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Martingale Representation in the Enlargement of the Filtration Generated by a Point Process

Paolo Di Tella^{1,2} and Monique Jeanblanc³

Abstract

Let X be a point process and let \mathbb{X} denote the filtration generated by X . In this paper we study martingale representation theorems in the filtration \mathbb{G} obtained as an initial and progressive enlargement of the filtration \mathbb{X} . In particular, the progressive enlargement is done by means of a whole point process H . We work here in full generality, without requiring any further assumption on the process H and we recover the special case in which \mathbb{X} is enlarged progressively by a random time τ .

Keywords: Point processes, martingale representation, progressive enlargement, initial enlargement, random measures.

1 Introduction

In this paper we study martingale representation theorems in the enlargement \mathbb{G} of the filtration \mathbb{X} generated by a point process X (see Theorem 3.6 below). The filtration \mathbb{G} here is obtained first enlarging \mathbb{X} initially by a sigma-field \mathcal{R} and then progressively by a whole point process H (and not only by a random time). In other words, \mathbb{G} is the smallest right-continuous filtration containing \mathcal{R} and such that X and H are adapted. We show that in \mathbb{G} all local martingales can be represented as a stochastic integral with respect to a compensated random measure and also as the sum of stochastic integrals with respect to *three fundamental martingales*. Due to the particular structure of the filtration \mathbb{G} , we work in full generality without requiring any additional condition on the point processes X and H . In this way, for the setting of point process, we generalize all the results from literature (which we describe below) about the propagation of martingale representation theorems to the enlarged filtration.

We recall that the multiplicity (see Davis and Varaiya [6]) or the spanning number (see Duffie [9]) of a filtration \mathbb{F} is the minimal number of locally square integrable orthogonal martingales which is necessary to represent all the square integrable martingales in \mathbb{F} .

It is well-known that the multiplicity of \mathbb{X} is equal to one. In some special cases we shall show that the multiplicity of the filtration \mathbb{G} is equal to two or three.

We now give an overview of the literature about the propagation of martingale representation theorems to the enlarged filtration. We denote now by \mathbb{F} a reference filtration and by \mathbb{G} an (initial or progressive) enlargement of \mathbb{F} .

¹Corresponding author.

²Address: Inst. für Mathematische Stochastik, TU Dresden, Zellescher Weg 12-14 01069 Dresden, Germany.

E-Mail: Paolo.Di_Tella@tu-dresden.de

³Address: Laboratoire de Mathématiques et Modélisation d'Évry (LaMME), UMR CNRS 8071, Université d'Évry-Val-d'Essonne, Université Paris Saclay, 23 Boulevard de France, 91037 Évry cedex, France.

E-Mail: monique.jeanblanc@uni-evry.fr

A local martingale M is said to possess the predictable representation property (from now on the PRP) with respect to \mathbb{F} if all \mathbb{F} -local martingales can be represented as a stochastic integral of predictable integrands with respect to M .

In [10], Fontana proved that the PRP of a d -dimensional local martingale M with respect to \mathbb{F} propagates to \mathbb{G} , if \mathbb{G} is the initial enlargement of \mathbb{F} by a random time τ which satisfies *Jacod's absolutely continuity hypothesis*. In particular, in [10] it is shown that the multiplicity of \mathbb{G} is also d .

If \mathbb{G} is the *progressive* enlargement of \mathbb{F} by a random time τ such that \mathbb{F} -martingales remain \mathbb{G} -semimartingales, first results were obtained by Barlow in [3], for the case of *honest times*. In [17], Kusuoka studied the propagation of the PRP if \mathbb{F} is a Brownian filtration and τ is a random time satisfying, among other additional conditions, the *immersion property*, i.e., τ is such that \mathbb{F} -martingales remain \mathbb{G} martingales, and *avoiding* \mathbb{F} -stopping times. In the case of [17] the multiplicity of \mathbb{G} is equal to two. In [14], Jeanblanc and Song considered more general random times and they proved that the multiplicity of \mathbb{G} usually increases by one and for the case of honest times by two. In [2], Aksamit, Jeanblanc and Rutkowski studied the propagation of the PRP to \mathbb{G} by a random time τ if \mathbb{F} is a Poisson filtration. The representation results obtained in [2] concerns however only a special class of \mathbb{G} -local martingales stopped at τ .

We stress that, if $\tau : \Omega \rightarrow (0, +\infty]$ is a random time, taking the point process $H = 1_{[\tau, +\infty)}$, we recover in the present paper the special case in which \mathbb{F} is progressively enlarged by τ . Furthermore, if $H = 1_{[\tau, +\infty)}$, the martingale representation theorems which we obtain here hold for every \mathbb{G} -local martingale and not only for \mathbb{G} -local martingales stopped at τ .

Another important (and more general than the PRP) martingale representation theorem is the so called *weak* representation property (from now on WRP) (see [11, Definition 13.13]). The propagation of the WRP to the progressive enlargement \mathbb{G} of \mathbb{F} by a random time τ has been established in [8, Theorem 5.3] under the assumptions of the immersion property and the avoidance of \mathbb{F} -stopping times.

If the filtration \mathbb{G} is obtained by \mathbb{F} by adding a *whole process* and not only a random time, the propagation of martingale representation theorems to \mathbb{G} has been up to now very little studied and all the existing results make use of independence assumptions.

For example, in [19], Xue considered two semimartingales X and Y possessing the WRP each in its own filtration and, under the assumptions of the independence and the quasi-left continuity of X and Y , the author shows that the WRP propagates to \mathbb{G} , the smallest right-continuous filtration with respect to which both X and Y are adapted.

Similarly, in [4], Calzolari and Torti consider two local martingales M and N possessing the PRP each in its own filtration and then they study the propagation of the PRP to \mathbb{G} . The assumptions in [4] imply that M and N are independent and that \mathcal{G}_0 is trivial. In [4] it is also shown that the multiplicity of \mathbb{G} is, in general, equal to three. In [5], the results of [4] are generalized to multidimensional martingales M and N .

The present paper has the following structure. In Section 2, we recall some basics on point processes, martingale theory and progressive enlargement of filtrations which are needed in this work. In Section 3, we establish the main results of the present paper, Theorem 3.6 below, about the propagation of the WRP and the PRP. Section 4 is devoted to the study of the orthogonality of the fundamental martingales which are used for the martingale representation in \mathbb{G} . In particular, as a consequence of Theorem 3.6, we generalize the results obtained in [4] on the martingale representation to the case of a non-trivial initial filtration in the context of point processes. In the appendix we discuss a lemma which will be useful to study the orthogonality of the involved martingales.

2 Basic Notions

Filtrations, martingales and increasing processes. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. For any càdlàg process X , we denote by ΔX its jump process $\Delta X_t = X_t - X_{t-}$ and we define $X_{0-} := X_0$ so that the jump process ΔX is equal to zero in $t = 0$.

We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ a right-continuous filtration and by $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{F})$) the σ -algebra of the \mathbb{F} -optional (resp. \mathbb{F} -predictable) sets of $\Omega \times \mathbb{R}_+$.

For an \mathbb{F} -adapted process X and an \mathbb{F} -stopping time σ , we denote by X^σ the process stopped at σ , that is, $X_t^\sigma := X_{t \wedge \sigma}$, $t \geq 0$.

An \mathbb{F} -adapted càdlàg process X is called *quasi-left continuous* if $\Delta X_\sigma = 0$ for every finite-valued and \mathbb{F} -predictable stopping time σ .

For $p \geq 1$, we denote by $\mathcal{H}^p(\mathbb{F})$ the space of \mathbb{F} -uniformly integrable martingales X such that $\|X\|_{\mathcal{H}^p} := \mathbb{E}[\sup_{t \geq 0} |X_t|^p]^{1/p} < +\infty$. Recall that $(\mathcal{H}^p, \|\cdot\|_{\mathcal{H}^p})$ is a Banach space. For $X \in \mathcal{H}^2(\mathbb{F})$, we also introduce the equivalent norm $\|X\|_2 := \mathbb{E}[X_\infty^2]^{1/2}$ and $(\mathcal{H}^2, \|\cdot\|_2)$ is a Hilbert space.

For each $p \geq 1$, the space $\mathcal{H}_{\text{loc}}^p(\mathbb{F})$ is introduced from $\mathcal{H}^p(\mathbb{F})$ by localization. We observe that $\mathcal{H}_{\text{loc}}^1(\mathbb{F})$ coincides with the space of *all* \mathbb{F} -local martingales (see [12, Lemma 2.38]). We denote by $\mathcal{H}_0^p(\mathbb{F})$ (resp., $\mathcal{H}_{\text{loc},0}^p(\mathbb{F})$) the subspace of martingales (resp., local martingales) $Z \in \mathcal{H}^p(\mathbb{F})$ (resp., $Z \in \mathcal{H}_{\text{loc}}^p(\mathbb{F})$) such that $Z_0 = 0$.

We recall that $X, Y \in \mathcal{H}_{\text{loc}}^1(\mathbb{F})$ are called *orthogonal* if $XY \in \mathcal{H}_{\text{loc},0}^1(\mathbb{F})$ holds.

For $X, Y \in \mathcal{H}_{\text{loc}}^2(\mathbb{F})$ we denote by $\langle X, Y \rangle$ the *predictable covariation* of X and Y . We recall that $XY - \langle X, Y \rangle \in \mathcal{H}_{\text{loc}}^1(\mathbb{F})$. Hence, if $X_0 Y_0 = 0$, then X and Y are orthogonal if and only if $\langle X, Y \rangle = 0$.

An \mathbb{R} -valued \mathbb{F} -adapted process X such that $X_0 = 0$ is called *increasing* if X is càdlàg and the paths $t \mapsto X_t(\omega)$ are non-decreasing, $\omega \in \Omega$. We denote by $\mathcal{A}^+ = \mathcal{A}^+(\mathbb{F})$ the space of \mathbb{F} -adapted integrable processes, that is, \mathcal{A}^+ is the space of increasing process X such that $\mathbb{E}[X_\infty] < +\infty$ (see [13, I.3.6]). We denote by $\mathcal{A}_{\text{loc}}^+ = \mathcal{A}_{\text{loc}}^+(\mathbb{F})$ the localized version of \mathcal{A}^+ .

Let $X \in \mathcal{A}^+$ and let $K \geq 0$ be a progressively measurable process. We denote by $K \cdot X = (K \cdot X_t)_{t \geq 0}$ the process defined by the (Lebesgue–Stieltjes) integral of K with respect to X , that is, $K \cdot X_t(\omega) := \int_0^t K_s(\omega) dX_s(\omega)$, if $\int_0^t K_s(\omega) dX_s(\omega)$ is finite-valued, for every $\omega \in \Omega$ and $t \geq 0$. Notice that $K \cdot X$ is an increasing process.

If $X \in \mathcal{A}_{\text{loc}}^+$, there exists a unique \mathbb{F} -predictable process $X^{p,\mathbb{F}} \in \mathcal{A}_{\text{loc}}^+$ such that $X - X^{p,\mathbb{F}} \in \mathcal{H}_{\text{loc}}^1(\mathbb{F})$ (see [13, Theorem I.3.17]). The process $X^{p,\mathbb{F}}$ is called the \mathbb{F} -compensator or the \mathbb{F} -dual predictable projection of X . We use the notation $\bar{X}^{\mathbb{F}} := X - X^{p,\mathbb{F}}$. If $X \in \mathcal{A}_{\text{loc}}^+$ has bounded jumps, then also $X^{p,\mathbb{F}}$ has bounded jumps and X is quasi-left continuous if and only if $X^{p,\mathbb{F}}$ is continuous (see [11, Corollary 5.28] for these results).

