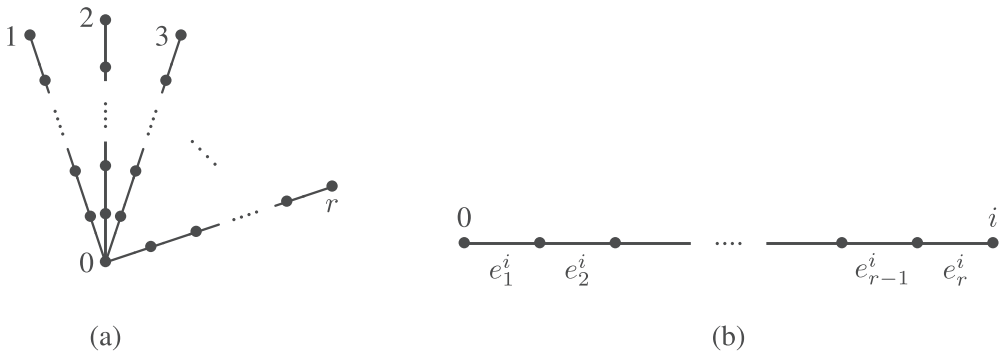

Fig. 1. (a) The graph G_r; (b) the ith branch of G_r.

3. Minimising the number of two-way calls

In this section we consider the epistemic gossip problem Gossip$_G(d)$. The non-epistemic version Gossip$_G(1)$ has been well studied. The minimal number of calls to obtain the solution of Gossip$_G(1)$ is either $2n-4$ if the graph G contains a quadrilateral (a cycle of length 4) as a subgraph, or $2n-3$ in the general case [15]. We first give a simple protocol for Gossip$_G(d)$ for any connected graph G before giving protocols requiring many less calls for special cases of G.

Proposition 1. *If the graph G is connected, then for $n \geq 2$ and $d \geq 1$, Gossip$_G(d)$ has a solution of length no greater than $d(2n-3)$ calls.*

Proof. Since G is connected, it has a spanning tree $\mathcal{T}$. Let vertex 1 be a leaf of $\mathcal{T}$, and let 2 be its neighbour. Let $\mathcal{T}_2$ be the subtree of $\mathcal{T}$ after deletion of node 1 and edge $\{1, 2\}$. The number of edges in tree $\mathcal{T}_2$ is $n-2$.

Consider the following protocol consisting of d rounds. In each round, first every vertex except 1 funnels its secrets to 2, then 1 and 2 swap information via the call C(1, 2), and then 2 funnels all secrets back to all vertices of $\mathcal{T}_2$. After m rounds: $K_{i_1} \cdots K_{i_m} s_j$ is true for all $i_1, \ldots, i_m, j$. So the goal is attained after d rounds. Since each round requires $2n-3$ calls, this gives a total of $d(2n-3)$ calls. ■

We now show that there exist graphs for which asymptotically we can do no better than the naive protocol described in the proof of Proposition 1.

Proposition 2. *There is a family of graphs for which Gossip$_G(d)$ requires $2dn - o(n)$ calls.*

Proof. Let G_r be the graph shown in Fig. 1(a), a spider graph with r branches each of length r, with leaves numbered from 1 to r. The ith branch of G_r is shown in Fig. 1(b). The total number of vertices n in G_r is $1 + r^2$. Thus $r = \sqrt{n-1}$.

We denote the sequence of edges $e_1^i e_2^i \ldots e_{r-1}^i$ by O_i, the reverse sequence of edges $e_{r-1}^i \ldots e_2^i e_1^i$ by I_i and the sequence of edges $O_i e_r^i I_i$ by R_i. By equating each edge $\{p, q\}$ with the corresponding action C(p, q), we can view O_i, I_i and R_i as sequences of calls.

Observation: To establish $K_{i_{d+1}} K_{i_d} \ldots K_{i_2} s_{i_1}$ (where $i_{d+1}, i_d, \ldots, i_2, i_1 \in \{1, \ldots, r\}$ and $i_k \neq i_{k+1}$ for $1 \leq k \leq d$), a plan π must necessarily contain the calls given by

$$e_r^{i_1} I_{i_1} R_{i_2} R_{i_3} \ldots R_{i_d} O_{i_{d+1}} e_r^{i_{d+1}},$$

in that order, since the secret s_{i_1} must pass from i_1 to i_{d+1} transiting by $i_2, \ldots, i_d$ (in this order).

Consider a minimum-length solution-plan π for Gossip$_G(d)$. Reading from the first call onwards, let i_1 be such that I_{i_1} is the last of $I_1, \ldots, I_r$ to occur as a subsequence of π. Then let i_2 be such that R_{i_2} is the last of $R_1, \ldots, R_{i_1-1}, R_{i_1+1}, \ldots, R_r$ to occur as a subsequence of π after this occurrence of I_{i_1}. Then let i_3 be such that R_{i_3} is the last of $R_1, \ldots, R_{i_2-1}, R_{i_2+1}, \ldots, R_r$ to occur as a subsequence of π after this occurrence of I_{i_2}. ... Then let i_d be such that R_{i_d} is the last of $R_1, \ldots, R_{i_{d-1}-1}, R_{i_{d-1}+1}, \ldots, R_r$ to occur as a subsequence of π after this occurrence of $I_{i_{d-1}}$. Finally, let i_{d+1} be such that $O_{i_{d+1}}$ is the last of $O_1, \ldots, O_{i_d-1}, O_{i_d+1}, \ldots, O_r$ to occur as a subsequence of π after this occurrence of R_{i_d}. Thus by the observation above and our choice of $i_1, i_2, \ldots$, π must contain each of I_i $(i = 1, \ldots, r)$, followed by each of R_i $(i = 1, \ldots, i_1 - 1, i_1 + 1, \ldots, r)$, followed by each of R_i $(i = 1, \ldots, i_2 - 1, i_2 + 1, \ldots, r)$, ... followed by each of R_i $(i = 1, \ldots, i_{d-1} - 1, i_{d-1} + 1, \ldots, r)$, followed by each of O_i $(i = 1, \ldots, i_d - 1, i_d + 1, \ldots, r)$. Knowing that $|I_i| = |O_i| = r - 1$ and $|R_i| = 2r - 1$ (for $i = 1, \ldots, r$), we can deduce that $|\pi| \geq r(r-1) + (d-1)(r-1)(2r-1) + (r-1)^2 = d(r-1)(2r-1)$. Since $r = \sqrt{n-1}$, we have $|\pi| \geq 2dn - 3d\sqrt{n-1} - d$. Thus any solution-plan for Gossip$_{G_r}(d)$ requires $2dn - \mathcal{O}(n)$ calls. ■

It turns out that, for $d \geq 2$, we may require considerably less than $d(2n-3)$ calls. We now show that there is a protocol which achieves $(d+1)(n-2)$ calls provided G contains the complete bipartite graph $K_{2,n-2}$ as a subgraph. This subsumes a previous result which was given only for the case of a complete graph G [17].

