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A true least action principle for damped motion

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Abstract. This work is a formulation of the least action principle for classical mechanical dissipative systems. We consider a whole conservative system composed of a damped moving body and its environment receiving the dissipated energy. This composite system has a conservative Hamiltonian $H = K_1 + V_1 + H_2$ where K_1 is the kinetic energy of the moving body, V_1 its potential energy and H_2 the energy of the environment. The Lagrangian is found to be $L = K_1 - V_1 - E_d$ where E_d is the energy dissipated from the moving body into the environment. The usual variation calculus of least action leads to the correct equation of the damped motion.

1. Introduction

The Least Action Principle (LAP) is one of the most valuable heritages from the classical mechanics[1-3]. The fact that the formulation of the whole classical physics as well as of the quantum theory in its path integral formalism[4]could be based on or related to this single mathematical rule gives to LAP a fundamental priority to all other visibly different principles, empirical laws and differential equations in different branches of physics. This priority of LAP has nourished two ambitions of physicists. The first one is the (rather controversial) effort to deepen the understanding of nature through this principle and to search for the fundamental meaning of its exceptional universality in physics[3,5,6]. The second one is to extend it to more domains such as thermodynamics and statistical mechanics (with the pioneer efforts of Boltzmann, Helmholtz and Hertz[7]), stochastic dynamics (e.g., large deviation theory[8] and stochastic mechanics[9]), and dissipative mechanical systems[10-12]. The present work follows this last effort to formulate LAP for classical mechanical dissipative systems.

LAP was originally formulated only for Hamiltonian system[2], i.e., the sum $H = K + V$ of kinetic energy K and potential energy V of the considered system satisfies the Hamiltonian equations. For Hamiltonian systems, any real trajectory between two given configuration points must satisfy the LAP, a vanishing first variation of the action A created by tiny deformation of the trajectory[2,6]:

$$\delta A = \delta \int_0^{t_b} L dt = \int_0^{t_b} \delta L dt = 0 \tag{1}$$

where the action $A = \int_0^{t_b} L dt$ is a time integral of the Lagrangian $L = K - V$ on the trajectory over a fixed time period t_b. One of the important results of this variational calculus is the

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Euler-Lagrange equation given by[6] (for one freedom x)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{2}$$

where $\dot{x}$ is the velocity. In many cases when H and L do not depend on time explicitly, a Hamiltonian system is energy conservative.

A problem of variation takes place for nonconservative systems whose dissipative force, say, f_d is artificially introduced into the equation of motion in this way $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = f_d$. This is equivalent to write $\int_0^{t_b}(\delta L + f_d \delta x)dt = 0$. However, it is impossible to define an action integral with a single (Lagrangian) function satisfying Eq.(2). Hence LAP is absent for dissipative systems.

There has been a longstanding effort to recover LAP for dissipative system[11]. Since all systems in nature are energy dissipative, this formulation is essential for this beautiful fundamental principle to be better understood, applied to a much wider range of systems and to be related to many other scientific principles relative to the energy dissipation[10,11,13].

As far as we know, the first variational calculus applied to damped motion dates back to Euler's work in 1744 for the brachistochrone (shortest time) problem with friction[13]. More recently, Rayleigh[15] has introduced a 'dissipative function' $D = \frac{1}{2}\zeta\dot{x}^2$, for the special case of the Stokes' law $f_d = -m\zeta\dot{x}$, to write $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial D}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$, where ζ is the drag constant and m the mass of the damped body. Nevertheless, LAP is not recovered since there is no Lagrangian for defining an action which satisfies Eq.(1). Other major propositions include the Bateman approach[16] to introduce complementary variables and equations, the definition of dissipative Lagrangian by multiplying the non dissipative one with an exponential factor $exp(\zeta t)$[17] where t is the time, the fractional derivative formulation[18], and the pseudo-Hamiltonian mechanics[19] where a parameter was introduced to characterize the degree of dissipation. The reader is referred to the reviews in [10,11,14,18,19] about the details of these propositions. In general, the Lagrangian in these solutions is not unique and has no energy connection like $L = K - V$ (see for instance the quasi-Lagrangian $L = e^{\zeta t}(K - V)$ and the corresponding quasi-Hamiltonian $H = e^{-\zeta t}K + e^{\zeta t}V$ for damped harmonic oscillator[17]). Hence no variational or optimal calculus was possible in general form[10,11,14].