A *point process* X with respect to \mathbb{F} is an \mathbb{N} -valued and \mathbb{F} -adapted increasing process such that $\Delta X \in \{0, 1\}$. Notice that a point process X is locally bounded and therefore $X \in \mathcal{A}_{\text{loc}}^+$.

Random measures. Let μ be a nonnegative random measure on $\mathbb{R}_+ \times E$ in the sense of [13, Definition II.1.3], where E coincides with \mathbb{R}^d or with a Borel subset of \mathbb{R}^d . We stress that we assume $\mu(\omega, \{0\} \times E) = 0$ identically.

We denote $\tilde{\Omega} := \Omega \times \mathbb{R}_+ \times E$, $\tilde{\mathcal{O}}(\mathbb{F}) := \mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(E)$ and $\tilde{\mathcal{P}}(\mathbb{F}) := \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(E)$.

Let W be an $\tilde{\mathcal{O}}(\mathbb{F})$ -measurable (resp. $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable) mapping from $\tilde{\Omega}$ into \mathbb{R} . We say that W is an \mathbb{F} -optional (resp. \mathbb{F} -predictable) function.

Let W be an \mathbb{F} -optional function. As in [13, II.1.5], we define

$$W * \mu(\omega)_t := \begin{cases} \int_{[0,t] \times E} W(\omega, t, x) \mu(\omega, dt, dx), & \text{if } \int_{[0,t] \times E} |W(\omega, t, x)| \mu(\omega, dt, dx) < +\infty; \\ +\infty, & \text{else.} \end{cases}$$

The process $W * \mu$ is finite-valued (and hence of finite variation) if and only if, for every $(\omega, t, x) \in \tilde{\Omega}$, $\int_{[0,t] \times E} |W(\omega, t, x)| \mu(\omega, dt, dx) < +\infty$ holds.

We say that μ is an \mathbb{F} -optional (resp. \mathbb{F} -predictable) random measure if $W * \mu$ is \mathbb{F} -optional (resp. \mathbb{F} -predictable), for every optional (resp. \mathbb{F} -predictable) function W .

Semimartingales. Let X be an \mathbb{R}^d -valued \mathbb{F} -semimartingale. We denote by μ^X the jump measure of X , that is,

$$\mu^X(\omega, dt, dx) = \sum_{s \geq 0} 1_{\{\Delta X_s(\omega) \neq 0\}} \delta_{(s, \Delta X_s(\omega))}(dt, dx),$$

where, here and in the whole paper, δ_a denotes the Dirac measure at point a (which can be n -dimensional, $n \geq 1$). From [13, Theorem II.1.16], μ^X is an *integer-valued random measure* with respect to \mathbb{F} (see [13, Definition II.1.13]). Thus, μ^X is, in particular, an \mathbb{F} -optional random measure. We recall that, according to [13, Definition III.1.23], μ^X is an \mathbb{R}^d -valued *marked point process* (with respect to \mathbb{F}) if $\mu^X(\omega; [0, t] \times \mathbb{R}^d) < +\infty$, for every $\omega \in \Omega$ and $t \in \mathbb{R}_+$.

By (B^X, C^X, ν^X) we denote the \mathbb{F} -predictable characteristics of X with respect to the \mathbb{R}^d -valued truncation function $h(x) := 1_{\{|x| \leq 1\}}x$ (see [13, Definition II.2.6]). We recall that ν^X is a predictable random measure characterized by the following two properties: For any \mathbb{F} -predictable mapping W such that $|W| * \mu^X \in \mathcal{A}_{\text{loc}}^+$, we have $|W| * \nu^X \in \mathcal{A}_{\text{loc}}^+$ and $W * \mu^X - W * \nu^X \in \mathcal{H}_{\text{loc}}^1(\mathbb{F})$.

For two \mathbb{F} -semimartingales X and Y , we denote by $[X, Y]$ the quadratic variation of X and Y :

$$[X, Y]_t := \langle X^c, Y^c \rangle_t + \sum_{s \leq t} \Delta X_s \Delta Y_s,$$

where X^c and Y^c denote the continuous local martingale part of X and Y , respectively. Recall that $X, Y \in \mathcal{H}_{\text{loc},0}^1(\mathbb{F})$ are orthogonal if and only if $[X, Y] \in \mathcal{H}_{\text{loc},0}^1(\mathbb{F})$.

For $X \in \mathcal{H}_{\text{loc}}^1(\mathbb{F})$ and $p \geq 1$, we denote by $L^p(X)$ the space of \mathbb{F} -predictable processes K such that $(K^2 \cdot [X, X])^{p/2} \in \mathcal{A}^+(\mathbb{F})$. The space $L_{\text{loc}}^p(X)$ is defined analogously but making use of $\mathcal{A}_{\text{loc}}^+(\mathbb{F})$ instead. For $K \in L_{\text{loc}}^1(X)$ we denote by $K \cdot X$ the stochastic integral of K with respect to X , which is a local martingale starting at zero and, by [12, Proposition 2.46 b)], $K \cdot X \in \mathcal{H}^p(\mathbb{F})$ if and only if $K \in L^p(X)$. We observe that if $X \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$ is of finite variation and $K \in L_{\text{loc}}^1(X)$, then the stochastic integral $K \cdot X$ coincides with the Stieltjes-Lebesgue integral, whenever this latter one exists and is finite. Sometimes, to stress the underlying filtration, we write $L_{\text{loc}}^p(X, \mathbb{F})$, $p \geq 1$.

Let $X \in \mathcal{H}_{\text{loc}}^1(\mathbb{F})$. Then we say that X possesses the *predictable representation property* (from now on the PRP) with respect to \mathbb{F} if for every $Y \in \mathcal{H}_{\text{loc}}^1(\mathbb{F})$ there exists $K \in L_{\text{loc}}^1(X, \mathbb{F})$ such that $Y = Y_0 + K \cdot X$.

The progressive enlargement by a process. Let H be a stochastic process and let $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ be the smallest right-continuous filtration generated by H , i.e., $\mathcal{H}_t := \bigcap_{s > t} \sigma(H_u, 0 \leq u \leq s)$, $t \geq 0$.

We consider a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and denote by $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ the *progressive enlargement* of \mathbb{F} by H : That is, \mathbb{G} is defined by

$$\mathcal{G}_t := \bigcap_{s > t} \tilde{\mathcal{G}}_s, \quad \tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \mathcal{H}_s, \quad s, t \geq 0. \quad (2.1)$$

In other words, \mathbb{G} is the *smallest* right-continuous filtration containing \mathbb{F} and \mathbb{H} .

As a special example of H , one can take the default process associated with a random time τ , i.e., $H_t(\omega) := 1_{[\tau, +\infty)}(\omega, t)$, where τ is a $(0, +\infty]$ -valued random variable. In this case, \mathbb{G} is called progressive enlargement of \mathbb{F} by τ and it is the smallest filtration satisfying the usual conditions containing \mathbb{F} and such that τ is a \mathbb{G} -stopping time.

Proposition 2.1. *Let H be a point process with respect to \mathbb{H} . Then H is a point process with respect to \mathbb{G} and therefore there exists the \mathbb{G} -dual predictable projection $H^{p, \mathbb{G}}$ of H . Moreover, every \mathbb{F} -point process X is a \mathbb{G} -point process and, hence, there exists the \mathbb{G} -dual predictable projection $X^{p, \mathbb{G}}$ of X .*

Proof. Let $\mathbb{A} \in \{\mathbb{F}, \mathbb{H}\}$ and let $Y = X$ if $\mathbb{A} = \mathbb{F}$ and $Y = H$ if $\mathbb{A} = \mathbb{H}$. The \mathbb{A} -point process Y has the representation $Y = \sum_{n=1}^{\infty} 1_{[\tau_n, +\infty)}$, where $\tau_n := \inf\{t \geq 0 : Y_t = n\}$ $n \geq 0$, is an \mathbb{A} -stopping time. Since Y is \mathbb{G} -adapted, τ_n is also a \mathbb{G} -stopping time and therefore Y is a point process also in the filtration \mathbb{G} . Since Y is locally bounded and increasing, we also have $Y \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$. In conclusion, there exists the \mathbb{G} -dual predictable projection $Y^{p, \mathbb{G}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$ of Y . The proof is complete. \square

3 Martingale Representation

Let X be a point process, let \mathcal{R} be a σ -field, called the *initial σ -field*, and let $\mathbb{X} = (\mathcal{X}_t)_{t \geq 0}$ denote the smallest right-continuous filtration with respect to which X is adapted, i.e.,

$$\mathcal{X}_t := \bigcap_{s > t} \sigma(X_u, 0 \leq u \leq s), \quad t \geq 0.$$

By $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ we denote the filtration given by $\mathcal{F}_t := \mathcal{R} \vee \mathcal{X}_t$. Clearly, X is a point process with respect to \mathbb{F} .

Lemma 3.1. (i) *The filtration \mathbb{F} is right-continuous.*

(ii) *The \mathbb{F} -local martingale $\bar{X}^{\mathbb{F}} := X - X^{p, \mathbb{F}}$ possesses the PRP with respect to \mathbb{F} .*

Proof. The statement (i) follows by [12, Proposition 3.39 a)] and the definition of \mathcal{F}_t . The statement (ii) is [13, Theorem III.4.37]. The proof is complete. \square

We remark that the filtration \mathbb{F} is the *initial enlargement* of the filtration \mathbb{X} with an arbitrary initial σ -field \mathcal{R} .

Let now H be a stochastic process and let \mathbb{H} denote the smallest right-continuous filtration with respect to which H is adapted. As in (2.1), we consider the progressive enlargement \mathbb{G} of \mathbb{F} by \mathbb{H} . We now state the following assumptions on H which will be in force for the remaining part of this paper.

Assumption 3.2. The process H is a point process with respect to the filtration \mathbb{H} .

We denote by $H^{p, \mathbb{H}}$ the \mathbb{H} -dual predictable projection of H . The local martingale $\bar{H}^{\mathbb{H}} := H - H^{p, \mathbb{H}}$ possesses the PRP with respect to the filtration \mathbb{H} .

In other words, we are in the following situation: The local martingale $\bar{X}^{\mathbb{F}}$ has the PRP with respect to \mathbb{F} and the local martingale $\bar{H}^{\mathbb{H}}$ has the PRP with respect to \mathbb{H} . We are now going to investigate the martingale representation in the filtration \mathbb{G} .

Because of Lemma 2.1, we know that in \mathbb{G} both X and H are point processes and that the \mathbb{G} -dual predictable projections $X^{p, \mathbb{G}}$ and $H^{p, \mathbb{G}}$ of X and H , respectively, exist.

Proposition 3.3. *Let X and H be two point processes with respect to \mathbb{F} and \mathbb{H} , respectively. Then, the processes $X - [X, H]$, $H - [X, H]$ and $[X, H]$ are point processes with respect to \mathbb{G} . Furthermore, they have pairwise no common jumps.*

Proof. By Proposition 2.1, X and H are \mathbb{G} -point processes. Therefore, in particular, $\Delta X, \Delta H \in \{0, 1\}$ holds. We show that $X - [X, H]$ is a point process with respect to \mathbb{G} . Clearly, this is a \mathbb{G} -adapted process and, using the definition of the quadratic variation, we get $X - [X, H] = \sum_{0 \leq s \leq \cdot} \Delta X_s (1 - \Delta H_s)$, which shows that $X - [X, H]$ is an \mathbb{N} -valued increasing process. Furthermore, we have $\Delta(X - [X, H]) = \Delta X (1 - \Delta H) \in \{0, 1\}$, showing that $X - [X, H]$ is a point process. The proof that $H - [X, H]$ and $[X, H]$ are \mathbb{G} -point processes is similar. It is clear that these processes have pairwise no common jumps. Indeed, for example, we have $[X - [X, H], [X, H]] = [X, [X, H]] - [[X, H], [X, H]] = [X, H] - [X, H] = 0$. The proof of the lemma is complete. \square

We now consider the \mathbb{R}^2 -valued \mathbb{G} -semimartingale $\tilde{X} = (X, H)^\top$. Then, the jump measure $\mu^{\tilde{X}}$ of \tilde{X} is an integer-valued random measure on $\mathbb{R}_+ \times E$, where $E = \{(1, 0); (0, 1); (1, 1)\}$. We then have $\tilde{\Omega} := \Omega \times \mathbb{R}_+ \times E$.