Proposition 3. *For $n \geq 4$, if the n-vertex graph G has $K_{2,n-2}$ as a subgraph, then Gossip$_G(d)$ has a solution of length $(d+1)(n-2)$.*

Proof. Suppose that the two parts of $K_{2,n-2}$ are $\{1, 2\}$, $\{3, \ldots, n\}$. We choose an arbitrary partition of the vertices $3, \ldots, n$ into two non-empty sets $L = \{3, \ldots, p\}$, $R = \{p+1, \ldots, n\}$.

Consider the protocol that is described by:

Odd passes: C(1, 3) C(1, 4) ... C(1, p) C(2, p+1) C(2, p+2) ... C(2, n)
Even passes: C(1, n) C(1, n−1) ... C(1, p+1) C(2, p) C(2, p−1) ... C(2, 3)

The odd passes are composed of C(1, x) for each $x \in L$ in increasing order of x, followed by C(2, y) for each $y \in R$ in increasing order of y; and the even passes are composed of C(1, y) for each $y \in R$ in decreasing order of y, followed by C(2, x) for each $x \in L$ in decreasing order of x. The length of this plan after $d + 1$ passes is $(d + 1)(|L| + |R|) = (d + 1)(n - 2)$. It therefore only remains to show that $(d + 1)$ passes are sufficient to establish all possible depth-d epistemic goals. An epistemic goal of the form $K_{i_1} \ldots K_{i_d} s_j$, for agents $i_1, \ldots, i_d, j$, has depth d. In particular, s_j has depth 0.

For $m \geq 1$, let H_m be the hypothesis that after m passes, for all depth $m - 1$ epistemic goals f, we have

$(K_1 f \vee K_n f) \wedge (K_2 f \vee K_p f)$ if m is odd
$(K_1 f \vee K_3 f) \wedge (K_2 f \vee K_{p+1} f)$ if m is even

It is not difficult to see that H_1 is true after the first pass. For $H_m \Rightarrow H_{m+1}$, suppose m is even. By H_m, after pass m, we have $K_1 f \vee K_3 f$ for all epistemic goals f of depth $m-1$. Thus the first call of pass $m+1$, C(1, 3), makes 1 and 3 know all epistemic goals of depth $m-1$. After C(1, p), 1 and p know that 1, 3, 4 ..., p know all epistemic goals of depth $m-1$. The same goes for 2: since we have $K_2 f \vee K_{p+1} f$ by H_m, after C(2, p+1), 2 and p+1 know all epistemic goals of depth $m-1$. At the end of pass $m+1$ (after C(2, n)), 2 and n know that 2, p+1, p+2, ..., n know all epistemic goals of depth $m-1$. Thus for any epistemic goals f of depth m, either 1 knows f or n knows f, and either 2 knows f or p knows f, that is, H_{m+1}. The reasoning is similar for m odd. The above plan therefore establishes, after $d + 1$ passes, all possible depth-d epistemic goals. ■

Detecting whether an arbitrary graph G has $K_{2,n-2}$ as a subgraph can clearly be achieved in polynomial time, since it suffices to test for each pair of vertices $\{i, j\}$ whether or not G contains all edges of the form $\{u, v\}$ ($u \in \{i, j\}$, $v \in \{1, \ldots, n\} \setminus \{i, j\}$).

We can, in fact, show that the solution plan given in the proof of Proposition 3 is optimal.

Theorem 1. *The number of calls required to solve Gossip$_G(d)$, for any graph G with $n \geq 2$ vertices, is at least $(d + 1)(n - 2)$.*

Proof. Consider any solution plan for Gossip$_G(d)$. Recall that the goal of Gossip$_G(d)$ is to establish T_{d+1}. We give a proof by induction. Suppose that at least $(r + 1)(n - 2)$ calls are required to establish T_{r+1}. This is true for $r = 1$ because it takes at least a sequence of $2n - 4$ calls to establish T_2 (each agent knows the secret of each other agent) [1,14,29].

For $r \geq 1$ and without loss of generality, suppose that before the last call to establish it, T_{r+1} was false because of lack of knowledge of agent j (i.e. $K_j T_r$ was false). By the induction hypothesis this is at least the $(r+1)(n-2)$th call. This call involves j and another agent, say i, and establishes not only T_{r+1}, but also $K_j T_{r+1}$ and $K_i T_{r+1}$. However, $\neg K_k T_{r+1}$ holds both before and after this call, for the agents k distinct from i and j. To establish T_{r+2}, it is necessary to distribute T_{r+1} from i and j to other agents and this takes at least $n-2$ calls. Hence, at least $(r+2)(n-2)$ calls are required in total to establish T_{r+2}. By induction on r, it takes at least a sequence of $(d+1)(n-2)$ calls to establish T_{d+1}. ■

4. One-way communications

We now consider a different version of the epistemic gossip problem, which we denote by Directional-gossip, in which communications are one-way (such as e-mails). In this case, the result of C(i, j) is that agent i shares all his knowledge with agent j but agent i receives no information from agent j. Indeed, to be consistent with communication by e-mail, in which the sender cannot be certain that an e-mail will be read by the receiver, we assume that after C(i, j), agent i does not even gain the knowledge that agent j knows the information that agent i has just sent in this call.

What is surprising is that the number of calls to solve Directional-gossip$_G(d)$ is very close to the number of calls required to solve Gossip$_G(d)$, differing by only $d + 1$ in the case of a complete graph G.

In the directional version, the graph of possible communications is now a directed graph G. Let $\overline{G}$ be the symmetric part of G, i.e., the graph with the same n vertices as the directed graph G but with an edge between i and j if and only if G contains the two directed edges (i, j) and (j, i). It is known that if the directed graph G is strongly connected, the minimal number of calls for Directional-gossip$_G(1)$ is $2n - 2$ [15]. We now generalise this to arbitrary d under an assumption about the graph $\overline{G}$. Just as in Gossip$_G(d)$, the goal formula in Directional-gossip$_G(d)$ is T_{d+1}.

