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## A time-dependent obstacle problem in linearised elasticity

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**Abstract.** In this paper we establish the existence of solutions to a time-dependent problem for a linearly elastic body subjected to a confinement condition, expressing that all the points of the deformed reference configuration remain confined in a prescribed half space. This problem takes the form of a set of hyperbolic variational inequalities posed over a nonempty, closed, and convex subset of the space  $H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ . The fact that any solution of the studied problem takes the form of a vector field, the generality of the confinement condition, and the choice of the function space where solutions are sought make the analysis substantially more complicated, thus requiring the adoption of new resolution strategies.

*Keywords:* Obstacle problem; time-dependent; penalty method; variational inequalities; linearised elasticity.

### 1. Geometrical preliminaries

For details about the classical notions of differential geometry recalled in this section, see, e.g. [1] or [2].

Latin indices, except when they are used for indexing sequences, take their values in the set  $\{1, 2, 3\}$ , and the summation convention with respect to repeated indices is systematically used in conjunction with this rule.

Given an open subset  $\Omega \subset \mathbb{R}^3$ , notations such as  $L^2(\Omega)$  and  $H^1(\Omega)$  denote the standard Lebesgue and Sobolev spaces. The notation  $\mathcal{D}(\Omega)$  designates the space of functions that are infinitely differentiable over  $\Omega$  and have a compact support in  $\Omega$ . The notation  $\|\cdot\|_X$  designates the norm of a vector space  $X$ . Spaces of vector-valued functions are denoted by boldface letters. Lebesgue-Bochner spaces

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(see, e.g., [3]) are designated by the notation  $L^p(0, T; X)$ , where  $1 \leq p \leq \infty$ ,  $T > 0$  and  $X$  is a Banach space satisfying the Radon-Nikodym property. The notation  $\mathcal{M}([0, T]; X)$  designates the space of  $X$ -valued measures defined over the compact interval  $[0, T]$ . The notation  $X^*$  designates the dual space of a vector space  $X$  and the notation  ${}_X \langle \cdot, \cdot \rangle_X$  denotes the duality pair between  $X^*$  and  $X$ . The notation  $\mathcal{C}^0([0, T]; X)$  denotes the space of  $X$ -valued continuous functions defined over the compact interval  $[0, T]$  and the special notation  $\langle \langle \cdot, \cdot \rangle \rangle_X$  denotes the duality pair between  $(\mathcal{C}^0([0, T]; X))^*$  and  $\mathcal{C}^0([0, T]; X)$ . The notations  $\dot{\eta}$  and  $\ddot{\eta}$  denote the first weak derivative with respect to  $t \in (0, T)$  and the second weak derivative with respect to  $t \in (0, T)$  of a scalar function  $\eta$  defined over the interval  $(0, T)$ . The notations  $\dot{\boldsymbol{\eta}}$  and  $\ddot{\boldsymbol{\eta}}$  denote the first weak derivative with respect to  $t \in (0, T)$  and the second weak derivative with respect to  $t \in (0, T)$  of a vector field  $\boldsymbol{\eta}$  defined over the interval  $(0, T)$ . A *domain*  $\Omega \subset \mathbb{R}^3$  is a nonempty, open, bounded and connected subset with Lipschitz continuous boundary  $\Gamma$ , the set  $\Omega$  being locally on the same side of  $\Gamma$ . The notation  $dx$  designates the *volume element* in  $\Omega$ , the symbol  $d\Gamma$  designates the *area element* along  $\Gamma$ . Finally, let  $\Gamma = \Gamma_0 \cup \Gamma_1$  be a  $d\Gamma$ -measurable portion of the boundary such that  $\Gamma_0 \cap \Gamma_1 = \emptyset$  and  $\text{area } \Gamma_0 > 0$ .

As a model of the three-dimensional “physical” space  $\mathbb{R}^3$ , we take a *real three-dimensional affine Euclidean space*, i.e., a set in which a point  $O$  has been chosen as the *origin* and with which a *real three-dimensional Euclidean space*, denoted  $\mathbb{E}^3$ , is associated. We equip  $\mathbb{E}^3$  with an *orthonormal basis* consisting of three vectors  $\mathbf{e}^i$ . The Euclidean inner product of two elements  $\mathbf{a}$  and  $\mathbf{b}$  of  $\mathbb{E}^3$  is denoted by  $\mathbf{a} \cdot \mathbf{b}$ ; the Euclidean norm of any  $\mathbf{a} \in \mathbb{E}^3$  is denoted by  $|\mathbf{a}|$ ; the Kronecker symbol is denoted by  $\delta^{ij}$ .

The definition of  $\mathbb{R}^3$  as an affine Euclidean space means that with any point  $x \in \mathbb{R}^3$  is associated an uniquely defined vector  $\mathbf{Ox} \in \mathbb{E}^3$ . The origin  $O \in \mathbb{R}^3$  and the orthonormal vectors  $\mathbf{e}^i \in \mathbb{E}^3$  together constitute a *Cartesian frame* in  $\mathbb{R}^3$  and the three components  $x_i$  of the vector  $\mathbf{Ox}$  over the basis formed by  $\mathbf{e}^i$  are called *Cartesian coordinates* of  $x \in \mathbb{R}^3$ , or the *Cartesian components* of  $\mathbf{Ox} \in \mathbb{E}^3$ . Once a Cartesian frame has been chosen, any point  $x \in \mathbb{R}^3$  may be thus *identified* with the vector  $\mathbf{Ox} = x_i \mathbf{e}^i \in \mathbb{E}^3$ . We then denote  $\partial_i = \partial/\partial x_i$ .

The set  $\bar{\Omega}$  is the *reference configuration* occupied by a *linearly elastic elastic body* in absence of applied body forces. We assume that  $\bar{\Omega}$  is a natural state, i.e., that the body is stress-free in this configuration. We also assume, following [4], that the constituting material is *isotropic, homogeneous, and linearly elastic*. Under these assumptions, the behaviour of the linearly elastic material is governed by its two *Lamé constants*  $\lambda \geq 0$  and  $\mu > 0$ . The positive constant  $\rho$  designates the *mass density* of the linearly elastic body per unit volume.

We also assume that the linearly elastic body to be subjected to *applied body forces* in its interior, whose density per unit volume is defined by means of its contravariant components  $f^i \in L^\infty(0, T; L^2(\Omega))$  over the vectors  $\mathbf{e}^i$ .

In what follows, “a.e.” stands for “almost everywhere”. Define the space

$$\mathbf{V}(\Omega) := \{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega); \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\},$$

and equip it with the norm

$$\|\mathbf{v}\|_{\mathbf{V}(\Omega)} := \left( \sum_i \|v_i\|_{H^1(\Omega)}^2 \right)^{1/2}.$$

Next, we define the three-dimensional elasticity tensor in Cartesian coordinates and we denote its components by  $A^{ijkl}$ . We recall that the contravariant components of this tensor are defined by (see, e.g., [4])

$$A^{ijkl} := \lambda \delta^{ij} \delta^{kl} + \mu (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}),$$

and that  $A^{ijkl} = A^{jikl} = A^{klij} \in C^1(\bar{\Omega})$ .

For each  $\mathbf{v} \in \mathbf{H}^1(\Omega)$  we consider the *linearised change of metric tensor*  $\mathbf{e}(\mathbf{v})$ , whose components  $e_{i\|j}(\mathbf{v})$  are defined by

$$e_{i\|j}(\mathbf{v}) := \frac{1}{2} (\partial_j v_i + \partial_i v_j) \in L^2(\Omega).$$

This tensor is symmetric, i.e.,  $e_{i\|j}(\mathbf{v}) = e_{j\|i}(\mathbf{v})$ , for all  $\mathbf{v} \in \mathbf{H}^1(\Omega)$ . Likewise, we can define the *time-dependent version of the linearised change of metric tensor* by considering the operator

$$\tilde{e}_{i\|j} : L^2(0, T; \mathbf{H}^1(\Omega)) \rightarrow L^2(0, T; L^2(\Omega))$$

defined by

$$\tilde{e}_{i\|j}(\mathbf{v})(t) = e_{i\|j}(\mathbf{v}(t)), \text{ for all } \mathbf{v} \in L^2(0, T; \mathbf{H}^1(\Omega)),$$

for almost all (“a.a.” in what follows)  $t \in (0, T)$ . It is easy to see that such an operator is well-defined, linear and continuous. It can be easily verified (cf., e.g., [5]) that the continuity constant is independent of  $t \in (0, T)$ .

To begin with, we state *Korn’s inequality in Cartesian coordinates* (see, e.g., Theorem 6.3-6 of [4]).

