NEW COHOMOLOGICAL INVARIANTS OF FOLIATIONS

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ABSTRACT. Given a smooth foliation on a closed manifold, basic forms are differential forms that can be expressed locally in terms of the transverse variables. The space of basic forms yields a differential complex, because the exterior derivative fixes this set. The basic cohomology is the cohomology of this complex, and this has been studied extensively. Given a Riemannian metric, the adjoint of the exterior derivative maps the orthogonal complement of the basic forms to itself, and we call the resulting cohomology the “antibasic cohomology”. Although these groups are defined using the metric, the dimensions of the antibasic cohomology groups are invariant under diffeomorphism and metric changes. If the underlying foliation is Riemannian, the groups are foliated homotopy invariants that are independent of basic cohomology and ordinary cohomology of the manifold. For this class of foliations we use the codifferential on basic forms to obtain the corresponding Laplace operator, develop its analytic properties, and prove a Hodge theorem. We then find some topological and geometric properties that impose restrictions on the antibasic Betti numbers.

1. Introduction

The ordinary Hodge decomposition theorem on a closed Riemannian manifold \((M, g)\) of dimension \(n\) gives an \(L^2\)-orthogonal decomposition of differential forms:

\[
\Omega^k(M) = \text{im}(d_{k-1}) \oplus \mathcal{H}^k \oplus \text{im}(\delta_{k+1}), \quad 0 \leq k \leq n,
\]

where \(d_k : \Omega^k(M) \to \Omega^{k+1}(M)\) is the exterior derivative with \(L^2\)-adjoint \(\delta_{k+1} : \Omega^{k+1}(M) \to \Omega^k(M)\), and where \(\mathcal{H}^k = \ker (\Delta_k)\) is the space of harmonic \(k\)-forms. Note also that

\[
\ker (d_k) = \text{im}(d_{k-1}) \oplus \mathcal{H}^k \quad \text{and} \quad \ker (\delta_k) = \mathcal{H}^k \oplus \text{im}(\delta_{k+1}).
\]

From this we get that the de Rham cohomology groups satisfy \(H^k(M) \cong \mathcal{H}^k\). Now, an alternative way of looking at this is to define a “new” de Rham homology \(H^k_b(M)\) using \(\delta\) instead of \(d\): \(\delta^2 = 0\), so

\[
H^k_b(M) = \frac{\ker (d_k)}{-\text{im}(\delta_{k+1})}, \quad 0 \leq k \leq n,
\]

is well-defined. By the equations above for \(\ker (\delta_k)\) and \(\ker (d_k)\), \(H^k_b(M) \cong H^k_d(M)\). So no one ever defines \(H^k_d(M)\) separately, because it does not provide anything new, and it seems to require a metric.

We consider however the situation where \(M\) is endowed with a smooth foliation \(\mathcal{F}\) of codimension \(q\). Many researchers have studied the properties of basic forms on foliations (see [13] for the original work and the expositions [15], [11], [18] and the references therein). Specifically, the basic forms are differential forms on \(M\) that locally depend only on the transverse variables. Because the exterior derivative preserves the set \(\Omega^*_b(M)\) of basic forms, one can define the basic cohomology groups as

\[
H^k_b(M, \mathcal{F}) := \frac{\ker \left( d : \Omega^k_b(M) \to \Omega^{k+1}_b(M) \right)}{-\text{im}(d : \Omega^{k-1}_b(M) \to \Omega^k_b(M))}, \quad 0 \leq k \leq q.
\]

These cohomology groups are invariants of the foliation and can in general be infinite dimensional even when \(M\) is compact. The isomorphism classes of these groups are invariant under any homotopy equivalence between foliations that preserve the leaves. For certain classes of foliations, such as Riemannian foliations, these cohomology groups are finite dimensional.

Since \(d\) preserves the basic forms as mentioned previously, the \(L^2\) adjoint \(\delta\) of \(d\) with respect to the inner product in \(L^2(\Omega^*_b(M))\) preserves the orthogonal complement \((\Omega^*_b(M))\perp\), and we denote the set of smooth...
forms in this subspace by \( \Omega^*_a(M,g) \), the set of “antibasic forms”. Because \( \delta^2 = 0 \) on this space, we may define the “antibasic cohomology groups” as

\[
H^k_a(M,F,g) := \frac{\ker(\delta : \Omega^k_a(M,g) \to \Omega^{k-1}_a(M,g))}{\text{im}(\delta : \Omega^{k+1}_a(M,g) \to \Omega^k_a(M,g))}, \quad 0 \leq k \leq n.
\]

We see that \( H^k_a(M,F,g) \) depends on the choice of \( g \), but we show that the isomorphism classes of these groups are independent of this choice (Theorem 2.1) and are in fact invariants of the smooth foliation structure (Corollary 2.2). For that reason, we henceforth remove the background metric \( g \) from the notation. Unlike the case of the de Rham cohomology of ordinary manifolds defined using \( \delta \) above, these cohomology groups provide new invariants of the foliation, independent of the basic and ordinary de Rham cohomology.

