On the time of existence of solutions of the Euler-Korteweg system

Corentin Audiard [*†‡]

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Abstract

Under a natural stability condition on the pressure, it is known that for small irrotational initial data, the solutions of the Euler-Korteweg system are global in time when the space dimension is at least 3. If the initial velocity has a small rotational part, we obtain a lower bound on the time of existence that depends only on the rotational part. In the zero vorticity limit we recover the previous global well-posedness result.
Independently of this analysis, we also provide (in a special case) a simple example of solution that blows up in finite time.

1 Introduction

The Cauchy problem for the Euler-Korteweg system reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + u \cdot \nabla u + \nabla g(\rho) = \nabla\left(K\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2 \right), \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+. \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

g is the pressure term, K the capillary coefficient, a smooth function $\mathbb{R}^{+*} \to \mathbb{R}^{+*}$. It appears in the litterature in various contexts depending on K. K constant has been largely investigated, see , and corresponds to capillary fluids. The important case where K is proportional to $1/\rho$ corresponds to the so called quantum fluids, the equations are then formally equivalent to the nonlinear Schrödinger equation

$$i\partial_t \psi + \Delta\psi = g(|\psi|^2)\psi, \tag{1.2}$$

through the so called Madelung transform $\psi = \sqrt{\rho}e^{i\varphi}$, $\nabla\varphi = u$. It is worth pointing out that even for a smooth solution of NLS the map $(\psi \to (\rho, u)$ is not well defined if ψ cancels (existence of vortices).
The main result on local well-posedness for the general Euler-Korteweg system is due to Benzoni-Danchin-Descombes [5], we shall use the following (slightly simpler) version:

[*]Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France
[†]CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France
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Theorem 1.1 ([5]). *For $(\rho_0 - \alpha, u_0) \in \mathcal{H}^{s+1}(\mathbb{R}^d)$, $s > d/2 + 1$, with $\mathcal{H}^s := H^{s+1} \times H^s$, there exists a unique solution $(\rho, u) \in (\alpha + C_t H^{s+1}) \times C_t H^s$ to (1.1), and it exists on $[0, T]$ if the following two conditions are satisfied*

1. $\inf_{\mathbb{R}^d \times [0,T]} \rho(x,t) > 0$,
2. $\int_0^T \|\Delta\rho(s)\|_\infty + \|\nabla u(s)\|_\infty ds$.

The original proof also shows that the time of existence of the solutions is of order at least $\ln(1/\|(\rho_0 - \alpha, u_0)\|_{H^{s+1} \times H^s})$. This rather small lower bound is due to the absence of assumptions on the pressure term which can cause exponentially growing instabilities. For stable pressure terms, this result was more recently sharpened by Benzoni and Chiron [7] who obtained the natural time $O(1/\|(\rho_0 - \alpha, u_0)\|_{H^{s+1} \times H^s})$.
In irrotational settings, the author proved with B.Haspot [3] that small initial data lead to a global solution under a natural stability assumption on g. The main focus of this paper is to describe more accurately the time of existence for small data that have a non zero rotational part.
We denote $\mathbb{Q} = \Delta^{-1}\nabla\mathrm{div}$ the projector on potential vector fields, $\mathbb{P} = I - \mathbb{Q}$ the projector on solenoidal vector fields. In this paper, we prove the following informally stated theorem (see theorems 4.1 and 5.1 for the precise statements):

Theorem 1.2. *Let $d \geq 3$, $\alpha > 0$ a positive constant such that $g'(\alpha) > 0$. For some function spaces X, Y, Z, if $\|\rho_0 - \alpha\|_X$, $\|u_0\|_Y$, $\|\mathbb{P}u_0\|_Z$ are small enough, then there exists $c(d, \alpha) > 0$ such that the time of existence of the solution to (1.1) is bounded from below by $c/\|\mathbb{P}u_0\|_Y$.*

Note that in the special case $\mathbb{P}u_0 = 0$, we recover the global well-posedness result from [3]. Before commenting the proof and sharpness of this result, let us give a bit more background on the well-posedness theory of the Euler-Korteweg system.

Weak solutions In the case of the quantum Navier-Stokes equations (K proportional to $1/\rho$ and addition of a viscosity term) the existence of global weak solutions has been obtained under various assumptions, an important breakthrough was obtained by Bresch et al [12], introducing what is now called the Bresh-Desjardins entropy, a key a priori estimate to construct global weak solutions by compactness methods.
The inviscid case is more intricate. As the existence of global strong solutions to (1.2) with a large range of nonlinearities is well-known, Antonelli-Marcati [1] managed to use the formal equivalence with (1.1) to construct global weak solutions, the main difficulty difficulty being to give a meaning to the Madelung transform in the vacuum region where ρ cancels, see also the review paper [13] for a simpler proof. Relative entropy methods have since been developed [17],[11] that should eventually lead to the existence of global weak solutions for more general capillary coefficients K. Noticeably, these methods allow solutions with vorticity.

Strong solutions As we mentioned, theorem 1.1 is the first well-posedness result in very general settings, an important idea due to Frédéric Coquel was to use a reformulation of the equations as a quasi-linear degenerate Schrödinger equation for which energy estimates in arbitrary high Sobolev spaces can be derived.

For quantum hydrodynamics ($K = 1/\rho$) in the long wave regime with irrotational speed, the time interval of existence was improved by Béthuel-Danchin-Smets [10] thanks to the use of Strichartz estimates. This approach is not tractable to the general case of system (1.1). Note however that the second aim in [10] (long wave limit) was recently studied in [7] where the authors study (1.1) in several long waves regimes and prove convergence to more classical equations such as Burgers, KdV or KP. Their analysis does not require the solutions to be irrotational.
The analogy with the Schrödinger equation was pushed further in [3] where the authors prove the existence of global strong solutions for small irotational data in dimension at least 3. As a byproduct of the proof, such solutions behave asymptotically as solutions of the linearized system near a constant density and zero speed, i.e. they "scatter". The strategy of proof was reminiscent of ideas developed by Gustavson, Nakanishi and Tsai [21] for the Gross-Pitaevskii equation, and more generally the method of space time resonance (see Germain-Masmoudi-Shatah [16] for a clear description) which has had prolific applications for nonlinear dispersive equations. To some extent the present paper is a continuation of such results for a mixed dispersive-transport system.

Travelling waves The system (1.1) being of dispersive nature, it is expectable that soliton like solutions exist, that is solutions that only depend on $x\cdot e-ct$ for some direction $e \in \mathbb{R}^d$ and speed c. In dimension 1, the existence of solitons (traveling waves with same limits at $\pm\infty$) and kinks (different limits at $\pm\infty$) was derived in [6] by ODE methods. A stability criterion à la Grillakis-Shatah-Strauss [18] was also exhibited. It is a stability of weak type, as it implies that the solution remains close to the soliton in a norm that does not give local well-posedness (stability "until possible blow up"). Still in dimension 1, the author proved the existence of multi-solitons type solutions, a first example of global solution in small dimension which is not an ODE solution. Finally, motivated by the scattering result [3] in dimension larger than 2, the author also proved in [2] the existence of small amplitude traveling waves in dimension 2, an obstruction to scattering.

Blow up To the best of our knowledge, blow up for the Euler-Korteweg system is a completely open problem. The formation of vacuum for NLS equations with non zero conditions at infinity is also not clearly understood. We construct in appendix 6 a solution to (1.1) (quantum case $K = 1/\rho$) that blows up in finite time. The construction is very simple, it relies on the existence of smooth solutions to (1.2) such that ψ vanishes at some time and the reversibility of (1.1).

The Euler-Korteweg system with a small vorticity To give some intuition of our approach it is useful to introduce the reformulation from [5] : set $\nabla l := w := \sqrt{K/\rho}\nabla\rho$, then for smooth solution without vacuum (1.1) is equivalent to the extended system

$$\left\{\begin{array}{lcl} \partial_t l + u\cdot\nabla l + a\mathrm{div}u &=& 0,\\ \partial_t w + \nabla(u\cdot w) + \nabla(a\mathrm{div}u) &=& 0,\\ \partial_t u + u\cdot\nabla u - w\cdot\nabla w - \nabla(a\mathrm{div}w) + g'w &=& 0, \end{array}\right. \tag{1.3}$$

with $a = \sqrt{\rho K}$ and the second equation is simply the gradient of the first one.
If u is irrotational, setting $z = u + iw$ we have using $\nabla \mathrm{div} z = \Delta z$

$$\partial_t z + ia\Delta z = \mathcal{N}(z),$$

despite the fact that $\mathcal{N}$ is a highly nonlinear term, the link with the Schrödinger equation is clear. This observation is the starting point of the analysis in []. Note that we have abusively neglected $g'(\rho)w$, which is at first order a linear term and thus must be taken into account for long time analysis.
If u is not potential, it is natural to write $u = \mathbb{P}u + \mathbb{Q}u$ and split the potential and the solenoidal part of the last equation. For v potential $v \cdot \nabla v = \frac{1}{2}\nabla |v|^2$, so the last two equations of (1.3) rewrite

$$\left\{ \begin{array}{lcl} \partial_t w + \nabla(u \cdot w) + \nabla(a \mathrm{div} \mathbb{Q}u) & = & 0, \\ \partial_t \mathbb{Q}u + \mathbb{Q}(u \cdot \nabla \mathbb{P}u + \mathbb{P}u \cdot \nabla \mathbb{Q}u) + \frac{1}{2}\nabla(|\mathbb{Q}u|^2 - |w|^2) - \nabla(a \mathrm{div} w) + g'w & = & 0, \\ \partial_t \mathbb{P}u + \mathbb{P}(u \cdot \nabla \mathbb{P}u + \mathbb{P}u \cdot \nabla \mathbb{Q}u) & = & 0. \end{array} \right. \tag{1.4}$$

The only important point is that the first two equations are still the same quasilinear Schrödinger equation (where the Schrödinger evolution causes some decay), coupled to $\mathbb{P}u$ and the evolution equation on $\mathbb{P}u$ has $\mathbb{P}u$ in factor of all its nonlinear terms.
A very simplified version of this dynamical system is the following ODE system

$$\left\{ \begin{array}{l} x' = -x + x^2 + y^2, \\ y' = y(x + y), \end{array} \right. \tag{1.5}$$

where one should think of x as $\mathbb{Q}u + iw$, y as $\mathbb{P}u$ and the linear evolution $x' = -x$ gives decay. The proof of the following elementary property is the guideline of this paper :

Proposition 1.3. *Assume $|x(0)| \leq \varepsilon$, $|y(0)| \leq \delta \leq \varepsilon$. Then for ε, δ small enough there exists $c > 0$ such that the solution of (1.5) exists on a time interval $[0, T]$ with $T \geq c/\delta$.*

Proof. We plug the ansatz

$$|x(t)| \leq \delta + 2\varepsilon e^{-t}, |y(t)| \leq 2\delta \tag{1.6}$$

in (1.5):

$$\begin{array}{rcl} |x(t)| & \leq & \varepsilon e^{-t} + \displaystyle\int_0^t e^{s-t}\big(2\delta^2 + 8\varepsilon^2 e^{-2s} + 4\delta^2\big)ds \leq (\varepsilon + 8\varepsilon^2)e^{-t} + 6\delta^2, \\ |y(t)| & \leq & \delta + \displaystyle\int_0^t 2\delta(\delta + 2\varepsilon e^{-s} + 2\delta)ds \leq \delta(1 + 6\delta t + 4\varepsilon) \end{array}$$

For $\varepsilon, \delta \leq 1/16$, $t \leq 1/(12\delta)$ we get

$$|x(t)| \leq \frac{3}{2}\varepsilon e^{-t} + \frac{3}{8}\delta, \ |y(t)| \leq \frac{7}{4}\delta,$$

so that a standard continuation argument ensures that the solution exists on $[0, 1/(12\delta)]$ and (1.6) is true on this interval. □

Of course, some difficulties arise in our case: first due to the quasi-linear nature of the problem, loss of derivatives are bound to arise. This is handled by a method well-understood since the work of Klainerman-Ponce [23], where one mixes dispersive (decay) estimates with high order energy estimates (see for example the introduction of [3] for a short description).
The second difficulty is more consequent and is due to some lack of integrability of the decay. Basically, we have $\|e^{it\Delta}\|_{L^p\to L^{p'}} \lesssim 1/t^{d(1/2-1/p)}$, which is weaker as the dimension decreases. Again, it was identified in [23] that this is not an issue for quasi-linear Schrödinger equations if $d \geq 5$, but the case $d < 5$ requires much more intricate (and recent) methods.
There has been an extremely abundant activity on global well-posedness for quasi-linear dispersive equations over the last decade. The method of space-time resonances initiated by Germain-Masmoudi-Shatah[16] and Gustavson-Nakanishi-Tsai [21] led to numerous improvements and outstanding papers, a recent prominent result being the global well-posedness of the capillary-gravity water waves in dimension 3 due to Deng-Ionescu-Pausader-Pusateri [14].
The issue of long time existence for coupled dispersive-transport equations is more scarce. Nevertheless it arises naturally in numerous physical problem, and has been treated at least in the case of the Euler-Maxwell system [22]. The strategy of proof in this references seems to be close to proposition 1.3, despite the considerable technical difficulties that are bound to arise.
It is worth pointing out that the time of existence is quite natural : it is related to the time of existence for $y' = y^2$, which is $1/y(0)$. It should be understood that the finite time of existence is due to the lack of control of the transport equation.

Organization of the article We define our notations, functional framework and recall some technical tools in section 2. Section 3 is devoted to some energy estimates for (1.1). The main H^n energy estimate is a modification of the arguments in [5] and is proved for completeness in the appendix A. As is common for dispersive equations, the proof of theorem 1.2 is more difficult in smaller dimensions. Here $d \geq 5$ is quite straightforward and is treated in section 4 while $d = 3$ is in section 5. $d = 4$ is similar to $d = 3$ but simpler,thus we do not detail this case. A large part of the analysis in dimension 3 builds upon previous results from [3], as such this part is not self-contained. The new difficulties are detailed, but the delicate estimates for the so-called "purely dispersive" quadratic nonlinearities is a bit redundant with [3] and are thus only partially carried out in the appendix B. We also construct in section 6 an example of solution which blows up in finite time. This construction relies on the Madelung transform and the finite time formation of vacuum for the Gross-Pitaevskii equation.

2 Notations and functional spaces

Constants and inequalities We will denote by C a constant used in the bootstrap argument of sections 4 and 5, it remains the same in the section. Constants that are allowed to change from line to line are rather denoted $C_1, C_2 \cdots$
We denote $a \lesssim b$ when there exists C_1 such that $a \leq C_1 b$, with C_1 a "constant" that depends in a clear way on the various parameters of the problem.

Functional spaces $L^p(\mathbb{R}^d), W^{k,p}(\mathbb{R}^d), W^{k,2} = H^k(\mathbb{R}^d), \dot{W}^{k,p}$ are the usual Lebesgue, Sobolev and homogeneous Sobolev spaces. $L^{p,q}$ is the Lorentz space obtained as an interpolation space

of L^{p_1}, L^{p_2} by real interpolation with parameter q, see [9].
$\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class, $\mathcal{S}'(\mathbb{R}^d)$ its dual, the space of tempered distribution.
If there is no ambiguity we drop the $(\mathbb{R}^d)$ reference. In our settings, ρ is one derivative more regular than u, therefore we define

$$\mathcal{H}^n = H^{n+1} \times H^n,\ \mathcal{W}^{k,p} = W^{k+1,p} \times W^{k,p}.$$

We recall the Sobolev embeddings

$$\forall, kp < d,\ \dot{W}^{k,p} \hookrightarrow L^q,\ \frac{1}{q} = \frac{1}{p} - \frac{k}{d},\ \forall\, kp > d,\ W^{k,p} \hookrightarrow C^0 \cap L^\infty,$$

the tame product estimate for $p, q, r > 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$

$$\|uv\|_{W^{k,r}} \lesssim \|u\|_{L^p}\|v\|_{W^{k,q}} + \|u\|_{W^{k,q}}\|v\|_{L^p}, \tag{2.1}$$

and the composition rule, for F smooth, $F(0) = 0$,

$$\forall\, u \in W^{k,p} \cap L^\infty, \|F(u)\|_{W^{k,p}} \leq C(\|u\|_\infty)\|u\|_{W^{k,p}} \tag{2.2}$$

Fourier and bilinear Fourier multiplier The Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ is denoted $\widehat{f}$ or $\mathcal{F}(f)$. A Fourier multiplier of symbol $m(\xi)$ with moderate growth acts on $\mathcal{S}$

$$m(D)f = \mathcal{F}^{-1}(m(\xi)\widehat{f}(\xi)),$$

with natural extensions for matrix valued symbols. A multiplier denoted $m(-\Delta)$ is of course the multiplier of symbol $m(|\xi|^2)$.
The Mihlin-Hörmander theorem (see [9]) states that for M large enough, if for any multi-index α with $|\alpha| \leq M,\ |\nabla^k m| \lesssim |\xi|^{-|\alpha|}$, then m acts continuously on L^p, $1 < p < \infty$.
A bilinear Fourier multiplier of symbol $B(\eta, \xi - \eta)$ acts on $\mathcal{S}^2$

$$B[f,g] = \mathcal{F}^{-1}\bigg(\int_{\mathbb{R}^d} B(\eta, \xi-\eta)\widehat{f}(\eta)\widehat{g}(\xi-\eta)d\eta\bigg) = \mathcal{F}^{-1}\bigg(\int_{\mathbb{R}^d} B(\zeta-\eta, \zeta)\widehat{f}(\zeta-\eta)\widehat{g}(\zeta)d\eta\bigg).$$

The Coifman-Meyer [24] theorem states that if $|\nabla^k B| \lesssim 1/(|\xi| + |\eta|)^k$ for sufficiently many k, then B is continuous $L^p \times L^q \to L^r$, $1/p + 1 + q = 1/r$, $p > 1,\ q, r \leq \infty$.
We denote $\nabla_\xi B$ the bilinear multiplier of symbol $\nabla_\xi B(\eta, \xi - \eta)$, and similarly for $\nabla_\eta B$.

