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Planar sampling sets for the short-time Fourier transform

Philippe Jaming* and Michael Speckbacher†

*Univ. Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251
F-33400, Talence, France*

Abstract

This paper considers the problem of restricting the short-time Fourier transform to sets of nonzero measure in the plane. Thereby, we study under which conditions one has a sampling set and provide estimates of the corresponding sampling bound. In particular, we give a quantitative estimate for the lower sampling bound in the case of Hermite windows and derive a sufficient condition for a large class of windows in terms of a certain planar density. On the way, we prove a Remez-type inequality for polyanalytic functions.

MSC2010: 42C40, 46E15, 46E20, 42C15

Keywords: short-time Fourier transform, concentration estimates, sampling sets, polyanalytic Bargmann-Fock spaces, irregular Gabor frames, Remez-type inequalities

1 Introduction

In this paper, we investigate the existence and behavior of lower norm bounds for the problem of restricting the short-time Fourier transform

$$V_g f(z) := \int_{\mathbb{R}} f(t)g(t-x)e^{-2\pi i\xi t} dt, \quad z = x + i\xi \in \mathbb{C},$$

to a domain in \mathbb{C} of nonzero Lebesgue measure. This can also be viewed as a planar subsampling problem or a concentration problem.

More precisely, we are looking for conditions on a measurable set $\Omega \subset \mathbb{C}$ to be a *sampling set* (or *dominating set*) for the short-time Fourier transform V_g in the sense that there exists a constant C depending only on g and Ω such that

$$\|V_g f\|_{L^p(\mathbb{C})} \leq C \|V_g f\|_{L^p(\Omega)}, \quad \forall f \in M^p(\mathbb{R}),$$

where $M^p(\mathbb{R})$ denotes the *modulation space* associated to $L^p(\mathbb{C})$, *i.e.* the space of tempered distributions whose short-time Fourier transform is contained in $L^p(\mathbb{C})$. Moreover, we

*philippe.jaming@math.u-bordeaux.fr

†michael.speckbacher@u-bordeaux.fr

intend to estimate the *sampling constant* C that appears in this inequality in terms of the window g and geometric properties of Ω .

The question of existence of such a sampling constant has been addressed in different contexts during the last decades. One of the first instances of such a problem is in the context of the Paley-Wiener space of functions whose Fourier transform is supported in $[-W/2, W/2]$. Here, given $W > 0$, the task is to determine sets $\Omega \subset \mathbb{R}$ and the constant C such that

$$\|f\|_{L^2(\mathbb{R})} \leq C \|f\|_{L^2(\Omega)}, \quad \forall f \in L^2(\mathbb{R}) \text{ satisfying } \widehat{f}(\xi) = 0, \text{ if } |\xi| > W/2. \quad (1.1)$$

This question has applications in signal processing, but also in control of PDEs (*see e.g.* [8]) and local solvability of PDEs (*see e.g.* [31, Theorem 10.10]). The first solution was given by the well-known Logvinenko-Sereda theorem (Panejah [34, 35], Kacnelson [26] and Logvinenko-Sereda [29]), and the sampling constant has since been improved by Kovrijkine [27] to an essentially optimal quantitative estimates, *see also* [37]. Later, this approach was adapted to derive estimates of the sampling constant for Bergman spaces [30], functions with compactly supported Fourier-Bessel transform [18], model spaces [23] and finite expansions on compact manifolds [33].

Since discrete sampling sets for the Paley-Wiener space are required to satisfy a certain density condition, it is not surprising that this remains true for non-discrete sampling sets. In particular, the existence of a sampling constant C in (1.1) is equivalent to the property that each interval of a given fixed size contains at least a minimum fraction of the sampling set. More precisely, the validity of (1.1) is equivalent to Ω being *relatively dense*.

Let us denote by $\mathbb{D}(z, R) \subset \mathbb{C}$ the disc of radius $R > 0$ centered at $z \in \mathbb{C}$. Recall that a measurable set $\Omega \subset \mathbb{C}$ is called (γ, R) -dense if

$$\gamma = \inf_{z \in \mathbb{C}} \frac{|\Omega \cap \mathbb{D}(z, R)|}{|\mathbb{D}(z, R)|} > 0, \quad (1.2)$$

and *relatively dense* if there exist $\gamma, R > 0$ such that Ω is (γ, R) -dense. Janson, Peetre and Rochberg [25] and Ortega-Cerdà [32] proved that $\Omega \subset \mathbb{C}$ is a sampling set for the Bargmann-Fock space of analytic functions if and only if Ω is relatively dense. As a direct consequence, this result settles the question of determining sampling sets for the short-time Fourier transform with Gaussian window. Later, Ascensi extended this characterization to a larger class of window functions that are nonzero almost everywhere and satisfy certain decay conditions [5, Section 6.1].

However, all results up to this date are non-quantitative and thus do not provide estimates of the sampling constant. Moreover, the equivalence between sampling sets and relatively dense sets cannot be extended to general window functions. Although each sampling set is necessarily relatively dense [5, Theorem 10], the opposite is not true. Take for example g, f to be compactly supported. Then $V_g f$ is supported on a strip $\mathfrak{S} = [-S, S] + i\mathbb{R}$ in phase space. Taking $\Omega = \mathbb{C} \setminus \mathfrak{S}$, we see that Ω cannot be a sampling set but for every $R > 2S$, there exists $\gamma > 0$ such that Ω is (γ, R) -dense.

It is the principal goal of this contribution to establish quantitative estimates of the sampling bounds in the case of Hermite functions and to investigate under which conditions a (γ, R) -dense set (at specific scales R) is also a sampling set.

Hermite functions as windows for the short-time Fourier transform are particularly interesting for multiple reasons. First, Hermite functions are best concentrated in the time-frequency plane in many ways and also appear as optimizers of uncertainty principles for the short-time Fourier transform (see e.g. [9]). Further, they are also eigenfunctions of localization operators with radial weights [12] which makes them particularly interesting for multitaper time-frequency analysis [7, 40]. There are further connections to planar point processes [22] and there is also a natural connection to coherent states on higher Landau levels [1], via function spaces of polyanalytic functions [2]. Recall for further use in this paper that a polyanalytic function of order n is a solution of the higher-order Cauchy-Riemann equation $(\bar{\partial})^{n+1}F(z) = 0$ and can be written in the form $F(z) = \sum_{k=0}^n F_k(z)\bar{z}^k$ with the F_k 's being analytic.

Our main result is the following. It gives quantitative estimates for the case of Hermite windows (see (2.5) for the definition) and can also be formulated in terms of true polyanalytic functions, see Corollary 4.1.

Theorem 1.1 *Let $1 \leq p < \infty$, $\gamma, R > 0$ and h_n be the n -th Hermite function. Then there exists $\eta = \eta(n, R)$, $\sigma = \sigma(n, R)$ and a numerical constant $C > 0$ such that if $\Omega \subset \mathbb{C}$ is (γ, R) -dense for some scale $R > 0$ and $f \in M^p(\mathbb{R})$, then*

$$\|V_{h_n}f\|_{L^p(\mathbb{C})} \leq \eta \left(\frac{\gamma}{C}\right)^{-\sigma} \|V_{h_n}f\|_{L^p(\Omega)}. \quad (1.3)$$

A more precise estimate of η and σ will be given below. The proof follows in parts the strategy of Kovrijkine for the Paley-Wiener space [27]. The two key ingredients of this proof are Bernstein's inequality and Remez' inequality. In the course of proving Theorem 1.1, we derive a Remez-type inequality for polyanalytic functions from a result on plurisubharmonic functions [10]. On the other hand, the lack of Bernstein's inequality in \mathbb{C} is overcome by a local reproducing formula for the short-time Fourier transform with Hermite functions [4], a generalization of Seip's formula for the Bargmann Fock space [38].

The Remez inequality for polyanalytic functions is new and of independent interest. Let us give the statement here in a slightly simplified form:

Theorem 1.2 *Let $R > 0$, $\Omega \subset \mathbb{D}(0, R)$ be measurable, and F be polyanalytic of order n in $\mathbb{D}(0, 5R)$. If we write $M := \sup_{\mathbb{D}(0, 4R)} |F|$, and assume that $m := |F(0)| > 0$ then, for $1 \leq p \leq \infty$,*

$$\|F\|_{L^p(\mathbb{D}(0, R))} \leq \left(\frac{\kappa|\mathbb{D}(0, \rho)|}{|\Omega|}\right)^{c[\ln \frac{M}{m} + \vartheta] + \frac{1}{p}} \|F\|_{L^p(\Omega)},$$

where the constants c, κ are absolute constants and ϑ depends on n only.

During the second part of this paper, we change perspective and consider general window functions $g \in L^2(\mathbb{R})$. Thereby, we give a partial answer to the following problem:

Does there exist $R(g) > 0$ such that any (γ, R) -dense set Ω is a sampling set for $R \leq R(g)$?