**Theorem 1.1.** *Let  $\Omega$  be a domain in  $\mathbb{R}^3$  and let  $\Gamma_0$  be a nonzero area subset of the whole boundary  $\Gamma$ . Then, there exists a constant  $C > 0$  such that*

$$C^{-1} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq \|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega)} \leq C \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)},$$

for all  $\mathbf{v} \in \mathbf{V}(\Omega)$ .

Various proofs have been given of this delicate inequality; see in particular [6], [7], [8], [9], page 110 of [10], Sect. 6.3 of [11]; in [12], Korn’s inequality is proved in the space  $W^{1,p}(\Omega)$ ,  $1 \leq p \leq \infty$ ; an elementary proof is given in [13] (see also Appendix (A) in [14]).

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## 2. A natural formulation of the time-dependent obstacle problem for a linearly elastic body

In this paper, we consider a specific obstacle problem for a linearly elastic body subjected to a confinement condition, expressing that any admissible displacement vector field  $v_i \mathbf{e}^i$ , must be such that all the points of the corresponding deformed configuration remain in a half-space of the form

$$\mathbb{H} := \{x \in \mathbb{R}^3; \mathbf{O}x \cdot \mathbf{q} \geq 0\},$$

where  $\mathbf{q}$  is a *nonzero vector* given once and for all. Let us denote by  $\mathbf{I}$  the identity mapping  $\mathbf{I} : \overline{\Omega} \rightarrow \mathbb{E}^3$  and let us assume that the *undeformed* reference configuration satisfies

$$\mathbf{I}(x) \cdot \mathbf{q} > 0, \text{ for all } x \in \overline{\Omega},$$

or, in other words, *there is no contact between the obstacle and the reference configuration when no applied body forces are acting on the reference configuration.*

The general confinement condition can be thus formulated as follows: any *admissible displacement vector field* must satisfy

$$(\mathbf{I}(x) + v_i(x) \mathbf{e}^i) \cdot \mathbf{q} \geq 0,$$

for all  $x \in \overline{\Omega}$  or, possibly, only for a.a.  $x \in \Omega$  when the covariant components  $v_i$  are required to belong to the Sobolev space  $H^1(\Omega)$ . The subset  $\mathbf{U}(\Omega)$  of admissible displacements thus takes the form

$$\begin{aligned} \mathbf{U}(\Omega) := \{ & \mathbf{v} = (v^i) \in \mathbf{H}^1(\Omega); \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0 \\ & \text{and } (\mathbf{I} + v_i \mathbf{e}^i) \cdot \mathbf{q} \geq 0 \text{ a.e. in } \Omega \}. \end{aligned}$$

We emphasise that the *vectorial* confinement condition above, which was originally considered in [15], [16], [17] and [18], considerably departs from the *scalar* conditions favoured by many authors (see, e.g., [19], [20] and [21]). Such a confinement condition renders the analysis substantially more difficult, as the constraint now bears on a vector field, the displacement vector field of the reference configuration, instead of on only a single component of this field.

A natural formulation of the corresponding time-dependent obstacle problem takes the form of a set of hyperbolic three-dimensional variational inequalities (“three-dimensional”, in the sense that they are posed over the three-dimensional subset  $\Omega$ ), which can be derived by slightly modifying the model proposed by Xiao in the papers [22], [23] and [24].

Let us introduce the problem  $\mathcal{P}(\Omega)$ , which constitutes the point of departure of our analysis.

Problem  $\mathcal{P}(\Omega)$ . Find  $\mathbf{u} = (u_i) : (0, T) \rightarrow \mathbf{V}(\Omega)$  such that

$$\begin{aligned} \mathbf{u} & \in L^\infty(0, T; \mathbf{U}(\Omega)), \\ \dot{\mathbf{u}} & \in L^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \ddot{\mathbf{u}} & \in \mathcal{M}([0, T]; \mathbf{L}^2(\Omega)), \end{aligned}$$

that satisfies the following variational inequalities

$$\begin{aligned} & 2\rho \langle \langle d\ddot{\mathbf{u}}(t), \mathbf{v}(t) - \mathbf{u}(t) \rangle \rangle_{\mathbf{L}^2(\Omega)} \\ & + \int_0^T \int_{\Omega} A^{ijkl} e_{k||l}(\mathbf{u}(t)) e_{i||j}(\mathbf{v}(t) - \mathbf{u}(t)) \, dx \, dt \\ & \geq \int_0^T \int_{\Omega} f^i(t)(v_i(t) - u_i(t)) \, dx \, dt, \end{aligned}$$

for all  $\mathbf{v} \in \mathcal{D}(0, T; \mathbf{V}(\Omega))$  such that  $\mathbf{v}(t) \in \mathbf{U}(\Omega)$  for a.a.  $t \in (0, T)$ , and that satisfies the following initial conditions

$$\begin{cases} \mathbf{u}(0) = \mathbf{u}_0, \\ \dot{\mathbf{u}}(0) = \mathbf{u}_1, \end{cases} \quad (2.1)$$

where we assume  $\mathbf{u}_0 = \mathbf{0}$ , and  $\mathbf{u}_1 = (u_{i,1}) \in \mathbf{L}^2(\Omega)$  is prescribed. ■

Observe that the “acceleration term” in Problem  $\mathcal{P}(\Omega)$  is described in terms of a vector-valued measure. Note in passing that the concept of solution of Problem  $\mathcal{P}(\Omega)$  is inspired by the one given on page 403 of [25]. The concept of *solution of Problem  $\mathcal{P}(\Omega)$*  will be thoroughly explained in the proof of Theorem 3.4, which constitutes the main result of this paper.

Let us now spend few words on the importance of the initial condition for the displacement, viz.,  $\mathbf{u}_0 = \mathbf{0}$ . It is clear that this assumption considerably restricts the applicability of the result stating the existence of solutions to Problem  $\mathcal{P}(\Omega)$  but such an assumption, however, plays an important role in the establishment of the existence of a *uniform upper bound* in some energy estimates we will recover later on. It is this very uniform upper bound that will ensure the validity of the confinement condition in the main existence theorem.

We recall a very important inequality which is used to study evolutionary problems: Gronwall’s inequality (see the seminal paper [26] and Theorem 1.1 in Chapter III of [27]).

**Theorem 2.1.** *Let  $T > 0$  and suppose that the function  $y : [0, T] \rightarrow \mathbb{R}$  is absolutely continuous and such that*

$$\frac{dy}{dt}(t) \leq a(t)y(t) + b(t), \quad \text{a.e. in } (0, T),$$

where  $a, b \in L^1(0, T)$  and  $a, b \geq 0$  a.e. in  $(0, T)$ . Then, it results

$$y(t) \leq \left[ y(0) + \int_0^t b(s) \, ds \right] e^{\int_0^t a(s) \, ds}, \quad \text{for all } t \in [0, T].$$

### 3. Proof of existence of solutions to Problem $\mathcal{P}(\Omega)$

Let us recall a compactness result proved by Simons (see, e.g., Corollary 4 of [28]), which will be used in what follows to recover the initial conditions. In what follows,

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the symbol " $\hookrightarrow$ " denotes a *continuous embedding*, whereas the symbol " $\hookrightarrow\hookrightarrow$ " denotes a *compact embedding*.

**Theorem 3.1.** *Let  $T > 0$  and let  $X, Y$  and  $Z$  be three Banach spaces such that*

$$X \hookrightarrow\hookrightarrow Y \hookrightarrow Z.$$

*Let  $(f_n)_{n=1}^\infty$  be a bounded sequence in  $L^\infty(0, T; X)$  and assume that the sequence  $(\dot{f}_n)_{n=1}^\infty$  is bounded in  $L^\infty(0, T; Z)$ . Then, there exists a subsequence, still denoted  $(f_n)_{n=1}^\infty$ , that converges in the space  $\mathcal{C}^0([0, T]; Y)$ .*

In what follows we identify the spaces  $L^2(\Omega)$  and  $\mathbf{L}^2(\Omega)$  with their respective dual spaces, and we equip them with the following inner products

$$\begin{aligned} (v, w) \in L^2(\Omega) \times L^2(\Omega) &\rightarrow \int_{\Omega} vw \, dx, \\ (\mathbf{v}, \mathbf{w}) \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) &\rightarrow \int_{\Omega} v_i w_i \, dx. \end{aligned}$$

We observe that the following chain of immersions holds

$$\mathbf{V}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega) \hookrightarrow\hookrightarrow \mathbf{V}^*(\Omega),$$

viz., following the notation of [29],  $(\mathbf{V}(\Omega), \mathbf{L}^2(\Omega), \mathbf{V}^*(\Omega))$  is an *evolution triple* (or *Gelfand triple*).