Potential and solenoidal fields Potential fields v are vector fields of the form $v = \nabla f,\ f : \mathbb{R}^d \to \mathbb{C}$, they satisfy

$$\mathrm{curl}(v) = (\partial_i v_j - \partial_j v_i)_{1 \leq i,j \leq d} = 0.$$

Solenoidal fields satisfy $\mathrm{div}(v) = \sum \partial_i v_i = 0$.
The projector on potential vector fields is the Fourier multiplier $\mathbb{Q} = \Delta^{-1}\nabla\mathrm{div}$, the projector on solenoidal vector fields is $\mathbb{P} = I_n - \mathbb{Q}$. According to Mihlin-Hörmander multiplier theorem, $\mathbb{P} = (-\Delta)^{-1}\nabla\mathrm{div}$ and $\mathbb{Q}$ act continuously $L^p \to L^p$, $1 < p < \infty$, and in the related Sobolev spaces.

Reformulation of the equations We denote $r = \rho - \alpha$, $r_0 = \rho_0 - \alpha$, $w := \nabla l := \sqrt{K/\rho}\nabla\rho$. According to [5], if $(r_0, u_0) \in \mathcal{H}^N$ with $N > d/2+1$, there exists a unique local solution to (1.1) such that $(\rho - \alpha, u) \in C_t\mathcal{H}^N$. For N large enough the solution is smooth so it is equivalent to work on the extended formulation (1.3).

Assumptions 2.1. *Up to a change of variables, we can assume*

1. $\alpha = 1$,
2. $a(1) = 1$,
3. $g'(1) = 2 > 0$.

Equations (1.4) read

$$\left\{\begin{array}{lcl} \partial_t w + \Delta\mathbb{Q}u & = & \nabla\big((1-a)\mathrm{div}\mathbb{Q}u - u\cdot w\big), \\ \partial_t \mathbb{Q}u + \mathcal{N}(\mathbb{Q}u, \mathbb{P}u, w) + (-\Delta + 2)w & = & \nabla\big((a-1)\mathrm{div}w\big) + (2-g')w, \\ \partial_t \mathbb{P}u + \mathbb{P}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u) & = & 0. \end{array}\right. \tag{2.3}$$

with $\mathcal{N}(\mathbb{Q}u, \mathbb{P}u, w) = \mathbb{Q}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u) + \frac{1}{2}\nabla(|\mathbb{Q}u|^2 - |w|^2)$.

Set $U = \sqrt{\dfrac{-\Delta}{2-\Delta}}$, $H = \sqrt{-\Delta(2-\Delta)}$, then $\psi := \mathbb{Q}u + iU^{-1}w$ satisfies

$$\left\{\begin{array}{lcl} \partial_t\psi - iH\psi & = & \mathcal{N}_1(\psi, \mathbb{P}u), \\ \partial_t\mathbb{P}u + \mathbb{P}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u) & = & 0. \end{array}\right. \tag{2.4}$$

with

$$\mathcal{N}_1 = \nabla((a-1)\mathrm{div}w) + (2-g')w + iU^{-1}\nabla\big((1-a)\mathrm{div}\mathbb{Q}u - u\cdot w\big) - \mathcal{N}. \tag{2.5}$$

Note that U^{-1} is singular, but we have for $1 < p < \infty$

$$\|U^{-1}w\|_{L^p} \sim \|\rho\|_{W^{1,p}},$$

therefore using the composition rule (2.2), at least when $\|(\rho - 1, u)\|_{W^{k,p}} << 1$ and k is large enough

$$\|\psi\|_{W^{k,p}} \sim \|(\rho - 1, u)\|_{\mathcal{W}^{k,p}}. \tag{2.6}$$

Dispersive estimates Dispersion estimates for the semi-group e^{itH} were obtained by Gustafson, Nakanishi and Tsai in [20], a version in Lorentz spaces follows from real interpolation as pointed out in [21].

Theorem 2.2 ([20][21])**.** *For $2 \le p \le \infty$, $s \in \mathbb{R}^+$, $U = \sqrt{-\Delta/(2-\Delta)}$, we have*

$$\|e^{itH}\varphi\|_{W^{s,p}} \lesssim \frac{\|U^{(d-2)(1/2-1/p)}\varphi\|_{W^{s,p'}}}{t^{d(1/2-1/p)}}, \tag{2.7}$$

and for $2 \le p < \infty$

$$\|e^{itH}\varphi\|_{L^{p,2}} \lesssim \frac{\|U^{(d-2)(1/2-1/p)}\varphi\|_{L^{p',2}}}{t^{d(1/2-1/p)}}. \tag{2.8}$$

Remark 1. The estimates from [20] actually involve Besov spaces $B^s_{p,2}$ instead of $W^{s,p}$, and are slightly better than (2.7) due to the embedding $B^s_{p,2} \subset W^{s,p}$, $B^s_{p',2} \supset W^{s,p'}$ (see [9] chapter 6).

3 Energy estimates

High total energy estimate The following energy estimate bounds all components of the solution (ρ, u).

Proposition 3.1. *We recall the notation $r = \rho - 1$. For $(r_0, u_0) \in \mathcal{H}^N$, N large enough, $\|r\|_{W^{2,\infty}}$ small enough,*

$$\|(r,u)(t)\|_{\mathcal{H}^N} \leq C\|(r_0,u_0)\|_{\mathcal{H}^N} exp\big(\int_0^t C\|(r,u)\|_{\mathcal{W}^{1,\infty}} ds\big),$$

with $C = C(\|(r,u)\|_{L^\infty \mathcal{H}^N})$ a locally bounded function.

The proof, not new, is postponed for completeness in appendix A

Low transport energy estimate

Proposition 3.2. *Let $\mathbb{P}u$ satisfy*

$$\partial_t \mathbb{P}u + \mathbb{P}(u \cdot \nabla \mathbb{P}u + \mathbb{P}u \cdot \nabla \mathbb{Q}u) = 0,$$

then for $p, q > 1$, $k \in \mathbb{N}$, $2k > d/q + 1$ we have the a priori estimate

$$\frac{d}{dt}\|\mathbb{P}u\|_{W^{2k,p}} \lesssim (\|\mathbb{P}u\|_{W^{2k,q}} + \|\mathbb{Q}u\|_{W^{2k,q}})\|\mathbb{P}u\|_{W^{2k,p}}. \tag{3.1}$$

Energy estimates for transport type equations are standard, see e.g. the textbook [4] chapter 3. Since the "transport" term is $\mathbb{P}(u \cdot \nabla \mathbb{P}u)$ rather than $u \cdot \nabla \mathbb{P}u$, we include a short self-contained proof.

Proof. Set $P_k = \Delta^k \mathbb{P}u$, then $\Delta^k \mathbb{P} = \Delta^k - \Delta^{k-1}\nabla \mathrm{div}$ is a differential operator of order $2k$ so that

$$\partial_t P_k + (u \cdot \nabla P_k) = R_k(\mathbb{P}u, \mathbb{Q}u),$$

Note that since $\Delta^k \mathbb{P}\mathbb{Q} = 0$ we have

$$R_k = -[\Delta^k \mathbb{P}, u \cdot \nabla]\mathbb{P}u - \Delta^k \mathbb{P}(\mathbb{P}u \cdot \nabla \mathbb{Q}u) = -[\Delta^k \mathbb{P}, u \cdot \nabla]u - [\Delta^k \mathbb{P}, \mathbb{P}u \cdot \nabla]\mathbb{Q}u.$$

We take the scalar product with $|P_k|^{p-2}P_k$ and integrate in space to get

$$\frac{d}{dt}\|P_k\|_p^p \lesssim \|\mathrm{div}(u)\|_\infty \|P_k\|_p + \|R_k|P_k|^{p-2}P_k\|_1.$$

Since $W^{2k,q} \subset W^{1,\infty}$, we are left to estimate terms of the form $\|\partial^\alpha \mathbb{P}u \partial^\beta v |P_k|^{p-1}\|_1$ with v a placeholder for $\mathbb{P}u$ or $\mathbb{Q}u$, $|\alpha| + |\beta| = 2k + 1$. For $1/p_1 + 1/p_2 = 1/p$ we have

$$\begin{aligned} \|\partial^\alpha \mathbb{P}u \partial^\beta v |P_k|^{p-1}\|_1 &\lesssim \|\mathbb{P}u\|_{W^{|\alpha|,p_1}} \|v\|_{W^{|\beta|,p_2}} \|\mathbb{P}u\|_{W^{2k,p}}^{p-1} \\ &\lesssim \|\mathbb{P}u\|_{W^{2k,p}} \|v\|_{W^{|\beta|,p_2}} \|\mathbb{P}u\|_{W^{2k,p}}^{p-1}, \end{aligned}$$

provided $1/p - (2k - |\alpha|)/d \leq 1/p_1 \leq 1/p$, which is equivalent to

$$0 \leq \frac{1}{p_2} \leq \frac{2k - |\alpha|}{d} = \frac{|\beta| - 1}{d}.$$

On the other hand the condition $W^{2k,q} \subset W^{|\beta|,p_2}$ is satisfied provided $\frac{1}{q} - \frac{2k-|\beta|}{d} \leq \frac{1}{p_2}$, the two conditions on p_2 lead to $1/q < (2k-1)/d$ which is the assumption. We conclude

$$\frac{d}{dt}\|P_k\|_p^p \lesssim (\|\mathbb{P}u\|_{W^{2k,q}} + \|\mathbb{Q}u\|_{W^{2k,q}})\|\mathbb{P}u\|_{W^{2k,p}}^p. \tag{3.2}$$

Taking the L^p norm in (4.3) and using the continuity of $\mathbb{P}: \ L^p \to L^p$ directly gives

$$\frac{d}{dt}\|\mathbb{P}u\|_p \lesssim (\|\mathbb{Q}u\|_\infty + \|\mathbb{P}u\|_\infty)\|\mathbb{P}u\|_{W^{1,p}} \lesssim (\|\mathbb{Q}u\|_{W^{2k,q}} + \|\mathbb{P}u\|_{W^{2k,q}})\|\mathbb{P}u\|_{W^{2k,p}}. \tag{3.3}$$

Summing (3.2) and (3.3) concludes. □

4 Well-posedness for $d \geq 5$

The main result of this section is the following :

Theorem 4.1. *Under assumptions 2.1, for $d \geq 5$, there exists $(\varepsilon_0, c, N, k) \in (\mathbb{R}^{+*})^2 \times \mathbb{N}^2$ such that for $\varepsilon \leq \varepsilon_0$, $\delta << \varepsilon$, if*

$$\|(\rho_0 - \alpha, u_0)\|_{\mathcal{H}^N \cap \mathcal{W}^{k,4/3}} \leq \varepsilon, \ \|\mathbb{P}u_0\|_{W^{k,4}} \leq \delta,$$

then the solution of (1.1) *exists on $[0,T]$ with*

$$T \geq \frac{c}{\delta}.$$

We recall that the system satisfied by $\psi = \mathbb{Q}u + iU^{-1}w$ and $\mathbb{P}u$ is (see (2.4))

$$\left\{ \begin{array}{lcl} \partial_t \psi - iH\psi & = & \mathcal{N}_1(\psi, \mathbb{P}u), \\ \partial_t \mathbb{P}u + \mathbb{P}(u \cdot \nabla \mathbb{P}u + \mathbb{P}u \cdot \nabla \mathbb{Q}u) & = & 0. \end{array} \right. \tag{4.1}$$

We will prove a priori estimates for the solution in a space where local well-posedness holds.

The bootstrap argument We shall prove the following property : for c, ε small enough, there exists $C > 0$ such that for $t \leq c/\delta$, if we have the estimates

$$\|\psi\|_{H^N} + \|\mathbb{P}u\|_{H^N} \leq C\varepsilon, \ \|\psi\|_{W^{k,4}} \leq C\delta + \frac{C\varepsilon}{(1+t)^{d/4}}, \ \|\mathbb{P}u\|_{W^{k,4}} \leq C\delta,$$

that we respectively name total energy, dispersive estimate and transport energy, then

$$\|\psi\|_{H^N} + \|\mathbb{P}u\|_{H^N} \leq C\varepsilon/2, \ \|\psi\|_{W^{k,4}} \leq C\delta/2 + \frac{C\varepsilon}{2(1+t)^{d/4}}, \ \|\mathbb{P}u\|_{W^{k,4}} \leq C\delta/2.$$

From now on, C is only used for the constant of the bootstrap argument, while other constants are labelled as $C_1, C_2 ...$ and can change from line to line.

The energy estimate Since $\|\psi\|_{H^N} \sim \|(\rho-1, \mathbb{Q}u)\|_{\mathcal{H}^N}$, the energy estimate of proposition 3.1 implies for $k > d/4+1$

$$\begin{aligned}\|\psi\|_{H^N} + \|\mathbb{P}u\|_{H^N} &\leq C_1\|z_0\|_{H^N}\exp\left(C_2\int_0^t \|\psi\|_{W^{k,4}} + \|\mathbb{P}u\|_{W^{k,4}}ds\right)\\ &\leq C_1\varepsilon\exp(C_2C\big(2\delta t + \varepsilon/(d/4-1)\big)).\end{aligned}$$

Take $C \geq 2C_1e^1$, for $t \leq c\delta$, ε, c small enough (depending on C) we have

$$\|\psi\|_{H^N} + \|\mathbb{P}u\|_{H^N} \leq C_1e^1\varepsilon \leq C\varepsilon/2. \tag{4.2}$$

The transport energy estimate We apply Proposition 3.2 with $p=q=4$, k even, $4k > d$, $t \leq c/\delta$

$$\begin{aligned}\frac{d}{dt}\|\mathbb{P}u\|_{W^{k,4}} &\lesssim (\|\mathbb{P}u\|_{W^{k,4}} + \|\mathbb{Q}u\|_{W^{k,4}})\|\mathbb{P}u\|_{W^{k,4}} \leq \left(C\delta + C\delta + \frac{C\varepsilon}{(1+t)^{d/4}}\right)C\delta\\ \Rightarrow \|\mathbb{P}u\|_{W^{k,4}} &\leq \delta\left(1 + 2C_1C^2c + \frac{C^2C_1\varepsilon}{d/4-1}\right) \leq 2\delta < C\delta/2,\end{aligned} \tag{4.3}$$

for c, ε small enough, $C > 2$.

The dispersive estimate The first equation in (2.4) rewrites

$$\psi(t) = e^{itH}\psi_0 + \int_0^t e^{i(t-s)H}\mathcal{N}_1(\psi, \mathbb{P}u)ds,$$

The linear evolution $e^{itH}\psi_0$ is estimated with the dispersive estimate (2.7) and Sobolev embeddings

$$\|e^{itH}\psi_0\|_{W^{k,4}} \lesssim \min(\|\psi_0\|_{W^{k,4/3}}/t^{d/4}, \|\psi_0\|_{H^{k+d/4}}) \lesssim \frac{\varepsilon}{(1+t)^{d/4}}. \tag{4.4}$$

The structure of the nonlinearity does not matter here, the only important points are

1. The presence of U^{-1} in $U^{-1}\nabla\big((1-a)\mathrm{div}\mathbb{Q}u\big)$ is not an issue since $U^{-1}\nabla = \sqrt{2-\Delta}\nabla/|\nabla$ is the composition of a smooth Fourier multiplier and the Riesz multiplier,
2. All nonlinear terms are at least quadratic, and involve derivatives of order at most 2.