We show that under the mild condition that the window function g is contained in the Sobolev space $H^1(\mathbb{R})$ (defined in (5.28)), there exists a scale R^* such that the above holds:

Theorem 1.3 *Let $g \in H^1(\mathbb{R})$ be compactly supported, then there exists $R(g)$ and $\kappa = \kappa(g)$ such that if $\Omega \subset \mathbb{C}$ is (γ, R) -dense at scale $R \leq R(g)$ and $f \in L^2(\mathbb{R})$, then*

$$\|V_g f\|_{L^2(\mathbb{C})} \leq \frac{\kappa}{\gamma^{1/2}} \|V_g f\|_{L^2(\Omega)}. \quad (1.4)$$

As an application, let us mention that, if $p \in \mathbb{C}[X, Y]$ is a polynomial in two variables, then for every $R > 0$, there exists ε, γ (depending only on R and p) such that the level set $\Omega := \{z \in \mathbb{C} : |p(z, \bar{z})| \geq \varepsilon\}$ is (γ, R) -dense (see e.g. [31, Section 10.4.2]). But then

$$\|V_g f\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon} \|p V_g f\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon} \|p V_g f\|_{L^2(\mathbb{C})}.$$

Together with Theorem 1.1 or Theorem 1.3, we obtain the following version of Heisenberg's inequality for the short-time Fourier transform (see e.g. [9, 20] for other versions of Heisenberg's inequality for the STFT):

Corollary 1.4 *Let $g \neq 0$ be either $g = h_n$ or $g \in H^1(\mathbb{R})$ with compact support. Let p be a polynomial of two variables. Then there exists a constant $C = C(g, p)$ such that, for every $f \in L^2(\mathbb{R})$*

$$\int_{\mathbb{C}} |p(z, \bar{z}) V_g f(z)|^2 dz \geq C \|f\|_2^2$$

where dz stands for the Lebesgue measure on \mathbb{C} .

Alternatively, sampling bounds can be obtained from upper bounds on the concentration problem for the complement of Ω . The use of large sieve methods has been introduced for the estimate of the sampling constant in (1.1) by Donoho-Logan [13]. This approach has recently been extended to the short-time Fourier transform with Hermite window, see [3, 4] and can also be adapted to finite spherical harmonic expansions on the sphere [24]. If the sets are “thicker” than generic dense sets, this leads to better constants than the one in this paper. Further results in this direction can be found in [16].

The paper is organized as follows. The next section is devoted to preliminaries. In Section 3, we focus on properties of polyanalytic functions. After some preliminary work (basic facts in Section 3.1 and a precised maximum principle in Section 3.2), we prove our Remez inequality for polyanalytic functions in Section 3.3. Section 4 is devoted to the proof of Theorem 1.1 while Section 5 is devoted to the proof of Theorem 1.3.

2 Preliminaries and Notation

Throughout this paper we will write $Q_R := \{x + i\xi \in \mathbb{C} : \max(|x|, |\xi|) \leq R/2\}$ for the square with sidelength R , $\mathbb{D}(z, R)$ for the disc in \mathbb{C} with radius $R > 0$ and center z ,

$B_{\mathbb{C}}(z, R)$ for the ball in \mathbb{C}^d , and $B_{\mathbb{R}}(z, R)$ for the restriction of $B_{\mathbb{C}}(z, R)$ to \mathbb{R}^d . Moreover, we use the following convention for the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t} dt,$$

and define the Hermite functions by

$$h_n(t) = c_n e^{\pi t^2} \left(\frac{d}{dt} \right)^n \left(e^{-2\pi t^2} \right), \quad (2.5)$$

where c_n is chosen such that $\|h_n\|_2 = 1$.

Let $z = x + i\xi \in \mathbb{C}$. The *time-frequency shift* $\pi(z)$ of a function $g : \mathbb{R} \rightarrow \mathbb{C}$ is defined as

$$\pi(z)g(t) := M_{\xi} T_x g(t) = e^{2\pi i \xi t} g(t - x),$$

where $T_x g(t) = g(t - x)$ denotes the translation, and $M_{\xi} g(t) = e^{2\pi i \xi t} g(t)$ the modulation operator. For $f, g \in L^2(\mathbb{R})$, the *short-time Fourier transform* of f with window g , defined as

$$V_g f(z) := \langle f, \pi(z)g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(t) \overline{g(t - x)} e^{-2\pi i \xi t} dt,$$

is a scalar multiple of an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{C})$. In particular, writing dz for the Lebesgue measure in \mathbb{C} ,

$$\int_{\mathbb{C}} |V_g f(z)|^2 dz = \|f\|_2^2 \|g\|_2^2, \quad \forall f, g \in L^2(\mathbb{R}).$$

Note that we may also define $V_g f(z) := \langle f, \pi(z)g \rangle_{L^2(\mathbb{R})}$ when $g \in \mathcal{S}(\mathbb{R})$ (the Schwartz class) and $f \in \mathcal{S}'(\mathbb{R})$ (a tempered distribution) and that $V_g f$ is then a locally bounded function.

As we derive lower bounds for general L^p -spaces, we need to recall the definition of modulation spaces which were introduced by Feichtinger [14]. Following [20], one can define the modulation space $M^p(\mathbb{R})$, $1 \leq p \leq \infty$ as the space of all tempered distributions f for which

$$\|f\|_{M^p(\mathbb{R})} := \|V_{h_0} f\|_{L^p(\mathbb{C})}$$

is finite. Note that $\|V_{h_n} f\|_{L^p(\mathbb{C})}$ is an equivalent norm on $M^p(\mathbb{R})$ for any $n \in \mathbb{N}_0$. For further reading on time-frequency analysis we refer to the standard textbooks [17, 20].

3 Polyanalytic Functions

3.1 Hermite Windows and Spaces of Polyanalytic Functions

A function $F : \mathbb{C} \rightarrow \mathbb{C}$ is called *polyanalytic of order n* if it satisfies the higher order Cauchy-Riemann equation $(\bar{\partial})^{n+1} F = 0$. In that case, F can be written as

$$F(z) = F(z, \bar{z}) = \sum_{k=0}^n F_k(z) \bar{z}^k, \quad (3.6)$$

where $F_0, \dots, F_n : \mathbb{C} \rightarrow \mathbb{C}$ are holomorphic functions.

The *true polyanalytic Bargmann transform* \mathcal{B}^{n+1} of a function $f \in L^2(\mathbb{R})$ is defined via the short-time Fourier transform of f using Hermite window h_n , see [2, Section 2.2]:

$$\mathcal{B}^{n+1}f(z) = V_{h_n}f(\bar{z})e^{\pi(z^2 - \bar{z}^2)/4}e^{\pi|z|^2/2}. \quad (3.7)$$

In particular, $|\mathcal{B}^{n+1}f(z)| = |V_{h_n}f(\bar{z})|e^{\pi|z|^2/2}$. The *polyanalytic Bargmann space* $\mathbf{F}_p^n(\mathbb{C})$ is defined as the space of all polyanalytic functions of order n such that

$$\|F\|_{\mathcal{L}^p}^p := \int_{\mathbb{C}} |F(z)|^p e^{-\pi p|z|^2/2} dz < \infty. \quad (3.8)$$

Moreover, the *true polyanalytic Bargmann space* $\mathcal{F}_p^n(\mathbb{C})$ is the subspace of $\mathbf{F}_p^n(\mathbb{C})$ consisting of all those functions F for which there exists an analytic function H such that

$$F(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} e^{\pi|z|^2} \left(\frac{d}{dz}\right)^n \left[e^{-\pi|z|^2} H(z) \right].$$

It was shown in [2, Section 3.2 and 3.3] that the images of the true polyanalytic Bargmann transform applied to the modulation spaces $M^p(\mathbb{R})$ are simply the true polyanalytic Bargmann spaces $\mathcal{F}_p^n(\mathbb{C})$, *i.e.* $\mathcal{B}^{n+1}(M^p(\mathbb{R})) = \mathcal{F}_p^n(\mathbb{C})$, and in particular, $\mathcal{B}^{n+1}(L^2(\mathbb{R})) = \mathcal{F}_2^n(\mathbb{C})$. Moreover, the polyanalytic Bargmann spaces can be written as the direct sum of the true polyanalytic Bargmann spaces, *i.e.* for $1 \leq p < \infty$

$$\mathbf{F}_p^n(\mathbb{C}) = \bigoplus_{k=0}^n \mathcal{F}_p^k(\mathbb{C}).$$

Let L_n be the n -th Laguerre polynomial given by the closed form $L_n(t) = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{k!} t^k$. In [4, Theorem 1] the following *local reproducing formula* is shown to hold for every $f \in M^\infty(\mathbb{R})$

$$V_{h_n}f(z) = \nu_n(R)^{-1} \int_{\mathbb{D}(z,R)} V_{h_n}f(w) \langle \pi(w)h_n, \pi(z)h_n \rangle dw, \quad (3.9)$$

where

$$\nu_n(R) := \int_0^{\pi R^2} L_n(t)^2 e^{-t} dt. \quad (3.10)$$

For the case $n = 0$, *i.e.* the case of the Gaussian window, this result can be deduced from Seip's local reproducing formula for the Bargmann-Fock space [38].