A common character of these previous works is to consider the damped body as an isolated body in the calculations. The problem is that a dissipative system is coupled to an environment and loses energy into the latter which becomes an integral part of the motion. As far as this lost energy is not considered, the Lagrangian of the 'isolated' body inevitably loses energy connection and generic optimal characters[10,11] as mentioned above.

The aim of this work is to recover LAP in a general way for dissipative system with an energy connected and unique Lagrangian which is related as usual to the conservative Hamiltonian by Legendre transformation. The three conventional formulations of analytical mechanics, i.e., the Hamiltonian, Lagrangian and the Hamilton-Jacobi equations, are all preserved.

2. The conservative Hamiltonian

Our basic idea is to consider the damped moving body (system 1) and its environment (system 2) as a whole conservative system. The total Hamiltonian includes the instantaneous kinetic and the potential energy of the body, as well as the mechanical energy transformed into heat or other forms of energy (noises, vibration, electromagnetic radiation etc.) into the environment. The body moves along the axis x with velocity $\dot{x}$. Its environment is composed of N bodies with position x_i and velocity $\dot{x}_i$ $(i = 1, 2, ..., N)$. The energy transfer from system 1 to system 2 occurs only through a friction force. The total Hamiltonian reads $H = K_1 + V_1 + K_2 + V_2 + H_{int}$ where $K_1 = \frac{1}{2}m\dot{x}^2$ is the kinetic energy and V_1 the potential energy of the system 1,

$K_2 = \frac{1}{2}\sum_i^N m_i\dot{x}_i(t)^2$ the kinetic energy and $V_2 = \sum_i^N v(x_i)$ the potential energy of system 2 with $v(x_i)$ the potential energy of the particle i, and H_{int} the interaction energy between the system 1 and 2. H_{int} is responsible for the friction law and determined by the coupling mechanism on the interface between the moving body and the environment. We can suppose that, for a limited time period of the motion under consideration, this interface (body's shape and size, body-environment distance, nature of the closest parts of the environment to the interface, etc.) and the friction law do not change significantly, hence the interaction mechanism should not change with the virtual variation of paths. In this case H_{int} can be neglected in the variational calculus and we only have $H = K_1 + K_2 + V_1 + V_2$ or $H = H_1 + H_2$ where $H_1 = K_1 + V_1$ is the total energy of the system 1 and $H_2 = K_2 + V_2$ that of the system 2.

It should be stressed that the impact of the thermal fluctuation in system 2 on system 1 should be neglected. The motion of the system 1 remains classical mechanical and deterministic. This is reasonable for a mechanical body which is much larger than the constituents of system 2 and has much larger energy variation during the motion than the energy fluctuation of the thermal motion in system 2.

3. Variational calculus of LAP for damped motion

The Lagrangian L of the whole conservative system can be defined by using the Legendre transformation $L = p\dot{x} + \sum_i^N p_i(t)\dot{x}_i(t) - H = K_1 - V_1 + 2K_2 - H_2$ where p is the momentum of system 1 and p_i the momentum of the particle i of the system 2. The corresponding action is given by

$$A = \int_0^{t_b} Ldt = \int_0^{t_b} (K_1 - V_1 + 2K_2 - H_2)dt. \tag{3}$$

H_2 can be expressed as a function of $x_i(t)$ and the velocity $\dot{x}_i(t)$ of the constituent parts of the system 2. Its general expression reads

$$H_2 = \sum_i^N \frac{1}{2}m\dot{x}_i^2(t) + V_2[x_1(t), x_2(t)...x_N(t)] \tag{4}$$

With this expression of H_2, the Lagrangian only depends on its variables at the time moment t, i.e.,

$$L = K_1(\dot{x}(t)) - V_1(x(t)) + 2K_2(\dot{x}_i(t)) - H_2(x_i(t), \dot{x}_i(t)) \quad i = 1, 2, ..., N \tag{5}$$

The usual variation calculus with a tiny variation $\delta x(t)$ of the path $x(t)$ damped body yields :

$$\begin{aligned} \delta A &= \int_0^{t_b} [\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x}]dt \\ &= \int_0^{t_b} [\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}]\delta x dt. \end{aligned} \tag{6}$$

where we have made a time integration by parts of $\delta\dot{x}$ with the conditions $\delta x(a) = \delta x(b) = 0$. The LAP requires $\delta A = 0$, which, due to the arbitrary nature of $\delta x(t)$, leads to

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0. \tag{7}$$

This is the differential equation for the damped motion of the system 1.