We only detail the estimate of $\mathbb{Q}(\mathbb{Q}u\cdot\nabla\mathbb{P}u)$ as the others can be done in a similar (simpler) way. Using the dispersion estimate and Sobolev embedding

$$\left\|\int_0^t e^{i(t-s)H}\mathbb{Q}(\mathbb{Q}u\cdot\nabla\mathbb{P}u)ds\right\|_{W^{k,4}} \lesssim \int_0^{t-1}\frac{\|\mathbb{Q}u\cdot\nabla\mathbb{P}u\|_{W^{k,4/3}}}{(t-s)^{d/4}}ds + \int_{t-1}^t \|\mathbb{Q}u\cdot\nabla\mathbb{P}u\|_{H^{k+d/4}}ds.$$

The product rules give

$$\begin{aligned}\|\mathbb{Q}u\cdot\nabla\mathbb{P}u\|_{W^{k,4/3}} &\lesssim \|\mathbb{Q}u\|_{L^4}\|\mathbb{P}u\|_{H^{k+1}} + \|\mathbb{Q}u\|_{W^{k,4}}\|\mathbb{P}u\|_{H^1} \leq 2C^2\left(\delta + \frac{\varepsilon}{(1+s)^{d/4}}\right)\varepsilon,\\ \|\mathbb{Q}u\cdot\nabla\mathbb{P}u\|_{H^{k+d/4}} &\lesssim \|\mathbb{Q}u\|_{W^{k+d/4,4}}\|\mathbb{P}u\|_{W^{1,4}} + \|\mathbb{Q}u\|_{L^4}\|\mathbb{P}u\|_{W^{k+1+d/4,4}}\\ &\lesssim \|\mathbb{Q}u\|_{H^N}\|\mathbb{P}u\|_{W^{k,4}} + \|\mathbb{Q}u\|_{W^{k,4}}\|\mathbb{P}u\|_{H^N}\\ &\leq C^2\varepsilon\delta + C^2\varepsilon\left(\delta + \frac{\varepsilon}{(1+s)^{d/4}}\right).\end{aligned}$$

The bootstrap assumption directly gives

$$\begin{aligned}\left\| \int_0^t e^{i(t-s)H}\mathbb{Q}(\mathbb{Q}u\cdot\nabla\mathbb{P}u)ds\right\|_{W^{k,4}} &\leq C_1C^2\left(\delta\varepsilon+\varepsilon^2\int_0^{t-1}\frac{1}{(1+s)^{d/4}(t-s)^{d/4}}ds\right)\\ &\quad +C_1C^2\varepsilon\left(\delta+\frac{\varepsilon}{(1+t)^{d/4}}\right)\\ &\leq C_2C^2\varepsilon\left(\delta+\frac{\varepsilon}{(1+t)^{d/4}}\right).\end{aligned}$$

We conclude by using (4.4), for C large enough, ε small enough

$$\|z(t)\|_{W^{k,4}}\leq\frac{C_0\varepsilon}{(1+t)^{d/4}}+C_1C^2\varepsilon\left(\delta+\frac{\varepsilon}{(1+t)^{d/4}}\right)\leq\frac{C}{2}\left(\delta+\frac{\varepsilon}{(1+t)^{d/4}}\right). \tag{4.5}$$

End of proof Putting together (4.2), (4.3) and (4.5), we see that as long as the solution exists and $t\leq c/\delta$, $\|z\|_{\mathcal{H}^N}$ remains small and ρ remains bounded away from 0. According to the blow up criterion the solution exists at least for $t\leq c/\delta$.

5 Well-posedness for $d = 3, 4$

This section is similar to the previous one but is significantly more technical. The low dimension version of theorem 4.1 reads

Theorem 5.1. *Under assumptions 2.1, for $d = 3, 4$, there exists $(\varepsilon, c, N, k) \in (\mathbb{R}^{+*})^2 \times \mathbb{N}^2$, $p > \frac{2d}{d-2}$ such that for $\delta << \varepsilon$, if*

$$\|(r,u_0)\|_{\mathcal{H}^N\cap\mathcal{W}^{k,p'}}+\||x|(r_0,\mathbb{Q}u_0)\|_{L^2}\leq\varepsilon,\ \|\mathbb{P}u_0\|_{W^{k,p'}\cap W^{k,p}}+\||x|\mathbb{P}u_0\|_{L^2}\leq\delta,$$

then the solution of (1.1) *exists on* $[0, T]$ *with*

$$T\geq\frac{c}{\delta}.$$

Remark 2. Unlike $d \geq 5$, one can not directly use the dispersive estimate to get closed bounds. This approach works for cubic and higher order nonlinearities, but not for quadratic terms. Therefore the emphasis is put here on how to control quadratic terms, while the analysis of higher order terms is much less detailed. We label such terms as "*cubic*" and they are generically denoted R. The fact that they include loss of derivatives is unimportant.

For $\psi : [0, T] \times \mathbb{R}^d \to \mathbb{C}^d$, and C a constant to choose later, we use the following notations:

$$\begin{aligned}\|\psi\|_{X(t)} &= \max\left(\|\psi(t)\|_{H^N}+\|xe^{-itH}\psi\|_{L^2},\ (1+t)^{3(1/2-1/p)}(\|\psi\|_{W^{k,p}}-C\delta)\right),\\ \|\psi\|_{X_T} &= \sup_{[0,T]}\|\psi\|_{X(t)}.\end{aligned}$$

For simplicity of notations, we only consider the (most difficult) case $d = 3$.

5.1 Preparation of the equations

We recall that the extended system is

$$\left\{\begin{array}{lcl} \partial_t\psi - iH\psi & = & \mathcal{N}_1(\psi, \mathbb{P}u) + R, \text{ R cubic,} \\ \partial_t \mathbb{P}u + \mathbb{P}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u) & = & 0, \end{array}\right. \tag{5.1}$$

$$\mathcal{N}_1 = \nabla((1-a)\mathrm{div}w) + (2-g')w - \frac{1}{2}\nabla(|\mathbb{Q}u|^2 - |w|^2) + iU^{-1}\nabla\big((1-a)\mathrm{div}\mathbb{Q}u - u\cdot w\big) \\ -\mathbb{Q}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u),$$

the first line of the nonlinearity $\mathcal{N}_1$ depends only on the dispersive variable ψ ("purely dispersive terms") while the second line contains interaction between ψ and the transport component $\mathbb{P}u$ ("dispersive-transport terms").
In order to apply the method of space-time resonances, it is useful that the Fourier transform of the purely dispersive nonlinear terms cancels at 0. As such, the real part $\nabla((1-a)\mathrm{div}w) + (2-g')w + \frac{1}{2}\nabla(|\mathbb{Q}u|^2 - |w|^2)$ is well prepared, but not the imaginary part $U^{-1}\nabla((1-a)\mathrm{div}\mathbb{Q}u)$. We refer to the discussion at the beginning of section 5 in [3] for a more detailed motivation. As in [3] (see also [21]) we use the following normal form transform:

Lemma 5.2. *For*

$$w_1 = w - \nabla\big(B[w,w] - B[\mathbb{Q}u, \mathbb{Q}u]\big).$$

with B the bilinear Fourier multiplier of symbol $\dfrac{a'(1)-1}{2(2+|\eta|^2)+|\xi-\eta|^2)}$. Then w_1 satisfies

$$\partial_t w_1 + \Delta\mathbb{Q}u = \nabla\, div((1-a)\mathbb{Q}u) + R, \tag{5.2}$$

where R contains cubic and higher order nonlinearities in $\mathbb{Q}u, \mathbb{P}u, l$.
Moreover, for any $T > 0$ the map $\psi = \mathbb{Q}u + iU^{-1}w \to \mathbb{Q}u + iU^{-1}w_1$ is bi-lipschitz on a neighbourhood of 0 in X_T, it is also bi-Lipschitz near 0 for the norm $\|\psi_0\|_{H^N\cap W^{k,p'}} + \||x|\psi_0\|_{L^2}$.

Proof. According to (2.3) w satisfies

$$\begin{array}{rcl} \partial_t w + \Delta\mathbb{Q}u & = & \nabla\big((1-a)\mathrm{div}\mathbb{Q}u\big) - \nabla(u\cdot w) \\ & = & \nabla\mathrm{div}\big((1-a)\mathbb{Q}u\big) + \nabla\big(\nabla a\cdot\mathbb{Q}u\big) - \nabla(\mathbb{Q}u\cdot w) - \nabla(\mathbb{P}u\cdot w) \\ & = & \nabla\mathrm{div}\big((1-a)\mathbb{Q}u\big) + \nabla\big((a'(1)-1)w\cdot\mathbb{Q}u\big) - \nabla(\mathbb{P}u\cdot w) + R, \end{array}$$

with $R = \nabla\big((\nabla a - a'(1)w)\cdot\mathbb{Q}u\big)$ a cubic term. Then w_1 satisfies

$$\begin{array}{rcl} \partial_t w_1 + \Delta\mathbb{Q}u & = & \nabla\mathrm{div}\big((1-a)\mathbb{Q}u\big) - \nabla(\mathbb{P}u\cdot w) + R \\ & & +\nabla\Big((a'(1)-1)w\cdot\mathbb{Q}u + 2B[w,\Delta\mathbb{Q}u] + 2B[(\Delta-2)w, \mathbb{Q}u]\Big) \\ & = & \nabla\mathrm{div}\big((1-a)\mathbb{Q}u\big) - \nabla(\mathbb{P}u\cdot w) + R, \end{array}$$

by construction of B (note that R includes now terms like $\nabla B[\mathbb{Q}(u\cdot\nabla\mathbb{P}u), \mathbb{Q}u]$ that have all a gradient in factor). The fact that $w \to w_1$ is bi-Lipschitz is proposition 5.4 and proposition 5.5 in [3]. □

Final form of the equations We define $b(\mathbb{Q}u, w) = B[w, w] - B[\mathbb{Q}u, \mathbb{Q}u]$ so that $w_1 = w - \nabla b(\mathbb{Q}u, w) = w - \nabla b(\mathbb{Q}u, w_1) + R$, R cubic. The new system on $\Psi = \mathbb{Q}u + iU^{-1}w_1$ and $\mathbb{P}u$ is

$$\left\{\begin{array}{lcl} \partial_t\Psi - iH\Psi &=& \nabla\Big((\Delta-2)b + (a-1)\mathrm{div}w_1 - \frac{1}{2}\big(|\mathbb{Q}u|^2 - |w_1|^2\big)\Big) + (2-g')w_1 \\ & & +iU^{-1}\nabla\mathrm{div}\big((1-a)\mathbb{Q}u\big) \\ & & -iU^{-1}\nabla(\mathbb{P}u\cdot w_1) - \mathbb{Q}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u) + R, \\ \partial_t\mathbb{P}u &=& -\mathbb{P}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u), \end{array}\right. \tag{5.3}$$

with R containing cubic terms.

Remark 3. Note that all cubic terms in (5.2) are gradients of the unknowns (see also the system (2.3)), therefore the change of variables $w_1 \to U^{-1}w_1$ creates no nonlinearities with singularity at low frequency. For example, the new term $\nabla B[\mathbb{Q}(u\cdot\nabla\mathbb{P}u), \mathbb{Q}u]$ becomes $U^{-1}\nabla B[\mathbb{Q}(u\cdot\nabla\mathbb{P}u), \mathbb{Q}u]$.

Note that $(1-a)\mathbb{Q}u$ is a quadratic term since at main order it is $-a'(1)l\mathbb{Q}u$, with $l = \Delta^{-1}\mathrm{div}w$.

Remark 4. An important consequence of lemma 5.2 is that it suffices to estimate Ψ instead of ψ, and the smallness of ψ_0 implies the smallness of Ψ_0.

According to the remark above, it is sufficient to prove the following :

Theorem 5.3. *Under assumptions* 2.1, *there exists* $\varepsilon, c, N, k \in (\mathbb{R}^{+*})^2 \times \mathbb{N}^2$, $p > 2d/(d-2)$ *such that for* $\delta << \varepsilon$, *if*

$$\|\Psi_0\|_{H^N\cap W^{k,p'}} + \||x|\Psi_0\|_{L^2} + \|\mathbb{P}u_0\|_{H^N} \le \varepsilon,\ \|\mathbb{P}u_0\|_{W^{k,p}\cap W^{k,p'}} + \||x|\mathbb{P}u_0\|_{L^2} \le \delta,$$

then the solution of (5.3) *exists on* $[0,T]$, $T \ge c/\delta$ *and* $\|\Psi\|_{X_T} \lesssim \varepsilon$.
This result implies theorem 5.1.

5.2 The bootstrap argument

A priori estimates The aim of this paragraph and the next one is to prove that for c, ε small enough, there exists $C > 0$ such that for $t \le c\delta$, if we have the following estimates

$$\left\{\begin{array}{ll} \|\Psi\|_{H^N} \le C\varepsilon & \text{(total energy)}, \\ \||x|e^{-itH}\Psi\|_{L^2} \le C\varepsilon,\ \|\Psi\|_{W^{k,p}} \le C\delta + \dfrac{C\varepsilon}{(1+t)^{d(1/2-1/p)}} & \text{(dispersive estimates)}, \\ \|\mathbb{P}u\|_{W^{k,q}} + \||x|\mathbb{P}u_0\|_{L^2} \le C\delta & \text{(transport energy)}, \end{array}\right. \tag{5.4}$$

then the same estimates hold with $C/2$ instead of C.

Remark 5. We point out that the bootstrap argument is slightly different from the one for theorem 4.1. Indeed in large dimension, we can propagate the a priori bounds (up to multiplicative constants independent of ε, δ) on a time c/δ while for $d = 3, 4$ the proof implies $c = O(\varepsilon)$. In other words, if $\|\Psi_0\|_{H^N} \le \varepsilon' < \varepsilon$ it is not clear if $\|\Psi(t)\|_{H^N} \lesssim \varepsilon'$ on $[0, c/\delta]$ with δ independent of ε', see remark 6 for technical details.

The dispersive estimates are significantly more difficult than for $d \ge 5$ and are detailed in paragraph 5.3.

The energy estimate This is the same argument as for $d \geq 5$, from proposition 3.1 and using $3(1/2 - 1/p) > 1$ (integrability of the decay)

$$\|\Psi\|_{H^N} + \|\mathbb{P}u\|_{H^N} \lesssim C_1\varepsilon \exp\left(C_2 C\left(2\delta t + \frac{\varepsilon}{3(1/2-1/p)-1}\right)\right),$$

so that for C large enough, ε, c small enough, $t \leq c/\delta$

$$\|\Psi\|_{H^N} + \|\mathbb{P}u\|_{H^N} \leq C\varepsilon/2. \tag{5.5}$$

The transport energy estimate The $W^{k,q}$ estimate is a consequence of proposition 3.2 as for $d \geq 5$: for k even large enough

$$\begin{aligned} \frac{d}{dt}\|\mathbb{P}u\|_{W^{k,p}} \leq C_1(\|\mathbb{P}u\|_{W^{k,p}} + \|\mathbb{Q}u\|_{W^{k,p}})\|\mathbb{P}u\|_{W^{k,p}} &\lesssim C_1 C^2\left(\delta + \frac{\varepsilon}{t^{1+}}\right)\delta \\ \Rightarrow \|\mathbb{P}u\|_{W^{k,p}} \leq C_1 C^2\delta(c + C_2\varepsilon) &\leq \frac{C}{2}\delta, \end{aligned} \tag{5.6}$$

for c, ε small enough. The same estimate (with indices $q \leq p$) applied again gives

$$\|\mathbb{P}u\|_{W^{k,q}} \leq C_1 \int_0^t (\|\mathbb{P}u\|_{W^{k,p}} + \|\mathbb{Q}u\|_{W^{k,p}})\|\mathbb{P}u\|_{W^{k,q}} ds \leq \frac{C\delta}{2}.$$

For the weighted estimate we follow a similar energy method. First multiply the equation on $\mathbb{P}u$ by x_j :

$$\begin{aligned} \partial_t(x_j\mathbb{P}u) + x_j\mathbb{P}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u) &= \partial_t(x_j\mathbb{P}u) + \mathbb{P}\big(u\cdot\nabla(x_j\mathbb{P}u) + x_j\mathbb{P}u\cdot\nabla\mathbb{Q}u\big) \\ &\quad +[x_j,\mathbb{P}](u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u) + \mathbb{P}\big([x_j, u\cdot\nabla]\mathbb{P}u\big) \\ &= = 0. \end{aligned}$$

The operator $[x_j, \mathbb{P}]$ is the Fourier multiplier of symbol $i\partial_{\xi_j}\mathbb{P}(\xi)$ which is dominated by $1/|\xi|$ therefore it is bounded $\dot{H}^{-1} \mapsto L^2$. From the embedding $\dot{H}^1 \subset L^6$, $[x_j, \mathbb{P}]$ is bounded $L^{6/5} \to L^2$. We deduce the following bound

$$\begin{aligned} \int_{\mathbb{R}^d} [x_j,\mathbb{P}](u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u)\cdot(x_j\mathbb{P}u)dx &\lesssim \|u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u\|_{L^{6/5}}\|x_j\mathbb{P}u\|_{L^2} \\ &\lesssim \|\mathbb{P}u\|_{W^{k,6/5}}\big(\|\mathbb{P}u\|_{W^{k,p}} + \|\mathbb{Q}u\|_{W^{k,p}}\big)\|x_j\mathbb{P}u\|_{L^2}. \end{aligned}$$