For a \mathbb{G} -predictable function W on $\tilde{\Omega}$, we introduce the \mathbb{G} -predictable processes $W(1, 0)$, $W(0, 1)$ and $W(1, 1)$ defined by

$$W_t(0, 1) := W(t, 0, 1), \quad W_t(1, 0) := W(t, 1, 0), \quad W_t(1, 1) := W(t, 1, 1), \quad t \geq 0.$$

In the next theorem we compute the jump measure $\mu^{\tilde{X}}$ of the \mathbb{R}^2 -valued semimartingale \tilde{X} and the \mathbb{G} -predictable compensator $\nu^{\tilde{X}}$ of $\mu^{\tilde{X}}$.

Theorem 3.4. *Let X be a point process with respect to \mathbb{F} and let H be a point process with respect to \mathbb{H} . Let \mathbb{G} be the progressive enlargement of \mathbb{F} by \mathbb{H} . Let us consider the \mathbb{R}^2 -valued \mathbb{G} -semimartingale $\tilde{X} = (X, H)^\top$. We then have:*

(i) *The jump measure $\mu^{\tilde{X}}$ of \tilde{X} on $\mathbb{R}_+ \times E$ is given by*

$$\begin{aligned} \mu^{\tilde{X}}(dt, dx_1, dx_2) &= d(X_t - [X, H]_t) \delta_{(1,0)}(dx_1, dx_2) \\ &\quad + d(H_t - [X, H]_t) \delta_{(0,1)}(dx_1, dx_2) + d[X, H]_t \delta_{(1,1)}(dx_1, dx_2). \end{aligned} \quad (3.1)$$

(ii) *The \mathbb{G} -predictable compensator $\nu^{\tilde{X}}$ of $\mu^{\tilde{X}}$ is given by*

$$\begin{aligned} \nu^{\tilde{X}}(dt, dx_1, dx_2) &= d(X_t - [X, H]_t)^{p, \mathbb{G}} \delta_{(1,0)}(dx_1, dx_2) \\ &\quad + d(H_t - [X, H]_t)^{p, \mathbb{G}} \delta_{(0,1)}(dx_1, dx_2) + d[X, H]_t^{p, \mathbb{G}} \delta_{(1,1)}(dx_1, dx_2). \end{aligned} \quad (3.2)$$

(iii) *The integer-valued random measure $\mu^{\tilde{X}}$ is an \mathbb{R}^2 -valued marked point process with respect to \mathbb{G} .*

Proof. We start verifying (i). By the definition of the jump measure of the semimartingale \tilde{X} , by direct

computation, we get:

$$\begin{aligned}
\mu^{\tilde{X}}(\omega, dt, dx_1, dx_2) &= \sum_{s>0} 1_{\{\Delta \tilde{X}_s(\omega) \neq 0\}} \delta_{(s, \Delta \tilde{X}_s(\omega))}(dt, dx_1, dx_2) \\
&= \sum_{s>0} 1_{\{\Delta X_s(\omega)=1\}} 1_{\{\Delta H_s(\omega)=0\}} \delta_{(s,1,0)}(dt, dx_1, dx_2) \\
&\quad + \sum_{s>0} 1_{\{\Delta X_s(\omega)=0\}} 1_{\{\Delta H_s(\omega)=1\}} \delta_{(s,0,1)}(dt, dx_1, dx_2) \\
&\quad + \sum_{s>0} 1_{\{\Delta X_s(\omega)=1\}} 1_{\{\Delta H_s(\omega)=1\}} \delta_{(s,1,1)}(dt, dx_1, dx_2) \\
&= \sum_{s>0} \Delta X_s(\omega)(1 - \Delta H_s(\omega)) \delta_{(s,1,0)}(dt, dx_1, dx_2) \\
&\quad + \sum_{s>0} \Delta H_s(\omega)(1 - \Delta X_s(\omega)) \delta_{(s,0,1)}(dt, dx_1, dx_2) \\
&\quad + \sum_{s>0} \Delta X_s(\omega) \Delta H_s(\omega) \delta_{(s,1,1)}(dt, dx_1, dx_2),
\end{aligned}$$

which is (3.1). We now come to (ii). Let us denote by ν the \mathbb{G} -predictable random measure on the right-hand side of (3.2) and let $\nu^{\tilde{X}}$ denote the \mathbb{G} -predictable compensator of $\mu^{\tilde{X}}$. We have to show that ν coincides (up to a null set) with $\nu^{\tilde{X}}$. To this goal, it is enough to consider a \mathbb{G} -predictable function W such that $|W| * \mu^{\tilde{X}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$ and show that $|W| * \nu \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$ and $W * \mu^{\tilde{X}} - W * \nu \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$. So, let W be a \mathbb{G} -predictable function such that $|W| * \mu^{\tilde{X}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$. By (i) we immediately get the identity

$$|W| * \mu^{\tilde{X}} = |W(1,0)| \cdot (X - [X,H]) + |W(0,1)| \cdot (H - [X,H]) + |W(1,1)| \cdot [X,H]. \quad (3.3)$$

We now denote by Y^1 , Y^2 and Y^3 the first, the second and the third term on the right-hand side of (3.3), respectively. By Lemma 3.3, $X - [X,H]$, $H - [X,H]$ and $[X,H]$ being point processes, we deduce from (3.3) that $Y^i \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$, $i = 1, 2, 3$, since $|W| * \mu^{\tilde{X}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$. Then, the \mathbb{G} -dual predictable projection $(Y^i)^{p,\mathbb{G}}$ of Y^i , $i = 1, 2, 3$, exists and it belongs to $\mathcal{A}_{\text{loc}}^+(\mathbb{G})$. Since $W(1,0)$, $W(0,1)$ and $W(1,1)$ are \mathbb{G} -predictable processes, the properties of the \mathbb{G} -dual predictable projection yield the identities $(Y^1)^{p,\mathbb{G}} = |W(1,0)| \cdot (X - [X,H])^{p,\mathbb{G}}$, $(Y^2)^{p,\mathbb{G}} = |W(0,1)| \cdot (H - [X,H])^{p,\mathbb{G}}$ and finally $(Y^3)^{p,\mathbb{G}} = |W(1,1)| \cdot [X,H]^{p,\mathbb{G}}$. By the definition of ν , we deduce

$$\begin{aligned}
|W| * \nu &= |W(1,0)| \cdot (X - [X,H])^{p,\mathbb{G}} + |W(0,1)| \cdot (H - [X,H])^{p,\mathbb{G}} + |W(1,1)| \cdot [X,H]^{p,\mathbb{G}} \\
&= (Y^1)^{p,\mathbb{G}} + (Y^2)^{p,\mathbb{G}} + (Y^3)^{p,\mathbb{G}}
\end{aligned} \quad (3.4)$$

showing that $|W| * \nu \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$ holds, since $(Y^i)^{p,\mathbb{G}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$, $i = 1, 2, 3$. It remains to show that $W * \mu^{\tilde{X}} - W * \nu \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$. By (3.3) and (3.4), it is clear that both the integrals $W * \mu^{\tilde{X}}$ and $W * \nu$ exist and are finite. The linearity of the integral with respect to the *integrator*, (i) and the definition of ν then yield

$$\begin{aligned}
W * \mu^{\tilde{X}} - W * \nu &= W(1,0) \cdot ((X - [X,H]) - (X - [X,H])^{p,\mathbb{G}}) \\
&\quad + W(0,1) \cdot ((H - [X,H]) - (H - [X,H])^{p,\mathbb{G}}) \\
&\quad + W(1,1) \cdot ([X,H] - [X,H]^{p,\mathbb{G}}).
\end{aligned}$$

Since, by the properties of the \mathbb{G} -dual predictable projection, each addend on the right-hand side of the previous formula belongs to $\mathcal{H}_{\text{loc}}^1(\mathbb{G})$, we deduce that $W * \mu^{\tilde{X}} - W * \nu \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$ too. Therefore,

we get that the \mathbb{G} -predictable measure ν coincides with the \mathbb{G} -predictable compensator $\nu^{\tilde{X}}$ of $\mu^{\tilde{X}}$. The proof of (ii) is complete. We now come to (iii). We observe that, from (i) and recalling that $E = \{(1,0); (0,1); (1,1)\}$, we get

$$\mu^{\tilde{X}}(\omega, [0, t] \times \mathbb{R}^2) = \mu^{\tilde{X}}(\omega, [0, t] \times E) = X_t(\omega) - [X, H]_t(\omega) + H_t(\omega) < +\infty, \quad \omega \in \Omega, \quad t \geq 0,$$

meaning that $\mu^{\tilde{X}}$ is an \mathbb{R}^2 -valued marked point process with respect to the filtration \mathbb{G} . The proof of the theorem is now complete. \square

We now denote by $\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_t)_{t \geq 0}$ the smallest right-continuous filtration such that $\mu^{\tilde{X}}$ is an optional random measure. By $\mathbb{G}^{\tilde{X}} = (\mathcal{G}_t^{\tilde{X}})_{t \geq 0}$ we indicate the smallest right-continuous filtration such that \tilde{X} is adapted. We then introduce $\mathbb{G}^* = (\mathcal{G}_t^*)_{t \geq 0}$ and $\mathbb{G}' = (\mathcal{G}_t')_{t \geq 0}$ by setting $\mathcal{G}_t^* := \mathcal{R} \vee \tilde{\mathcal{G}}_t$ and $\mathcal{G}_t' := \mathcal{R} \vee \mathcal{G}_t^{\tilde{X}}$, $t \geq 0$, respectively, where, we recall, \mathcal{R} denotes an initial σ -field. From [12, Proposition 3.39 a)], we know that \mathbb{G}^* is a right-continuous filtration.

