Classifying (Weak) Coideal Subalgebras of Weak Hopf C ¦ -Algebras

Leonid Vainerman, Jean-Michel Vallin

Classifying (Weak) Coideal Subalgebras of Weak Hopf C^*-Algebras

Leonid Vainerman Jean-Michel Vallin

Dedicated to the Memory of Etienne Blanchard

Abstract

We develop a general approach to the problem of classification of weak coideal C^*-subalgebras of weak Hopf C^*-algebras. As an example, we consider weak Hopf C^*-algebras and their weak coideal C^*-subalgebras associated with Tambara Yamagami categories.

Contents

[1]AMS Subject Classification [2010]: Primary 16T05, Secondary 18D10, Tertiary 46L05.
[2] Keywords : Coactions and corepresentations of quantum groupoids, C^*-categories, reconstruction theorem.

1 Introduction

It is known that any finite tensor category equipped with a fiber functor to the category of finite dimensional vector spaces is equivalent to the representation category of some Hopf algebra - see, for example, [5], Theorem 5.3.12. But many tensor categories do not admit a fiber functor, so they cannot be presented as representation categories of Hopf algebras. On the other hand, T. Hayashi [7] showed that any fusion category always admits a tensor functor to the category of bimodules over some semisimple (even commutative) algebra. Using this, it was proved in [7], [15], [17] that any fusion category is equivalent to the representation category of some algebraic structure generalizing Hopf algebras called a *weak Hopf algebra* [2] or a *finite quantum groupoid* [14]. The main difference between weak and usual Hopf algebra is that in the former the coproduct Δ is not necessarily unital.

Apart from tensor categories, weak Hopf algebras have interesting applications to the subfactor theory. In particular, for any finite index and finite depth II_1-subfactor $N \subset M$, there exists a weak Hopf C^*-algebra $\mathfrak{G}$ such that the corresponding Jones tower can be expressed in terms of crossed products of N and M with $\mathfrak{G}$ and its dual. Moreover, there is *a Galois correspondence* between intermediate subfactors in this Jones tower and coideal C^*-subalgebras of $\mathfrak{G}$ - see [13]. This motivates the study of coideal C^*-subalgebras of weak Hopf C^*-algebras which is the subject of the present paper. The abbreviation WHA will always mean a weak Hopf C^*-algebra.

A coideal C^*-subalgebra is a special case of the notion of a $\mathfrak{G}$-C^*-algebra, which is, by definition, a unital C^* algebra A equipped with a *coaction* $\mathfrak{a}$ of a WHA $\mathfrak{G} = (B, \Delta, S, \varepsilon)$. More exactly, we will use the following

Definition 1.1 *A weak right coideal C^*-subalgebra of B is a right $\mathfrak{G}$-C^*-algebra $(A, \mathfrak{a})$ with a C^*-algebra inclusion $i : A \mapsto B$ (not necessarily unital) satisfying $\Delta = (i \otimes id_B)\mathfrak{a}$. One can think of A as of a C^*-subalgebra of B such that $\mathfrak{a} = \Delta$. If i is unital, we call A a coideal C^*-subalgebra of B.*

For the sake of brevity, we will call a (weak) coideal C^*-subalgebra a (weak) coideal of B. Note that if $\mathfrak{G}$ is a usual Hopf C^*-algebra, then one can prove that necessarily $1_A = 1_B$, so weak and usual coideals coincide.

It was shown in [20] that any $\mathfrak{G}$-C^*-algebra $(A, \mathfrak{a})$ corresponds to a pair $(\mathcal{M}, M)$, where $\mathcal{M}$ is a module category with a generator M over the category of unitary corepresentations of $\mathfrak{G}$.

In Preliminaries we recall definitions and facts needed for the exact formulation of this result expressed in Theorem 2.9. Note that similar categorical duality for compact quantum group coactions was obtained earlier in [4], [11].

Section 3 is devoted to necessary conditions which a pair $(\mathcal{M}, M)$ satisfies if $(A, \mathfrak{a})$ is an indecomposable (weak) coideal.

In Sections 4 and 5 the above mentioned general approach is applied to the problem of classification of $\mathfrak{G}$-algebras and weak coideals of WHA's associated with a concrete class of fusion categories - Tambara-Yamagami categories $\mathcal{TY}(G, \chi, \tau)$ [18].

Recall that simple objects of $\mathcal{TY}(G, \chi, \tau)$ are exactly the elements of a finite abelian group G and one separate element m satisfying the fusion rule $g \cdot h = gh,\ g \cdot m = m \cdot g = m,\ m^2 = \sum\limits_{g \in G} g,\ g^* = -g,\ m = m^*\ (g, h \in G)$. These categories are parameterized by non degenerate symmetric bicharacters $\chi : G \times G \to \mathbb{C}\backslash\{0\}$ and $\tau = \pm|G|^{-1/2}$. For any subset $K \subset G$, we shall denote $K^\perp := \{g \in G | \chi(k, g) = 1,\ \forall k \in K\}$.

The Hayashi's reconstruction theorem allows to construct a WHA $\mathfrak{G}_{\mathcal{TY}}$ associated with $\mathcal{TY}(G, \chi, \tau)$ - see [10]. We recall this construction in slightly different form in Subsection 4.1. Then, using the methods elaborated in [6], [8], [9], we classify in Subsection 4.2 indecomposable module categories over representations of $\mathfrak{G}_{\mathcal{TY}}$, their autoequivalences and generators. Together with the above mentioned results this leads to the following classification theorem:

Theorem 1.2 *There are two types of isomorphism classes of indecomposable finite dimensional $\mathfrak{G}_{\mathcal{TY}}$-$C^*$-algebras:*

(i) *those parameterized by pairs $(K, \{m_\lambda\}^{orb})$, where $K < G$ and $\{m_\lambda\}^{orb}$ is the orbit of a nonzero collection $\{m_\lambda \in \mathbb{Z}_+ | \lambda \in G/K\}$ under the action of the group of translations on G/K.*

(ii) *those parameterized by pairs $(K, (\{m_\lambda\}, \{m_\mu\})^{orb})$, where $K < G$ and $(\{m_\lambda\}, \{m_\mu\})^{orb}$ is the orbit of a nonzero double collection $(\{m_\lambda \in \mathbb{Z}_+ | \lambda \in G/K\}, \{m_\mu \in \mathbb{Z}_+ | \mu \in G/K^\perp\})$ under the action of:*

a) the group of translations on $G/K \times G/K^\perp$ if $K \neq K^\perp$;

b) the semi-direct product $(G/K \times G/K) \underset{\sigma}{\ltimes} \mathbb{Z}_2$ generated by translations on $G/K \times G/K$ and the flip $\sigma : (\{m_\lambda\}, \{m_\mu\}) \leftrightarrow (\{m_\mu\}, \{m_\lambda\})$ if $K = K^\perp$.

Finally, Section 5 is devoted to the classification of indecomposable (weak) coideals of $\mathfrak{G}_{\mathcal{TY}}$. Their classification is given by the following

Theorem 1.3 *Isomorphism classes of indecomposable weak coideals of $\mathfrak{G}_{\mathcal{TY}}$ are parameterized by pairs $(K,(Z_0,Z_1)^{orb})$, where K is a subgroup of G and $(Z_0,Z_1)^{orb}$ is the orbit of a nonempty subset $(Z_0,Z_1)\subset G/K\times G/K^{\perp}$ such that either $|Z_0|\leqslant 1$ or $|Z_1|\leqslant 1$, under the action of:*

a) the group of translations on $G/K\times G/K^{\perp}$ if $K\neq K^{\perp}$;

b) the semi-direct product $(G/K\times G/K)\underset{\sigma}{\ltimes}\mathbb{Z}_2$ generated by translations on $G/K\times G/K$ and the flip $\sigma:(Z_0,Z_1)\leftrightarrow(Z_1,Z_0)$ if $K=K^{\perp}$.

Given a subgroup $K<G$, the isomorphism classes containing coideals correspond exactly to the following orbits:

when $K\neq K^{\perp}$, to the four orbits $\{(\lambda,\varnothing)/\lambda\in G/K\}$, $\{(\varnothing,\mu),/\mu\in G/K^{\perp}\}$, $\{(G/K,\mu)/\mu\in G/K^{\perp}\}$, $\{(\lambda,G/K^{\perp})/\lambda\in G/K\}$,

when $K=K^{\perp}$, to the two orbits $\{(\lambda,\varnothing)\cup(\varnothing,\lambda),/\lambda\in G/K\}$ and $\{(G/K,\lambda)\cup(\lambda,G/K)/\lambda\in G/K\}$.

In fact, we give an explicit construction of representatives of all isomorphism classes of indecomposable finite dimensional $\mathfrak{G}_{\mathcal{TY}}$-$C^*$-algebras and indecomposable (weak) coideals of $\mathfrak{G}_{\mathcal{TY}}$.

Our references are: to [5] for tensor categories, to [12] for C^*-tensor categories and to [14] for weak Hopf algebras (finite quantum groupoids).

2 Preliminaries

2.1 Weak Hopf C^*-algebras

A *weak Hopf C^*-algebra* (WHA) $\mathfrak{G}=(B,\Delta,S,\varepsilon)$ is a finite dimensional C^*-algebra B with the comultiplication $\Delta:B\to B\otimes B$, counit $\varepsilon:B\to\mathbb{C}$, and antipode $S:B\to B$ such that (B,Δ,ε) is a coalgebra and the following axioms hold for all $b,c,d\in B$:

(1) Δ is a (not necessarily unital) $*$-homomorphism :

$$\Delta(bc)=\Delta(b)\Delta(c),\quad \Delta(b^*)=\Delta(b)^*,$$

(2) The unit and counit satisfy the identities (we use the Sweedler leg notation $\Delta(c)=c_1\otimes c_2$, $(\Delta\otimes id_B)\Delta(c)=c_1\otimes c_2\otimes c_3$ etc.):

$$\begin{aligned}\varepsilon(bc_1)\varepsilon(c_2d) &= \varepsilon(bcd),\\ (\Delta(1)\otimes 1)(1\otimes\Delta(1)) &= (\Delta\otimes id_B)\Delta(1),\end{aligned}$$

(3) S is an anti-algebra and anti-coalgebra map such that

$$\begin{aligned} m(id_B \otimes S)\Delta(b) &= (\varepsilon \otimes id_B)(\Delta(1)(b \otimes 1)), \\ m(S \otimes id_B)\Delta(b) &= (id_B \otimes \varepsilon)((1 \otimes b)\Delta(1)), \end{aligned}$$

where m denotes the multiplication.

The right hand sides of two last formulas are called *target* and *source counital maps* ε_t and ε_s, respectively. Their images are unital C^*-subalgebras of B called *target* and *source counital subalgebras* B_t and B_s, respectively.

The dual vector space $\hat{B}$ has a natural structure of a weak Hopf C^*-algebra $\hat{\mathfrak{G}} = (\hat{B}, \hat{\Delta}, \hat{S}, \hat{\varepsilon})$ given by dualizing the structure operations of B:

$$\begin{aligned} < \varphi\psi,\, b > &= < \varphi \otimes \psi,\, \Delta(b) >, \\ < \hat{\Delta}(\varphi),\, b \otimes c > &= < \varphi,\, bc >, \\ < \hat{S}(\varphi),\, b > &= \overline{< \varphi,\, S(b) >}, \\ < \phi^*, b > &= \overline{< \varphi,\, S(b)^* >}, \end{aligned}$$

for all $b, c \in B$ and $\varphi, \psi \in \hat{B}$. The unit of $\hat{B}$ is ε and the counit is 1.

The counital subalgebras commute elementwise, we have $S \circ \varepsilon_s = \varepsilon_t \circ S$ and $S(B_t) = B_s$. We say that B is *connected* if $B_t \cap Z(B) = \mathbb{C}$ (where $Z(B)$ is the center of B), coconnected if $B_t \cap B_s = \mathbb{C}$, and *biconnected* if both conditions are satisfied.