Using an integration by parts

$$\begin{aligned} \int_{\mathbb{R}^d} \mathbb{P}\big(u\cdot\nabla(x_j\mathbb{P}u)\big)\cdot x_j\mathbb{P}u dx &= \int_{\mathbb{R}^d} \big(u\cdot\nabla(x_j\mathbb{P}u)\big)\cdot([\mathbb{P},x_j]\mathbb{P}u + x_j\mathbb{P}u)dx \\ &= \int_{\mathbb{R}^d} -\frac{\mathrm{div}\mathbb{Q}u}{2}\big(|x_j\mathbb{P}u|^2 + (x_j\mathbb{P}u)[\mathbb{P},x_j]\mathbb{P}u\big) \\ &\qquad -u\cdot\nabla([x_j,\mathbb{P}]\mathbb{P}u)\cdot x_j\mathbb{P}u dx \\ &\lesssim \|\mathbb{Q}u\|_{W^{k,p}}(\|x_j\mathbb{P}u\|_{L^2} + \|\mathbb{P}u\|_{L^{6/5}})\|x_j\mathbb{P}u\|_{L^2} \\ &\quad +(\|\mathbb{P}u\|_{W^{k,p}} + \|\mathbb{Q}u\|_{W^{k,p}})\|\mathbb{P}u\|_{L^2}\|x_j\mathbb{P}u\|_{L^2}. \end{aligned}$$

Similarly

$$\begin{aligned}\int_{\mathbb{R}^d} x_j\mathbb{P}u\cdot\nabla\mathbb{Q}u\cdot(x_j\mathbb{P}u)dx &\leq \|\mathbb{Q}u\|_{W^{k,p}}\|x_j\mathbb{P}u\|_{L^2}^2,\\ \int_{\mathbb{R}^d}\mathbb{P}([x_j,u\cdot\nabla]\mathbb{P}u)\cdot x_j\mathbb{P}udx &= -\int_{\mathbb{R}^d}\mathbb{P}(u_j\mathbb{P}u)\cdot x_j\mathbb{P}udx\\ &\lesssim \big(\|\mathbb{P}u\|_{W^{k,p}}+\|\mathbb{Q}u\|_{W^{k,p}}\big)\|\mathbb{P}u\|_{L^2}\|x_j\mathbb{P}u\|_{L^2}.\end{aligned}$$

From these estimates we deduce

$$\frac{d}{dt}\|x_j\mathbb{P}u\|_{L^2}^2\leq C\|x_j\mathbb{P}u\|_{L^2}\big(\|\mathbb{Q}u\|_{W^{k,p}}+\|\mathbb{P}u\|_{W^{k,p}\cap W^{k,q}}+\|x_j\mathbb{P}u\|_{L^2}\big)^2,$$

which readily yields by integration in time and the bootstrap assumption (5.4)

$$\|x_j\mathbb{P}u\|_{L^2}\leq C_1C^2(\delta t+\varepsilon)\delta\leq\frac{C\delta}{2}. \tag{5.7}$$

5.3 The dispersive estimates

We start from (5.3) that reads $\partial_t\Psi = iH\psi+\mathcal{D}(\Psi)+\mathcal{T}(\Psi,\mathbb{P}u)+R$, with $\mathcal{D}$ the first two lines of nonlinear terms (quadratic dispersive terms), $\mathcal{T}$ the third line (dispersive-transport, and transport-transport) and R cubic. Equivalently

$$\Psi(t)=e^{itH}\Psi_0+\int_0^t e^{i(t-s)H}\big(\mathcal{D}(\Psi)+\mathcal{T}(\Psi,\mathbb{P}u)+R\big)(s)ds$$

The linear part is not difficult to control:

$$\||x|e^{-itH}e^{itH}\Psi_0\|_{L^2}=\||x|\Psi_0\|_{L^2}, \tag{5.8}$$

$$\|e^{itH}\Psi_0\|_{W^{k,p}}\lesssim\frac{\|\Psi_0\|_{H^N\cap W^{k,p'}}}{(1+t)^{3(1/2-1/p)}}. \tag{5.9}$$

The terms in $\mathcal{D}$ and $\mathcal{T}$ are not estimated exactly similarly. Basically the control of $\mathcal{D}$ is quite difficult, but amounts to a straightforward modification of the estimates in [3], while $\mathcal{T}$ is new but a bit easier to control. For completeness, the key arguments to estimate $\mathcal{D}$ are provided in the appendix B.
The nonlinearity $\mathcal{T}$ contains four terms that are all very similar. For conciseness we only detail how to estimate $U^{-1}\nabla(\mathbb{P}u\cdot w_1)$, which contains all the difficulties of the other terms plus a singular factor U^{-1}. Finally, R contains cubic terms easier to control. To fix ideas, we estimate the term $U^{-1}\nabla B[\mathbb{Q}(u\cdot\nabla\mathbb{P}u),\mathbb{Q}u]$ that appears in the proof of lemma 5.2).

Weighted bounds

Quadratic term We shall detail the estimate of $xe^{-itH}\int_0^t e^{i(t-s)H}U^{-1}\nabla(\mathbb{P}u\cdot\Psi)ds$. Since $w_1 = U(\Psi - \overline{\Psi})/2$, we have

$$U^{-1}\nabla(\mathbb{P}u\cdot w_1) = U^{-1}\nabla\bigg(\mathbb{P}u\cdot U\frac{\Psi-\overline{\Psi}}{2}\bigg),$$

so we define $\varphi = U\Psi$ and consider the term $U^{-1}\nabla(\mathbb{P}u\cdot\varphi)$. The weighted estimate amounts to control

$$\bigg\|\nabla_\xi\bigg(\int_{[0,t]\times\mathbb{R}^d} i\xi e^{-isH(\xi}U^{-1}(\xi)\widehat{\mathbb{P}u}(\xi-\eta)\cdot\widehat{\varphi}(\eta)d\eta\bigg)ds\bigg\|_2,$$

therefore setting $m(\xi,s) = i\partial_{\xi_j}(\xi U^{-1}(\xi)e^{-isH(\xi)})$

$$\begin{aligned} x_j e^{-itH}\int_0^t U^{-1}\nabla(\mathbb{P}u\cdot\varphi)ds &= \mathcal{F}^{-1}\bigg(\int_0^t\int_{\mathbb{R}^d} m(\xi,s)\widehat{\mathbb{P}u}(\xi-\eta)\cdot\widehat{\varphi}(\eta)d\eta ds\bigg) \\ &\quad + \int_0^t e^{-isH}U^{-1}\nabla\big(x_j\mathbb{P}u\cdot\varphi\big)ds. \end{aligned}$$

We have $m = i\partial_{\xi_j}(\xi U^{-1})e^{-isH} + \xi U^{-1}e^{-isH}s\partial_{\xi_j}H = m_1 + m_2$. From elementary computations $m_1 = m_3\sqrt{1+|\xi|^2}/|\xi|$ with m_3 a bounded multiplier, therefore it is continuous $W^{1,6/5}\to L^2$ and from Minkowski's inequality

$$\bigg\|\int_0^t m_1(D)(\mathbb{P}u\cdot\varphi)ds\bigg\|_{L^2} \lesssim \int_0^t \|\mathbb{P}u\cdot\varphi\|_{W^{1,6/5}}ds \lesssim \int_0^t \|\mathbb{P}u\|_{W^{1,6/5}}\|\varphi\|_{W^{k,p}}ds, \tag{5.10}$$

similarly $m_2 \lesssim (1+|\xi|)^2 s$ so

$$\bigg\|\int_0^t m_2(D)(\mathbb{P}u\cdot\varphi)ds\bigg\|_{L^2} \lesssim \int_0^t \|\mathbb{P}u\cdot\varphi\|_{H^2}ds \lesssim \int_0^t s\|\mathbb{P}u\|_{H^2}\|\varphi\|_{W^{k,p}}ds. \tag{5.11}$$

Next we use a frequency truncation $\chi(D)$, with $\chi\in C_c^\infty,\ \chi\equiv 1$ near 0, and split

$$\int_0^t e^{-isH}U^{-1}\nabla\big(x_j\mathbb{P}u\cdot\varphi\big)ds = \int_0^t e^{-isH}(\chi+1-\chi)U^{-1}\nabla\big(x_j\mathbb{P}u\cdot\varphi\big)ds.$$

The low frequency part is estimated using the boundedness of $\chi U^{-1}\nabla:\ L^{6/5}\to L^2$

$$\bigg\|\int_0^t e^{-isH}\chi U^{-1}\nabla\big(x_j\mathbb{P}u\cdot\varphi\big)ds\bigg\|_2 \lesssim \int_0^t \|x_j\mathbb{P}u\cdot\varphi\|_{6/5}ds \lesssim \int_0^t \|x_j\mathbb{P}u\|_2\|\varphi\|_3 ds. \tag{5.12}$$

For the high frequency part, we use that $(1-\chi)U^{-1}$ is a bounded multiplier, the identity

$$\nabla(x_j\mathbb{P}u\cdot\varphi) = (\mathbb{P}u\cdot\varphi)e_j + (\nabla\varphi)\cdot(x_j\mathbb{P}u) + \nabla(\mathbb{P}u)\cdot\big([x_j, Ue^{isH}]e^{-isH}\Psi + Ue^{isH}(x_je^{-isH}\Psi)\big),$$

and the bound $|[x_j, Ue^{isH}]|(\xi)\lesssim (1+s)(1+|\xi|)$, so

$$\begin{aligned} \bigg\|\int_0^t e^{-isH}(1-\chi)U^{-1}\nabla\big(x_j\mathbb{P}u\cdot\varphi\big)ds\bigg\|_2 &\lesssim \int_0^t (\|\mathbb{P}u\|_2 + \|x_j\mathbb{P}u\|_2)\|\varphi\|_{W^{k,p}} \\ &\quad + \|\mathbb{P}u\|_{H^1}(1+s)\|\Psi\|_{W^{k,p}} \\ &\quad + \|\mathbb{P}u\|_{W^{k,p}}\|x_je^{-isH}\Psi\|_2 ds. \end{aligned} \tag{5.13}$$

From estimates (5.10),(5.11),(5.12),(5.13) and the bootstrap assumptions (5.4) we get for c, ε small enough, $t \leq c/\delta$

$$\begin{aligned}\left\| x_j e^{-itH} \int_0^t e^{i(t-s)H} U^{-1} \nabla(\mathbb{P}u \cdot U\Psi) ds \right\|_2 &\leq C^2 C_1 \int_0^t (1+s)\delta \left(\delta + \frac{\varepsilon}{(1+s)^{3(1/2-1/p)}} \right) + \delta\varepsilon ds \\ &\leq C^2 C_1 (c^2 + c\varepsilon). \end{aligned} \tag{5.14}$$

Remark 6. The weighted estimate is the only point in the proof where we need $c \lesssim \varepsilon$. More precisely, it is due to the commutator term $[x_j, e^{-isH}] = s(\nabla H)$ which causes a strong loss of decay in the estimate (5.11).

Cubic term From similar computations, we end up estimating terms like

$$\int_0^t e^{-isH} U^{-1} \nabla B[\mathbb{Q}(u \cdot \nabla \mathbb{P}u), e^{isH} x_j e^{-isH} \mathbb{Q}u] d\eta, \tag{5.15}$$

$$\int_0^t e^{-isH} s(\partial_j H) U^{-1} \nabla B[\mathbb{Q}(u \cdot \nabla \mathbb{P}u), \mathbb{Q}u] ds ... \tag{5.16}$$

For the first one, since the symbol of B is $(a'(1) - 1)/(2(2 + |\eta|^2 + |\xi - \eta|^2))$) we may use the boundedness of the bilinear multiplier ∇B

$$\begin{aligned}\left\| \int_0^t e^{-isH} U^{-1} \nabla B[\mathbb{Q}(u \cdot \nabla \mathbb{P}u), e^{isH} x_j e^{-isH} \mathbb{Q}u] d\eta \right\|_2 &\lesssim \int_0^t \|u \cdot \nabla \mathbb{P}u\|_\infty \|x_j e^{-isH} \mathbb{Q}u\|_2 ds \\ &\lesssim \int_0^t \|u\|_{W^{k,p}} \|\mathbb{P}u\|_{W^{k,p}} \|x_j e^{-isH} \mathbb{Q}u\|_2 ds \\ &\lesssim \int_0^t C^3 \left(\delta + \frac{\varepsilon}{(1+s)^{3(1/2-1/p)}} \right)^2 \varepsilon ds \\ &\lesssim C^3(\delta^2 t + \varepsilon^2)\varepsilon, \end{aligned} \tag{5.17}$$

similarly for (5.16)

$$\begin{aligned}\left\| \int_0^t e^{-isH} s(\partial_j H) U^{-1} \nabla B[\mathbb{Q}(u \cdot \nabla \mathbb{P}u), \mathbb{Q}u] ds \right\|_2 &\lesssim \int_0^t s \|u\|_{W^{k,p}} \|\mathbb{Q}u\|_{W^{k,p}} \|\mathbb{P}u\|_{W^{k,p}} ds \\ &\lesssim \int_0^t C^3 s \left(\delta + \frac{\varepsilon}{(1+s)^{3(1/2-1/p)}} \right)^3 ds \\ &\lesssim C^3(t^2\delta^3 + \varepsilon^3). \end{aligned} \tag{5.18}$$

(5.17) and (5.18) are clearly more than enough to close the weighted estimate.

Closing the bound The estimates (5.8) (linear), quadratic (5.14) (quadratic, see also the appendix for the purely dispersive terms) and (5.17) ,(5.18) (cubic) lead to

$$\|x e^{-itH} \Psi(t)\|_{L^2} \leq C_1 \varepsilon + C^2 C_1 (c^2 + c\varepsilon), \tag{5.19}$$

which gives the first part of the dispersive estimate by choosing C large enough, $c = \varepsilon$ small enough.

Bounds in $W^{k,p}$ The computations are done "up to choosing k, N larger".

Quadratic term As previously we focus on $U^{-1}\nabla(\mathbb{P}u \cdot U\Psi) := U^{-1}\nabla(\mathbb{P}u \cdot \varphi)$. We can assume $t \geq 2$, indeed for $t \leq 2$ by Sobolev's embedding and for N large enough

$$\begin{aligned} \| \int_0^t e^{i(-s)H} U^{-1}\nabla(\mathbb{P}u \cdot \varphi) ds\|_{W^{k,p}} &\lesssim \int_0^t \|\nabla(\mathbb{P}u \cdot \varphi)\|_{H^{N-1}} ds \\ &\lesssim \|\mathbb{P}u\|_{L^\infty([0,2],H^N)} \|\varphi\|_{L^\infty([0,2],H^N)} \leq C\varepsilon^2. \end{aligned}$$

Since $p > 6$, $3(1/2 - 1/p) := 1 + \gamma > 1$. Minkowski's inequality and the dispersive estimate imply

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)H} U^{-1}\nabla(\mathbb{P}u \cdot \varphi) ds \right\|_{W^{k,p}} &\lesssim \int_0^{t-1} \frac{\|\mathbb{P}u \cdot \varphi\|_{W^{k+1,p'}}}{(t-s)^{1+\gamma}} ds + \int_{t-1}^t \|\mathbb{P}u \cdot \varphi\|_{H^{k+2}} ds \\ &\lesssim \int_0^{t-1} \frac{\|\mathbb{P}u\|_{W^{k,p'}} \|\varphi\|_{H^N} + \|\mathbb{P}u\|_{H^{k+1}} \|\varphi\|_{W^{k,q}}}{(t-s)^{1+\gamma}} ds \\ &\quad + \int_{t-1}^t \|\mathbb{P}u\|_{W^{k,p}} \|\varphi\|_{H^N} + \|\mathbb{P}u\|_{H^N} \|\varphi\|_{W^{k,p}} ds, \end{aligned}$$

with $1/q + 1/2 = 1/p'$. By interpolation, the bootstrap assumption gives with $\theta/p + (1-\theta)/2 = 1/q$

$$\begin{aligned} \|\mathbb{P}u\|_{H^{k+1}} \leq C\varepsilon^{1/(N-k)} \delta^{1-1/(N-k)}, \ \|\varphi\|_{W^{k,q}} &\leq \|\varphi\|^\theta_{W^{k,p}} \|\varphi\|^{1-\theta}_{H^k} \\ &\leq C\left(\delta + \frac{\varepsilon}{(1+s)^{1+\gamma}}\right)^\theta \varepsilon^{1-\theta}. \end{aligned}$$