3.2 Maximum Modulus Principle for Polyanalytic Functions

For our proof of the Remez-type inequality for polyanalytic functions (Theorem 3.6), we need the maximum modulus principle for polyanalytic functions, see Balk [6, Theorem 1.5].

Lemma 3.1 (Balk) *If F is a polyanalytic function of order n in a disc $\mathbb{D}(0, \lambda R)$ for some $\lambda > 1$ and $M := \sup_{\mathbb{D}(0, \lambda R)} |F|$, then there exists $D_n = D_n(\lambda) > 0$, only depending on λ , such that*

$$\sup_{z \in \mathbb{D}(0, R)} |\bar{z}^k F_k(z)| \leq D_n M, \quad k = 0, \dots, n,$$

where D_n is given by

$$D_n := \left(\frac{2\lambda}{\lambda - 1} \right)^{n+2} (n+2)^{(n+2)^2}. \quad (3.11)$$

This lemma is due to Balk, up to the dependence (3.11) of the constant D_n on n which is needed in our results. As the proof by Balk leaves out technical details, we now precise his arguments which lead also to (3.11). As for the proof of the maximum modulus principle for holomorphic functions, the main tool is a Cauchy formula. To state this formula, let us recall that a polyanalytic function F of order n is called *reduced* if it can be written as

$$F(z) = \sum_{k=0}^n H_k(z) |z|^{2k},$$

where H_k is a holomorphic function. If F is a reduced polyanalytic function it satisfies a Cauchy-type formula [6, Section 1.3, (11)].

Lemma 3.2 *Let F be a reduced polyanalytic function of order n in $\mathbb{D}(0, R)$, $0 < R_0 < R_1 < \dots < R_n < R$, and let $\Gamma_k := \{z : |z| = R_k\}$. For every $z \in \mathbb{D}(0, R_0)$, F satisfies*

$$F(z) = \frac{1}{2\pi i} \sum_{k=0}^n P_k(|z|^2) \int_{\Gamma_k} \frac{F(t)}{t - z} dt, \quad (3.12)$$

where P_k is a polynomial given by

$$P_k(t) := \prod_{j \neq k} \frac{R_j^2 - t}{R_j^2 - R_k^2}.$$

This Cauchy formula will allow us in Lemma 3.3 to estimate $|\bar{z}^n F_n(z)|$, the term of highest degree in \bar{z} in (3.6), as well as the difference $|F(z) - \bar{z}^n F_n(z)|$. The remaining of the proof of the maximum principle will then consist of applying Lemma 3.3 iteratively.

Lemma 3.3 *If F is a polyanalytic function of order n in $\mathbb{D}(0, \lambda R)$, for some $\lambda > 1$, and $M = \sup_{\mathbb{D}(0, \lambda R)} |F|$, then for every $z \in D(0, (1 + \lambda)R/2)$ it holds*

$$|\bar{z}^n F_n(z)| \leq M \left(\frac{2\lambda(n+2)}{\lambda - 1} \right)^{n+2}, \quad (3.13)$$

and

$$|F(z) - \bar{z}^n F_n(z)| \leq 2M \left(\frac{2\lambda(n+2)}{\lambda - 1} \right)^{n+2}. \quad (3.14)$$

Proof: First, observe that $z^n F(z)$ is a reduced polyanalytic function satisfying $\bar{\partial}^n(z^n F(z)) = z^n(\bar{\partial})^n F(z) = n!z^n F_n(z)$. If we choose $R_k = \frac{R}{2}(1 + \lambda + (\lambda - 1)\frac{k+1}{n+2})$, then $(1 + \lambda)R/2 < R_0 < R_1 < \dots < R_n < \lambda R$ and

$$|R_j^2 - R_k^2| = |(R_j + R_k)(R_j - R_k)| \geq 2 \cdot \frac{(1 + \lambda)R}{2} \cdot \frac{(\lambda - 1)R}{2(n + 2)} \geq \frac{\lambda(\lambda - 1)R^2}{2(n + 2)}.$$

As P_k is a polynomial of degree n with leading coefficient $(-1)^n / \prod_{j \neq k} (R_j^2 - R_k^2)$, it follows that

$$P_k^{(n)}(z) = \frac{(-1)^n n!}{\prod_{j \neq k} (R_j^2 - R_k^2)}.$$

and consequently that $|P_k^{(n)}| \leq n! \left(\frac{2(n+2)}{\lambda(\lambda-1)R^2} \right)^n$. By (3.12), we may thus write

$$\begin{aligned} |\bar{z}^n F_n(z)| &= \frac{1}{n!} |(\bar{\partial})^n(z^n F(z))| \leq \frac{1}{2\pi n!} \sum_{k=0}^n |(\bar{\partial})^n P_k(|z|^2)| \int_{\Gamma_k} \frac{|t^n F(t)|}{|t - z|} dt \\ &\leq \frac{1}{2\pi n!} \sum_{k=0}^n |z|^n |P_k^{(n)}(|z|^2)| \int_{\Gamma_k} \frac{|t^n F(t)|}{|t - z|} dt \\ &\leq \frac{1}{2\pi} \sum_{k=0}^n (\lambda R)^n \left(\frac{2(n+2)}{\lambda(\lambda-1)R^2} \right)^n \int_{\Gamma_k} \frac{(\lambda R)^n M}{|t - z|} dt \\ &\leq \sum_{k=0}^n \left(\frac{2\lambda(n+2)}{\lambda-1} \right)^{n+1} M = (n+1) \left(\frac{2\lambda(n+2)}{\lambda-1} \right)^{n+1} M \\ &\leq \left(\frac{2\lambda(n+2)}{\lambda-1} \right)^{n+2} M, \end{aligned}$$

as $|z - t| \geq \frac{R(\lambda-1)}{2(n+2)}$, $|\Gamma_k| < 2\pi\lambda R$, and $\lambda/(\lambda-1) \geq 1$. Equation (3.14) then follows from $|F(z) - \bar{z}^n F_n(z)| \leq M + |z^n F_n(z)|$, (3.13) and the fact that $\lambda(n+2)/(\lambda-1) > 1$. \square

We can now complete the proof of the maximum principle.

Proof of Lemma 3.1 : As $\tilde{F}(z) := F(z) - F_n(z)\bar{z}^n$ is a polyanalytic function of order $n-1$ with

$$\sup_{\mathbb{D}(0, (1+\lambda)R/2)} |\tilde{F}| \leq 2M \left(\frac{2\lambda(n+2)}{\lambda-1} \right)^{n+2},$$

we can reapply Lemma 3.3 with $\lambda_1 = (1 + \lambda)/2$ to obtain

$$|z^{n-1} F_{n-1}(z)| = |z^{n-1} \tilde{F}_{n-1}(z)| \leq 2M \left(\frac{2\lambda_1(n+1)}{\lambda_1-1} \right)^{n+1} \left(\frac{2\lambda(n+2)}{\lambda-1} \right)^{n+2},$$

and

$$|F(z) - \bar{z}^{n-1} F_{n-1}(z) - \bar{z}^n F_n(z)| \leq 4M \left(\frac{2\lambda_1(n+1)}{\lambda_1-1} \right)^{n+1} \left(\frac{2\lambda(n+2)}{\lambda-1} \right)^{n+2},$$

for every $z \in \mathbb{D}(0, (1 + \lambda_1)R/2)$. Iterating this argument with $\lambda_l = \frac{1+\lambda_{l-1}}{2} = 1 + \frac{\lambda-1}{2^l}$ and setting $\lambda_0 = \lambda$ then yields

$$\begin{aligned} |\bar{z}^{n-k} F_{n-k}(z)| &\leq 2^k M \prod_{l=0}^k \left(\frac{2\lambda_l(n+2-l)}{\lambda_l-1} \right)^{n+2-l} \leq 2^n M \prod_{l=0}^n \left(\frac{2\lambda_0(n+2-l)}{\lambda_n-1} \right)^{n+2} \\ &= 2^n M \left(\frac{2^{n+1}\lambda(n+2)!}{\lambda-1} \right)^{n+2} \leq M \left(\frac{2^{n+2}\lambda(n+2)!}{\lambda-1} \right)^{n+2} \\ &\leq M \left(\frac{2\lambda}{\lambda-1} (n+2)^{n+2} \right)^{n+2}, \end{aligned}$$

where we used that $\lambda_n - 1 = \frac{\lambda-1}{2^n}$ and that $2^{n-1}n! \leq n^n$. This is precisely the statement of Lemma 3.1. \square

3.3 A Remez-Type Inequality for Polyanalytic Functions

In this section, we derive a Remez-type inequality for polyanalytic functions in Theorem 3.6 which is a precised version of Theorem 1.2 from the introduction. The proof requires further preparatory steps. The first one is an observation that a polyanalytic function is a restriction of an holomorphic function of two complex variables to \mathbb{R}^2 . Indeed, let F be a polyanalytic function of order n written as in (3.6):

$$F(z) = F(z, \bar{z}) = \sum_{k=0}^n F_k(z) \bar{z}^k.$$

The function

$$\Phi(F)(z_1, z_2) := \sum_{k=0}^n F_k(z_1 + iz_2)(z_1 - iz_2)^k, \quad (z_1, z_2) \in \mathbb{C}^2, \quad (3.15)$$

is a holomorphic function in two complex variables. If we set $z = x + iy$, $z_1 = x$ and $z_2 = y$ then $F(z) = F(x + iy) = \Phi(F)(x, y) = \Phi(F)(z_1, z_2)$, i.e. $F = \Phi(F)|_{\mathbb{R} \times \mathbb{R}}$. Moreover, $\ln |\Phi(F)|$ is plurisubharmonic, see [28, Section 2.2, p. 85]. Our estimate relies on the the following result about plurisubharmonic functions, see [10, Theorem 1.2].