In order to make calculation with Eq.(7), H_2 must be written as an explicit function of x and $\dot{x}$. For this purpose, we suppose that H_2 changes in time only due to the energy dissipation of the system 1. We can write $H_2(t) = H_2(t_a) + E_d(x(t))$ where $H_2(t_a)$ is the energy of the system

2 at the initial time t_a (a constant for the motion hereafter), and $E_d(x(t))$ the energy received from system 1 through the work of the friction force $f_d(x, \dot{x})$ from the initial moment t_a to a moment t ($0 \leq t \leq t_b$):

$$E_d(x(t)) = -\int_{x_a}^{x(t)} f_d(x(\tau), \dot{x}(\tau))dx(\tau) \tag{8}$$

where τ is any time moment between $t_a = 0$ and t, and $dx(\tau)$ a small displacement along the path at the time τ. The Lagrangian can be written as $L(x(t), \dot{x}(t)) = K_1(\dot{x}(t)) + 2K_2(\dot{x}_i(t)) - V_1(x(t)) - E_d(x(t))$ where the constant $H_2(t_a)$ is dropped. Introducing this Lagrangian into Eq.(7) and considering the fact that K_1 and K_2 are not explicit function of $x(t)$ and that K_2, V_1 and E_d are not explicit function of $\dot{x}(t)$ at time t, one gets

$$\frac{d}{dt}\frac{\partial K_1}{\partial \dot{x}} + \frac{\partial V_1}{\partial x} + \frac{\partial E_d}{\partial x} = 0. \tag{9}$$

Considering $f_d(x(t), \dot{x}(t)) = -\frac{\partial E_d(x(t))}{\partial x(t)} = -\frac{\partial}{\partial x(t)} \int_0^{x(t)} f(x(\tau), \dot{x}(\tau))dx(\tau)$ [20], we straightforwardly obtain

$$m\ddot{x} = -\frac{\partial V_1}{\partial x} + f_d, \tag{10}$$

the Newtonian equation for the damped motion with the friction f_d.

To see the second formulism of classical mechanics, the Hamiltonian equations, we calculate the total time derivative of H reads $\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p}$. Compare this to the same derivative made with the Legendre transformation $H = p\dot{x} - L$, i.e., $\frac{dH}{dt} = \dot{p}\dot{x} + \dot{p}\ddot{x} - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial x}\dot{x} - \frac{\partial L}{\partial \dot{x}}\ddot{x}$ and considering the Euler-Lagrange equation Eq.(7), we get, besides $\frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$, the following Hamiltonian equations [6] :

$$\dot{p} = -\frac{\partial H}{\partial x}, \dot{x} = \frac{\partial H}{\partial p} \tag{11}$$

from which the conservation of total energy H can be calculated [2].

To see the third formalism of classical mechanics, the Hamilton-Jacobi equation, we relax t_b in the time integral of action or consider the integral as indefinite, and compute $L = \frac{dA}{de} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x}\dot{x}$. Thanks to Eq.(7) and the Legendre transformation, we can get $p = \frac{\partial A}{\partial t}$ and the Hamilton-Jacobi equation :

$$\frac{\partial A}{\partial t} + H = 0 \tag{12}$$

for the whole system.

4. Concluding remarks

In summary, we have proposed a formulation of the least action principle for damped motion with an universal, energy connected and unique Lagrangian, thanks to the model of a conservative system composed of the damped moving body and of its environment. We hope that these results are helpful for further study of the relations between the fundamental principles of Lagrangian/Hamiltonian mechanics and the variational principles relative to energy dissipation [23-27]. It is shown that the three conventional formulations of analytical mechanics, i.e., the Hamiltonian, Lagrangian and the Hamilton-Jacobi equations, are all preserved.