The antipode S is unique, invertible, and satisfies $(S \circ *)^2 = id_B$. We will only consider *regular* quantum groupoids, i.e., such that $S^2|_{B_t} = id$. In this case, there exists a canonical positive element H in the center of B_t such that S^2 is an inner automorphism implemented by $G = HS(H)^{-1}$, i.e., $S^2(b) = GbG^{-1}$ for all $b \in B$. The element G is called the canonical group-like element of B, it satisfies the relation $\Delta(G) = (G \otimes G)\Delta(1) = \Delta(1)(G \otimes G)$.

There exists a unique positive functional h on B, called a *normalized Haar measure* such that

$$(id_B \otimes h)\Delta = (\varepsilon_t \otimes h)\Delta, \quad h \circ S = h, \quad h \circ \varepsilon_t = \varepsilon, \quad (id_B \otimes h)\Delta(1_B) = 1_B.$$

We will dehote by H_h the GNS Hilbert space generated by B and h and by $\Lambda_h : B \to H_h$ the corresponding GNS map.

2.2 Unitary representations and corepresentations of a weak Hopf C^*-algebra

Let $\mathfrak{G} = (B, \Delta, S, \varepsilon)$ be a weak Hopf C^*-algebra. We denote by $\varepsilon_t, \varepsilon_s$ its target and source counital maps, by B_t and B_s its target and source subalgebras, respectively, and by G its canonical group-like element. We also denote by h the normalized Haar measure of $\mathfrak{G}$.

Any object of the category $URep(\mathfrak{G})$ of unitary representations of $\mathfrak{G}$ is a left B-module of finite rank such that the underlying vector space is a Hilbert space H with a scalar product $< \cdot, \cdot >$ satisfying

$$< b \cdot v, w >=< v, b^* \cdot w >, \quad \text{for all} \quad v, w \in H,\ b \in B.$$

$URep(\mathfrak{G})$ is a semisimple category whose morphisms are B-linear maps and simple objects are irreducible B-modules. One defines the tensor product of two objects $H_1, H_2 \in URep(\mathfrak{G})$ as the Hilbert subspace $\Delta(1_B) \cdot (H_1 \otimes H_2)$ of the usual tensor product together with the action of B given by Δ. Here we use the fact that $\Delta(1_B)$ is an orthogonal projection.

Tensor product of morphisms is the restriction of the usual tensor product of B-module morphisms. Let us note that any $H \in URep(\mathfrak{G})$ is automatically a B_t-bimodule via $z \cdot v \cdot t := zS(t) \cdot v,\ \forall z, t \in B_t, v \in E$, and that the above tensor product is in fact $\otimes_{B_t}$, moreover the B_t-bimodule structure for $H_1 \otimes_{B_t} H_2$ is given by $z \cdot \xi \cdot t = (z \otimes S(t)) \cdot \xi,\ \forall z, t \in B_t, \xi \in H_1 \otimes_{B_t} H_2$. The above tensor product is associative, so the associativity isomorphisms are trivial. The unit object of $URep(\mathfrak{G})$ is B_t with the action of B given by $b \cdot z := \varepsilon_t(bz),\ \forall b \in B, z \in B_t$ and the scalar product $< z, t >= h(t^* z)$.

For any morphism $f : H_1 \to H_2$, let $f^* : H_2 \to H_1$ be the adjoint linear map: $< f(v), w >=< v, f^*(w) >,\ \forall v \in H_1, w \in H_2$. Clearly, f^* is B-linear, $f^{**} = f$, $(f \otimes_{B_t} g)^* = f^* \otimes_{B_t} g^*$, and $End(H)$ is a C^*-algebra, for any object H. So $URep(\mathfrak{G})$ is a finite C^*-multitensor category ($\mathbf{1}$ can be decomposable).

The conjugate object for any $H \in URep(\mathfrak{G})$ is the dual vector space $\hat{H}$ naturally identified ($v \mapsto \overline{v}$) with the conjugate Hilbert space $\overline{H}$ with the action of B defined by $b \cdot \overline{v} = \overline{G^{1/2} S(b)^* G^{-1/2} \cdot v}$, where G is the canonical group-like element of $\mathfrak{G}$. Then the rigidity morphisms defined by

$$R_H(1_B) = \Sigma_i (G^{1/2} \cdot \overline{e}_i \otimes_{B_t} \cdot e_i),\ \overline{R}_H(1_B) = \Sigma_i (e_i \otimes_{B_t} G^{-1/2} \cdot \overline{e}_i), \qquad (1)$$

where $\{e_i\}_i$ is any orthogonal basis in H, satisfy all the needed properties - see [3], 3.6. Also, it is known that the B-module B_t is irreducible if and only if $B_s \cap Z(B) = \mathbb{C}1_B$, i.e., if $\mathfrak{G}$ is connected. So that, we have

Proposition 2.1 *$URep(\mathfrak{G})$ is a rigid finite C^*-multitensor category with trivial associativity constraints. It is C^*-tensor if and only if $\mathfrak{G}$ is connected.*

Definition 2.2 *A* **right unitary corepresentation** *U of $\mathfrak{G}$ on a Hilbert space H_U is a partial isometry $U \in B(H_U) \otimes B$ such that:*

(i) $(id \otimes \Delta)(U) = U_{12}U_{13}$.

(ii) $(id \otimes \varepsilon)(U) = id$.

If U and V are two right corepresentations on Hilbert spaces H_U and H_V, respectively, a morphism between them is a bounded linear map $T \in B(H_U, H_V)$ such that $(T \otimes 1_B)U = V(T \otimes 1_B)$. We denote by $UCorep(\mathfrak{G})$ the category whose objects are unitary corepresentations on **finite dimensional** *vector spaces with morphisms as above.*

If $\mathfrak{G}$ is coconnected (i.e., if $B_t \cap B_s = \mathbb{C}1_B$), $UCorep(\mathfrak{G})$ is a rigid C^*-tensor category with trivial associativities isomorphic to $URep(\hat{\mathfrak{G}})$. Namely, any H_U is a right B-comodule via $v \mapsto U(v \otimes 1_B)$, therefore, automatically a (B_s, B_s)-bimodule. Then tensor product $U \bigoplus V := U_{13}V_{23}$ acts on $H_U \otimes_{B_s} H_V$, the unit object $U_\varepsilon \in B(B_s) \otimes B$ is defined by $z \otimes b \mapsto \Delta(1_B)(1_B \otimes zb)$, $\forall z \in B_s, b \in B$, and the rigidity morphisms related to the conjugate $\overline{U}$ of an object U which acts on the conjugate Hilbert space $\overline{H}_U$ of H_U, are

$$R_U(1_B) = \Sigma_i(\hat{G}^{1/2} \cdot \overline{e}_i \otimes_{B_s} e_i),\ \overline{R}_U(1_B) = \Sigma_i(e_i \otimes_{B_s} \hat{G}^{-1/2} \cdot \overline{e}_i), \qquad (2)$$

where $\{e_i\}_i$ is any orthogonal basis in H_U. We denote by Ω an exhaustive set of representatives of the equivalence classes of irreducibles in $UCorep(\mathfrak{G})$.

Denote H_{U^x} by H^x, then $U^x = \oplus_{i,j} m^x_{i,j} \otimes U^x_{i,j}$, where $m^x_{i,j}$ are the matrix units of $B(H^x)$ with respect to some orthogonal basis $\{e_i\} \in H^x$ and $U^x_{i,j}$ are the corresponding matrix coefficients of U^x. Recall that $B = \oplus_{x \in \Omega} B_{U^x}$, where $B_{U^x} = Span(U^x_{i,j})$.

2.3 The Hayashi's fiber functor and reconstruction theorem.

Let $\mathcal{C}$ be a rigid finite C^*-tensor category and $\Omega = Irr(\mathcal{C})$ be an exhaustive set of representatives of equivalence classes of its simple objects. Let R be the C^*-algebra $R = \mathbb{C}^\Omega = \bigoplus_{x \in \Omega} \mathbb{C}p_x$, where $p_x = p_x^*$ are mutually orthogonal idempotents: $p_x p_y = \delta_{x,y} p_x$, for all $x, y \in \Omega$. Let us define a functor $\mathcal{H}$ from $\mathcal{C}$

to the category $Corr_f(R)$ of finite dimensional Hilbert R-bimodules (called R-correspondences) by:

$$\mathcal{H}(x) = H^x = \bigoplus_{y,z\in\Omega} Hom(z, y\otimes x), \quad \text{for every } x \in \Omega,$$

where $Hom(x, y)$ is the vector space of morphisms $x \to y$. The R-bimodule structure on H^x is given by:

$$p_y \cdot H^x \cdot p_z = Hom(z, y \otimes x), \quad \text{for all} \quad x, y, z \in \Omega.$$

If $f \in Hom(x, y)$, then $\mathcal{H}(f) : H^x \to H^y$ is defined by:

$$\mathcal{H}(f)(g) = (id_z \otimes f) \cdot g, \quad \text{for any } z, t \in \Omega \text{ and } g \in p_z \cdot H^x \cdot p_t.$$

The tensor structure of $\mathcal{H}$ is a family of natural isomorphisms $\mathcal{H}_{x,y} : H^x \otimes H^y \to H^x \underset{R}{\otimes} H^y$ defined by:

$$\mathcal{H}_{x,y}(v \otimes w) = a_{z,x,y} \cdot (v \otimes id_y) \cdot w \in p_z \cdot H^{(x\otimes y)} \cdot p_s, \tag{3}$$

for all $v \in p_z \cdot H^x \cdot p_t, w \in p_t \cdot H^y \cdot p_s, z, s, t \in \Omega$. Here $a_{z,x,y}$ are the associativity isomorphisms of $\mathcal{C}$.

We define the scalar product on H^x as follows. If $x, y, z \in \Omega$ and $f, g \in Hom(z, y \otimes x)$, then $g^* \in Hom(y \otimes x, z)$ and $g^* \cdot f \in End(z) = \mathbb{C}$, so one can put $< f, g >_x = g^* \cdot f$. The subspaces $Hom(z, y \otimes x)$ are declared to be orthogonal, so $H^x \in Corr_f(R)$. Dually, $\overline{H}^x \in Corr_f(R)$ via $z_1 \cdot \overline{v} \cdot z_2 = \overline{z_2^* \cdot v \cdot z_1^*}$, for all $z_1, z_2 \in R, v \in H^x$. Now one can check that $\mathcal{H}$ is a unitary tensor functor in the sense of [12] 2.1.3.

Theorem 2.3 *(a C^*-version of the Hayashi's theorem -see [7], [16])*

Let $\mathcal{C}$ be a rigid finite C^*-tensor category, $\Omega = Irr(\mathcal{C})$ and $\mathcal{H} : \mathcal{C} \to Corr_f(R)$ be the Hayashi's functor, where $R = \mathbb{C}^{|\Omega|}$. Then the vector space

$$B = \bigoplus_{x\in\Omega} H_x \otimes \overline{H}_x, \tag{4}$$

has a regular biconnected weak Hopf C^*-algebra structure $\mathfrak{G}$ such that $\mathcal{C} \cong UCorep(\mathfrak{G})$ as rigid C^*-tensor categories.