Theorem 3.4 (Brudnyi) *Let $a > 1$ be some fixed constant chosen so that the ball $B_{\mathbb{R}}(x, t)$ satisfies $B_{\mathbb{R}}(x, t) \subset B_{\mathbb{C}}(x, at) \subset B_{\mathbb{C}}(0, 1)$ and let $r > 1$. If $h : \mathbb{C}^d \rightarrow \mathbb{R}$ is plurisubharmonic and satisfies*

$$\sup_{B_{\mathbb{C}}(0, r)} h = 0 \quad \text{and} \quad \sup_{B_{\mathbb{C}}(0, 1)} h \geq -1,$$

then there exist constants $c = c(a, r) > 0$ and $\kappa = \kappa(d) \geq 1$ such that the inequality

$$\sup_{B_{\mathbb{R}}(x, t)} h \leq c \ln \left(\frac{\kappa |B_{\mathbb{R}}(x, t)|}{|\Omega|} \right) + \sup_{\Omega} h, \quad (3.16)$$

holds for every measurable set $\Omega \subset B_{\mathbb{R}}(x, t)$ of positive measure.

Using Lemma 3.1, we can now estimate the supremum of $|\Phi(F)|$ in terms of a supremum of $|F|$.

Lemma 3.5 *If F is a polyanalytic function of order n with $M := \sup_{\mathbb{D}(0,4R)} |F|$, then*

$$\sup_{B_{\mathbb{C}}(0,2R)} |\Phi(F)(z_1, z_2)| \leq (4(n+2))^{(n+2)^2} M. \quad (3.17)$$

Proof: First, note that if $(z_1, z_2) \in B_{\mathbb{C}}(0, 2R)$, then

$$|z_1 \pm iz_2|^2 \leq (|z_1| + |z_2|)^2 \leq 2(|z_1|^2 + |z_2|^2) \leq 8R^2.$$

As F_k is analytic, it follows by the maximum modulus principle that it attains its maximum in $\mathbb{D}(0, \sqrt{8}R)$ at the boundary $\partial\mathbb{D}(0, \sqrt{8}R)$ and consequently that

$$\sup_{\mathbb{D}(0, \sqrt{8}R)} |F_k(z)|(\sqrt{8}R)^k = \sup_{\mathbb{D}(0, \sqrt{8}R)} |F_k(z)z^k| \leq D_n M,$$

with Lemma 3.1. Here, $D_n = D_n(4/\sqrt{8}) = D_n(\sqrt{2})$. We then conclude that

$$\begin{aligned} \sup_{B_{\mathbb{C}}(0,2R)} |\Phi(F)(z_1, z_2)| &\leq \sum_{k=0}^n \sup_{B_{\mathbb{C}}(0,2R)} |F_k(z_1 + iz_2)| \sup_{B_{\mathbb{C}}(0,2R)} |z_1 - iz_2|^k \\ &\leq \sum_{k=0}^n \sup_{\mathbb{D}(0, \sqrt{8}R)} |F_k(z)|(\sqrt{8}R)^k \\ &\leq (n+1)D_n M \leq (n+1)8^{n+2}(n+2)^{(n+2)^2} M \\ &\leq (4(n+2))^{(n+2)^2} M, \end{aligned}$$

where we used that $2\sqrt{2}/(\sqrt{2}-1) < 8$ and $(n+1)8^{n+2} \leq 4^{n(n+2)}8^{n+2} = 4^{(n+2)^2}$. \square

We are now in position to prove the Remez inequality for polyanalytic functions which is a precised version of Theorem 1.2 from the introduction:

Theorem 3.6 *Let $0 < \rho \leq R$, and $\Omega \subset \mathbb{D}(0, R)$ be measurable, and F be polyanalytic of order n in $\mathbb{D}(0, 5R)$. If we write $M := \sup_{\mathbb{D}(0,4R)} |F|$, and assume that $m := |F(0)| > 0$, then*

$$\sup_{z \in \mathbb{D}(0, \rho)} |F(z)| \leq \left(\frac{\kappa |\mathbb{D}(0, \rho)|}{|\Omega|} \right)^{c[\ln \frac{M}{m} + (n+2)^2 \ln 4(n+2)]} \sup_{z \in \Omega} |F(z)|, \quad (3.18)$$

where the constant c can be chosen so that it only depends on the fraction ρ/R , and κ is the constant in Brudnyi's theorem for $d = 2$.

Further, for $1 \leq p < \infty$,

$$\left(\int_{\mathbb{D}(0, \rho)} |F(z)|^p dz \right)^{\frac{1}{p}} \leq \left(\frac{\kappa |\mathbb{D}(0, \rho)|}{|\Omega|} \right)^{c[\ln \frac{M}{m} + (n+2)^2 \ln 4(n+2)] + \frac{1}{p}} \left(\int_{\Omega} |F(z)|^p dz \right)^{\frac{1}{p}}. \quad (3.19)$$

Proof: Let $\alpha := |\Phi(F)(0, 0)|$ and $\beta := \sup_{B_{\mathbb{C}}(0, 2R)} |\Phi(F)|$. The function

$$h(z_1, z_2) := \frac{1}{\ln \frac{\beta}{\alpha}} \ln \frac{|\Phi(F)(\lambda R z_1, \lambda R z_2)|}{\beta},$$

$1 < \lambda < 2$, is plurisubharmonic and satisfies

$$h(0, 0) = \frac{1}{\ln \frac{\beta}{\alpha}} \ln \frac{|\Phi(F)(0, 0)|}{\beta} = \frac{\ln \frac{\alpha}{\beta}}{\ln \frac{\beta}{\alpha}} = -1,$$

and

$$\sup_{B_{\mathbb{C}}(0, \frac{2}{\lambda})} h = \frac{1}{\ln \frac{\beta}{\alpha}} \sup_{B_{\mathbb{C}}(0, 2R)} \ln \frac{|\Phi(F)|}{\beta} = 0.$$

Hence, the assumptions of Theorem 3.4 are satisfied with $r = \frac{2}{\lambda} > 1$, $d = 2$ and $a = 1/t$. Choosing $t = \frac{\rho}{\lambda R} < 1$, it therefore follows that there exist $c = c(\frac{\lambda R}{\rho}, \frac{2}{\lambda})$ and $\kappa = \kappa(2)$ such that

$$\begin{aligned} \sup_{\mathbb{D}(0, \rho)} \frac{1}{\ln \frac{\beta}{\alpha}} \ln \frac{|F|}{\beta} &= \sup_{\mathbb{D}(0, \lambda R t)} \frac{1}{\ln \frac{\beta}{\alpha}} \ln \frac{|F|}{\beta} = \sup_{B_{\mathbb{R}}(0, \lambda R t)} \frac{1}{\ln \frac{\beta}{\alpha}} \ln \frac{|\Phi(F)|}{\beta} \\ &= \sup_{B_{\mathbb{R}}(0, t)} \frac{1}{\ln \frac{\beta}{\alpha}} \ln \frac{|\Phi(F)(\lambda R \cdot)|}{\beta} \\ &\leq c \ln \left(\frac{\kappa |B_{\mathbb{R}}(0, t)|}{|\Omega / \lambda R|} \right) + \sup_{\Omega / \lambda R} \frac{1}{\ln \frac{\beta}{\alpha}} \ln \frac{|\Phi(F)(\lambda R \cdot)|}{\beta} \\ &= c \ln \left(\frac{\kappa \lambda^2 R^2 |\mathbb{D}(0, t)|}{|\Omega|} \right) + \sup_{\Omega} \frac{1}{\ln \frac{\beta}{\alpha}} \ln \frac{|\Phi(F)|}{\beta} \\ &= c \ln \left(\frac{\kappa |\mathbb{D}(0, \lambda R t)|}{|\Omega|} \right) + \sup_{\Omega} \frac{1}{\ln \frac{\beta}{\alpha}} \ln \frac{|F|}{\beta}. \end{aligned}$$

Taking the exponential function of both sides of the inequality and recalling the definition of t yields

$$\left(\sup_{\mathbb{D}(0, \rho)} \frac{|F|}{\beta} \right)^{\frac{1}{\ln \frac{\beta}{\alpha}}} \leq \left(\frac{\kappa |\mathbb{D}(0, \rho)|}{|\Omega|} \right)^c \left(\sup_{\Omega} \frac{|F|}{\beta} \right)^{\frac{1}{\ln \frac{\beta}{\alpha}}},$$

and consequently

$$\sup_{\mathbb{D}(0, \rho)} |F| \leq \left(\frac{\kappa |\mathbb{D}(0, \rho)|}{|\Omega|} \right)^{c \ln \frac{\beta}{\alpha}} \sup_{\Omega} |F|.$$

This concludes the proof of (3.18) when observing that $\alpha = m$ and that $\beta \leq (4(n+2))^{(n+2)^2} M$ by Lemma 3.5.