We are interested in whether these new foliation invariants give obstructions to certain types of geometric structures on the manifolds and foliations. In Theorem 2.3 we show that if the foliation is codimension one on a connected manifold, and if the mean curvature form of the normal bundle is everywhere nonzero, then \( H^0_a(M) = \{0\} \), and \( H^j_a(M) = H^j(M) \) for \( j \geq 1 \).

Starting with Section 3, we consider the case of Riemannian foliations, where the normal bundle carries a holonomy invariant metric; c.f. [14], [11], [18]. As is customary, we choose a bundle-like metric, one such that the leaves of the foliation are locally equidistant. In this particular case, the geometry forces many consequences for the antibasic cohomology. One crucial property of Riemannian foliations that allows us to proceed with analysis is that the \( L^2 \) orthogonal projection \( P_a \) to all forms to basic forms preserves smoothness. This was shown in [12], and it is false in general for non-Riemannian foliations (see Example 9.4). As a consequence, it is also true that the \( L^2 \) orthogonal projection \( P_a \) to all forms to antibasic forms preserves smoothness. In Proposition 3.1, we derive explicit formulas for the commutators \([d,P_a]\) and \([\delta,P_a]\), which are zeroth order operators that are in general not pseudodifferential. These formulas allow us to express the antibasic Laplacian \( \Delta_a = (P_a (d + \delta) P_a)^2 \) in terms of elliptic operators on all forms in Theorem 4.1. That is, \( \Delta_a \) can be written in terms of the ordinary Laplacian \( \Delta \) on \( M \) by the formula

\[
\Delta_a = (\Delta + \delta P_b \varepsilon^* + P_b \varepsilon^* \delta) P_a,
\]

where \( \varepsilon^* \) is a zeroth order differential operator determined by the geometry of the foliation and defined explicitly in Proposition 5.1.

Because \( \Delta_a \) and \( D_a = P_a (d + \delta) P_a \) are similar to elliptic differential operators but are in general not pseudodifferential, we do not necessarily expect them to satisfy the usual properties of Laplace and Dirac operators. However, in Section 5, we are able to show many of the functional analysis results with a few modifications. Specifically, we prove a version of Gårding’s Inequality (Lemma 5.2), the elliptic estimates (Lemma 5.3), and the essential self-adjointness of both \( D_a \) and \( \Delta_a \) (Corollary 5.12). Also, we show that elliptic regularity holds (Proposition 5.13), and finally we prove the spectral theorem (Theorem 5.17) for \( \Delta_a = D_a^2 \) and \( D_a \), showing that there exists a complete orthonormal basis of \( L^2(\Omega^*_a(M)) \) consisting of smooth eigenforms of \( D_a \), and the eigenvalues of \( \Delta_a \) have finite multiplicity and accumulate only at \( +\infty \). In all of these cases, the proofs are a bit more complicated than usual because of the antibasic projection and the issue of operators not being pseudodifferential.

In Section 6, we are able to prove the Hodge theorem and decomposition (Theorem 6.1 and Corollary 6.3) for the antibasic forms, again only for the Riemannian foliation case. For these foliations, there is an alternate way of expressing the antibasic cohomology, using \( d_a = P_a d P_a \) as a differential. Then it turns out that if \( f : (M,F) \to (M',F') \) is a foliated map, which takes leaves into leaves, then \( P_a f^* P_a' \) induces a linear map on \( d_a \)-cohomology. We show that for Riemannian foliations, the isomorphism classes of antibasic cohomology groups are foliated homotopy invariants; see Theorem 6.6 and Corollary 6.7. We do know in general that the antibasic Betti numbers are foliated diffeomorphism invariants, but it is an open question whether they are foliated homotopy invariants; see Problem 1 and the preceding discussion.

In Section 7 we prove identities for antibasic cohomology in special cases. If the foliation is Riemannian, then \( H^0(M) \cong H^0_0(M,F) \oplus H^0(M,F) \) and

\[
\dim H^1(M) \leq \dim H^1_b(M,F) + \dim H^1_a(M,F);
\]

see Proposition 7.5 and Proposition 7.6. If in addition the normal bundle is involutive, then for all \( k \),

\[
H^k(M) \cong H^k_b(M,F) \oplus H^k_a(M,F);
\]

and
by Proposition 7.1. In the special case where the Riemannian foliation is the set of orbits of a connected, compact Lie group of isometries, we show that antibasic cohomology can be computed using only the subspace of invariant differential forms; see Proposition 7.8.

