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# Backward stochastic Volterra integral equations with jumps in a general filtration.

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## Abstract

In this paper, we study backward stochastic Volterra integral equations introduced in [36] and extend the existence, uniqueness or comparison results for general filtration (not only Brownian-Poisson setting) and  $L^p$ -data with  $p < 2$ . Moreover the time regularity of the solution is explored, which is also new in this jump setting.

**2010 Mathematics Subject Classification.** 45D99, 60H20, 60H99.

**Keywords.** Backward Volterra integral equation, general filtration,  $L^p$ -solution, jumps, time regularity.

## Introduction

In this paper, we deal with a backward stochastic Volterra integral equations (BSVIE in short) of the following type<sup>1</sup>:

$$\begin{aligned} Y(t) = & \Phi(t) + \int_t^T f(t, s, Y(s), Z(t, s), Z(s, t), U(t, s), U(s, t)) ds - \int_t^T Z(t, s) dW_s \\ & - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM(t, s). \end{aligned} \quad (1)$$

The unknown processes, called an adapted solution, are the quadruplet  $(Y, Z, U, M)$  valued in  $\mathbb{R}^{d+(d \times k)+d+d}$  such that  $Y(\cdot)$  is  $\mathbb{F}$ -adapted and  $(Z(t, \cdot), U(t, \cdot), M(t, \cdot))$  are  $\mathbb{F}$ -progressively measurable for almost all  $t \in [0, T]$ .  $f$  is called the *generator* or the driver of the BSVIE and  $\Phi$  is the *free term* (or sometimes the terminal condition).

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<sup>1</sup>In the whole paper,  $dM(t, s)$  is the integration w.r.t. the second time parameter  $s$  where  $t$  is fixed. In particular  $\int_u^v dM(t, s) = M(t, v) - M(t, u)$  is the increment of  $M(t, \cdot)$  between the time  $u$  and  $v$ .

A particular case of the preceding BSVIE is:

$$\begin{aligned}
Y(t) = & \xi + \int_t^T f(s, Y(s), Z(s), U(s)) ds - \int_t^T Z(s) dW_s \\
& - \int_t^T \int_{\mathcal{E}} U(s, e) \tilde{\pi}(de, ds) - (M(T) - M(t)).
\end{aligned} \tag{2}$$

This equation is a backward stochastic differential equation (BSDE for short) and since the seminal paper [24], it has been intensively studied (see among many others [8, 10, 26, 32]). Since we are interesting here by  $L^p$ -solution for BSVIE, let us mention also [6, 16, 18] which deal with  $L^p$ -solution for BSDE in a general filtration.

The goal of this paper is to generalize or to adapt some results concerning BSVIEs. To our best knowledge, [20, 36, 37] were the first papers dealing with BSVIEs and the authors considered the following class of BSVIEs:

$$Y(t) = \Phi(t) + \int_t^T f(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s) dW_s. \tag{3}$$

They proved existence and uniqueness of the solution  $(Y, Z)$  (M-solution in [37]) under the natural Lipschitz continuity regularity of  $f$  and square integrability condition on the data. The extension to  $L^p$ -solution ( $1 < p < 2$ ) has been done in [33]. In these four papers, the filtration is generated by the Brownian motion  $W$ . In [35], the authors introduced the jump component  $\tilde{\pi}$ . In the filtration generated by  $W$  and  $\tilde{\pi}$ , they consider:

$$\begin{aligned}
Y(t) = & \Phi(t) + \int_t^T f(t, s, Y(s), Z(t, s), U(t, s)) ds \\
& - \int_t^T Z(t, s) dW_s - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds)
\end{aligned}$$

and prove existence and uniqueness of the solution in the  $L^2$  setting. The result has been extended in [23, 30] (see also [21] for the Lévy case).

Another real issue for BSVIE concerns the comparison principle. In the BSDE theory, comparison principle holds under quite general conditions (see e.g. [16, 18, 26]). Roughly speaking, the comparison result is proved by a linearization procedure and by an explicit form for the solution of a linear BSDE. For BSVIE, these arguments fail and comparison is a difficult problem. The paper [34] is the most relevant paper on this topic. It provides comparison results and gives several counter-examples where comparison principle fails.

Our initial motivation comes from the existence of a solution for a singular BSDE (see [17, 27]). A key ingredient to overcome the singularity is an a priori estimate. If the generator is bounded from above by a polynomial function of  $y$ , this a priori estimate is obtained using classical results concerning BSDEs (or some particular control as in [1]). In our forthcoming paper we want to remove this polynomial growth assumption on  $f$ . In general the a priori estimate on the solution takes the following form:

$$\forall t \in [0, T), \quad \mathcal{Y}(t) = \frac{1}{T-t} \mathbb{E} \left[ \int_t^T \Gamma \left( \frac{T-t}{\eta_s} \right) ds \middle| \mathcal{F}_t \right].$$

Formally  $\mathcal{Y}$  is the first component of the BSVIE: for  $0 \leq t \leq \tau < T$

$$\begin{aligned} \mathcal{Y}(t) &= \Phi(t) + \int_t^\tau \left( -\frac{1}{T-s} \mathcal{Y}(s) + \gamma(t, s) + g(s, Z(t, s), U(t, s)) \right) ds \\ &\quad - \int_t^\tau \widehat{Z}(t, s) dW_s - \int_t^\tau \int_{\mathcal{E}} \widehat{U}(t, s, e) \widetilde{\pi}(de, ds) - \int_t^\tau d\widehat{M}(t, s). \end{aligned}$$

To get the desired a priori estimate, we need some new existence and comparison results on BSVIEs. This will be a new application for BSVIEs (see the previous papers for other applications of BSVIEs).

### Main contributions

Let us outline the main contributions of our paper compared to the existing literature. First of all our paper generalizes the preceding results (of course only some of them) since we allow for a more general filtration. This is the reason of the presence of the additional martingale term  $M$  in (1).

Moreover we provide existence and uniqueness of M-solutions in  $L^2$ -space of (1). We also extend this result to  $L^p$ -spaces,  $p > 1$  for BSVIE of the form

$$\begin{aligned} Y(t) &= \Phi(t) + \int_t^T f(t, s, Y(s), Z(t, s), U(t, s)) ds - \int_t^T Z(t, s) dW_s \\ &\quad - \int_t^T \int_{\mathcal{E}} U(t, s, e) \widetilde{\pi}(de, ds) - \int_t^T dM(t, s). \end{aligned} \quad (4)$$

The reason why we do not consider the full generality here, is due the lack of symmetry for  $(t, s) \mapsto Z(t, s)$  when  $p \neq 2$ . Indeed since  $Z$  is integrated w.r.t. the Brownian motion  $W$ , the natural norm on  $Z$  is

$$\mathbb{E} \int_0^T \left[ \left( \int_0^T |Z(t, s)|^2 ds \right)^{\frac{p}{2}} \right] dt < +\infty,$$

which is symmetric w.r.t.  $(t, s)$  only for  $p = 2$ . Let us also mention that in the case where the generator depends on the stochastic integrand w.r.t. a Poisson random measure, the case when  $p < 2$  has to be handled carefully. Indeed in this case Burkholder-Davis-Gundy inequality with  $p/2 < 1$  does not apply and the  $L^{p/2}$ -norm of the predictable projection cannot be controlled by the  $L^{p/2}$ -norm of the quadratic variation (see [19] and the discussion in [18]). To the best of our knowledge, there is no existence and uniqueness result for BSVIEs with  $L^p$  coefficients in a general filtration.

Another contribution is the study of the regularity of the map  $t \mapsto Y(t)$ . For the solution of the BSDE (2), from the càdlàg regularity of all martingales,  $Y$  inherits the same time regularity. For BSVIE, we only require that the paths of  $Y$  are in  $\mathbb{L}^2(0, T)$ . Essentially because we assume that  $\Phi$  and  $t \mapsto f(t, \dots)$  are also only in  $\mathbb{L}^2(0, T)$ . In [37], it is proved that under weak regularity conditions on the data, then the solution of (3)  $t \mapsto Y(t)$  is continuous from  $[0, T]$  to  $\mathbb{L}^2(\Omega)$ . Let us stress that the Malliavin calculus is used to control the  $Z(s, t)$  term in the generator. Similarly we show that the paths  $t \in [0, T] \mapsto Y(t) \in \mathbb{L}^p(\Omega)$  of the

solution of the BSVIE (4) are càdlàg if roughly speaking  $\Phi$  and  $t \mapsto f(t, \dots)$  satisfy the same property.

However this first property does not give a.s. continuity of the paths of  $Y$  in general. Getting an almost sure continuity is a more challenging issue and is proved in [35] for the BSVIE (3) when  $f$  does not depend on  $Z(s, t)$ , assuming a Hölder continuity property of  $t \mapsto f(t, \dots)$  for a constant  $\Phi(t) = \xi$ . To understand the difficulty, let us recall that if  $f$  does not depend on  $y$ , the solution  $Y$  of the BSVIE (4) is obtained by the formula:  $Y(t) = \lambda(t, t)$  where  $\lambda(t, \cdot)$  is the solution of the related BSDE parametrized by  $t$ . In the Brownian setting, a.s.  $s \mapsto \lambda(t, s)$  is continuous. Using the Kolmogorov continuity criterion, the authors show that  $(t, s) \mapsto \lambda(t, s)$  is bi-continuous, which leads to a continuous version of  $Y$ . Our goal is to extend this point and to obtain that a.s. the paths of  $Y$  are càdlàg if we know the regularity of  $\Phi$  and  $f$  w.r.t.  $t$ . To our best knowledge, there is not equivalent result to the Kolmogorov criterion for càdlàg paths. Hence we assume that the free term  $\Phi$  and the generator  $f$  are Hölder continuous. Thus we sketch the arguments of [35] to obtain the desired result. To summarize, we are only able to prove that  $Y$  is a.s. càdlàg if the data are Hölder continuous, meaning that the jumps only come from the martingale parts in the BSVIE. Relaxing the regularity of the data is still an open question.

Finally we study two classical properties concerning BSVIE. The duality principle holds in our setting provided we know that the solution  $X$  of the forward SVIE is itself càdlàg (see [28]). Note the importance of the time regularity here. In the BSDE theory, comparison principle holds under quite general conditions (see e.g. [8, 16, 18, 26]). Roughly speaking, the comparison result is proved by a linearization procedure and by an explicit form for the solution of a linear BSDE. For BSVIE, these arguments fail and comparison is a difficult issue. The paper [34] is the most relevant paper on this topic. It provides comparison results and gives several counter-examples where comparison principle fails. The comparison results of [34] can be extended to our setting. Of course all their counter-examples are still valid in our case; thereby we do not have intrinsically better results. Somehow we show that the additional martingale terms do not destroy the comparison result.

We point out that all results proved in this paper are used in the study of singular BSDEs with general generator.

## Decomposition of the paper

The paper is decomposed as follows. In the first section, we give the mathematical setting. In the second part, we prove existence and uniqueness of the M-solution of the BSVIE (1). The proof follows exactly the scheme of [37, Section 3] with the required modifications due to  $U$  and  $M$ . Even if we present all results in a self-contained way, some details are skipped here. The third section is devoted to the time regularity of the solution. In the last section, we show that the duality principle and some comparison results of [34] can be quite easily extended to our setting. We also provide a half-explicit formula for linear w.r.t.  $Z$  and  $U$  BSVIEs.

# 1 Setting and notations

Along this paper, we consider a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$ . The filtration  $\mathbb{F}$  is supposed to be complete and right continuous. Without loss of generality we suppose that all semimartingales are càdlàg<sup>2</sup>, that is they have a.s. right continuous paths with left limits. This probability space supports a  $k$ -dimensional Brownian motion  $W$  and a compensated Poisson random measure  $\tilde{\pi}$ . More precisely there is a Poisson random measure  $\pi$  with intensity  $\mu(de)dt$  on the space  $\mathcal{E} \subset \mathbb{R}^m \setminus \{0\}$ . The measure  $\mu$  is  $\sigma$ -finite on  $\mathcal{E}$  such that

$$\int_{\mathcal{E}} (1 \wedge |e|^2) \mu(de) < +\infty.$$

The compensated Poisson random measure  $\tilde{\pi}(de, dt) = \pi(de, dt) - \mu(de)dt$  is a martingale w.r.t. the filtration  $\mathbb{F}$ .

**Remark 1** *All results obtained in this paper can be generalized to the case where the compensator of  $\pi$  is random and equivalent to the measure  $\mu \otimes dt$  with a bounded density for example (see the introduction of [2] for example). Nevertheless since we refer to [16, 18] for the existence and uniqueness of the solution of BSDE, we keep this setting for  $\pi$ .*

On  $\mathbb{R}^d$ ,  $|\cdot|$  denotes the Euclidean norm and  $\mathbb{R}^{d \times k}$  is identified with the space of real matrices with  $d$  rows and  $k$  columns. If  $z \in \mathbb{R}^{d \times k}$ , we have  $|z|^2 = \text{Trace}(zz^*)$ . For any metric space  $G$ ,  $\mathcal{B}(G)$  is the Borel  $\sigma$ -field.

Let us first recall some standard notations.

- If  $M$  is a  $\mathbb{R}^d$ -valued  $\mathbb{F}$ -martingale in  $\mathcal{M}$ , the bracket process  $[M]_t$  is

$$[M]_t = \sum_{i=1}^d [M^i]_t,$$

where  $M^i$  is the  $i$ -th component of the vector  $M$ . For  $s \leq t$ ,  $[M]_{s,t}$  denotes the difference  $[M]_t - [M]_s$ .

- $\mathcal{P}$  is the predictable  $\sigma$ -field on  $\Omega \times [0, T]$  and  $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathcal{E})$ . On  $\tilde{\Omega} = \Omega \times [0, T] \times \mathcal{E}$ , a function that is  $\tilde{\mathcal{P}}$ -measurable, is called predictable.
- $G_{loc}(\mu)$  is the set of predictable functions  $\psi$  on  $\tilde{\Omega}$  such that for any  $t \geq 0$  a.s.

$$\int_0^t \int_{\mathcal{E}} (|\psi_s(e)|^2 \wedge |\psi_s(e)|) \mu(de) < +\infty.$$

- $\mathcal{M}_{loc}$ : the set of càdlàg local martingales orthogonal to  $W$  and  $\tilde{\pi}$ . If  $M \in \mathcal{M}_{loc}$  then

$$[M, W^i]_t = 0, 1 \leq i \leq k \quad [M, \tilde{\pi}(A, \cdot)]_t = 0$$

for all  $A \in \mathcal{B}(\mathcal{E})$ . In other words,  $\mathbb{E}(\Delta M * \pi | \tilde{\mathcal{P}}) = 0$ , where the product  $*$  denotes the integral process (see II.1.5 in [12]).

$\mathcal{M}$  is the subspace of  $\mathcal{M}_{loc}$  of martingales.

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<sup>2</sup>French acronym for right continuous with left limit

- $\mathbb{D}^p(0, T)$  is the space of all  $\mathbb{F}$ -adapted càdlàg processes  $X$  such that  $\mathbb{E} \left( \sup_{t \in [0, T]} |X_t|^p \right) < +\infty$ .
- $\mathbb{H}^p(0, T)$  is the subspace of all predictable processes  $X$  such that  $\mathbb{E} \left[ \left( \int_0^T |X_t|^2 dt \right)^{p/2} \right] < +\infty$ .
- $\mathbb{M}^p(0, T)$  is the space of all martingales such that  $\mathbb{E} \left[ ([M]_T)^{p/2} \right] < +\infty$ . The space  $\mathbb{M}^p(0, T) \cap \mathcal{M}$  is denoted by  $\mathbb{M}^{p, \perp}(0, T)$ .
- $\mathbb{L}_\pi^p(0, T) = \mathbb{L}_\pi^p(\Omega \times [0, T] \times \mathcal{E})$  is the set of processes  $\psi \in G_{loc}(\mu)$  such that

$$\mathbb{E} \left[ \left( \int_0^T \int_{\mathcal{E}} |\psi_s(e)|^2 \pi(de, ds) \right)^{p/2} \right] < +\infty.$$

- $\mathbb{L}_\mu^p = \mathbb{L}^p(\mathcal{E}, \mu; \mathbb{R}^d)$  is the set of measurable functions  $\psi : \mathcal{E} \rightarrow \mathbb{R}^d$  such that  $\|\psi\|_{\mathbb{L}_\mu^p}^p = \int_{\mathcal{E}} |\psi(e)|^p \mu(de) < +\infty$ .
- $\mathbb{S}^p(0, T) = \mathbb{D}^p(0, T) \times \mathbb{H}^p(0, T) \times \mathbb{L}_\pi^p(0, T) \times \mathbb{M}^{p, \perp}(0, T)$ .

The next result is proved in [18].

**Proposition 1** *Assume that*

- *For any  $(y, z, \psi)$ ,  $f(\cdot, y, z, \psi)$  is progressively measurable.*
- *Uniform Lipschitz regularity: there exists a constant  $K$  such that for any  $\omega, t$  and  $y, y'$  in  $\mathbb{R}^d$ , for any  $z, z'$  in  $\mathbb{R}^{d \times k}$  and  $\psi, \psi'$  in  $\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2$  (this space is defined below)*

$$|f(\omega, t, y, z, \psi) - f(\omega, t, y', z', \psi')| \leq K \left( |y - y'| + |z - z'| + \|\psi - \psi'\|_{\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2} \right).$$

If

$$\mathbb{E} \left( |\xi|^p + \int_0^T |f(t, 0, 0, 0)|^p dt \right) < +\infty,$$

there exists a unique solution  $(Y, Z, U, M)$  in  $\mathbb{S}^p(0, T)$  to the BSDE (2). Moreover for some constant  $C = C_{p, K, T}$

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^p + \left( \int_0^T |Z_t|^2 dt \right)^{p/2} + \left( \int_0^T \int_{\mathcal{E}} |U_s(e)|^2 \pi(de, ds) \right)^{p/2} + ([M]_T)^{p/2} \right] \\ & \leq C \mathbb{E} \left[ |\xi|^p + \left( \int_0^T |f(r, 0, 0, 0)|^p dr \right)^p \right]. \end{aligned}$$

Concerning BSVIE, we take the same notations as in [37] and thus we skip some details (see [37, Section 2.1] for interesting readers). For  $0 \leq R \leq S \leq T$  we denote

$$\begin{aligned} \Delta[R, S] &= \{(t, s) \in [R, S]^2, R \leq s \leq t \leq S\}, \\ \Delta^c[R, S] &= \{(t, s) \in [R, S]^2, R \leq t < s \leq S\} = [R, S]^2 \setminus \Delta[R, S]. \end{aligned}$$

In  $R = 0$  and  $S = T$ , we simply write:  $\Delta = \Delta[0, T]$  and  $\Delta^c = \Delta^c[0, T]$ . For any  $p, q$  in  $[0, +\infty)$ ,  $H = \mathbb{R}^d$  or  $\mathbb{R}^{d \times k}$ , and  $S \in [0, T]$ ,

$$\begin{aligned} \mathbb{L}_{\mathcal{F}_S}^p(\Omega) &= \{\xi : \Omega \rightarrow H, \xi \text{ is } \mathcal{F}_S\text{-measurable, } \mathbb{E}(|\xi|^p) < +\infty\}, \\ \mathbb{L}_{\mathcal{F}_S}^p(\Omega; \mathbb{L}^q(0, T)) &= \left\{ \phi : (0, T) \times \Omega \rightarrow H, \mathcal{B}([0, T]) \otimes \mathcal{F}_S\text{-measurable with} \right. \\ &\quad \left. \mathbb{E} \left( \int_0^T |\phi(t)|^q dt \right)^{\frac{p}{q}} < +\infty \right\}, \\ \mathbb{L}_{\mathcal{F}_S}^q(0, T; \mathbb{L}_{\mathcal{F}_S}^p(\Omega)) &= \left\{ \phi : (0, T) \times \Omega \rightarrow H, \mathcal{B}([0, T]) \otimes \mathcal{F}_S\text{-measurable with} \right. \\ &\quad \left. \int_0^T (\mathbb{E}|\phi(t)|^p dt)^{\frac{q}{p}} < +\infty \right\}, \end{aligned}$$

We identify

$$\mathbb{L}_{\mathcal{F}_S}^p(\Omega; \mathbb{L}^p(0, T)) = \mathbb{L}_{\mathcal{F}_S}^p(0, T; \mathbb{L}_{\mathcal{F}_S}^p(\Omega)) = \mathbb{L}_{\mathcal{F}_S}^p(0, T).$$

The cases where  $p$  or  $q$  are equal to  $\infty$  can be defined in a similar way. When adaptiveness is required, we replace the subscript  $\mathcal{F}_S$  by  $\mathbb{F}$ . The above spaces are for the free term  $\Phi(\cdot)$  (for which the  $\mathbb{F}$ -adaptiveness is not required) and for  $Y(\cdot)$  (for which  $\mathbb{F}$ -adaptiveness is required). Sometimes we also use the subscript  $\mathcal{P}$  if we require predictability.