Let us also recall a result on vector-valued measures proved by Zinger in the paper [30] (see also, e.g., page 182 of [31], and page 380 of [32]).

**Theorem 3.2.** *Let  $\omega$  be a compact Hausdorff space and let  $X$  be a Banach space satisfying the Radon-Nikodym property. Let  $\mathcal{F}$  be the collection of Borel sets of  $\omega$ .*

*There exists an isomorphism between  $(\mathcal{C}^0(\omega; X))^*$  and the space of the regular Borel measures with finite variation taking values in  $X^*$ . In particular, for each  $F \in (\mathcal{C}^0(\omega; X))^*$ , there exists a unique regular Borel measure  $\mu : \mathcal{F} \rightarrow X^*$  in  $\mathcal{M}(\omega; X^*)$  with finite variation such that*

$$\langle\langle \alpha, F \rangle\rangle_X = \int_{\omega} X^* \langle d\mu, \alpha \rangle_X,$$

for all  $\alpha \in \mathcal{C}^0(\omega; X)$ .

Let us denote the *penalty parameter* by  $\kappa$  and let us introduce the corresponding *penalised problem*  $\mathcal{P}(\kappa; \Omega)$ . We resort to a penalty scheme to prove the existence of solutions of Problem  $\mathcal{P}(\Omega)$ , since such a scheme is very advantageous in the context of numerical approximations.

Problem  $\mathcal{P}(\kappa; \Omega)$ . Find  $\mathbf{u}_\kappa = (u_{i,\kappa}) : (0, T) \rightarrow \mathbf{V}(\Omega)$  such that

$$\begin{aligned} \mathbf{u}_\kappa &\in L^\infty(0, T; \mathbf{V}(\Omega)), \\ \dot{\mathbf{u}}_\kappa &\in L^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \ddot{\mathbf{u}}_\kappa &\in L^\infty(0, T; \mathbf{V}^*(\Omega)), \end{aligned}$$

that satisfies the variational inequalities

$$\begin{aligned} & 2\rho_{\mathbf{V}^*(\Omega)} \langle \ddot{u}_{i,\kappa}(t) \mathbf{e}^i, v_j \mathbf{e}^j \rangle_{\mathbf{V}(\Omega)} \\ & + \int_{\Omega} A^{ijkl} e_{k||i}(\mathbf{u}_{\kappa}(t)) e_{i||j}(\mathbf{v}) \, dx \\ & - \frac{1}{\kappa} \int_{\Omega} (\{[\mathbf{I} + u_{i,\kappa}(t) \mathbf{e}^i] \cdot \mathbf{q}\}^-) (v_i \mathbf{e}^i \cdot \mathbf{q}) \, dx \\ & = \int_{\Omega} f^i(t) v_i \, dx, \end{aligned}$$

for all  $\mathbf{v} \in \mathbf{V}(\Omega)$ , in the sense of distributions in  $(0, T)$ , and that satisfies the initial conditions (2.1).  $\blacksquare$

We first prove, by Galerkin method, that Problem  $\mathcal{P}(\kappa; \Omega)$  admits a solution.

**Theorem 3.3.** *Problem  $\mathcal{P}(\kappa; \Omega)$  admits a solution.*

**Proof.** The proof is carried out via a Galerkin argument and is subdivided into three steps.

(a) *Construction of a Galerkin approximation.* In order to construct such a scheme, we rely on the fact that  $\mathbf{V}(\Omega)$  is an infinite dimensional separable Hilbert space which is also dense in  $\mathbf{L}^2(\Omega)$ , in order to infer the existence of an orthogonal basis  $(\mathbf{w}^k)_{k=1}^{\infty}$  of the space  $\mathbf{V}(\Omega)$ , whose elements also constitute a Hilbert basis of the space  $\mathbf{L}^2(\Omega)$ .

The existence of such a basis is assured by the spectral theorem (Theorem 6.2-1 of [33]). For each positive integer  $m \geq 1$ , we denote by  $\mathbf{E}^m$  the following  $m$ -dimensional linear hull

$$\mathbf{E}^m := \text{Span} (\mathbf{w}^k)_{k=1}^m \subset \mathbf{V}(\Omega) \subset \mathbf{L}^2(\Omega).$$

Since each element of this Hilbert basis is independent of the variable  $t$ , we have that  $\mathbf{w}^k \in L^{\infty}(0, T; \mathbf{V}(\Omega))$  for each integer  $1 \leq k \leq m$ .

We now discretise Problem  $\mathcal{P}(\kappa; \Omega)$  and, in order to keep the notation simple, we drop the dependence of the vector fields entering the variational equations on the penalty parameter  $\kappa$ . Observe that the duality pair between  $\mathbf{E}^m$  and its dual coincides with the inner product of  $\mathbf{L}^2(\Omega)$  introduced beforehand.

Problem  $\mathcal{P}_m(\kappa; \Omega)$ . Find functions  $c_k : [0, T] \rightarrow \mathbb{R}$ ,  $1 \leq k \leq m$ , such that

$$\mathbf{u}^m(t) := \sum_{k=1}^m c_k(t) \mathbf{w}^k, \text{ for a.a. } t \in (0, T),$$

and satisfying the following *penalised variational equations* a.e. in  $(0, T)$



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$$\begin{aligned}
 & 2\rho \int_{\Omega} \ddot{u}_i^m(t) w_i^p \, dx + \int_{\Omega} A^{ijkl} e_{k||l}(\mathbf{u}^m(t)) e_{i||j}(\mathbf{w}^p) \, dx \\
 & - \frac{1}{\kappa} \int_{\Omega} (\{[\mathbf{I} + u_i^m(t) \mathbf{e}^i] \cdot \mathbf{q}\}^-) (w_i^p \mathbf{e}^i \cdot \mathbf{q}) \, dx \\
 & = \int_{\Omega} f^i(t) w_i^p \, dx,
 \end{aligned}$$

for each integer  $1 \leq p \leq m$ .

Such a function  $\mathbf{u}^m$  must satisfy, in addition, the following initial conditions

$$\begin{aligned}
 \mathbf{u}^m(0) &= \mathbf{u}_0^m, \\
 \dot{\mathbf{u}}^m(0) &= \mathbf{u}_1^m, \\
 \mathbf{u}_0^m &\in \mathbf{V}(\Omega) \quad \text{and} \quad \mathbf{u}_0^m \rightarrow \mathbf{u}_0 \text{ in } \mathbf{V}(\Omega) \text{ as } m \rightarrow \infty, \\
 \mathbf{u}_1^m &\in \mathbf{L}^2(\Omega) \quad \text{and} \quad \mathbf{u}_1^m \rightarrow \mathbf{u}_1 \text{ in } \mathbf{L}^2(\Omega) \text{ as } m \rightarrow \infty,
 \end{aligned} \tag{3.1}$$

where the initial data  $\mathbf{u}_0^m$  and  $\mathbf{u}_1^m$  are, respectively, the projections of  $\mathbf{u}_0$  and  $\mathbf{u}_1$  onto the finite dimensional space  $\mathbf{E}^m$ . Clearly, the assumption  $\mathbf{u}_0 = \mathbf{0}$  implies  $\mathbf{u}_0^m = \mathbf{0}$ , and  $\mathbf{u}_1^m$  can be expanded as follows (cf., e.g., Theorem 4.9-1 of [34])

$$\mathbf{u}_1^m = \sum_{k=1}^m \left( \int_{\Omega} u_{i,1} w_i^k \, dx \right) \mathbf{w}^k.$$

■

Since the elements of the Hilbert basis do not depend on the time variable we can take the coefficients  $c_k$  as well as their derivatives outside the integral sign, getting a  $m \times m$  nonlinear system of second order ordinary differential equations with respect to the variable  $t$ . Such a system can be rewritten in the form

$$\begin{aligned}
 2\rho \ddot{\mathbf{C}}(t) &= \left( - \int_{\Omega} A^{ijkl} e_{k||l}(\mathbf{w}^r) e_{i||j}(\mathbf{w}^p) \, dx \right)_{p,r=1}^m \mathbf{C}(t) \\
 &+ \frac{1}{\kappa} \left( \int_{\Omega} (\{[\mathbf{I} + (\mathbf{C}(t) \cdot (w_i^1 \dots w_i^m)) \mathbf{e}^i] \cdot \mathbf{q}\}^-) (w_i^p \mathbf{e}^i \cdot \mathbf{q}) \, dx \right)_{p=1}^m \\
 &+ \left( \int_{\Omega} f^i(t) w_i^p \, dx \right)_{p=1}^m
 \end{aligned} \tag{3.2}$$

where  $\mathbf{C}(t) := (c_1(t) \dots c_m(t))$ .