The L^p inequality (3.19) follows from standard techniques: write

$$K = c \left[\ln \frac{M}{m} + (n+2)^2 \ln 4(n+2) \right]$$

and, for $\theta > 0$, define the set

$$A_\theta := \{\zeta \in \mathbb{D}(0, \rho) : |F(\zeta)| < \left(\frac{\theta}{\kappa|\mathbb{D}(0, \rho)|}\right)^K \sup_{z \in \mathbb{D}(0, \rho)} |F(z)|\}.$$

Taking $\Omega = A_\theta$ in (3.18) gives

$$\left(\frac{|A_\theta|}{\kappa|\mathbb{D}(0, \rho)|}\right)^K \sup_{z \in \mathbb{D}(0, \rho)} |F(z)| \leq \sup_{z \in A_\theta} |F(z)| < \left(\frac{\theta}{\kappa|\mathbb{D}(0, \rho)|}\right)^K \sup_{z \in \mathbb{D}(0, \rho)} |F(z)|.$$

Consequently, $|A_\theta| < \theta$. Now, let $\Omega \subset \mathbb{D}(0, \rho)$ and $\theta = \frac{|\Omega|}{2}$ so that $\frac{|\Omega|}{2} \leq |\Omega \cap A_{|\Omega|/2}^c|$. But then

$$\begin{aligned} \int_{\Omega} |F(z)|^p dz &\geq \int_{\Omega \cap A_{|\Omega|/2}^c} |F(z)|^p dz \\ &\geq |\Omega \cap A_{|\Omega|/2}^c| \left(\frac{|\Omega|}{2\kappa|\mathbb{D}(0, \rho)|}\right)^{pK} \sup_{z \in \mathbb{D}(0, \rho)} |F(z)|^p \\ &\geq \frac{|\Omega|}{2|\mathbb{D}(0, \rho)|} \left(\frac{|\Omega|}{2\kappa|\mathbb{D}(0, \rho)|}\right)^{pK} |\mathbb{D}(0, \rho)| \sup_{z \in \mathbb{D}(0, \rho)} |F(z)|^p \\ &\geq \left(\frac{|\Omega|}{2\kappa|\mathbb{D}(0, \rho)|}\right)^{pK+1} \int_{\mathbb{D}(0, \rho)} |F(z)|^p dz \end{aligned}$$

since $\kappa \geq 1$. Here we used the definition of $A_{|\Omega|/2}$ in the second inequality. \square

4 Lower Sampling Bounds for the STFT with Hermite Windows

This section is devoted to the proof of Theorem 1.1. Before going into the details, let us shortly state a direct consequence of this result for functions in the true polyanalytic Bargmann spaces \mathcal{F}_p^n .

Corollary 4.1 *Let $1 \leq p < \infty$, and $F \in \mathcal{F}_p^n(\mathbb{C})$. If $\Omega \subset \mathbb{C}$ is (γ, R) -dense for some scale $R > 0$, then there exists $\eta = \eta(n, R)$ and $\sigma = \sigma(n, R)$ and a numerical constant $C > 0$ such that*

$$\|F\|_{\mathcal{L}^p(\mathbb{C})} \leq \eta \left(\frac{\gamma}{C}\right)^{-\sigma} \|F\|_{\mathcal{L}^p(\Omega)}. \quad (4.20)$$

Proof: The result follows from Theorem 1.1 once we recall that $|\mathcal{B}^{n+1}f(z)|e^{-\pi|z|^2/2} = |V_{h_n}f(\bar{z})|$, $\mathcal{B}^{n+1}(M^p(\mathbb{R})) = \mathcal{F}_p^n(\mathbb{C})$, and the definition of $\mathcal{L}^p(\Omega)$ in (3.8). \square

Let us shortly sketch the overall strategy of the proof of Theorem 1.1 which requires several auxiliary lemmas. First, we use the connection with polyanalytic Bargmann spaces to derive a Remez-type inequality for the short-time Fourier transform with Hermite windows in

Lemma 4.2. The exponent in the Remez-type inequality depends on the function f and the point $w \in \mathbb{C}$. So in order to eliminate this dependence we restrict the domain of integration to a set $W_{\theta,R}$ for which we show that the exponent is uniformly bounded (Lemma 4.4) and more than half of the energy of $V_{h_n}f$ is concentrated on $W_{\theta,R}$ (Lemma 4.3).

Note that if we set $\rho = R$, and make the particular choice $\lambda = \sqrt{2}$ in Theorem 3.6, it follows that the constant $c = c(\sqrt{2}, \sqrt{2})$ is independent of R .

Lemma 4.2 *Let $w \in \mathbb{C}$, $\Omega \subset \mathbb{D}(w, R)$ be measurable, $1 \leq p \leq \infty$, $f \in M^p(\mathbb{R})$ such that $m := |V_{h_n}f(w)| > 0$, and*

$$\delta := \int_{\mathbb{D}(w, 5R)} |V_{h_n}f(z)| dz.$$

There exists numerical constants $c > 0$ and $\kappa \geq 1$ (independent of R , n and f) such that

$$\|V_{h_n}f\|_{L^p(\mathbb{D}(w, R))} \leq e^{\frac{\pi}{2}R^2} \left(\frac{\kappa |\mathbb{D}(0, R)|}{|\Omega|} \right)^{K(R, n, \delta, m)} \|V_{h_n}f\|_{L^p(\Omega)}, \quad (4.21)$$

where

$$K(R, n, \delta, m) := c \left[\ln \frac{\delta}{m} + 8\pi R^2 + \ln(\nu_n(R)^{-1}) + (n+2)^2 \ln 4(n+2) \right] + \frac{1}{p}$$

and $\nu_n(R)$ is given by (3.10).

Proof: As $|V_{h_n}f(z-w)| = |V_{h_n}\pi(w)f(z)|$, we may without loss of generality assume that $w = 0$. Set $M := \sup_{\mathbb{D}(0, 4R)} |\mathcal{B}^{n+1}f|$. From (3.7) we know that $m = |V_{h_n}f(0)| = |\mathcal{B}^{n+1}f(0)|$ and that

$$|V_{h_n}f(z)| = |\mathcal{B}^{n+1}f(\bar{z})| e^{-\pi|z|^2/2}. \quad (4.22)$$

Let us first estimate M in terms of the quantity δ . By the local reproducing formula (3.9), we have that

$$\begin{aligned} M &= \sup_{z \in \mathbb{D}(0, 4R)} |\mathcal{B}^{n+1}f(z)| = \sup_{z \in \mathbb{D}(0, 4R)} |V_{h_n}f(z) e^{\pi|z|^2/2}| \\ &\leq e^{8\pi R^2} \sup_{z \in \mathbb{D}(0, 4R)} \nu_n(R)^{-1} \int_{\mathbb{D}(z, R)} |V_{h_n}f(w) \langle \pi(w)h_n, \pi(z)h_n \rangle| dw \\ &\leq e^{8\pi R^2} \nu_n(R)^{-1} \int_{\mathbb{D}(0, 5R)} |V_{h_n}f(w)| dw = e^{8\pi R^2} \nu_n(R)^{-1} \delta. \end{aligned} \quad (4.23)$$

Let $\Omega^* := \{\bar{z} : z \in \Omega\}$. As $\mathcal{B}^{n+1}f$ is polyanalytic of order n , we may use Theorem 3.6 and (4.22) to show that

$$\begin{aligned} \|V_{h_n}f\|_{L^p(\mathbb{D}(0, R))} &= \|\mathcal{B}^{n+1}f(\cdot) e^{-\pi|\cdot|^2/2}\|_{L^p(\mathbb{D}(0, R))} \leq \|\mathcal{B}^{n+1}f\|_{L^p(\mathbb{D}(0, R))} \\ &\leq \left(\frac{\kappa |\mathbb{D}(0, R)|}{|\Omega^*|} \right)^{c \left[\ln \frac{M}{m} + (n+2)^2 \ln 4(n+2) \right] + \frac{1}{p}} \|\mathcal{B}^{n+1}f\|_{L^p(\Omega^*)} \\ &\leq e^{\frac{\pi}{2}R^2} \left(\frac{\kappa |\mathbb{D}(0, R)|}{|\Omega|} \right)^{c \left[\ln \frac{M}{m} + (n+2)^2 \ln 4(n+2) \right] + \frac{1}{p}} \|\mathcal{B}^{n+1}f(\cdot) e^{-\pi|\cdot|^2/2}\|_{L^p(\Omega^*)}. \end{aligned}$$

Using again (4.22), we obtain $\|\mathcal{B}^{n+1}f(\cdot)e^{-\pi|\cdot|^2/2}\|_{L^p(\Omega^*)} = \|V_{h_n}f\|_{L^p(\Omega)}$ and the result follows once we plug in the estimate from (4.23). \square

As said, we may now eliminate the dependence of $K(R, n, \delta, m)$ on f (via δ and m).