To control the martingale terms in the BSVIE, we need other spaces. We define for any  $p, q \geq 1$

$$\mathbb{L}^q(S, T; \mathbb{M}^p(S, T))$$

the set of processes  $M(\cdot, \cdot)$  such that for almost all  $t \in [0, T]$ ,  $M(t, \cdot)$  belongs to  $\mathbb{M}^p(S, T)$  and

$$\int_S^T \left[ \mathbb{E}([M(t, \cdot)]_{S, T})^{\frac{p}{2}} \right]^{\frac{q}{p}} dt < +\infty.$$

In the particular case where  $M(t, \cdot) = \int_S^{\cdot} Z(t, s) dW_s$ , then  $M \in \mathbb{L}^q(S, T; \mathbb{M}^p(S, T))$  is equivalent to

$$Z \in \mathbb{L}^q(S, T; \mathbb{L}_p^p(\Omega; \mathbb{L}^2(S, T))) = \mathbb{L}^q(S, T; \mathbb{H}^p(S, T)),$$

that is  $Z$  belongs to the set of all processes  $Z : [S, T]^2 \times \Omega \rightarrow \mathbb{R}^k$  such that for almost all  $t \in [S, T]$ ,  $Z(t, \cdot) \in \mathbb{H}^p(S, T) = \mathbb{L}_p^p(\Omega; \mathbb{L}^2(S, T))$  satisfying

$$\int_S^T \left[ \mathbb{E} \left( \int_S^T |Z(t, s)|^2 ds \right)^{\frac{p}{2}} \right]^{\frac{q}{p}} dt < +\infty.$$

Let us also consider the case:

$$N(t, s) = \int_S^s \int_{\mathcal{E}} \psi(t, u, e) \tilde{\pi}(de, du), \quad t \geq S, \quad s \geq S,$$

where  $\psi(t, \cdot, \cdot) \in G_{loc}(\mu)$ . It follows that the compensator is given by

$$[N(t, \cdot)]_{S, s} = \int_S^s \int_{\mathcal{E}} |\psi(t, u, e)|^2 \pi(de, du).$$



Thus  $N \in \mathbb{L}^q(S, T; \mathbb{M}^p(S, T))$  if and only if

$$\psi \in \mathbb{L}^q(S, T; \mathbb{L}_\pi^p(S, T)),$$

namely  $\psi$  is in the set of all processes  $\psi : [S, T]^2 \times \mathcal{E} \rightarrow H$  such that for almost all  $t \in [S, T]$ ,  $\psi(t, \cdot, \cdot) \in \mathbb{L}_\pi^p(S, T)$  verifying

$$\int_S^T \left[ \mathbb{E} \left( \int_S^T \int_{\mathcal{E}} |\psi(t, s, e)|^2 \pi(de, ds) \right)^{\frac{p}{2}} \right]^{\frac{q}{p}} dt < +\infty.$$

Sometimes to lighten the notations, we define the martingale

$$M^\sharp(t, u) = \int_0^u Z(t, s) dW_s + \int_0^u \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) + M(t, u)$$

on  $[S, T] \times [S, T]$ , such that the BSVIE (1) becomes

$$Y(t) = \Phi(t) + \int_t^T f(t, s, Y(s), Z(t, s), Z(s, t), U(t, s), U(s, t)) ds - \left( M^\sharp(t, T) - M^\sharp(t, t) \right).$$

Note that due to the orthogonality of the components of  $M^\sharp$ , for any  $p > 1$ , there exist two universal constants  $c_p$  and  $C_p$  such that

$$\begin{aligned} & c_p \mathbb{E} \left[ \left( \int_S^T |Z(t, s)|^2 ds \right)^{\frac{p}{2}} + \left( \int_S^T \int_{\mathcal{E}} |U(t, s, e)|^2 \pi(de, ds) \right)^{\frac{p}{2}} + \left( [M(t, \cdot)]_{S, T} \right)^{\frac{p}{2}} \right] \\ & \leq \mathbb{E} \left[ \left( [M^\sharp(t, \cdot)]_{S, T} \right)^{\frac{p}{2}} \right] \\ & \leq C_p \mathbb{E} \left[ \left( \int_S^T |Z(t, s)|^2 ds \right)^{\frac{p}{2}} + \left( \int_S^T \int_{\mathcal{E}} |U(t, s, e)|^2 \pi(de, ds) \right)^{\frac{p}{2}} + \left( [M(t, \cdot)]_{S, T} \right)^{\frac{p}{2}} \right]. \end{aligned}$$

And  $M^\sharp$  belongs to  $\mathbb{L}^q(S, T; \mathbb{M}^p(S, T))$  if and only if the triplet  $(Z, U, M)$  is in  $\mathbb{L}^q(0, T; \mathbb{H}^p(0, T)) \times \mathbb{L}^q(0, T; \mathbb{L}_\pi^p(0, T)) \times \mathbb{L}^q(0, T; \mathbb{M}^{p, \perp}(0, T))$ .

Finally we define

$$\begin{aligned} \mathfrak{M}^p(0, T) &= \mathbb{L}^p(0, T; \mathbb{H}^p(0, T)) \times \mathbb{L}^p(0, T; \mathbb{L}_\pi^p(0, T)) \times \mathbb{L}^p(0, T; \mathbb{M}^{p, \perp}(0, T)) \\ \mathcal{H}^p(0, T) &= \mathbb{L}_{\mathbb{R}}^p(0, T) \times \mathbb{L}^p(0, T; \mathbb{H}^p(0, T)) \times \mathbb{L}^p(0, T; \mathbb{L}_\pi^p(0, T)) \times \mathbb{L}^p(0, T; \mathbb{M}^{p, \perp}(0, T)) \\ &= \mathbb{L}_{\mathbb{R}}^p(0, T) \times \mathfrak{M}^p(S, T). \end{aligned}$$

with the naturally induced norm.

The notion of solution of a BSVIE is the following.

**Definition 1 (Adapted solution)** *A quadruple  $(Y, Z, U, M) \in \mathcal{H}^p(0, T)$  is called an adapted solution of the BSVIE (1) if the equation is satisfied a.s. for almost all  $t \in [0, T]$ .*

As it pointed out in [37], uniqueness of an adapted solution fails. Roughly speaking, there is a additional freedom on  $\Delta$ . To avoid this problem, the next definition is formulated in [37].

**Definition 2 (M-solution)** Let  $S \in [0, T]$ . A quadruple  $(Y, Z, U, M) \in \mathcal{H}^1(S, T)$  is called an adapted M-solution of (1) on  $[S, T]$  if (1) holds in the usual Itô sense for almost all  $t \in [S, T]$  and, in addition, the following holds: for a.e.  $t \in [S, T]$

$$Y(t) = \mathbb{E}[Y(t)|\mathcal{F}_S] + \int_S^t Z(t, s)dW_s + \int_S^t \int_{\mathcal{E}} U(t, s, e)\tilde{\pi}(de, ds) + \int_S^t dM(t, s). \quad (5)$$

Note that we keep the notion of M-solution of [37], where the letter M stands for “a martingale representation” for  $Y(t)$  to determine  $Z(\cdot, \cdot)$ ,  $U(\cdot, \cdot)$  and  $M(\cdot, \cdot)$  on  $\Delta[S, T]$ . It should not be confused with the orthogonal martingale part  $M$ .

As in [37], any M-solution on  $[S, T]$  is also a M-solution on  $[\bar{S}, T]$  with  $\bar{S} \in (S, T)$ . Moreover if  $(Y, Z, U, M)$  is a solution of the BSDE (2) in  $\mathbb{S}^p(0, T)$ , then by the martingale representation ([12, Lemma III.4.24]), we have

$$Y(t) = \mathbb{E}(Y(t)) + \int_0^t \zeta(t, s)dW_s + \int_0^t \int_{\mathcal{E}} v(t, s, e)\tilde{\pi}(de, ds) + m(t, s),$$

where  $(\zeta, v, m) \in \mathbb{L}^p(0, T; \mathbb{L}_{\mathbb{F}}^2(0, T)) \times \mathbb{L}^p(0, T; \mathbb{L}_{\pi}^2(0, T)) \times \mathbb{L}^p(0, T; \mathbb{M}^{2,\perp}(0, T))$ . Thus we define

$$(Z(t, s), U(t, s, e), M(t, s)) = \begin{cases} (\zeta(t, s), v(t, s, e), m(t, s)), & (t, s) \in \Delta \\ (Z(s), U(s), M(s)), & (t, s) \in \Delta^c \end{cases}$$

and we get that  $(Y, Z, U, M)$  is an adapted M-solution of the BSVIE (2) on  $[0, T]$ , and in fact it is the unique solution (see Theorem 2)

To complete this presentation, let us recall some facts concerning the Poisson integral. From the *Burkholder-Davis-Gundy inequality* (see [29, Theorem 48]), for all  $p \in [1, \infty)$  there exist two universal constants  $c_p$  and  $C_p$  (not depending on  $M$ ) such that for any càdlàg  $\mathbb{F}$ -martingale  $M(\cdot)$  and for any  $T \geq 0$

$$c_p \mathbb{E} \left( [M]_T^{p/2} \right) \leq \mathbb{E} \left[ \left( \sup_{t \in [0, T]} |M(t)| \right)^p \right] \leq C_p \mathbb{E} \left( [M]_T^{p/2} \right). \quad (6)$$

In particular (6) means that the Poisson martingale  $N$  is well-defined on  $[0, T]$  (see Chapter II in [11]) provided we can control the expectation of  $[N]_T^{p/2}$  for some  $p \geq 1$ .

From the *Bichteler-Jacod inequality* (see for example [22]), we distinguish the two cases:  $p \geq 2$  and  $p < 2$ .

- Assume that  $p \geq 2$ . If  $\mathbb{E}([N]_T^{p/2}) < +\infty$ , then  $\mathbb{P} \otimes \text{Leb}$ -a.s. on  $\Omega \times [0, T]$ ,  $\psi(t, \cdot)$  is in  $\mathbb{L}_{\mu}^2$ . Hence the generator of our BSVIE can be defined on  $\mathbb{L}_{\mu}^2$ .
- But if  $p < 2$ ,  $\mathbb{P} \otimes \text{Leb}$ -a.s. on  $\Omega \times [0, T]$ ,  $\psi(t, \cdot)$  is in  $\mathbb{L}_{\mu}^p + \mathbb{L}_{\mu}^2$  if again  $\mathbb{E}([N]_T^{p/2}) < +\infty$ . Moreover  $\psi_t$  is also in  $\mathbb{L}_{\mu}^1 + \mathbb{L}_{\mu}^2$ . Thereby for  $p < 2$ , our generator is defined on  $\mathbb{L}_{\mu}^1 + \mathbb{L}_{\mu}^2$  (for the definition of the sum of two Banach spaces, see for example [15]).

See [18, Section 1] for details on this point. In particular for any  $\psi \in G_{loc}(\mu)$  and  $N$  defined by

$$N_t = \int_0^t \int_{\mathcal{U}} \psi_s(u)\tilde{\pi}(du, ds), t \geq 0,$$

if  $p \geq 2$ , there exist two universal constants  $\kappa_p$  and  $K_p$  such that

$$\kappa_p \left[ \mathbb{E} \left( [N]_T^{p/2} \right) \right] \leq \mathbb{E} \left( \int_0^T \|\psi_t\|_{\mathbb{L}_\mu^2}^2 dt \right)^{p/2} \leq K_p \left[ \mathbb{E} \left( [N]_T^{p/2} \right) \right]. \quad (7)$$

But if  $1 < p < 2$ , we only have the existence of a universal constant  $K_{p,T}$  such that

$$\mathbb{E} \left[ \int_0^T \|\psi_s\|_{\mathbb{L}_\mu^p + \mathbb{L}_\mu^2}^p ds \right] \leq K_{p,T} \mathbb{E} \left( [N]_T^{p/2} \right). \quad (8)$$

And it holds that  $\mathbb{L}_\mu^p + \mathbb{L}_\mu^2 \subset \mathbb{L}_\mu^1 + \mathbb{L}_\mu^2$ .

## 2 Existence and uniqueness of an adapted M-solution.

Let us precise the assumptions on the generator  $f$  of the BSVIE (1). We suppose that  $f$  is defined on  $\Omega \times \Delta^c \times \mathbb{R}^{d \times (1+2k)} \times (\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2)^2$  and we assume that for any fixed  $(t, y, z, \zeta, u, \nu)$  the process  $f(t, \cdot, y, z, \zeta, u, \nu)$  is progressively measurable. Moreover the next condition holds:

**(H1)** *There exists a constant  $K$  such that for any  $(\omega, t, s), (y, \bar{y}), (z, \bar{z}), (\zeta, \bar{\zeta}), (u, \bar{u}), (\nu, \bar{\nu})$  we have*

$$\begin{aligned} & |f(\omega, t, s, y, z, \zeta, u, \nu) - f(\omega, t, s, \bar{y}, \bar{z}, \bar{\zeta}, \bar{u}, \bar{\nu})| \\ & \leq K(|y - \bar{y}| + |z - \bar{z}| + |\zeta - \bar{\zeta}| + \|u - \bar{u}\|_{\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2} + \|\nu - \bar{\nu}\|_{\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2}). \end{aligned}$$

To simplify the notation in the sequel

$$f^0(t, s) = f(t, s, 0, 0, 0, 0, 0).$$

Our result can be extended to non Lipschitz-continuous w.r.t.  $y$  driver  $f$  as in [35] (using a concave function  $\rho$ ) or if  $K$  becomes a function of  $(\omega, t, s)$  with a suitable integrability condition (see [37] Condition (3.13)), since the arguments remain almost the same.

Let us emphasize that we do not require any regularity of the paths  $t \mapsto Y(t)$ , which are a priori not continuous nor càdlàg. The component  $Y$  is only supposed to be in  $\mathbb{L}_{\mathbb{F}}^p(0, T)$ .

### 2.1 Preliminary results

First we consider for any  $R$  and  $S$  in  $[0, T)$  a driver  $h : \Omega \times [S, T] \times [R, T] \times \mathbb{R}^{d \times k} \times (\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2) \rightarrow \mathbb{R}^d$  such that **(H1)** holds and for some  $p > 1$

$$\mathbb{E} \int_S^T \left( \int_R^T |h(t, s, 0, 0)| ds \right)^p dt < +\infty. \quad (9)$$

Then let us define the BSDE on  $[R, T]$  parameterized by  $t \in [S, T]$ :

$$\begin{aligned} \lambda(t, r) = & \Phi(t) + \int_r^T h(t, s, z(t, s), u(t, s)) ds - \int_r^T z(t, s) dW_s \\ & - \int_r^T \int_{\mathcal{E}} u(t, s, e) \tilde{\pi}(de, ds) - \int_r^T dm(t, s), \end{aligned} \quad (10)$$

where  $t \in [S, T]$  and  $r \in [R, T]$ . From Proposition 1, for any  $\Phi(\cdot) \in \mathbb{L}_{\mathcal{F}_T}^p(S, T)$  the previous BSDE has a unique solution  $(\lambda(t, \cdot), z(t, \cdot), u(t, \cdot), m(t, \cdot))$  in  $\mathbb{S}^p(R, T)$  and for a.e.  $t \in [S, T]$

$$\begin{aligned} & \mathbb{E} \left[ \sup_{r \in [R, T]} |\lambda(t, r)|^p + \left( \int_R^T |z(t, r)|^2 dr \right)^{p/2} + ([m(t, \cdot)]_{R, T})^{p/2} \right. \\ & \left. + \left( \int_R^T \int_{\mathcal{E}} |u(t, r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \leq C \mathbb{E} \left[ |\Phi(t)|^p + \left( \int_R^T |h(t, r, 0, 0)| dr \right)^p \right]. \end{aligned} \quad (11)$$

Moreover we have a stability result for BSDEs ([16, Lemma 5 and proof of Proposition 2] for  $p \geq 2$ , [18, Proposition 3]). Let  $(\bar{\Phi}, \bar{h})$  be a couple of data each satisfying the above assumption **(H1)** and the required integrability conditions on the data. Let  $(\bar{\lambda}(t, \cdot), \bar{z}(t, \cdot), \bar{u}(t, \cdot), \bar{m}(t, \cdot))$  the solution of the BSDE (10) with data  $(\bar{\Phi}, \bar{h})$ . Define

$$(\widehat{\lambda}(t, \cdot), \widehat{z}(t, \cdot), \widehat{u}(t, \cdot), \widehat{m}(t, \cdot)) = (\lambda(t, \cdot) - \bar{\lambda}(t, \cdot), z(t, \cdot) - \bar{z}(t, \cdot), u(t, \cdot) - \bar{u}(t, \cdot), m(t, \cdot) - \bar{m}(t, \cdot)).$$

Then there exists a constant  $C$  depending on  $p, K$  and  $T$ , such that

$$\begin{aligned} & \mathbb{E} \left[ \sup_{r \in [R, T]} |\widehat{\lambda}(t, r)|^p + \left( \int_R^T |\widehat{z}(t, r)|^2 dr \right)^{p/2} + ([\widehat{m}(t, \cdot)]_{R, T})^{p/2} \right. \\ & \left. + \left( \int_R^T \int_{\mathcal{E}} |\widehat{u}(t, r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \\ & \leq C \mathbb{E} \left[ |\Phi(t) - \bar{\Phi}(t)|^p + \left( \int_R^T |h(t, r, z(t, r), u(t, r)) - \bar{h}(t, r, z(t, r), u(t, r))| dr \right)^p \right]. \end{aligned} \quad (12)$$

Let us derive two consequences, one for *stochastic Fredholm integral equation* (SFIE in abbreviated form) and one for a particular BSVIE.

Indeed let us fix  $r = S \in [R, T]$  and define for  $t \in [R, S]$  and  $s \in [S, T]$ :

$$\psi^S(t) = \lambda(t, S), \quad Z(t, s) = z(t, s), \quad U(t, s) = u(t, s), \quad M(t, s) = m(t, s).$$

Then Equation (10) becomes a SFIE: for  $t \in [R, S]$

$$\begin{aligned} \psi^S(t) &= \Phi(t) + \int_S^T h(t, s, Z(t, s), U(t, s)) ds - \int_S^T Z(t, s) dW_s \\ &\quad - \int_S^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_S^T dM(t, s). \end{aligned} \quad (13)$$

**Lemma 1** *If (9) holds and if  $\Phi \in \mathbb{L}_{\mathcal{F}_T}^p(S, T)$ , then the SFIE (13) has a unique solution such that  $\psi^S$  belongs to  $\mathbb{L}_{\mathcal{F}_S}^p(R, S)$  and*

$$(Z, U, M) \in \times \mathbb{L}^p(R, S; \mathbb{H}^2(S, T)) \times \mathbb{L}^p(R, S; \mathbb{L}_{\pi}^2(S, T)) \times \mathbb{L}^p(R, S; \mathbb{M}^{2, \perp}(S, T))$$

and for  $t \in [R, S]$

$$\begin{aligned} & \mathbb{E} \left[ |\psi^S(t)|^p + \left( \int_S^T |Z(t, r)|^2 dr \right)^{p/2} + ([M(t, \cdot)]_{S, T})^{p/2} \right. \\ & \left. + \left( \int_S^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \leq C \mathbb{E} \left[ |\Phi(t)|^p + \left( \int_S^T |h(t, r, 0, 0)| dr \right)^p \right]. \end{aligned} \quad (14)$$

Note that here  $\psi^S(t)$  is only required to be  $\mathcal{F}_S$ -measurable for almost all  $t$  and not  $\mathbb{F}$ -adapted.