It can be observed that

$$|b^- - a^-| \leq |b - a|, \quad \text{for all } a, b \in \mathbb{R}. \tag{3.3}$$

As a result, the right hand side of (3.2) is Lipschitz continuous in  $\mathbb{R}^m$  uniformly with respect to  $t$ , since it does not explicitly depend on  $t$ . By the Cauchy-Lipschitz theorem (cf., e.g., Theorem 3.8-1 of [34]), we deduce that for each integer  $m \geq 1$

there exists a unique global solution  $\mathbf{u}^m$  to Problem  $\mathcal{P}_m(\kappa; \Omega)$ , defined a.e. over the interval  $(0, T)$ , such that

$$\begin{aligned} \mathbf{u}^m &\in L^\infty(0, T; \mathbf{E}^m), \\ \dot{\mathbf{u}}^m &\in L^\infty(0, T; \mathbf{E}^m), \\ \ddot{\mathbf{u}}^m &\in L^\infty(0, T; \mathbf{E}^m). \end{aligned} \quad (3.4)$$

(b) *Energy estimates for the Galerkin scheme.* Let us multiply the variational equations in Problem  $\mathcal{P}_m(\kappa; \Omega)$  by  $\dot{c}_k(t)$ , with  $0 < t < T$ , and sum with respect to  $k$  varying in the discrete set  $\{1, \dots, m\}$ . As a result, we obtain that the penalised variational equations in Problem  $\mathcal{P}_m(\kappa; \Omega)$  take the form

$$\begin{aligned} &\rho \frac{d}{dt} \int_{\Omega} \dot{u}_i^m(t) \dot{u}_i^m(t) dx \\ &\quad + \frac{1}{2} \frac{d}{dt} \int_{\Omega} A^{ijkl} e_{k||l}(\mathbf{u}^m(t)) e_{i||j}(\mathbf{u}^m(t)) dx \\ &\quad + \frac{1}{2\kappa} \frac{d}{dt} \left( \int_{\Omega} \left( \{[\mathbf{I} + u_i^m(t) \mathbf{e}^i] \cdot \mathbf{q}\}^- \right)^2 dx \right) \\ &= \int_{\Omega} f^i(t) \dot{u}_i^m(t) dx, \end{aligned} \quad (3.5)$$

for a.a.  $t \in (0, T)$ .

Observe that the differentiation of the negative part is obtained as a result of the same computational steps as in Stampacchia's theorem (cf., e.g., [35]), together with an application of Theorem 8.28 of [3]. The change in sign of the penalty term is due to the properties of Heavyside function.

Carrying out an integration over the interval  $(0, t)$ , where  $0 < t \leq T$ , changes (3.5) into

$$\begin{aligned} &\rho \int_{\Omega} \dot{u}_i^m(t) \dot{u}_i^m(t) dx + \frac{1}{2} \int_{\Omega} A^{ijkl} e_{k||l}(\mathbf{u}^m(t)) e_{i||j}(\mathbf{u}^m(t)) dx \\ &\quad + \frac{1}{2\kappa} \int_{\Omega} \left( \{[\mathbf{I} + u_i^m(t) \mathbf{e}^i] \cdot \mathbf{q}\}^- \right)^2 dx \\ &= \rho \int_{\Omega} u_{i,1}^m u_{i,1}^m dx + \frac{1}{2} \int_{\Omega} A^{ijkl} e_{k||l}(\mathbf{u}_0^m) e_{i||j}(\mathbf{u}_0^m) dx \\ &\quad + \frac{1}{2\kappa} \int_{\Omega} \left( \{[\mathbf{I} + u_i^m(0) \mathbf{e}^i] \cdot \mathbf{q}\}^- \right)^2 dx \\ &\quad + \int_0^t \int_{\Omega} f^i(\tau) \dot{u}_i^m(\tau) dx d\tau. \end{aligned} \quad (3.6)$$

Let us observe that the second and the third integral in the left hand side of (3.6) vanish, since  $\mathbf{u}_0 = \mathbf{0}$  and since  $\mathbf{I} \cdot \mathbf{q} > 0$  in  $\bar{\Omega}$ . By Cauchy-Schwarz inequality, there

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exists a constant  $C > 0$  such that

$$\begin{aligned}
 & \int_0^t \int_{\Omega} f^i(\tau) \dot{u}_i^m(\tau) \, dx \, d\tau \\
 & \leq \left( \int_0^T \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)}^2 \, dt \right)^{1/2} \left( \int_0^t \|\dot{\mathbf{u}}^m(\tau)\|_{\mathbf{L}^2(\Omega)}^2 \, d\tau \right)^{1/2} \\
 & \leq \frac{1}{2} \left( \int_0^T \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)}^2 \, dt + \int_0^t \|\dot{\mathbf{u}}^m(\tau)\|_{\mathbf{L}^2(\Omega)}^2 \, d\tau \right).
 \end{aligned} \tag{3.7}$$

By the uniform positive-definiteness of the elasticity tensor  $(A^{ijkl})$ , Korn's inequality (Theorem 1.1), (3.6), and (3.7), we obtain that there exists a real constant  $\tilde{C} > 0$  independent of  $\mathbf{u}^m$  (and so independent of  $t$ ,  $m$  and  $\kappa$ ) for which the following estimate holds

$$\begin{aligned}
 & \frac{1}{\tilde{C}} \left\{ \|\dot{\mathbf{u}}^m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{u}^m(t)\|_{\mathbf{V}(\Omega)}^2 \right\} \\
 & \quad + \frac{1}{\kappa \tilde{C}} \left\| \{[\mathbf{I} + u_i^m(t) \mathbf{e}^i] \cdot \mathbf{q}\}^- \right\|_{\mathbf{L}^2(\Omega)}^2 \\
 & \leq \|\mathbf{u}_1\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{f}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}^2 \\
 & \quad + \int_0^t \left\{ \|\dot{\mathbf{u}}^m(\tau)\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{u}^m(\tau)\|_{\mathbf{V}(\Omega)}^2 \right\} \, d\tau \\
 & \quad + \frac{1}{\kappa} \int_0^t \left\| \{[\mathbf{I} + u_i^m(\tau) \mathbf{e}^i] \cdot \mathbf{q}\}^- \right\|_{\mathbf{L}^2(\Omega)}^2 \, d\tau.
 \end{aligned} \tag{3.8}$$

An application of the Gronwall's inequality (Theorem 2.1) with  $a \equiv \tilde{C} > 0$  and

$$b \equiv \tilde{C} \left( \|\mathbf{u}_1\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{f}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}^2 \right) \geq 0$$

gives the following uniform upper bound

$$\begin{aligned}
 & \int_0^t \left\{ \|\dot{\mathbf{u}}^m(\tau)\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{u}^m(\tau)\|_{\mathbf{V}(\Omega)}^2 \right\} \, d\tau + \frac{1}{\kappa} \int_0^t \left\| \{[\mathbf{I} + u_i^m(\tau) \mathbf{e}^i] \cdot \mathbf{q}\}^- \right\|_{\mathbf{L}^2(\Omega)}^2 \, d\tau \\
 & \leq \tilde{C} T e^{\tilde{C} T} \left\{ \|\mathbf{u}_1\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{f}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}^2 \right\} =: c(\mathbf{u}_1, \mathbf{f}).
 \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
 & (\mathbf{u}^m)_{m=1}^\infty \text{ is uniformly bounded in } L^\infty(0, T; \mathbf{V}(\Omega)), \\
 & (\dot{\mathbf{u}}^m)_{m=1}^\infty \text{ is uniformly bounded in } L^\infty(0, T; \mathbf{L}^2(\Omega)),
 \end{aligned} \tag{3.9}$$

and, moreover, by (3.8), there exists a positive uniform constant  $L$  such that

$$0 \leq \left\| \{[\mathbf{I} + u_i^m \mathbf{e}^i] \cdot \mathbf{q}\}^- \right\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leq \sqrt{L\kappa}. \tag{3.10}$$