Proposition 4. *For all $d \geq 1$, if $\overline{G}$ contains a Hamiltonian path, then any instance of Directional-gossip$_G(d)$ has a solution of length no greater than $(d + 1)(n - 1)$.*

Proof. We give a protocol which establishes T_{d+1}. Without loss of generality, suppose that the Hamiltonian path in $\overline{G}$ is $1, 2, \ldots, n$. Consider the plan consisting of $d+1$ passes according to the protocol described by:

Odd passes: C(i, i+1) (for $i = 1, \ldots, n-1$)
Even passes: C(i+1, i) (for $i = n-1, \ldots, 1$)

We show by a simple inductive proof that this protocol is correct for any $d \geq 1$. Recall that T_r is the conjunction of $K_{i_1} \ldots K_{i_{r-1}} s_{i_r}$ for all $i_1, \ldots, i_r \in \{1, \ldots, n\}$. Consider the hypothesis H(r): at the end of pass r, if r is odd we have $K_n T_r$ and if r is even we have $K_1 T_r$. Clearly, H(1) is true since at the end of the first pass agent n knows all the secrets s_i ($i = 1, \ldots, n$). If r is odd and H(r) holds, then at the end of pass $r+1$, all agents know T_r and furthermore agent 1 knows this (i.e. $K_1 T_{r+1}$). A similar argument shows that $H(r) \Rightarrow H(r+1)$ when r is even. By induction, H(r) holds for all $r = 1, \ldots, d+1$. For $K_n T_r$ or $K_1 T_r$ to hold, we must have T_r (by the truth axiom for knowledge). Thus after $d+1$ passes, and $(d+1)(n-1)$ calls, we achieve the goal T_{d+1}. ■

We now show that the solution plan given in the proof of Proposition 4 is optimal even for a complete symmetric digraph G.

Theorem 2. *The number of calls required to solve Directional-gossip$_G(d)$, for any digraph G with $n \geq 2$ vertices, is at least $(d+1)(n-1)$.*

Proof. Consider any solution plan for Directional-gossip$_G(d)$. The goal of Directional-gossip$_G(d)$ is to establish T_{d+1} (the conjunction of $K_{i_1} \ldots K_{i_d} s_{i_{d+1}}$ for all $i_1, \ldots, i_{d+1} \in \{1, \ldots, n\}$). Consider the following claims (for $1 \leq r \leq d$):

C1(r) after $r(n-1)-1$ calls no agent knows T_r.

C2(r) after $r(n-1)$ calls at most one agent knows T_r.

C3(r) at least $(r+1)(n-1)$ calls are required to establish T_{r+1}.

C1(1) is true because T_1 is the conjunction of all the secrets s_j and no agent can know all the secrets after only $n-2$ calls since after $n-2$ calls, there is necessarily some agent who has not communicated his secret to anyone. Let $r \in \{1, \ldots, d\}$. We will show C1(r) $\Rightarrow$ C2(r) $\Rightarrow$ C3(r) $\Rightarrow$ C1($r+1$).

C1(r) $\Rightarrow$ **C2**(r) Straightforward, since during one call only one agent gains knowledge.

C2(r) $\Rightarrow$ **C3**(r) Suppose that C2(r) holds, i.e. after $r(n-1)$ calls at most one agent knows T_r. This means that the other $n-1$ agents require some information in order to know T_r. Hence we require at least $n-1$ other calls, i.e. $(r+1)(n-1)$ calls in total, to establish T_{r+1}.

C3(r) $\Rightarrow$ **C1**($r+1$) Suppose C3(r) is true and C1($r+1$) is false. Then we require at least $(r+1)(n-1)$ calls to establish T_{r+1} but after $(r+1)(n-1)-1$ calls some agent i knows T_{r+1}. This cannot be the case: by the truth axiom, agent i cannot know something which is false.

This completes the proof by induction that at least $(d+1)(n-1)$ calls are required to establish T_{r+1}, since this corresponds exactly to C3(d). ■

Proposition 5. *For $d \geq 2$, if an instance of Directional-gossip$_G(d)$ has a solution of length $(d+1)(n-1)$ then $\overline{G}$ contains a Hamiltonian path.*

Proof. Suppose that a protocol of length $(d+1)(n-1)$ exists. From the proof of Theorem 2 we can deduce that, for $1 \leq r \leq d$, after $r(n-1)$ calls, exactly one agent i knows T_r: C2(r) tells us that at most one agent knows T_r and the proof of C2(r) $\Rightarrow$ C3(r) shows that one agent must know T_r for us to obtain an optimal number of calls. Similarly, after $(r+1)(n-1)$ calls, exactly one agent j knows T_{r+1}. Let S be the sequence of $n-1$ intervening calls between these two states and let G_S be the directed graph whose arcs correspond to the calls in S. Since T_{r+1} is the conjunction of $K_m T_r$ for $m = 1, \ldots, n$, each agent $m \neq i$ must receive a call during S (in order for m to learn T_r) and each agent $m \neq j$ must make a call during S (in order for j to learn $K_m T_r$). Indeed, in order for j to learn $K_m T_r$, there must be a path in the directed graph G_S from each agent $m \neq j$ to j. The union of these paths forms a subgraph H_S of G_S. The undirected graph corresponding to H_S is connected and has at most $n-1$ edges (since G_S has $n-1$ arcs). But a connected graph with n vertices must have at least $n-1$ edges, and so $H_S = G_S$. Furthermore, a connected graph with n vertices and $n-1$ edges is a tree, so we can deduce that H_S is a tree with root j. The leaves of H_S are those agents that do not receive a call during S. But, recall that each agent $m \neq i$ must receive a call during S. Hence there is only one leaf in H_S, namely i. A tree with a single leaf is necessarily a path. Thus there is a Hamiltonian path (from i to j) in $\overline{G}$. ■

An interesting question is the computational complexity of the problem of finding an optimal protocol for a graph G given as input.

Corollary 1. *For $d \geq 2$, the problem of deciding the existence of a protocol using $(d+1)(n-1)$ calls for Directional-gossip$_G(d)$, where the graph G is given as input, is NP-complete.*

Proof. NP-membership is obvious. NP-hardness follows from Propositions 4, 5 and the well-known fact that the problem of determining the existence of a Hamiltonian path in a graph is NP-complete [13]. ■

5. Parallel communications

We now go back to classic two-way communications. An interesting variant, which we call Parallel-gossip$_G(d)$, is to consider time steps instead of calls, and thus suppose that in each time step several calls are executed in parallel. However, each agent can only make one call in any given time step. For Parallel-gossip$_G(1)$ on a complete graph G, if the number of agents n is even, the time taken (in number of steps) is $\lceil \log_2 n \rceil$, and if n is odd, it is $\lceil \log_2 n \rceil + 1$ [2,26,20]. We now generalise this to the case of arbitrary epistemic depth d.