Again let us consider (10), but fix  $R = S$  and define  $Y(t) = \lambda(t, t)$ ,  $t \in [S, T]$ ,  $Z(t, s) = z(t, s)$ ,  $U(t, s, e) = u(t, s, e)$ ,  $M(t, s) = m(t, s)$  for  $(t, s) \in \Delta^c[S, T]$ . Equation (10) becomes

$$\begin{aligned} Y(t) &= \Phi(t) + \int_t^T h(t, s, Z(t, s), U(t, s)) ds - \int_t^T Z(t, s) dW_s \\ &\quad - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM(t, s), \end{aligned} \quad (15)$$

which is a special case of (1) where  $h$  does not depend on  $y$ ,  $\zeta$  and  $\nu$ . We can define  $Z$ ,  $U$  and  $M$  on  $\Delta[S, T]$  by the martingale representation:

$$Y(t) = \mathbb{E}[Y(t) | \mathcal{F}_S] + \int_S^t Z(t, s) dW_s + \int_S^t \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) + \int_S^t dM(t, s).$$

Let us stress that  $z(t, s)$ ,  $Z(t, s)$ ,  $u(t, s)$ ,  $U(t, s)$ ,  $m(t, s)$  and  $M(t, s)$  might be different for  $(t, s) \in \Delta[S, T]$ . We obtain immediately the next result.

**Lemma 2** *If (9) holds and if  $\Phi \in \mathbb{L}_{\mathcal{F}_T}^p(S, T)$ , then the BSVIE (15) has a unique adapted  $M$ -solution in  $\mathcal{H}^p(S, T)$  and for  $t \in [S, T]$ :*

$$\begin{aligned} &\mathbb{E} \left[ |Y(t)|^p + \left( \int_S^T |Z(t, r)|^2 dr \right)^{p/2} + ([M(t, \cdot)]_{S, T})^{p/2} \right. \\ &\quad \left. + \left( \int_S^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \leq C \mathbb{E} \left[ |\Phi(t)|^p + \left( \int_S^T |h(t, r, 0, 0)| dr \right)^p \right]. \end{aligned} \quad (16)$$

Moreover we have a stability result for this BSVIE. Let  $(\bar{\Phi}, \bar{h})$  be a couple of data each satisfying the above assumption **(H1)** and the same integrability conditions. Let  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{M})$  the solution of the BSVIE (15) with data  $(\bar{\Phi}, \bar{h})$ . Define

$$(\hat{Y}, \hat{Z}, \hat{U}, \hat{M}) = (Y - \bar{Y}, Z - \bar{Z}, U - \bar{U}, M - \bar{M}).$$

Then there exists a constant  $C$  depending on  $p$ ,  $K$  and  $T$ , such that for  $t \in [S, T]$

$$\begin{aligned} &\mathbb{E} \left[ |\hat{Y}(t)|^p + \left( \int_S^T |\hat{Z}(t, r)|^2 dr \right)^{p/2} + ([\hat{M}(t, \cdot)]_{S, T})^{p/2} \right. \\ &\quad \left. + \left( \int_S^T \int_{\mathcal{E}} |\hat{U}(t, r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \leq C \mathbb{E} \left[ |\Phi(t) - \bar{\Phi}(t)|^p \right. \\ &\quad \left. + \left( \int_S^T |h(t, r, Z(t, r), U(t, r)) - \bar{h}(t, r, Z(t, r), U(t, r))| dr \right)^p \right]. \end{aligned} \quad (17)$$

In the rest of this section, we distinguish the particular case  $p = 2$ . The reason can be understood just by considering the term  $Z$ . Indeed if  $Z \in \mathbb{L}^p(0, T; \mathbb{H}^p(0, T))$  then

$$\mathbb{E} \int_0^T \left[ \int_0^T |Z(t, r)|^2 dr \right]^{\frac{p}{2}} dt < +\infty.$$

The two time variables  $t$  and  $r$  don't play the same role and the integrability property is not the same w.r.t.  $t$  or w.r.t.  $r$ , except if  $p = 2$ . Thereby in the BSVIE (1), we can use both  $Z(t, s)$  and  $Z(s, t)$  if  $p = 2$ . For  $p \neq 2$ , we will assume that  $f$  does not depend on  $Z(s, t)$  nor on  $U(s, t)$ . The extension to the general case seems difficult to prove and is left for further research. We also point out that  $p \geq 2$  implies that  $\mathbb{L}^p(S, T; \mathbb{H}^p(R, T)) \subset \mathbb{L}^2(S, T; \mathbb{H}^2(R, T))$  and  $\mathbb{L}^p(S, T; \mathbb{L}_\pi^p(R, T)) \subset \mathbb{L}^2(S, T; \mathbb{L}_\pi^2(R, T))$ :

$$\mathbb{E} \int_S^T \left( \int_R^T |Z(t, r)|^2 dr \right) dt \leq C \left( \mathbb{E} \int_S^T \left[ \left( \int_R^T |Z(t, r)|^2 dr \right)^{\frac{p}{2}} \right] dt \right)^{\frac{2}{p}}$$

and

$$\mathbb{E} \int_S^T \|U(t, \cdot)\|_{\mathbb{L}_\pi^2(R, T)}^2 dt \leq C \left[ \mathbb{E} \int_S^T \left( \|U(t, \cdot)\|_{\mathbb{L}_\pi^2(R, T)}^2 \right)^{\frac{p}{2}} dt \right]^{\frac{2}{p}}.$$

For  $1 < p < 2$ , this property fails.

## 2.2 The case $p = 2$

Now we come back to the BSVIE (1).

**Theorem 1** *Assume that  $\Phi \in \mathbb{L}_{\mathcal{F}_T}^2(0, T)$ , that **(H1)** holds<sup>3</sup> and*

$$\mathbb{E} \int_0^T \left( \int_t^T |f^0(t, s)| ds \right)^2 dt < +\infty. \quad (18)$$

*Then the BSVIE (1) has a unique adapted  $M$ -solution  $(Y, Z, U, M)$  in  $\mathcal{H}^2(0, T)$  on  $[0, T]$ . Moreover for any  $S \in [0, T]$*

$$\begin{aligned} \|(Y, Z, U, M)\|_{\mathcal{H}^2(S, T)}^2 &= \mathbb{E} \left[ \int_S^T |Y(t)|^2 dt + \int_S^T \left( \int_S^T |Z(t, r)|^2 dr \right) dt \right. \\ &\quad \left. + \int_S^T \left( \|U(t, \cdot)\|_{\mathbb{L}_\pi^2(S, T)}^2 \right) dt + \int_S^T ([M(t, \cdot)]_{S, T}) dt \right] \\ &\leq C \mathbb{E} \left[ \int_S^T |\Phi(t)|^2 dt + \int_S^T \left( \int_t^T |f^0(t, r)| dr \right)^2 dt \right]. \end{aligned} \quad (19)$$

Recall that

$$\|U(t, \cdot)\|_{\mathbb{L}_\pi^2(S, T)}^2 = \int_S^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr).$$

**Proof.** Let us stress again that we follow the scheme of the proof of [37, Theorem 3.7] and that  $p = 2$  allows us to use Inequality (7), that is the ‘‘equivalence’’ between  $\mathbb{L}_\mu^2$  and  $\mathbb{L}_\pi^2$ .

**Step 1.** For any  $S \in [0, T]$ , let us consider the space  $\widehat{\mathcal{H}}^2(S, T)$  be the space of all  $(y, z, u, m)$  in  $\mathcal{H}^2(S, T)$  such that for a.e.  $t \in [S, T]$  a.s.

$$y(t) = \mathbb{E}[y(t)|\mathcal{F}_S] + \int_S^t z(t, s) dW_s + \int_S^t \int_{\mathcal{E}} u(t, s, e) \widetilde{\pi}(de, ds) + \int_S^t dm(t, s).$$

<sup>3</sup>The  $\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2$  norm in **(H1)** is replaced by the  $\mathbb{L}_\mu^2$  norm in this case.

From this representation, Doob's martingale inequality and the Burkholder-Davis-Gundy inequality, we have for  $t \in [S, T]$

$$\mathbb{E} \left[ \left( \int_S^t |z(t, r)|^2 dr \right) + ([m(t, \cdot)]_{S, t}) + \left( \int_S^t \int_{\mathcal{E}} |u(t, r, e)|^2 \pi(de, dr) \right) \right] \leq C \mathbb{E} |y(t)|^2.$$

Using (7) we deduce that there exists a constant  $C$  such that

$$\mathbb{E} \left[ \left( \int_S^t \|u(t, r, \cdot)\|_{\mathbb{L}_\mu^2}^2 dr \right) \right] \leq C \mathbb{E} |y(t)|^2.$$

We take on  $\widehat{\mathcal{H}}^2(S, T)$  the following norm:

$$\begin{aligned} \|(y, z, u, m)\|_{\widehat{\mathcal{H}}^2(S, T)}^2 &= \mathbb{E} \left[ \int_S^T |y(t)|^2 dt + \int_S^T \left( \int_t^T |z(t, r)|^2 dr \right) dt \right. \\ &\quad \left. + \int_S^T \left( \int_t^T \|u(t, r)\|_{\mathbb{L}_\mu^2}^2 dr \right) dt + \int_S^T ([m(t, \cdot)]_{t, T}) dt \right]. \end{aligned}$$

The same arguments as [37, Inequality (3.48)] show that

$$\begin{aligned} \|(y, z, u, m)\|_{\widehat{\mathcal{H}}^2(S, T)}^2 &\leq \mathbb{E} \left[ \int_S^T |y(t)|^2 dt + \int_S^T \left( \int_S^T |z(t, r)|^2 dr \right) dt \right. \\ &\quad \left. + \int_S^T \left( \int_S^T \|u(t, r)\|_{\mathbb{L}_\mu^2}^2 dr \right) dt + \int_S^T ([m(t, \cdot)]_{S, T}) dt \right] \\ &\leq (C + 1) \|(y, z, u, m)\|_{\widehat{\mathcal{H}}^2(S, T)}^2. \end{aligned} \tag{20}$$

Hence we have an equivalent norm for  $\widehat{\mathcal{H}}^2(S, T)$ .

Fix  $\Phi \in \mathbb{L}_{\mathcal{F}_T}^2(S, T)$  and  $(y, \zeta, \nu, m) \in \widehat{\mathcal{H}}^2(S, T)$  and consider the BSVIE on  $[S, T]$

$$\begin{aligned} Y(t) &= \Phi(t) + \int_t^T f(t, s, y(s), Z(t, s), \zeta(s, t), U(t, s), \nu(s, t)) ds - \int_t^T Z(t, s) dW_s \\ &\quad - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM(t, s), \end{aligned} \tag{21}$$

We want to apply Lemma 2. If

$$h(t, s, z, \psi) = f(t, s, y(s), z, \zeta(s, t), \psi, \nu(s, t)),$$

we need to check that

$$\mathbb{E} \left[ \left( \int_S^T |h(t, r, 0, 0)| dr \right)^2 \right] < +\infty.$$

The Lipschitz property of  $f$  leads to

$$|h(t, r, 0, 0)| \leq |f^0(t, r)| + K|y(r)| + K|\zeta(r, t)| + K\|\nu(r, t)\|_{\mathbb{L}_\mu^2}$$

thus by Hölder's inequality

$$\begin{aligned}
\mathbb{E} \left[ \left( \int_S^T |h(t, r, 0, 0)| dr \right)^2 \right] &\leq C \mathbb{E} \left[ \left( \int_S^T |f^0(t, r)| dr \right)^2 \right] + CK^2 \mathbb{E} \left[ \left( \int_S^T |y(r)| dr \right)^2 \right] \\
&\quad + CK^2 \mathbb{E} \left[ \left( \int_S^T |\zeta(r, t)| dr \right)^2 \right] + CK^2 \mathbb{E} \left[ \left( \int_S^T \|\nu(r, t)\|_{\mathbb{L}_\mu^2} dr \right)^2 \right] \\
&\leq C \mathbb{E} \left[ \left( \int_S^T |f^0(t, r)| dr \right)^2 \right] + CK^2 \mathbb{E} \left[ \int_S^T |y(r)|^2 dr \right] \\
&\quad + CK^2 \mathbb{E} \left[ \left( \int_S^T |\zeta(r, t)|^2 dr \right) \right] + CK^2 \mathbb{E} \left[ \left( \int_S^T \|\nu(r, t)\|_{\mathbb{L}_\mu^2}^2 dr \right) \right] < +\infty.
\end{aligned}$$

Here we use the ‘‘symmetry’’ between the two time variables when  $p = 2$ . Thus the BSVIE (21) has a unique adapted M-solution  $(Y, Z, U, M) \in \mathcal{H}^2(S, T)$  and for any  $t \in [S, T]$

$$\begin{aligned}
\mathbb{E} \left[ |Y(t)|^2 + \left( \int_t^T |Z(t, r)|^2 dr \right) + ([M(t, \cdot)]_{t, T}) + \left( \int_t^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right) \right] \\
\leq C \mathbb{E} \left[ |\Phi(t)|^2 + \left( \int_t^T |f(t, r, y(r), 0, \zeta(r, t), 0, \nu(r, t))| dr \right)^2 \right] \\
\leq C \mathbb{E} \left[ |\Phi(t)|^2 + \left( \int_t^T |f^0(t, r)| dr \right)^2 + \int_t^T |y(r)|^2 dr \right. \\
\quad \left. + \left( \int_t^T |\zeta(r, t)|^2 dr \right) + \left( \int_t^T \|\nu(r, t)\|_{\mathbb{L}_\mu^2}^2 dr \right) \right].
\end{aligned}$$

Then using (7) we deduce that

$$\begin{aligned}
\mathbb{E} \left[ \int_S^T |Y(t)|^2 dt + \int_S^T \left( \int_t^T |Z(t, r)|^2 dr \right) dt + \int_S^T ([M(t, \cdot)]_{t, T}) dt \right. \\
\quad \left. + \int_S^T \left( \int_t^T \|U(t, r)\|_{\mathbb{L}_\mu^2}^2 dr \right) dt \right] \\
\leq C \mathbb{E} \left[ \int_S^T |\Phi(t)|^2 dt + \int_S^T \left( \int_t^T |f^0(t, r)| dr \right)^2 dt + \int_S^T |y(r)|^2 dr \right. \\
\quad \left. + \int_S^T \left( \int_t^T |\zeta(r, t)|^2 dr \right) dt + \int_S^T \left( \int_t^T \|\nu(r, t)\|_{\mathbb{L}_\mu^2}^2 dr \right) dt \right]. \quad (22)
\end{aligned}$$

Therefore  $(Y, Z, U, M) \in \widehat{\mathcal{H}}^2(S, T)$ . In other words we have a map  $\Theta$  from  $\widehat{\mathcal{H}}^2(S, T)$  to  $\widehat{\mathcal{H}}^2(S, T)$ . Moreover arguing as in [37], we obtain that for  $(y, \zeta, \nu, m)$  and  $(\bar{u}, \bar{\zeta}, \bar{\nu}, \bar{m})$  in  $\widehat{\mathcal{H}}^2(S, T)$ , if  $(Y, Z, U, M)$  and  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{M})$  are the solutions of the BSVIE (21), then from



(17) the difference satisfies:

$$\begin{aligned}
& \mathbb{E} \left[ |\widehat{Y}(t)|^2 + \left( \int_S^T |\widehat{Z}(t,r)|^2 dr \right) + \left( [\widehat{M}(t, \cdot)]_{S,T} \right) \right. \\
& \quad \left. + \left( \int_S^T \int_{\mathcal{E}} |\widehat{U}(t,r,e)|^2 \pi(de, dr) \right) \right] \\
& \leq C \mathbb{E} \left[ \left( \int_S^T K \left( |\widehat{y}(r)| + |\widehat{\zeta}(t,r)| + \|\widehat{\nu}(t,r)\|_{\mathbb{L}_\mu^2} \right) dr \right)^2 \right] \\
& \leq CK^2(T-S) \mathbb{E} \left[ \left( \int_S^T \left( |\widehat{y}(r)|^2 + |\widehat{\zeta}(t,r)|^2 + \|\widehat{\nu}(t,r)\|_{\mathbb{L}_\mu^2}^2 \right) dr \right) \right].
\end{aligned}$$

For  $T-S$  sufficiently small, this map is a contraction and thus it admits a unique fixed point  $(Y, Z, U, M) \in \widehat{\mathcal{H}}^2(S, T)$  which is the unique adapted M-solution of (1) on  $[S, T]$ . Moreover Estimate (19) holds. This step determines the values  $(Y(t), Z(t, s), U(t, s), M(t, s))$  for  $(t, s) \in [S, T] \times [S, T]$ .