Since the following direct sum decomposition holds true

$$\mathbf{V}(\Omega) = \mathbf{E}^m \oplus (\mathbf{E}^m)^\perp,$$

we get that for any  $\mathbf{v} \in \mathbf{V}(\Omega)$ , with  $\|\mathbf{v}\|_{\mathbf{V}(\Omega)} \leq 1$  and a.a.  $t \in (0, T)$ , the variational equations in Problem  $\mathcal{P}_m(\kappa; \Omega)$  give

$$\begin{aligned} |\mathbf{v}^*(\Omega) \langle \ddot{u}_i^m(t) \mathbf{e}^i, v_j \mathbf{e}^j \rangle_{\mathbf{V}(\Omega)}| &\leq \|\mathbf{f}\|_{L^\infty(0, T; \mathbf{L}^2(\Omega))} + C \|\mathbf{u}^m\|_{L^\infty(0, T; \mathbf{V}(\Omega))} \\ &+ \frac{1}{\kappa} \left\| \{[\mathbf{I} + u_i^m \mathbf{e}^i] \cdot \mathbf{q}\}^- \right\|_{L^\infty(0, T; \mathbf{L}^2(\Omega))}, \end{aligned}$$

and, by (3.9) and (3.10), we thus infer that there exists a constant  $C_\kappa > 0$ , independent of  $m$ , such that

$$\|\ddot{\mathbf{u}}^m\|_{L^\infty(0, T; \mathbf{V}^*(\Omega))} \leq C_\kappa. \quad (3.11)$$

(c) *Passage to the limit and retrieval of Problem  $\mathcal{P}(\kappa; \Omega)$ .* By (3.9), (3.10) and (3.11) we can infer that there exist subsequences, still denoted  $(\mathbf{u}^m)_{m=1}^\infty$ ,  $(\dot{\mathbf{u}}^m)_{m=1}^\infty$  and  $(\ddot{\mathbf{u}}^m)_{m=1}^\infty$  such that the following convergences take place

$$\begin{aligned} \mathbf{u}^m &\overset{*}{\rightharpoonup} \mathbf{u}_\kappa, & \text{in } L^\infty(0, T; \mathbf{V}(\Omega)) \text{ as } m \rightarrow \infty, \\ \dot{\mathbf{u}}^m &\overset{*}{\rightharpoonup} \dot{\mathbf{u}}_\kappa, & \text{in } L^\infty(0, T; \mathbf{L}^2(\Omega)) \text{ as } m \rightarrow \infty, \\ \ddot{\mathbf{u}}^m &\overset{*}{\rightharpoonup} \ddot{\mathbf{u}}_\kappa, & \text{in } L^\infty(0, T; \mathbf{V}^*(\Omega)) \text{ as } m \rightarrow \infty, \\ \kappa^{-1} \{[\mathbf{I} + u_i^m \mathbf{e}^i] \cdot \mathbf{q}\}^- &\overset{*}{\rightharpoonup} \chi_\kappa, & \text{in } L^\infty(0, T; \mathbf{L}^2(\Omega)) \text{ as } m \rightarrow \infty. \end{aligned} \quad (3.12)$$

By the Sobolev embedding theorem (Theorem 10.1.20 of [29]), we obtain

$$\begin{aligned} \mathbf{u}^m &\rightharpoonup \mathbf{u}_\kappa, & \text{in } \mathcal{C}^0([0, T]; \mathbf{L}^2(\Omega)) \text{ as } m \rightarrow \infty, \\ \dot{\mathbf{u}}^m &\rightharpoonup \dot{\mathbf{u}}_\kappa, & \text{in } \mathcal{C}^0([0, T]; \mathbf{V}^*(\Omega)) \text{ as } m \rightarrow \infty, \end{aligned} \quad (3.13)$$

An application of Theorem 8.28 of [3] to the fourth convergence of the process (3.12) gives

$$\kappa^{-1} \{[\mathbf{I} + u_i^m \mathbf{e}^i] \cdot \mathbf{q}\}^- \rightharpoonup \chi_\kappa, \quad \text{in } L^2((0, T) \times \Omega) \text{ as } m \rightarrow \infty. \quad (3.14)$$

By (3.3), the first convergence of (3.13) and the weak convergence (3.14), Theorem 8.28 of [3] and Theorem 8.62 of [3], we are in a position to apply Theorem 9.13-2 of [34] (where the involved monotone operator is nothing but the *negative part operator*) and, so, to obtain

$$\chi_\kappa = \kappa^{-1} \{[\mathbf{I} + u_{i, \kappa} \mathbf{e}^i] \cdot \mathbf{q}\}^- \in L^2((0, T) \times \Omega). \quad (3.15)$$

We now verify that  $\mathbf{u}_\kappa$  is a solution to the penalised variational equations in Problem  $(\mathcal{P}(\kappa; \Omega))$ . Let  $\psi \in \mathcal{D}(0, T)$  and let  $\mu \geq 1$  be any integer. For each  $m \geq \mu$ , we have

$$\begin{aligned} &2\rho \int_0^T \int_\Omega \ddot{u}_i^m(t) v_i \, dx \psi(t) \, dt \\ &+ \int_0^T \int_\Omega A^{ijkl} e_{k||l}(\mathbf{u}^m(t)) e_{i||j}(\mathbf{v}) \, dx \psi(t) \, dt \\ &- \frac{1}{\kappa} \int_0^T \int_\Omega \left( \{[\mathbf{I} + u_i^m(t) \mathbf{e}^i] \cdot \mathbf{q}\}^- \right) (v_i \mathbf{e}^i \cdot \mathbf{q}) \, dx \psi(t) \, dt \\ &= \int_0^T \int_\Omega f^i(t) v_i \, dx \psi(t) \, dt, \end{aligned} \quad (3.16)$$

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for all  $\mathbf{v} \in \mathbf{E}^\mu$ .

Keeping in mind (3.3), the convergence process (3.12), (3.15), the arbitrariness of  $\psi \in \mathcal{D}(0, T)$ , as well as the fact that

$$\overline{\bigcup_{\mu \geq 1} \mathbf{E}^\mu}^{\|\cdot\|_{\mathbf{V}(\Omega)}} = \mathbf{V}(\Omega),$$

a passage to the limit as  $m \rightarrow \infty$  in (3.16) shows that  $\mathbf{u}_\kappa$  is a solution to the penalised variational equations in Problem  $\mathcal{P}(\kappa; \Omega)$ .

The last thing that we have to check is the validity of the initial conditions for  $\mathbf{u}_\kappa$ . Let us introduce the operator  $\mathbf{L}_0 : \mathcal{C}^0([0, T]; \mathbf{L}^2(\Omega)) \rightarrow \mathbf{L}^2(\Omega)$  defined in a way such that  $\mathbf{L}_0(\mathbf{v}) := \mathbf{v}(0)$ . Such an operator  $\mathbf{L}_0$  turns out to be linear and continuous and, therefore, by the first convergence of (3.13), we immediately get that

$$\mathbf{u}_\kappa(0) = \mathbf{0}, \quad \text{in } \mathbf{L}^2(\Omega).$$

Similarly, let us introduce the operator  $\mathbf{L}_1 : \mathcal{C}^0([0, T]; \mathbf{V}^*(\Omega)) \rightarrow \mathbf{V}^*(\Omega)$  defined in a way such that  $\mathbf{L}_1(\mathbf{v}) := \mathbf{v}(0)$ . Such an operator  $\mathbf{L}_1$  turns out to be linear and continuous and, therefore, by the second convergence of (3.13), we get that

$$\mathbf{u}_1^m \rightharpoonup \dot{\mathbf{u}}_\kappa(0), \quad \text{in } \mathbf{V}^*(\Omega).$$

By definition,  $\mathbf{u}_1^m$  is the projection of  $\mathbf{u}_1$  onto  $\mathbf{E}^m$ . We thus get that  $\mathbf{u}_\kappa$  is a solution of Problem  $\mathcal{P}(\kappa; \Omega)$ . This completes the proof.  $\square$

We are now in a position to prove the existence of solutions of Problem  $\mathcal{P}(\Omega)$ .

**Theorem 3.4.** *Problem  $\mathcal{P}(\Omega)$  admits a solution.*

**Proof.** By the energy estimate (3.8), it can be easily observed that there exists a positive constant  $c = c(\mathbf{u}_1, \mathbf{f})$  such that

$$\begin{aligned} & \frac{1}{\bar{C}} \left\{ \|\dot{\mathbf{u}}_\kappa\|_{L^\infty(0, T; \mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_\kappa\|_{L^\infty(0, T; \mathbf{V}(\Omega))}^2 \right\} \\ & + \frac{1}{\kappa \bar{C}} \left\| \{[\mathbf{I} + u_{i, \kappa} \mathbf{e}^i] \cdot \mathbf{q}\}^- \right\|_{L^\infty(0, T; \mathbf{L}^2(\Omega))}^2 \\ & \leq c(\mathbf{u}_1, \mathbf{f}), \end{aligned}$$

remains valid for any solution  $\mathbf{u}_\kappa$ , i.e., the sequences  $(\mathbf{u}_\kappa)_{\kappa > 0}$  and  $(\dot{\mathbf{u}}_\kappa)_{\kappa > 0}$  are uniformly bounded in  $L^\infty(0, T; \mathbf{V}(\Omega))$  and  $L^\infty(0, T; \mathbf{L}^2(\Omega))$ , respectively.