The case of Riemannian flows is investigated in Section 8. In this setting, we are able to characterize $H^1_a(M, \mathcal{F})$. We prove in Proposition 8.1 that when the flow is taut, meaning that there exists a metric for which the leaves are minimal,

$$\dim (H^{r+1}_a(M, \mathcal{F})) \geq \dim (H^r_a(M, \mathcal{F}))$$

for $r \geq 0$. In the particular case where $H^1(M) = \{0\}$ and $M$ is connected, we get $H^1_a(M) \cong \mathbb{R}$ always (Theorem 8.4). On the other hand, if $M$ is connected and the flow is nontaut, we have that $H^1_a(M) \cong \{0\}$. In Section 8, we compute the antibasic cohomology of specific foliations in low dimensions. These examples include Riemannian and non-Riemannian foliations and illustrate the results we have proved. For two of the foliations, $P_\alpha$ does not preserve smoothness, but the antibasic cohomology groups can still be computed. Of particular note are Example 9.2 and Example 9.5, where all the Betti numbers and basic Betti numbers of the two foliations are the same, but the antibasic Betti numbers are different. Thus, antibasic cohomology groups do indeed provide smooth foliation invariants that are independent of ordinary and basic de Rham cohomology.

2. Basic and antibasic cohomology of foliations

Let $(M, \mathcal{F})$ be a smooth foliation of codimension $q$ and dimension $p$ (i.e. the dimension of $M$ is $n = p + q$). The subspace $\Omega_b(M) \subset \Omega(M)$ of basic differential forms is defined as

$$\Omega_b(M) = \{ \beta \in \Omega(M) : X_\beta = 0, X_\beta d\beta = 0 \text{ for all } X \in \Gamma(TF) \}$$

where $X_\beta$ denotes interior product with $X$. Since $d$ maps basic forms to themselves, we may compute the basic cohomology

$$H^k_b(M, \mathcal{F}) := \frac{\ker (d : \Omega^k_b(M) \to \Omega^{k+1}_b(M))}{\im (d : \Omega^{k-1}_b(M) \to \Omega^k_b(M))}$$

for $0 \leq k \leq q$. These vector spaces are smooth invariants of the foliation; in fact their dimensions are topological invariants (see [8]). In general, $H^k_b(M, \mathcal{F})$ need not be finite dimensional, unless there are topological restrictions (such as the existence of a bundle-like metric), and even when such restrictions apply, Poincaré duality is not satisfied except in special cases, such as when the foliation is taut and Riemannian. Much work on these cohomology groups has been done (c.f. [1, 11, 18, 2, 5, 6], and the associated references).

Suppose next that $M$ is endowed with a Riemannian metric $g$. A metric on the bundle of differential forms is induced from $g$, and in fact for any $\alpha, \beta \in \Omega^r(M)$,

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta,$$

where $*$ is the Hodge star operator. The formal adjoint $\delta$ of $d$ with respect to this metric satisfies

$$\delta = (-1)^{nr+n+1} * d * = (-1)^r *^{-1} d *$$

on $\Omega^r(M)$. We let the smooth part of the $L^2$-orthogonal complement of $\Omega^r_b(M)$ be

$$\Omega^r_a(M, g) := \Omega^r_b(M)^\perp = \{ \alpha \in \Omega^r(M) : \langle \alpha, \beta \rangle = 0 \text{ for all } \beta \in \Omega^r_b(M) \}$$

the space of antibasic $r$-forms. Observe that for all $\beta \in \Omega^r_a(M), \alpha \in \Omega^r_a(M, g)$,

$$0 = \langle d\beta, \alpha \rangle = \langle \beta, \delta \alpha \rangle,$$

so that $\delta$ preserves the antibasic forms, and again $\delta^2 = 0$. We now define the antibasic cohomology groups $H^r_a(M, \mathcal{F}, g)$ for $0 \leq r \leq n$ by

$$H^r_a(M, \mathcal{F}, g) := \frac{\ker (\delta : \Omega^r_a(M, g) \to \Omega^{r-1}_a(M, g))}{\im (\delta : \Omega^{r+1}_a(M, g) \to \Omega^r_a(M, g))}.$$