Note that $\theta = 1/(p/2 - 1)$ with $p > 6$ thus $\theta < 1/2$. We choose $6 < p < 8$, so that $\theta > 1/3$, from elementary computations, and for $t \geq 2$

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)H} U^{-1}\nabla(\mathbb{P}u \cdot \varphi) ds \right\|_{W^{k,p}} &\leq C_1 C^2 \int_0^{t-1} \frac{\delta\varepsilon}{(t-s)^{1+\gamma}} + \frac{\delta^{1-\frac{1}{N-k}+\theta} \varepsilon^{1-\theta+\frac{1}{N-k}}}{(t-s)^{1+\gamma}} \\ &\qquad + \frac{\delta^{1-\frac{1}{N-k}} \varepsilon^{1+\frac{1}{N-k}}}{(t-s)^{1+\gamma}(1+s)^{\theta(1+\gamma)}} ds \\ &\quad + C_1 C^2 \left(\delta\varepsilon + \frac{\varepsilon^2}{(1+t)^{1+\gamma}}\right) \\ &\leq C_1 C^2 \left(\delta\varepsilon + \frac{\varepsilon^2}{(1+t^2)^{1+\gamma}} + \frac{\delta^{1-\frac{1}{N-k}} \varepsilon^{1+\frac{1}{N-k}}}{t^{\theta(1+\gamma)}}\right) \\ &\leq C_1 C^2 \left(\delta\varepsilon + \frac{\varepsilon^2}{(1+t^2)^{1+\gamma}} + \delta\varepsilon + \frac{\varepsilon^2}{t^{\frac{N-k}{3}}}\right). \end{aligned} \tag{5.20}$$

Cubic term As for the quadratic terms, we split the integral on $[0, t-1] \cup [t-1, t]$. The integral on $[t-1, t]$ is easily controlled, for the other part, using again the boundedness of ∇B,

and choosing $1/q = 1/p' - 1/2$

$$\begin{aligned}\left\| \int_0^{t-1} e^{i(t-s)H} U^{-1} \nabla B[\mathbb{Q}(u\cdot\nabla\mathbb{P}u), \mathbb{Q}u] ds \right\|_{W^{k,p}} &\lesssim \int_0^{t-1} \frac{\|u\cdot\nabla\mathbb{P}u\|_{W^{k,q}} \|\mathbb{Q}u\|_{H^N}}{(t-s)^{1+\gamma}} ds \\ &\lesssim \int_0^{t-1} \frac{\|u\|_{W^{k,p}} \|\mathbb{P}u\|_{H^N}}{(t-s)^{1+\gamma}} \|\mathbb{Q}u\|_{H^N} ds \\ &\lesssim C^3\varepsilon^2\left(\delta + \frac{\varepsilon}{(1+t)^{1+\gamma}}\right).\end{aligned}$$

Closing the bound From (5.9), (5.20) and the cubic estimates above we deduce

$$\|\Psi(t)\|_{W^{k,p}} \leq \frac{C_1\varepsilon}{(1+t)^{1+\gamma}} + C^2C_1\varepsilon\left(\delta + \frac{\varepsilon}{(1+t)^{1+\gamma}}\right) \leq \frac{C}{2}\left(\delta + \frac{\varepsilon}{(1+t)^{1+\gamma}}\right). \tag{5.21}$$

End of proof As for theorem 4.1, we close the bootstrap argument thanks to the energy estimate (5.5), the transport energy estimates (5.6),(5.7), and the dispersive estimates (5.19), (5.21).

6 An example of blow up

We consider in this section the special case of quantum fluid, where K is proportional to $1/\rho$. More precisely, if ψ is a smooth solution of

$$i\partial_t\psi + \Delta\psi = \frac{g(|\psi|^2)\psi}{2}, \tag{6.1}$$

that does not cancel, the so-called Madelung transform $\psi = \sqrt{\rho}e^{i\phi}$, $u = \nabla\phi$ is well defined and (ρ, u) satisfy

$$\left\{\begin{array}{lcl} \partial_t\rho + \mathrm{div}(\rho u) & = & 0, \\ \partial_t u + u\cdot\nabla u + \nabla g(\rho) & = & 2\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \end{array}\right. \tag{6.2}$$

As pointed out in the review article [13], the Madelung transform is a major tool to study nonlinear Schrödinger equations with non zero boundary conditions at infinity, with a (technical but important) drawback that it becomes singular in presence of vacuum, that is when ρ vanishes. Cancellation of ψ is often labelled as vortex formation in the framework of NLS. We construct here in dimension one an example of solution such that vacuum appears in finite time.

Proposition 6.1. *Let ψ_0 real valued such that*

$$1 - \psi_0 \in \mathcal{S}(\mathbb{R}^d),\ \psi_0 > 0 \text{ on } \mathbb{R}^d \setminus \{0\},\ \psi_0(0) = 0,\ \Delta\psi_0(0) \neq 0.$$

Then there exists a local solution to (6.1) *with $\psi|_{t=0} = \psi_0$, and $T > 0$ such that $\psi(x,t) > 0$ on $]0,T] \times \mathbb{R}^d$.*
Consequently, there exists a solution to (6.2) *that blows up in finite time.*

Proof. Since $1-\psi_0$ is smooth, the existence of a smooth solution to (6.1) is a consequence of the standard theory for NLS equations. From direct computations

$$\partial_t|\psi|^2 = -2\mathrm{Im}(\overline{\psi}\Delta\psi), \tag{6.3}$$
$$\partial_t^2|\psi|^2 = 2|\Delta\psi|^2 - \mathrm{Re}\big(g\overline{\psi}\Delta\psi + \overline{\psi}\Delta^2\psi - \overline{\psi}\Delta(g\psi)\big). \tag{6.4}$$

Since ψ_0 is real valued, we deduce

$$\forall\, x\in\mathbb{R}^d,\ \partial_t|\psi(x,0)|^2 = 0, \partial_t^2|\psi|^2(0,0) = 2|\Delta\psi_0(0)|^2 > 0. \tag{6.5}$$

By continuity, there exists $\alpha>0$ such that $\partial_t^2|\psi(x,t)|^2\geq\alpha$ on a neighbourhood U of $(x,t)=(0,0)$, we deduce by Taylor expansion

$$\forall\,(x,t)\in U,\ |\psi(x,t)|^2 = |\psi(x,0)|^2 + \int_0^t (t-s)\partial_t^2|\psi(x,s)|^2 ds \geq \alpha t^2/2.$$

Now by continuity, for $(x,t)\in U^c$, t small enough, $\psi(x,t)$ does not vanish hence for t small enough, $\psi(\cdot,t)>0$ on $\mathbb{R}^d$. Thanks to the reversibility[1] of the equations, starting with initial data $\psi(\cdot,t)$ and going backwards in time provides a solution of (6.1) that cancels at $x=0$ in a finite time T^*. The (inverse)Madelung transform $\psi\to(\rho,u)=\big(|\psi|^2,\ \mathrm{Im}(\frac{\overline{\psi}\nabla\psi}{|\psi|^2})\big)$ then gives a smooth solution of (6.2) initially without vacuum, but with formation of vacuum at $x=0, t=T^*$. This implies blow up of u according to the method of characteristics: define $X(t)$ as the flow associated to u, $X'(t)=u(t,X(t))$, we have

$$\frac{d}{dt}\rho(t,X(t)) = -\rho\mathrm{div}u,$$

hence $\rho(t,X(t))=\rho_0(X(0))e^{-\int_0^t \mathrm{div}u ds}$, the cancellation of ρ implies $\|u\|_{L^1_{T^*}W^{1,\infty}}=\infty$. □

Remark 7. The blow up is not linked to vorticity, indeed the initial data ψ_0 is real positive, thus its index is zero.

A The total energy estimate

This section is devoted to the proof of proposition 3.1. This is essentially a variation on the estimates in [5], that we include here for self-containedness.
We define $z=u+iw$ so that according to (1.3), (ρ,z) satisfy

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t z + u\cdot\nabla z + i\nabla z\cdot w + i\nabla(a\mathrm{div}z) + \nabla(g(\rho)) = 0. \end{cases} \tag{A.1}$$

A direct energy method where one takes the scalar product of the second equation with z and integrate causes loss of derivatives due to the term $i\nabla z\cdot w$. The remedy is done in two times : first use a gauge $\varphi_n(\rho)$ and derive an energy bound $\frac{d}{dt}\int|\mathbb{Q}(\varphi_n\Delta^n z)|^2dx$ for $n\in\mathbb{N}$, this estimate contains a loss of derivatives, but an other gauge estimate on $\mathbb{P}(\phi_n\Delta^n z)$ for an appropriate choice of ϕ_n compensates exactly the loss.

[1] The map $\psi(t)\to: \overline{\psi}(-t)$ leaves the solution set invariant, or equivalently $(\rho,u)(t)\to(\rho,-u)(-t)$

In what follows, R stands for a nonlinear term (quadratic of higher) that contains only derivatives of z of order at most $2n$, and is thus without loss of derivatives, I_R is an integrated term which is dominated by $\|(r,u)\|_{\mathcal{W}^{1,\infty}}\|(r,u)\|^2_{\mathcal{H}^{2n}}$.
We will need the following lemma :

Lemma A.1 ([8], lemma 3.1)**.** *For $Z \in C^1(\mathbb{R}^d,\mathbb{C})$, $W \in C^1(\mathbb{R}^d,\mathbb{R}^d)$, with limit 0 at infinity,*

$$2iIm\int_{\mathbb{R}^d} Z^*\cdot\nabla_0 Z\cdot W dx = \int Z^*\cdot curlW\cdot Z dx.$$

In particular, if W is a gradient, the integral is 0.

Equation on $\varphi_n(\rho)\Delta^n z$ We recall $a=\sqrt{\rho K}$, $w=\sqrt{K/\rho}\nabla\rho$, and start from

$$\partial_t z + u\cdot\nabla z + i\nabla z\cdot w + i\nabla(a\mathrm{div}z) + g'w = 0.$$

Apply $\varphi_n\Delta^n$ together with the commutator identity

$$\nabla(\Delta^n(a\mathrm{div}z)) = \nabla(a\mathrm{div}\Delta^n z) + \nabla\big((2n\nabla a)\cdot\nabla\mathrm{div}\Delta^{n-1}z\big) + R,$$

and $\varphi_n\nabla(a\mathrm{div}\Delta^n z) = \nabla(a\mathrm{div}(\varphi_n\Delta^n z)) - a\nabla\varphi_n\mathrm{div}\Delta^n z - a\nabla\Delta^n z\cdot\nabla\varphi_n + R$,

$$\begin{aligned}\partial_t(\varphi_n\Delta^n z) + u\cdot\nabla(\varphi_n\Delta^n z) + i\varphi_n\nabla(\Delta^n z)\cdot w + i\nabla(a\mathrm{div}(\varphi_n\Delta^n z)) + g'\varphi_n\Delta^n w\\ +2in\varphi_n\nabla(\nabla a\cdot\nabla\mathrm{div}\Delta^{n-1}z) - ia(\nabla\varphi_n)\mathrm{div}\Delta^n z - ia\nabla\Delta^n z\cdot\nabla\varphi_n = R,\end{aligned}$$

so using $\mathrm{div}=\mathrm{div}\circ\mathbb{Q}$ and $\nabla\mathrm{div}\Delta^{n-1}z=\Delta^n\mathbb{Q}z$

$$\begin{aligned}\partial_t(\varphi_n\Delta^n z) + u\cdot\nabla(\varphi_n\Delta^n z) + i\nabla(a\mathrm{div}(\varphi_n\Delta^n z)) + g'\varphi_n\Delta^n w\\ +i\varphi_n\nabla(\Delta^n z)\cdot w + 2in\varphi_n\nabla(\Delta^n\mathbb{Q}z)\cdot\nabla a - ia(\nabla\varphi_n)\mathrm{div}\Delta^n\mathbb{Q}z - ia\nabla\Delta^n z\cdot\nabla\varphi_n &= R\end{aligned} \quad (A.2)$$

The loss of derivative is caused by the left hand side of the second line. For $\varphi_n = a^n\sqrt{\rho}$, and denoting $\nabla_0 := \nabla - I_d\mathrm{div}$, we find

$$\begin{aligned}&a^n\sqrt{\rho}\nabla(\Delta^n z)\cdot w + 2na^n\sqrt{\rho}\nabla(\Delta^n\mathbb{Q}z)\cdot\nabla a - a\nabla(a^n\sqrt{\rho})\mathrm{div}\Delta^n\mathbb{Q}z - a\nabla\Delta^n z\cdot\nabla(a^n\sqrt{\rho})\\ &= \frac{a^{n+1}}{\sqrt{\rho}}\nabla(\Delta^n z)\cdot\nabla\rho + 2na^n a'\sqrt{\rho}\nabla(\Delta^n\mathbb{Q}z)\cdot\nabla\rho - \Big(na^n a'\sqrt{\rho}+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\mathrm{div}(\Delta^n\mathbb{Q}z)\nabla\rho\\ &\qquad - \Big(na^n a'\sqrt{\rho}+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla\Delta^n z\cdot\nabla\rho\\ &= \Big(na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla_0\Delta^n\mathbb{Q}z\cdot\nabla\rho + \Big(-na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla\Delta^n\mathbb{P}z\cdot\nabla\rho. \quad (A.3)\end{aligned}$$

We write $na^n\sqrt{\rho}a' + \frac{a^{n+1}}{2\sqrt{\rho}} = \varphi_n(na' + a/(2\rho))$, commute φ_n with ∇_0, then we use that for $n\geq 1$, $\mathbb{Q}\Delta^n$ is a differential operator of order $2n$, so $\varphi_n\mathbb{Q}\Delta^n\cdot = \mathbb{Q}\Delta^n(\varphi_n\cdot) + [\varphi_n,\mathbb{Q}\Delta^n]\cdot = \mathbb{Q}\big(\varphi_n\Delta^n\cdot\big) + P\cdot$ with P a differential operator of order $2n-1$, therefore

$$\Big(na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla_0\Delta^n\mathbb{Q}z\cdot\nabla\rho = \Big(na'+\frac{a}{2\rho}\Big)\nabla_0\mathbb{Q}(\varphi_n\Delta^n z)\cdot\nabla\rho + R. \quad (A.4)$$

Plugging (A.3) and (A.4) in (A.2) we get

$$\begin{aligned}\partial_t(\varphi_n\Delta^n z) &+ u\cdot\nabla(\varphi_n\Delta^n z)+g'\varphi_n\Delta^n w+i\nabla(a\mathrm{div}(\varphi_n\Delta^n z)) \qquad (A.5)\\ &= -i\Big(na'+\frac{a}{2\rho}\Big)\nabla_0\mathbb{Q}(\varphi_n\Delta^n z)\cdot\nabla\rho-i\Big(-na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla\Delta^n\mathbb{P}z\cdot\nabla\rho+R.\end{aligned}$$

Note that $g'\varphi_n\Delta^n w = g'(1)\varphi_n(1)\Delta^n w + R = 2\Delta^n w + R$ is without loss of derivatives but contains a linear term that can not be neglected for long time dynamics.