Moreover, the uniform boundedness of the sequence  $(\dot{\mathbf{u}}_\kappa)_{\kappa > 0}$  in  $L^1(0, T; \mathbf{V}^*(\Omega))$  can be obtained by considering, for a.a.  $0 < t < T$ , the partial differential equation associated with Problem  $\mathcal{P}(\kappa; \Omega)$ , i.e.,

$$2\rho \ddot{\mathbf{u}}_\kappa(t) + A\mathbf{u}_\kappa(t) - \frac{1}{\kappa} N\mathbf{u}_\kappa(t) = \mathbf{f}(t), \quad \text{in } \mathbf{V}^*(\Omega), \quad (3.17)$$

where the operator  $A : \mathbf{V}(\Omega) \rightarrow \mathbf{V}^*(\Omega)$  defined by

$$\mathbf{v}^*(\Omega) \langle A\mathbf{u}, \mathbf{v} \rangle_{\mathbf{V}(\Omega)} := \int_{\Omega} A^{ijkl} e_{k||l}(\mathbf{u}) e_{i||j}(\mathbf{v}) \, dx, \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}(\Omega),$$

is linear and continuous.

Similarly, we define the nonlinear operator  $N : \mathbf{V}(\Omega) \rightarrow \mathbf{V}^*(\Omega)$  as

$$\mathbf{V}^*(\Omega) \langle N\mathbf{u}, \mathbf{v} \rangle_{\mathbf{V}(\Omega)} := \int_{\Omega} \left( \{[\mathbf{I} + u_i \mathbf{e}^i] \cdot \mathbf{q}\}^- \right) (v_i \mathbf{e}^i \cdot \mathbf{q}) \, dx, \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}(\Omega).$$

Let us prove the uniform boundedness of the sequence  $(N\mathbf{u}_\kappa)_{\kappa>0}$  by observing that

$$\begin{aligned} & \frac{1}{\kappa} \int_0^T \left( \sup_{\substack{\mathbf{v} \in \mathbf{V}(\Omega) \\ \|\mathbf{v}\|_{\mathbf{V}(\Omega)} \leq 1}} \left| \mathbf{V}^*(\Omega) \langle N\mathbf{u}_\kappa(t), \mathbf{v} \rangle_{\mathbf{V}(\Omega)} \right| \right) dt \\ & \leq \|\mathbf{f}\|_{L^\infty(0,T; \mathbf{L}^2(\Omega))} + \|\mathbf{u}_\kappa\|_{L^\infty(0,T; \mathbf{V}(\Omega))} \\ & \quad + \sup_{\substack{\mathbf{v} \in \mathbf{V}(\Omega) \\ \|\mathbf{v}\|_{\mathbf{V}(\Omega)} \leq 1}} \left| \mathbf{V}^*(\Omega) \langle \dot{\mathbf{u}}_\kappa(T) - \dot{\mathbf{u}}_\kappa(0), \mathbf{v} \rangle_{\mathbf{V}(\Omega)} \right|, \end{aligned}$$

where the last term in the right hand side derives from an application of Corollary 10.1.26 of [29]. We make use of this strategy to gain insight into a uniform bound for the nonlinear term, since nothing is known about the boundedness of the sequence  $(\ddot{\mathbf{u}}_\kappa)_{\kappa>0}$  yet.

Observe that, by Theorem 3.2, the following chain of embeddings holds

$$\begin{aligned} L^\infty(0, T; \mathbf{V}(\Omega)) & \hookrightarrow L^\infty(0, T; \mathbf{L}^2(\Omega)) \hookrightarrow L^\infty(0, T; \mathbf{V}^*(\Omega)) \hookrightarrow L^1(0, T; \mathbf{V}^*(\Omega)) \\ & \hookrightarrow (L^\infty(0, T; \mathbf{V}^*(\Omega)))^* \hookrightarrow (\mathcal{C}^0([0, T]; \mathbf{L}^2(\Omega)))^* \cong \mathcal{M}([0, T]; \mathbf{L}^2(\Omega)). \end{aligned}$$

An application of (3.12) and (3.13) thus gives that the sequence  $(N\mathbf{u}_\kappa)_{\kappa>0}$  is bounded in  $L^1(0, T; \mathbf{V}(\Omega))$ . As a result, it turns out that

$$(\ddot{\mathbf{u}}_\kappa)_{\kappa>0} \text{ is bounded in } (\mathcal{C}^0([0, T]; \mathbf{L}^2(\Omega)))^*.$$

Hence, up to passing to a subsequence, we get that the following convergence process takes place

$$\begin{aligned} \mathbf{u}_\kappa & \overset{*}{\rightharpoonup} \mathbf{u}, \quad \text{in } L^\infty(0, T; \mathbf{V}(\Omega)) \text{ as } \kappa \rightarrow 0, \\ \dot{\mathbf{u}}_\kappa & \overset{*}{\rightharpoonup} \dot{\mathbf{u}}, \quad \text{in } L^\infty(0, T; \mathbf{L}^2(\Omega)) \text{ as } \kappa \rightarrow 0, \\ \ddot{\mathbf{u}}_\kappa & \overset{*}{\rightharpoonup} \tilde{\mathbf{u}}, \quad \text{in } (\mathcal{C}^0([0, T]; \mathbf{L}^2(\Omega)))^* \text{ as } \kappa \rightarrow 0. \end{aligned} \tag{3.18}$$

We immediately deduce, by Theorem 3.2, that there exists a unique vector-valued measure  $\boldsymbol{\mu} \in \mathcal{M}([0, T]; \mathbf{L}^2(\Omega))$  such that

$$\langle \langle \tilde{\mathbf{u}}(t), \boldsymbol{\sigma}(t) \rangle \rangle_{\mathbf{L}^2(\Omega)} = \int_0^T \int_{\Omega} d\mu_i(t) \sigma_i(t) \, dx \, dt,$$

for all  $\boldsymbol{\sigma} \in \mathcal{C}^0([0, T]; \mathbf{L}^2(\Omega))$ . Clearly, the vector-valued measure  $\boldsymbol{\mu}$  is regular (cf., e.g., [31]).

By Theorem 3.1, the following convergence holds, up to passing to a subsequence

$$\mathbf{u}_\kappa \rightarrow \mathbf{u}, \quad \text{in } \mathcal{C}^0([0, T]; \mathbf{L}^2(\Omega)). \tag{3.19}$$

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Besides, by (3.10) we have

$$0 \leq \left\| \{[\mathbf{I} + u_{i,\kappa} \mathbf{e}^i] \cdot \mathbf{q}\}^- \right\|_{L^\infty(0,T;L^2(\Omega))} \leq \sqrt{L\kappa}. \quad (3.20)$$

Consequently, by (3.3), (3.19) and (3.20), we get

$$\{[\mathbf{I} + u_i(t) \mathbf{e}^i] \cdot \mathbf{q}\}^- = 0, \quad \text{for a.a. } t \in (0, T), \quad (3.21)$$

i.e.,  $\mathbf{u}(t) \in \mathbf{U}(\Omega)$ , for a.a.  $t \in (0, T)$ .

Given any  $\mathbf{v} \in \mathcal{D}(0, T; \mathbf{V}(\Omega))$  such that  $\mathbf{v}(t) \in \mathbf{U}(\Omega)$  for a.a.  $t \in (0, T)$ , we use  $(\mathbf{v} - \mathbf{u}_\kappa)$  as a test function in the variational equations of Problem  $\mathcal{P}(\kappa; \Omega)$ , getting

$$\begin{aligned} & 2\rho \langle \langle \ddot{u}_{i,\kappa}(t) \mathbf{e}^i, (v_j(t) - u_{j,\kappa}(t)) \mathbf{e}^j \rangle \rangle_{L^2(\Omega)} \\ & + \int_0^T \int_\Omega A^{ijkl} e_{k||l}(\mathbf{u}_\kappa(t)) e_{i||j}(\mathbf{v}(t) - \mathbf{u}_\kappa(t)) \, dx \, dt \\ & - \frac{1}{\kappa} \int_0^T \int_\Omega (\{[\mathbf{I} + u_{i,\kappa}(t) \mathbf{e}^i] \cdot \mathbf{q}\}^-) ((v_i(t) - u_{i,\kappa}(t)) \mathbf{e}^i \cdot \mathbf{q}) \, dx \, dt \\ & = \int_0^T \int_\Omega f^i(t) (v_i(t) - u_{i,\kappa}(t)) \, dx \, dt. \end{aligned} \quad (3.22)$$

Besides, we observe that the third integral term of (3.22) is such that

$$\frac{1}{\kappa} \int_0^T \int_\Omega \{[\mathbf{I} + u_{i,\kappa}(t) \mathbf{e}^i] \cdot \mathbf{q}\}^- ((v_i(t) - u_{i,\kappa}(t)) \mathbf{e}^i \cdot \mathbf{q}) \, dx \, dt \geq 0, \quad (3.23)$$

for all  $\kappa > 0$ , since  $\mathbf{v}(t) \in \mathbf{U}(\Omega)$  for a.a.  $t \in (0, T)$ .