Lemma 4.3 *Let $1 \leq p < \infty$, and $\theta, R > 0$. Define the set $W_{\theta, R} \subset \mathbb{C}$ by*

$$W_{\theta, R} := \left\{ z \in \mathbb{C} : |V_{h_n}f(z)|^p \leq \frac{\theta}{|\mathbb{D}(0, R)|} \int_{\mathbb{D}(z, R)} |V_{h_n}f(w)|^p dw \right\}.$$

Then the following inequality holds

$$\int_{W_{\theta, R}^c} |V_{h_n}f(z)|^p dz \geq (1 - \theta) \int_{\mathbb{C}} |V_{h_n}f(z)|^p dz, \quad \forall f \in M^p(\mathbb{R}). \quad (4.24)$$

Proof: By definition of $W_{\theta, R}$ and Fubini's theorem we can write

$$\begin{aligned} \int_{W_{\theta, R}^c} |V_{h_n}f(z)|^p dz &\leq \int_{W_{\theta, R}^c} \frac{\theta}{|\mathbb{D}(0, R)|} \int_{\mathbb{D}(z, R)} |V_{h_n}f(w)|^p dw dz \\ &= \frac{\theta}{|\mathbb{D}(0, R)|} \int_{\mathbb{C}} |V_{h_n}f(w)|^p \int_{W_{\theta, R}^c} \chi_{\mathbb{D}(w, R)}(z) dz dw \\ &\leq \theta \int_{\mathbb{C}} |V_{h_n}f(w)|^p dw, \end{aligned}$$

as claimed. \square

Lemma 4.4 *Let $K(R, n, \delta, m)$ be as defined in Lemma 4.2. If $1 \leq p < \infty$ and $w \in W_{1/2^p, 5R}^c$, then there exists a constant $C > 0$ such that for $b(R, n)$ being defined as*

$$\sigma(n, R) := C (R^2 + \ln(\nu_n(R)^{-1}) + n^2 \ln n + 1), \quad (4.25)$$

one has

$$K(R, n, \delta, m) \leq b(R, n),$$

which is independent of f .

Proof: It is enough to show that δ/m is bounded independently of f . By the definition of $W_{\theta, 5R}^c$ it follows by Hölder's inequality that if $w \in W_{1/2^p, 5R}^c$ we have

$$\begin{aligned} \delta^p &= \left(\int_{\mathbb{D}(w, 5R)} |V_{h_n}f(z)| dz \right)^p \leq |\mathbb{D}(0, 5R)|^{p/q} \int_{\mathbb{D}(w, 5R)} |V_{h_n}f(z)|^p dz \\ &< \frac{|\mathbb{D}(0, 5R)|^{1+p/q}}{1/2^p} |V_{h_n}f(w)|^p = 2^p |\mathbb{D}(0, 5R)|^p m^p. \end{aligned}$$

Consequently, $\frac{\delta}{m} < 2|\mathbb{D}(0, 5R)| = 50\pi R^2$, and

$$\begin{aligned} K(R, n, \delta, m) &\leq c [\ln 50\pi R^2 + 8\pi R^2 + \ln(\nu_n(R)^{-1}) + (n+2)^2 \ln 4(n+2)] + \frac{1}{p} \\ &\leq c \left(\tilde{C}R^2 + \ln(\nu_n(R)^{-1}) + (n+2)^2 \ln 4(n+2) \right) + \frac{1}{p} \\ &\leq C (R^2 + \ln(\nu_n(R)^{-1}) + n^2 \ln n + 1), \end{aligned}$$

as claimed. \square

We are now in position to prove Theorem 1.1 which we restate here in a more precise form:

Theorem 4.5 *Let $1 \leq p < \infty$, $f \in M^p(\mathbb{R})$. If $\Omega \subset \mathbb{C}$ is (γ, R) -dense for some scale $R > 0$, then there exists a numerical constant $C > 0$ such that*

$$\|V_{h_n} f\|_{L^p(\mathbb{C})} \leq \eta(n, R) \left(\frac{\gamma}{C}\right)^{-\sigma(n, R)} \|V_{h_n} f\|_{L^p(\Omega)}, \quad (4.26)$$

where $\sigma(n, R)$ is given in (4.25), and $\eta(n, R) := \frac{R^2}{\nu_n(R)} C^{R^2+1}$.

Proof: By Fubini's theorem and Theorem 3.19 we may derive

$$\begin{aligned} \int_{\Omega} |V_{h_n} f(z)|^p dz &\geq \int_{\Omega} |V_{h_n} f(z)|^p \frac{1}{|\mathbb{D}(0, R)|} \int_{W_{1/2^p, 5R}^c} \chi_{\mathbb{D}(z, R)}(w) dw dz \\ &= \int_{W_{1/2^p, 5R}^c} \frac{1}{|\mathbb{D}(0, R)|} \int_{\Omega \cap \mathbb{D}(w, R)} |V_{h_n} f(z)|^p dz dw \\ &\geq \frac{e^{-\pi p R^2/2}}{|\mathbb{D}(0, R)|} \int_{W_{1/2^p, 5R}^c} \left(\frac{|\Omega \cap \mathbb{D}(w, R)|}{2\kappa |\mathbb{D}(w, R)|} \right)^{p \cdot K(R, n, \delta, m)} \int_{\mathbb{D}(w, R)} |V_{h_n} f(z)|^p dz dw. \end{aligned}$$

Now, Lemma 4.4 allows to estimate $K(R, n, \delta, m) \leq \sigma(n, R)$ which is independent of f and w . As Ω is (γ, R) -dense, it follows that

$$\int_{\Omega} |V_{h_n} f(z)|^p dz \geq \frac{e^{-\pi p R^2/2}}{|\mathbb{D}(0, R)|} \left(\frac{\gamma}{2\kappa}\right)^{p \cdot \sigma(n, R)} \int_{W_{1/2^p, 5R}^c} \int_{\mathbb{D}(w, R)} |V_{h_n} f(z)|^p dz dw. \quad (4.27)$$

It remains to estimate the double integral on the right hand side. Hölder's inequality, the

local reproducing formula (3.9), and Lemma 4.3 give

$$\begin{aligned}
\int_{W_{1/2^p, 5R}^c} \int_{\mathbb{D}(w, R)} |V_{h_n} f(z)|^p dz dw &\geq \int_{W_{1/2^p, 5R}^c} \int_{\mathbb{D}(w, R)} |V_{h_n} f(z) \langle \pi(z) h_n, \pi(w) h_n \rangle|^p dz dw \\
&\geq \int_{W_{1/2^p, 5R}^c} |\mathbb{D}(0, R)|^{-p/q} \left| \int_{\mathbb{D}(w, R)} V_{h_n} f(z) \langle \pi(z) h_n, \pi(w) h_n \rangle dz \right|^p dw \\
&= |\mathbb{D}(0, R)|^{-p/q} \int_{W_{1/2^p, 5R}^c} \nu_n(R)^p |V_{h_n} f(w)|^p dw \\
&\geq |\mathbb{D}(0, R)|^{-p/q} \nu_n(R)^p (1 - 2^{-p}) \int_{\mathbb{C}} |V_{h_n} f(z)|^p dz \\
&\geq \left(\frac{\nu_n(R)}{2|\mathbb{D}(0, R)|^{1/q}} \right)^p \int_{\mathbb{C}} |V_{h_n} f(z)|^p dz,
\end{aligned}$$

where we used $1 - 2^{-p} \geq 2^{-p}$ in the final step. Plugging this into (4.27) finally yields

$$\int_{\Omega} |V_{h_n} f(z)|^p dz \geq \left(\frac{e^{-\pi R^2/2} \nu_n(R)}{2|\mathbb{D}(0, R)|} \right)^p \left(\frac{\gamma}{2\kappa} \right)^{p \cdot \sigma(n, R)} \int_{\mathbb{C}} |V_{h_n} f(z)|^p dz$$