Explicitly, if $v, w \in H^x, g, h \in H^y$ and $\{e_j^x\}$ is an orthonormal basis in H^x, for all $x, y \in \Omega$, then:

$$(w \otimes \overline{v})_x \cdot (g \otimes \overline{h})_y = (\mathcal{H}_{x,y}(w \otimes g) \otimes \overline{\mathcal{H}_{x,y}(v \otimes h)})_{x \otimes y} \in H^{(x \otimes y)} \otimes \overline{H^{(x \otimes y)}} \quad (5)$$

$$\Delta(w \otimes \overline{v}) = \bigoplus_j (w \otimes \overline{e_j^x})_x \otimes (e_j^x \otimes \overline{v})_x, \quad (6)$$

$$\varepsilon(w \otimes \overline{v}) = < w, v >_x . \quad (7)$$

Now define an antipode and an involution. Consider the natural isomorphisms $\Phi_x : H^x \to \overline{H}^{x^*}$ and $\Psi_x : \overline{H}^x \to H^{x^*}$, where x^* is the dual of $x \in \Omega$:

$$\Phi_x(v) = (id_y \otimes \overline{R_x}^*) \cdot a_{y,x,x^*} \cdot (v \otimes id_{x^*}), \Psi_x(\overline{v}) = (\overline{v} \otimes id_{x^*}) \cdot a^{-1}_{y,x,x^*} \cdot (id_y \otimes \overline{R_x}), \quad (8)$$

where $x, y, z \in \Omega$, we identify y with $y \otimes 1$, $v \in p_y \cdot H^x \cdot p_z$, $\overline{R_x}$ and a_{y,x,x^*} are, respectively, the rigidity morphisms and the associativities in $\mathcal{C}$. Then:

$$S(w \otimes \overline{v}) = \Psi_x(\overline{v}) \otimes \Phi_x(w), \quad (9)$$

$$(w \otimes \overline{v})^* = w^\natural \otimes \overline{v}^\flat, \text{ where } w^\natural = \Psi_x(\overline{w}),\ \overline{v}^\flat = \Phi_x(v). \quad (10)$$

Any H^x is a right B-comodule via

$$\mathfrak{a}_x(v) = \sum_j e_j^x \otimes \overline{e_j^x} \otimes v, \quad \text{where} \quad v \in H_x,$$

one checks that it is unitary which gives the equivalence $\mathcal{C} \cong UCorep(\mathfrak{G})$.

The algebra of the dual quantum groupoid $\hat{\mathfrak{G}}$ is

$$\hat{B} = \bigoplus_{x \in \Omega} B(H_x), \quad (11)$$

the duality is given, for all $x \in \Omega, A \in B(H_x), v, w \in H_x$ by:

$$< A, w \otimes \overline{v} > = < Aw, v >_x .$$

$\hat{B}$ is clearly a C^*-algebra with the obvious matrix product and involution,

Notations 2.4 *For all $x, y \in \Omega$ and all $v \in H^x$, $w \in H^y$, we denote:*

$$v \circ w = \mathcal{H}_{x,y}(v \otimes_R w)$$

Remark 2.5 *Let* 0 *be the unit element of* $\mathcal{C}$*, and* $H^0 := \bigoplus_{x\in\Omega} Hom(x,x)$*, then using (3) and (8) it is easy to check that* $(H^0, \circ, \sharp)$ *is a commutative* C^**-algebra and if, for all* $x \in \Omega$*,* v_x^0 *is a normalized vector in* $Hom(x,x)$*, then* $(v_x^0)_{x\in\Omega}$ *is a basis of mutually orthogonal projections in* H^0*.*

Remark 2.6 *Let* $\mathcal{C}$ *be a rigid finite* C^**-tensor category and* $\mathcal{F} : \mathcal{C} \to Corr_f(R)$ *be a unitary tensor functor, where* R *is a finite dimensional unital* C^**-algebra. Then there exists [17] a regular biconnected finite quantum groupoid* $\mathfrak{G}$ *with* $B_t \cong B_s \cong R$ *such that* $\mathcal{C} \cong UCorep(\mathfrak{G})$ *as* C^**-tensor categories. For any fixed* $\mathcal{C}$*, the set of* C^**-algebras* R *for which the above mentioned functor* $\mathcal{F}$ *exists, contains at least* $R = \mathbb{C}^{|\Omega|}$ *(where* $\Omega = Irr(\mathcal{C})$*), then* $\mathcal{F} = \mathcal{H}$*. In general, this set contains several elements, and the corresponding WHAs are called Morita equivalent.*

In particular, if the above set of functors contains a **fiber** *functor* $\mathcal{F} : \mathcal{C} \to Hilb_f$*, i.e.,* $R = \mathbb{C}$*, the corresponding quantum groupoids are Morita equivalent to a usual* C^**-Hopf algebra.*

2.4 Coactions.

Definition 2.7 *A right coaction of a WHA* $\mathfrak{G}$ *on a unital* $*$*-algebra* A*, is a* $*$*-homomorphism* $\mathfrak{a} : A \to A \otimes B$ *such that:*

1) $(\mathfrak{a} \otimes i)\mathfrak{a} = (id_A \otimes \Delta)\mathfrak{a}$*.*

2) $(id_A \otimes \varepsilon)\mathfrak{a} = id_A$*.*

3) $\mathfrak{a}(1_A) \in A \otimes B_t$*.*

One also says that $(A, \mathfrak{a})$ *is a* $\mathfrak{G}$*-*$*$*-algebra.*

If A is a C^*-algebra, then $\mathfrak{a}$ is automatically continuous, even an isometry.

There are $*$-homomorphism $\alpha : B_s \to A$ and $*$-antihomomorphism $\beta : B_s \to A$ with commuting images defined by $\alpha(x)\beta(y) := (id_A \otimes \varepsilon)[(1_A \otimes x)\mathfrak{a}(1_A)(1_A \otimes y)]$, for all $x, y \in B_s$. We also have $\mathfrak{a}(1_A) = (\alpha \otimes id_B)\Delta(1_B)$,

$$\mathfrak{a}(\alpha(x)a\beta(y)) = (1_A \otimes x)\mathfrak{a}(a)(1_A \otimes y), \tag{12}$$

and

$$(\alpha(x) \otimes 1_B)\mathfrak{a}(a)(\beta(y) \otimes 1_B) = (1_A \otimes S(x))\mathfrak{a}(a)(1_A \otimes S(y)). \tag{13}$$

The set $A^{\mathfrak{a}} = \{a \in A | \mathfrak{a}(a) = \mathfrak{a}(1_A)(a \otimes 1_B)\}$ is a unital $*$-subalgebra of A (it is a unital C^*-subalgebra of A when A is a C^*-algebra) commuting pointwise with $\alpha(B_s)$. A coaction $\mathfrak{a}$ is called *ergodic* if $A^{\mathfrak{a}} = \mathbb{C}1_A$.

Definition 2.8 *A $\mathfrak{G}-C^*$-algebra $(A,\mathfrak{a})$ is said to be indecomposable if it cannot be presented as a direct sum of two $\mathfrak{G}-C^*$-algebras.*

It is easy to see that $(A,\mathfrak{a})$ is indecomposable if and only if $Z(A)\cap A^{\mathfrak{a}}=\mathbb{C}1_A$. Clearly, any ergodic $\mathfrak{G}-C^*$-algebra is indecomposable.

For any $(U,H_U)\in UCorep(\mathfrak{G})$, we define the *spectral subspace* of A corresponding to (U,H_U) by

$$A_U:=\{a\in A|\mathfrak{a}(a)\in\mathfrak{a}(1_A)(A\otimes B_U)\}.$$

Let us recall the properties of the spectral subspaces:

(i) All A_U are closed.

(ii) $A=\oplus_{x\in\Omega}A_{U^x}$.

(iii) $A_{U^x}A_{U^y}\subset\oplus_z A_{U^z}$, where z runs over the set of all irreducible direct summands of $U^x\bigoplus U^y$.

(iv) $\mathfrak{a}(A_U)\subset\mathfrak{a}(1_A)(A_U\otimes B_U)$ and $A_{\overline{U}}=(A_U)^*$.

(v) A_ε is a unital C^*-algebra.

2.5 Categorical duality.

Let us recall the main result of [20]:

Theorem 2.9 *Given a regular coconnected WHA $\mathfrak{G}$, the following two categories are equivalent:*

(i) The category of unital $\mathfrak{G}$-C^*-algebras with unital $\mathfrak{G}$-equivariant $*$-homomorphisms as morphisms.

(ii) The category of pairs $(\mathcal{M},M)$, where $\mathcal{M}$ is a left module C^*-category with trivial module associativities over the C^*-tensor category $UCorep(\mathfrak{G})$ and M is a generator in $\mathcal{M}$, with equivalence classes of unitary module functors respecting the prescribed generators as morphisms.

In particular, given a unital $\mathfrak{G}$-C^*-algebra A, one constructs the C^*-category $\mathcal{M}=\mathcal{D}_A$ of finitely generated right Hilbert A-modules which are equivariant, that is, equipped with a compatible right coaction [1]. Any its object is automatically a (B_s,A)-bimodule, and the bifunctor $U\boxtimes X:=H_U\otimes_{B_s}X\in\mathcal{D}_A$, for all $U\in UCorep(\mathfrak{G})$ and $X\in\mathcal{D}_A$, turns $\mathcal{D}_A$ into a left module C^*-category over $UCorep(\mathfrak{G})$ with generator A and trivial associativities.

Vice versa, if a pair $(\mathcal{M},M)$ is given, the construction of a $\mathfrak{G}$-C^*-algebra $(A,\mathfrak{a})$ contains the following steps. First, denote by R the unital C^*-algebra

$End(M)$ and consider the functor $F : \mathcal{C} \to Corr(R)$ defined on the objects by $F(U) = Hom_{\mathcal{M}}(M, U \boxtimes M)\ \forall U \in \mathcal{C}$. Here $X = F(U)$ is a right R-module via the composition of morphisms, a left R-module via $rX = (id \otimes r)X$, the R-valued inner product is given by $< X, Y >= X^*Y$, the action of F on morphisms is defined by $F(T)X = (T \otimes id)X$. The weak tensor structure of F (in the sense of [11]) is given by $J_{X,Y}(X \otimes Y) = (id \otimes Y)X$, for all $X \in F(U), Y \in F(V), U, V \in UCorep(\mathfrak{G})$.

Then consider two vector spaces:

$$A = \bigoplus_{x \in \Omega} A_{U^x} := \bigoplus_{x \in \Omega} (F(U^x) \otimes \overline{H^x}) \tag{14}$$

and

$$\tilde{A} = \bigoplus_{U \in \|UCorep(\mathfrak{G})\|} A_U := \bigoplus_{U \in \|UCorep(\mathfrak{G})\|} (\overline{F(U)} \otimes H_U), \tag{15}$$

where $F(U) = \bigoplus_i F(U_i)$ corresponds to the decomposition $U = \bigoplus U_i$ into irreducibles, and $\|UCorep(\mathfrak{G})\|$ is an exhaustive set of representatives of the equivalence classes of objects in $UCorep(\mathfrak{G})$ (these classes constitute a countable set). $\tilde{A}$ is a unital associative algebra with the product

$$(X \otimes \overline{\xi})(Y \otimes \overline{\eta}) = (id \otimes Y)X \otimes (\overline{\xi} \otimes_{B_s} \overline{\eta}),\ \forall (X \otimes \overline{\xi}) \in A_U, (Y \otimes \overline{\eta}) \in A_V,$$

and the unit

$$1_{\tilde{A}} = id_M \otimes \overline{1_B}.$$

Note that $(id \otimes Y)X = J_{X,Y}(X \otimes Y) \in F(U \oplus V)$. Then, for any $U \in UCorep(G)$, choose isometries $w_i : H_i \to H_U$ defining the decomposition of U into irreducibles, and construct the projection $p : \tilde{A} \to A$ by

$$p(X \otimes \overline{\xi}) = \Sigma_i (F(w_i^*)X \otimes \overline{w_i^* \xi}),\ \forall (X \otimes \overline{\xi}) \in A_U, \tag{16}$$

which does not depend on the choice of w_i. Then A is a unital $*$-algebra with the product $x \cdot y := p(xy)$, for all $x, y \in A$ and the involution $x^* := p(x^\bullet)$, where $(X \otimes \overline{\xi})^\bullet := (id \otimes X^*)F(\overline{R}_U) \otimes \overline{\hat{G}^{1/2}\xi}$, for all $\xi \in H_U, X \in F(U), U \in UCorep(\mathfrak{G})$. Here $\overline{R}_U$ is the rigidity morphism from (2). Finally, the map

$$\mathfrak{a}(X \otimes \overline{\xi}_i) = X \otimes \Sigma_j (\overline{\xi}_j \otimes U^x_{j,i}), \tag{17}$$

where $\{\xi_i\}$ is an orthogonal basis in H^x and ($U^x_{i,j}$ are the matrix elements of U^x in this basis, is a right coaction of $\mathfrak{G}$ on A. Moreover, A admits a unique C^*-completion $\overline{A}$ such that $\mathfrak{a}$ extends to a continuous coaction of $\mathfrak{G}$ on it.