**Step 2.** We use the martingale representation theorem to define  $(Z, U, M)$  on  $[S, T] \times [R, S]$  for any  $R \in [0, S]$ . Indeed since  $\mathbb{E}[Y(t)|\mathcal{F}_S] \in \mathbb{L}^2(S, T; \mathbb{L}_{\mathcal{F}_S}^2(\Omega))$ , there exists a unique triple  $(Z, U, M)$  in  $\mathbb{L}^2(S, T; \mathbb{H}^2(R, S)) \times \mathbb{L}^2(S, T; \mathbb{L}_\pi^2(R, S)) \times \mathbb{L}^2(S, T; \mathbb{M}^{2,\perp}(R, S))$  such that for  $t \in [S, T]$ :

$$\mathbb{E}[Y(t)|\mathcal{F}_S] = \mathbb{E}[Y(t)|\mathcal{F}_R] + \int_R^S Z(t, s) dW_s + \int_R^S \int_{\mathcal{E}} U(t, s, e) \widetilde{\pi}(de, ds) + \int_R^S dM(t, s),$$

and

$$\begin{aligned}
& \mathbb{E} \left[ \left( \int_R^S |Z(t, r)|^2 dr \right) + \left( \int_R^S \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right) + ([M(t, \cdot)]_{R,S}) \right] \\
& \leq C \mathbb{E}[Y(t)]^2.
\end{aligned}$$

Thus together with the first step, we have defined  $(Z, U, M)$  for  $(t, s) \in [S, T] \times [R, T]$  and

$$\begin{aligned}
& \mathbb{E} \left[ \int_S^T \left( \int_R^T |Z(t, r)|^2 dr \right) dt + \int_S^T \left( \int_R^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right) dt \right. \\
& \quad \left. + \int_S^T ([M(t, \cdot)]_{R,T}) dt \right] \\
& \leq C \mathbb{E} \left[ \int_S^T |\Phi(t)|^2 dt + \int_S^T \left( \int_t^T |f^0(t, r)| dr \right)^2 dt \right]. \tag{23}
\end{aligned}$$

**Step 3.** From the two previous steps, for  $(t, s) \in [R, S] \times [S, T]$ , the values of  $Y(s)$  and  $(Z(s, t), U(s, t))$  are already defined. Thus let us consider

$$f^S(t, s, z, u) = f(t, s, Y(s), z, Z(s, t), u, U(s, t)), \quad (t, s, z, u) \in [R, S] \times [S, T] \times \mathbb{R}^k \times \mathbb{L}_\mu^2,$$

and the SFIE

$$\begin{aligned}
\psi^S(t) &= \Phi(t) + \int_S^T f^S(t, s, Z(t, s), U(t, s)) ds - \int_S^T Z(t, s) dW_s \\
&\quad - \int_S^T \int_{\mathcal{E}} U(t, s, e) \widetilde{\pi}(de, ds) - \int_S^T dM(t, s).
\end{aligned}$$

From Lemma 1, this equation has a unique solution  $(\psi^S, Z, U, M)$  such that for  $t \in [R, S]$

$$\begin{aligned}
& \mathbb{E} \left[ |\psi^S(t)|^2 + \left( \int_S^T |Z(t, r)|^2 dr \right) \right. \\
& \quad \left. + ([M(t, \cdot)]_{S, T}) + \left( \int_S^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right) \right] \\
& \leq C \mathbb{E} \left[ |\Phi(t)|^2 + \left( \int_S^T |f^S(t, r, 0, 0)| dr \right)^2 \right] \\
& = C \mathbb{E} \left[ |\Phi(t)|^2 + \left( \int_S^T |f(t, r, Y(r), 0, Z(r, t), 0, U(r, t))| dr \right)^2 \right] \\
& \leq C \mathbb{E} \left[ |\Phi(t)|^2 + \left( \int_S^T |f^0(t, r)| dr \right)^2 + \int_S^T |Y(r)|^2 dr \right. \\
& \quad \left. + \left( \int_S^T |Z(r, t)|^2 dr \right) + \left( \int_S^T \|U(r, t)\|_{\mathbb{L}_\mu^2}^2 dr \right) \right].
\end{aligned}$$

Hence using (7), (22) and (23), we obtain:

$$\begin{aligned}
& \mathbb{E} \left[ \int_R^S |\psi^S(t)|^2 dt + \int_R^S \left( \int_S^T |Z(t, r)|^2 dr \right) dt + \int_R^S ([M(t, \cdot)]_{S, T}) dt \right. \\
& \quad \left. + \int_R^S \left( \int_S^T \|U(t, r)\|_{\mathbb{L}_\mu^2}^2 dr \right) dt \right] \\
& \leq C \mathbb{E} \left[ \int_R^S |\Phi(t)|^2 dt + \int_R^S \left( \int_S^T |f^0(t, r)| dr \right)^2 dt + \int_S^T |Y(r)|^2 dr \right] \\
& \quad + C \mathbb{E} \left[ \int_R^S \left( \int_S^T |Z(r, t)|^2 dr \right) dt + \int_R^S \left( \int_S^T \|U(r, t)\|_{\mathbb{L}_\mu^2}^2 dr \right) dt \right] \\
& \leq C \mathbb{E} \left[ \int_R^T |\Phi(t)|^p dt + \int_R^T \left( \int_t^T |f^0(t, r)| dr \right)^2 dt \right]. \tag{24}
\end{aligned}$$

Hence we have defined  $(Z, U, M)$  for  $(t, s) \in [R, S] \times [S, T]$ , and by the definition of  $f^S$ , for  $t \in [R, S]$

$$\begin{aligned}
\psi^S(t) &= \Phi(t) + \int_S^T f(t, s, Y(s), Z(t, s), Z(s, t), U(t, s), U(s, t)) ds - \int_S^T Z(t, s) dW_s \\
& \quad - \int_S^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_S^T dM(t, s). \tag{25}
\end{aligned}$$

**Step 4.** Let us summarize what we have after these three steps.  $Y$  is uniquely defined on  $[S, T]$  (from Step 1) and  $(Z, U, M)$  are uniquely determined on  $[S, T] \times [R, T]$  (from Steps 1 and 2) and on  $[R, S] \times [S, T]$  (from Steps 1 and 3). Let us now solve (1) on  $[R, S]^2$ . Consider

$$\begin{aligned}
Y(t) &= \psi^S(t) + \int_t^S f(t, s, Y(s), Z(t, s), Z(s, t), U(t, s), U(s, t)) ds - \int_t^S Z(t, s) dW_s \\
& \quad - \int_t^S \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^S dM(t, s).
\end{aligned}$$

It is a BSVIE with terminal condition  $\psi^S \in \mathbb{L}_{\mathcal{F}_S}^2(R, S)$  and generator  $f$ . As in the first step, this BSVIE has a unique solution in  $\mathcal{H}^2(R, S)$  provided that  $S - R > 0$  is small enough. Now for  $t \in [R, S]$  from the expression (25) of  $\psi^S$ , we obtain that

$$\begin{aligned} Y(t) &= \Phi(t) + \int_t^T f(t, s, Y(s), Z(t, s), Z(s, t), U(t, s), U(s, t)) ds - \int_t^T Z(t, s) dW_s \\ &\quad - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM(t, s). \end{aligned}$$

Moreover again by the arguments as in the first step

$$\begin{aligned} &\mathbb{E} \left[ \int_R^S |Y(t)|^2 dt + \int_R^S \left( \int_R^S |Z(t, r)|^2 dr \right) dt + \int_R^S ([M(t, \cdot)]_{R, S}) dt \right. \\ &\quad \left. + \int_R^S \left( \int_R^S \|U(t, r)\|_{\mathbb{L}_\mu^2}^2 dr \right) dt \right] \\ &\leq C \mathbb{E} \left[ \int_R^S |\Phi(t)|^2 dt + \int_R^S \left( \int_t^S |f^0(t, r)| dr \right)^2 dt \right] \\ &\leq C \mathbb{E} \left[ \int_R^T |\Phi(t)|^2 dt + \int_R^T \left( \int_t^T |f^0(t, r)| dr \right)^2 dt \right]. \end{aligned}$$

From this inequality together with (19) on  $[S, T]$ , (23) and (24), we proved that the BSVIE (1) has a unique adapted M-solution  $(Y, Z, U, M)$  in  $\mathcal{H}^p(R, T)$  on  $[R, T]$  with the estimate (19) on  $[R, T]$ .

**Step 5.** The conclusion of the proof is done by induction since the time intervals  $[S, T]$  (Step 1) and  $[R, S]$  (Step 4) are determined by absolute constants depending only on the Lipschitz constant  $K$  of  $f$  in Condition **(H1)** and on the time horizon  $T$ . □

Note that from (7), concerning  $U$ , Estimate (19) is completely equivalent to: for any  $S \in [0, T]$

$$\mathbb{E} \left[ \int_S^T \left( \int_S^T \|U(t, r)\|_{\mathbb{L}_\mu^2}^2 dr \right) dt \right] \leq C \mathbb{E} \left[ \int_S^T |\Phi(t)|^2 dt + \int_S^T \left( \int_t^T |f^0(t, r)| dr \right)^2 dt \right].$$

### 2.3 The case $p \neq 2$

Here we consider the special case (4):

$$\begin{aligned} Y(t) &= \Phi(t) + \int_t^T f(t, s, Y(s), Z(t, s), U(t, s)) ds - \int_t^T Z(t, s) dW_s \\ &\quad - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM(t, s). \end{aligned}$$

In the Brownian-Poisson  $L^2$ -setting, it was already studied in [35]. And if  $f$  does not depend on  $Y$ , Lemma 2 gives existence and uniqueness of an adapted M-solution in  $\mathcal{H}^p(0, T)$ .

**Theorem 2** For  $p > 1$ , assume that  $\Phi \in \mathbb{L}_{\mathcal{F}_T}^p(0, T)$ , that **(H1)** holds and

$$\mathbb{E} \int_0^T \left( \int_t^T |f^0(t, s)| ds \right)^p dt < +\infty. \quad (26)$$

Then the BSVIE (4) has a unique adapted  $M$ -solution  $(Y, Z, U, M)$  in  $\mathcal{H}^p(0, T)$  on  $[0, T]$ . Moreover for any  $S \in [0, T]$

$$\begin{aligned} \|(Y, Z, U, M)\|_{\mathcal{H}^p(S, T)}^p &= \mathbb{E} \left[ \int_S^T |Y(t)|^p dt + \int_S^T \left( \int_S^T |Z(t, r)|^2 dr \right)^{p/2} dt \right. \\ &\quad \left. + \int_S^T \left( \|U(t, \cdot)\|_{\mathbb{L}_\pi^2(S, T)}^2 \right)^{p/2} dt + \int_S^T ([M(t, \cdot)]_{S, T})^{p/2} dt \right] \\ &\leq C \mathbb{E} \left[ \int_S^T |\Phi(t)|^p dt + \int_S^T \left( \int_t^T |f^0(t, r)| dr \right)^p dt \right]. \end{aligned} \quad (27)$$

Recall that

$$\|U(t, \cdot)\|_{\mathbb{L}_\pi^2(S, T)}^2 = \int_S^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr).$$

**Proof.** Here we adapt the proof of Theorem 1. Let us stress that  $p \geq 2$  allows us to use Inequality (7), that is the “equivalence” between  $\mathbb{L}_\mu^2$  and  $\mathbb{L}_\pi^2$ .

**Step 1.** Now the set  $\widehat{\mathcal{H}}^p(S, T)$  is the space of all  $(y, z, u, m)$  in  $\mathcal{H}^p(S, T)$  such that for a.e.  $t \in [S, T]$  a.s.

$$y(t) = \mathbb{E}[y(t)|\mathcal{F}_S] + \int_S^t z(t, s) dW_s + \int_S^t \int_{\mathcal{E}} u(t, s, e) \tilde{\pi}(de, ds) + \int_S^t dm(t, s).$$

We take on  $\widehat{\mathcal{H}}^p(S, T)$  the following norm:

$$\begin{aligned} \|(y, z, u, m)\|_{\widehat{\mathcal{H}}^p(S, T)}^p &= \mathbb{E} \left[ \int_S^T |y(t)|^p dt + \int_S^T \left( \int_t^T |z(t, r)|^2 dr \right)^{p/2} dt \right. \\ &\quad \left. + \int_S^T \left( \|u(t, r)\|_{\mathbb{L}_\pi^2(t, T)}^2 \right)^{p/2} dt + \int_S^T ([m(t, \cdot)]_{t, T})^{p/2} dt \right]. \end{aligned}$$

or using (7), if  $p \geq 2$ :

$$\begin{aligned} \|(y, z, u, m)\|_{\widehat{\mathcal{H}}^p(S, T)}^p &= \mathbb{E} \left[ \int_S^T |y(t)|^p dt + \int_S^T \left( \int_t^T |z(t, r)|^2 dr \right)^{p/2} dt \right. \\ &\quad \left. + \int_S^T \left( \int_t^T \|u(t, r)\|_{\mathbb{L}_\mu^2}^2 dr \right)^{p/2} dt + \int_S^T ([m(t, \cdot)]_{t, T})^{p/2} dt \right]. \end{aligned}$$

The norm equivalence (20) becomes:

$$\begin{aligned} \|(y, z, u, m)\|_{\mathfrak{M}^p(S, T)}^p &\leq \mathbb{E} \left[ \int_S^T |y(t)|^p dt + \int_S^T \left( \int_S^T |z(t, r)|^2 dr \right)^{p/2} dt \right. \\ &\quad \left. + \int_S^T \left( \|u(t, \cdot)\|_{\mathbb{L}_\pi^2(S, T)}^2 \right)^{p/2} dt + \int_S^T ([m(t, \cdot)]_{S, T})^{p/2} dt \right] \\ &\leq (C_p + 1) \|(y, z, u, m)\|_{\mathfrak{M}^p(S, T)}^p. \end{aligned} \quad (28)$$

And again for  $p \geq 2$ ,  $\|u(t, \cdot)\|_{\mathbb{L}_\pi^2(S, T)}^2$  can be replaced by  $\int_S^T \|u(t, r)\|_{\mathbb{L}_\mu^2}^2 dr$  in the previous estimates (with the suitable modifications of the constant  $C_p$ ).

If  $\Phi \in \mathbb{L}_{\mathcal{F}_T}^p(S, T)$  and  $(y, \zeta, \nu, m) \in \widehat{\mathcal{H}}^p(S, T)$ , we consider the BSVIE on  $[S, T]$ , equivalent to (21):

$$\begin{aligned} Y(t) &= \Phi(t) + \int_t^T f(t, s, y(s), Z(t, s), U(t, s)) ds - \int_t^T Z(t, s) dW_s \\ &\quad - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM(t, s), \end{aligned} \quad (29)$$

From Lemma 2 this BSVIE has a unique adapted M-solution  $(Y, Z, U, M) \in \mathcal{H}^p(S, T)$  and for any  $t \in [S, T]$

$$\begin{aligned} &\mathbb{E} \left[ |Y(t)|^p + \left( \int_t^T |Z(t, r)|^2 dr \right)^{p/2} + ([M(t, \cdot)]_{t, T})^{p/2} + \left( \int_t^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \\ &\leq C \mathbb{E} \left[ |\Phi(t)|^p + \left( \int_t^T |f^0(t, r)| dr \right)^p + \int_t^T |y(r)|^p dr \right]. \end{aligned} \quad (30)$$

Therefore  $(Y, Z, U, M) \in \widehat{\mathcal{H}}^p(S, T)$ . In other words we have a map  $\Theta$  from  $\widehat{\mathcal{H}}^p(S, T)$  to  $\widehat{\mathcal{H}}^p(S, T)$  and we show that for  $T - S$  sufficiently small, this map is a contraction and thus it admits a unique fixed point  $(Y, Z, U, M) \in \widehat{\mathcal{H}}^p(S, T)$  which is the unique adapted M-solution of (4) on  $[S, T]$ . Moreover Estimate (27) holds. This step determines the values  $(Y(t), Z(t, s), U(t, s), M(t, s))$  for  $(t, s) \in [S, T] \times [S, T]$ .

Before going on in the proof, let us explain why we cannot deal directly with the general BSVIE (1). Indeed we should add in (30) the terms

$$\mathbb{E} \left[ \int_S^T \left( \int_t^T |\zeta(r, t)|^2 dr \right)^{p/2} dt + \int_S^T \left( \int_t^T \|\nu(r, t)\|_{\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2}^2 dr \right)^{p/2} dt \right].$$

However our assumptions do not imply that these integrals are finite.

**Step 2.** This part is the same as for  $p = 2$ . There exists a unique triple  $(Z, U, M)$  in  $\mathbb{L}^p(S, T; \mathbb{H}^p(R, S)) \times \mathbb{L}^p(S, T; \mathbb{L}_\pi^p(R, S)) \times \mathbb{L}^p(S, T; \mathbb{M}^{p, \perp}(R, S))$  such that for  $t \in [S, T]$ :

$$\mathbb{E}[Y(t)|\mathcal{F}_S] = \mathbb{E}[Y(t)|\mathcal{F}_R] + \int_R^S Z(t, s) dW_s + \int_R^S \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) + \int_R^S dM(t, s),$$

and

$$\begin{aligned} &\mathbb{E} \left[ \left( \int_R^S |Z(t, r)|^2 dr \right)^{p/2} + \left( \int_R^S \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right)^{p/2} + ([M(t, \cdot)]_{R, S})^{p/2} \right] \\ &\leq C \mathbb{E}|Y(t)|^p. \end{aligned}$$

Thus together with the first step, we have defined  $(Z, U, M)$  for  $(t, s) \in [S, T] \times [R, T]$  and

$$\begin{aligned} & \mathbb{E} \left[ \int_S^T \left( \int_R^T |Z(t, r)|^2 dr \right)^{p/2} dt + \int_S^T \left( \int_R^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right)^{p/2} dt \right. \\ & \quad \left. + \int_S^T ([M(t, \cdot)]_{R, T})^{p/2} dt \right] \\ & \leq C \mathbb{E} \left[ \int_S^T |\Phi(t)|^p dt + \int_S^T \left( \int_t^T |f^0(t, r)| dr \right)^p dt \right]. \end{aligned} \quad (31)$$

**Step 3.** We define

$$f^S(t, s, z, u) = f(t, s, Y(s), z, u), \quad (t, s, z, u) \in [R, S] \times [S, T] \times \mathbb{R}^k \times (\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2),$$

and from Lemma 1, the SFIE

$$\begin{aligned} \psi^S(t) &= \Phi(t) + \int_S^T f^S(t, s, Z(t, s), U(t, s)) ds - \int_S^T Z(t, s) dW_s \\ &\quad - \int_S^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_S^T dM(t, s). \end{aligned}$$

has a unique solution  $(\psi^S, Z, U, M)$  such that for  $t \in [R, S]$

$$\begin{aligned} & \mathbb{E} \left[ |\psi^S(t)|^p + \left( \int_S^T |Z(t, r)|^2 dr \right)^{p/2} \right. \\ & \quad \left. + ([M(t, \cdot)]_{S, T})^{p/2} + \left( \int_S^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \\ & \leq C \mathbb{E} \left[ |\Phi(t)|^p + \left( \int_S^T |f^S(t, r, 0, 0)| dr \right)^p \right] \\ & = C \mathbb{E} \left[ |\Phi(t)|^p + \left( \int_S^T |f(t, r, Y(r), 0, 0)| dr \right)^p \right] \\ & \leq C \mathbb{E} \left[ |\Phi(t)|^p + \left( \int_S^T |f^0(t, r)| dr \right)^p + \int_S^T |Y(r)|^p dr \right]. \end{aligned}$$

Hence using (30) we obtain:

$$\begin{aligned} & \mathbb{E} \left[ \int_R^S |\psi^S(t)|^p dt + \int_R^S \left( \int_S^T |Z(t, r)|^2 dr \right)^{p/2} dt + \int_R^S ([M(t, \cdot)]_{S, T})^{p/2} dt \right. \\ & \quad \left. + \int_R^S \left( \int_S^T \|U(t, r)\|_{\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2}^2 dr \right)^{p/2} dt \right] \\ & \leq C \mathbb{E} \left[ \int_R^S |\Phi(t)|^p dt + \int_R^S \left( \int_S^T |f^0(t, r)| dr \right)^p dt + \int_S^T |Y(r)|^p dr \right] \\ & \leq C \mathbb{E} \left[ \int_R^T |\Phi(t)|^p dt + \int_R^T \left( \int_t^T |f^0(t, r)| dr \right)^p dt \right]. \end{aligned} \quad (32)$$

Hence we have defined  $(Z, U, M)$  for  $(t, s) \in [R, S] \times [S, T]$ , and (25) holds:

$$\begin{aligned} \psi^S(t) &= \Phi(t) + \int_S^T f(t, s, Y(s), Z(t, s), U(t, s)) ds - \int_S^T Z(t, s) dW_s \\ &\quad - \int_S^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_S^T dM(t, s). \end{aligned} \quad (33)$$

As in the first step and for the same reason, the general driver of the BSVIE (1) can not be handled here.

**Step 4.** We can copy the fourth step of the case  $p = 2$ , but now the estimate becomes:

$$\begin{aligned} &\mathbb{E} \left[ \int_R^S |Y(t)|^p dt + \int_R^S \left( \int_R^S |Z(t, r)|^2 dr \right)^{p/2} dt + \int_R^S ([M(t, \cdot)]_{R, S})^{p/2} dt \right. \\ &\quad \left. + \int_R^S \left( \|U(t, \cdot)\|_{\mathbb{L}_{\tilde{\pi}}^2(R, S)}^2 \right)^{p/2} dt \right] \\ &\leq C \mathbb{E} \left[ \int_R^S |\Phi(t)|^p dt + \int_R^S \left( \int_t^S |f^0(t, r)| dr \right)^p dt \right] \\ &\leq C \mathbb{E} \left[ \int_R^T |\Phi(t)|^p dt + \int_R^T \left( \int_t^T |f^0(t, r)| dr \right)^p dt \right]. \end{aligned}$$

From this inequality together with (27) on  $[S, T]$ , (31) and (32), we proved that the BSVIE (1) has a unique adapted M-solution  $(Y, Z, U, M)$  in  $\mathcal{H}^p(R, T)$  on  $[R, T]$  with the estimate (27) on  $[R, T]$ .