as claimed. \square

Remark 1 *The previous result is most likely not optimal with respect to most parameters, however the behavior with respect to R cannot be improved (up to numerical constants). Indeed, if $f = h_0$, then $|V_{h_n} h_0(z)| = c_n |z|^n e^{-\pi|z|^2/2}$, so that for $\Omega_R = \mathbb{C} \setminus \mathbb{D}(0, R/\sqrt{2})$,*

$$\int_{\Omega_R} |V_{h_n} h_0|^p dz = 2\pi c_n \int_{R/\sqrt{2}}^{\infty} t^{n+1} e^{-\pi p t^2/2} dt = \frac{c_n}{\pi^{n/2}} \left(\frac{2}{p} \right)^{n/2+1} \int_{\pi p R^2/4}^{\infty} t^{n/2} e^{-t} dt,$$

which yields

$$\frac{\int_{\Omega_R} |V_{h_n} h_0|^p dz}{\int_{\mathbb{C}} |V_{h_n} h_0|^p dz} = \beta_n \Gamma \left(\frac{pn+2}{2}, \frac{\pi p R^2}{4} \right)^{1/p} \sim \beta_n R^n e^{-\frac{\pi}{4} p R^2} = e^{-C_1 R^2 + \mathcal{O}(\ln R)},$$

where $\beta_n, C_1 > 0$ are independent of R and $\Gamma(s, x)$ is the incomplete Γ function. Here we used that $\Gamma(s, x) \sim x^{s-1} e^{-x}$, as $x \rightarrow \infty$.

On the other hand, $\inf_{z \in \mathbb{C}} \frac{|\Omega_R \cap \mathbb{D}(z, R)|}{|\mathbb{D}(z, R)|} = \frac{1}{2}$ i.e. Ω_R is $(1/2, R)$ -dense and one can show that $\nu_n(R) \rightarrow 1$, as $R \rightarrow \infty$. Hence,

$$\eta(n, R)^{-1} (2C)^{-\sigma(n, R)} \sim e^{-C_2 R^2 + \mathcal{O}(\ln R)},$$

for some constant $C_2 > 0$. Both expressions thus have the same leading term.

5 Sufficient Density Conditions for General Windows

5.1 Irregular Gabor Frames

Let $\Gamma = \{z_i\}_{i \in \mathcal{I}} \subset \mathbb{C}$ be discrete. A collection $\{\pi(z_i)g\}_{i \in \mathcal{I}}$ of time-frequency shifts of a window $g \in L^2(\mathbb{R})$ is called a *Gabor frame*, if there exist constants $A, B > 0$, called the frame bounds, such that

$$A\|f\|^2 \leq \sum_{i \in \mathcal{I}} |\langle f, \pi(z_i)g \rangle|^2 \leq B\|f\|^2, \quad \forall f \in L^2(\mathbb{R}).$$

If only the right inequality is satisfied, then $\{\pi(z_i)g\}_{i \in \mathcal{I}}$ is called a *Gabor Bessel sequence*. A discrete set $\Gamma \subset \mathbb{C}$ is said to be *uniformly separated* if $\inf\{|z - y| : z, y \in \Gamma, z \neq y\} > 0$ and *relatively uniformly separated* if it is the union of finitely many uniformly separated sets. The *lower Beurling density* is defined as

$$D^-(\Gamma) := \liminf_{R \rightarrow \infty} \inf_{z \in \mathbb{C}} \frac{\#\{\Gamma \cap z + Q_R\}}{R^2}.$$

The density theorem for Gabor frames is originally due to Ramanathan and Steger [36]. For our purposes, we need a slightly different formulation which can be found in [11].

Lemma 5.1 *If Γ generates a Gabor frame, then Γ is relatively uniformly separated and $D^-(\Gamma) \geq 1$.*

There is only little literature on irregular Gabor frames, *i.e.* Gabor frames that are not necessarily generated by a lattice of time-frequency shifts. Coorbit theory [15, 19], on the one hand, guarantees the frame property for "nice" windows and irregular sampling points with a sufficiently high density. However, the results do not provide any estimate of how to choose the density and how the frame bounds behave. Gröchenig on the other hand derived quantitative results in [21]. The choice of the sampling sets however do not leave enough freedom for our purposes in this section. It is a result by Sun and Zhou [39, Lemma 2.6] that has both ingredients: quantitative estimates of the frame bounds and enough freedom in choosing the sampling points. Recall that the Sobolev space $H^1(\mathbb{R})$ is defined as

$$H^1(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + |\xi|^2) |\widehat{f}(\xi)|^2 d\xi < \infty \right\}. \quad (5.28)$$

Theorem 5.2 (Sun & Zhou) *Let $g, tg \in H^1(\mathbb{R})$ and let $R > 0$ be such that*

$$\Delta := \frac{2R}{\pi} \left(\|g'\|_2 + \|tg\|_2 + \frac{2R}{\pi} \|tg'\|_2 \right) < \|g\|_2. \quad (5.29)$$

Moreover, let Q_n be a collection of squares with side length $R_n \leq R$ such that $\bigcup_{n \in \mathbb{N}} Q_n = \mathbb{C}$, and $|Q_n \cap Q_m| = 0$, whenever $n \neq m$. Then for any $z_n \in Q_n$, $\{R_n \pi(z_n)g\}_{n \in \mathbb{N}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A \geq (\|g\|_2 - \Delta)^2$ and $B \leq (\|g\|_2 + \Delta)^2$.

Note that Theorem 2.8 in [39] gives a more detailed picture of the frame structure for this class of windows. The full generality of the result is however not needed for our purposes in Section 5.2. In the following, we restate Theorem 5.2 in a simplified version that will be sufficient for our needs.

Corollary 5.3 *Let $g \in H^1(\mathbb{R})$ be compactly supported in $[-S, S]$, and $R < \min\left(\frac{\pi\|g\|_2}{4\|g'\|_2}, \frac{1}{2S}\right)$. If $|x_{n,m} - Rn| < R/2$, and $|\xi_{n,m} - Rm| < R/2$, then $\{\pi(z_{n,m})g\}_{n,m \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with*

$$\frac{1}{3R^2} \left(\|g\|_2 - \frac{4R}{\pi} \|g'\|_2 \right)^2 \|f\|_2^2 \leq \sum_{n,m \in \mathbb{Z}} |V_g f(z_{n,m})|^2 \leq \frac{2}{R^2} \left(\|g\|_2 + \frac{2R}{\pi} \|g'\|_2 \right)^2 \|f\|_2^2. \quad (5.30)$$

Proof: First note that $g \in H^1(\mathbb{R})$ compactly supported implies that $tg \in H^1(\mathbb{R})$. Using the assumption $R < 1/2S$, Δ can be estimated as

$$\begin{aligned} \Delta &= \frac{2R}{\pi} \left(\|g'\|_2 + \|tg\|_2 + \frac{2R}{\pi} \|tg'\|_2 \right) \leq \frac{2R}{\pi} \left(\|g'\|_2 + S\|g\|_2 + \frac{1}{\pi} \|g'\|_2 \right) \\ &\leq \frac{2R}{\pi} \frac{\pi+1}{\pi} \|g'\|_2 + \frac{1}{\pi} \|g\|_2. \end{aligned}$$

Hence, $\Delta < \|g\|_2$ if

$$R < \frac{\pi\|g\|_2}{4\|g'\|_2} < \frac{\pi(\pi-1)\|g\|_2}{2(\pi+1)\|g'\|_2},$$

as $1/2 < \frac{\pi-1}{\pi+1}$. Using (5.29) and the estimate for Δ , the upper frame bound is estimated by

$$B \leq \frac{1}{R^2} (\|g\|_2 + \Delta)^2 \leq \frac{1}{R^2} \left(\frac{\pi+1}{\pi} \right)^2 \left(\|g\|_2 + \frac{2R}{\pi} \|g'\|_2 \right)^2 \leq \frac{2}{R^2} \left(\|g\|_2 + \frac{2R}{\pi} \|g'\|_2 \right)^2.$$

For the lower frame bound, we have

$$\begin{aligned} A &\geq \frac{1}{R^2} (\|g\|_2 - \Delta)^2 = \frac{1}{R^2} \left(\frac{\pi-1}{\pi} \right)^2 \left(\|g\|_2 - \frac{2R}{\pi} \frac{\pi+1}{\pi-1} \|g'\|_2 \right)^2 \\ &\geq \frac{1}{3R^2} \left(\|g\|_2 - \frac{4R}{\pi} \|g'\|_2 \right)^2 \end{aligned}$$

as claimed. \square

5.2 Sufficient Density Conditions for Sampling Sets

For the rest of this paper let us slightly change the definition of (γ, R) -dense sets. Instead of discs we now use squares to define such sets. In particular, a set $\Omega \subset \mathbb{C}$ is called (γ, R) -dense if

$$\gamma := \inf_{z \in \mathbb{C}} \frac{|\Omega \cap z + Q_R|}{R^2} > 0. \quad (5.31)$$

Our main goal in this section is to determine under which conditions on the window g one has that a (γ, R) -dense set at small enough scales R is a sampling set. It is therefore irrelevant which definition of density we use as every (γ, R) -dense set in the sense of (1.2) is a $(\gamma/2, R)$ -dense set in the sense of (5.31) and every (γ, R) -dense set in the sense of (5.31) is a $(\gamma/2, \sqrt{2}R)$ -dense set in the sense of (1.2).