**Theorem 2.1.** Let $\mathcal{F}$ be a smooth foliation on a smooth (not necessarily compact) manifold $M$ that is endowed with a metric $g$. The isomorphism classes of the groups $H^r_a(M, \mathcal{F}, g)$ do not depend on the choice of $g$ and are thus invariants of $(M, \mathcal{F})$. 
Lemma 2.4. Henceforth we will denote \( \Omega^r (M) \) by \( A^r \) and \( B^r \) on \( \Omega^r (M) \) by
\[
A^r : = *^{-1} *' : \Omega^r (M) \to \Omega^r (M), \\
B^r : = *' *^{-1} : \Omega^r (M) \to \Omega^r (M).
\]
Then observe that
\[
A^r B^r = *^{-1} *' * *^{-1} = \text{identity}, \\
\text{and also } B^r A^r \text{ is the identity. Thus we also have that}
\]
\[
A^r = *^{-1} *' = * (\ast')^{-1}, \\
B^r = *' *^{-1} = (\ast')^{-1} *.
\]
With these definitions,
\[
*' = A^r B^r \ast .
\]
Then on \( \Omega^r (M) \), the formal adjoint of \( d \) in the \( g' \) metric is
\[
\delta' = \pm A^{n-r+1} d *A^r \\
= B^{r-1} \delta A^r.
\]
Then we check
\[
0 = (\delta')^2 = B^{r-2} \delta A^{r-1} B^{r-1} \delta A^r \\
= B^{r-2} \delta^2 A^r.
\]
Consider the map on differential \( r \)-forms given by \( \psi \mapsto \psi' = (A^r)^{-1} \psi = B^r \psi \), which is an isomorphism. Then we see that
\[
\delta' \psi' = B^{r-1} \delta A^r (A^r)^{-1} \psi \\
= B^{r-1} (\delta \psi) = (\delta \psi)'.
\]
Restricting now to the foliation case and the antibasic forms, we must determine if \( B^r \) maps the \( g \)-antibasic \( r \)-forms to the \( g' \)-antibasic \( r \) forms. We check this by taking any \( g \)-antibasic \( r \)-form \( \psi \) and any basic form \( \beta \):
\[
0 = \langle \beta, \psi \rangle = \int_M \beta \wedge *\psi \\
= \int_M \beta \wedge *\psi = \int_M \beta \wedge *' (\ast')^{-1} * \psi \\
= \int_M \beta \wedge * B^r \psi = \langle \beta, B^r \psi \rangle'.
\]
Hence \( B^r \) maps the \( g \)-basic forms to the \( g' \)-antibasic forms. By the above, \( \delta' (B^r \psi) = (\delta \psi)' \) for antibasic \( r \)-forms \( \psi \), so that \( B^r (\ker \delta) = \ker \delta' \) and \( B^r (\im \delta) = \im \delta' \), so that the antibasic cohomology groups corresponding to \( g \) and \( g' \) are isomorphic through the map \( [\psi] \mapsto [B^r \psi]' \). \( \square \)

Corollary 2.2. Suppose that \((M, F)\) is a smooth foliation of (not necessarily compact) smooth manifold \( M \). Suppose that \( F : M \to M' \) is a diffeomorphism, and let \( F' \) be the foliation induced on \( M' \). Then for any two metrics \( g, g' \) on \( M \) and \( M' \), respectively, \( H^*_a (M, F, g) \cong H^*_a (M', F', g) \). Thus, the isomorphism class of \( H^*_a (M, F, g) \) is a smooth foliation invariant.

Proof. Given the setting as above, observe that \( F^* g \) is another metric on \( M \). By construction and the theorem above, \( H^*_a (M', F', g) \cong H^*_a (M', F, F^* g) \cong H^*_a (M, F, g) \). \( \square \)

Notation 2.3. Henceforth we will denote \( \Omega^r_a (M) = \Omega^r_a (M, g) \) and \( H^*_a (M, F, g) \cong H^*_a (M, F, g) \), with the particular background metric \( g \) understood.

Lemma 2.4. Let \((M, F)\) be a smooth foliation of codimension \( q \) on a closed manifold \( M \) with any Riemannian metric. Then \( H^k_a (M, F) = H^k (M) \) for \( k > q \), and \( H^q_a (M, F) \) is isomorphic to a subspace of \( H^q (M) \).
Proof. Since $\Omega^k_\alpha (M) = \Omega^k (M)$ for $k > q$, $H^k_\alpha (M, \mathcal{F}) = H^k (M)$ for $k > q$. We also have

$$H^k_\alpha (M, \mathcal{F}) = \frac{\ker (\delta |_{\Omega^k(M)})}{\im (\delta |_{\Omega^{k+1}(M)})} = \frac{\ker (\delta |_{\Omega^k(M)})}{\im (\delta |_{\Omega^{k+1}(M)})} \subseteq \frac{\ker (\delta |_{\Omega^0(M)})}{\im (\delta |_{\Omega^1(M)})} = H^q (M).$$

\[\square\]

In the case of codimension 1 foliations, we can say more.