**Step 5.** Rigorously the same. □

## 2.4 Stability result and some extensions

The stability result holds in our setting. Let  $\bar{\Phi} \in L_{\mathcal{F}_T}^2(0, T)$  and  $\bar{f} : \Omega \times [0, T] \times \mathbb{R}^{d+2(d \times k)} \times (\mathbb{L}_{\mu}^2)^2 \rightarrow \mathbb{R}^d$  satisfy **(H1)** and (18)

$$\mathbb{E} \int_0^T \left( \int_t^T |\bar{f}^0(t, s)| ds \right)^2 dt < +\infty.$$

Let  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{M})$  in  $\mathcal{H}^2(0, T)$  be the unique adapted M-solution of the BSVIE (1) with data  $\bar{\Phi}$  and  $\bar{f}$  (Theorem 1). Then for any  $S \in [0, T]$

$$\begin{aligned} &\mathbb{E} \left[ \int_S^T |Y(t) - \bar{Y}(t)|^2 dt + \int_S^T \left( \int_S^T |Z(t, r) - \bar{Z}(t, s)|^2 dr \right) dt \right. \\ &\quad \left. + \int_S^T ([M(t, \cdot) - \bar{M}(t, \cdot)]_{S, T}) dt + \int_S^T \left( \|U(t, \cdot) - \bar{U}(t, \cdot)\|_{\mathbb{L}_{\tilde{\pi}}^2(S, T)}^2 dr \right) dt \right] \\ &\leq C \mathbb{E} \left[ \int_S^T |\Phi(t) - \bar{\Phi}(t)|^2 dt \right. \\ &\quad \left. + \int_S^T \left( \int_t^T |f(t, r, Y(r), Z(t, r), Z(r, t), U(t, r), U(r, t)) \right. \right. \\ &\quad \left. \left. - \bar{f}(t, r, Y(r), Z(t, r), Z(r, t), U(t, r), U(r, t))| dr \right)^2 dt \right]. \end{aligned} \quad (34)$$

In the case of Theorem 2, we also have a similar estimate:

$$\begin{aligned}
& \mathbb{E} \left[ \int_S^T |Y(t) - \bar{Y}(t)|^p dt + \int_S^T \left( \int_S^T |Z(t,r) - \bar{Z}(t,s)|^2 dr \right)^{\frac{p}{2}} dt \right. \\
& \quad \left. + \int_S^T ([M(t,\cdot) - \bar{M}(t,\cdot)]_{S,T})^{\frac{p}{2}} dt + \int_S^T \left( \int_S^T \|U(t,r) - \bar{U}(t,r)\|_{\mathbb{L}_\mu^2}^2 dr \right)^{\frac{p}{2}} dt \right] \\
& \leq C \mathbb{E} \left[ \int_S^T |\Phi(t) - \bar{\Phi}(t)|^p dt \right. \\
& \quad \left. + \int_S^T \left( \int_t^T |f(t,r,Y(r),Z(t,r),Z(r,t),U(t,r),U(r,t)) \right. \right. \\
& \quad \quad \left. \left. - \bar{f}(t,r,Y(r),Z(t,r),Z(r,t),U(t,r),U(r,t))| dr \right)^p dt \right]. \quad (35)
\end{aligned}$$

In both cases, the proof is based on the same arguments given in [37] (see Equation (3.71) in particular) and we skip it here.

The next results are already proved in [37, Corollary 3.8 and Proposition 3.9]. We extend them to our setting (the proof is the same, thus is omitted here). Under the conditions of Theorems 1 or 2, if  $(Y, Z, U, M)$  is the unique M-solution of the BSVIE (1), then for all  $S \in [0, T]$

$$\begin{aligned}
\Phi^S(t) &= \Phi(t) + \int_S^T f(t,s,Y(s),Z(t,s),Z(s,t),U(t,s),U(s,t)) ds - \int_S^T Z(t,s) dW_s \\
&\quad - \int_S^T \int_{\mathcal{E}} U(t,s,e) \tilde{\pi}(de, ds) - \int_S^T dM(t,s).
\end{aligned}$$

is  $\mathcal{F}_S$ -measurable for almost all  $t \in [0, S]$ . Now for any  $t \in [0, T]$ , let  $(\lambda^t(\cdot), z^t(\cdot), u^t(\cdot), m^t(\cdot))$  be the adapted solution of the BSDE: for  $r \in [t, T]$

$$\begin{aligned}
\lambda^t(r) &= \Phi(t) + \int_r^T f(t,s,Y(s),z^t(s),Z(s,t),u^t(s),U(s,t)) ds - \int_r^T z^t(s) dW_s \\
&\quad - \int_r^T \int_{\mathcal{E}} u^t(s,e) \tilde{\pi}(de, ds) - \int_r^T dm^t(s).
\end{aligned}$$

Define  $\bar{Y}(t) = \lambda^t(t)$  for  $t \in [0, T]$  and  $\bar{Z}(t,s) = z^t(s)$ ,  $\bar{U}(t,s) = u^t(s)$  and  $\bar{M}(t,s) = m^t(s)$  for  $(t,s) \in \Delta^c[0, T]$ . For  $(t,s) \in \Delta[0, T]$ ,  $(\bar{Z}, \bar{U}, \bar{M})$  is defined through

$$\bar{Y}(t) = \mathbb{E}(\bar{Y}(t)) + \int_0^t \bar{Z}(t,s) dW_s + \int_0^t \int_{\mathcal{E}} U(t,s,e) \tilde{\pi}(de, ds) + \int_0^t dM(t,s).$$

Then  $\bar{Y} = Y$  and  $\bar{Z}(t,s) = Z(t,s)$ ,  $\bar{U}(t,s) = U(t,s)$  and  $\bar{M}(t,s) = M(t,s)$  for  $(t,s) \in \Delta^c[0, T]$ .

### 3 Time regularity

Up to now the component  $Y$  is only supposed to be in  $\mathbb{L}_{\mathbb{F}}^p(0, T)$  and thereby we don't know any regularity property for the paths  $t \mapsto Y(t)$ . If  $Y$  solves the BSDE (2), then it has



the same regularity as the martingale part, thus a.s. it is a càdlàg process. For a BSVIE it is more delicate. In [37, Theorem 4.2], the author shows that in the Brownian setting the BSVIE

$$Y(t) = \Phi(t) + \int_t^T f(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s) dW_s$$

is continuous in  $\mathbb{L}^2(\Omega)$ , which does not mean that  $Y$  has a.s. continuous paths. Of course since  $t$  appears in the generator  $f$  and in the free term  $\Phi$ , we have to add some property on  $t \mapsto \Phi(t)$  and  $t \mapsto f(t, \dots)$ .

Let us describe several sets for the process  $Y$ .

$$\begin{aligned} D([0, T]; \mathbb{L}_{\mathbb{F}}^p(\Omega)) &= \left\{ \phi \in \mathbb{L}^\infty(0, T; \mathbb{L}_{\mathbb{F}}^p(\Omega)), \phi(t) \text{ is } \mathbb{F} - \text{adapted}, \right. \\ &\quad \left. \phi(\cdot) \text{ is càdlàg from } [0, T] \text{ to } \mathbb{L}_{\mathbb{F}}^p(\Omega). \right\}, \\ D^\sharp([0, T]; \mathbb{L}_{\mathbb{F}}^p(\Omega)) &= \left\{ \phi \in D([0, T]; \mathbb{L}_{\mathbb{F}}^p(\Omega)), \phi(\cdot) \text{ is càdlàg paths a.s.} \right\}, \\ \mathbb{D}_{\mathbb{F}}^p(0, T) &= \mathbb{D}^p(0, T) = \left\{ \phi \in D^\sharp([0, T]; \mathbb{L}_{\mathbb{F}}^p(\Omega)), \mathbb{E} \left[ \sup_{t \in [0, T]} |\phi(t)|^p \right] < +\infty \right\}. \end{aligned}$$

When only measurability is required, we replace the subscript  $\mathbb{F}$  by  $\mathcal{F}_S$ . If we want to deal with continuity, then we replace  $D$  (resp.  $\mathbb{D}$ ) by  $C$  (resp.  $\mathbb{C}$ ) (see [37, Section 2.1]). Coming back to a generic martingale  $M(t, \cdot)$ , we also define

$$\mathbb{L}^p(\Omega; D([S, T]; \mathbb{M}^p(S, T)))$$

as the set of all  $M \in \mathbb{L}^\infty(S, T; \mathbb{M}^p(S, T))$  such that  $t \mapsto M(t, \cdot)$  is càdlàg from  $[S, T]$  to  $\mathbb{M}^p(0, T)$  and

$$\mathbb{E} \left( \sup_{t \in [S, T]} [M(t, \cdot)]_{S, T} \right)^{\frac{p}{2}} < +\infty.$$

Again if  $M(t, \cdot)$  is a Brownian martingale, then  $M \in \mathbb{L}^p(\Omega; D([S, T]; \mathbb{M}^p(S, T)))$  if and only if  $Z \in \mathbb{L}^p(\Omega; D([S, T]; \mathbb{H}^p(S, T)))$  and if  $N(t, \cdot)$  is a Poisson martingale, then  $N \in \mathbb{L}^p(\Omega; D([S, T]; \mathbb{M}^p(S, T)))$  is equivalent to  $\psi \in \mathbb{L}^p(\Omega; D([S, T]; \mathbb{L}_\pi^2(S, T)))$ .

In [37], to obtain the time regularity for the BSVIE (3), the author uses the Malliavin derivative to control the term  $Z(s, t)$  in the generator (see [37, Theorems 4.1 and 4.2]). Hence to apply the same arguments, we should use the Malliavin calculus in the presence of jumps (see e.g. [5, 9]). This point is left as future research and to avoid this machinery, let us study the BSVIE (4)

$$\begin{aligned} Y(t) &= \Phi(t) + \int_t^T f(t, s, Y(s), Z(t, s), U(t, s)) ds - \int_t^T Z(t, s) dW_s \\ &\quad - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM(t, s). \end{aligned}$$

Our condition **(H1)** becomes:

**(H2)** Let  $f : \Omega \times \Delta^c \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \times (\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2) \rightarrow \mathbb{R}^d$  be a measurable function such that  $s \mapsto f(t, s, y, z, \psi)$  is  $\mathbb{F}$ -adapted for all  $(t, y, z, \psi)$ ,  $(y, z, \psi) \mapsto f(t, s, y, z, \psi)$  is Lipschitz continuous, uniformly in  $(\omega, t, s)$  (see **(H1)**).

The integrability condition (26) is replaced by the stronger one:

$$\sup_{t \in [0, T]} \mathbb{E} \left[ \left( \int_t^T |f^0(t, s)| ds \right)^p \right] < +\infty. \quad (36)$$

Finally instead of  $\Phi \in \mathbb{L}_{\mathcal{F}_T}^p(0, T)$ , we assume also that

$$\sup_{t \in [0, T]} \mathbb{E} [|\Phi(t)|^p] < +\infty. \quad (37)$$

If  $(Y, Z, U, M)$  solves the BSVIE (4), then taking  $h(t, s, z, \psi) = f(t, s, Y(s), z, \psi)$  and using the estimate (16) of Lemma 2, we have:

$$\begin{aligned} & \mathbb{E} \left[ |Y(t)|^p + \left( \int_S^T |Z(t, r)|^2 dr \right)^{p/2} + ([M(t, \cdot)]_{S, T})^{p/2} \right. \\ & \left. + \left( \int_S^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \leq C \mathbb{E} \left[ |\Phi(t)|^p + \left( \int_S^T |h(t, r, 0, 0)| dr \right)^p \right]. \end{aligned}$$

Since  $f$  is Lipschitz continuous, taking  $S = t$ , the Gronwall inequality leads to

$$\begin{aligned} & \mathbb{E} \left[ |Y(t)|^p + \left( \int_t^T |Z(t, r)|^2 dr \right)^{p/2} + ([M(t, \cdot)]_{t, T})^{p/2} \right. \\ & \left. + \left( \int_t^T \int_{\mathcal{E}} |U(t, r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \\ & \leq C \mathbb{E} \left[ |\Phi(t)|^p + \left( \int_t^T |\Phi(r)| dr \right)^p + \left( \int_t^T |f^0(t, r)| dr \right)^p \right]. \end{aligned}$$

Under our stronger integrability conditions on  $\Phi$  and  $f^0$ , we obtain an stronger estimate on  $(Z, U, M)$ :

$$\sup_{t \in [0, T]} \mathbb{E} \left[ \left( \int_t^T |Z(t, r)|^2 dr \right)^{p/2} + ([M(t, \cdot)]_{t, T})^{p/2} + \left( \|U(t, \cdot)\|_{\mathbb{L}_\pi^2(t, T)}^2 \right)^{p/2} \right] < +\infty.$$

This property is important to get the càdlàg in mean property of  $Y$ .

**Lemma 3** Assume that **(H2)** holds. Then the solution of the BSVIE (4) satisfies: for any

$(t, t') \in [S, T]$  and if  $t_\star = t \wedge t'$  and  $t^\star = t \vee t'$ :

$$\begin{aligned}
& \mathbb{E} [|Y(t) - Y(t')|^p] \\
& + \mathbb{E} \left[ \left( \int_S^{t_\star} |Z(t, r) - Z(t', r)|^2 dr \right)^{p/2} + \left( \int_{t^\star}^T |Z(t, r) - Z(t', r)|^2 dr \right)^{p/2} \right] \\
& + \mathbb{E} \left[ \left( [M(t, \cdot) - M(t', \cdot)]_{S, t_\star} \right)^{p/2} + \left( [M(t, \cdot) - M(t', \cdot)]_{t^\star, T} \right)^{p/2} \right] \\
& + \mathbb{E} \left[ \left( \|U(t, \cdot) - U(t', \cdot)\|_{\mathbb{L}_\pi^2(S, t_\star)}^2 \right)^{p/2} + \left( \|U(t, \cdot) - U(t', \cdot)\|_{\mathbb{L}_\pi^2(t^\star, T)}^2 \right)^{p/2} \right] \\
& \leq C \mathbb{E} [|\Phi(t) - \Phi(t')|^p] + C \mathbb{E} \left[ \left( \int_{t_\star}^{t^\star} |h(t, r, Z(t, r), \psi(t, r))| dr \right)^p \right] \\
& + C \mathbb{E} \left[ \left( \int_{t_\star}^{t^\star} |Z(t, r)|^2 dr \right)^{p/2} \right] + C \mathbb{E} \left[ \left( \|U(t, \cdot)\|_{\mathbb{L}_\pi^2(t_\star, t^\star)}^2 \right)^{p/2} \right] \\
& + C \mathbb{E} \left[ \left( [M(t, \cdot)]_{t_\star, t^\star} \right)^{p/2} \right] \\
& + C \mathbb{E} \left[ \left( \int_{t^\star}^T |h(t, r, Z(t, r), \psi(t, r)) - h(t', r, Z(t, r), \psi(t, r))| dr \right)^p \right].
\end{aligned}$$

**Proof.** We first consider the BSVIE (15)

$$\begin{aligned}
Y(t) &= \Phi(t) + \int_t^T h(t, s, Z(t, s), U(t, s)) ds - \int_t^T Z(t, s) dW_s \\
&\quad - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM(t, s).
\end{aligned}$$

We take  $t, t'$  in  $[S, T]$  and w.l.o.g. let  $S \leq t \leq t' \leq T$ . Applying (12) to the solution of the BSDE with parameter  $t$ , we obtain:

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{r \in [t', T]} |\lambda(t, r) - \lambda(t', r)|^p + \left( \int_{t'}^T |z(t, r) - z(t', r)|^2 dr \right)^{p/2} \right. \\
& \quad \left. + \left( [m(t, \cdot) - m(t', \cdot)]_{t', T} \right)^{p/2} + \left( \int_{t'}^T \int_{\mathcal{E}} |u(t, r, e) - u(t', r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \\
& \leq C \mathbb{E} \left[ |\Phi(t) - \Phi(t')|^p + \left( \int_{t'}^T |h(t, r, z(t, r), u(t, r)) - h(t', r, z(t, r), u(t, r))| dr \right)^p \right] \\
& = C \mathbb{E} \left[ |\Phi(t) - \Phi(t')|^p + \left( \int_{t'}^T |h(t, r, Z(t, r), \psi(t, r)) - h(t', r, Z(t, r), \psi(t, r))| dr \right)^p \right]
\end{aligned}$$

Remark that

$$\begin{aligned}
\mathbb{E} [|Y(t) - Y(t')|^p] &= \mathbb{E} [|\lambda(t, t) - \lambda(t', t')|^p] \\
&\leq C\mathbb{E} [|\lambda(t, t) - \lambda(t, t')|^p] + C\mathbb{E} \left[ \sup_{r \in [t', T]} |\lambda(t, r) - \lambda(t', r)|^p \right] \\
&\leq C\mathbb{E} \left[ \left( \int_t^{t'} |h(t, r, Z(t, r), \psi(t, r))| dr \right)^p \right] + C\mathbb{E} \left[ \left( \int_t^{t'} |Z(t, r)|^2 dr \right)^{p/2} \right] \\
&\quad + C\mathbb{E} \left[ \left( \int_t^{t'} |\psi(t, r)|_{\mathbb{L}_\pi^2}^2 dr \right)^{p/2} \right] + C\mathbb{E} \left[ ([M(t, \cdot)]_{t, t'})^{p/2} \right] \\
&\quad + C\mathbb{E} \left[ \sup_{r \in [t', T]} |\lambda(t, r) - \lambda(t', r)|^p \right].
\end{aligned}$$

Moreover the notion of M-solution (Equation (5)) implies that

$$\begin{aligned}
Y(t) - Y(t') - \mathbb{E} [Y(t) - Y(t') | \mathcal{F}_S] &= \int_S^t (Z(t, r) - Z(t', r)) dW_r \\
&\quad + \int_S^t \int_{\mathcal{E}} (U(t, r, e) - U(t', r, e)) \tilde{\pi}(de, dr) + \int_S^t d(M(t, r) - M(t', r)) \\
&\quad + \int_t^{t'} Z(t', r) dW_r + \int_t^{t'} \int_{\mathcal{E}} U(t', r, e) \tilde{\pi}(de, dr) + \int_t^{t'} dM(t', r).
\end{aligned}$$

Using BDG's inequality, we get that

$$\begin{aligned}
&\mathbb{E} \left[ \left( \int_S^t |Z(t, r) - Z(t', r)|^2 dr \right)^{p/2} \right] + \mathbb{E} \left[ ([M(t, \cdot) - M(t', \cdot)]_{S, t})^{p/2} \right] \\
&\quad + \mathbb{E} \left[ \left( \int_S^t \int_{\mathcal{E}} |U(t, r, e) - U(t', r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \\
&\quad + \mathbb{E} \left[ \left( \int_t^{t'} |Z(t', r)|^2 dr \right)^{p/2} \right] + \mathbb{E} \left[ ([M(t', \cdot)]_{t, t'})^{p/2} \right] \\
&\quad + \mathbb{E} \left[ \left( \int_t^{t'} \int_{\mathcal{E}} |U(t', r, e)|^2 \pi(de, dr) \right)^{p/2} \right] \\
&\leq C\mathbb{E} [|Y(t) - Y(t') - \mathbb{E} [Y(t) - Y(t') | \mathcal{F}_S]|^p] \leq C\mathbb{E} [|Y(t) - Y(t')|^p].
\end{aligned}$$

Combining the previous inequalities, we obtain the desired result for the BSVIE (15). For the BSVIE (4), we apply the preceding arguments using the generator  $h(t, s, z, \psi) = f(t, s, Y(s), z, \psi)$ .  $\square$

From this lemma, it is possible to deduce that  $Y$  belongs to  $D([0, T]; \mathbb{L}_{\mathbb{R}}^p(\Omega))$ , provided that we have regularity assumption on  $t \mapsto \Phi(t)$  and  $t \mapsto f(t, s, y, z, \psi)$ , as in [37, Theorem 4.2] in the continuous setting. Note that the estimate on  $(Z, U, M)$  is crucial here. Let us emphasize again that it does not mean that  $Y$  is in  $D^\sharp([0, T]; \mathbb{L}_{\mathbb{R}}^p(\Omega))$ ; in other words we do not deduce that a.s. the paths are càdlàg.