Lemma 3.5. *The following statements hold:*

- (i) *The filtration $\mathbb{G}^{\tilde{X}}$ coincides with $\tilde{\mathbb{G}}$.*
- (ii) *The filtration \mathbb{G}' coincides with \mathbb{G}^* .*
- (iii) *The filtration \mathbb{G}' (and hence the filtration \mathbb{G}^*) coincides with \mathbb{G} .*

Proof. We start showing (i). First of all we observe that the inclusion $\tilde{\mathbb{G}} \subseteq \mathbb{G}^{\tilde{X}}$ holds. Indeed, by definition, $\mathbb{G}^{\tilde{X}}$ is right-continuous and \tilde{X} is a $\mathbb{G}^{\tilde{X}}$ -semimartingale. Since $\mu^{\tilde{X}}$ is the jump measure of \tilde{X} , by [13, Proposition II.1.16], $\mu^{\tilde{X}}$ is an integer-valued random measure with respect to $\mathbb{G}^{\tilde{X}}$ (see [13, Definition II.1.13]). Hence, $\mu^{\tilde{X}}$ is, in particular, a $\mathbb{G}^{\tilde{X}}$ -optional random measure. We now show the converse inclusion, i.e., $\mathbb{G}^{\tilde{X}} \subseteq \tilde{\mathbb{G}}$. From (3.1), it follows that

$$X = (1_{\{x_1=1, x_2=0\}} + 1_{\{x_1=1, x_2=1\}}) * \mu^{\tilde{X}}, \quad H = (1_{\{x_1=0, x_2=1\}} + 1_{\{x_1=1, x_2=1\}}) * \mu^{\tilde{X}}.$$

Clearly, $(\omega, t, x_1, x_2) \longrightarrow 1_{\{x_1=1, x_2=0\}}$, $(\omega, t, x_1, x_2) \longrightarrow 1_{\{x_1=0, x_2=1\}}$ and $(\omega, t, x_1, x_2) \longrightarrow 1_{\{x_1=1, x_2=1\}}$ are $\tilde{\mathbb{G}}$ -optional functions. Therefore, X and H are $\tilde{\mathbb{G}}$ -optional processes. So, since $\tilde{\mathbb{G}}$ is right continuous by definition, this yields $\mathbb{G}^{\tilde{X}} \subseteq \tilde{\mathbb{G}}$ and the proof of (i) is complete. We now come to (ii). By definition and (i) we have $\mathcal{G}_t^* = \mathcal{R} \vee \tilde{\mathcal{G}}_t = \mathcal{R} \vee \mathcal{G}_t^{\tilde{X}} = \mathcal{G}_t'$, for every $t \geq 0$. Hence, we also have $\mathbb{G}^* = \mathbb{G}'$ and the proof of (ii) is complete. We now come to (iii). To this aim, we recall that \mathbb{G} is the smallest right continuous filtration containing \mathbb{F} and \mathbb{H} . We now show the inclusion $\mathbb{G} \subseteq \mathbb{G}'$. We observe that, because of (ii), the filtration \mathbb{G}' is right-continuous, \mathbb{G}^* being right-continuous. Furthermore, $\mathcal{G}_t^{\tilde{X}}$ contains $\sigma(X_u, 0 \leq u \leq t)$ and $\sigma(H_u, 0 \leq u \leq t)$, for every $t \geq 0$. So, by the right-continuity of $\mathbb{G}^{\tilde{X}}$, we also get that $\mathbb{G}^{\tilde{X}}$ contains both \mathbb{X} and \mathbb{H} . But then \mathbb{G}' is right-continuous and it contains \mathcal{R} , \mathbb{X} and \mathbb{H} . Therefore, since by definition \mathbb{F} is the smallest right-continuous filtration containing \mathbb{X} and \mathcal{R} , we get $\mathbb{F} \subseteq \mathbb{G}'$. This yields the inclusion $\mathbb{G} \subseteq \mathbb{G}'$, \mathbb{G}' being right-continuous and containing both \mathbb{F} and \mathbb{H} . We now verify the converse inclusion, i.e., $\mathbb{G}' \subseteq \mathbb{G}$. Clearly, \mathbb{G} contains both \mathcal{R} and \mathbb{X} , since the inclusion $\mathbb{F} \subseteq \mathbb{G}$ holds. Furthermore, by definition, \mathbb{G} contains \mathbb{H} . So, by the right-continuity of \mathbb{G} , we deduce that \mathbb{G} contains $\mathbb{G}^{\tilde{X}}$ and \mathcal{R} . On the other side, \mathbb{G}' is the smallest right-continuous filtration containing both $\mathbb{G}^{\tilde{X}}$ and \mathcal{R} . This yields $\mathbb{G}' \subseteq \mathbb{G}$. Hence, $\mathbb{G} = \mathbb{G}'$ and by (ii) we also have $\mathbb{G} = \mathbb{G}^*$. The proof of the lemma is now complete. \square

We now come to the martingale representation result in the enlarged filtration \mathbb{G} . First, we introduce the following \mathbb{G} -local martingales:

$$Z^1 := (X - [X, H]) - (X - [X, H])^{p, \mathbb{G}}, \quad (3.5)$$

$$Z^2 := (H - [X, H]) - (H - [X, H])^{p, \mathbb{G}}, \quad (3.6)$$

$$Z^3 := [X, H] - [X, H]^{p, \mathbb{G}}. \quad (3.7)$$

We observe that the local martingales Z^1 , Z^2 and Z^3 are locally bounded. Indeed, by Proposition 3.3, we have that Z^i is a compensated point process and, therefore, Z^i has bounded jumps, $i = 1, 2, 3$. In particular, we have $Z^i \in \mathcal{H}_{\text{loc}}^p(\mathbb{G})$ for every $p \geq 1$, $i = 1, 2, 3$.

Theorem 3.6. (i) Let $Y \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$. Then, there exists a \mathbb{G} -predictable function W such that $|W| * \mu^{\tilde{X}}$ belongs to $\mathcal{A}_{\text{loc}}^+(\mathbb{G})$ and

$$Y = Y_0 + W * \mu^{\tilde{X}} - W * \nu^{\tilde{X}}. \quad (3.8)$$

(ii) Let $Y \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$. Then Y has the following representation:

$$Y = Y_0 + K^1 \cdot Z^1 + K^2 \cdot Z^2 + K^3 \cdot Z^3, \quad K^i \in L_{\text{loc}}^1(Z^i, \mathbb{G}), \quad i = 1, 2, 3. \quad (3.9)$$

Proof. We first verify (i). Because of Theorem 3.4 (iii), the random measure $\mu^{\tilde{X}}$ is a marked point process with respect to the filtration \mathbb{G} . Furthermore, by Lemma 3.5 (iii), the filtration \mathbb{G} coincides with \mathbb{G}^* . Hence, we can apply [13, Theorem III.4.37], to get that all \mathbb{G} -local martingales Y have the representation in (3.8). The proof of (i) is complete.

We now come to (ii). Let $Y \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$. By (i), there exists a \mathbb{G} -predictable function W such that $|W| * \mu^{\tilde{X}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$ and $Y = Y_0 + W * \mu^{\tilde{X}} - W * \nu^{\tilde{X}}$. Since $|W| * \mu^{\tilde{X}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$, we clearly get that the \mathbb{G} -predictable mappings $W^{j,k} := W 1_{\{x_1=j, x_2=k\}}$, $(j, k) \in E$ (recall $E = \{(1, 0); (0, 1); (1, 1)\}$), satisfy $|W^{j,k}| * \mu^{\tilde{X}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$. Therefore, we obtain $W^{j,k} * \mu^{\tilde{X}} - W^{j,k} * \nu^{\tilde{X}} = W^{j,k} * (\mu^{\tilde{X}} - \nu^{\tilde{X}}) \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$. From (3.1) and (3.2), the relations

$$W^{1,0} * (\mu^{\tilde{X}} - \nu^{\tilde{X}}) = W(1, 0) \cdot Z^1, \quad (3.10)$$

$$W^{0,1} * (\mu^{\tilde{X}} - \nu^{\tilde{X}}) = W(0, 1) \cdot Z^2, \quad (3.11)$$

$$W^{1,1} * (\mu^{\tilde{X}} - \nu^{\tilde{X}}) = W(1, 1) \cdot Z^3 \quad (3.12)$$

hold. So from (3.8), Theorem 3.4 and (3.10), (3.11), (3.12), we get

$$\begin{aligned} Y &= Y_0 + W * \mu^{\tilde{X}} - W * \nu^{\tilde{X}} \\ &= Y_0 + W^{1,0} * (\mu^{\tilde{X}} - \nu^{\tilde{X}}) + W^{0,1} * (\mu^{\tilde{X}} - \nu^{\tilde{X}}) + W^{1,1} * (\mu^{\tilde{X}} - \nu^{\tilde{X}}) \\ &= Y_0 + W(1, 0) \cdot Z^1 + W(0, 1) \cdot Z^2 + W(1, 1) \cdot Z^3. \end{aligned}$$

We now define the \mathbb{G} -predictable processes $K^1 := W(1, 0)$, $K^2 := W(0, 1)$ and $K^3 := W(1, 1)$. It remains to show that $K^i \in L_{\text{loc}}^1(Z^i, \mathbb{G})$, $i = 1, 2, 3$. To this aim, we observe that, by Theorem 3.4 and (3.5), (3.6), (3.7), the following estimates hold:

$$\begin{aligned} \sum_{0 \leq s \leq \cdot} |K_s^1 \Delta Z_s^1| &\leq |W^{1,0}| * \mu^{\tilde{X}} + |W^{1,0}| * \nu^{\tilde{X}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G}), \\ \sum_{0 \leq s \leq \cdot} |K_s^2 \Delta Z_s^2| &\leq |W^{0,1}| * \mu^{\tilde{X}} + |W^{0,1}| * \nu^{\tilde{X}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G}), \\ \sum_{0 \leq s \leq \cdot} |K_s^3 \Delta Z_s^3| &\leq |W^{1,1}| * \mu^{\tilde{X}} + |W^{1,1}| * \nu^{\tilde{X}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G}). \end{aligned}$$

So, [11, Theorem 9.5.1], yields $K^i \in L^1_{\text{loc}}(Z^i, \mathbb{G})$, $i = 1, 2, 3$. The proof of (ii) is complete. The proof of the theorem is now complete. \square

Theorem 3.6 shows that the martingale representation property of the process X in \mathbb{F} propagates to \mathbb{G} . This martingale representation result is obtained in full generality, without requiring any further condition on the point processes X and H . Additionally, the progressive enlargement \mathbb{G} is obtained adding to the filtration \mathbb{F} a whole process and not, as commonly done in the literature, only a random time.

Remark 3.7 (Comparison with the existing literature). We now review some results from the literature which are generalized by Theorem 3.6, at least in the special case of point processes.

- For a process Y we denote by \mathbb{F}^Y the smallest right-continuous filtration such that Y is adapted. In [4], Calzolari and Torti have proven the following result (see Theorem 4.5 therein): Let $M \in \mathcal{H}^2(\mathbb{F}^M)$ and let $N \in \mathcal{H}^2(\mathbb{F}^N)$. Let M have the PRP with respect to \mathbb{F}^M and let N have the PRP with respect to \mathbb{F}^N . Let \mathcal{F}_0^M and \mathcal{F}_0^N be trivial. Let $\mathbb{G} = \mathbb{F}^M \vee \mathbb{F}^N$ and assume that:
 - (1) $M, N \in \mathcal{H}^2(\mathbb{G})$, that is, \mathbb{F}^M and \mathbb{F}^N are both immersed in \mathbb{G} .
 - (2) M and N are orthogonal in \mathbb{G} , that is, $[M, N] \in \mathcal{H}^1_{\text{loc}}(\mathbb{G})$ -local martingale.

Then \mathbb{F}^M and \mathbb{F}^N are independent, M , N and $[M, N]$ are pairwise orthogonal martingales in $\mathcal{H}^2(\mathbb{G})$ and every $Y \in \mathcal{H}^2(\mathbb{G})$ has the representation

$$Y = Y_0 + K^1 \cdot M + K^2 \cdot N + K^3 \cdot [M, N], \quad K^1 \in L^2(M, \mathbb{G}), K^2 \in L^2(N, \mathbb{G}), K^3 \in L^2([M, N], \mathbb{G}).$$

In the setting of point processes, Theorem 3.6 generalizes [4, Theorem 4.5] since we do not assume the double immersion of \mathbb{F} and \mathbb{H} in \mathbb{G} , hence, $X - X^{p, \mathbb{F}}$ and $H - H^{p, \mathbb{H}}$ will not be, in general, \mathbb{G} -local martingales. Moreover, we do not assume that the \mathbb{G} -local martingales $M = X - X^{p, \mathbb{G}}$ and $N = H - H^{p, \mathbb{G}}$ are orthogonal in \mathbb{G} , nor the triviality of \mathcal{F}_0 and, nevertheless, we show that the PRP propagates to \mathbb{G} . Notice that the main idea here is to use the special structure of the process X and H which ensures that, independently of the filtration, they can be always compensated to local martingales.