Energy estimate for $\mathbb{Q}\varphi_n\Delta^n z$ Take the scalar product of (A.5) with $\mathbb{Q}\varphi_n\Delta^n z$, integrate in space, and use lemma A.1

$$\begin{aligned}&\frac{d}{2dt}\int|\mathbb{Q}\varphi_n\Delta^n z|^2dx+\mathrm{Re}\int\big(u\cdot\nabla(\varphi_n\Delta^n z)+2\Delta^n w\big)\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}dx\\ =\mathrm{Im}\int\overline{\mathbb{Q}\varphi_n\Delta^n z}\cdot&\Big[\Big(na'+\frac{a}{2\rho}\Big)\nabla_0\mathbb{Q}(\varphi_n\Delta^n z)+\Big(-na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla\Delta^n\mathbb{P}z\Big]\cdot\nabla\rho dx+I_R\\ &=\mathrm{Im}\int\overline{\mathbb{Q}\varphi_n\Delta^n z}\cdot\Big(-na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla\Delta^n\mathbb{P}z\cdot\nabla\rho dx+I_R, \qquad (A.6)\end{aligned}$$

where $I_R=\int R\cdot\mathbb{Q}(\varphi_n\Delta^n z)dx$, for more details on the generic estimate I_R we refer to [3]. The right hand side is an unavoidable loss of derivative, the second term on the left hand side rewrites with the convention of summation on repeated indices, and using $\partial_j(\mathbb{Q}v)_i=\partial_i(\mathbb{Q}v)_j$

$$\begin{aligned}\int u_j\partial_j(\varphi_n\Delta^n z_i)\overline{(\mathbb{Q}\varphi_n\Delta^n z)_i})dx &= -\int\mathrm{div}(u)\varphi_n\Delta^n z\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}+u_j\varphi_n\Delta^n z_i\partial_j(\overline{\mathbb{Q}\varphi_n\Delta^n z})_i dx\\ &= -\int\mathrm{div}(u)\varphi_n\Delta^n z\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}+\frac{\mathrm{div}u}{2}|\mathbb{Q}\varphi_n\Delta^n z|^2\\ &\qquad +u_j(\mathbb{P}\varphi_n\Delta^n z)_i\partial_i\overline{(\mathbb{Q}\varphi_n\Delta^n z)}_j dx\\ &= -\int\mathrm{div}(u)\varphi_n\Delta^n z\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}+\frac{\mathrm{div}u}{2}|\mathbb{Q}\varphi_n\Delta^n z|^2\\ &\qquad +(\mathbb{P}\varphi_n\Delta^n z)\cdot\nabla u\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}dx\\ &= I_R.\end{aligned}$$

To summarize

$$\begin{aligned}\frac{d}{2dt}\int|\mathbb{Q}\varphi_n\Delta^n z|^2dx &+ \mathrm{Re}\int 2\Delta^n w\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}dx \qquad (A.7)\\ &= \mathrm{Im}\int\overline{\mathbb{Q}\varphi_n\Delta^n z}\cdot\Big(-na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla\Delta^n\mathbb{P}z\cdot\nabla\rho dx+I_R, \qquad (A.8)\end{aligned}$$

Energy estimate for $\mathbb{P}(\phi_n\Delta^n z)$ and compensated loss Let $\phi_n(\rho)$ be a second gauge. Following the same computations that led to (A.2)

$$\begin{aligned}&\partial_t(\phi_n\Delta^n z)+u\cdot\nabla(\phi_n\Delta^n z)+i\nabla(\phi_n\Delta^n z)\cdot w+i\nabla(a\mathrm{div}(\phi_n\Delta^n z))+2g'\phi_n\Delta^n w\\ &\quad+2in\phi_n\nabla(\Delta^n\mathbb{Q}z)\cdot\nabla a-ia(\nabla\phi_n)\mathrm{div}\Delta^n\mathbb{Q}z-ia\nabla\Delta^n z\cdot\nabla\phi_n=R.\end{aligned}$$

We take the scalar product with $\mathbb{P}\phi_n\Delta^n z$ and integrate in space, the first two terms are

$$\begin{aligned}\mathrm{Re}\int \big(\partial_t(\phi_n\Delta^n z) &+ u\cdot\nabla(\phi_n\Delta^n z)\big)\cdot\overline{(\mathbb{P}\phi_n\Delta^n z)} \\ &= \frac{1}{2}\frac{d}{dt}\int|\mathbb{P}\phi_n\Delta^n z|^2dx - \mathrm{Re}\int\frac{\mathrm{div}u}{2}|\mathbb{P}\phi_n\Delta^n z|^2 - \mathbb{Q}\phi_n\Delta^n z\cdot\nabla u\cdot\mathbb{P}\phi_n\Delta^n z\,dx \\ &= \frac{1}{2}\frac{d}{dt}\int|\mathbb{P}\phi_n\Delta^n z|^2dx + I_R.\end{aligned}$$

Most of the other terms are actually neglectible

$$\int\overline{(\mathbb{P}\phi_n\Delta^n z)}_i\partial_i(\phi_n\Delta^n z_j)w_j dx = -\int\overline{(\mathbb{P}\varphi_n\Delta^n z)}_i\varphi_n\Delta^n z_j\partial_i w_j dx = I_R,$$

and from the same computation $\int\overline{\mathbb{P}\phi_n\Delta^n z}\cdot(2in\phi_n\nabla(\Delta^n\mathbb{Q}z)\cdot\nabla a - ia\nabla\Delta^n z\cdot\nabla\phi_n)dx = I_R$. We are only left with

$$\mathrm{Im}\int\mathrm{div}(\Delta^n\mathbb{Q}z)a\nabla\phi_n\cdot\overline{\mathbb{P}\phi_n\Delta^n z}dx = -\mathrm{Im}\int a(\mathbb{Q}\Delta^n z)\cdot\nabla\overline{(\mathbb{P}\phi_n\Delta^n z)}\cdot\nabla\phi_n dx + I_R,$$

therefore

$$\frac{1}{2}\frac{d}{dt}\int|\mathbb{P}\phi_n\Delta^n z|^2dx = \int a(\mathbb{Q}\Delta^n z)\nabla\overline{(\mathbb{P}\Delta^n z)}\cdot(\phi_n\nabla\phi_n)dx + I_R. \tag{A.9}$$

Sum (A.8) and (A.9)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\mathbb{Q}\varphi_n\Delta^n z|^2+|\mathbb{P}\phi_n\Delta^n z|^2dx &+ \mathrm{Re}\int 2\Delta^n w\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}dx \\ &= \mathrm{Im}\int\varphi_n\Big(-na^n\sqrt{\rho}a' + \frac{a^{n+1}}{2\sqrt{\rho}}\Big)\overline{\mathbb{Q}\Delta^n z}\cdot\nabla\mathbb{P}\Delta^n z\cdot\nabla\rho dx \\ &\quad + \mathrm{Im}\int a\phi_n\phi_n'(\mathbb{Q}\Delta^n z)\cdot\nabla\overline{(\mathbb{P}\Delta^n z)}\cdot\nabla\rho dx + I_R.\end{aligned} \tag{A.10}$$

It is now apparent that the right choice for ϕ_n is a function such that

$$a\phi_n\phi_n' = \varphi_n\Big(-na^n\sqrt{\rho}a' + \frac{a^{n+1}}{2\sqrt{\rho}}\Big) \Leftrightarrow (\phi_n^2)' = -2na^{2n-1}\rho a' + a^{2n},$$

and which is positive close to $\rho = 1$, of course there exists such functions.

Correction of the linear drift There only remains to cancel the "linear" term

$$\mathrm{Re}\int 2\Delta^n w\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}dx = \int 2\Delta^n\nabla\rho\cdot\mathbb{Q}\Delta^n u dx + I_R.$$

We apply Δ^n to the mass conservation equation, multiply by $\Delta^n\rho$ and integrate,

$$\begin{aligned}\int\Delta^n\big(\partial_t\rho + \mathrm{div}(\rho u)\big)\Delta^n\rho dx &= \frac{1}{2}\frac{d}{dt}\int(\Delta^n\rho)^2dx + \int\rho\mathrm{div}(\Delta^n u)\Delta^n\rho dx \\ &= \frac{1}{2}\frac{d}{dt}\int(\Delta^n\rho)^2dx - \int\mathbb{Q}(\Delta^n u)\nabla\Delta^n\rho dx.\end{aligned} \tag{A.11}$$

Therefore adding (A.10) to two times (A.11) we obtain

$$\frac{1}{2}\frac{d}{dt}\int|\mathbb{Q}\varphi_n\Delta^n z|^2+|\mathbb{P}\phi_n\Delta^n z|^2+2|\Delta^n\rho|^2dx = I_R. \tag{A.12}$$

Conclusion By integration of (A.12) we find

$$\|\mathbb{Q}\varphi_n\Delta^n z\|_{L^2}^2+\|\mathbb{P}\phi_n\Delta^n z\|_{L^2}^2+2\|\Delta^n r\|_{L^2}^2\lesssim\int_0^t\|z\|_{\mathcal{H}^{2n}}^2\|z\|_{\mathcal{W}^{1,\infty}}ds. \tag{A.13}$$

Note that for $n=0$, we have $\varphi_0=\phi_0=\sqrt{\rho}$, therefore

$$\|z\|_{\mathcal{H}^0}\sim\|\mathbb{Q}\varphi_0\Delta^0 z\|_{L^2}^2+\|\mathbb{P}\phi_0\Delta^0 z\|_{L^2}^2+2\|r\|_{L^2}^2\lesssim\int_0^t\|z\|_{\mathcal{H}^0}^2\|z\|_{\mathcal{W}^{1,\infty}}ds,$$

(the estimate is actually a conservation of energy, see [5] paragraph 3.1).
Moreover for $n\geq 1$, $\mathbb{Q}\varphi_n\Delta^n z=\mathbb{Q}[\varphi_n,\Delta]\Delta^{n-1}z+[\mathbb{Q}\Delta,\varphi_n]\Delta^{n-1}z+\varphi_n\mathbb{Q}\Delta^n z=R+\varphi_n\mathbb{Q}\Delta^n z$, with $\|R\|_{H^2n}=O(\|\rho-1\|_{W^{2,\infty}}\|z\|_{H^{2n}})$ and the same observation stands for $\mathbb{P}\phi_n\Delta^n z$, thus

$$\|\mathbb{Q}\varphi_n\Delta^n z\|_{L^2}+\|\mathbb{P}\phi_n\Delta^n z\|_{L^2}=\|\Delta^n z\|_{L^2}+o(\|z\|_{H^{2n-1}}).$$

Using (A.13) for $n=0,N$ we conclude

$$\|z\|_{\mathcal{H}^{2N}}\sim\|z(t)\|_{H^{2N}}^2+\|\rho(t)-1\|_{L^2}^2\lesssim\int_0^t\|z(s)\|_{\mathcal{H}^{2N}}^2\|z(s)\|_{\mathcal{W}^{1,\infty}}ds.$$

B Control of the quadratic dispersive terms

The key result in [3] was the uniform bounds on $t\geq 0$

$$\|xe^{-itH}\Psi\|_{L^2}\lesssim\varepsilon,\ \|\Psi\|_{W^{k,p}}\lesssim\varepsilon/(1+t)^\gamma,$$

for irrotational initial data (that is $\mathbb{P}u=0$). Actually in [3] since $\mathbb{Q}u+iw=u+iw=\nabla(\phi+r)$, it was more convenient to work on $\widetilde{\Psi}=U\phi+r$. This difference causes merely a shift in regularity indices as $\|\widetilde{\Psi}\|_{H^N\times W^{k,p}}\sim\|\Psi\|_{H^{N-1}\times W^{k-1,p}}$.
We summarize here the arguments that can be used as a blackbox to obtain the bounds of the bootstrap argument (5.4). A few estimates are performed, but since the detailed analysis would be quite lengthy and is basically a repetition mutatis mutandis of the arguments in [3], we choose to only sketch the argument and point to the appropriate section of [3] when needed.

Generic nonlinearity According to (5.3), and linearizing $a-1=a'(1)l+R$, with R quadratic in l, $2w-g'w=\nabla(2l-g'(l))=\nabla(g''(1)l^2/2+O(l^3))$, the quadratic purely dispersive nonlinearity is

$$\nabla\bigg((\Delta-2)b+a'(1)l\mathrm{div}w_1-\frac{1}{2}\big(|\mathbb{Q}u|^2-|w_1|^2\big)\bigg)-g''(1)\nabla(l^2/2)-iU^{-1}\nabla\mathrm{div}\big(a'(1)l\mathbb{Q}u\big). \tag{B.1}$$

Following [3] we denote $\Psi^\pm$ as a placeholder for Ψ or $\overline{\Psi}$. Since $\mathbb{Q}u=(\Psi^+-\Psi^-)/2$, $w_1=U(\Psi^+-\Psi^-)/(2i)$, $l=\Delta^{-1}U\mathrm{div}(\Psi^+-\Psi^-)/(2i)+R$, R quadratic (see the change of variables of lemma 5.2), all quadratic nonlinearities can be written as nonlinearities in $\Psi^\pm$. Their precise form does not really matter, the main point is that they all take the form

$$\nabla B_g[\Psi^\pm,\Psi^\pm]=\nabla\mathcal{F}^{-1}\bigg(\int_{\mathbb{R}^d}\widehat{\Psi^\pm}\cdot B_g(\eta,\xi-\eta)\cdot\widehat{\Psi^\pm}d\eta\bigg), \tag{B.2}$$

with B_g a matrix valued symbol that can be for example $\frac{(|\xi|^2+2)(a'(1)-1)}{2(2+|\eta|^2+|\xi-\eta|^2)}$, $a'(1)\frac{U(\eta)\eta}{|\eta|^2}\otimes(\xi-\eta)$, $-iU^{-1}(\xi)\frac{U(\eta)\eta^t}{|\eta|^2}\otimes\xi$...

The method of space time resonances We denote $\widetilde{\Psi^\pm}=\mathcal{F}\big((e^{-itH})^\pm\big)\Psi)^\pm$. We recall that the equation (5.3) reads

$$\partial_t\Psi - iH\Psi = \mathcal{N}(\Psi,\mathbb{Q}u) = \mathcal{D}(\Psi)+\mathcal{T}(\Psi,\mathbb{P}u),$$

where $\mathcal{D}$, resp. $\mathcal{T}$, correspond to the purely dispersive, resp. dispersive transport terms. Let $B[\Psi^\pm,\Psi^\pm]$ a generic nonlinearity, the Duhamel formula leads to terms

$$\mathcal{F}\bigg(e^{-itH}\int_0^t e^{i(t-s)H}B_g[\Psi^\pm,\Psi^\pm]ds\bigg)=\int_0^t\int_{\mathbb{R}^d}e^{-is\Omega_{\pm\pm}}\widetilde{\Psi^\pm}(\eta)\cdot B_g(\eta,\xi-\eta)\cdot\widetilde{\Psi^\pm}(\xi-\eta)d\eta ds, \tag{B.3}$$

where $\Omega_{\pm\pm}(\xi,\eta)=H(\xi)\mp H(\eta)\mp H(\xi-\eta)$. The estimates do not require to split the various cases $\Omega_{++},\Omega_{-,+}\cdots$, so we write (if not ambiguous) Ω instead of $\Omega_{\pm,\pm}$.
Since $\partial_s(e^{-isH}\Psi)=e^{-isH}(\mathcal{D}(\Psi)+\mathcal{T}(\Psi,\mathbb{P}u)$, an integration by parts in s "improves" the nonlinearity which becomes cubic. Similarly from the identity

$$e^{-is\Omega}=\frac{\nabla_\eta\Omega}{-is|\nabla_\eta\Omega|^2}\cdot\nabla_\eta e^{-is\Omega},$$

an integration by parts in η leads to a gain of decay of $1/s$. Of course these integrations by parts are fruitful only if Ω,$|\nabla_\eta\Omega|$ do not cancel (resp. no time resonances and no space resonances), this leads to define the space-time resonant set as $\{(\xi,\eta):\ \Omega=0\}\cap\{(\xi,\eta):\ \nabla_\eta\Omega=0\}$. The so-called method of space-time resonances simply consists in splitting the phase space in time non resonant and space non resonant regions and do the integration by parts accordingly.
Some difficulties are that the space-time resonant region is actually quite large, as one can check that in the case of Ω_{-+} it is $\{(\xi,\eta):\ \xi=0\}$, thus a subspace of dimension 3 in $\mathbb{R}^6$. A second issue is that the symbol $H(\xi)=|\xi|\sqrt{2+|\xi|^2}$ is similar to $\sqrt{2}|\xi|$ at low frequencies (wave-like), so that for ε,η small $\Omega_{-+}(\varepsilon\eta,\eta)\sim-3\varepsilon|\eta|^3/(2\sqrt{2})$. This third order cancellation is worse than for the Schrödinger equation, and prevents any use of the Coifman-Meyer theorem. Instead, we use the following rough multiplier lemma due to Guo and Pausader (inspired by lemma 10.1 in [21])

Lemma B.1 ([19]). *For $0\le s\le d/2$, let $\|B\|_{[B^s]}=\min\big(\|B(\eta,\xi-\eta)\|_{\widetilde{L^\infty_\xi}\dot B^s_{2,1,\eta}},\|B(\xi-\zeta,\zeta)\|_{\widetilde{L^\infty_\xi}\dot B^s_{2,1,\zeta}}\big)$. For q_1,q_2 such that $2\le q_2,q_2'\le\frac{2d}{d-2s}$ and $\frac{1}{q_2}+\frac12=\frac{1}{q_1}+\frac12-\frac{s}{d}$, then*

$$\|B[f,g]\|_{L^{q_1}}\lesssim\|B\|_{[B^s]}\|f\|_{L^{q_2}}\|g\|_{L^2}.$$

Moreover, for $2\le q_1,q_2,q_3\le 2d(d-2s)$, and $\frac{1}{q_3}+\frac12-\frac{s}{d}=\frac{1}{q_1}+\frac{1}{q_2}$

$$\|B[f,g]\|_{L^{q_3}}\lesssim\|B\|_{[B^s]}\|f\|_{L^{q_1}}\|g\|_{L^{q_2}}.$$

The black box We use the following arguments directly taken from [3]: let $(\chi^a)_{a\in 2^{\mathbb{Z}}}$ a dyadic partition of unity, $\mathrm{supp}(\chi^a)\subset\{|\xi|\sim a\}$, for $a,b,c\in(2^{\mathbb{Z}})^3$, B_g a symbol associated to one of the nonlinearities, a function $\Phi(\xi,\eta)$ splits the phase space in time non resonant and space non resonant regions (in a sense to precise in lemma B.2), and we define the frequency localized symbols

$$B^{a,b,c,T}=\Phi\chi^a(\xi)\chi^b(\eta)\chi^c(\zeta)B_g,\ B^{a,b,c,X}=(1-\Phi)\chi^a\chi^b\chi^cB_g=\chi^a\chi^b\chi^cB^X, \tag{B.4}$$

with $\zeta=\xi-\eta$. Note that due to the relation $|\zeta|=|\xi-\eta|$, $B^{a,b,c}$ is zero except if $b\lesssim a\sim c, a\lesssim b\sim c$, $c\lesssim b\sim a$.