Proposition 6. *For $n \geq 2$, if the n-vertex graph G has the complete bipartite graph $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ as a subgraph, then Parallel-gossip$_G(d)$ has a solution with $d(\lceil \log_2 n \rceil - 1) + 1$ time steps if n is even, or $d\lceil \log_2 n \rceil + 1$ time steps if n is odd.*

Proof. Suppose that G has $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ as a subgraph. Then we can partition the vertex set of G into two subsets V_1 and V_2 of size $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, respectively, such that G has an edge $\{i, j\}$ for each $i \in V_1$ and $j \in V_2$. We can number agents by elements of the ring $\mathbb{Z}_n = \{1, \ldots, n\}$ so that for all $i \in \mathbb{Z}$, $2i + 1 \in V_1$ and $2i + 2 \in V_2$, where arithmetic here and throughout the proof is modulo n. We consider separately the cases n even and n odd. For even n, consider the protocol:

First pass:
For each step s from 1 to $\lceil \log_2 n \rceil$: $\forall i \in \{0, \ldots, (\frac{n}{2} - 1)\}$, C$(2i + 1, 2i + 2^s)$
Subsequent passes:
Reorder even agents according to the permutation π given by
$\pi(2i + 2^{\lceil \log_2 n \rceil}) = 2i + 2$;
Proceed as in the first pass but only for steps s from 2 to $\lceil \log_2 n \rceil$

The first pass of this protocol is illustrated in Fig. 2 for $n = 14$. Calls are represented by a line joining two agents.

In the first pass, because of the calls C$(2i + 1, 2i + 2)$, the first step establishes for all $i \in \mathbb{Z}$, $K_{2i+1}s_{2i+2}$ and $K_{2i+2}s_{2i+1}$. Suppose that after step s, for all $i \in \mathbb{Z}$, we have the conjunction of $K_{2i+1}s_j$ and $K_{2i+2^s}s_j$ for all $j \in \{2i + 1, \ldots, 2i + 2^s\}$. We have just seen that this is true for $s = 1$ (given that each agent knows his own secret). In particular, if we replace i by $i + 2^{s-1}$ we have $K_{2i+2^s+1}s_j$ and $K_{2i+2^{s+1}}s_j$ for all $j \in \{2i+2^s + 1, \ldots, 2i+2^{s+1}\}$. At step $s+1$, we make the calls C$(2i + 1, 2i + 2^{s+1})$ for all $i \in \mathbb{Z}$, which establishes $K_{2i+1}s_j$ and $K_{2i+2^{s+1}}s_j$ for all $j \in \{2i+1, \ldots, 2i+2^{s+1}\}$. By induction on s, it is easily seen that after $\lceil \log_2 n \rceil$ steps, for all $i \in \mathbb{Z}$, we have $K_{2i+1}s_j$ and $K_{2i+2}s_j$ for all $j \in \mathbb{Z}_n$. This means that at the end of the first pass $\forall i, j \in \{2i + 1, \ldots, 2i + 2^{s+1}\}$, $K_i s_j$.

Let T_r be the conjunction of $K_{j_1} \ldots K_{j_{r-1}} s_{j_r}$ for all $j_1, \ldots, j_r \in \mathbb{Z}_n$. We have just seen that after the first pass T_2 is true. Suppose that at the end of pass r, T_{r+1} is true. For the next pass $r + 1$, C$(2i + 1, 2i + 2^{\lceil \log_2 n \rceil})$ are the calls in last step of the previous pass r. Hence, after reordering even agents so that $2i + 2^{\lceil \log_2 n \rceil}$ replaces $2i + 2$, we already have for all $i \in \mathbb{Z}$, $K_{2i+1}K_{2i+2}T_r$ and $K_{2i+2}K_{2i+1}T_r$. We then proceed as for the first pass replacing s_j by $K_j T_r$ to establish T_{r+2} in $\lceil \log_2 n \rceil - 1$ more steps.

It therefore takes d passes to establish all possible depth-d epistemic goals T_{d+1}. The first pass takes $\lceil \log_2 n \rceil$ steps and the next $d - 1$ passes $\lceil \log_2 n \rceil - 1$ steps, making a total of $d(\lceil \log_2 n \rceil - 1) + 1$ steps.

For odd n, one can place the first $2^{\lfloor \log_2 n \rfloor}$ agents in a subset V_{first}, the others being in a subset V_{last} (see the example in Fig. 3 for $n = 13$). Consider the protocol:

Preliminary step:
Each agent in $V_1 \cap V_{last}$ calls one agent in $V_2 \cap V_{first}$,
and each agent in $V_2 \cap V_{last}$ calls one agent in $V_1 \cap V_{first}$.
Subsequent passes:
Proceed in V_{first} as for the first pass of even case in $\mathbb{Z}/2^{\lfloor \log_2 n \rfloor}\mathbb{Z}$;
Each agent in $V_1 \cap V_{last}$ calls one agent in $V_2 \cap V_{first}$,
and each agent in $V_2 \cap V_{last}$ calls one agent in $V_1 \cap V_{first}$.

A typical pass of this protocol is illustrated in Fig. 3. The preliminary step is the step on the right of this figure.

In the preliminary step, all agents $i_1, \ldots, i_m \in V_{last}$ distribute their knowledge to some agents $j_1, \ldots, j_m \in V_{first}$. Hence, after this step we have $K_{j_k}s_{i_k}$ for all $k \in \{1, \ldots, m\}$. For each subsequent pass r, it takes $\lfloor \log_2 n \rfloor = \lceil \log_2 n \rceil - 1$ steps to distribute knowledge from all agents in V_{first} (hence, in V_{last} too because of the previous step) and establish $K_j T_r$ for all $j \in V_{first}$. Then agents $j_1, \ldots, j_m \in V_{first}$ respectively call the agents $i_1, \ldots, i_m \in V_{last}$ in one more step to establish T_{r+1}. These last calls also establish $K_{j_k}K_{i_k}T_r$ for all $k \in \{1, \ldots, m\}$ if necessary for the next pass $r + 1$.