- In [8, Theorem 5.3] it was shown that if X is an \mathbb{R}^d -dimensional semimartingale possessing the weak representation property with respect to a filtration \mathbb{F} and \mathbb{G} is the progressive enlargement of \mathbb{F} by a random time τ such that \mathbb{F} is immersed in \mathbb{G} and τ avoids \mathbb{F} stopping times, that is, $\mathbb{P}[\tau = \sigma < +\infty] = 0$, for every \mathbb{F} -stopping time σ , then the semimartingale (X, H) possesses the weak representation property with respect to \mathbb{G} , where $H = 1_{[\tau, +\infty)}$ (see [8, Definition 3.1]). Theorem 3.6 above generalizes [8, Theorem 5.3] because, in the special case in which $H = 1_{[\tau, +\infty)}$ and X is a point process, it shows that the weak representation property propagates to \mathbb{G} , without any further assumption on τ .
- In [2] Aksamit, Jeanblanc and Rutkowski considered the case of a Poisson process X with respect to \mathbb{X} and enlarged \mathbb{X} progressively by a random time τ . They obtained a predictable representation result for a particular class of \mathbb{G} -martingales stopped at τ . Theorem 3.6 generalizes the results of [2] first because \mathbb{X} is generated by an arbitrary point process, it is initially enlarged to \mathbb{F} and then progressively enlarged by a whole process H and not only by a random time. Second because we obtain a predictable representation result which is valid for every \mathbb{G} -local martingale and not only for \mathbb{G} -local martingales stopped at τ .

- In [19], Xue has shown that the weak representation property of a *quasi-left continuous* semimartingale X with respect to a filtration \mathbb{F} propagates to the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where \mathbb{H} is a filtration independent of \mathbb{F} which supports a quasi-left continuous semimartingale Y possessing the weak representation property with respect to \mathbb{H} . If X and H are point processes, Theorem 3.6 generalizes the work of Xue because we require neither the independence of \mathbb{F} and \mathbb{H} nor the quasi-left continuity of X and H .

Our next aim is to study the *stable subspaces* generated by the local martingales Z^1, Z^2 and Z^3 in $\mathcal{H}_0^p(\mathbb{G})$, for $p \geq 1$. We refer to [12, Chapter IV] for this topic and we now shortly recall some basics about stable subspaces.

Let \mathcal{H} be a linear subspace of $(\mathcal{H}_0^p(\mathbb{G}), \|\cdot\|_{\mathcal{H}^p})$. Then \mathcal{H} is called a *stable subspace* of $\mathcal{H}_0^p(\mathbb{G})$ if \mathcal{H} is closed in $(\mathcal{H}_0^p(\mathbb{G}), \|\cdot\|_{\mathcal{H}^p})$ and *stable under stopping*, that is, for every \mathbb{G} -stopping time σ and $Z \in \mathcal{H}$, then $Z^\sigma \in \mathcal{H}$.

For a family $\mathcal{Z} \subseteq \mathcal{H}_{\text{loc},0}^p(\mathbb{G})$, the stable subspace generated by \mathcal{Z} in $\mathcal{H}_0^p(\mathbb{G})$ is denoted by $\mathcal{L}_{\mathbb{G}}^p(\mathcal{Z})$ and it is defined as the closure in $(\mathcal{H}_0^p(\mathbb{G}), \|\cdot\|_{\mathcal{H}^p})$ of the set of all stochastic integrals with respect to Z , that is, of the set $\{K \cdot Z, Z \in \mathcal{Z}, K \in L^p(Z, \mathbb{G})\}$. Notice that $\mathcal{L}_{\mathbb{G}}^p(\mathcal{Z})$ is the smallest stable subspace of $\mathcal{H}_0^p(\mathbb{G})$ containing the set $\mathcal{S} := \{Z^\sigma, \sigma \text{ } \mathbb{G}\text{-stopping time} : Z^\sigma \in \mathcal{H}^p(\mathbb{G})\}$ (see [12, Definition 4.4] and the subsequent comment therein).

We stress that, since Z^i is locally bounded, $i = 1, 2, 3$, it follows that $\mathcal{L}_{\mathbb{G}}^p(\mathcal{Z}) \neq \emptyset$, for every $p \geq 1$, where $\mathcal{Z} := \{Z^1, Z^2, Z^3\}$.

Corollary 3.8. *Let $p \geq 1$ and let $\mathcal{Z} := \{Z^1, Z^2, Z^3\}$.*

- (i) *The identity $\mathcal{L}_{\mathbb{G}}^p(\mathcal{Z}) = \mathcal{H}_0^p(\mathbb{G})$ holds.*
- (ii) *If furthermore $p = 2$ and the local martingales $Z^1, Z^2, Z^3 \in \mathcal{H}_{\text{loc},0}^2(\mathbb{G})$ are pairwise orthogonal, then every $Y \in \mathcal{H}^2(\mathbb{G})$ can be represented as*

$$Y = Y_0 + K^1 \cdot Z^1 + K^2 \cdot Z^2 + K^3 \cdot Z^3, \quad K^i \in L^2(Z^i, \mathbb{G}), \quad i = 1, 2, 3,$$

and this is an orthogonal decomposition of Y in $(\mathcal{H}^2(\mathbb{G}), \|\cdot\|_2)$.

Proof. We first verify (i). To this aim we show that every bounded $Y \in \mathcal{H}_{\text{loc},0}^1(\mathbb{G})$ which is orthogonal to \mathcal{Z} vanishes. Then, [12, Theorem 4.11 (b)] yields $\mathcal{L}_{\mathbb{G}}^1(\mathcal{Z}) = \mathcal{H}_0^1(\mathbb{G})$. Let now $Y \in \mathcal{H}_{\text{loc},0}^1(\mathbb{G})$ be bounded and orthogonal to \mathcal{Z} . Then, $[Y, Z^i] \in \mathcal{H}_{\text{loc},0}^1(\mathbb{G})$ holds, for $i = 1, 2, 3$. By Theorem 3.6 (ii), we can represent Y as in (3.9). Hence,

$$[Y, Y] = K^1 \cdot [Y, Z^1] + K^2 \cdot [Y, Z^2] + K^3 \cdot [Y, Z^3], \quad K^i \in L_{\text{loc}}^1(Z^i, \mathbb{G}). \quad (3.13)$$

Now, using that Y is bounded by a constant, say $C > 0$, and that $K^i \in L_{\text{loc}}^1(Z^i, \mathbb{G})$, we obtain

$$(K^i)^2 \cdot [[Y, Z^i], [Y, Z^i]] = \sum (K^i \Delta Y \Delta Z^i)^2 \leq 4C^2 (K^i)^2 \cdot [Z^i, Z^i] \in \mathcal{A}_{\text{loc}}^+(\mathbb{G}), \quad i = 1, 2, 3.$$

This shows that each addend on the right-hand side of (3.13) belongs to $\mathcal{H}_{\text{loc},0}^1(\mathbb{G})$. Therefore, we also have $[Y, Y] \in \mathcal{H}_{\text{loc},0}^1(\mathbb{G})$, implying that Y is orthogonal to itself. Hence, by [13, Lemma 4.13], we get that Y vanishes. This implies $\mathcal{L}_{\mathbb{G}}^1(\mathcal{Z}) = \mathcal{H}_0^1(\mathbb{G})$. Applying now [12, Theorem 4.67], if $Y \in \mathcal{H}_{\text{loc},0}^1(\mathbb{G})$ is orthogonal to \mathcal{Z} (and not necessarily bounded), then Y vanishes. Hence, since for every $p \geq 1$ we have $\mathcal{Z} \subseteq \mathcal{H}_{\text{loc},0}^p(\mathbb{G})$, [12, Theorem 4.11 (b)] yields $\mathcal{L}_{\mathbb{G}}^p(\mathcal{Z}) = \mathcal{H}_0^p(\mathbb{G})$, for every $p \geq 1$. This completes the proof of (i). We now come to (ii). If Z^1, Z^2 and Z^3 are pairwise orthogonal, then $\mathcal{L}_{\mathbb{G}}^2(\mathcal{Z}) = \mathcal{L}_{\mathbb{G}}^2(Z^1) \oplus \mathcal{L}_{\mathbb{G}}^2(Z^2) \oplus \mathcal{L}_{\mathbb{G}}^2(Z^3)$ (see [12, Remark 4.36]). Then, since we have $Y - Y_0 \in \mathcal{H}_0^2(\mathbb{G})$ and, by (i), $\mathcal{L}_{\mathbb{G}}^2(\mathcal{Z}) = \mathcal{H}_0^2(\mathbb{G})$, we deduce (ii) from [12, Theorem 4.6]. The proof of the corollary is complete. \square

According to Lemma A.1 (iv) and (v), since by Proposition 3.3 $X - [X, H]$, $H - [X, H]$ and $[X, H]$ have pairwise no common jumps, then Z^1 , Z^2 and Z^3 are pairwise orthogonal, if and only if $(X - [X, H])^{p, \mathbb{G}}$, $(H - [X, H])^{p, \mathbb{G}}$ and $[X, H]^{p, \mathbb{G}}$ have pairwise no common jumps, which is equivalent to $[Z^i, Z^j] = 0$, $i \neq j$.

We conclude this section giving a more suggestive interpretation of Corollary 3.8 (i). Let us regard the family $\mathcal{Z} = \{Z^1, Z^2, Z^3\} \subseteq \mathcal{H}_{\text{loc}, 0}^1$ as the *multidimensional local martingale* $Z := (Z^1, Z^2, Z^3)^\top$. Since $Z^i \in \mathcal{H}_{\text{loc}}^p(\mathbb{G})$ for every $p \geq 1$, combining Corollary 3.8 and [12, Theorem 4.60], yields that every $Y \in \mathcal{H}^p(\mathbb{G})$ can be represented as

$$Y = Y_0 + K \cdot Z, \quad K \in \bar{\mathcal{L}}^p(Z, \mathbb{G}),$$

where we now make use of the *multidimensional stochastic integral* with respect to Z and $\bar{\mathcal{L}}^p(Z, \mathbb{G})$ denotes the space of *multidimensional* predictable integrands for the local-martingale Z (see [12, Eq. (4.59)]).

4 Predictable Representation with respect to Orthogonal Martingales

We recall that X denotes a point process and \mathcal{R} an initial σ -field. The filtration \mathbb{X} is the smallest right-continuous filtration such that X is adapted and \mathbb{F} denotes the initial enlargement of \mathbb{X} by \mathcal{R} . We also consider another point process H with respect to \mathbb{H} , i.e., the smallest right-continuous filtration such that H is adapted, and denote by \mathbb{G} the progressive enlargement of \mathbb{F} by H .

In this section we discuss some cases in which the martingale representation in \mathbb{G} yields with respect to orthogonal local martingales.

In Subsection 4.1 we study the situation of the independent enlargement, that is, the filtrations \mathbb{F} and \mathbb{H} are assumed to be independent. In this particular case, as a consequence of Theorem 3.6, we obtain a martingale representation result with respect to the pairwise orthogonal local martingales $\bar{X}^{\mathbb{G}}$, $\bar{H}^{\mathbb{G}}$ and $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}]$ (see Section 2 for this notation). In other words, we show that, for the case of point processes, [4, Theorem 4.5] is a direct consequence of Theorem 3.6. We stress that in the general case, without the further assumption on the independence of \mathbb{F} and \mathbb{H} , the process $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}]$ will not be a \mathbb{G} -local martingale, in general.

In Subsection 4.2 we consider the case in which \mathbb{F} is progressively enlarged by a random time τ , that is, $H = 1_{[\tau, +\infty)}$ and we do not require the independence of the filtrations \mathbb{F} and \mathbb{H} . We then investigate the orthogonality of the martingales Z^1 , Z^2 and Z^3 introduced in (3.5), (3.6) and (3.7). We recall that, because of Lemma 3.3 (v), the local martingales Z^1 , Z^2 and Z^3 are pairwise orthogonal, if and only if $(X - [X, H])^{p, \mathbb{G}}$, $(H - [X, H])^{p, \mathbb{G}}$ and $[X, H]^{p, \mathbb{G}}$ have pairwise no common jumps. Therefore, we will concentrate on sufficient conditions on τ which ensure that these \mathbb{G} -dual predictable projections have pairwise no common jumps.