In [35, Theorem 2.4], a.s. continuity of  $Y$  is proved in the Brownian setting and if the generator of the previous BSVIE does not depend on  $Z(s, t)$ , namely for the BSVIE:

$$Y(t) = \Phi(t) + \int_t^T f(t, s, Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW_s.$$

Our aim now is to extend this property for the BSVIE (4), assuming that  $\Phi$  and  $f$  are Hölder continuous w.r.t.  $t$ . Before we state the next result, which is the same as [35, Lemma 3.1]:

**Lemma 4** *Let us assume that for  $\Phi \in \mathbb{L}_{\mathcal{F}_T}^p(0, T)$ , for  $f = \{f(t, s), 0 \leq t \leq s \leq T\}$  such that*

$$\mathbb{E} \int_0^T \int_0^T |f(t, s)|^p \mathbf{1}_{s \geq t} ds < +\infty,$$

*and for some  $(Z, U, M)$  is in  $\mathfrak{M}^p(0, T)$ , we have for almost all  $t \in [0, T]$*

$$\begin{aligned} Y(t) &= \Phi(t) + \int_t^T f(t, s)ds + \int_t^T Z(t, s)dW_s + \int_t^T \int_{\mathcal{E}} U(t, s, e)\tilde{\pi}(de, ds) + \int_t^T dM(t, s) \\ &= \Phi(t) + \int_t^T f(t, s)ds + \left( M^\sharp(t, T) - M^\sharp(t, t) \right). \end{aligned}$$

*Then*

$$e^{\beta t} |Y(t)|^p \leq e^{\beta T} \mathbb{E}^{\mathcal{F}_t} |\Phi(t)|^p + \left( \frac{p-1}{\beta} \right)^{p-1} \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} |f(t, s)|^p ds.$$

*And for  $p = 2$  we also control  $(Z, U, M)$ , namely*

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} |Z(t, s)|^2 ds + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} d[M(t, \cdot)]_{0, s} + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} \int_{\mathcal{E}} |U(t, s, e)|^2 \pi(de, ds) \\ &\leq e^{\beta T} \mathbb{E}^{\mathcal{F}_t} |\Phi(t)|^2 + \frac{1}{\beta} \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} |f(t, s)|^2 ds. \end{aligned}$$

**Proof.** Fix one  $t \in [0, T]$  such that the equation is satisfied and define on  $[t, T]$

$$\begin{aligned} X_t(u) &= Y(t) - \int_t^u f(t, s)ds - (M^\sharp(t, u) - M^\sharp(t, t)) \\ &= \mathbb{E} \left[ Y(t) + \int_t^T f(t, s)ds \middle| \mathcal{F}_u \right] + \int_u^T f(t, s)ds. \end{aligned}$$

The process  $X_t = \{X_t(u), u \in [t, T]\}$  is a càdlàg semimartingale. And by Doob's martingale inequality, we have

$$\begin{aligned} \mathbb{E} \left[ \sup_{u \in [t, T]} |X_t(u)|^p \right] &\leq C \mathbb{E} \left[ |X_t(T)|^p + \left( \int_t^T |f(t, s)| ds \right)^p \right] \\ &\leq C \mathbb{E} \left[ |Y(t)|^p + \left( \int_t^T |f(t, s)| ds \right)^p \right]. \end{aligned}$$

First assume that  $p = 2$  (at least  $p \geq 2$ ). Using Itô's formula for  $u \mapsto |X_t(u)|^2 e^{\beta(u-t)}$ , on  $[t, T]$  we obtain that:

$$\begin{aligned} & |Y(t)|^2 + \int_t^T e^{\beta(s-t)} |Z(t, s)|^2 ds \\ & + \int_t^T e^{\beta(s-t)} \int_{\mathcal{E}} |U(t, s, e)|^2 \pi(de, ds) + \int_t^T e^{\beta(s-t)} d[M(t, \cdot)]_{0,s} \\ & \leq |X_t(T)|^2 e^{\beta(T-t)} + 2 \int_t^T e^{\beta(s-t)} X_t(s) f(t, s) ds - \beta \int_t^T |X_t(s)|^2 e^{\beta(s-t)} ds \\ & + 2 \int_t^T e^{\beta(s-t)} X_t(s) dM^\sharp(t, s). \end{aligned}$$

From our hypotheses and the control of  $u \mapsto X_t(u)$ , the martingale terms are true martingales. Thus taking the conditional expectation w.r.t.  $\mathcal{F}_t$  and using Young's inequality give the desired control for  $p = 2$ .

Now suppose only that  $p > 1$ . We apply again Itô's formula (cf. [26, Corollary 2.30] or [16, Corollary 1]), to  $e^{\beta(s-t)} |X_t(s)|^p$  on  $[t, T]$  and we obtain that for all  $u \in [t, T]$ :

$$\begin{aligned} |X_t(u)|^p e^{\beta(u-t)} & \geq |Y(t)|^p + \beta \int_t^u |X_t(s)|^p e^{\beta(s-t)} ds \\ & - p \int_t^u e^{\beta(s-t)} |X_t(s)|^{p-1} \check{X}_t(s) \mathbf{1}_{X_t(s) \neq 0} f(t, s) ds \\ & - p \int_t^u e^{\beta(s-t)} |X_t(s)|^{p-1} \check{X}_t(s) \mathbf{1}_{X_t(s) \neq 0} dM^\sharp(t, s) \\ & + \int_t^u e^{\beta(s-t)} \int_{\mathcal{E}} [|X_t(s-) + U_t(s, e)|^p - |X_t(s-)|^p - p|X_t(s-)|^{p-1} \check{X}_t(s-) \psi_t(s, e)] \pi(de, ds) \\ & + \sum_{t < s \leq u} e^{\beta(s-t)} [|X_t(s-) + \Delta M(t, s)|^p - |X_t(s-)|^p - p|X_t(s-)|^{p-1} \check{X}_t(s-) \Delta M(t, s)] \\ & + c(p) \int_t^u e^{\beta(s-t)} |X_t(s)|^{p-2} \mathbf{1}_{X_t(s) \neq 0} |Z(t, s)|^2 ds \\ & + c(p) \int_t^u e^{\beta(s-t)} |X_t(s)|^{p-2} \mathbf{1}_{X_t(s) \neq 0} d[M(t, \cdot)]_{0,s}^c. \end{aligned}$$

with  $c(p) = (p/2)((p-1) \wedge 1)$  and  $\check{x} = |x|^{-1} x \mathbf{1}_{x \neq 0}$ . Thus for  $u = T$ , taking the conditional expectation w.r.t.  $\mathcal{F}_t$  and using Young's inequality lead to the control on  $Y$ . Note that it seems to be much more difficult to get something similar for  $(Z, U, M)$  if  $p \neq 2$ . This achieves the proof of the Lemma.  $\square$

Now we have the next regularity result, which is the extension of [35, Theorem 2.4].

**Theorem 3** *Suppose that the generator satisfies: for some  $\alpha \in (0, 1/2]$  and  $K \geq 0$ , for all  $(y, z, \psi)$  and all  $0 \leq t, t' \leq s \leq T$*

$$|f(t, s, y, z, \psi) - f(t', s, y, z, \psi)| \leq K|t - t'|^\alpha, \quad \mathbb{E} [|\Phi(t) - \Phi(t')|^\varrho] \leq K|t - t'|^{\alpha\varrho}.$$

Moreover for some  $1/\alpha < \varrho$ ,

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_t^T |f^0(t, s)|^2 ds \right|^{\varrho/2} \right] < +\infty.$$

Then the solution  $Y$  of the BSVIE (4) has a càdlàg version, still denoted by  $Y$ .

**Proof.** We follow the scheme of the proof of [35, Theorem 2.4].

*Step 1.* We consider for a fixed  $t$  in  $[0, T]$ :

$$X_t(u) = \mathbb{E} \left[ \Phi(t) + \int_t^T f(t, s) ds \middle| \mathcal{F}_u \right], \quad u \in [0, T].$$

From our assumption,  $u \mapsto X_t(u)$  is a càdlàg  $L^p$ -martingale. For  $0 \leq t \leq t' \leq T$ , Doob's martingale inequality implies for any  $1 < \varrho \leq 2$

$$\begin{aligned} \mathbb{E} \left[ \sup_{u \in [0, T]} |X_t(u) - X_{t'}(u)|^\varrho \right] &\leq C \mathbb{E} [|X_t(T) - X_{t'}(T)|^\varrho] \\ &\leq C \mathbb{E} [|\Phi(t) - \Phi(t')|^\varrho] \\ &\quad + C \mathbb{E} \left[ \left| \int_t^{t'} f(t, s) ds \right|^\varrho + \left| \int_{t'}^T |f(t, s) - f(t', s)| ds \right|^\varrho \right]. \end{aligned}$$

Hölder's inequality leads to:

$$\mathbb{E} \left[ \left| \int_t^{t'} f(t, s) ds \right|^\varrho \right] \leq |t' - t|^{\varrho/2} \mathbb{E} \left[ \left| \int_t^{t'} |f(t, s)|^2 ds \right|^{\varrho/2} \right].$$

Hence from our setting we have:

$$\mathbb{E} \left[ \sup_{u \in [0, T]} |X_t(u) - X_{t'}(u)|^\varrho \right] \leq C |t - t'|^{\alpha \varrho}.$$

Since  $\alpha \varrho > 1$ , if we consider  $X = (X_t, t \in [0, T])$  as a process with values in the Skorohod space  $D([0, T]; \mathbb{R}^d)$  equipped with the uniform norm, which is a complete metric space, then we can apply the Kolmogorov continuity criterion (see [29, IV. Corollary 1] or [31, Theorem I.2.1]): there is a continuity version of  $t \in [0, T] \mapsto X_t \in D([0, T]; \mathbb{R}^d)$ . In particular a.s.  $t \mapsto Y(t) := X_t(t)$  is càdlàg:

$$\begin{aligned} |Y(t) - Y(t')| &\leq |X_t(t) - X_t(t')| + |X_t(t') - X_{t'}(t')| \\ &\leq |X_t(t) - X_t(t')| + \sup_{s \in [0, T]} |X_t(s) - X_{t'}(s)|. \end{aligned}$$

Note that

$$\begin{aligned} X_t(u) &= X_t(0) + \int_0^u Z(t, s) dW_s + \int_0^u \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) + M(t, u) \\ &= X_t(0) + M^\sharp(t, u). \end{aligned}$$

Using the BDG inequality, we have

$$\begin{aligned} \mathbb{E} \left[ \left( \left[ M^\sharp(t, \cdot) - M^\sharp(t', \cdot) \right]_{0, T} \right)^{\varrho/2} \right] &\leq C \mathbb{E} \left[ \left| M^\sharp(t, T) - M^\sharp(t', T) \right|^\varrho \right] \\ &\leq C (\mathbb{E} [|X_t(0) - X_{t'}(0)|^\varrho] + \mathbb{E} [|X_t(T) - X_{t'}(T)|^\varrho]) \\ &\leq C |t - t'|^{\alpha \varrho}. \end{aligned}$$

And

$$\begin{aligned} \mathbb{E} \left[ \left( \left[ M^\sharp(0, \cdot) \right]_{0,T} \right)^{\varrho/2} \right] &\leq C \mathbb{E} \left[ \left| M^\sharp(0, T) \right|^\varrho \right] \leq C \mathbb{E} \left[ |X_0(T) - X_0(0)|^\varrho \right] \\ &\leq C \mathbb{E} \left( |\Phi(0)|^\varrho + \left( \int_0^T |f(0, s)| ds \right)^\varrho \right) \leq C. \end{aligned}$$

Let us recall that the space  $\mathbb{M}^2(0, T)$  is a Banach space (see [7], Section VII.3 (98.1)-(98.2) or [29], Section V.2). If we consider  $t \mapsto M^\sharp(t, \cdot)$ , this map defined on  $[0, T]$  takes values in the space  $\mathbb{M}^2(0, T)$ . We can apply the Kolmogorov continuity criterion (see again [31, Theorem I.2.1]) in order to have:

$$\mathbb{E} \left[ \left( \sup_{t \in [0, T]} \left[ M^\sharp(t, \cdot) \right]_{0,T} \right)^{\varrho/2} \right] \leq C. \quad (38)$$

*Step 2.* Assume that  $f$  depends only on  $z$  and  $\psi$ . Let us define  $Z_0(t, s) \equiv 0$ ,  $U_0(t, s) \equiv 0$  and recursively for  $n \geq 1$ :

$$\begin{aligned} Y_n(t) &= \Phi(t) + \int_t^T f(t, s, Z_{n-1}(t, s), U_{n-1}(t, s)) ds - \int_t^T Z_n(t, s) dW_s \\ &\quad - \int_t^T \int_{\mathcal{E}} U_n(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM_n(t, s). \end{aligned}$$

We can argue exactly as in [35] in order to prove that for any  $n \geq 1$ :

$$\begin{aligned} \mathbb{E} \left[ \left( \left[ M_n^\sharp(t, \cdot) - M_n^\sharp(t', \cdot) \right]_{0,T} \right)^{\varrho/2} \right] &\leq C_n |t - t'|^{\alpha \varrho}, \\ \mathbb{E} \left[ \left( \sup_{t \in [0, T]} \left[ M_n^\sharp(t, \cdot) \right]_{0,T} \right)^{\varrho/2} \right] &\leq C_n, \\ t \mapsto Y_n(t) &\text{ is càdlàg.} \end{aligned}$$

Let us now prove the convergence of  $Y_n$ . Using Lemma 4 with  $p = 2$ , we obtain

$$\begin{aligned} &e^{\beta t} |Y_{n+1}(t) - Y_n(t)|^2 + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} |Z_{n+1}(t, s) - Z_n(t, s)|^2 ds \\ &\quad + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} \|U_{n+1}(t, s, \cdot) - U_n(t, s, \cdot)\|_{\mathbb{L}_\mu^2}^2 ds \\ &\quad + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} d[M_{n+1}(t, \cdot) - M_n(t, \cdot)]_{0,s} \\ &\leq \frac{1}{\beta} \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} |f(t, s, Z_n(t, s), U_n(t, s)) - f(t, s, Z_{n-1}(t, s), U_{n-1}(t, s))|^2 ds \\ &\leq \frac{2K^2}{\beta} \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} \left( |Z_n(t, s) - Z_{n-1}(t, s)|^2 + \|U_n(t, s, \cdot) - U_{n-1}(t, s, \cdot)\|_{\mathbb{L}_\mu^2}^2 \right) ds. \end{aligned}$$



Taking  $\beta$  equal to  $4K^2$  and iterating the previous inequality, we have

$$\begin{aligned}
& e^{\beta t} |Y_{n+1}(t) - Y_n(t)|^2 + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} |Z_{n+1}(t, s) - Z_n(t, s)|^2 ds \\
& + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} \|U_{n+1}(t, s, \cdot) - U_n(t, s, \cdot)\|_{\mathbb{L}_\mu^2}^2 ds \\
& + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} d[M_{n+1}(t, \cdot) - M_n(t, \cdot)]_{0,s} \\
& \leq \frac{1}{2^n} \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} \left( |Z_1(t, s) - Z_0(t, s)|^2 + \|U_1(t, s, \cdot) - U_0(t, s, \cdot)\|_{\mathbb{L}_\mu^2}^2 \right) ds.
\end{aligned}$$

First taking the expectation and integrating w.r.t  $t \in [0, T]$ , we deduce the convergence of  $(Z_n, U_n, M_n)$  in  $\mathfrak{M}^2(0, T)$ . Then

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_{n+1}(t) - Y_n(t)|^\varrho \right] \\
& \leq \frac{1}{2^{n\varrho/2}} \mathbb{E} \left[ \sup_{t \in [0, T]} \left( \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} \left( |Z_1(t, s)|^2 + \|U_1(t, s, \cdot)\|_{\mathbb{L}_\mu^2}^2 ds \right) \right)^{\varrho/2} \right] \\
& \leq \frac{1}{2^{n\varrho/2}} \mathbb{E} \left[ \sup_{t \in [0, T]} (\mathbb{E}^{\mathcal{F}_t} \xi)^{\varrho/2} \right]
\end{aligned}$$

where

$$\xi = \sup_{t \in [0, T]} \int_t^T e^{\beta s} \left( |Z_1(t, s)|^2 + \|U_1(t, s, \cdot)\|_{\mathbb{L}_\mu^2}^2 \right) ds.$$

From (38),  $\mathbb{E}(\xi^{\varrho/2}) < +\infty$  and  $t \mapsto \mathbb{E}^{\mathcal{F}_t}(\xi)$  is a martingale. By Doob's maximal inequality

$$\begin{aligned}
\mathbb{E} \left[ \sup_{t \in [0, T]} |Y_{n+1}(t) - Y_n(t)|^\varrho \right] & \leq \frac{1}{2^{n\varrho/2}} \mathbb{E} \left[ \sup_{t \in [0, T]} (\mathbb{E}^{\mathcal{F}_t} \xi)^{\varrho/2} \right] \\
& \leq C \frac{e^{\beta\varrho T}}{2^{n\varrho/2}} \mathbb{E} \left[ (\xi)^{\varrho/2} \right] \leq \frac{C}{2^{n\varrho/2}}
\end{aligned}$$

where the constant  $C$  does not depend on  $n$ . Thus there exists a càdlàg adapted process  $Y$  such that

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_{n+1}(t) - Y(t)|^\varrho \right] = 0.$$

And we deduce immediately that the limit is the unique solution of the BSVIE

$$\begin{aligned}
Y(t) & = \Phi(t) + \int_t^T f(t, s, Z(t, s), U(t, s)) ds - \int_t^T Z(t, s) dW_s \\
& \quad - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM(t, s).
\end{aligned}$$

*Step 3.* Assume that  $f$  depends now also on  $y$ . Let us define  $Y_0(t) \equiv 0$  and for  $n \geq 1$ :

$$\begin{aligned}
Y_n(t) & = \Phi(t) + \int_t^T f(t, s, Y_{n-1}(s), Z_n(t, s), U_n(t, s)) ds - \int_t^T Z_n(t, s) dW_s \\
& \quad - \int_t^T \int_{\mathcal{E}} U_n(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM_n(t, s).
\end{aligned}$$

We know that  $t \mapsto Y_n(t)$  is càdlàg. Using again Lemma 4 with  $p = 2$ , we obtain:

$$\begin{aligned} & e^{\beta t} |Y_n(t)|^2 + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} |Z_n(t, s)|^2 ds + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} \|U_n(t, s, \cdot)\|_{\mathbb{L}_\mu^2}^2 ds \\ & + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} d[M_n(t, \cdot)]_{0, s} \leq e^{\beta T} \mathbb{E}^{\mathcal{F}_t} |\Phi(t)|^2 \\ & + \frac{4K^2}{\beta} \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} \left( |f^0(t, s)|^2 + |Y_{n-1}(s)|^2 + |Z_n(t, s)|^2 + \|U_n(t, s, \cdot)\|_{\mathbb{L}_\mu^2}^2 \right) ds. \end{aligned}$$

Thus for  $\beta = 8K^2$ , we have:

$$\begin{aligned} & e^{\beta t} |Y_n(t)|^2 + \frac{1}{2} \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} \left( |Z_n(t, s)|^2 + \|U_n(t, s, \cdot)\|_{\mathbb{L}_\mu^2}^2 \right) ds \\ & + \frac{1}{2} \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} d[M_n(t, \cdot)]_{0, s} \\ & \leq e^{\beta T} \mathbb{E}^{\mathcal{F}_t} |\Phi(t)|^2 + \frac{1}{2} \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} \left( |f^0(t, s)|^2 + |Y_{n-1}(s)|^2 \right) ds. \end{aligned}$$

Set

$$h_n(t) = \sup_{1 \leq k \leq n} \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_k(s)|^\varrho \right].$$