It takes one preliminary step and d passes of $\lceil \log_2 n \rceil$ steps to establish all possible depth-d epistemic goals T_{d+1}, which makes a total of $d\lceil \log_2 n \rceil + 1$ steps. ■

It is worth pointing out that determining whether a n-vertex graph G has the complete bipartite graph $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ as a subgraph can be achieved in polynomial time. To see this, firstly observe that any pair of vertices i, j of G which are not joined by an edge must be in the same part in the complete bipartite graph. In $O(n^2)$ time, we can partition the vertices of G into subsets $S_1, \ldots, S_r$ such that vertices i, j not joined by an edge in G belong to the same set S_t (for some $1 \leq t \leq r$). It only

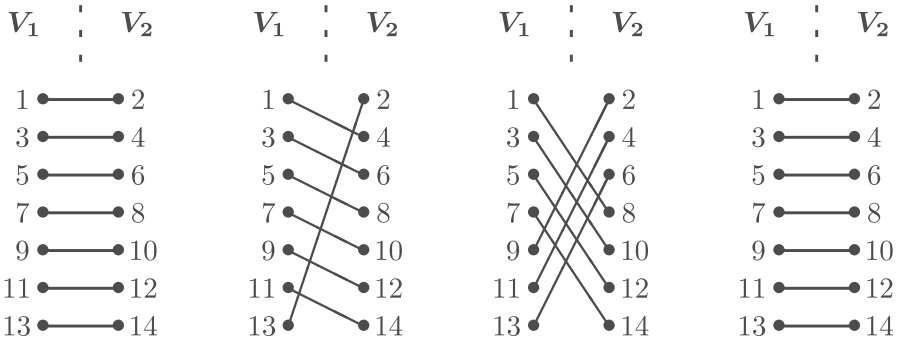

Fig. 2. The four steps in the first pass of the parallel protocol for $n = 14$.

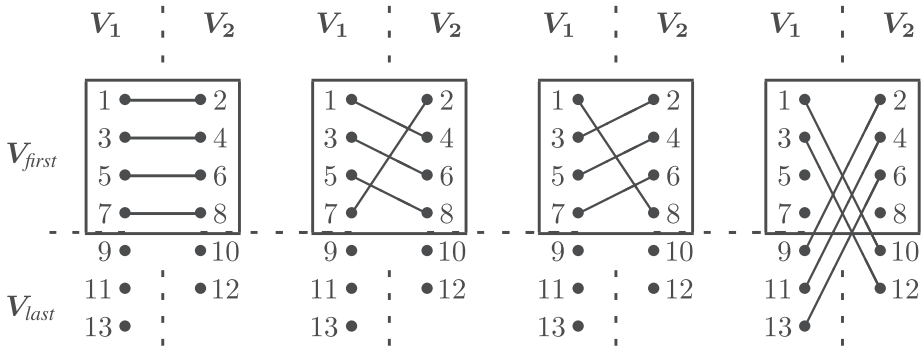

Fig. 3. The four steps in each pass of the parallel protocol for $n = 13$. The box identifies V_{first}. The step on the right also occurs on its own as a preliminary step.

remains to test whether it is possible to partition the numbers $|S_1|, \ldots, |S_r|$ into two sets whose sums are $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$. This partition problem can be solved by dynamic programming in $O(r(|S_1| + \cdots + |S_r|))$ time and space, which is at worst quadratic since $r \leq n$ and $|S_1| + \cdots + |S_r| = n$ [21]. On the other hand, it is known that deciding whether Parallel-gossip(1) (the problem in which the graph G is part of the input) can be solved in a given number of steps is NP-complete [23].

We now show that the solution plans given in the proof of Proposition 6 are optimal in the number of steps.

Theorem 3. *The number of steps required to solve Parallel-gossip$_G(d)$ for any graph G with $n \geq 2$ vertices is at least $d(\lceil \log_2 n \rceil - 1) + 1$ if n is even, or $d\lceil \log_2 n \rceil + 1$ if n is odd.*

Proof. The proof is similar to that of Theorem 1. Consider any solution plan for Parallel-gossip$_G(d)$. Recall that T_r denotes the conjunction of $K_{i_1} \ldots K_{i_{r-1}} s_{i_r}$ for all $i_1, \ldots, i_r \in \{1, \ldots, n\}$.

We give a proof by induction. For even n, suppose that at least $r(\lceil \log_2 n \rceil - 1) + 1$ steps are required to establish T_{r+1}. This is true for $r = 1$ because it takes at least a sequence of $\lceil \log_2 n \rceil$ steps of calls for the secrets of any agent to reach n agents (thus establishing T_2) [2,26,20]. For $r \geq 2$ and without loss of generality, suppose that before the last step to establish it, T_{r+1} was false because of lack of knowledge of agent j (i.e. $K_j T_r$ was false). By induction hypothesis this is at least the $(r(\lceil \log_2 n \rceil - 1))$th step. A call in this step involves j and another agent, say i, and establishes not only T_{r+1}, but also $K_j T_{r+1}$ and $K_i T_{r+1}$. However, $\neg K_k T_{r+1}$ holds both before and after this step, for the agents k distinct from i and j. To establish T_{r+2}, it is necessary to distribute T_{r+1} from i and j to all other agents and this takes at least $\lceil \log_2 n \rceil - 1$ steps (since each step can at most double the number m of agents knowing T_{r+1} and thus $\lceil \log_2(n/2) \rceil = \lceil \log_2 n \rceil - 1$ steps are required to go from $m = 2$ to $m = n$). Hence, at least $(r + 1)(\lceil \log_2 n \rceil - 1) + 1$ steps are required to establish T_{r+2}. By induction on r, we obtain the lower bound $d(\lceil \log_2 n \rceil - 1) + 1$.

For odd n, the proof is similar but at least one more step is required for each epistemic level r because at least one agent does not communicate his knowledge in the first step to establish T_{r+1}. Hence, it takes at least a sequence of $\lceil \log_2 n \rceil + 1$ steps for knowledge from all n agents to reach each other, and the lower bound is $d\lceil \log_2 n \rceil + 1$. ■

We note that it can happen that increasing the number of agents (and hence the number of secrets) leads to less steps. Consider the concrete example of 7 or 8 agents. By Proposition 6 and Theorem 3, the number of steps decreases from $3d+1$ to $2d + 1$ when the number of agents increases from 7 to 8. We can explain this by the fact that in the case of an odd number of agents, during each step there is necessarily one agent who is not communicating. By adding an extra agent, we can actually achieve a larger number of calls in a fewer number of steps.

6. One-way parallel communications

We now consider the combination of parallel and one-way communication which we denote by Parallel-directional-gossip(d). In the following, we define the *Fibonacci* sequence by $F_0 = 0, F_1 = 1$ and for $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$. We denote the golden ratio by $\varphi = \frac{1+\sqrt{5}}{2}$. We only need to consider the case $n \geq 4$ since for $n \leq 3$ only one communication is possible in each step and so the problem reduces to the sequential one-way case of Section 4.