Lemma B.2. *For $a,b,c\in\mathbb{Z}^3$, let*

$$\mathcal{B}^{a,b,c,T}:=\frac{B^{a,b,c,T}}{\Omega},\ \mathcal{B}_1^{a,b,c,X}=\frac{\nabla_\eta\Omega B^{a,b,c,X}}{|\nabla_\eta\Omega|^2},\ \mathcal{B}_2^{a,b,c,X}=\nabla_\eta\cdot\mathcal{B}_1^{a,b,c,X},$$

$m=\min(a,b,c)$, $M=\max(a,b,c), l=\min(b,c)$. *For $0<s<2$, we have*

$$\begin{aligned}
&\text{if } M\gtrsim 1,\ \|\mathcal{B}^{a,b,c,T}\|_{[B^s]}\lesssim\frac{\langle M\rangle l^{3/2-s}}{\langle a\rangle}, && \|\mathcal{B}_1^{a,b,c,X}\|_{[B^s]}\lesssim\tfrac{\langle M\rangle^2 l^{3/2-s}}{\langle a\rangle},\\
& && \|\mathcal{B}_2^{a,b,c,X}\|_{[B^s]}\lesssim\tfrac{\langle M\rangle^2 l^{1/2-s}}{\langle a\rangle},\\
&\text{if } M<<1,\ \|\mathcal{B}^{a,b,c,T}\|_{[B^s]}\lesssim M^{-s}l^{1/2-s}, && \|\mathcal{B}_1^{a,b,c,X}\|_{[B^s]}\lesssim M^{1-s}l^{3/2-s},\\
& && \|\mathcal{B}_2^{a,b,c,X}\|_{[B^s]}\lesssim M^{-s}l^{1/2-s},
\end{aligned}$$

Lemma B.3. *We have for $t\geq 0$*

$$\|U^{-1}\Psi\|_{L^6}\lesssim\frac{1}{t^{3/5}}(\|xe^{-itH}\Psi\|_{L^2}+\|\Psi\|_{H^1}). \tag{B.5}$$

Proof. By interpolation and the dispersion estimate 2.2

$$\begin{aligned}
\|U^{-1}\Psi\|_{L^6}\leq\|U^{-1/3}\Psi\|_6^{3/5}\|U^{-2}\Psi\|_{L^6} &\lesssim \left(\frac{\|e^{-itH}\Psi\|_{L^{6/5,2}}}{t}\right)^{3/5}\left(\|U^{-1}\Psi\|_{L^2}+\|\Psi\|_{H^1}\right)^{2/5}\\
&\lesssim \left(\frac{\|xe^{-itH}\Psi\|_2}{t}\right)^{3/5}\left(\|xe^{-itH}\Psi\|_2+\|\Psi\|_{H^1}\right)^{2/5}.
\end{aligned}$$

□

Control of the purely dispersive quadratic terms in $W^{k,p}$ This (long) paragraph is devoted the bootstrap of the $W^{k,p}$ estimate. We focus on control of space non resonant and time non resonant terms.

Control of time non resonant terms in $W^{k,p}$ Integrating by parts in s, the frequency localized Duhamel terms of (B.3) lead to the following quantities

$$I^{a,b,c,T}:=\int_0^t e^{i(t-s)H}\left(\mathcal{B}^{a,b,c,T}[\mathcal{N}^\pm,\psi^\pm]+\mathcal{B}^{a,b,c,T}[\psi^\pm,\mathcal{N}^\pm]\right)ds-\left[e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\Psi^\pm,\Psi^\pm]\right]_0^t. \tag{B.6}$$

Consider for example $\int_0^{t-1} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{D}^\pm+\mathcal{T}^\pm,\Psi^\pm]ds$, $b\lesssim a\sim c$. We choose p,N such that $1/2-2\gamma>0$, $N-k-1/2+\gamma>0$ (this corresponds to p close enough to 6 and N large enough) and apply lemma B.1 with $s=1+\gamma$,

$$\begin{aligned}\Big\|\nabla^{k_1}\int_0^{t-1}\sum_{b\lesssim a\sim c}\mathcal{B}^{a,b,c,T}\|[\mathcal{D}^\pm,\Psi^\pm]ds\Big\|_p &\lesssim \int_0^{t-1}\sum_{b\lesssim a\sim c\le 1}\frac{ab\|\mathcal{B}^{a,b,c,T}\|_{[B^{1+\gamma}]}\|U^{-1}\mathcal{D}\|_2\|U^{-1}\Psi\|_2}{(t-s)^{1+\gamma}}\\ &\quad+\sum_{b\lesssim a\sim c,\ c\ge 1}\frac{c^{-N+k_1}\|\mathcal{B}^{a,b,c,T}\|_{[B^{1+\gamma}]}\|\mathcal{D}\|_2\|\Psi\|_{H^N}}{(t-s)^{1+\gamma}}ds\\ &\lesssim \int_0^{t-1}\sum_{b\lesssim a\sim c\le 1}\frac{aba^{-(1+\gamma)}b^{1/2-(1+\gamma)}\|U^{-1}\mathcal{D}\|_2\|U^{-1}\Psi\|_2}{(t-s)^{1+\gamma}}\\ &\quad+\sum_{b\lesssim a\sim c,\ c\ge 1}\frac{b^{3/2-(1+\gamma)}\|\mathcal{D}\|_2\|\Psi\|_{H^N}}{a^{N-k_1}N(t-s)^{1+\gamma}}ds\\ &\lesssim \int_0^{t-1}\frac{\|U^{-1}\mathcal{D}\|_2(\|U^{-1}\Psi\|_2+\|\Psi\|_{H^N})}{(t-s)^{1+\gamma}}ds,\end{aligned}$$

Then $\|U^{-1}\Psi\|_2\lesssim\|e^{-itH}|\Delta|^{-1/2}\Psi\|_2+\|\Psi\|_2\lesssim\|e^{-itH}\Psi\|_{L^{6/5,2}}+\|\Psi\|_2\lesssim\|xe^{-itH}\Psi\|_2+\|\Psi\|_2$, where $L^{6/5,2}$ is the Lorentz space, and we used the generalized Hölder inequality $L^{6/5,2}\times L^{3,\infty}\subset L^2$. On the other hand since ∇ is in factor of all purely dispersive quadratic nonlinearities (see equation (B.1)), it compensates the singular factor U^{-1}, and one easily gets

$$\|U^{-1}\mathcal{D}\|_{L^2}\lesssim\|\Psi\|_{W^{2,4}}^2\lesssim\|\Psi\|_{H^2}^{\frac{1+4\gamma}{4+4\gamma}}\|\Psi\|_{W^{2,p}}^{\frac{3}{4+4\gamma}}.$$

The bootstrap assumption gives the bound

$$\begin{aligned}C^3\int_0^{t-1}\frac{\varepsilon^{1/2}\big(\delta+\varepsilon/(1+s)^{1+\gamma}\big)^{\frac{3}{2(1+\gamma)}}\varepsilon}{(t-s)^{1+\gamma}}ds &\lesssim C^3\varepsilon^{3/2}\bigg(\delta^{\frac{3}{2(1+\gamma)}}+\frac{\varepsilon^{\frac{3}{2(1+\gamma)}}}{t^{\min(1+\gamma,3/(2(1+\gamma)))}}\bigg)\\ &\le C^3\varepsilon^{3/2}\bigg(\delta+\frac{\varepsilon}{t^{1+\gamma}}\bigg),\end{aligned}$$

for γ small enough. Like the case $d\ge 5$, the estimate of $\int_{t-1}^t$ is simpler, so is the estimate of $[e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\Psi^\pm,\Psi^\pm]]_0^{t-1}$, and the cases $a\lesssim b\sim c$, $c\lesssim b\sim a$ are similar. More detailed computations can be found in [3] paragraph 6.1.2 where the only difference is that instead of $\|\Psi\|_{W^{k,p}}\le C(\delta+\varepsilon/(1+t)^{1+\gamma})$, the boostrap assumption is $\|\Psi\|_{W^{k,p}}\le C\varepsilon/(1+t)^{1+\gamma}$. Omitting these computations, to summarize,

$$\Big\|\int_0^t e^{i(t-s)H}\sum_{a,b,c}\mathcal{B}^{a,b,c,T}[\mathcal{D}^\pm,\psi^\pm]-[e^{-i(t-s)H}\mathcal{B}^{a,b,c,T}[\Psi^\pm,\Psi^\pm]]_0^t\Big\|_{W^{k,p}}\lesssim C^3\varepsilon\bigg(\delta+\frac{\varepsilon}{(1+t)^{1+\gamma}}\bigg).\tag{B.7}$$

There remains to bound the new term $B[\mathcal{T}(\Psi)^\pm,\Psi^\pm]$: for $b\lesssim a\sim c\le 1$

$$\|\nabla_1^k\int_0^{t-1}e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}(\Psi)^\pm,\Psi^\pm]ds\|_{L^p}\lesssim\int_0^{t-1}\frac{a^{-\gamma}b^{1/2-\gamma}}{(t-s)^{1+\gamma}}\|U^{-1}\mathcal{T}\|_2\|U^{-1}\Psi\|_2ds,$$

and for $c \geq 1$

$$\big\|\nabla_1^k \int_0^{t-1} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}(\Psi)^{\pm},\Psi^{\pm}]ds\big\|_{L^p} \lesssim \int_0^{t-1} \frac{\langle a\rangle^k b^{3/2-(1+\gamma)}}{(t-s)^{1+\gamma}\langle a\rangle^{N+1}}\|\mathcal{T}\Psi\|_2\|\Psi\|_2 ds.$$

We deduce by summation

$$\big\| \sum_{b\lesssim a\sim c} \int_0^{t-1} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^{\pm},\Psi^{\pm}]ds\big\|_{W^{k,p}} \lesssim \int_0^{t-1} \frac{\|U^{-1}\mathcal{T}\|_2\|U^{-1}\Psi\|_2}{(t-s)^{1+\gamma}}ds + \int_0^{t-1} \frac{\|\mathcal{T}\|_2\|\Psi\|_{H^N}}{(t-s)^{1+\gamma}}ds$$

Unlike the purely dispersive nonlinearity, the transport-dispersive nonlinearity is not well-prepared, let us recall it is

$$\mathcal{T} = -iU^{-1}\nabla(\mathbb{P}u\cdot w_1) - \mathbb{Q}\big(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u\big),$$

but the factor $\mathbb{P}u$ is much more favourable thus we can simply apply the following estimates

$$\begin{aligned}\|U^{-1}\mathcal{T}\|_2 &\lesssim \|\mathcal{T}\|_{W^{1,6/5}} \lesssim \|\mathbb{P}u\|_{W^{k,6/5}}(\|\Psi\|_{H^N} + \|\mathbb{P}u\|_{H^N}),\\ \|\mathcal{T}\|_2 &\lesssim \|\mathbb{P}u\|_2(\|\Psi\|_{H^N} + \|\mathbb{P}u\|_{H^N}),\end{aligned}$$

combined with the bootstrap assumption (5.4) we find

$$\big\| \sum_{b\lesssim a\sim c} \int_0^{t-1} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^{\pm},\Psi^{\pm}]ds\big\|_{W^{k,p}} \lesssim C^3\int_0^{t-1}\frac{\varepsilon^2\delta}{(t-s)^{1+\gamma}}ds \lesssim C^3\varepsilon^2\delta. \tag{B.8}$$

The integral over $[t-1,t]$ is estimated in the same spirit : for $c\leq 1$

$$\begin{aligned}\big\| \int_{t-1}^{t} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^{\pm},\Psi^{\pm}]ds\big\|_{W^{k,p}} &\lesssim \int_{t-1}^{t}\|\mathcal{B}^{a,b,c,T}[\mathcal{T}^{\pm},\Psi^{\pm}]\big\|_{H^{k+2}}ds\\ &\lesssim \int_{t-1}^{t} ab\|\mathcal{B}^{a,b,c,T}\|_{[B^{1+\gamma}]}\|U^{-1}\mathcal{T}\|_2\|U^{-1}\Psi\|_2 ds,\end{aligned}$$

for $c\geq 1$

$$\begin{aligned}\big\| \int_{t-1}^{t} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^{\pm},\Psi^{\pm}]ds\big\|_{W^{k,p}} &\lesssim \int_{t-1}^{t}\|\mathcal{B}^{a,b,c,T}[\mathcal{T}^{\pm},\Psi^{\pm}]\big\|_{H^{k+2}}ds\\ &\lesssim \int_{t-1}^{t} \frac{\|\mathcal{B}^{a,b,c,T}\|_{[B^{1+\gamma}]}}{c^{N-k-2}}\|\mathcal{T}\|_2\|\Psi\|_{H^N}ds.\end{aligned}$$

As previously, we have $\|U^{-1}\mathcal{T}\|_2 + \|\mathcal{T}\|_2 \lesssim \big(\|\mathbb{P}u\|_{W^{k,6/5}} + \|\mathbb{P}u\|_{H^k}\big)\big(\|\mathbb{P}u\|_{H^N} + \|\Psi\|_{H^N}\big)$ and $\sum_{b\lesssim a\sim c\leq 1} ab\mathcal{B}^{a,b,c,T}\|_{[B^{1+\gamma}]} + \sum_{b\lesssim a\sim c,\ c\geq 1}\frac{\mathcal{B}^{a,b,c,T}\|_{[B^{1+\gamma}]}}{c^{N-k-2}} < \infty$, thus

$$\big\| \int_{t-1}^{t} \sum_{b\lesssim a\sim c} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^{\pm},\Psi^{\pm}]ds\big\|_{W^{k,p}} \lesssim C^3\int_{t-1}^{t}\varepsilon^2\delta ds = C^3\varepsilon^2\delta. \tag{B.9}$$

Putting together (B.7),(B.8),(B.9),

$$\big\|\sum_{a,b,c} I^{a,b,c,T}\big\|_{W^{k,p}} \lesssim C^3\varepsilon\bigg(\delta + \frac{\varepsilon}{(1+t)^{1+\gamma}}\bigg). \tag{B.10}$$

Control of $I^{a,b,c,X}$ in $W^{k,p}$ Integration by parts in η does not require to handle the nonlinear term $\mathcal{T}$ which, in this appendix, is the only novelty compared to [3].
If necessary, we simply reproduce the argument with the minor modifications:
Since control for t small just follows from the H^N bounds, we focus on $t \geq 1$, and the integral over $[1, t-1]$.

Frequency splitting

Since we only control $xe^{-itH}z$ in $L^\infty L^2$, in order to handle the loss of derivatives we follow the idea from [15] which corresponds to distinguish low and high frequencies with a threshold frequency depending on t. Let $\theta \in C_c^\infty(\mathbb{R}^+)$, $\theta|_{[0,1]} = 1$, $\mathrm{supp}(\theta) \subset [0,2]$, $\Theta(t) = \theta(\frac{|D|}{t^\nu})$, $\nu > 0$ small to choose later. For any quadratic term $B_g[z,z]$, we write

$$B_g[z^\pm, z^\pm] = \overbrace{B_g[(1-\Theta(t))z^\pm, z^\pm] + B_g[\Theta(t)z^\pm, (1-\Theta)(t)z^\pm]}^{\text{high frequencies}} + \overbrace{B_g[\Theta(t)z^\pm, \Theta(t)z^\pm]}^{\text{low frequencies}}.$$

High frequencies

Using the dispersion estimate 2.2, product estimates and Sobolev embedding we have for $\frac{1}{p_1} = \frac{1+\gamma}{3}$ and for any quadratic term $B^X[\Psi^\pm, \Psi^\pm]$:

$$\begin{aligned}
&\left\| \int_1^{t-1} e^{i(t-s)H} \big(U B_g[(1-\Theta(t))\Psi^\pm, \Psi^\pm] + U B_j[\Theta(t)z, (1-\Theta)(t)\Psi^\pm] \big) ds \right\|_{W^{k,p}} \\
&\leq \int_1^{t-1} \frac{1}{(t-s)^{1+\gamma}} \|\Psi\|_{W^{k+2,p_1}} \|(1-\Theta(s))\Psi\|_{H^{k+2}} ds \qquad \text{(B.11)} \\
&\leq \int_1^{t-1} \frac{1}{(t-s)^{1+\gamma}} \|\Psi\|_{H^N}^2 \frac{1}{s^{\nu(N-2-k)}} ds,
\end{aligned}$$

choosing N large enough so that $\nu(N-2-k) \geq 1+\gamma$, we obtain a bound $C_1 C^2 \varepsilon^2 / t^{1+\gamma}$.