Remark 2.10 *1) We say that a $UCorep(\mathfrak{G})$-module category is indecomposable if it is not equivalent to a direct sum of two nontrivial $UCorep(\mathfrak{G})$-module subcategories. Theorem 2.9 implies that a $\mathfrak{G}-C^*$-algebra $(A,\mathfrak{a})$ is indecomposable if and only if the $UCorep(\mathfrak{G})$-module category $\mathcal{M}$ is indecomposable.*

2) Let I be a unital right coideal $*$-subalgebra of B. Then $I_{U^x}=I\cap B_{U^x}$ and $F(U^x)$ can be identified with a Hilbert subspace of H^x $(\forall x\in\Omega)$ and the coaction is the restriction of Δ.

Example 2.11 *The C^*-algebra B with coproduct Δ viewed as $\mathfrak{G}$-C^*-algebra, corresponds to the $UCorep(\mathfrak{G})$-module C^*-category $\mathcal{C}orr_f(B_s)$ with generator $M=B_s$: for any element $U\in UCorep(\mathfrak{G})$ and $N\in Corr_f(B_s)$, one defines $U\boxtimes N:=F(U)\otimes_{B_s}N$, where the functor $F: UCorep(\mathfrak{G})\to Corr_f(B_s)$ $(F(U)=H_U)$ is the forgetful functor. Indeed, identifying $\mathcal{M}(B_s,H_U)$ with H_U, we get an isomorphism of the algebra $\tilde{A}$ constructed from the pair $(\mathcal{M},M)$ onto $\tilde{B}=\bigoplus\limits_U(H_U\otimes\overline{H}_U)$ and then an isomorphism $A\cong B=\bigoplus\limits_{x\in\hat{G}}(H_x\otimes\overline{H}_x)$ such that $p:\tilde{A}\to A$ turns into the map $\tilde{B}\to B$ sending $\xi\otimes\overline{\eta}\in H_U\otimes\overline{H}_U$ into the matrix coefficient $U_{\xi,\eta}$.*

3 Classifying Indecomposable Weak Coideals

If $dim(A)<\infty$, we have the following remarks.

Remark 3.1 *If $(A,\mathfrak{a})$ is a finite dimensional $\mathfrak{G}-C^*$-algebra, then $\mathcal{M}=\mathcal{D}_A$ is a semisimple C^*-category. Indeed, $dim(Hom_{\mathcal{M}}(\mathcal{E},\mathcal{E}))<\infty$, for any $\mathcal{E}\in D_A$ which is finitely generated. Then the proof of [4], Proposition 3.9 applies. As A is a generator of $\mathcal{M}$, the set $\{M_\lambda|\lambda\in\Lambda\}$ of its (classes of) simple objects is finite and we have the corresponding fusion rule*

$$U_x\boxtimes M_\lambda=\sum_\mu n^\mu_{x,\lambda}M_\mu,\ \text{where } x\in\Omega, n^\mu_{x,\lambda}=dim(Hom_{\mathcal{M}}(U_x\boxtimes M_\lambda,M_\mu))\in\mathbb{Z}_+. \tag{18}$$

The associativity and the unit object conditions mean, respectively, that

$$\sum_{z\in\Omega}c^z_{x,y}n^\rho_{z,\lambda}=\sum_{\mu\in\Lambda}n^\rho_{x,\mu}n^\mu_{y,\lambda},\ \text{and } n^\rho_{1,\lambda}=\delta_{\rho,\lambda},\quad\forall x,y\in\Omega,\rho,\lambda\in\Lambda, \tag{19}$$

where $c^z_{x,y}$ are the fusion coefficients of $\mathcal{C}=UCorep(\mathfrak{G})$. Proposition 7.1.6 of [5] gives $n^\mu_{x,\lambda}=n^\lambda_{x^*,\mu}$, for all $\lambda,\mu\in\Lambda,x\in\Omega$.

Remark 3.2 *If A is a coideal of B, then, due to [19], Theorem 1.1, there is an inclusion $j : \mathcal{M} \mapsto \mathcal{C}$ such that*

$$j(M) = \underset{x\in\Omega}{\oplus} U^x, \tag{20}$$

where $\mathcal{M}$ is the left $\mathcal{C}$-module category with generator M coming from $(A, \Delta|_A)$ and $\mathcal{C} = UCorep(\mathfrak{G})$ is viewed as a $\mathcal{C}$-module category with generator the $\underset{x\in\Omega}{\oplus} U^x$.

If Λ is the set of irreducibles of $\mathcal{M}$ (we denote them by M_λ), we can write $j(M_\lambda) = \underset{x\in\Omega}{\Sigma} a_{\lambda,x}U^x$, for all $\lambda \in \Lambda$, where $a_{\lambda,x} \in \mathbb{Z}_+$. Writing $M = \underset{\lambda\in\Lambda}{\Sigma} m_\lambda M_\lambda$, we must have:

$$\underset{\lambda\in\Lambda}{\Sigma} m_\lambda a_{\lambda,x} = 1, \quad \textit{for all} \quad x \in \Omega. \tag{21}$$

Recall that due to the reconstruction theorem for $\mathfrak{G}$, any $H^x(x \in \Omega)$ is the direct sum of 1-dimensional subspaces $Hom(z, y \otimes x)$, where $y, z \in \Omega$ are such that $z \subset (y \otimes x)$. In particular, $H^0 = \underset{z\in\Omega}{\oplus} Hom(z, z)$ (where 0 denotes the trivial corepresentation of $\mathfrak{G}$); we will denote by v_z^0 a norm one vector generating $Hom(z, z)$ viewed as a subspace of H^0.

The following lemma allows to select weak coideals of B from all $\mathfrak{G} - C^*$-algebras.

Lemma 3.3 *Let us fix a $UCorep(\mathfrak{G})$-module category $\mathcal{M}$ and a generator M in it, and let $(A, \mathfrak{a})$ be a $\mathfrak{G}$-algebra constructed from this data using the weak tensor functor $(F, J_{U,V})$. Then:*

a) $(A, \mathfrak{a})$ is a weak coideal of B if and only if each $F(U^x)$ can be identified with a subspace $X^x \subset H^x$ such that the map $\zeta \mapsto \zeta^\natural = \Psi_x(\overline{\zeta})$ sends X^x onto $X^{\overline{x}} \cong F(\overline{U^x})$ and $J_{U^x,U^y} = \mathcal{H}_{x,y}$, for all $x, y \in \Omega$.

b) X^0 is a C^*-subalgebra of H^0. The unit of X^0 is $v_\Gamma^0 := \underset{x\in\Gamma}{\oplus} v_x^0$, where $\Gamma \subset \Omega$ is some nonempty subset. $A = \underset{x\in\Omega}{\oplus}(X^x \otimes \overline{H}^x)$ is a coideal if and only if $\Gamma = \Omega$.

c) A weak coideal $A = \underset{x\in\Omega}{\oplus}(X^x \otimes \overline{H}^x)$ is decomposable if and only if $Z(A)$ contains an element of the form $p = v_{\Gamma_0}^0 \otimes v_\Omega^0$, where Γ_0 is a proper nonempty subset of Γ.

d) For any two identifications, $F(U^x) \cong X^x$ and $F(U^x) \cong \tilde{X}^x, \forall x \in \Omega$, satisfying the above mentioned conditions, the corresponding weak coideals $\underset{x\in\Omega}{\oplus}(X^x \otimes \overline{H}^x)$ and $\underset{x\in\Omega}{\oplus}(\tilde{X}^x \otimes \overline{H}^x)$ are isomorphic as $\mathfrak{G}$-C^*-algebras.

Proof. a) If $(A, \mathfrak{a})$ is a weak coideal of B, then $A_U \subset B_U$, for any $U \in UCorep(\mathfrak{G})$. Indeed, by [20], Proposition 3.17 $A_U = \{a \in A | \Delta(a) \in \Delta(1_A)(A \otimes B_U)\}$, but $\Delta(1_A) = \Delta(1_B)(1_A \otimes 1_B)$, hence $\Delta(a) \in \Delta(1_B)(A \otimes B_U) \subset \Delta(1_B)(B \otimes B_U)$, so that $A_U \subset B_U$. It follows from [20], Theorem 4.12 and Theorem 2.21, respectively, that $A_{U^x} \cong F(U^x) \otimes \overline{H^x}$ and $B_{U^x} \cong H^x \otimes \overline{H^x}$, so the above inclusions mean that $F(U^x) \subset H^x$, for all $x \in \Omega$.

The multiplication in A is the restriction of that in B, therefore, comparing the formulas (15) and [20], (16) and using the relation $J_{U^x,U^y}(X \otimes_{B_s} Y) = (id \otimes Y)X (\forall X \in F(U^x), Y \in F(U^y))$, we have $J_{U^x,U^y} = \mathcal{H}_{x,y}$.

The involution in A sends $X \otimes \overline{\eta}$ onto $X^* \otimes (\overline{\eta})^\flat$ (see Subsection 2.4) and is the restriction of that in B, the last one is defined by $\zeta \otimes \overline{\eta} \mapsto \zeta^\natural \otimes (\overline{\eta})^\flat \ \forall \zeta, \eta \in H^x, x \in \Omega$. Then for $X \in F(U^x) \subset H^x$ we have $X^* = X^\natural$.

Conversely, suppose that $F(U^x) \subset H^x$ and $J_{U^x,U^y} = \mathcal{H}_{x,y}$, for all $x, y \in \Omega$. It follows from the argument above that the multiplication in A is the restriction of that in B. Next, compare the formulas [20], (29) for $\mathfrak{a}$ and [20], (14) for Δ. Since $B_{U^x} = H^x \otimes \overline{H}^x$, for any U^x - see [20], (12), the matrix coefficient $U^x_{\zeta,\eta}$ with respect to a basis $\{\zeta^x\}$ of H^x can be identified with $\zeta^x \otimes \eta^x$, for all $x \in \Omega$. Now it is clear that $\mathfrak{a}$ is the restriction of Δ. Finally, putting $X^* = X^\natural$ for any $X \in F(U^x)$ and using the fact that $F(U^x)^\natural = F(\overline{U^x})$, one checks that $(A, \mathfrak{a})$ is a coideal of B.

b) By Remark 2.5, $H^0 = \underset{x \in \Omega}{\oplus} \mathbb{C} v^0_x$ is a commutative unital C^*-algebra, v^0_x $(x \in \Omega)$ are mutually orthogonal projections, and if $A = \underset{x \in \Omega}{\oplus} (X^x \otimes \overline{H}^x)$ is a weak coideal of B, then X^0 is a C^*-subalgebra of H^0. Its spectral mutually orthogonal projectors are $v^0_{\Gamma_i}$, where $\Gamma_i \subset \Omega$ $(i = 1, ..., k_0 = dim(X^0))$ are disjoint subsets of Ω, the unit of X^0, i.e., the image of $F(id_M)$, is v^0_Γ, where $\Gamma = \sqcup_{i=1}^{k_0} \Gamma_i$. As $1_A = v^0_\Gamma \otimes \overline{v}^0_\Omega$ and $1_B = v^0_\Omega \otimes \overline{v}^0_\Omega$, A is a coideal if and only if $\Gamma = \Omega$.

c) One checks that $B_t = H^0 \otimes \overline{v}^0_\Omega$ and that any nontrivial orthoprojector $p \in [Z(A) \cap B_t]$ gives a decomposition $A = pA \oplus (1-p)A$ into the direct sum of two weak coideals of B. As $1_A = v^0_\Gamma \otimes \overline{v}^0_\Omega$, p must be of the form $v^0_{\Gamma_0} \otimes \overline{v}^0_\Omega$, where Γ_0 is a proper nonempty subset of Γ.

d) The two $\mathfrak{G}$-C^*-algebras are isomorphic because they correspond to the same couple $(\mathcal{M}, M)$. ▫

Corollary 3.4 *It follows from the definition of the functor F that $X^0 = F(U^0) = End_{\mathcal{M}}(M)$. This finite dimensional C^*-algebra is commutative due to the statement b) which is only possible if $m_\lambda \in \{0, 1\}$ for all $\lambda \in \Lambda$.*