Proposition 7. *For $n \geq 4$, if the n-vertex graph $\overline{G}$ has the complete bipartite graph $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ as a subgraph, then Parallel-directional-gossip(d) has a solution with $d(\lceil \log_\varphi n \rceil + 2) + 1$ time steps.*

Proof. Suppose that $\overline{G}$ has $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ as a subgraph. So we can partition the vertex set of G into two subsets V_1 and V_2 of size $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, respectively, such that $\overline{G}$ has an edge $\{i, j\}$ for each $i \in V_1$ and $j \in V_2$. We can number agents by elements of the ring $\mathbb{Z}_n = \{1, \ldots, n\}$ so that $V_1 = \{2i + 1 : i = 0, \ldots, \lceil n/2 \rceil - 1\}$ and $V_2 = \{2i + 2 : i = 0, \ldots, \lfloor n/2 \rfloor - 1\}$, where arithmetic here and throughout the proof is modulo n. We consider separately the cases n even and n odd. For even n, consider the protocol:

Preliminary step:
  For each $i \in \{0, \ldots, (\frac{n}{2} - 1)\}$, $\mathrm{C}(2i + 1, 2i + 2)$.
First pass:
  For each step s from 1 to $\lceil \log_\varphi n \rceil$: $\forall i \in \{0, \ldots, (\frac{n}{2} - 1)\}$,
    if s is odd, $\mathrm{C}(2i + 2, 2(i + F_{s-1}) + 1)$;
    if s is even, $\mathrm{C}(2i + 1, 2(i + F_{s-1}) + 2)$.
Subsequent passes:
  If $\lceil \log_\varphi n \rceil$ is odd, reorder all agents according to the
      permutation π_1 given by $\pi_1(2(i + F_{\lceil \log_\varphi n \rceil + 1}) + 1) = 2i + 2$
      and $\pi_1(2i + 2) = 2i + 1$;
  If $\lceil \log_\varphi n \rceil$ is even, reorder even agents according to the
      permutation π_2 given by $\pi_2(2(i + F_{\lceil \log_\varphi n \rceil + 1}) + 2) = 2i + 2$;
  Proceed as in the first pass for steps s from 1 to $\lceil \log_\varphi n \rceil$.

Because of the calls $\mathrm{C}(2i + 1, 2i + 2)$, the preliminary step establishes for all $i \in \mathbb{Z}$, $K_{2i+2}s_{2i+1}$. Furthermore, in the first pass, because of the $\mathrm{C}(2i + 2, 2i + 1)$, the first step establishes for all $i \in \mathbb{Z}$, $K_{2i+1}s_{2i+2}$. Suppose that after odd step $s = 2k + 1$, for all $i \in \mathbb{Z}$, we have the conjunction of $K_{2i+1}s_j$ for all $j \in \{2(i - F_{s+1}) + 3, \ldots, 2i + 2\}$ together with $K_{2i+2}s_j$ for all $j \in \{2(i - F_s) + 3, \ldots, 2i + 2\}$. We have just seen that this is true for $s = 1$ (given that each agent knows their own secret). In particular, if we replace i by $i - F_s$ we have $K_{2(i-F_s)+1}s_j$ for all $j \in \{2(i - F_{s+2}) + 3, \ldots, 2(i - F_s) + 2\}$. At even step $s + 1$, we make the calls $\mathrm{C}(2i + 1, 2(i + F_s) + 2)$ for all $i \in \mathbb{Z}$, which are exactly the same as $\mathrm{C}(2(i - F_s) + 1, 2i + 2)$ for all $i \in \mathbb{Z}$, and this establishes $K_{2i+2}s_j$ for all $j \in \{2(i - F_{s+2}) + 3, \ldots, 2i + 2\}$. In particular, if we replace i by $i - F_{s+1}$ we have $K_{2(i-F_{s+1})+2}s_j$ for all $j \in \{2(i - F_{s+3}) + 3, \ldots, 2(i - F_{s+1}) + 2\}$. At odd step $s + 2$, we make the calls $\mathrm{C}(2i + 2, 2(i + F_{s+1}) + 1)$ for all $i \in \mathbb{Z}$, which are exactly the same as $\mathrm{C}(2(i - F_{s+1}) + 2, 2i + 2)$ for all $i \in \mathbb{Z}$, and this establishes $K_{2i+1}s_j$ for all $j \in \{2(i - F_{s+3}) + 3, \ldots, 2i + 2\}$.

It is known that:

$$F_s = \frac{\varphi^s - (-\varphi)^{-s}}{\sqrt{5}} \sim \frac{\varphi^s}{\sqrt{5}} = \varphi^{s - \log_\varphi \sqrt{5}}$$

This value is greater than n when $s \geq \log_\varphi n + \log_\varphi \sqrt{5}$. Hence, by induction on s, it is easily seen that after the preliminary step and $\lceil \log_\varphi n \rceil + 2$ more steps, for all $i \in \mathbb{Z}$, we have $K_{2i+1}s_j$ and $K_{2i+2}s_j$ for all $j \in \mathbb{Z}_n$. This means that at the end of the first pass $\forall i, j \in \{2i + 1, \ldots, 2(i + F_{s-1} + 1)\}$, $K_i s_j$.

Let T_r be the conjunction of $K_{j_1} \ldots K_{j_{r-1}} s_{j_r}$ for all $j_1, \ldots, j_r \in \mathbb{Z}_n$. We have just seen that after the first pass T_2 is true. Suppose that at the end of pass r, T_{r+1} is true. Consider the next pass $r + 1$. On the one hand, if $\lceil \log_\varphi n \rceil$ is odd, the calls in last step of the previous pass r are $\mathrm{C}(2i + 2, 2(i + F_{\lceil \log_\varphi n \rceil + 1}) + 1)$. Hence, after reordering odd agents so that $2(i + F_{\lceil \log_\varphi n \rceil + 1}) + 1$ replaces $2i + 1$, and swapping V_1 and V_2, we already have for all $i \in \mathbb{Z}$, $K_{2i+2}K_{2i+1}T_r$. On the other hand, if $\lceil \log_\varphi n \rceil$ is even, $\mathrm{C}(2i + 1, 2(i + F_{\lceil \log_\varphi n \rceil + 1}) + 2)$ are the calls in last step of the previous pass r. Hence, after reordering even agents so that $2(i + F_{\lceil \log_\varphi n \rceil + 1}) + 2$ replaces $2i + 2$, we already have for all $i \in \mathbb{Z}$, $K_{2i+2}K_{2i+1}T_r$. We then proceed as for the first pass, replacing s_j by $K_j T_r$ to establish T_{r+2} in $\lceil \log_\varphi n \rceil + 2$ more steps.