4.1 The independent enlargement

Let \mathbb{F} and \mathbb{H} be independent, that is, the process H is independent of $\{\mathcal{R}, X\}$. We observe that from [18, Theorem 1], the filtration $\mathbb{F} \vee \mathbb{H}$ is right-continuous. Therefore, $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ holds. Furthermore, since by the assumed independence, \mathbb{F} -local martingales and \mathbb{H} -local martingales remain \mathbb{G} -local martingales, we obtain that $X^{p, \mathbb{G}} = X^{p, \mathbb{F}}$ and $H^{p, \mathbb{G}} = H^{p, \mathbb{H}}$. Hence, the identities $\bar{X}^{\mathbb{G}} = \bar{X}^{\mathbb{F}}$ and $\bar{H}^{\mathbb{G}} = \bar{H}^{\mathbb{H}}$ hold.

We stress that the local martingales $\bar{X}^{\mathbb{G}}$ and $\bar{H}^{\mathbb{G}}$ have bounded jumps and hence they are locally bounded.

Lemma 4.1. *Let \mathbb{F} and \mathbb{H} be independent.*

(i) *The local martingales $\bar{X}^{\mathbb{G}}$ and $\bar{H}^{\mathbb{G}}$ are orthogonal, i.e., $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$. Additionally, $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}]$ is locally bounded.*

(ii) *The local martingales $\bar{X}^{\mathbb{G}}$, $\bar{H}^{\mathbb{G}}$ and $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}]$ are pairwise orthogonal.*

Proof. We first show (i). To this aim it is sufficient to verify that $\bar{X}^{\mathbb{G}} \bar{H}^{\mathbb{G}} \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$ or, equivalently, that $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$. Since $\bar{X}^{\mathbb{G}} = \bar{X}^{\mathbb{F}} \in \mathcal{H}_{\text{loc}}^2(\mathbb{F})$, we find a sequence $(\tau_n)_{n \geq 0}$ of \mathbb{F} -stopping times localizing $\bar{X}^{\mathbb{G}} = \bar{X}^{\mathbb{F}}$ to $\mathcal{H}^2(\mathbb{F})$. Analogously, we find a sequence $(\sigma_n)_{n \geq 0}$ of \mathbb{H} -stopping times localizing $\bar{H}^{\mathbb{G}} = \bar{H}^{\mathbb{H}}$ to $\mathcal{H}^2(\mathbb{H})$. By the independence of \mathbb{F} and \mathbb{H} , the process $(\bar{X}^{\mathbb{G}})^{\tau_n} (\bar{H}^{\mathbb{G}})^{\sigma_n}$ belongs to $\mathcal{H}^2(\mathbb{G})$. Therefore, we get $[(\bar{X}^{\mathbb{G}})^{\tau_n}, (\bar{H}^{\mathbb{G}})^{\sigma_n}] \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$ and, by Kunita–Watanabe’s inequality (see [12, Theorem 2.32]), this is a process of *integrable variation*. Therefore, from the estimate $\mathbb{E}[\sup_{t \geq 0} |[(\bar{X}^{\mathbb{G}})^{\tau_n}, (\bar{H}^{\mathbb{G}})^{\sigma_n}]_t|] \leq \mathbb{E}[\text{Var}([(X^{\mathbb{G}})^{\tau_n}, (\bar{H}^{\mathbb{G}})^{\sigma_n}])_{\infty}] < +\infty$, where $\text{Var}(\cdot)$ denotes the total-variation process, we get $[(\bar{X}^{\mathbb{G}})^{\tau_n}, (\bar{H}^{\mathbb{G}})^{\sigma_n}] \in \mathcal{H}^1(\mathbb{G})$. By the properties of the covariation, we deduce the identity $[(\bar{X}^{\mathbb{G}})^{\tau_n}, (\bar{H}^{\mathbb{G}})^{\sigma_n}] = [\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}]_{\tau_n \wedge \sigma_n}$. Hence, $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$ and a localizing sequence is given by $(\rho_n)_{n \geq 0}$, $\rho_n := \tau_n \wedge \sigma_n$, $n \geq 1$. Notice that $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}]$ is locally bounded because $\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}$ are locally bounded and the identity $\Delta[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] = \Delta \bar{X}^{\mathbb{G}} \Delta \bar{H}^{\mathbb{G}}$ holds. The proof of (i) is complete. For (ii) we refer to the proof of [5, Proposition 3.4]. The proof of the lemma is complete. \square

Theorem 4.2. *Let \mathbb{F} and \mathbb{H} be independent. Then every $Y \in \mathcal{H}^2(\mathbb{G})$ can be represented as follow:*

$$Y = Y_0 + K^1 \cdot \bar{X}^{\mathbb{G}} + K^2 \cdot \bar{H}^{\mathbb{G}} + K^3 \cdot [\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] \quad (4.1)$$

where $K^1 \in L^2(\bar{X}^{\mathbb{G}}, \mathbb{G})$, $K^2 \in L^2(\bar{H}^{\mathbb{G}}, \mathbb{G})$ and $K^3 \in L^2([\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}], \mathbb{G})$. Additionally, (4.1) is an orthogonal decomposition of Y in $(\mathcal{H}^2(\mathbb{G}), \|\cdot\|_2)$.

Proof. First, using [13, Proposition I.4.49 (b)], we compute

$$\begin{aligned} [\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] &= [X, H] - [X^{p, \mathbb{G}}, H] - [X, H^{p, \mathbb{G}}] + [X^{p, \mathbb{G}}, H^{p, \mathbb{G}}] \\ &= [X, H] - [X^{p, \mathbb{G}}, H] - \Delta H^{p, \mathbb{G}} \cdot \bar{X}^{\mathbb{G}} \\ &= [X, H] - [X^{p, \mathbb{G}}, H^{p, \mathbb{G}}] + [X^{p, \mathbb{G}}, H^{p, \mathbb{G}}] - [X^{p, \mathbb{G}}, H] - \Delta H^{p, \mathbb{G}} \cdot \bar{X}^{\mathbb{G}} \\ &= [X, H] - [X^{p, \mathbb{G}}, H^{p, \mathbb{G}}] - \Delta X^{p, \mathbb{G}} \cdot \bar{H}^{\mathbb{G}} - \Delta H^{p, \mathbb{G}} \cdot \bar{X}^{\mathbb{G}} \end{aligned} \quad (4.2)$$

Because of Lemma 4.1, $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$ holds. Hence, Lemma A.1 (iii) yields the identity $[X, H]^{p, \mathbb{G}} = [X^{p, \mathbb{G}}, H^{p, \mathbb{G}}]$. Therefore, since $Z^3 = [X, H] - [X, H]^{p, \mathbb{G}}$, from the previous computation and from (3.5) and (3.6) we obtain

$$Z^3 = [\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] + \Delta X^{p, \mathbb{G}} \cdot \bar{H}^{\mathbb{G}} + \Delta H^{p, \mathbb{G}} \cdot \bar{X}^{\mathbb{G}} \quad (4.3)$$

$$Z^1 = [\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] + \Delta X^{p, \mathbb{G}} \cdot \bar{H}^{\mathbb{G}} + (1 - \Delta H^{p, \mathbb{G}}) \cdot \bar{X}^{\mathbb{G}} \quad (4.4)$$

$$Z^2 = [\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] + (1 - \Delta X^{p, \mathbb{G}}) \cdot \bar{H}^{\mathbb{G}} + \Delta H^{p, \mathbb{G}} \cdot \bar{X}^{\mathbb{G}}. \quad (4.5)$$

Let us denote $\mathcal{Z} := \{Z^1, Z^2, Z^3\}$ and $\mathcal{X} := \{\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}, [\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}]\}$. We are now going to show the inclusion $\mathcal{L}_{\mathbb{G}}^2(\mathcal{Z}) \subseteq \mathcal{L}_{\mathbb{G}}^2(\mathcal{X})$. To this aim, it is enough to verify the inclusion $\mathcal{R} \subseteq \mathcal{L}_{\mathbb{G}}^2(\mathcal{X})$, where we define

$$\mathcal{R} := \{(Z^i)^{\sigma}, \quad \sigma \text{ } \mathbb{G}\text{-stopping time} : (Z^i)^{\sigma} \in \mathcal{H}^2(\mathbb{G}), i = 1, 2, 3\}.$$

Indeed, by the property of the stable subspaces, $\mathcal{L}_{\mathbb{G}}^2(\mathcal{Z})$ is the smallest stable subspace of $\mathcal{H}_0^2(\mathbb{G})$ containing \mathcal{R} . Let now σ be an arbitrary \mathbb{G} -stopping time such that the stopped process $(Z^3)^\sigma$ belongs to $\mathcal{H}^2(\mathbb{G})$. Notice that such stopping times exist, because $Z^3 \in \mathcal{H}_{\text{loc}}^2(\mathbb{G})$, Z^3 being locally bounded. From (4.3), since by Lemma 4.1 (iii) the locally bounded \mathbb{G} -local martingales $\bar{X}^{\mathbb{G}}$, $\bar{H}^{\mathbb{G}}$ and $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}]$ are pairwise orthogonal, we get

$$\begin{aligned} \langle (Z^3)^\sigma, (Z^3)^\sigma \rangle = \\ 1_{\llbracket 0, \sigma \rrbracket} \cdot \langle [\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}], [\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] \rangle + 1_{\llbracket 0, \sigma \rrbracket} (\Delta X^{p, \mathbb{G}})^2 \cdot \langle \bar{H}^{\mathbb{G}}, \bar{H}^{\mathbb{G}} \rangle + 1_{\llbracket 0, \sigma \rrbracket} (\Delta H^{p, \mathbb{G}})^2 \cdot \langle \bar{X}^{\mathbb{G}}, \bar{X}^{\mathbb{G}} \rangle. \end{aligned}$$

Since $(Z^3)^\sigma \in \mathcal{H}_0^2(\mathbb{G})$, it follows that $\langle (Z^3)^\sigma, (Z^3)^\sigma \rangle \in \mathcal{A}^+(\mathbb{G})$. From this we obtain the following inclusions: $1_{\llbracket 0, \sigma \rrbracket} \in L^2([\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}], \mathbb{G})$, $1_{\llbracket 0, \sigma \rrbracket} \Delta X^{p, \mathbb{G}} \in L^2(\bar{H}^{\mathbb{G}}, \mathbb{G})$ and $1_{\llbracket 0, \sigma \rrbracket} \Delta H^{p, \mathbb{G}} \in L^2(\bar{X}^{\mathbb{G}}, \mathbb{G})$. Hence, we have $(Z^3)^\sigma \in \mathcal{L}_{\mathbb{G}}^2(\mathcal{Z})$, for every \mathbb{G} -stopping time σ such that $(Z^3)^\sigma$ belongs to $\mathcal{H}^2(\mathbb{G})$. Analogously, we see that the same holds also for Z^1 and Z^2 . Hence, we get the inclusion $\mathcal{S} \subseteq \mathcal{L}_{\mathbb{G}}^2(\mathcal{Z})$ and so the inclusion $\mathcal{L}_{\mathbb{G}}^2(\mathcal{Z}) \subseteq \mathcal{L}_{\mathbb{G}}^2(\mathcal{X})$ holds. From Corollary 3.8 (i), we have $\mathcal{L}_{\mathbb{G}}^2(\mathcal{Z}) = \mathcal{H}_0^2(\mathbb{G})$. Therefore, we obtain the identity $\mathcal{L}_{\mathbb{G}}^2(\mathcal{X}) = \mathcal{H}_0^2(\mathbb{G})$. By the pairwise orthogonality of $\bar{X}^{\mathbb{G}}$, $\bar{H}^{\mathbb{G}}$ and $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}]$ we have $\mathcal{L}_{\mathbb{G}}^2(\mathcal{X}) = \mathcal{L}_{\mathbb{G}}^2(\bar{H}^{\mathbb{G}}) \oplus \mathcal{L}_{\mathbb{G}}^2(\bar{X}^{\mathbb{G}}) \oplus \mathcal{L}_{\mathbb{G}}^2([\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}])$ (see [12, Remark 4.36]). This, recalling [12, Theorem 4.6], shows (4.1). The proof of the theorem is now complete. \square