Then

$$\begin{aligned} h_n(t) & \leq \sup_{1 \leq k \leq n} \mathbb{E} \left[ \left( \sup_{s \in [t, T]} e^{\beta s} |Y_k(s)|^2 \right)^{\varrho/2} \right] \\ & \leq C \mathbb{E} \left[ \left( \sup_{s \in [t, T]} \mathbb{E}^{\mathcal{F}_s} |\Phi(s)|^2 \right)^{\varrho/2} \right] + C \mathbb{E} \left[ \left( \sup_{s \in [t, T]} \mathbb{E}^{\mathcal{F}_s} \int_s^T e^{\beta u} |f^0(s, u)|^2 du \right)^{\varrho/2} \right] \\ & + C \sup_{1 \leq k \leq n} \mathbb{E} \left[ \left( \sup_{s \in [t, T]} \mathbb{E}^{\mathcal{F}_s} \int_s^T e^{\beta u} |Y_{k-1}(u)|^2 du \right)^{\varrho/2} \right] \\ & \leq C \mathbb{E} \left[ \left( \sup_{s \in [t, T]} \mathbb{E}^{\mathcal{F}_s} \sup_{r \in [0, T]} |\Phi(r)|^2 \right)^{\varrho/2} \right] \\ & + C \mathbb{E} \left[ \left( \sup_{s \in [t, T]} \mathbb{E}^{\mathcal{F}_s} \sup_{r \in [t, T]} \int_r^T e^{\beta u} |f^0(r, u)|^2 du \right)^{\varrho/2} \right] \\ & + C \sup_{1 \leq k \leq n} \mathbb{E} \left[ \left( \sup_{s \in [t, T]} \mathbb{E}^{\mathcal{F}_s} \int_t^T e^{\beta u} |Y_{k-1}(u)|^2 du \right)^{\varrho/2} \right]. \end{aligned}$$

By Doob's maximal inequality

$$\begin{aligned} h_n(t) & \leq C \mathbb{E} \left( \sup_{s \in [t, T]} |\Phi(s)|^\varrho \right) + C \mathbb{E} \left[ \left( \sup_{s \in [t, T]} \int_s^T |f^0(s, u)|^2 du \right)^{\varrho/2} \right] \\ & + C \sup_{1 \leq k \leq n} \mathbb{E} \left[ \left( \int_t^T |Y_{k-1}(u)|^2 du \right)^{\varrho/2} \right]. \end{aligned}$$

Since  $\rho > 2$ , by Jensen's inequality,

$$h_n(t) \leq C + C \int_t^T |h_n(u)| du.$$

Gronwall's inequality leads to

$$\sup_{1 \leq k \leq n} \mathbb{E} \left[ \sup_{s \in [0, T]} |Y_k(s)|^\rho \right] \leq C$$

for any  $n$ , that is

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{s \in [0, T]} |Y_k(s)|^\rho \right] \leq C.$$

By Lemma 4, we also have for almost all  $t \in [0, T]$

$$e^{\beta t} |Y_n(t) - Y_m(t)|^2 \leq C \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s} |Y_{n-1}(s) - Y_{m-1}(s)|^2 ds.$$

Define

$$h(t) = \limsup_{m, n \rightarrow +\infty} \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_n(s) - Y_m(s)|^2 \right].$$

Arguing as above, with Fatou's lemma and the preceding uniform (in  $n$  and  $s$ ) estimate, we have

$$h(t) \leq C \int_t^T h(s) ds \implies h(t) = 0.$$

Hence there is a càdlàg adapted process  $Y$  such that

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[ \sup_{s \in [0, T]} |Y_n(s) - Y(s)|^2 \right] = 0.$$

And from the above estimate, we have

$$\mathbb{E} \left[ \sup_{s \in [0, T]} |Y(s)|^\rho \right] \leq \infty.$$

This achieves the proof of this theorem.  $\square$

Let us emphasize that our conditions imply that a.s.  $t \mapsto \Phi(t)$  is continuous. Straightforward adaptations of the steps 1 and 3 (and the second part of Lemma 4) show that it is possible to extend this point to the  $L^p$ -setting if the generator  $f$  depends only on  $(t, s, y)$ , but not on  $z$  nor  $u$ . We only need to suppose that  $\alpha \in (0, 1/q]$  where  $q$  is the Hölder conjugate of  $p$ .

## 4 Duality and comparison principles for BSVIE and linear BSVIE.

In this section we study several properties of the solution of a BSVIE.

## 4.1 A duality result

The *duality principle* of linear stochastic integral equations ([36, Section 4]) plays an important role for comparison principle or optimal control problem (see [37, Section 5]). This result is based on the notion of FSVIE (see among many others [3, 4, 13, 14, 25, 28]). In [36, 37], the next FSVIE is considered: for  $t \in [0, T]$

$$X(t) = \Psi(t) + \int_0^t A_0(t, s)X(s)ds + \int_0^t \sum_{i=1}^k A_i(t, s)X(s)dW_i(s),$$

where  $A_i(\cdot, \cdot) \in \mathbb{L}^\infty([0, T]; \mathbb{L}_{\mathbb{F}}^\infty(0, T; \mathbb{R}^{d \times d}))$  for  $i = 0, 1, \dots, k$ . It means that  $A_i : \Omega \times [0, T]^2 \rightarrow \mathbb{R}^{d \times d}$  is bounded,  $\mathcal{F}_T \otimes \mathcal{B}([0, T]^2)$ -measurable and for almost all  $t \in [0, T]$ ,  $A_i(t, \cdot)$  is  $\mathcal{F}$ -adapted. Then for any  $\Psi \in \mathbb{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d)$ , there exists a unique solution  $X$  in  $\mathbb{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d)$ .

Here we consider the extension: for  $t \in [0, T]$

$$\begin{aligned} X(t) &= \Psi(t) + \int_0^t A_0(t, s)X(s)ds + \int_0^t \sum_{i=1}^k A_i(t, s)X(s-)dW_i(s) \\ &+ \int_0^t \int_{\mathcal{E}} B(t, s, e)X(s-)\tilde{\pi}(de, ds). \end{aligned} \quad (39)$$

We keep the same conditions on the  $A_i$ . We assume that  $B : \Omega \times [0, T]^2 \times \mathcal{E} \rightarrow \mathbb{R}^d$  is bounded and such that for almost all  $t \in [0, T]$ ,  $B(t, \cdot, \cdot)$  is in  $G_{loc}(\mu)$ . Since we are interesting by càdlàg processes  $X$ , we use the setting of [28, Condition 4.1]. Hence we also suppose that  $A_i$  and  $B$  are differentiable w.r.t.  $t$  with a bounded derivative (uniformly in  $(\omega, t, s)$ ). Thus we can apply [28, Theorem 4.3]: if  $\Psi$  is a càdlàg process, then there exists a unique càdlàg solution  $X$  of the preceding FSVIE. The key point is that  $X$  is a càdlàg process, hence for a.e.  $t \in [0, T]$ ,  $X(t) = X(t-)$ .

**Lemma 5** *Let  $\Psi(\cdot) \in \mathbb{L}_{\mathbb{F}}^2(0, T; \mathbb{R}^d) \cap \mathbb{D}^2(0, T)$  and  $\Phi(\cdot) \in \mathbb{L}^2((0, T) \times \Omega; \mathbb{R}^d)$ . Let  $X \in \mathbb{L}^2(0, T; \mathbb{R}^d)$  be the càdlàg solution of the linear FSVIE (39). We also consider the BSVIE:*

$$\begin{aligned} Y(t) &= \Phi(t) - \int_t^T Z(t, s)dW_s - \int_t^T \int_{\mathcal{E}} U(t, s, e)\tilde{\pi}(de, ds) - \int_t^T dM(t, s) \\ &+ \int_t^T \left[ A_0(s, t)^\top Y(s) + \sum_{i=1}^k A_i(s, t)^\top Z_i(s, t) + \int_{\mathcal{E}} B(s, t, e)^\top U(s, t, e)\mu(de) \right] ds. \end{aligned}$$

Then

$$\mathbb{E} \int_0^T \langle \Psi(t), Y(t) \rangle dt = \mathbb{E} \int_0^T \langle X(t), \Phi(t) \rangle dt.$$

**Proof.** The arguments are the same as [23, 37].

$$\begin{aligned}
& \mathbb{E} \int_0^T \langle \Psi(t), Y(t) \rangle dt \\
&= \mathbb{E} \int_0^T \langle X(t), Y(t) \rangle dt - \mathbb{E} \int_0^T \int_0^t \langle A_0(t, s) X(s), Y(t) \rangle ds dt \\
&\quad - \mathbb{E} \int_0^T \int_0^t \sum_{i=1}^k \langle A_i(t, s) X(s-), Y(t) \rangle dW_i(s) dt \\
&\quad - \mathbb{E} \int_0^T \int_0^t \int_{\mathcal{E}} \langle B(t, s, e) X(s-), Y(t) \rangle \tilde{\pi}(de, ds) dt.
\end{aligned}$$

Now by Fubini's theorem

$$\mathbb{E} \int_0^T \int_0^t \langle A_0(t, s) X(s), Y(t) \rangle ds dt = \mathbb{E} \int_0^T \left\langle X(t), \int_t^T A_0(s, t)^\top Y(s) ds \right\rangle dt.$$

Since  $(Y, Z, U, M)$  is a M-solution,

$$Y(t) = \mathbb{E}(Y(t)) + \int_0^t Z(t, s) dW_s + \int_0^t \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) + \int_0^t dM(t, s).$$

Let us plug this into the Brownian integral and use the orthogonality of  $W$ ,  $\tilde{\pi}$  and  $M$ :

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_0^t \sum_{i=1}^k \langle A_i(t, s) X(s-), Y(t) \rangle dW_i(s) dt \\
&= \mathbb{E} \int_0^T \int_0^t \sum_{i=1}^k \langle A_i(t, s) X(s-), Z_i(t, s) \rangle ds dt \\
&= \mathbb{E} \int_0^T \left\langle X(t-), \int_t^T \sum_{i=1}^k A_i(s, t)^\top Z_i(s, t) ds \right\rangle dt.
\end{aligned}$$

For the Poisson integral we obtain:

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_0^t \int_{\mathcal{E}} \langle B(t, s, e) X(s-), Y(t) \rangle \tilde{\pi}(de, ds) dt \\
&= \mathbb{E} \int_0^T \left\langle X(t-), \int_t^T \int_{\mathcal{E}} B(s, t, e)^\top U(s, t, e) \mu(de) ds \right\rangle dt.
\end{aligned}$$

Recall that  $X(t) = X(t-)$  for almost every  $t \in [0, T]$ . Then we have

$$\begin{aligned}
& \mathbb{E} \int_0^T \langle \Psi(t), Y(t) \rangle dt = \mathbb{E} \int_0^T \left\langle X(t), \left[ \Phi(t) - \int_t^T dM^\sharp(t, s) \right] \right\rangle dt \\
&= \mathbb{E} \int_0^T \langle X(t), \Phi(t) \rangle dt
\end{aligned}$$

and the desired result is proved.  $\square$

Let us emphasize that the role of the càdlàg property of  $X$  is important here. Thus it should be possible to relax the regularity assumption on the coefficients  $A_i$  or  $B$  of the

FSVIE. But as for a BSVIE, the regularity of the paths of  $X$  is not a direct property nor an easy stuff.

Let us finish this part with some formula for linear BSVIE in dimension one. Contrary to the linear BSDE, we do not have in general an explicit form for the first component of the solution. Let us consider a particular case where the generator is of the form:

$$f(t, s, y, z, \psi) = g(t, s, y) + h(s)z + \int_{\mathcal{E}} \psi(e)\kappa(s, e)\mu(de), \quad (40)$$

Thus we consider the BSVIE:

$$\begin{aligned} Y(t) &= \Phi(t) + \int_t^T \left[ g(t, s, Y(s)) + h(s)Z(t, s) + \int_{\mathcal{E}} \kappa(s, e)U(t, s)\mu(de) \right] ds \\ &\quad - \int_t^T Z(t, s)dW_s - \int_t^T \int_{\mathcal{E}} U(t, s, e)\tilde{\pi}(de, ds) - \int_t^T dM(t, s). \end{aligned} \quad (41)$$

**Lemma 6** *Assume that  $h$  is a bounded process and that there exists a constant  $C_0$  such that  $\mathbb{P} \otimes \text{Leb} \otimes \mu$ -a.e.,  $-1 < C_0 \leq \kappa(t, e)$  and  $|\kappa(t, e)| \leq \vartheta(e)$  where  $\vartheta$  belongs to the space  $\mathbb{L}_{\mu}^{\infty} \cap \mathbb{L}_{\mu}^2$ . We also suppose that  $f$  satisfies **(H1)** and that the required integrability conditions on  $\Phi$  and  $g(\cdot, \cdot, 0)$  hold. Then there exists a probability  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  such that a.s. for any  $t \in [0, T]$*

$$Y(t) = \mathbb{E}^{\mathbb{Q}} \left[ \Phi(t) + \int_t^T g(t, s, Y(s))ds \middle| \mathcal{F}_t \right].$$

**Proof.** We know that the solution  $Y$  can be represented with the formula  $Y(t) = \lambda(t, t)$  where  $\lambda$  is the first part of the solution of the parametrized BSDE (10): for a fixed  $t \in [0, T]$  and  $t \leq r \leq T$

$$\begin{aligned} \lambda(t, r) &= \Phi(t) + \int_r^T [g(t, s, Y(s)) + h(s, 0, z(t, s), u(t, s))] ds \\ &\quad - \int_r^T z(t, s)dW_s - \int_r^T \int_{\mathcal{E}} u(t, s, e)\tilde{\pi}(de, ds) - \int_r^T dm(t, s) \\ &= \Phi(t) + \int_r^T g(t, s, Y(s))ds + \int_r^T \beta(t, s)z(t, s)ds \\ &\quad + \int_r^T h(s, 0, 0, u(t, s))ds \\ &\quad - \int_r^T z(t, s)dW_s - \int_r^T \int_{\mathcal{E}} u(t, s, e)\tilde{\pi}(de, ds) - \int_r^T dm(t, s). \end{aligned}$$

Let us write this BSDE in a different way:

$$\begin{aligned} \lambda(t, r) &= \Phi(t) + \int_r^T [g(t, s, Y(s))] ds \\ &\quad + \int_r^T h(s)z(t, s)ds - \int_r^T z(t, s)dW_s \\ &\quad + \int_r^T \int_{\mathcal{E}} \kappa_s(e)u(t, s)\mu(de)ds - \int_r^T \int_{\mathcal{E}} u(t, s, e)\tilde{\pi}(de, ds) - \int_r^T dm(t, s). \end{aligned}$$

If we consider the martingale

$$N(r) = \int_0^r h(s) dW_s + \int_0^r \int_{\mathcal{E}} \kappa(s, e) \tilde{\pi}(de, ds)$$

and if we denote by  $E$  the solution of the linear SDE:

$$dE_r = E_{r-} \left[ h(r) dW_r + \int_{\mathcal{E}} \kappa(r, e) \tilde{\pi}(de, dr) \right] = E_{r-} dN(r),$$

then  $E$  is the Doléans-Dade exponential of  $N$ . From our assumption, if we define  $\mathbb{Q} = E_T \mathbb{P}$ ,  $\mathbb{Q}$  is a probability measure equivalent to  $\mathbb{P}$ . Moreover the Girsanov theorem implies that

$$\begin{aligned} & \int_r^T h(s) z(t, s) ds - \int_r^T z(t, s) dW_s \\ & + \int_r^T \int_{\mathcal{E}} \kappa(s, e) u(t, s) \mu(de) ds - \int_r^T \int_{\mathcal{E}} u(t, s, e) \tilde{\pi}(de, ds) - \int_r^T dm(t, s) \end{aligned}$$

is a  $\mathbb{Q}$ -martingale. Taking conditional expectation w.r.t.  $\mathcal{F}_r$  under probability measure  $\mathbb{Q}$  leads to: for any  $r \in [t, T]$

$$\lambda(t, r) = \mathbb{E}^{\mathbb{Q}} \left[ \Phi(t) + \int_r^T g(t, s, Y(s)) ds \middle| \mathcal{F}_r \right].$$

In particular for  $r = t$ , we get the desired result.  $\square$

Let us emphasize that the probability measure  $\mathbb{Q}$  depends only on  $h$  and  $\kappa$ . If we consider a more general linear BSVIE, then  $\mathbb{Q}$  may depend on  $t \in [0, T]$  and thus we cannot obtain a similar result with these arguments.

## 4.2 Comparison principle

In this section, the dimension  $d$  is equal to one. Let us remark that the result of [34, Proposition 3.3] still holds in our setting since it is based on the comparison principle for BSDEs and this property has been proved in [16, 18]. In other words if we consider the BSVIE (15) where the generator  $f$  does not depend on  $y$ ,  $\zeta$  and  $\theta$ :

$$\begin{aligned} Y(t) &= \Phi(t) + \int_t^T f(t, s, Z(t, s), U(t, s)) ds - \int_t^T Z(t, s) dW_s \\ &\quad - \int_t^T \int_{\mathcal{E}} U(t, s, e) \tilde{\pi}(de, ds) - \int_t^T dM(t, s). \end{aligned}$$

the comparison principle holds.

**Proposition 2** For  $i = 0, 1$ , let  $f^i : \Omega \times \Delta^c \times \mathbb{R}^k \times (\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2) \rightarrow \mathbb{R}$  satisfy **(H1)**. Moreover

$$f^0(t, s, z, \psi) \leq f^1(t, s, z, \psi), \quad \forall (t, z, \psi) \in [0, s] \times \mathbb{R}^k \times (\mathbb{L}_\mu^1 + \mathbb{L}_\mu^2), \quad a.s., a.e. s \in [0, T].$$

Then for any  $\Phi^i \in \mathbb{L}_{\mathcal{F}_T}^p(0, T)$  with  $\Phi^0(t) \leq \Phi^1(t)$  a.s.,  $t \in [0, T]$ , the solutions  $(Y^i, Z^i, \psi^i, M^i) \in \mathcal{H}^p[0, T]$  of (15) verify

$$Y^0(t) \leq Y^1(t), \quad a.s., t \in [0, T].$$

**Proof.** Let us consider  $\lambda^i(t, \cdot)$  solution of the parametrized BSDE (10) with data  $(\Phi^i, f^i)$ . From our setting and the standard comparison principle for BSDEs (see [8, Theorem 3.2.1], [26, Proposition 5.32] or [16, 18]), we obtain that a.s. for any  $s \in [t, T]$ ,  $\lambda^0(t, s) \leq \lambda^1(t, s)$ . Sending  $s$  to  $t$ , since  $Y^i(t) = \lambda^i(t, t)$ , we obtain the desired result.  $\square$

Nevertheless to extend this result for generators depending also on  $y$ , some extra monotonicity conditions are required. For example in [34, Theorem 3.4],  $f$  is supposed to be bounded (from above or from below) by a non decreasing w.r.t.  $y$  generator. The arguments used in [34] can be extended easily to our jump setting for  $p = 2$ . However for  $p < 2$ , we have a priori the same difficulty In the proof and the extension of Theorem 3. Moreover in the in progress framework of singular BSDEs, this monotone condition w.r.t.  $y$  is not verified. Thereby we do not develop this point here.

If this ‘‘monotone’’ assumption does not hold, as in [34, Theorem 3.9], the generator  $f$  is given by (40), namely is linear w.r.t.  $z$  and  $\psi$ :

$$f(t, s, y, z, \psi) = g(t, s, y) + h(s)z + \int_{\mathcal{E}} \psi(e)\kappa(s, e)\mu(de),$$

where  $h$  is a bounded process and  $\kappa : \Omega \times [0, T] \times \mathcal{E} \rightarrow \mathbb{R}$  is progressively measurable and such that  $\mathbb{P} \otimes Leb \otimes \mu$ -a.e.,  $-1 \leq \kappa(t, e)$  and  $|\kappa(t, e)| \leq \vartheta(e)$  where  $\vartheta$  belongs to the space  $\mathbb{L}_{\mu}^{\infty} \cap \mathbb{L}_{\mu}^2$ . Our comparison result is a extension to the jump case of [34, Theorems 3.8 and 3.9] for BSVIE (4) where  $f$  is given by (40), that is  $f$  is linear w.r.t.  $(z, \psi)$ .