It therefore takes d passes to establish the depth-d epistemic goal T_{d+1}. After the preliminary step, the next d passes take $\lceil \log_\varphi n \rceil$ steps, making a total of $d(\lceil \log_\varphi n \rceil + 2) + 1$ steps.

For odd n, let $m = 2\lceil n/4 \rceil$. We place the first m agents in a subset V^{odd}_{first}, the others being in a subset V^{odd}_{last}. Furthermore, we place the last m agents in a subset V^{even}_{first}, the others being in a subset V^{even}_{last} and reorder agents in these two latter sets following the permutation π given by $\pi(j) = n + 1 - j$ for all $j \in \mathbb{Z}_n$. Consider the protocol:

Preliminary step:
Each agent in $V_1 \cap V^{odd}_{last}$ calls one agent in $V_2 \cap V^{odd}_{first}$,
and each agent in $V_2 \cap V^{odd}_{last}$ calls one agent in $V_1 \cap V^{odd}_{first}$
Subsequent passes:
Proceed in V^{p}_{first} as for the preliminary step and
the first pass of even case in $\mathbb{Z}_m$,
where $p \in \{odd, even\}$ is the parity of current pass;
Each agent in $V_1 \cap V^{\overline{p}}_{last}$ calls one agent in $V_2 \cap V^{\overline{p}}_{first}$,
and each agent in $V_2 \cap V^{\overline{p}}_{last}$ calls one agent in $V_1 \cap V^{\overline{p}}_{first}$,
where $\overline{p}$ is the parity of next pass.

In the preliminary step, all agents $i_1, \ldots, i_{n-m} \in V^{odd}_{last}$ distribute their knowledge to some agents $j_1, \ldots, j_{n-m} \in V^{odd}_{first}$ (since $|V^{odd}_{last}| < |V^{odd}_{first}|$, note that there are some agents $j_{n-m+1}, \ldots, j_m$ who do not communicate in this step). Hence, after this step we have $K_{j_k} s_{i_k}$ for all $k \in \{1, \ldots, n-m\}$. Observe that $m = 2\lceil n/4 \rceil \leq \frac{n+3}{2}$ for all $n \in \mathbb{N}$. Recall that F_s is greater than $\frac{n+3}{2} \geq m$ when $s \geq log_\varphi \frac{n+3}{2} + log_\varphi \sqrt{5}$. For $n \geq 7$, $log_\varphi(1 + \frac{3}{n}) + log_\varphi \frac{\sqrt{5}}{2} < 1$, so $log_\varphi(n+3) + log_\varphi \frac{\sqrt{5}}{2} < log_\varphi n + 1 \leq \lceil \log_\varphi n \rceil + 1 = s$. For the case $n = 5$, we have $\lceil log_\phi n \rceil + 1 = 5$ and $F_5 = 5 > m = 4$. Thus $s = \lceil log_\phi n \rceil + 1$ implies $F_s > m$ for all odd $n > 4$. Hence, for each subsequent pass r of parity p, it takes $\lceil \log_\varphi n \rceil + 1$ steps to distribute knowledge from all agents in V^{p}_{first} (hence, in $V^{\overline{p}}_{last}$ too because $V^{\overline{p}}_{last} \subset V^{p}_{first}$) and establish $K_j T_r$ for all $j \in V^{p}_{first}$. Then all agents $j'_1, \ldots, j'_{n-m} \in V^{\overline{p}}_{last} \subset V^{p}_{first}$ respectively call some agents $i'_1, \ldots, i'_{n-m} \in V^{\overline{p}}_{first}$ in one more step to establish T_{r+1}. These last calls also establish $K_{i'_k} K_{j'_k} T_r$ for all $k \in \{1, \ldots, n-m\}$ if necessary for the next pass $r+1$.

It takes one preliminary step and d passes of $\lceil \log_\varphi n \rceil + 2$ steps to establish the depth-d epistemic goal T_{d+1}, which makes a total of $d(\lceil \log_\varphi n \rceil + 2) + 1$ steps. ■

In the proof of Proposition 7, for even n, we stated that $\lceil \log_\varphi n \rceil + 2 \geq log_\varphi n + log_\varphi \sqrt{5}$. Notice that when $log_\varphi n - \lfloor log_\varphi n \rfloor \leq 0.32$ then $\lceil \log_\varphi n \rceil + 1 \geq log_\varphi n + log_\varphi \sqrt{5}$. We can then obtain the goal in d less steps, making a total of $d(\lceil log_\varphi n \rceil + 1) + 1$ time steps. This result follows immediately by the same proof as for Proposition 7.

Again, it is known that deciding whether Parallel-directional-gossip(1) (the problem in which the digraph G is part of the input) can be solved in a given number of steps is NP-complete [23].

7. Discussion and conclusion

We consider the epistemic gossip problem to be a foundation on which to base the study of richer epistemic planning problems involving, for example, communication actions with preconditions involving the contents of the messages received by the agent or negative goals to model applications in which certain agents must not learn certain secrets. Previous work on temporal planning may help to provide a more realistic model of communication actions in which, for example, the length of a call is a function of the quantity of information exchanged, and correct communication during a telephone call requires concurrency of the speaking and listening actions of the two agents [7,6]. In separate work we have shown that when allowing negative goals, it is NP-complete to decide the existence of a solution plan even at epistemic depth 1 [5].

Restricting our attention to the epistemic version of the classical gossip problem in which all positive epistemic goals of depth d must be attained, we have generalised many results from the classical gossip problem to the epistemic version. We have shown that for a complete graph G, no protocol exists which solves $\text{Gossip}_G(d)$ in less than $(d+1)(n-2)$ calls. This was known to be true for $d = 1$ [1,14]. For any graph G containing $K_{2,n-2}$ as a subgraph, we have given an optimal protocol which uses exactly this number of calls. In the case of one-way communications, we have again generalised the optimal protocol from the classical gossip problem to the epistemic version. This protocol requires only $(d+1)(n-1)$ calls. When calls can be performed in parallel, and the aim is to minimise the number of such parallel steps rather than the number of calls, we have again generalised the optimal protocol from the classical gossip problem to the epistemic version. In this case, only $O(d \log n)$ steps are required.