Remark 4.3 (The multiplicity of \mathbb{G} in the independent enlargement). We remark that, because of the independence of \mathbb{F} and \mathbb{H} , we have the identities $\bar{X}^{\mathbb{G}} = \bar{X}^{\mathbb{F}}$ and $\bar{H}^{\mathbb{G}} = \bar{H}^{\mathbb{H}}$. Theorem 4.2 shows that every $Y \in \mathcal{H}^2(\mathbb{G})$ can be represented as a stochastic integral with respect to the pairwise orthogonal local martingales $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}]$, $\bar{X}^{\mathbb{G}}$, $\bar{H}^{\mathbb{G}}$. Therefore, we conclude that, in general, the multiplicity of the filtration \mathbb{G} is equal to *three*. However, if $X^{p, \mathbb{G}} = X^{p, \mathbb{F}}$ or $H^{p, \mathbb{G}} = H^{p, \mathbb{H}}$ is continuous, i.e., if X or H is quasi-left continuous in \mathbb{G} , then $[X^{p, \mathbb{G}}, H^{p, \mathbb{G}}] = 0$ and the multiplicity of \mathbb{G} is equal to two. Indeed, since $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$, Lemma A.1 (iii) implies that $[X, H] = 0$. Hence, X and H have no common jumps and Lemma A.1 (iv) yields $[\bar{X}^{\mathbb{G}}, \bar{H}^{\mathbb{G}}] = 0$.

4.2 The Progressive Enlargement by a Random Time

Let \mathbb{F} denote the initial enlargement of \mathbb{X} by \mathcal{R} . In this subsection we state the following further assumptions:

(A1) The process H is given by $H = 1_{\llbracket \tau, +\infty \rrbracket}$, where $\tau : \Omega \rightarrow (0, +\infty]$ is a *random time*.

(A2) The process X is quasi-left continuous with respect to \mathbb{F} , i.e., $X^{p, \mathbb{F}}$ is continuous.

We stress that in this subsection we do not assume that the random time τ and the filtration \mathbb{F} are independent.

Let ${}^{o, \mathbb{F}}(H)$ denote the \mathbb{F} -optional projection of H . We set $A := {}^{o, \mathbb{F}}(H)$. The process A is càdlàg and satisfies $A_t = \mathbb{P}[\tau > t | \mathcal{F}_t]$, because of the properties of the \mathbb{F} -optional projection. In particular, A is a càdlàg \mathbb{F} -supermartingale of class (D), which is called the *Azéma supermartingale* of H . We recall that A_- does not vanish on $[0, \tau]$ while A does not vanish on $[0, \tau)$.

From [15, Proposition 5.9.4.5], we have

$$H_t^{p, \mathbb{G}} = \int_0^{\tau \wedge t} \frac{1}{A_{s-}} dH_s^{p, \mathbb{F}}, \quad t \in [0, T], \quad (4.6)$$

We now state the first result about the orthogonality in this subsection: Notice that for this the quasi-left continuity of X with respect to \mathbb{F} is not required. We recall that τ is said to avoid \mathbb{F} -stopping times if $\mathbb{P}[\tau = \sigma < +\infty] = 0$ for every \mathbb{F} -stopping time σ .

Proposition 4.4. *Let τ avoid \mathbb{F} -stopping times and X be an arbitrary point process with respect to \mathbb{F} (non-necessarily quasi-left continuous). Then $[X, H] = Z^3 = 0$, $Z^1 = \bar{X}^{p, \mathbb{G}}$, $Z^2 = \bar{H}^{p, \mathbb{G}}$ and $[Z^1, Z^2] = 0$.*

Proof. Because of the avoidance assumption, we have $[X, H] = 0$ and hence $Z^3 = 0$. So, $Z^1 = \bar{X}^{\mathbb{G}}$ and $Z^2 = \bar{H}^{\mathbb{G}}$ hold. Furthermore, by (4.6), $H^{p, \mathbb{F}}$ being continuous because of the avoidance assumption (see [15, p. 324]), we also get that $H^{p, \mathbb{G}}$ is continuous. So, $[X, H]^{p, \mathbb{G}} = [X^{p, \mathbb{G}}, H^{p, \mathbb{G}}] = 0$. In conclusion, Lemma A.1 (iii) and (iv) yield $[Z^1, Z^2] = 0$. The proof is complete. \square

In the remaining part of this section we investigate the pairwise orthogonality of Z^1 , Z^2 and Z^3 without requiring that τ avoids \mathbb{F} -stopping times.

Notice that $\bar{X}^{\mathbb{F}} \in \mathcal{H}_{\text{loc}}^2(\mathbb{F})$ and if furthermore X is quasi-left continuous with respect to \mathbb{F} , then $X^{p, \mathbb{F}}$ is continuous and we have the identity

$$\langle \bar{X}^{\mathbb{F}}, \bar{X}^{\mathbb{F}} \rangle^{\mathbb{F}} = X^{p, \mathbb{F}}. \quad (4.7)$$

Indeed, by the continuity of $X^{p, \mathbb{F}}$ and the properties of the predictable covariation, since $X = [X, X]$, we have

$$\langle \bar{X}^{\mathbb{F}}, \bar{X}^{\mathbb{F}} \rangle^{\mathbb{F}} = [\bar{X}^{\mathbb{F}}, \bar{X}^{\mathbb{F}}]^{p, \mathbb{F}} = [X, X]^{p, \mathbb{F}} = X^{p, \mathbb{F}}.$$

Lemma 4.5. *Let Assumptions (A1) and (A2) hold.*

(i) *There exists an \mathbb{F} -predictable process K such that*

$$[X, H]^{p, \mathbb{G}} = 1_{[0, \tau]} A_-^{-1} K \cdot X^{p, \mathbb{F}}. \quad (4.8)$$

In particular, $[X, H]$ is a quasi-left continuous process.

(ii) *The process X^τ is quasi-left continuous in \mathbb{G} .*

Proof. We first verify (i). We clearly have $[X, H] = \Delta X \cdot H = \Delta X_\tau H$. We can then apply [2, Lemma A.1] to deduce

$$[X, H]^{p, \mathbb{G}} = 1_{[0, \tau]} A_-^{-1} \cdot [X, H]^{p, \mathbb{F}} = 1_{[0, \tau]} A_-^{-1} \cdot [\bar{X}^{\mathbb{F}}, H]^{p, \mathbb{F}},$$

where in the last-but-one equality we have used that $X^{p, \mathbb{F}}$ is continuous. By [2, Corollary 2.1], there exists $Y \in \mathcal{H}_{\text{loc}}^2(\mathbb{F})$ such that

$$1_{[0, \tau]} A_-^{-1} \cdot [\bar{X}^{\mathbb{F}}, H]^{p, \mathbb{F}} = 1_{[0, \tau]} A_-^{-1} \cdot \langle \bar{X}^{\mathbb{F}}, Y \rangle^{\mathbb{F}}.$$

But $\bar{X}^{\mathbb{F}}$ has the PRP with respect to \mathbb{F} (see Lemma 3.1 (ii)). Therefore, there exists an \mathbb{F} -predictable process $K \in L_{\text{loc}}^2(\bar{X}^{\mathbb{F}}, \mathbb{F})$ such that $Y = Y_0 + K \cdot \bar{X}^{\mathbb{F}}$. Hence, $\langle \bar{X}^{\mathbb{F}}, Y \rangle^{\mathbb{F}} = K \cdot \langle \bar{X}^{\mathbb{F}}, \bar{X}^{\mathbb{F}} \rangle^{\mathbb{F}}$ and (4.8) follows by (4.7). By (4.8) we in particular get that $[X, H]^{p, \mathbb{G}}$ is continuous, meaning that $[X, H]$ is quasi-left continuous. The proof of (i) is complete. We now verify (ii). Clearly, X^τ is a point process and $(X^\tau)^{p, \mathbb{G}} = (1_{[0, \tau]} \cdot X)^{p, \mathbb{G}} = (X^{p, \mathbb{G}})^\tau$ holds. By [16, Proposition 2.1], we know that there exists $Y \in \mathcal{H}_{\text{loc}}^2(\mathbb{F})$ such that

$$\hat{X}_t = \bar{X}_{t \wedge \tau}^{\mathbb{F}} - \int_0^{t \wedge \tau} \frac{1}{A_s^-} d\langle \bar{X}^{\mathbb{F}}, Y \rangle_s^{\mathbb{F}}$$

belongs to $\mathcal{H}_{\text{loc}}^1(\mathbb{G})$, where Y is the same local martingale as in the proof of (i). Therefore, by (4.7), we get:

$$\widehat{X}_t = X_{t \wedge \tau} - \int_0^{t \wedge \tau} \left(1 + \frac{K_s}{A_{s-}}\right) dX_s^{p, \mathbb{F}}.$$

The process $B := \int_0^{\cdot \wedge \tau} \left(1 + \frac{K_s}{A_{s-}}\right) dX_s^{p, \mathbb{F}}$ is predictable and of locally-integrable variation. Because of the uniqueness of the compensator for processes of locally integrable variation (see [13, Theorem I.3.18]), we get $B = (X^\tau)^{p, \mathbb{G}}$. Since $X^{p, \mathbb{F}}$ is a continuous process, X being quasi-left continuous in \mathbb{F} by assumption, we deduce that $(X^{p, \mathbb{G}})^\tau$ is continuous as well. The proof of the lemma is complete. \square

In the next theorem we give sufficient conditions for the orthogonality of the local martingales Z^1 , Z^2 and Z^3 .

Theorem 4.6. *Let Assumptions (A1) and (A2) hold.*

(i) *The local martingales Z^1 and Z^2 are orthogonal to Z^3 . If furthermore $H^{p, \mathbb{G}}$ is continuous, then Z^1 and Z^2 are orthogonal as well and hence Z^1 , Z^2 and Z^3 are pairwise orthogonal.*

(ii) *The local martingales $(Z^1)^\tau$, Z^2 and Z^3 are pairwise orthogonal.*

Proof. We first verify (i). We define the processes $Y = X - [X, H]$ and $Z = [X, H]$. Then $\Delta Y \Delta Z = 0$. From Lemma 4.5, we know that $Z^{p, \mathbb{G}}$ is continuous. So $\Delta Y^{p, \mathbb{G}} \Delta Z^{p, \mathbb{G}} = 0$. Applying Lemma A.1 (iv) and (v) we deduce $[Z^1, Z^3] = 0$. Hence, Z^1 and Z^3 are orthogonal. The proof of the orthogonality of Z^2 and Z^3 is completely analogous. If we now assume that $H^{p, \mathbb{G}}$ is continuous, then Lemma 4.5 (i) implies that $(H - [X, H])^{p, \mathbb{G}}$ is continuous as well. Hence, Lemma A.1 (v) with $Y = Z^1$ and $Z = Z^2$ yields the orthogonality of Z^1 and Z^2 , since $X - [X, H]$ and $H - [X, H]$ have no common jumps. We now come to (ii). By (i), it is enough to verify that $(Z^1)^\tau$ and Z^2 are orthogonal. We have $[X, H]^\tau = [X, H]$. Hence, $(X - [X, H])^\tau$ is a point process and $(X^{p, \mathbb{G}})^\tau - [X, H]^{p, \mathbb{G}}$ is its compensator which, because of Lemma 4.5 (ii), is continuous. Since $Y = X^\tau - [X, H]$ and $Z = H - [X, H]$ have no common jumps and $\Delta Y^p \Delta Z^p = 0$, by Lemma A.1 (v), we get that $(Z^1)^\tau$ and Z^2 are orthogonal. The proof of the theorem is complete. \square

We stress that the assumption on the continuity of $H^{p, \mathbb{G}}$ made in Theorem 4.6 is equivalent to the following property: $\mathbb{P}[\tau = \sigma < +\infty] = 0$ for every \mathbb{F} -predictable stopping time σ (see [16, p.65]), meaning that τ avoids \mathbb{F} -predictable stopping times, i.e., τ is \mathbb{G} -totally inaccessible.