Let us write $Q_R(n, m) := Rn + iRm + Q_R$. We are now able to establish a connection between irregular Gabor frames and sampling sets. Theorem 1.3 from the introduction will then be deduced from the following result:

Theorem 5.4 *Let $g \in L^2(\mathbb{R})$, and $\Omega \subset \mathbb{C}$ be measurable.*

(i) *If there exists $R > 0$, such that $\{\pi(z_{n,m})g\}_{n,m \in \mathbb{Z}}$ is a frame for every choice $z_{n,m} \in Q_R(n, m)$ with a lower frame bound A independent of the particular choice of sampling points, then every (γ, R) -dense set Ω is a sampling set. The sampling bound satisfies $C \leq \frac{\|g\|_2^2}{AR^2} \gamma^{-1}$.*

If in addition, g is such that every relatively uniformly separated sequence of points in \mathbb{C} generates a Gabor Bessel sequence, then the following statements hold true.

(ii) *If there exists $R > 0$ such that every (γ, R) -dense set Ω is a sampling set with sampling bound $C \leq K\gamma^{-1}$, then $\{\pi(z_{n,m})g\}_{n,m \in \mathbb{Z}}$ is a frame for every choice $z_{n,m} \in Q_{R/2}(n, m)$ with lower frame bound $A \geq \frac{\|g\|_2^2}{KR^2}$.*

(iii) *There is no scale $R > 0$ and $\varepsilon > 0$ such that the sampling bound satisfies $C \leq K\gamma^{-1+\varepsilon}$ for every (γ, R) -dense set Ω .*

Proof: *Ad (i):* Let $f \in L^2(\mathbb{R})$ be fixed and let Ω be a (γ, R) -dense set. For every $n, m \in \mathbb{Z}$ there exist $z_{n,m} \in \Omega \cap Q_R(n, m)$ such that

$$|V_g f(z_{n,m})|^2 \leq \frac{1}{|\Omega \cap Q_R(n, m)|} \int_{\Omega \cap Q_R(n, m)} |V_g f(z)|^2 dz.$$

If A is the uniform lower frame bound, then

$$\begin{aligned} \|V_g f\|_{L^2(\mathbb{C})}^2 &= \|g\|_2^2 \|f\|_2^2 \leq \frac{\|g\|_2^2}{A} \sum_{n,m \in \mathbb{Z}} |V_g f(z_{n,m})|^2 \\ &\leq \frac{\|g\|_2^2}{AR^2} \sum_{n,m \in \mathbb{Z}} \frac{R^2}{|\Omega \cap Q_R(n, m)|} \int_{\Omega \cap Q_R(n, m)} |V_g f(z)|^2 dz \\ &\leq \frac{\|g\|_2^2}{AR^2} \gamma^{-1} \sum_{n,m \in \mathbb{Z}} \int_{\Omega \cap Q_R(n, m)} |V_g f(z)|^2 dz \\ &= \frac{\|g\|_2^2}{AR^2} \gamma^{-1} \int_{\Omega} |V_g f(z)|^2 dz. \end{aligned}$$

Ad (ii): Let $z_{n,m} \in Q_{R/2}(n, m)$ and choose Ω_μ such that $\Omega_\mu \cap Q_{R/2}(n, m)$ is an open neighborhood of $z_{n,m}$ and that $|\Omega_\mu \cap Q_{R/2}(n, m)| = \mu R^2$. It then follows that Ω_μ is

a (γ, R) -dense set with $\mu \leq \gamma$, since every square $z + Q_R$ contains at least one square $Q_{R/2}(n, m)$. As $C \leq K\gamma^{-1}$, we have

$$\begin{aligned} \int_{\mathbb{C}} |V_g f(z)|^2 dz &\leq K\gamma^{-1} \int_{\Omega_\mu} |V_g f(z)|^2 dz \leq K\mu^{-1} \int_{\Omega_\mu} |V_g f(z)|^2 dz \\ &= K \sum_{n, m \in \mathbb{Z}} \mu^{-1} \int_{\Omega_\mu \cap Q_{R/2}(n, m)} |V_g f(z)|^2 dz. \end{aligned}$$

Moreover, since

$$\begin{aligned} \frac{1}{\mu} \int_{\Omega_\mu \cap Q_{R/2}(n, m)} |V_g f(z)|^2 dz &= \frac{R^2}{|\Omega_\mu \cap Q_{R/2}(n, m)|} \int_{\Omega_\mu \cap Q_{R/2}(n, m)} |V_g f(z)|^2 dz \\ &\leq R^2 \sup_{z \in Q_{R/2}(n, m)} |V_g f(z)|^2, \end{aligned}$$

it follows by the assumption on g that, when considering the limit $\mu \rightarrow 0$, we may apply the dominated convergence theorem to obtain

$$\begin{aligned} \|g\|_2^2 \|f\|_2^2 &= \int_{\mathbb{C}} |V_g f(z)|^2 dz \leq K \lim_{\mu \rightarrow 0} \sum_{n, m \in \mathbb{Z}} \frac{1}{\mu} \int_{\Omega_\mu \cap Q_{R/2}(n, m)} |V_g f(z)|^2 dz \\ &= K \sum_{n, m \in \mathbb{Z}} \lim_{\mu \rightarrow 0} \frac{R^2}{|\Omega_\mu \cap Q_{R/2}(n, m)|} \int_{\Omega_\mu \cap Q_{R/2}(n, m)} |V_g f(z)|^2 dz \\ &= R^2 K \sum_{n, m \in \mathbb{Z}} |V_g f(z_{n, m})|^2. \end{aligned}$$

Ad (iii): Assume to the contrary, that there exist $R, \varepsilon > 0$ such that $C \leq K\gamma^{-1+\varepsilon} \leq K\gamma^{-1}$. In particular, the assumption of (ii) is satisfied and an arbitrary choice $z_{n, m} \in Q_{R/2}(n, m)$ generates a Gabor frame. On the other hand, repeating the calculations of the proof of (ii) with $K\gamma^{1-\varepsilon}$ instead of $K\gamma$ gives

$$\frac{\|g\|_2^2}{KR^2} \gamma^{-\varepsilon} \|f\|^2 \leq \sum_{n, m} |V_g f(z_{n, m})|^2,$$

for any $\gamma \in (0, 1]$. Taking the limit $\gamma \rightarrow 0$ then shows that $z_{n, m} \in Q_{R/2}(n, m)$ cannot generate a Gabor frame, a contradiction. \square

The necessary density condition for Gabor frames in Lemma 5.1 now shows that the sampling bound C is bounded by γ^{-1} only for small scales R .

Corollary 5.5 *If there exists $R > 2$ such that every (γ, R) -dense set Ω is a sampling set, then there exists no constant $K > 0$ such that the sampling constant is bounded by $K\gamma^{-1}$. In particular, $C \geq K\gamma^{-1-\varepsilon}$ for some $\varepsilon > 0$.*

Proof: Assume to the contrary, that there exists $K > 0$ such that the sampling bound satisfies $C \leq K\gamma^{-1}$. Then, by Theorem 5.4 (ii), it follows that $\{M_{Rm/2}T_{Rn/2}g\}_{n,m \in \mathbb{Z}}$ is a frame. The set $\Gamma = \{Rn/2 + iRm/2\}_{n,m \in \mathbb{Z}}$ set has lower Beurling density $D^-(\Gamma) = 4/R^2$. Now since $R > 2$, it follows that $D^-(\Gamma) < 1$ which, by Lemma 5.1, contradicts the assumption that $\{M_{Rm/2}T_{Rn/2}g\}_{n,m \in \mathbb{Z}}$ is a frame. \square

We can now prove Theorem 1.3 from the introduction in a more precise form:

Corollary 5.6 *Let $g \in H^1(\mathbb{R})$ be supported in $[-S, S]$. If*

$$R < \min \left(\frac{\pi \|g\|_2}{4 \|g'\|_2}, \frac{1}{2S} \right),$$

then every (γ, R) -dense set $\Omega \subset \mathbb{C}$ is a sampling set. In particular, for every $f \in L^2(\mathbb{R})$, one has

$$\int_{\mathbb{C}} |V_g f(z)|^2 dz \leq \frac{3}{\gamma} \left(1 - \frac{4 \|g'\|_2}{\pi \|g\|_2} R \right)^{-2} \int_{\Omega} |V_g f(z)|^2 dz. \quad (5.32)$$

Proof: This result follows directly from Theorem 5.4 (i) once one observes that the sampling constant is less than $\frac{\|g\|_2^2}{AR^2}\gamma^{-1}$, where A is the lower frame bound given in (5.30). \square

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