There remain interesting open problems concerning the optimisation version of the epistemic gossip problem: given any graph G, determine the minimum number of calls required to attain a set of goals. The computational complexity of the problem of minimising the number of calls (whether two-way or one-way) to solve $\text{Gossip}_G(d)$ on an arbitrary graph G (given as input) is still open for $d > 1$.

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References

[1] B. Baker, R. Shostak, Gossips and telephones, Discrete Math. 2 (3) (1972) 191–193, URL: http://dx.doi.org/10.1016/0012-365X(72)90001-5.
[2] A. Bavelas, Communication patterns in task-oriented groups, J. Acoust. Soc. Am. 22 (6) (1950) 725–730.
[3] R.T. Bumby, A problem with telephones, SIAM, J. Algebraic Discrete Methods 2 (1) (1981) 13–18.
[4] G.J. Chang, Y. Tsay, The partial gossiping problem, Discrete Math. 148 (1–3) (1996) 9–14, URL: http://dx.doi.org/10.1016/0012-365X(94)00257-J.
[5] M.C. Cooper, A. Herzig, F. Maffre, F. Maris, P. Régnier, Simple epistemic planning: generalised gossiping, 2016, CoRR, abs/1606.03244, URL: http: //arxiv.org/abs/1606.03244.
[6] M.C. Cooper, F. Maris, P. Régnier, Managing temporal cycles in planning problems requiring concurrency, Comput. Intell. 29 (1) (2013) 111–128, URL: http://dx.doi.org/10.1111/j.1467-8640.2012.00430.x.
[7] W. Cushing, S. Kambhampati, Mausam, D.S. Weld, When is temporal planning really temporal?, in: Manuela M. Veloso (Ed.), IJCAI 2007, Proceedings of the 20th International Joint Conference on Artificial Intelligence, Hyderabad, India, January 6–12, 2007, pp. 1852–1859, URL: http://dli.iiit.ac.in/ijcai/IJCAI-2007/PDF/IJCAI07-299.pdf.
[8] H. van Ditmarsch, J. van Eijck, P. Pardo, R. Ramezanian, F. Schwarzentruber, Epistemic protocols for dynamic gossip, J. Appl. Logic 20 (2017) 1–31, URL: http://dx.doi.org/10.1016/j.jal.2016.12.001.
[9] P.T. Eugster, R. Guerraoui, A. Kermarrec, L. Massoulié, Epidemic information dissemination in distributed systems, IEEE Comput. 37 (5) (2004) 60–67, URL: http://dx.doi.org/10.1109/MC.2004.1297243.
[10] G. Fertin, Hierarchical broadcast and gossip networks, Inf. Process. Lett. 73 (3–4) (2000) 131–136, URL: http://dx.doi.org/10.1016/S0020-0190(00)00014-4.
[11] G. Fertin, A study of minimum gossip graphs, Discrete Math. 215 (2000) 33–57, URL: http://dx.doi.org/10.1016/S0012-365X(99)00227-7.
[12] P. Fraigniaud, J.G. Peters, Minimum linear gossip graphs and maximal linear (delta, k)-gossip graphs, Networks 38 (3) (2001) 150–162, URL: http://dx.doi.org/10.1002/net.1033.
[13] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman, 1979.
[14] A. Hajnal, E. Milner, E. Szemerédi, A Cure for the telephone disease, Canad. Math. Bull. 15 (1972) 447–450.
[15] F. Harary, A.J. Schwenk, The communication problem on graphs and digraphs, J. Franklin Inst. B 297 (6) (1974) 491–495, URL: http://dx.doi.org/10.1016/0016-0032(74)90126-4.
[16] S.M. Hedetniemi, S.T. Hedetniemi, A.L. Liestman, A survey of gossiping and broadcasting in communication networks, Networks 18 (4) (1988) 319–349, URL: http://dx.doi.org/10.1002/net.3230180406.
[17] A. Herzig, F. Maffre, How to share knowledge by gossiping, AI Commun. 30 (1) (2017) 1–17, URL: http://dx.doi.org/10.3233/AIC-170723.
[18] S. Khuller, Y.A. Kim, Y.J. Wan, On generalized gossiping and broadcasting (extended abstract), in: G.D. Battista, U. Zwick (Eds.), Algorithms - ESA 2003, 11th Annual European Symposium, Budapest, Hungary, September 16–19, 2003, Proceedings, in: Lecture Notes in Computer Science, vol. 2832, Springer, 2003, pp. 373–384, URL: http://dx.doi.org/10.1007/978-3-540-39658-1_35.
[19] D.J. Kleitman, J.B. Shearer, Further gossip problems, Discrete Math. 30 (2) (1980) 151–156, URL: http://dx.doi.org/10.1016/0012-365X(80)90116-8.
[20] W. Knödel, New gossips and telephones, Discrete Math. 13 (1) (1975) 95, URL: http://www.sciencedirect.com/science/article/pii/0012365X75900904.
[21] R.E. Korf, Multi-Way number partitioning, in: Craig Boutilier (Ed.), IJCAI 2009, Proceedings of the 21st International Joint Conference on Artificial Intelligence, Pasadena, California, USA, July 11–17, 2009, 2009, pp. 538–543, http://ijcai.org/papers09/Papers/IJCAI09-096.pdf.
[22] D.W. Krumme, Reordered gossip schemes, Discrete Math. 156 (1–3) (1996) 113–140, URL: http://dx.doi.org/10.1016/0012-365X(94)00302-Y.
[23] D.W. Krumme, G. Cybenko, K.N. Venkataraman, Gossiping in minimal time, SIAM J. Comput. 21 (1) (1992) 111–139, URL: http://dx.doi.org/10.1137/0221010.
[24] R. Labahn, The telephone problem for trees, Elektron. Inf.verarb. Kybern. 22 (9) (1986) 475–485.
[25] R. Labahn, Some minimum gossip graphs, Networks 23 (4) (1993) 333–341, URL: http://dx.doi.org/10.1002/net.3230230416.
[26] H.G. Landau, The distribution of completion times for random communication in a task-oriented group, Bull. Math. Biophys. 16 (3) (1954) 187–201.
[27] D. Liben-Nowell, Gossip is synteny: incomplete gossip and the syntenic distance between genomes, J. Algorithms 43 (2) (2002) 264–283, URL: http://dx.doi.org/10.1016/S0196-6774(02)00006-8.
[28] A.L. Liestman, D.S. Richards, Perpetual gossiping, Parallel Process. Lett. 3 (1993) 347–355, URL: http://dx.doi.org/10.1142/S0129626493000381.
[29] R. Tijdeman, On a telephone problem, Nieuw Arch. Wiskd. 19 (3) (1971) 118–192.