4 Weak Hopf C^*-Algebras related to Tambara-Yamagami categories

4.1 Tambara-Yamagami categories

These categories denoted by $\mathcal{TY}(G,\chi,\tau)$ (G is a finite group; we consider them only over $\mathbb{C}$) are $\mathbb{Z}_2$-graded fusion categories whose 0-component is Vec_G - the category of finite dimensional G-graded vector spaces with trivial associativities (its simple objects are $g \in G$) and 1-component is generated by single simple object m. The Grothendieck ring of $\mathcal{TY}(G,\chi,\tau)$ is isomorphic to the $\mathbb{Z}_2$-graded fusion ring $\mathcal{TY}_G = \mathbb{Z}G \oplus \mathbb{Z}\{m\}$ such that $g \cdot m = m \cdot g = m$, $m^2 = \sum\limits_{g\in G} g$, $m = m^*$. These categories exist if and only if G is abelian, they are parameterized by non degenerate symmetric bicharacters $\chi : G \times G \to \mathbb{C}\backslash\{0\}$ and $\tau = \pm|G|^{-1/2}$ - see [18], [5], Example 4.10.5. The associativities $\phi(U,V,W) : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ are

$$\phi(g,h,k) = id_{g+h+k}, \quad \phi(g,h,m) = id_m, \quad \phi(m,g,h) = id_m,$$

$$\phi(g,m,h) = \chi(g,h)id_m, \quad \phi(g,m,m) = \bigoplus_{h\in G} id_h, \quad \phi(m,m,g) = \bigoplus_{h\in G} id_h,$$

$$\phi(m,g,m) = \bigoplus_{h\in G} \chi(g,h)id_h, \quad \phi(m,m,m) = (\tau\chi(g,h)^{-1}id_m)_{g,h},$$

where $g,h,k \in G$. The unit isomorphisms are trivial. $\mathcal{TY}(G,\chi,\tau)$ becomes a C^*-tensor category when $\chi : G \times G \to T = \{z \in \mathbb{C} | |z| = 1\}$, from now on we assume that this is the case. The dual objects are: $g^* = -g$, for all $g \in G$, and $m^* = m$. The rigidity morphisms are defined by $R_g : 0 \stackrel{id_0}{\to} g^* \otimes g$, $\overline{R_g} : 0 \stackrel{id_0}{\to} g \otimes g^*$, $R_m = \tau|G|^{1/2}\iota$, and $\overline{R_m} = |G|^{1/2}\iota$, where $\iota : 0 \to m \otimes m$ is the inclusion. Then $dim_q(g) = 1$, for all $g \in G$, and $dim_q(m) = \sqrt{|G|}$.

Let us apply Theorem 2.3 to the category $\mathcal{TY}(G,\chi,\tau)$ in order to construct a biconnected regular WHA $\mathfrak{G}_{\mathcal{TY}} = (B,\Delta,S,\varepsilon)$ with $UCorep(\mathfrak{G}_{\mathcal{TY}}) \cong \mathcal{TY}(G,\chi,\tau)$. The Hayashi's functor $\mathcal{H} : \mathcal{TY}(G,\chi,\tau) \to Corr_f(R)$, where C^*-algebra $R := End(\bigoplus\limits_{x\in\Omega} x) \cong \mathbb{C}^{|G|+1}$, was constructed in [10]. Denoting $\Omega_g = \Omega := G \sqcup \{m\}$ and $\Omega_m := G \sqcup \overline{G}$, where $g \in G$ and $\overline{G}$ is the second copy of G, one easily computes that $H^g \cong \mathbb{C}^{|G|+1}$, for all $g \in G$ and $H^m :\cong \mathbb{C}^{2|G|}$.

Let us fix a basis $\{v^x_y\}(y \in \Omega_x)$ in each H^x ($x \in \Omega$) choosing a norm one vector in every 1-dimensional vector subspace: $v^g_h \in Hom(h,(h-g)\otimes g)$, $v^g_m \in Hom(m, m \otimes g)$, $v^m_g \in Hom(m, g \otimes m)$, and $v^m_{\overline{g}} \in Hom(g, m \otimes m)$, where $g \in G$.

Lemma 4.1 *Using notations (10) and 2.4, for all $g,h,k\in G$, $x\in\Omega$, one has:*

$$v_k^g\circ v_x^h=\delta_{x,h+k}v_{h+k}^{g+h},\ v_m^g\circ v_x^h=\delta_{x,m}v_m^{g+h},$$
$$v_k^m\circ v_x^g=\delta_{x,m}\chi(g,k)v_k^m,\ v_{\overline{k}}^m\circ v_x^g=\delta_{x,g+k}v_{\overline{g+k}}^m,$$
$$v_x^g\circ v_{\overline{k}}^m=\delta_{x,m}\chi(g,k)v_{\overline{k}}^m,\ v_x^g\circ v_k^m=\delta_{x,k}v_{k-g}^m,$$
$$v_h^m\circ v_{\overline{k}}^m=v_k^{k-h},\ v_{\overline{h}}^m\circ v_k^m=\delta_{h,k}\tau\sum_{g\in G}\chi(g,h)^{-1}v_m^g.$$
$$(v_k^g)^\sharp=v_{k-g}^{-g},(v_m^g)^\sharp=v_m^{-g},(v_g^m)^\sharp=|G|^{1/2}v_{\overline{g}}^m,(v_{\overline{g}}^m)^\sharp=\tau^{-1}|G|^{1/2}v_g^m$$

Proof. For equations related to product $\circ$, these are computations made in [10] 2.1.5, where $\mathcal{H}_{x,y}$ must be replaced by $\mathcal{F}_{x,y}^{-1}$. Moreover, in the case of $\mathcal{TY}(G,\chi,\tau)$, the formulas of (8) imply that the isomorphisms $\Phi_x:H^x\to\overline{H^{x^*}}$ and $\Psi_x:\overline{H^x}\to H^{x^*}$ $(x\in\Omega)$ of Theorem 2.3 are given, for all $g,h\in G$, by:

$$\Phi_g(v_h^g)=\overline{v_{h-g}^{-g}},\ \Phi_g(v_m^g)=\overline{v_m^{-g}},\ \Phi_m(v_g^m)=|G|^{-1/2}\overline{v_{\overline{g}}^m},\ \Phi_m(v_{\overline{g}}^m)=\tau|G|^{-1/2}\overline{v_g^m},$$

$$\Psi_g(\overline{v_h^g})=v_{h-g}^{-g},\ \Psi_g(\overline{v_m^g})=v_m^{-g},\ \Psi_m(\overline{v_g^m})=|G|^{1/2}v_{\overline{g}}^m,\ \Psi_m(\overline{v_{\overline{g}}^m})=\tau^{-1}|G|^{1/2}v_g^m,$$

which implies, by (10) the formulas for involution $\sharp$. □

Now the whole structure of a WHA $\mathfrak{G}_{\mathcal{TY}}$ is given by formulas (4), (5), (6), and (7). It was shown in [10] that this WHA is isomorphic to its dual whose C^*-algebra $\hat{B}\cong\bigoplus_{x\in\Omega}B(H^x)$. This implies that $URep(\mathfrak{G}_{\mathcal{TY}})\cong UCorep(\mathfrak{G}_{\mathcal{TY}})$.

The isomorphisms $\Phi_x:H^x\to\overline{H^{x^*}}$ and $\Psi_x:\overline{H^x}\to H^{x^*}$ $(x\in\Omega)$ of Theorem 2.3 are now given by:

$$\Phi_g(v_h^g)=\overline{v_{h-g}^{-g}},\ \Phi_g(v_m^g)=\overline{v_m^{-g}},\ \Phi_m(v_g^m)=|G|^{-1/2}\overline{v_{\overline{g}}^m},\ \Phi_m(v_{\overline{g}}^m)=\tau|G|^{-1/2}\overline{v_g^m},$$

$$\Psi_g(\overline{v_h^g})=v_{h-g}^{-g},\ \Psi_g(\overline{v_m^g})=v_m^{-g},\ \Psi_m(\overline{v_g^m})=|G|^{1/2}v_{\overline{g}}^m,\ \Psi_m(\overline{v_{\overline{g}}^m})=\tau^{-1}|G|^{1/2}v_g^m,$$

which implies, for all $g,h,k\in G$:

$$S(v_h^g\otimes\overline{v_k^g})=v_{k-g}^{-g}\otimes\overline{v_{h-g}^{-g}},\ S(v_h^g\otimes\overline{v_m^g})=v_m^{-g}\otimes\overline{v_{h-g}^{-g}},\ S(v_m^g\otimes\overline{v_h^g})=v_{h-g}^{-g}\otimes\overline{v_m^{-g}},$$

$$S(v_m^g\otimes\overline{v_m^g})=v_m^{-g}\otimes\overline{v_m^{-g}},\ S(v_g^m\otimes\overline{v_h^m})=v_{\overline{h}}^m\otimes\overline{v_{\overline{g}}^m},\ S(v_g^m\otimes\overline{v_{\overline{h}}^m})=\tau^{-1}(v_h^m\otimes\overline{v_{\overline{g}}^m}),$$
$$S(v_{\overline{g}}^m\otimes\overline{v_h^m})=\tau(v_{\overline{h}}\otimes\overline{v_g^m}),\ S(v_{\overline{g}}^m\otimes\overline{v_{\overline{h}}^m})=v_h^m\otimes\overline{v_g^m}$$

and

$$(v_h^g \otimes \overline{v_k^g})^* = v_{h-g}^{-g} \otimes \overline{v_{k-g}^{-g}},\ (v_h^g \otimes \overline{v_m^g})^* = v_{h-g}^{-g} \otimes \overline{v_m^{-g}},\ (v_m^g \otimes \overline{v_h^g})^* = v_m^{-g} \otimes \overline{v_{h-g}^{-g}},$$

$$(v_m^g \otimes \overline{v_m^g})^* = v_m^{-g} \otimes \overline{v_m^{-g}},\ (v_g^m \otimes \overline{v_{\overline{h}}^m})^* = v_{\overline{g}}^m \otimes \overline{v_{\overline{h}}^m},\ (v_g^m \otimes \overline{v_{\overline{h}}^m})^* = \tau(v_{\overline{g}}^m \otimes \overline{v_h^m}),$$

$$(v_{\overline{g}}^m \otimes \overline{v_h^m}) = \tau^{-1}(v_g^m \otimes \overline{v_{\overline{h}}^m}),\ (v_{\overline{g}}^m \otimes \overline{v_{\overline{h}}^m}) = v_g^m \otimes \overline{v_h^m}.$$

We have $B = \bigoplus_{g \in G} B^g \oplus B^m$, where $B^g \cong M_{|G|+1}(\mathbb{C}),\ \forall g \in G,\ B^m \cong M_{2|G|}(\mathbb{C})$

4.2 Classification of Indecomposable Finite Dimensional $\mathfrak{G}_{\mathcal{TY}}$-$C^*$-algebras

Let us first recall the following well known (see, for instance, [5], 7.4)

Lemma 4.2 *Equivalence classes of left indecomposable Vec_G-module categories (G is abelian) are parameterized by couples (K, ϕ), where K is a stabilizing subgroup of G and $\phi \in H^2(K, \mathbb{C}^\times)$. The set of irreducibles of such a category $\mathcal{M}(K, \phi)$ is $\Lambda_K = G/K$ and ϕ defines the associativities. The corresponding fusion rule is $g \boxtimes \lambda := g + \lambda,\ \forall g \in G,\ \lambda \in G/K$.*

Although $\mathcal{C} = UCorep(\mathfrak{G}_{\mathcal{TY}}) \cong \mathcal{TY}(G, \chi, \tau)$, these categories have different associativities, so we cannot apply directly the classification of module categories from [8], Section 9. We will use the reasonings similar to those from [6], [8] and [9]. The category $\mathcal{C}$ is $\mathbb{Z}_2$-graded, i.e., $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$, where $\mathcal{C}_0 \cong Vec_G$ (both these categories have trivial associativities) and $\mathcal{C}_1$ is generated by a single simple object U^m. Indecomposable $\mathcal{C}_0$-module categories with trivial associativities are parameterized by their stabilizer subgroups $K < G$, they correspond to Vec_G-module categories of the form $\mathcal{M}(K, \mathbf{1})$, where $\mathbf{1}$ is the trivial cocycle. Let us denote them by $\mathcal{M}(K)$.