By virtue of the continuity of the mappings  $\tilde{e}_{i||j}$  and the convergence process (3.18) we get that, for all  $\mathbf{v} \in L^2(0, T; \mathbf{V}(\Omega))$ , the mapping

$$\mathbf{u} \in L^2(0, T; \mathbf{V}(\Omega)) \rightarrow \int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i||j}(\mathbf{u})(t) e_{k||l}(\mathbf{v}) \, dx \, dt$$

is linear and continuous. An application of the convergence process (3.18) thus gives

$$\int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i||j}(\mathbf{u}_\kappa)(t) e_{k||l}(\mathbf{v}) \, dx \, dt \rightarrow \int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i||j}(\mathbf{u})(t) e_{k||l}(\mathbf{v}) \, dx \, dt, \quad (3.24)$$

as  $\kappa \rightarrow 0$ .

By the convergence process (3.18), and (3.19) we infer the following convergence

$$\langle \langle \ddot{u}_{i,\kappa}(t) \mathbf{e}^i, (v_j(t) - u_{j,\kappa}(t)) \mathbf{e}^j \rangle \rangle \rightarrow \langle \langle \tilde{u}_i(t) \mathbf{e}^i, (v_j(t) - u_j(t)) \mathbf{e}^j \rangle \rangle, \quad (3.25)$$

as  $\kappa \rightarrow 0$ .

By the convergence process (3.18), and the continuity of the bilinear form

$$(\mathbf{u}, \mathbf{v}) \in L^2(0, T; \mathbf{V}(\Omega)) \times L^2(0, T; \mathbf{V}(\Omega)) \rightarrow \int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i||j}(\mathbf{u})(t) \tilde{e}_{k||l}(\mathbf{v})(t) \, dx \, dt,$$

we obtain, in particular, that

$$\int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i||j}(\mathbf{u})(t) \tilde{e}_{k||l}(\mathbf{u})(t) \, dx \, dt \leq \liminf_{\kappa \rightarrow 0} \int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i||j}(\mathbf{u}_\kappa)(t) \tilde{e}_{k||l}(\mathbf{u}_\kappa)(t) \, dx \, dt. \quad (3.26)$$

Combining (3.23), (3.24), (3.25), and (3.26), we immediately deduce that the limit  $\mathbf{u}$  is a solution to the variational inequalities in Problem  $(\mathcal{P}(\Omega))$ .

We can observe that, by the convergence process (3.18), the vector-valued measure  $\boldsymbol{\mu} \in \mathcal{M}([0, T]; \mathbf{L}^2(\Omega))$  can be interpreted as the acceleration of the limit displacement  $\mathbf{u}$ . Indeed, by the classical definition of weak derivative, we have that

$$\int_0^T \dot{\mathbf{u}}_\kappa(t) \varphi'(t) dt = - \int_0^T \ddot{\mathbf{u}}_\kappa(t) \varphi(t) dt, \quad \text{for all } \varphi \in \mathcal{D}(0, T).$$

By the properties of Lebesgue-Bochner integrals we have that, for all  $\mathbf{v} \in \mathbf{L}^2(\Omega)$  and all  $\varphi \in \mathcal{D}(0, T)$ , it results

$$\int_\Omega \int_0^T \dot{u}_{i,\kappa}(t) (\varphi'(t) v_i) dt dx = - \int_\Omega \int_0^T \ddot{u}_{i,\kappa}(t) (\varphi(t) v_i) dt dx,$$

so that, letting  $\kappa \rightarrow 0$  (see Comment 3 of Chapter 4 of [36]) gives

$$\begin{aligned} \int_\Omega \left( \int_0^T \dot{u}_i(t) \varphi'(t) dt \right) v_i dx &= \int_0^T \left( \int_\Omega \dot{u}_i(t) v_i dx \right) \varphi'(t) dt \\ &= - \langle \langle \ddot{\mathbf{u}}(t), \varphi(t) \mathbf{v} \rangle \rangle_{\mathbf{L}^2(\Omega)} = - \int_0^T \int_\Omega d\mu_i(t) (v_i \varphi(t)) dt, \end{aligned}$$

where the first equality holds by Fubini's theorem, the second equality holds by Theorem 3.2, the third convergence of the process (3.18) and the definition of weak derivative, and, finally, the last equality holds true by Theorem 3.2.

To sum up, we have obtained that

$$\int_0^T \int_\Omega \dot{u}_i(t) v_i dx \varphi'(t) dt = - \int_0^T \int_\Omega d\mu_i(t) v_i \varphi(t) dt,$$

for all  $\varphi \in \mathcal{D}(0, T)$  and all  $\mathbf{v} \in \mathbf{L}^2(\Omega)$ .

We can thus regard the vector-valued measure  $\boldsymbol{\mu}$  as the second weak derivative with respect to  $t \in (0, T)$  of the limit displacement  $\mathbf{u}$  obtained via the process (3.18). This justifies the following change in the notation

$$\boldsymbol{\mu} = \ddot{\mathbf{u}},$$

and the *symbol*  $\ddot{\mathbf{u}}$  is now an element of  $\mathcal{M}([0, T]; \mathbf{L}^2(\Omega))$ .

In conclusion, we have shown that  $\mathbf{u}$  is in the set  $\mathbf{U}(\Omega)$  and that satisfies the variational inequalities in Problem  $\mathcal{P}(\Omega)$ , namely,

$$\begin{aligned} &2\rho \langle \langle d\ddot{\mathbf{u}}(t), \mathbf{v}(t) - \mathbf{u}(t) \rangle \rangle_{\mathbf{L}^2(\Omega)} \\ &+ \int_0^T \int_\Omega A^{ijkl} e_{k||l}(\mathbf{u}(t)) e_{i||j}(\mathbf{v}(t) - \mathbf{u}(t)) dx dt \\ &\geq \int_0^T \int_\Omega f^i(t) (v_i(t) - u_i(t)) dx dt, \end{aligned}$$

for all  $\mathbf{v} \in \mathcal{D}(0, T; \mathbf{V}(\Omega))$  such that  $\mathbf{v}(t) \in \mathbf{U}(\Omega)$  for a.a.  $t \in (0, T)$ .



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The last thing to check is the validity of the initial conditions for  $\mathbf{u}$ . Let us introduce the operator  $\mathbf{L}_0 : \mathcal{C}^0([0, T]; \mathbf{L}^2(\Omega)) \rightarrow \mathbf{L}^2(\Omega)$  defined in a way such that  $\mathbf{L}_0(\mathbf{v}) := \mathbf{v}(0)$ . Such an operator turns out to be linear and continuous and we then immediately obtain  $\mathbf{u}(0) = \mathbf{0}$ .

For what concerns the initial condition for the first derivative of  $\mathbf{u}$  with respect to  $t$ , we present a new argument, different from the ones used in [19], [20], and [21]. For sake of clarity, we present all the computations in detail. Observe that the assumption  $\mathbf{I} \cdot \mathbf{q} > 0$  in  $\bar{\Omega}$  gives

$$\int_{\Omega} (\mathbf{I} \cdot \mathbf{q}) \varphi \, dx > 0,$$

for all  $\varphi \in \mathcal{D}(\Omega)$  such that  $\varphi \geq 0$  a.e. in  $\Omega$ , with  $\varphi \not\equiv 0$ . By virtue of (3.19) and the fact that  $\mathbf{u}_{\kappa}(0) = \mathbf{0}$ , for any given  $\varphi \in \mathcal{D}(\Omega)$  such that  $\varphi \geq 0$  a.e. in  $\Omega$ , with  $\varphi \not\equiv 0$ , we deduce the existence of a positive number  $t_{0,\kappa}(\varphi)$  such that

$$\int_{\Omega} ([\mathbf{I} + u_{i,\kappa}(t)\mathbf{e}^i] \cdot \mathbf{q}) \varphi \, dx > 0,$$

for all  $0 \leq t \leq t_{0,\kappa}(\varphi)$ .

Assume, furthermore, that the following “uniformity property” holds: *The number*

$$\bar{t}_0 := \inf_{\kappa > 0} \inf_{\substack{\varphi \in \mathcal{D}(\Omega) \\ \varphi \geq 0 \\ \varphi \not\equiv 0}} t_{0,\kappa}(\varphi)$$

is  $> 0$ .