Low frequencies

We estimate now quadratic term of the form $B^{a,b,c,X}[\Theta\Psi^\pm, \Theta\Psi^\pm]$ wich leads to consider:

$$\mathcal{F} I_3^{a,b,c,X} = e^{itH(\xi)} \int_1^{t-1} \int_{\mathbb{R}^N} \left((e^{-is\Omega} B^{a,b,c,X}(\eta, \xi-\eta) \widetilde{\Theta\Psi^\pm}(s,\eta) \widetilde{\Theta\Psi^\pm}(s, \xi-\eta) \right) d\eta ds,$$

with $\Omega = H(\xi) \mp H(\eta) \mp H(\xi-\eta)$. Using $e^{-is\Omega} = \frac{i\nabla_\eta \Omega}{s|\nabla_\eta \Omega|^2} \cdot \nabla_\eta e^{-is\Omega}$ and denoting $Ri = \frac{\nabla}{|\nabla|}$ the Riesz operator, $\Theta'(t) := \theta'(\frac{|D|}{t^\delta})$, $J = e^{itH} x e^{-itH}$, an integration by part in η gives:

$$\begin{aligned} I_3^{a,b,c,X} =& - \mathcal{F}^{-1}(e^{itH(\xi)}\Big(\int_1^{t-1} \frac{1}{s}\int_{\mathbb{R}^N} \big(e^{-is\Omega(\xi,\eta)}\mathcal{B}_{1,j}^{a,b,c,X}(\eta,\xi-\eta)\cdot\nabla_\eta[\Theta\widetilde{\Psi^\pm}(\eta)\Theta\widetilde{\Psi^\pm}(\xi-\eta)] \\ &\qquad + \mathcal{B}_{2,j}^{a,b,c,X}(\eta,\xi-\eta)\widetilde{\Theta\Psi^\pm}(\eta)\widetilde{\Theta\Psi^\pm}(\xi-\eta)d\eta\big)ds\Big) \\ =& -\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\Big(\mathcal{B}_{1,j}^{a,b,c,X}[\Theta(s)(Jz)^\pm,\Theta(s)\Psi^\pm] - \mathcal{B}_{1,j}^{a,b,c,X}[\Theta(s)\Psi^\pm,\Theta(s)(Jz)^\pm] \\ &\qquad + \mathcal{B}_{2,j}^{a,b,c,X}[\Theta(s)\Psi^\pm,\Theta(s)\Psi^\pm]\Big)ds \\ &-\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\Big(\mathcal{B}_{1,j}^{a,b,c,X}[\frac{1}{s^\delta}Ri\,\Theta'(s)\Psi^\pm,\Theta(s)\Psi^\pm] \\ &\qquad - \mathcal{B}_{1,j}^{a,b,c,X}[\Theta(s)\Psi^\pm,\frac{1}{s^\delta}Ri\Theta'(s)\Psi^\pm]\Big)ds. \end{aligned} \tag{B.12}$$

where we recall:

$$\mathcal{B}_1^{a,b,c,X} = \frac{\nabla_\eta\Omega}{|\nabla_\eta\Omega|^2}B^{a,b,c,X},\ \mathcal{B}_2^{a,b,c,X} = \nabla_\eta\cdot\mathcal{B}_1^{a,b,c,X}.$$

We now use these estimates to bound the first term of (B.12). There are three areas to consider: $b \lesssim c \sim a,\ c \lesssim c \lesssim a \sim b,\ a \lesssim b \sim c$.

Estimates for quadratic terms involving $\mathcal{B}_1^{a,b,c,X}$ In the case $c \lesssim a \sim b$, let $\varepsilon_1 > 0$ to be fixed later. Using Minkowski's inequality, dispersion and the rough multiplier theorem B.1 with $s = 1+\varepsilon_1$, $\frac{1}{q} = 1/2 + (\gamma-\varepsilon_1)/3$, $s = 4/3$, $\frac{1}{q_1} = 7/18 + \gamma/3$ for $a \geq 1$, $0 \leq k_1 \leq k$ we obtain

$$\begin{aligned} &\big\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{c\lesssim a\sim b}\mathcal{B}_1^{a,b,c,X}[\Theta(s)(Jz)^\pm,\Theta(s)z^\pm]ds\big\|_{L^p} \\ &\lesssim \int_1^{t-1}\frac{1}{s(t-s)^{1+3\varepsilon}}\sum_{c\lesssim a\sim b\leq 1}\|\mathcal{B}_1^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\|\Theta(s)Jz\|_{L^2}\|\Theta(s)z]\|_{L^q} \\ &\qquad + \sum_{c\lesssim a\sim b,\ 1\lesssim a\lesssim s^\nu} a^k\|\mathcal{B}_1^{a,b,c,X}\|_{[B^{4/3}]}\|\Theta(s)Jz\|_{L^2}\|\Theta(s)z]\|_{L^{q_1}}\big)ds \\ &\lesssim \int_1^{t-1}\frac{1}{s(t-s)^{1+3\varepsilon}}\Big(\sum_{a\lesssim 1}\sum_{c\lesssim a\sim b}\|\mathcal{B}_1^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\|\Theta(s)Jz\|_{L^2}\|\Theta(s)z]\|_{L^q} \\ &\qquad + \sum_{1\lesssim a\lesssim s^\nu}a^k\sum_{c\lesssim a\sim b}\|\mathcal{B}_1^{a,b,c,X}\|_{[B^{4/3}]}\|\Theta(s)Jz\|_{L^2}\|\Theta(s)z]\|_{L^{q_1}}\Big)ds \end{aligned}$$

Using lemma B.2 and interpolation we have for $\varepsilon_1 < 1/4$ and $\varepsilon_1 - \gamma > 0$,

$$\sum_{a\lesssim 1}\sum_{c\lesssim a\sim b}\|\mathcal{B}_1^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\lesssim \sum_{a\lesssim 1}a^{1-(1+\varepsilon_1)}\sum_{c\lesssim a}c^{\frac{3}{2}-(1+\varepsilon_1)}\lesssim 1,$$

$$\|\psi(s)\|_{L^q}\lesssim \|\psi(s)\|_{L^p}^{\frac{\epsilon_1-\gamma}{1+\gamma}}\|\psi(s)\|_{L^2}^{1-\frac{\epsilon_1-\gamma}{1+\gamma}}\lesssim C\varepsilon^{1-\frac{\epsilon_1-\gamma}{1+\gamma}}\left(\delta+\frac{\varepsilon}{(1+s)^{1+\gamma}}\right)^{\frac{\varepsilon_1-\gamma}{1+\gamma}}.$$

In high frequencies we have:

$$\sum_{1\lesssim a\lesssim s^\nu}a^k\sum_{c\lesssim a\sim b}\frac{\langle M\rangle^2 c^{3/2-4/3}}{\langle a\rangle}\lesssim s^{\nu(k+7/6)},\ \|\psi(s)\|_{L^{q_1}}\lesssim \varepsilon^{\frac{2+6\gamma}{3+3\gamma}}\left(\delta+\frac{\varepsilon}{(1+s)^{1+\gamma}}\right)^{\frac{1-3\gamma}{3+3\gamma}}.$$

Finally we conclude that if $\min\big(\varepsilon_1-2\gamma, 1/3-2\gamma-\nu(k+7/6)\big)\geq 0$ (this choice is possible provided γ and ν are small enough):

$$\begin{aligned}
&\Big\|\int_1^{t-1}\frac{1}{s}e^{-i(t-s)H}\sum_{c\lesssim a\sim b}\mathcal{B}_1^{a,b,c,X}[\Theta(s)(Jz)^\pm,\Theta(s)z^\pm]ds\Big\|_{W^{k,p}}\\
&\qquad\lesssim\int_1^{t-1}\frac{C^2\varepsilon}{s(t-s)^{1+\gamma}}\varepsilon^{1-\frac{\varepsilon_1-\gamma}{1+\gamma}}\left(\delta+\frac{\varepsilon}{(1+s)^{1+\gamma}}\right)^{\frac{\varepsilon_1-\gamma}{1+\gamma}}\\
&\qquad\qquad+\frac{C^2\varepsilon s^{\nu(7k+6)}}{s(t-s)^{1+\gamma}}\varepsilon^{\frac{2+6\gamma}{3+3\gamma}}\left(\delta+\frac{\varepsilon}{(1+s)^{1+\gamma}}\right)^{\frac{1-3\gamma}{3+3\gamma}}ds\\
&\qquad\lesssim\frac{C^2\varepsilon^2}{t^{1+\gamma}}+\frac{C^2\varepsilon^{2-\frac{\varepsilon_1-\gamma}{1+\gamma}}\delta^{\frac{\varepsilon_1-\gamma}{1+\gamma}}}{t}+\frac{C^2\varepsilon^{\frac{5+9\gamma}{3+3\gamma}}\delta^{\frac{1-3\gamma}{3+3\gamma}}}{t^{1-\nu(7k+6)}}\\
&\qquad\lesssim\frac{C^2\varepsilon^2}{t^{1+\gamma}}+\frac{C^2\varepsilon^2}{t^{\frac{1+\gamma}{1+2\gamma-\varepsilon_1}}}+C^2\varepsilon\delta+\frac{C^2\varepsilon^2}{t^{(1-\nu(7k+6))\frac{3+3\gamma}{2+6\gamma}}}\\
&\qquad\lesssim\frac{C^2\varepsilon^2}{t^{1+\gamma}}+C^2\varepsilon\delta.
\end{aligned}\tag{B.13}$$

The cases $b\lesssim c\sim a$ are very similar. The term $\nabla^{k_1}\displaystyle\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\mathcal{B}_1^{a,b,c,X}[\Theta(s)z^\pm,\Theta(s)(Jz)^\pm]ds$ is symmetric while the terms

$$\begin{aligned}
\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\big(&\mathcal{B}_1^{a,b,c,X}[\frac{1}{s^\delta}\frac{\nabla}{|\nabla|}\Theta'(s)z^\pm,\Theta(s)z^\pm]\\
&-\mathcal{B}_1^{a,b,c,X}[\Theta(s)z^\pm,\frac{1}{s^\delta}\frac{\nabla}{|\nabla|}\Theta'(s)z^\pm]\big)ds\|_{L^p},
\end{aligned}$$

are simpler since there is no weighted term Jz involved.

Estimates for quadratic terms involving $\mathcal{B}_2^{a,b,c,X}$ The last term to consider is

$$\Big\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{a,b,c}\mathcal{B}_2^{a,b,c,X}[\Theta(s)z^\pm,\Theta(s)z^\pm]ds\Big\|_{L^p}.$$

Let us focus on the case $b \lesssim a \sim c$. We use the same indices as for $\mathcal{B}_1^{a,b,c,X}$: $s = 1+\varepsilon_1$, $\frac{1}{q} = 1/2 + (\gamma-\varepsilon_1)/3$, $\frac{1}{q_1} = 7/18 + \gamma/3$,

$$\begin{aligned}
&\big\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{b\lesssim a\sim c}\mathcal{B}_2^{a,b,c,X}[\Theta(s)\Psi^{\pm},\Theta(s)\Psi^{\pm}]ds\big\|_{L^p}\\
&\lesssim \int_1^{t-1}\frac{1}{s(t-s)^{1+\gamma}}\bigg(\sum_{a\leq 1}\sum_{b\lesssim a\sim c}U(b)U(c)\|\mathcal{B}_2^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\|U^{-1}\Theta(s)\Psi\|_{L^2}\|U^{-1}\Theta(s)\Psi]\|_{L^q}\\
&\qquad\qquad + \sum_{1\leq a\lesssim s^{\nu}}a^k\sum_{b\lesssim a\sim c}\frac{U(b)}{\langle c\rangle^k}\|\mathcal{B}_2^{a,b,c,X}\|_{[B^{4/3}]}\|U^{-1}\Theta(s)\Psi\|_{L^2}\|\langle\nabla\rangle^k\Theta(s)\Psi]\|_{L^{q_1}}\bigg)ds
\end{aligned} \tag{B.14}$$

According to lemma B.2, we have for the first sum (provided $\varepsilon_1 < 1/4$):

$$\sum_{a\leq 1}\sum_{b\lesssim c\sim a}U(b)U(c)\|\mathcal{B}_2^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\lesssim\sum_{a\leq 1}\sum_{b\lesssim c\sim a}b^{1/2-\varepsilon_1}a^{\varepsilon_1}\lesssim 1.$$

and according to proposition B.3 and the bootstrap assumption (5.4)

$$\begin{aligned}
\|U^{-1}\Psi(s)\|_{L^2} &\lesssim \|\psi\|_X,\\
\|U^{-1}\Psi(s)\|_{L^q} &\lesssim \|U^{-1}\Psi\|_{L^2}^{1-\varepsilon_1+\gamma}\|U^{-1}\Psi\|_{L^6}^{\varepsilon_1-\gamma}\\
&\lesssim \frac{\|xe^{-itH}\Psi\|_2^{1-\varepsilon_1+\gamma}\big(\|xe^{-itH}\Psi\|_2+\|\Psi\|_{H^1}\big)^{\varepsilon_1-\gamma}}{s^{\frac{3(\varepsilon_1-\gamma)}{5}}}\\
&\lesssim \frac{C\varepsilon}{s^{\frac{3(\varepsilon_1-\gamma)}{5}}}.
\end{aligned}$$

Now for $M\gtrsim 1$

$$\sum_{1\leq a\lesssim s^{\nu}}a^k\sum_{b\lesssim c\sim a}\frac{U(b)\langle M\rangle^2 b^{1/2-4/3}}{\langle a\rangle\langle c\rangle^k}\lesssim\sum_{1\leq a\lesssim s^{\nu}}a\lesssim s^{\nu},\quad \|\psi(s)\|_{W^{k,q_1}}\lesssim\varepsilon^{\frac{2+6\gamma}{3+3\gamma}}\bigg(\delta+\frac{\varepsilon}{(1+s)^{1+\gamma}}\bigg)^{\frac{1-3\gamma}{3+3\gamma}}.$$

We inject these estimates in (B.14) and from the same computations as for (B.13) we find that if $\min\big(3(\varepsilon_1-\gamma)/5, 1/3-\gamma-\nu\big)\geq\gamma$,

$$\begin{aligned}
&\big\|\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{b\lesssim c\sim a}\mathcal{B}_2^{a,b,c,X}[\Theta(s)\Psi^{\pm},\Theta(s)\Psi^{\pm}]ds\big\|_{W^{k,p}}\\
&\qquad\lesssim\int_1^{t-1}\frac{C^2\varepsilon^2}{(t-s)^{1+\gamma}s^{1+\frac{3(\varepsilon_1-\gamma)}{5}}}+\frac{C^2\varepsilon^{1+\frac{2+6\gamma}{3+3\gamma}}}{(t-s)^{1+\gamma}}\bigg(\delta+\frac{\varepsilon}{(t-s)^{1+\gamma}}\bigg)^{\frac{1-3\gamma}{3+3\gamma}}ds\\
&\qquad\lesssim\frac{C^2\varepsilon^2}{t^{1+\gamma}}+C^2\varepsilon\delta.
\end{aligned} \tag{B.15}$$

The two other cases $c\lesssim a\sim b$ and $a\lesssim b\sim c$ can be treated in a similar way.
From (B.13),(B.15)

$$\big\|\sum_{a,b,c}I^{a,b,c,X}\big\|_{W^{k,p}}\lesssim C^2\varepsilon\bigg(\delta+\frac{\varepsilon}{(1+t)^{1+\gamma}}\bigg) \tag{B.16}$$

Conclusion From (B.10) and (B.16), we have

$$\Big\| \int_0^1 e^{i(t-s)H}\mathcal{D}(\Psi)ds\Big\|_{W^{k,p}} \lesssim C^2\varepsilon\bigg(\delta + \frac{\varepsilon}{(1+t)^{1+\gamma}}\bigg). \tag{B.17}$$

Higher order (cubic and quartic) terms are easier to control, we refer to [3] paragraph 5.2, we conclude

$$\bigg\|e^{itH}\Psi_0\| + \int_0^t e^{i(t-s)H}\mathcal{N}(\Psi,\mathbb{P}u)ds\bigg\|_{W^{k,p}} \leq \frac{C_1\varepsilon}{(1+t)^{1+\gamma}} + C_1C^2\varepsilon\bigg(\delta + \frac{\varepsilon}{(1+t)^{1+\gamma}},\bigg). \tag{B.18}$$

so that choosing C large enough, ε small enough we have as expected

$$\|\Psi\|_{W^{k,p}} \leq \frac{C}{2}(\delta + \varepsilon/(1+t)^{1+\gamma}).$$

Control of the purely quadratic terms in the weighted norm We refer to the paragraph 6.2 in [3], which can be applied with the same "routine" modifications as for the $W^{k,p}$ estimates.

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