Let $Y \in \mathcal{H}^2(\mathbb{G})$ and let X be quasi-left continuous with respect to \mathbb{F} . Combining Corollary 3.8 (ii) and Theorem 4.6 (ii) we get

$$Y^\tau = Y_0 + K^1 \cdot (Z^1)^\tau + K^2 \cdot Z^2 + K^3 \cdot Z^3, \quad K^i \in L^2(Z^i, \mathbb{G}), \quad i = 1, 2, 3, \quad (4.9)$$

and this is an orthogonal representation in $(\mathcal{H}^2(\mathbb{G}), \|\cdot\|_2)$.

Remark 4.7 (The multiplicity of \mathbb{G}). Because of Corollary 3.8 and Theorem 4.6 (ii) we see that, if $H^{p, \mathbb{G}}$ is continuous, then the local martingales Z^1 , Z^2 and Z^3 are orthogonal and the multiplicity of the filtration \mathbb{G} is equal to three. From Proposition 4.4, we see that, if τ avoids \mathbb{F} -stopping times then the multiplicity of \mathbb{G} is equal to two. Finally, from (4.9), we see that the space $\mathcal{H}_\tau^2(\mathbb{G})$ of square integrable martingale stopped at τ has always an integral representation with respect to three orthogonal martingales. So, following [2, Remark 3.3], we can say that the multiplicity of the class $\mathcal{H}_\tau^2(\mathbb{G})$ in \mathbb{G} is, in general, equal to three.

The quasi-left continuity of X . We conclude this subsection with a short discussion about the quasi-left continuity of X with respect to \mathbb{F} , which played a crucial role here.

Let \mathbb{A} be a right-continuous filtration. First, we stress that the quasi-left continuity of an \mathbb{A} -adapted càdlàg process Y is a property of the process but also of the filtration \mathbb{A} . The intuition of this fact is the following: Since in the enlargement of \mathbb{A} there will be more predictable times than in \mathbb{A} , we cannot expect, in general, that the quasi-left continuity of Y with respect to \mathbb{A} is preserved in the enlargement of \mathbb{A} . The following simple example shows that this intuition is indeed correct.

Counterexample 4.8. Let X be a Poisson process with respect to \mathbb{X} and let $(\tau_n)_{n \geq 1}$ be the sequence of jump-times of \mathbb{X} . The process X is quasi-left continuous in \mathbb{X} but it is not quasi-left continuous in the filtration \mathbb{F} obtained enlarging \mathbb{X} initially by the initial σ -field $\mathcal{R} = \sigma(\tau_1)$. Also, X is not quasi-left continuous in the progressively enlarged filtration \mathbb{G} obtained enlarging \mathbb{X} progressively by the random time $\tau = \frac{1}{2}(\tau_1 + \tau_2)$, since τ_2 is predictable in \mathbb{G} and it is *charged* by X .

We have assumed in this subsection 4.2 that the process X is quasi-left continuous with respect to the filtration \mathbb{F} , which is obtained from the filtration \mathbb{X} by an initial enlargement with respect to the initial σ -field \mathcal{R} .

Clearly, if \mathcal{R} is trivial, then the quasi-left continuity of X with respect to \mathbb{F} is the same as the quasi-left continuity with respect to \mathbb{X} . However, in general, it is more interesting to start assuming the quasi-left continuity of X with respect to \mathbb{X} and then to initially enlarge \mathbb{X} by \mathcal{R} in such a way to preserve the quasi-left continuity of X in \mathbb{F} . As an example in which this is possible, we consider the case of a quasi-left continuous point process X with respect to the filtration \mathbb{X} and then we enlarge \mathbb{X} initially with a nonnegative random variable L (i.e., $\mathcal{R} = \sigma(L)$) which satisfies *Jacod's absolute continuity hypothesis* (see, e.g., [1, Section 4.4]). Indeed, in this special but important case, from [1, Theorem 4.25] and the PRP of $\bar{X}^{\mathbb{X}}$ with respect to \mathbb{X} , we deduce as in Lemma 4.5 that $X^{p,\mathbb{F}}$ is continuous or, equivalently, that X is quasi-left continuous with respect to \mathbb{F} .

A A lemma on the orthogonality of compensated point processes

In this appendix we give a lemma which is helpful to investigate whether two compensated point processes are orthogonal local martingales.

Let \mathbb{F} be an *arbitrary* right-continuous filtration. We recall that two processes A and B in $\mathcal{A}_{\text{loc}}^+(\mathbb{F})$ are called *associated* if they have the same compensator, that is $A^p = B^p$. Notice that A and B are associated if and only if $A - B \in \mathcal{H}_{\text{loc}}^1$ (see [7, Theorem V.38]).

Lemma A.1. *Let Y and Z be point processes with respect to \mathbb{F} and let Y^p and Z^p denote the \mathbb{F} -dual predictable projection of Y and Z , respectively. Let us denote $\bar{Y} := Y - Y^p$ and $\bar{Z} := Z - Z^p$.*

- (i) *The processes $[Y^p, Z]$, $[Y, Z^p]$ and $[Y^p, Z^p]$ are locally bounded and increasing, hence they belong to $\mathcal{A}_{\text{loc}}^+$.*
- (ii) *The processes $[Y^p, Z]$ and $[Y, Z^p]$ are associated and their compensator is $[Y^p, Z^p]$.*
- (iii) *The process $[\bar{Y}, \bar{Z}]$ belongs to $\mathcal{H}_{\text{loc}}^1$ if and only if $[Y, Z]^p = [Y^p, Z^p]$.*
- (iv) *Let $\Delta Y \Delta Z = 0$. Then $[\bar{Y}, \bar{Z}] \in \mathcal{H}_{\text{loc}}^1$ if and only if $[\bar{Y}, \bar{Z}] = 0$.*
- (v) *Let $\Delta Y \Delta Z = 0$. Then $[\bar{Y}, \bar{Z}] \in \mathcal{H}_{\text{loc}}^1$ if and only if $\Delta Y^p \Delta Z^p = 0$.*

Proof. We first verify (i). We only show that $[Y, Z^p]$ is a locally bounded increasing process, the proof for $[Y^p, Z]$ and $[Y^p, Z^p]$ being completely analogous. We have $\Delta[Y, Z^p] = \Delta Y \Delta Z^p \geq 0$, because Y and Z^p are both increasing. Since $[Y, Z^p] = \sum_{s \leq \cdot} \Delta Y_s \Delta Z_s^p$, we obtain that $[Y, Z^p]$ is an increasing process.

Furthermore, since Y and Z^p have bounded jumps, $[Y, Z^p]$ has bounded jumps too. Hence, it is a locally bounded process. The proof of (i) is complete.

We now come to (ii). By [13, Proposition I.4.49 a)] we have $[Y^p, Z] = \Delta Y^p \cdot Z$ and $[Y, Z^p] = \Delta Z^p \cdot Y$. Then, since ΔY^p is a predictable process, we have $[Y^p, Z]^p = (\Delta Y^p \cdot Z)^p = \Delta Y^p \cdot Z^p = [Y^p, Z^p]$, where in the last equality we again used [13, Proposition I.4.49 a)]. Analogously, we get $[Y, Z^p]^p = [Y^p, Z^p]$ and the proof of (ii) is complete.

We now show (iii). First, we compute

$$[\bar{Y}, \bar{Z}] = [Y, Z] - [Y^p, Z] - [Y, Z^p] + [Y^p, Z^p]. \quad (\text{A.1})$$

By (A.1) and (ii), since $[Y^p, Z^p] - [Y, Z^p] \in \mathcal{H}_{\text{loc}}^1$, we get $[\bar{Y}, \bar{Z}] \in \mathcal{H}_{\text{loc}}^1$ if and only if $[Y, Z] - [Y^p, Z] \in \mathcal{H}_{\text{loc}}^1$. But this is the case if and only if $[Y, Z]$ and $[Y^p, Z]$ are associated processes thus, by (ii), if and only if $[Y, Z]^p = [Y^p, Z^p]$ holds. This shows (iii).

We now verify (iv). It is enough to show that if $[\bar{Y}, \bar{Z}] \in \mathcal{H}_{\text{loc}}^1$, then $[\bar{Y}, \bar{Z}] = 0$. Therefore, let $[\bar{Y}, \bar{Z}] \in \mathcal{H}_{\text{loc}}^1$ and $\Delta Y \Delta Z = 0$. By (iii), we get $[Y^p, Z^p] = 0$, since $[Y, Z] = 0$ by assumption. Hence, by (ii), we obtain $[Y, Z^p], [Y^p, Z] \in \mathcal{H}_{\text{loc}}^1$ implying that $[Y, Z^p] = [Y^p, Z] = 0$ since these are increasing processes starting at zero. Then (iii) follows immediately from (A.1).

We now verify (v). Let us assume $\Delta Y \Delta Z = 0$. Then $[Y, Z]^p = 0$. So, by (iii), $[\bar{Y}, \bar{Z}] \in \mathcal{H}_{\text{loc}}^1$ if and only if $[Y^p, Z^p] = 0$ or, equivalently, if and only if $\Delta Y^p \Delta Z^p = 0$. This shows (v). The proof of the lemma is now complete. \square

Notice that the relation $[Y, Z]^p = [Y^p, Z^p]$ of Lemma A.1 (iii) does not hold, in general. For a counterexample we consider the case in which Y is a homogeneous Poisson process with parameter $\lambda > 0$ and $Z = Y$. Then $[Y, Y]_t^p = Y_t^p = \lambda t$, while $[Y^p, Y^p] = 0$.

By Lemma A.1 (v), if $\Delta Y \Delta Z = 0$, then $[\bar{Y}, \bar{Z}] \in \mathcal{H}_{\text{loc}}^1$ if and only if $\Delta Y^p \Delta Z^p = 0$. However, the condition $\Delta Y \Delta Z = 0$ is not sufficient to ensure $\Delta Y^p \Delta Z^p = 0$, in general, as the following counterexample shows.

Counterexample A.2.¹ Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be such that \mathcal{F}_t is trivial for $t < 1$ and $\mathcal{F}_t = \mathcal{F}$ for $t \geq 1$. Then \mathbb{F} is obviously a right-continuous filtration. Let $A, B \in \mathcal{F}$ be disjoint. Then the processes Y and Z defined by $Y_t = 1_A 1_{\{t \geq 1\}}$ and $Z_t = 1_B 1_{\{t \geq 1\}}$, respectively, are point processes with respect to \mathbb{F} and they satisfy $\Delta Y \Delta Z = 0$. For the dual predictable projections Y^p and Z^p we have $Y_t^p = \mathbb{P}[A] 1_{\{t \geq 1\}}$ and $Z_t^p = \mathbb{P}[B] 1_{\{t \geq 1\}}$, respectively, and they have a common jump in $t = 1$.

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¹This counterexample has been suggested to us by Yuliya Mishura and Alexander Gushchin.

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