Then, according to [8], any indecomposable $\mathcal{C}$-module category $\mathcal{M}$ is either indecomposable over $\mathcal{C}_0$ (we say that it is of type **(I)**, it is then of the form $\mathcal{M}(K)$) or equivalent to $\mathcal{M}(K_0) \oplus \mathcal{M}(K_1)$, where K_0 and K_1 are subgroups of G (they can be equal) - a category of type **(D)**.

Moreover, $\mathcal{C}_1$ is an invertible $\mathcal{C}_0$-bimodule category, so one can define an action of $\mathbb{Z}_2 =< \sigma >$ on the set of (equivalence classes) indecomposable semisimple $\mathcal{C}_0$-module categories: $\sigma \cdot \mathcal{M}(K) := \mathcal{C}_1 \boxtimes \mathcal{M}(K)$.

Notations 4.3 *For $K < G,\ \rho \in \hat{K}$ denote $K_\rho^\perp = \{g \in G | \chi(g, k) = \rho(-k),\ \forall k \in K\}$. If $\rho = \mathbf{1}$ is trivial, denote $K_{\mathbf{1}}^\perp$ by $K^\perp$. Note that $\hat{K} \cong G/K^\perp$.*

Lemma 4.4 *For any $K < G$, we have $\sigma \cdot \mathcal{M}(K) = \mathcal{M}(K^{\perp})$.*

Proof. Adopting the strategy of the proof of [8], Lemma 30 to our context, let $A_K = \bigoplus_{k \in K} H^k$ be an algebra in the category $\mathcal{C}_0$ - the analog of the algebra $\mathbb{C}K$ in Vec_G. Viewed as a usual C^*-algebra, A_K has the following minimal central orthoprojectors:

$$P_\lambda = \frac{1}{|K|} \sum_{g \in \lambda} v_g^0, \quad P_\rho = \frac{1}{|K|} \sum_{k \in K} \rho(k) v_m^k \quad (\lambda \in G/K,\ \rho \in \hat{K})$$

So indecomposable right A_K-modules with support in $\mathcal{C}_0$ are: $V_\lambda = Vec\{v_\lambda^k | k \in K\}$ with the action $v_\lambda^k \circ H^h = v_\lambda^{h+k}$ $(h, k \in K)$ and $V_\rho = \mathbb{C} \sum_{k \in K} \rho(k) v_m^k$ with the action $(\sum_{k \in K} \rho(k) v_m^k) \circ H^h = \rho(-h) \sum_{k \in K} \rho(k) v_m^k$ $(h \in H)$, where we denote $v_\lambda^x := \sum_{g \in \lambda} v_g^x$ for any $x \in \Omega$. In both cases the stabilizer subgroup is K.

Then the category $\mathcal{C}_1 \boxtimes \mathcal{M}(K)$ can be described as the category of right A_K-modules in $\mathcal{C}$ with support in $\mathcal{C}_1$ which are of the form $H^m \otimes_R V_\lambda = Vec\{v_p^m | p \in \lambda\}$ with the action $v_p^m \circ H^h = v_{\overline{p+h}}^m$ $(p \in \lambda, h \in K)$ and $H^m \otimes_R V_\rho = Vec\{v_r^m | r \in K_\rho^\perp\}$ with the action $v_r^m \circ H^h = \chi(h, r) v_r^m$ $(r \in H_\rho^\perp, h \in H)$.

In order to determine the stabilizer of $H^m \otimes_R V_\lambda$, we calculate, as in the proof of [8], Lemma 30, for all $g \in G$ the modules $H^g \otimes_R (H^m \otimes_R V_\lambda) = Vec\{\chi(g, p) v_p^m | p \in \lambda\}$ with the action $\chi(g, p) v_p^m \circ H^h = \chi(g, -h)\chi(g, p + h) v_{\overline{p+h}}^m$ $(p \in \lambda, h \in K)$. Therefore, the stabilizer is $K^\perp$.

Similarly, we calculate for all $g \in G$ the modules $H^g \otimes_R (H^m \otimes_R V_\rho) = Vec\{v_{r-g}^m | r \in K_\rho^\perp\}$, but $r - g \in K_\rho^\perp$ is equivalent to $g \in K^\perp$. ▫

Thus, in case **(I)** necessarily $K = K^\perp$, so $|G|$ must be a square, and $\Lambda = G/K$. In case **(D)** $\mathcal{M} \cong \mathcal{M}(K) \oplus \mathcal{M}(K^\perp)$ and $\Lambda = G/K \sqcup G/K^\perp$.

Corollary 4.5 *The fusion rules for indecomposable $UCorep(\mathfrak{G}_{\mathcal{TY}})$-module categories are: $U^g \boxtimes M_\lambda = M_{g+\lambda}$ $(\forall g \in G,\ M_\lambda \in Irr(\mathcal{M}))$ in all cases and:*

$$\textit{For } \mathcal{M} = \mathcal{M}(K):\ U^m \boxtimes M_\lambda = \sum_{\mu \in G/K} M_\mu,\ \textit{where } M_\lambda, M_\mu \in Irr(\mathcal{M}(K))$$

For $\mathcal{M} = \mathcal{M}(K) \oplus \mathcal{M}(K^\perp)$:

$$U^m \boxtimes M_\lambda = \sum_{\mu \in G/K^\perp} M_\mu, \quad U^m \boxtimes M_\mu = \sum_{\lambda \in G/K} M_\lambda,$$

where M_λ $(\lambda \in G/K)$ and M_μ $(\mu \in G/K^\perp)$ are in $Irr(\mathcal{M})$.

Proof. A priori, we have the following fusion rules with U^m:

$$\text{For } \mathcal{M} = \mathcal{M}(K): \ U^m \boxtimes M_\lambda = \sum_{\mu \in G/K} n_\lambda^\mu M_\mu, \ (M_\lambda, M_\mu \in Irr(\mathcal{M}(K)), \ n_\lambda^\mu \in \mathbb{Z}_+)$$

For $\mathcal{M} = \mathcal{M}(K) \oplus \mathcal{M}(K^\perp)$:

$$U^m \boxtimes M_\lambda = \sum_{\mu \in G/K^\perp} m_\lambda^\mu M_\mu, \quad U^m \boxtimes M_\mu = \sum_{\lambda \in G/K} m_\mu^\lambda M_\lambda,$$

where M_λ $(\lambda \in G/K)$, M_μ $(\mu \in G/K^\perp)$ are in $Irr(\mathcal{M})$ and $m_\lambda^\mu, m_\mu^\lambda \in \mathbb{Z}_+$.

The relations of the type $(U^g \otimes U^m) \boxtimes M_\lambda = U^g \boxtimes (U^m \boxtimes M_\lambda)$, $(U^m \otimes U^g) \boxtimes M_\lambda = U^m \boxtimes (U^g \boxtimes M_\lambda)$ and similar relations with M_μ show that $n_\lambda^\mu, m_\lambda^\mu$ and m_μ^λ do not depend on λ and μ. Then it remains to apply again U^m to the above equalities and to use the last remark, the relation $U^m \otimes U^m = \sum_{g \in G} U^g$ and the fact that $|G| = |K||K^\perp|$. ▫

Corollary 4.6 *Any object $M = \bigoplus_{\lambda \in \Lambda} m_\lambda M_\lambda$ of an indecomposable semisimple $UCorep(\mathfrak{G}_{\mathcal{TY}})$-module category is a generator. Indeed, Corollary 4.5 shows that already any $\mathcal{M}_\lambda$ is a generator.*

Therefore, the set of all couples $(\mathcal{M}, M)$ is parameterized:

in case **(I)** *by couples $(K, \{m_\lambda | \lambda \in G/K\})$, where $K = K^\perp < G$ and $m_\lambda \in \mathbb{Z}_+$ are such that at least one $m_\lambda > 0$.*

in case **(D)** *by triples $(K, \{m_\lambda^0 | \lambda \in G/K\}, \{m_\mu^1 | \mu \in G/K^\perp\})$, where $K < G$ and $m_\lambda^0, m_\mu^1 \in \mathbb{Z}_+$ are such that at least one of them is nonzero.*

Lemma 4.7 *The group $Aut(\mathcal{M})$ of autoequivalences of an indecomposable semisimple $UCorep(\mathfrak{G}_{\mathcal{TY}})$-module category $\mathcal{M}$ with trivial associativities is as follows:*

(1) In case **(I)** *for any $\phi \in Aut(\mathcal{M})$, there exists a unique $p \in G/K$ such that $\phi(M_\lambda) = M_{p+\lambda}$, for all $\lambda \in G/K$, so $Aut(\mathcal{M}) \cong G/K$.*

(2) In case **(D)** *and:*

a) $K \neq K^\perp$, for all $\phi \in Aut(\mathcal{M})$, there exists a unique $(p_0, p_1) \in G/K \times G/K^\perp$ such that $\phi(M_\lambda) = M_{p_0+\lambda}$ and $\phi(M_\mu) = M_{p_1+\mu}$ for all $\lambda \in G/K, \mu \in G/K^\perp$, so $Aut(\mathcal{M})) \cong G/K \times G/K^\perp$.

b) $K = K^\perp$, $Aut(\mathcal{M})$, viewed as a bijection of $G/K \times G/K$ on itself, is generated by translations of irreducibles (M_λ, M_μ) by elements $(p_0, p_1) \in$

$G/K \times G/K$ and the flip $(M_\lambda, M_\mu) \mapsto (M_\mu, M_\lambda)$. Therefore, $Aut(\mathcal{M}) \cong (G/K \times G/K) \underset{\sigma}{\ltimes} \mathbb{Z}_2$, where σ is the flip of $G/K \times G/K$.

Proof. (1) By definition of ϕ, we must have $\phi(U^g \boxtimes M_\lambda) = U^g \boxtimes \phi(M_\lambda)$, for all $g \in G, \lambda \in G/K$. Then, putting $M_p = \phi(M_K)$, we have the needed formula for ϕ. Conversely, it is easy to check that for such a ϕ we have $\phi(U^x \boxtimes M_\lambda) = U^x \boxtimes \phi(M_\lambda)$, for all $x \in \Omega,\ \lambda \in G/K$.

(2a) As $\mathcal{M} = \mathcal{M}(K) \oplus \mathcal{M}(K^\perp)$ and $\mathcal{M}(K) \ncong \mathcal{M}(K^\perp)$, the above result applies to the corresponding restrictions of ϕ.

(2b) Now the above mentioned components have equal rights, so ϕ can permute them and we are done. □

Corollary 4.6 implies that any object $M = \underset{\lambda \in \Lambda}{\oplus} m_\lambda M_\lambda$ of a module category $\mathcal{M}$ as above can be identified either with a collection $\{m_\lambda | \lambda \in G/K\}$ or with a double collection $(\{m_\lambda | \lambda \in G/K\}, \{m_\mu | \mu \in G/K^\perp\})$, where $m_\lambda, m_\mu \in \mathbb{Z}_+$.

These considerations and Theorem 2.9 prove Theorem 1.2.

Remark 4.8 *Let us compute the dimensions of the spectral subspaces of a finite dimensional $\mathfrak{G}_{\mathcal{TY}}$-$C^*$-algebra (A, α). By Theorem 2.9, given a C^*-module category $\mathcal{M}$ over $UCorep(\mathfrak{G}_{\mathcal{TY}})$ with a generator $M = \underset{\lambda \in \lambda}{\oplus} m_\lambda M_\lambda$, we have $A_{U^x} = F(U^x) \otimes \overline{H}_x$ $(\forall x \in \Omega)$, where $F : UCorep(\mathfrak{G}_{\mathcal{TY}}) \to Corr_f(R)$ is the functor defined by $F(U^x) := Hom(M, U^x \boxtimes M)$, $R = End(M)$. Clearly,*

$$X^x := F(U^x) \cong \underset{\lambda, \rho \in \lambda}{\oplus} m_\lambda m_\rho Hom(M_\lambda, U^x \boxtimes M_\rho).$$

As $Hom(M_\lambda, U^g \boxtimes M_\rho) = \delta_{\lambda, g\cdot\rho}\mathbb{C},\ \forall \lambda, \rho \in \lambda$, we have $dim(X^g) = \underset{\rho \in \lambda}{\sum} m_\rho m_{g\cdot\rho}$.