This work is presented in term of friction as the coupling between the moving body and its environment. But the formulation is not limited to friction. This is an advantage of the

substitution of H_2 by E_d which is written as a function of the damped motion without involving the details of the motions of the tiny particles in the environment. Hence the coupling can be any mechanism depending on the motion and dissipating energy from it. For instance, the emission of electromagnetic wave or light into the void (environment) from an accelerated charged body, or emission of sound by a vibrating body.

Finally we would like to mention that, although the term "least action" is used here for historical reason, the stationarity $\delta A = 0$ is not necessarily a minimum. The nature of the stationarity (minimum, maximum or inflection) of A has been addressed in our recent work [22] by numerical simulation of damped motion and of variational analysis, in which the action A_{op} along the optimal path is calculated and compared to the actions of many paths around the optimal one. One of the conclusions is that the stationarity of A_{op} undergoes a transition from minimum to maximum when drag constant ζ and the dissipated energy increase. Further investigation is necessary to confirm this evolution of action stationarity.

References

[1] Maupertuis P L M, 1750 *Essai de cosmologie*(Berlin) ; 1744 *Accord de différentes lois de la nature qui avaient jusqu'ici paru incompatibles Mém. As. Sc. Paris* p. 417;1746 *Les lois de mouvement et du repos, déduites d'un principe de métaphysique Mém. Ac. Berlin* p. 267
[2] Arnold V I 1989 *Mathematical methods of classical mechanics*(New York: Springer-Verlag)
[3] Stoltzner M 2003 *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*,**34** 285
[4] Feynman R P and Hibbs A R 1965 *Quantum mechanics and path integrals*(New York: McGraw-Hill Publishing Company)
[5] Mach E 1987 *La mécanique, exposé historique et critique de son développement*(Paris:Edition Jacques Gabay)
[6] Lanczos C 1986 *The variational principles of mechanics*(New York: Dover Publication)
[7] I. Stengers 2003 ?*Cosmopolitiques I, Ed. La d—couverte*(Paris)
[8] Freidlin M I and Wentzell A D 1984 *Random perturbation of dynamical systems*(New York: Springer-Verlag)
[9] Yasue K 1981 *J. Math. Phys.*, **22** 110
[10] Sieniutycz S and Farkas H 2005 *Variational and extremum principles in macroscopic systems*(Elsevier)
[11] Vujanovic B D and Jones S E 1989 *Variational methods in nonconservative phenomena*(New York: Academic Press Inc.)
[12] Herrera L, Nunez L, Patino A and Rago H 1986 *A variational principle and the classical and quantum mechanics of the damped harmonic oscillator*, **54** 273
[13] Goldstine H H 1980 *A history of the calculus of variations from the 17th through the 19th century*(New York: Springer-Verlag)
[14] Gray C G 2009 *Scolarpedia* **4** 8291
[15] Goldstein H *Classical Mechanic*(Addison-Wesley: Mass)
[16] Bateman H 1931 *Phy. Rev.* **38** 815
[17] Sanjuan 1995 *Journal of Sound and Vibration* **185** 734
[18] Riewe F 1997 *Phys. Rev. E* **55** 3581.
[19] Duffin R J1962 *Arch. Rat. Mech. Anal.* **9** 309
[20] Weisstein E W *Second Fundamental Theorem of Calculus, From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.comSecondFundamentalTheoremofCalculus.htm*l
[21] Alembert J D 1990 *Traité de dynamique* (Sceaux: Editions Jacques Cabay)
[22] Hyon Y, Kwak D and Liu C 2009 *Energetic variational approach in complex fluids: Maximum dissipation principle, Discrete And Continuous Dynamical Systems* **26** 1291-1304
[23] Onsager L 1931 *Phys. Rev.* **37** 405
[24] Eylink G L 2008 *Physica D* **14**
[25] Prados A, Lasanta A and Hurtado P I 2011 *Phys. Rev. lett.* **107** 140601
[26] Koide T and Kodama T 2012 *J. Phys. A: Math. Theor.***45** 255204