Let  $t_{0,\kappa} := \inf_{\substack{\varphi \in \mathcal{D}(\Omega) \\ \varphi \geq 0 \\ \varphi \not\equiv 0}} t_{0,\kappa}(\varphi)$  and observe that, by virtue of the “uniformity property”, we have

$$\int_{\Omega} ([\mathbf{I} + u_{i,\kappa}(t)\mathbf{e}^i] \cdot \mathbf{q}) \varphi \, dx > 0, \quad \text{for all nonzero nonnegative } \varphi \in \mathcal{D}(\Omega),$$

for all  $0 \leq t \leq t_{0,\kappa}$ .

By the “uniformity property”, we obtain

$$[\mathbf{I} + u_{i,\kappa}(t)\mathbf{e}^i] \cdot \mathbf{q} > 0 \quad \text{a.e. in } \Omega,$$

for all  $0 \leq t \leq \bar{t}_0$ , and for all  $\kappa > 0$ .

As a result, for a.a.  $0 < t < \bar{t}_0$ , the equation (3.17) takes the simpler form

$$2\rho \ddot{\mathbf{u}}_{\kappa}(t) + A\mathbf{u}_{\kappa}(t) = \mathbf{f}(t), \quad \text{in } \mathbf{V}^*(\Omega), \quad (3.27)$$

since we have  $N\mathbf{u}_{\kappa}(t) = \mathbf{0}$  in  $\mathbf{V}^*(\Omega)$ , for a.a.  $0 < t < \bar{t}_0$ , for all  $\kappa > 0$ .

Since  $\mathbf{f} = (f^i) \in L^\infty(0, T; \mathbf{L}^2(\Omega))$ , we deduce that  $(\ddot{\mathbf{u}}_{\kappa})_{\kappa > 0}$  is bounded in  $L^\infty(0, \bar{t}_0; \mathbf{V}^*(\Omega))$  and, up to extracting a subsequence, we get that the following convergence takes place as  $\kappa \rightarrow 0$

$$\ddot{\mathbf{u}}_{\kappa} \rightharpoonup \ddot{\mathbf{u}}, \quad \text{in } L^2(0, \bar{t}_0; \mathbf{V}^*(\Omega)). \quad (3.28)$$

Hence, by the convergence process (3.28) and the Sobolev embedding theorem (Theorem 10.1.20 of [29]), the following convergence holds

$$\dot{\mathbf{u}}_\kappa \rightharpoonup \dot{\mathbf{u}}, \quad \text{in } \mathcal{C}^0([0, \bar{t}_0]; \mathbf{V}^*(\Omega)). \quad (3.29)$$

Let us thus introduce the operator

$$\bar{\mathbf{L}}_1 : \mathcal{C}^0([0, \bar{t}_0]; \mathbf{V}^*(\Omega)) \rightarrow \mathbf{V}^*(\Omega)$$

defined in a way such that  $\bar{\mathbf{L}}_1(\mathbf{v}) := \mathbf{v}(0)$ , for all  $\mathbf{v} \in \mathcal{C}^0([0, \bar{t}_0]; \mathbf{V}^*(\Omega))$ . Such an operator  $\bar{\mathbf{L}}_1$  is linear and continuous and, by the convergence (3.29) and the reflexivity of the space  $\mathbf{V}^*(\Omega)$ , we are in a position to recover the initial condition  $\dot{\mathbf{u}}(0) = \mathbf{u}_1$ .

In conclusion, we have shown that  $\mathbf{u}$  is a solution of Problem  $\mathcal{P}(\Omega)$  and the proof is thus complete.  $\square$

#### 4. About the uniqueness of the solution

To conclude the investigation, we observe that the following phenomenon that occurs in the early stage. We can show that, by virtue of the condition  $\mathbf{I} \cdot \mathbf{q} > 0$  in  $\bar{\Omega}$ , the equation

$$2\rho\ddot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}(t) = \mathbf{f}(t), \quad \text{in } \mathbf{V}^*(\Omega), \quad (4.1)$$

admits a unique solution, for a.a.  $0 < t < \bar{t}_0$ . In this direction, we follow [37] (Theorem 4, Section 7.2).

To see this, let us show that the only solution to the initial value problem

$$\begin{aligned} 2\rho\ddot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}(t) &= \mathbf{0}, & \text{in } \mathbf{V}^*(\Omega), & \text{for a.a. } 0 < t < \bar{t}_0, \\ \mathbf{u}(0) &= \mathbf{0}, \\ \dot{\mathbf{u}}(0) &= \mathbf{0}, \end{aligned} \quad (4.2)$$

is  $\mathbf{u} \equiv \mathbf{0}$ . To this aim, for any fixed  $0 \leq s \leq \bar{t}_0$ , let us define the function

$$\mathbf{v}(t) := \begin{cases} \int_t^s \mathbf{u}(\tau) \, d\tau, & 0 \leq t \leq s, \\ \mathbf{0} & , s < t \leq \bar{t}_0, \end{cases}$$

a.e. in  $\Omega$ , with  $\mathbf{v} \in \mathcal{C}^0([0, \bar{t}_0]; \mathbf{V}(\Omega))$ . Since  $\dot{\mathbf{u}}(0) = \mathbf{0} = \mathbf{v}(s)$ , an application of the integration by parts formula (Corollary 10.1.26 of [29]) gives

$$\int_0^s \left\{ -2\rho \mathbf{v}^*(\Omega) \langle \dot{\mathbf{u}}_i(t) \mathbf{e}^i, \dot{\mathbf{v}}_j(t) \mathbf{e}^j \rangle_{\mathbf{V}(\Omega)} + \int_{\Omega} A^{ijkl} e_{k||l}(\mathbf{u}(t)) e_{i||j}(\mathbf{v}(t)) \, dx \right\} dt = 0.$$

Since  $\dot{\mathbf{v}}(t) = -\mathbf{u}(t)$ , for all  $0 \leq t \leq s$ , the latter formula becomes

$$\int_0^s \left\{ 2\rho \mathbf{v}^*(\Omega) \langle \dot{\mathbf{u}}_i(t) \mathbf{e}^i, \mathbf{u}_j(t) \mathbf{e}^j \rangle_{\mathbf{V}(\Omega)} + \int_{\Omega} A^{ijkl} e_{k||l}(\dot{\mathbf{v}}(t)) e_{i||j}(\mathbf{v}(t)) \, dx \right\} dt = 0.$$

Again, by integration by parts formula (Corollary 10.1.26 of [29]), we get

$$\begin{aligned} & \int_0^s \frac{d}{dt} \left( \rho \|\mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 - \frac{1}{2} \int_{\Omega} A^{ijkl} e_{k||l}(\mathbf{v}(t)) e_{i||j}(\mathbf{v}(t)) \, dx \right) dt \\ &= \rho \|\mathbf{u}(s)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} A^{ijkl} e_{k||l}(\mathbf{v}(0)) e_{i||j}(\mathbf{v}(0)) \, dx = 0. \end{aligned}$$

We thus infer,  $\|\mathbf{u}(s)\|_{\mathbf{L}^2(\Omega)} = 0$ , for all  $0 \leq s \leq \bar{t}_0$ . By the arbitrariness of  $s$ , we conclude that the solution  $\mathbf{u}$  is uniquely defined in the interval  $[0, \bar{t}_0]$ .

In conclusion, all the solutions to Problem  $\mathcal{P}(\Omega)$  coincide in the interval  $[0, \bar{t}_0]$ .

### 5. A sufficient condition to the “uniformity property”: The case of “dormant” forces

Let us recall the uniformity property that we used to prove Theorem 3.4: *The number*

$$\bar{t}_0 := \inf_{\kappa > 0} \inf_{\substack{\varphi \in \mathcal{D}(\Omega) \\ \varphi \geq 0 \\ \varphi \not\equiv 0}} t_{0,\kappa}(\varphi)$$

is  $> 0$ .

In this section we identify a simple sufficient condition that insures the validity of such a property. In what follows, apart from the already assumed condition  $\mathbf{u}_0 = \mathbf{0}$ , we also assume  $\mathbf{u}_1 = \mathbf{0}$ . An applied body force is said to be “dormant” if there exists a number  $0 < \tau_0 < T$  such that

$$f^i(t) = 0 \in L^2(\Omega),$$

for a.a.  $0 < t < \tau_0$ . As a result, if the applied body force  $\mathbf{f} = (f^i)$  under consideration is “dormant”, then, any number between 0 and  $\tau_0$  verifies the “uniformity property”.

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