Now, in case **(I)**, *$Hom(M_\lambda, U^m \boxtimes M_\rho) \equiv \mathbb{C}$, so $dim(X^m) = \underset{\lambda, \rho \in G/K}{\sum} m_\lambda m_\rho$.*

And in case **(D)**, *$Hom(M_\lambda, U^m \boxtimes M_\rho) = 0$ when $\lambda, \rho \in G/K$ or $\lambda, \rho \in G/K^\perp$, and $Hom(M_\lambda, U^x \boxtimes M_\rho) = \mathbb{C}$ otherwise. So, $dim(X^m) = 2 \underset{\lambda \in G/K}{\sum} m_\lambda \times$ $\times \underset{\rho \in G/K^\perp}{\sum} m_\rho$. Therefore, in case* **(D)**, *$dim X^m$ must be even.*

5 Indecomposable Weak Coideals of $\mathfrak{G}_{TY}$

We begin the classification of indecomposable weak coideals of $\mathfrak{G}_{TY}$ by giving a canonical basis for them.

Notations 5.1 *For all $g \in G$ and $X \subset G \sqcup \{m\}$, let us denote:*

$$v_X^g = \sum_{x \in X} v_x^g, \quad v_X^m = \sum_{g \in G \cap X} v_g^m, \quad v_{\overline{X}}^m = \sum_{g \in G \cap X} v_{\overline{g}}^m.$$

Lemma 5.2 *Let A be a weak coideal of B. Then:*

a) For any $g \in G$ such that $X^g \neq \{0\}$, there exists a subset $I^g \subset I^0 = \{\Gamma_i | i = 1, 2, ..., k_0\}$ of cardinality k_g and a set of vectors $\{v_{\Gamma_i}^g(\Theta^g) | i \in I^g\}$ which is a basis of X^g, where Θ^g is a map from $\Gamma^g = \sqcup_{i \in I^g} \Gamma_i$ to $\mathbb{T} := \{z \in \mathbb{C} | |z| = 1\}$.

b) If $X^m \neq \{0\}$, then $v_m^0 \in X^0$, so we can chose $\{m\} \in I^0$, and there exists a subset $I^m \subset I^0 \backslash \{m\}$ of cardinality k_m and a basis of X^m of the form $\{v_{\Gamma_i}^m(\Theta^m), v_{\overline{\Gamma_i}}^m(\Theta^m) | i \in I^m\}$, where Θ^m is a map from $\Gamma^m = \sqcup_{i \in I^m} \Gamma_i$ to $\mathbb{T} := \{z \in \mathbb{C} | |z| = 1\}$. If $k_m = k_0 - 1$, this weak coideal is indecomposable.

Proof. a) Let $v^g = \sum_{x \in \Omega} a_x v_x^g$ be a nonzero vector from X^g. Then $v^g = v^g \circ v_\Gamma^0 = \sum_{i \in I^0} v^g(\Gamma_i)$, where $v^g(\Gamma_i) = \sum_{x \in \Gamma_i} a_x v_x^g$. Hence $X^g = \bigoplus_{i \in I^g} X_{\Gamma_i}^g$, where $X_{\Gamma_i}^g$ $(i \in I^g)$ are subspaces of X^g containing $v^g(\Gamma_i) \neq 0$. We have:

$$v^g(\Gamma_i)^\sharp \circ v^g(\Gamma_i) = \sum_{x \in \Gamma_i} |a_x|^2 v_x^0 = C v_{\Gamma_i}^0, \quad \text{where} \quad C > 0.$$

Let $w^g(\Gamma_i) = \sum_{y \in \Gamma_i} b_y v_y^g \in X_{\Gamma_i}^g$ be another vector with $|b_y| \equiv 1$, then:

$$\tilde{v}^g(\Gamma_i)^\sharp \circ w^g(\Gamma_i) = \sum_{x \in \Gamma_i} \overline{a_x} b_x v_x^0 = D v_{\Gamma_i}^0,$$

where $|D| = 1$. Then $b_x = D a_x$ for all $x \in \Gamma_i$ which shows that any $X_{\Gamma_i}^g (i \in I^g)$ is generated by a unique, up to a scalar $D \in \mathbb{T}$, vector as above. We fix such elements and denote them by $v_{\Gamma_i}^g(\Theta^g)$, the map Θ^g being defined by the coefficients of the chosen elements.

b) Let $X^m \neq \{0\}$ and let $v^m = \sum_{g \in G} a_g v_g^m + \sum_{h \in G} b_h v_{\overline{h}}^m$ be its nonzero vector. Then $(v^m)^\sharp := \Psi_m(\overline{v}^m) = |G|^{1/2} (\sum_{g \in G} \overline{a}_g v_{\overline{g}}^m + \sum_{h \in G} \overline{b}_h \tau^{-1} v_h^m)$. Next, we compute:

$$v^m \circ (v^m)^\sharp = |G|^{1/2} (\sum_{g,k \in G} a_g \overline{a}_k v_k^{k-g} + \sum_{p,h \in G} \overline{|b_h|^2 \chi(p,h)} v_m^p)$$

and similarly

$$(v^m)^\sharp \circ v^m = |G|^{1/2} (\tau \sum_{g,p \in G} |a_g|^2 \overline{\chi(p,g)} v_m^p + \tau^{-1} \sum_{h,k \in G} \overline{b}_k b_h v_h^{h-k}).$$

Hence, the components of index p of these vectors are:

$$[v^m \circ (v^m)^\sharp]_p = |G|^{1/2}(\sum_{g\in G} a_g \overline{a}_{p+g} v^p_{p+g} + \sum_{h\in G} |b_h|^2 \overline{\chi(p,h)} v^p_m)$$

and similarly

$$[(v^m)^\sharp \circ v^m]_p = |G|^{1/2}(\tau \sum_{g\in G} |a_g|^2 \overline{\chi(p,g)} v^p_m + \tau^{-1} \sum_{k\in G} \overline{b}_k b_{p+k} v^p_{p+k}).$$

In particular, the components of index 0 of these vectors are:

$$[v^m \circ (v^m)^\sharp]_0 = |G|^{1/2}(\sum_{g\in G} |a_g|^2 v^0_g + (\sum_{h\in G} |b_h|^2) v^0_m)$$

and similarly

$$[(v^m)^\sharp \circ v^m]_0 = |G|^{1/2}(\tau(\sum_{g\in G} |a_g|^2) v^0_m + \tau^{-1} \sum_{k\in G} |b_k|^2 v^0_k).$$

Since at least one of a_g or b_h is nonzero, it follows that $v^0_m \in X^0$, so we can chose $\{m\} \in I^0$. Further:

$$v^0_m \circ v^m = \sum_{h\in G} b_h v^m_{\overline{h}} \in X^m, \quad v^m \circ v^0_m = \sum_{g\in G} a_g v^m_g \in X^m,$$

which shows that $v^0_m \notin Z(A)$ and that $X^m = X^m_1 \oplus X^m_2$, where the subspaces $X^m_1, X^m_2 \subset X^m$ consist, respectively, of vectors of the form $\sum_{g\in G} a_g v^m_g$ and $\sum_{h\in G} b_h v^m_{\overline{h}}$. As $(X^m_1)^\sharp = X^m_2$ and $(X^m_2)^\sharp = X^m_1$, $dim(X^m)$ must be even.

Now, the relations $v^0_{\Gamma_i} \circ v^m = \sum_{g\in\Gamma_i} a_g v^m_g := w^m_{\Gamma_i}$ show that X^m has a basis of the form $\{w^m_{\Gamma_i}, (w^m_{\Gamma_i})^\sharp | i \in I^m\}$, and using the same reasoning as in part a), one can normalize: $w^m_{\Gamma_i} = v^m_{\Gamma_i}(\Theta^m)$. Finally, if $k_m = k_0 - 1$, there is no a combination of $v^0_{\Gamma_i}$ which would commute with all $v^m_{\Gamma_i}(\Theta^m)$, so A is indecomposable. □

Corollary 3.4 implies that for weak coideals we have $m_\lambda \in \{0,1\}$ for all $\lambda \in \Lambda$, so that the generator M can be identified either with a nonempty subset $Z \subset G/K$ or with a couple of subsets $(Z_0, Z_1) \subset G/K \times G/K^\perp$, at least one of which is nonempty.

5.1 The case $A^m = \{0\}$

Remark 5.3 *Let A be an indecomposable weak coideal such that $dim(X^m) = 0$. Then either the set I^0 consists of only one subset $\tilde{\Gamma} \subset \Omega$ containing $\{m\}$ (so that $dim(X^0) = 1$) or does not contain subset $\tilde{\Gamma} \subset \Omega$ containing $\{m\}$.*

Indeed, if $\tilde{\Gamma} \in I^0$, *it suffices to show that* $v^0_{\tilde{\Gamma}}$ *commutes with any basis element* $v^g_{\Gamma_i}(\Theta^g)$ *of* X. *Using the fact that no more than one element of* I^0 *can contain* $\{m\}$ *as well as Lemma 5.2, one can see that for any* $\Gamma_i \in I^0 \backslash \{\tilde{\Gamma}\}$ *we have* $v^0_{\tilde{\Gamma}} \circ v^g_{\Gamma_i}(\Theta^g) = v^g_{\Gamma_i}(\Theta^g) \circ v^0_{\tilde{\Gamma}} = 0$ *and* $v^0_{\tilde{\Gamma}} \circ v^g_{\tilde{\Gamma}}(\Theta^g) = v^g_{\tilde{\Gamma}}(\Theta^g) \circ v^0_{\tilde{\Gamma}} = v^g_{\tilde{\Gamma}}(\Theta^g)$. *It follows that either the basis of* X *consists only from vectors of the form* $v^g_{\tilde{\Gamma}}(\Theta^g)$ *or does not contain such vectors at all.*

The equality $dim X^0 = 1$ implies $M = M_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Then $dim F(U^g) = 1$ if $g \in K$ and $dim F(U^g) = 0$ otherwise. This gives a unique, up to isomorphism of $\mathfrak{G}$-C^*algebras, connected coideal $I^m_K = \bigoplus\limits_{k \in K} (\mathbb{C} v^k_m \otimes \overline{H}^k)$.

Now suppose that $\Gamma_i \subset G,\ \forall i \in I^0$. As $dim F(U^m) = 0$, M is supported only on G/K or only on $G/K^{\perp}$. Let us consider the first of these cases, the second one is completely similar. Identify the generator M with a nonempty subset $Z \subset G/K$. The following example shows that any such Z gives rise to an indecomposable weak coideal of $\mathfrak{G}_{\mathcal{TY}}$.

Example 5.4 *Let* Z *be a nonempty subset of* G/K, *then Remark 4.8 gives* $dim X^g = \sum\limits_{\lambda \in G/K} m_\lambda m_{g+\lambda} = |Z \cap (g + Z)|$.