**Proposition 3** *Consider two drivers  $g^i : \Omega \times \Delta \times \mathbb{R} \rightarrow \mathbb{R}$  satisfying **(H1)**. We suppose that a.s. for a.e.  $s \in [0, T]$  and for any  $0 \leq t \leq \tau \leq s$  and any  $y \in \mathbb{R}$ :*

$$g^1(t, s, y) - g^0(t, s, y) \geq g^1(\tau, s, y) - g^0(\tau, s, y) \geq 0. \quad (42)$$

Moreover for either  $i = 0$  or  $i = 1$

$$(g^i(t, s, y) - g^i(t, s, y'))(y - y') \geq (g^i(\tau, s, y) - g^i(\tau, s, y'))(y - y') \quad (43)$$

again a.s. for a.e.  $s \in [0, T]$  and for any  $0 \leq t \leq \tau \leq s$  and any  $y, y'$  in  $\mathbb{R}$ . Furthermore there exists a continuous nondecreasing function  $\rho : [0, T] \rightarrow [0, +\infty)$  with  $\rho(0) = 0$  such that a.s. for a.e.  $s \in [0, T]$  and for any  $0 \leq t, t' \leq s$

$$|g^i(t, s, y) - g^i(t, s, y') - g^i(t', s, y) + g^i(t', s, y')| \leq \rho(|t - t'|) \times |y - y'|. \quad (44)$$

If a.s. for  $0 \leq t \leq \tau \leq T$ ,

$$\Phi^1(t) - \Phi^0(t) \geq \Phi^1(\tau) - \Phi^0(\tau) \geq 0, \quad (45)$$

then the corresponding solutions of the BSVIEs (4) with generator  $f^i$  given by (40) with  $g^i$  instead of  $g$ , verify for any  $t \in [0, T]$ :

$$Y^1(t) \geq \bar{Y}^0(t), \quad a.s.$$



**Proof.** Let us first copy the arguments of the proof of [34, Theorem 3.9]. Suppose that  $g^0$  is differentiable and (43) holds for  $i = 0$ . Then we have

$$\begin{aligned}
Y^1(t) - Y^0(t) &= \Phi^1(t) - \Phi^0(t) + \int_t^T [g^1(t, s, Y^1(s)) - g^0(t, s, Y^1(s))] ds \\
&+ \int_t^T A(t, s)(Y^1(s) - Y^0(s))ds + \int_t^T h(s)(Z^1(t, s) - Z^0(t, s))ds \\
&+ \int_t^T \int_{\mathcal{E}} \kappa(s, e)(\psi^1(t, s, e) - \psi^0(t, s, e))\mu(de) \\
&- \int_t^T (Z^1(t, s) - Z^0(t, s))dW_s - \int_t^T \int_{\mathcal{E}} (\psi^1(t, s, e) - \psi^0(t, s, e))\tilde{\pi}(de, ds) \\
&- \int_t^T d(M^1(t, s) - M^0(t, s))
\end{aligned} \tag{46}$$

where

$$A(t, s) = \frac{g^0(t, s, Y^1(s)) - g^0(t, s, Y^0(s))}{Y^1(s) - Y^0(s)} \mathbf{1}_{Y^1(s) \neq Y^0(s)}, \quad 0 \leq t \leq s \leq T.$$

From **(H1)**,  $A$  is a bounded process. In other words we want to prove that the solution of the BSVIE

$$\begin{aligned}
\Delta Y(t) &= \Delta \Phi(t) + \int_t^T A(t, s)\Delta Y(s)ds - \int_t^T d\Delta M(t, s) \\
&+ \int_t^T h(s)\Delta Z(t, s)ds - \int_t^T \Delta Z(t, s)dW_s \\
&+ \int_t^T \int_{\mathcal{E}} \kappa(s, e)\Delta \psi(t, s, e)\mu(de) - \int_t^T \int_{\mathcal{E}} \Delta \psi(t, s, e)\tilde{\pi}(de, ds)
\end{aligned}$$

satisfies: for any  $t \in [0, T]$ , a.s.  $\Delta Y(t) \geq 0$ . Here

$$\Delta \Phi(t) = \Phi^1(t) - \Phi^0(t) + \int_t^T [g^1(t, s, Y^1(s)) - g^0(t, s, Y^1(s))] ds.$$

From our assumptions, for  $0 \leq t \leq \tau \leq T$ ,  $\Delta \Phi(t) \geq \Delta \Phi(\tau) \geq 0$ ,  $A(t, s) - A(\tau, s) \geq 0$  and

$$|A(t, s) - A(t', s)| \leq \rho(|t - t'|).$$

Now we can follow the proof of [34, Theorem 3.8]. We consider a partition  $\Pi = \{t_k, 0 \leq k \leq N\}$  of  $[0, T]$  and assume first that

$$A^\Pi(t, s) = \sum_{k=1}^N A(t_{k-1}, s)\mathbf{1}_{(t_{k-1}, t_k \wedge s]}(t), \quad \Phi^\Pi(t) = \sum_{k=1}^N \phi_k \mathbf{1}_{(t_{k-1}, t_k]}(t)$$

where  $A^\Pi$  still satisfies  $A^\Pi(t, s) - A^\Pi(\tau, s) \geq 0$  and  $\phi_k$  are  $\mathcal{F}_T$ -measurable r.v. such that

$$\phi_1 \geq \phi_2 \geq \dots \geq \phi_{N-1} \geq \phi_N \geq 0.$$

Let  $(Y^\Pi(\cdot), Z^\Pi(\cdot, \cdot), U^\Pi(\cdot, \cdot), M^\Pi(\cdot, \cdot))$  be the solution of the BSVIE:

$$\begin{aligned}
Y^\Pi(t) &= \Phi^\Pi(t) + \int_t^T A^\Pi(t, s)Y^\Pi(s)ds + \int_t^T h(s)Z^\Pi(t, s)ds + \int_t^T \int_{\mathcal{E}} \kappa(s, e)\psi^\Pi(t, s, e)\mu(de) \\
&- \int_t^T Z^\Pi(t, s)dW_s - \int_t^T \int_{\mathcal{E}} \psi^\Pi(t, s, e)\tilde{\pi}(de, ds) - \int_t^T dM^\Pi(t, s).
\end{aligned} \tag{47}$$

If we consider the BSDE

$$\begin{aligned} Y_N(t) &= \phi_N + \int_t^T \left[ A(t_{N-1}, s)Y_N(s) + h(s)Z_N(s) + \int_{\mathcal{E}} \kappa(s, e)U_N(s, e)\mu(de) \right] ds \\ &\quad - \int_t^T Z_N(s)dW_s - \int_t^T \int_{\mathcal{E}} U_N(s, e)\tilde{\pi}(de, ds) - \int_t^T dM_N(s) \end{aligned}$$

then for  $t_{N-1} < t \leq s \leq T$ ,

$$(Y_N(s), Z_N(s), U_N(s, e), M_N(s)) = (Y^\Pi(s), Z^\Pi(t, s), U^\Pi(t, s, e), M^\Pi(t, s))$$

solves the BSVIE (47) on the interval  $(t_{N-1}, t_N]$ . By the comparison principle for BSDE, we have a.s. for any  $s \in (t_{N-1}, t_N]$ ,  $Y^\Pi(s) = Y_N(s) \geq 0$ . Since all martingales are càdlàg processes,

$$\begin{aligned} Y^\Pi(t_{N-1}^+) &= \phi_N + \int_{t_{N-1}}^T \left[ A(t_{N-1}, s)Y^\Pi(s) + h(s)Z_N(s) + \int_{\mathcal{E}} \kappa(s, e)U_N(s, e)\mu(de) \right] ds \\ &\quad - \int_{t_{N-1}}^T Z_N(s)dW_s - \int_{t_{N-1}}^T \int_{\mathcal{E}} U_N(s, e)\tilde{\pi}(de, ds) - \int_{t_{N-1}}^T dM_N(s) \geq 0. \end{aligned}$$

Now the BSVIE on  $(t_{N-2}, t_{N-1}]$  can be written as follows:

$$\begin{aligned} Y^\Pi(t) &= \phi_{N-1} + \int_t^T A(t_{N-2}, s)Y^\Pi(s)ds + \int_t^T h(s)Z^\Pi(t, s)ds - \int_t^T Z^\Pi(t, s)dW_s \\ &\quad + \int_t^T \int_{\mathcal{E}} \kappa(s, e)\psi^\Pi(t, s, e)\mu(de) - \int_t^T \int_{\mathcal{E}} \psi^\Pi(t, s, e)\tilde{\pi}(de, ds) - \int_t^T dM^\Pi(t, s) \\ &= \phi_{N-1} - \phi_N + Y^\Pi(t_{N-1}^+) + \int_{t_{N-1}}^T [A(t_{N-2}, s) - A(t_{N-1}, s)]Y^\Pi(s)ds \\ &\quad + \int_{t_{N-1}}^T h(s) [Z^\Pi(t, s) - Z_N(s)] ds - \int_{t_{N-1}}^T (Z^\Pi(t, s) - Z_N(s))dW_s \\ &\quad + \int_{t_{N-1}}^T \int_{\mathcal{E}} \kappa(s, e) [\psi^\Pi(t, s, e) - U_N(s, e)] \mu(de) \\ &\quad - \int_{t_{N-1}}^T \int_{\mathcal{E}} [\psi^\Pi(t, s, e) - U_N(s, e)] \tilde{\pi}(de, ds) - \int_{t_{N-1}}^T d(M^\Pi(t, s) - M_N(s)) \\ &\quad + \int_t^{t_{N-1}} A(t_{N-2}, s)Y^\Pi(s)ds + \int_t^{t_{N-1}} h(s)Z^\Pi(t, s)ds - \int_t^{t_{N-1}} Z^\Pi(t, s)dW_s \\ &\quad + \int_t^{t_{N-1}} \int_{\mathcal{E}} \kappa_s(e)\psi^\Pi(t, s, e)\mu(de) - \int_t^{t_{N-1}} \int_{\mathcal{E}} \psi^\Pi(t, s, e)\tilde{\pi}(de, ds) - \int_t^{t_{N-1}} dM^\Pi(t, s). \end{aligned}$$

We consider the terminal condition

$$\xi_N = \phi_{N-1} - \phi_N + Y^\Pi(t_{N-1}^+) + \int_{t_{N-1}}^T [A(t_{N-2}, s) - A(t_{N-1}, s)]Y^\Pi(s)ds$$

and the solution  $(\tilde{Y}_N, \tilde{Z}_N, \tilde{U}_N, \tilde{M}_N)$  of the linear BSDE on  $[t_{N-1}, T]$ :

$$\begin{aligned} \tilde{Y}_N(t) &= \xi_N + \int_t^T h(s)\tilde{Z}_N(s)ds - \int_t^T \tilde{Z}_N(s)dW_s \\ &\quad + \int_t^T \int_{\mathcal{E}} \kappa(s, e)\tilde{U}_N(s, e)\mu(de) - \int_t^T \int_{\mathcal{E}} \tilde{U}_N(s, e)\tilde{\pi}(de, ds) - \int_t^T d\tilde{M}_N(s). \end{aligned}$$

By our conditions,  $\xi_N$  is non-negative and thus a.s.  $\tilde{Y}_N(t) \geq 0$  on  $[t_{N-1}, T]$ . By uniqueness of adapted solutions to the BSVIE, we have

$$Z^\Pi(t, s) = Z_N(s) + \tilde{Z}_N(s), \quad U^\Pi(t, s) = U_N(s) + \tilde{U}_N(s), \quad M^\Pi(t, s) = M_N(s) + \tilde{M}_N(s)$$

for  $(t, s) \in (t_{N-2}, t_{N-1}] \times (t_{N-1}, t_N]$  and our previous BSVIE becomes

$$\begin{aligned} Y^\Pi(t) &= \tilde{Y}_N(t_{N-1}) + \int_t^{t_{N-1}} A(t_{N-2}, s) Y^\Pi(s) ds \\ &+ \int_t^{t_{N-1}} h(s) Z^\Pi(t, s) ds - \int_t^{t_{N-1}} Z^\Pi(t, s) dW_s \\ &+ \int_t^{t_{N-1}} \int_{\mathcal{E}} \kappa(s, e) \psi^\Pi(t, s, e) \mu(de) - \int_t^{t_{N-1}} \int_{\mathcal{E}} \psi^\Pi(t, s, e) \tilde{\pi}(de, ds) - \int_t^{t_{N-1}} dM^\Pi(t, s). \end{aligned}$$

Again we solve the BSDE:

$$\begin{aligned} Y_{N-1}(t) &= \tilde{Y}_N(t_{N-1}) + \int_t^T \left[ A(t_{N-2}, s) Y_{N-1}(s) + h(s) Z_{N-1}(s) + \int_{\mathcal{E}} \kappa(s, e) U_{N-1}(s, e) \mu(de) \right] ds \\ &- \int_t^T Z_{N-1}(s) dW_s - \int_t^T \int_{\mathcal{E}} U_{N-1}(s, e) \tilde{\pi}(de, ds) - \int_t^T dM_{N-1}(s) \end{aligned}$$

on  $[t_{N-2}, t_{N-1}]$  and by uniqueness and the comparison principle for BSDE, we have

$$Y^\Pi(t) = Y_{N-1}(t) \geq 0, \quad t \in [t_{N-2}, t_{N-1}].$$

By induction we obtain that  $Y^\Pi(t) \geq 0$ ,  $t \in [0, T]$ . The stability estimate for BSVIE and the time regularity condition on  $A$  imply that

$$\lim_{\|\Pi\| \rightarrow 0} \sup_{t \in [0, T]} \mathbb{E} |Y^\Pi(t) - Y(t)|^p = 0.$$

The non-negativity of  $Y$  follows directly.  $\square$

In [34, Section 3.2], the authors also study the comparison principle for M-solution of a BSVIE of the form:

$$Y(t) = \Phi(t) + \int_t^T (g(t, s, Y(s)) + C(s)Z(s, t)) ds - \int_t^T dM^\sharp(t, s).$$

Up to some technical conditions, the proof of their results is based on the duality principle (see Section 4.1). The extension to the jump case can be proved following their scheme ([34, Theorems 3.12 and 3.13]).

## References

- [1] S. Ankirchner, M. Jeanblanc, and T. Kruse. BSDEs with Singular Terminal Condition and a Control Problem with Constraints. *SIAM J. Control Optim.*, 52(2):893–913, 2014.
- [2] D. Becherer. Bounded solutions to backward SDE's with jumps for utility optimization and indifference hedging. *Ann. Appl. Probab.*, 16(4):2027–2054, 2006.

- [3] M. A. Berger and V. J. Mizel. Volterra equations with Itô integrals. I. *J. Integral Equations*, 2(3):187–245, 1980.
- [4] M. A. Berger and V. J. Mizel. Volterra equations with Itô integrals. II. *J. Integral Equations*, 2(4):319–337, 1980.
- [5] N. Bouleau and L. Denis. *Dirichlet forms methods for Poisson point measures and Lévy processes*, volume 76 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2015. With emphasis on the creation-annihilation techniques.
- [6] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux, and L. Stoica.  $L^p$  solutions of backward stochastic differential equations. *Stochastic Process. Appl.*, 108(1):109–129, 2003.
- [7] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel. Théorie des martingales*. Hermann, Paris, 1980. Chapitres V à VIII.
- [8] Ł. Delong. *Backward stochastic differential equations with jumps and their actuarial and financial applications*. European Actuarial Academy (EAA) Series. Springer, London, 2013. BSDEs with jumps.
- [9] G. Di Nunno, B. Øksendal, and F. Proske. *Malliavin calculus for Lévy processes with applications to finance*. Universitext. Springer-Verlag, Berlin, 2009.
- [10] N. El Karoui, S.G. Peng, and M.C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, 7(1):1–71, 1997.
- [11] J. Jacod. *Calcul stochastique et problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.
- [12] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.
- [13] A. M. Kolodii. Existence of solutions of stochastic Volterra integral equations. In *Theory of random processes, No. 11*, pages 51–57. “Naukova Dumka”, Kiev, 1983.
- [14] A. M. Kolodii. Existence of solutions of stochastic integral equations of Itô-Volterra type with locally integrable and continuous trajectories. In *Theory of random processes, No. 12*, pages 32–40. “Naukova Dumka”, Kiev, 1984.
- [15] S. G. Kreĭn, Yu. Ī. Petunĭn, and E. M. Semĕnov. *Interpolation of linear operators*, volume 54 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, R.I., 1982. Translated from the Russian by J. Szűcs.
- [16] T. Kruse and A. Popier. Bsdés with monotone generator driven by brownian and poisson noises in a general filtration. *Stochastics*, 88(4):491–539, 2016.
- [17] T. Kruse and A. Popier. Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting. *Stochastic Processes and their Applications*, 126(9):2554 – 2592, 2016.

- [18] T. Kruse and A. Popier.  $L^p$ -solution for BSDEs with jumps in the case  $p < 2$ : corrections to the paper ‘BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration. *Stochastics*, 89(8):1201–1227, 2017.
- [19] E. Lenglart, D. Lépingle, and M. Pratelli. Présentation unifiée de certaines inégalités de la théorie des martingales. In *Seminar on Probability, XIV (Paris, 1978/1979) (French)*, volume 784 of *Lecture Notes in Math.*, pages 26–52. Springer, Berlin, 1980. With an appendix by Lenglart.
- [20] J. Lin. Adapted solution of a backward stochastic nonlinear Volterra integral equation. *Stochastic Anal. Appl.*, 20(1):165–183, 2002.
- [21] W. Lu. Backward stochastic Volterra integral equations associated with a Levy process and applications. *ArXiv e-prints*, June 2011.
- [22] C. Marinelli and M. Röckner. On maximal inequalities for purely discontinuous martingales in infinite dimensions. In *Séminaire de Probabilités XLVI*, volume 2123 of *Lecture Notes in Math.*, pages 293–315. Springer, Cham, 2014.
- [23] L. Overbeck and J. A. L. Röder. Path-dependent backward stochastic Volterra integral equations with jumps, differentiability and duality principle. *Probab. Uncertain. Quant. Risk*, 3:Paper No. 4, 37, 2018.
- [24] E. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.
- [25] E. Pardoux and P. Protter. Stochastic Volterra equations with anticipating coefficients. *Ann. Probab.*, 18(4):1635–1655, 1990.
- [26] E. Pardoux and A. Rascanu. *Stochastic Differential Equations, Backward SDEs, Partial Differential Equations*, volume 69 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, 2014.
- [27] A. Popier. Backward stochastic differential equations with singular terminal condition. *Stochastic Process. Appl.*, 116(12):2014–2056, 2006.
- [28] P. E. Protter. Volterra equations driven by semimartingales. *Ann. Probab.*, 13(2):519–530, 1985.
- [29] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability.
- [30] Y. Ren. On solutions of backward stochastic Volterra integral equations with jumps in Hilbert spaces. *J. Optim. Theory Appl.*, 144(2):319–333, 2010.
- [31] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

- [32] R. Situ. *Theory of stochastic differential equations with jumps and applications*. Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005. Mathematical and analytical techniques with applications to engineering.
- [33] T. Wang.  $L^p$  solutions of backward stochastic Volterra integral equations. *Acta Math. Sin. (Engl. Ser.)*, 28(9):1875–1882, 2012.
- [34] T. Wang and J. Yong. Comparison theorems for some backward stochastic Volterra integral equations. *Stochastic Process. Appl.*, 125(5):1756–1798, 2015.
- [35] Z. Wang and X. Zhang. Non-Lipschitz backward stochastic Volterra type equations with jumps. *Stoch. Dyn.*, 7(4):479–496, 2007.
- [36] J. Yong. Backward stochastic Volterra integral equations and some related problems. *Stochastic Process. Appl.*, 116(5):779–795, 2006.
- [37] J. Yong. Well-posedness and regularity of backward stochastic Volterra integral equations. *Probab. Theory Related Fields*, 142(1-2):21–77, 2008.