Put $X^g = Vec\{v^g_\lambda | \lambda \in Z \cap (g + Z)\}$ *and* $X^m = \{0\}$. *For any* $v^g_\lambda \in X^g$ $(g \in G)$, *we have* $(v^g_\lambda)^\sharp = v^{-g}_{\lambda - g} \in X^{-g}$. *Indeed, as* $\lambda \in Z \cap (g + Z)$, *there is* $\lambda' \in Z$ *such that* $\lambda = g + \lambda'$, *so* $\lambda - g = \lambda' \in Z$. *Clearly,* $(\lambda - g) \in Z - g$, *hence* $(\lambda - g) \in Z \cap (Z - g)$. *We also have:*

$$v^g_\lambda \circ v^h_\mu = \delta_{\mu, h+\lambda} v^{g+h}_\mu \in X^{g+h} \quad \textit{for all} \quad v^g_\lambda \in X^g,\ v^g_\mu \in X^h\ (g, h \in G).$$

Indeed, as $\lambda \in Z \cap (g + Z)$, $\mu \in Z \cap (h + Z)$, *there are* λ', $\mu' \in Z$ *such that* $\lambda = g + \lambda'$, $\mu = h + \mu'$, *so the above product is nonzero if and only if* $\mu = h + \lambda = h + g + \lambda' \in g + h + Z$. *Since* $\mu \in Z$, *it follows that* $\mu \in Z \cap (g + h + Z)$. *Thus, Lemma 3.3, a) implies that the family* $\{X^x | x \in \Omega\}$ *generates a weak coideal* $A \subset B$ *with unit* $1_A = v^0_L \otimes \overline{v}^0_\Omega$, *where* $L := \bigcup\limits_{\lambda \in Z} \lambda$.

Remark 5.5 *A is never a coideal but when* $|Z| = 1$ *it is isomorphic to a connected coideal* $I^\Omega_K = \bigoplus\limits_{k \in K} (\mathbb{C}(\sum\limits_{x \in \Omega} v^h_x) \otimes \overline{H}^k)$, *which is the "right sided" version of the left coideal* I_K *from [10].* I^Ω_K *is also isomorphic to* I^m_K *above. If* $|Z| > 1$, A *is also indecomposable because for an arbitrary proper subset* $Z_0 \subset Z$, *the element* $\sum\limits_{\lambda \in Z_0} v^0_\lambda$ *does not commute with any* v^g_μ $(\mu \in Z_0,\ g \notin K)$.

It follows from Remark 4.8 that $\mathfrak{G}_{\mathcal{TY}}$-$C^*$-algebras (A,α) with $A^m=\{0\}$ can be only of type (D).

We can summarize the above considerations as follows:

Proposition 5.6 *Isomorphism classes of indecomposable weak coideals A of $\mathfrak{G}_{\mathcal{TY}}$ with $A^m=\{0\}$ are parameterized by couples (K,Z^{orb}), where $K<G$ and Z^{orb} is the orbit of a nonempty subset $Z\subset G/K$ or $Z\subset G/K^{\perp}$ under the action of the group of the translations on G/K (resp., on $G/K^{\perp}$). A is isomorphic to a coideal if and only if $|Z|=1$.*

5.2 The case $A^m\neq\{0\}$

Proposition 5.7 *There is no weak coideals of $\mathfrak{G}_{\mathcal{TY}}$ corresponding to module categories $\mathcal{M}$ with $\Lambda=G/K$.*

Proof. Let A be such a weak coideal and M be the corresponding generator identified with the subset Z of G/K. Then $k_0=dim(X^0)=|Z|$ and $dim(X^m)=|Z|^2$. In terms of Lemma 5.2, b) we have $dim(X^m)=2k_m$, where $k_m\leqslant k_0-1$, so that $|Z|^2\leqslant 2(|Z|-1)$ which is only possible if $|Z|=1$. But then $dim(X^m)=1$ - contradicts to the fact that $dim(X^m)$ must be even. ▫

Proposition 5.8 *Let A be a weak coideal of $\mathfrak{G}_{\mathcal{TY}}$ corresponding to a module category $\mathcal{M}$ with $\Lambda=G/K\sqcup G/K^{\perp}$ and a generator M defined by a nonempty subset $(Z_0,Z_1)\subset G/K\times G/K^{\perp}$. Then either $|Z_0|=1$ or $|Z_1|=1$.*

Proof. We have $k_0=dim(X^0)=|Z_0|+|Z_1|$ and $dim(X^m)=2|Z_0||Z_1|$. In terms of Lemma 5.2, b) we have $dim(X^m)=2k_m$, where $k_m\leqslant k_0-1$, so $|Z_0||Z_1|\leqslant|Z_0|+|Z_1|-1$ from where either $|Z_0|=1$ or $|Z_1|=1$. ▫

The following example shows that any such set (Z_0,Z_1) gives rise to an indecomposable weak coideal of $\mathfrak{G}_{\mathcal{TY}}$.

Example 5.9 *Let Z be a nonempty subset of G/K and $\rho_0\in G/K^{\perp}$. For the generator corresponding to $Z\sqcup\rho_0$ we have $dimX^m=2|Z|$, $dimX^g=|Z\cap(g+Z)|$ if $g\notin K^{\perp}$ and $dimX^g=|Z\cap(g+Z)|+1$ if $g\in K^{\perp}$.*

Put $X^m=Vec\{v^m_\lambda,v^m_{\overline{\mu}}|\lambda,\mu\in Z\}$, $X^g=Vec\{v^g_m,v^g_\lambda|\lambda\in Z\cap(g+Z)\}$ if $g\in K^{\perp}$ and $X^g=Vec\{v^g_\lambda|\lambda\in Z\cap(g+Z=\}$ if $g\notin K^{\perp}$. The next relations, where $g,h\in G,k,l\in K^{\perp},\lambda,\mu\in Z,u(\lambda)$ is a representative of the coset λ, show that the family $\{X^x|x\in\Omega\}$ satisfies the conditions a) of Lemma 3.3:

$$(v^m_\lambda)^\sharp=|G|^{1/2}v^m_{\overline{\lambda}},\ (v^m_{\overline{\lambda}})^\sharp=\tau^{-1}|G|^{1/2}v^m_\lambda,\ (v^k_m)^\sharp=v^{-k}_m,\ (v^g_\lambda)^\sharp=v^{-g}_{\lambda-g},$$

$$v_m^k \circ v_m^l = v_m^{k+l},\ v_\lambda^g \circ v_\mu^h = \delta_{\mu,g+\lambda} v_\mu^{g+h},\ v_m^k \circ v_\lambda^h = v_\lambda^h \circ v_m^k = v_m^k \circ v_\lambda^m = 0,$$

$$v_m^k \circ v_{\overline{\lambda}}^m = \chi(k, u(\lambda)) v_{\overline{\lambda}}^m, v_{\overline{\lambda}}^m \circ v_m^k = 0,\ v_\lambda^m \circ v_m^k = \chi(u(\lambda), k) v_\lambda^m,$$

$$v_\lambda^g \circ v_\mu^m = \delta_{\lambda,\mu} v_{\lambda-g}^m,\ v_{\overline{\lambda}}^m \circ v_\mu^g = \delta_{\lambda,\mu} v_{\overline{g+\lambda}}^m,\ v_\lambda^m \circ v_\mu^g = v_\lambda^g \circ v_{\overline{\mu}}^m = v_\lambda^m \circ v_\mu^m = v_{\overline{\lambda}}^m \circ v_{\overline{\mu}}^m = 0,$$

and finally, using the fact that $\sum_{k \in K} \chi(g,k) = |K|$ *if* $g \in K^\perp$ *and is* 0 *otherwise:*

$$v_\lambda^m \circ v_{\overline{\mu}}^m = \sum_{g \in (\mu - \lambda)} v_\mu^g,\ v_{\overline{\lambda}}^m \circ v_\mu^m = \tau |K| \delta_{\lambda,\mu} \sum_{k \in K^\perp} v_m^k.$$

So, this family generates an indecomposable weak coideal $A \subset B$, $1_A = (v_m^0 + v_L^0) \otimes \overline{v}_\Omega^0$, *where* $L = \bigcup_{\lambda \in Z} \lambda$. *$A$ is a coideal if and only if $L = G$ in which case it is the analog of the left connected coideal J_K constructed in [10].*

Now we can summarize the above considerations as follows:

Proposition 5.10 *Isomorphism classes of indecomposable weak coideals A of $\mathfrak{G}_{\mathcal{TY}}$ with $A^m \neq \{0\}$ are parameterized by pairs $(K, (Z_0, Z_1)^{orb})$, where $K < G$ and $(Z_0, Z_1)^{orb}$ is the orbit of a subset $(Z_0, Z_1) \subset G/K \times G/K^\perp$ such that $min\{|Z_0|, |Z_1|\} = 1$ under the action of:*

a) the group $G/K \times G/K^\perp$ by translations, if $K \neq K^\perp$;

b) the semi direct product $(G/K \times G/K) \underset{\sigma}{\ltimes} \mathbb{Z}_2$ generated by the group $G/K \times G/K$ acting by translations and the flip $\sigma : (Z_0, Z_1) \mapsto (Z_1, Z_0)$ if $K = K^\perp$.

A is isomorphic to a coideal if and only if either $Z_0 = G/K$ or $Z_1 = G/K^\perp$.

Finally, Theorem 1.3 follows from Propositions 5.6 and 5.10.

References

[1] Saad Baaj and Georges Skandalis. C^*-algèbres de Hopf et théorie de Kasparov équivariante. *K-Theory*, 2(6):683–721, 1989. 12

[2] Gabriella Böhm, Florian Nill, and Kornél Szlachányi. Weak Hopf algebras. I. Integral theory and C^*-structure. *J. Algebra*, 221(2):385–438, 1999. 3

[3] Gabriella Böhm and Kornél Szlachányi. Weak Hopf algebras. II. Representation theory, dimensions, and the Markov trace. *J. Algebra*, 233(1):156–212, 2000. 7

[4] Kenny De Commer and Makoto Yamashita. Tannaka-Kreĭn duality for compact quantum homogeneous spaces. I. General theory. *Theory Appl. Categ.*, 28:No. 31, 1099–1138, 2013. 4, 14

[5] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015. 3, 5, 14, 17, 19

[6] Cesar Galindo. Clifford theory for graded fusion categories. *Israel J. Math.*, 192(2):841–867, 2012. 4, 19

[7] T. Hayashi. A canonical tannaka duality for semi finite tensor categories. Preprint, math.QA/9904073, 1999. 3, 9

[8] Ehud Meir and Evgeny Musicantov. Module categories over graded fusion categories. *Journal of Pure and Applied Algebra*, (216):2449–2466, 2012. 4, 19, 20

[9] Adriana Mejia Castano and Martin Mombelli. Equivalence classes of exact module categories over graded tensor categories. Preprint,arXiv: 1711.06118v3 [math.QA], 2019. 4, 19

[10] Camille Mevel. Exemples et applications des groupoides quantiques finis, https://tel.archives-ouvertes.fr/tel-00498884/document. Thèse, Université de Caen, 2010. 4, 17, 18, 25, 27

[11] Sergey Neshveyev. Duality theory for nonergodic actions. *Münster J. Math.*, 7(2):413–437, 2014. 4, 13

[12] Sergey Neshveyev and Lars Tuset. *Compact quantum groups and their representation categories*, volume 20 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2013. 5, 9

[13] Dmitri Nikshych and Leonid Vainerman. A Galois correspondence for II_1 factors and quantum groupoids. *J. Funct. Anal.*, 178(1):113–142, 2000. 3

[14] Dmitri Nikshych and Leonid Vainerman. Finite quantum groupoids and their applications. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 211–262. Cambridge Univ. Press, Cambridge, 2002. 3, 5

[15] Viktor Ostrik. Module categories, weak Hopf algebras and modular invariants. *Transform. Groups*, 8:177–206, 2003. 3

[16] Hendryk Pfeiffer. Tannaka-Kreĭn reconstruction and a characterization of modular tensor categories. *J. Algebra*, 321(12):3714–3763, 2009. 9

[17] K. Szlachányi. Finite quantum groupoids and inclusions of finite type. *Fields Inst.Commun.*, 30:314–343. AMS Providence, RI, 2001. 3, 11

[18] Daisuke Tambara and Shigeru Yamagami. Tensor categories with fusion rules of self-duality for finite abelian groups. *J. Algebra*, 209(2):692–707, 1998. 4, 17

[19] Leonid Vainerman and Jean-Michel Vallin. On Z/2Z-extensions of pointed fusion categories. *Banach Center Publications*, 98:343–366, 2012. 15

[20] Leonid Vainerman and Jean-Michel Vallin. Tannaka-Krein reconstruction for coactions of finite quantum groupoids. *Methods Funct. Anal. Topology*, 23(1):76–107, 2017. 3, 12, 16