**Proposition 2.5.** Let $M$ be a closed connected Riemannian manifold with codimension 1 foliation $\mathcal{F}$. Assume that the mean curvature form of the normal bundle is everywhere nonzero. Then the only basic functions on $M$ are constants, $H^0_\alpha (M) = \{0\}$, $H^0_\beta (M, \mathcal{F}) = \mathbb{R}$, and $H^j_\beta (M, \mathcal{F}) = H^j (M)$, $H^j_\beta (M, \mathcal{F}) = \{0\}$ for $j \geq 1$.

**Proof.** Since the normal bundle $(T\mathcal{F})^\perp$ has rank 1, it is involutive. Let $\nu$ be the transverse volume form of $\mathcal{F}$. Note that $T\mathcal{F} = \ker \nu$. By Rummel’s formula

$$dv = -\kappa_N \wedge \nu,$$

where $\kappa_N$ is the mean curvature 1-form of $(T\mathcal{F})^\perp$. By assumption, $\kappa_N$ is nonzero everywhere. Observe that any one-form may be written $\beta = \alpha \nu + \gamma$, where $\alpha$ is a function and $\gamma$ is orthogonal to $\nu$. Note that a one-form $\beta$ is basic if and only if $X_\beta \beta = 0$ and $X_\beta d\beta = 0$ for all $X \in \ker \nu$. The first condition implies $\beta = \alpha \nu$, and the second condition implies

$$0 = X_\beta (da \wedge \nu + b \nu) = X_\beta (da \wedge \nu - a \kappa_N \wedge \nu)$$

which implies

$$X_\beta (da - a \kappa_N) = 0$$

for all $X \in \Gamma (T\mathcal{F})$, or

$$da = a \kappa_N + b \nu$$

for some function $b$. Since $\kappa_N \neq 0$ and is orthogonal to $\nu$, the maximum and minimum of the function $\alpha$ on $M$ must occur when $a = 0$, so $\alpha = 0$. Thus, there are no nonzero basic one-forms, so that $\Omega^1_\beta (M) = \Omega^1 (M)$. Every function $f$ on $M$ can be written as $f = c + \delta \alpha$ for some one-form $\alpha$ and constant $c$ by the Hodge theorem. Since $\alpha$ and $\delta \alpha$ are necessarily antibasic, we have the natural decomposition of $f$ into its basic and antibasic components $c$ and $\delta \alpha$. Therefore, every antibasic function is $\delta \alpha$-exact, and every basic function is constant, so we have $H^0_\alpha (M, \mathcal{F}) = \{0\}$, $H^0_\beta (M, \mathcal{F}) = \mathbb{R}$, and $H^j_\beta (M, \mathcal{F}) = H^j (M)$, $H^j_\beta (M, \mathcal{F}) = \{0\}$ for $j \geq 1$, because $\Omega^j_\beta (M) = \Omega^j (M)$ for $j \geq 1$. \[\square\]

**Remark 2.6.** In the next section, we consider the case of Riemannian foliations. Codimension one Riemannian foliations always have $\kappa_N = 0$, so the proof of the previous proposition does not apply. Indeed, it is not true that there are no basic one-forms, since the transverse volume form $\nu$ is always a basic one-form. Also, it is quite possible that there are nonconstant basic functions. The cohomological facts in this case are only different in degree 1: $H^0_\alpha (M, \mathcal{F}) = \{0\}$, $H^0_\beta (M, \mathcal{F}) = \mathbb{R}$, $H^1_\beta (M, \mathcal{F}) = \mathbb{R}$, $H^1 (M) \cong H^1_\alpha (M, \mathcal{F}) \oplus H^1_\beta (M, \mathcal{F})$, and $H^j_\beta (M, \mathcal{F}) = H^j (M)$, $H^j_\beta (M, \mathcal{F}) = \{0\}$ for $j \geq 2$.

Given two smooth foliations $(M, \mathcal{F})$ and $(M', \mathcal{F}')$, a map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is called foliated if $f$ maps the leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}'$, which implies $f_*(T\mathcal{F}) \subset T\mathcal{F}'$. It follows that the basic forms on $(M', \mathcal{F}')$ pull back to basic forms on $(M, \mathcal{F})$. Two foliated maps $f, g : (M, \mathcal{F}) \to (M', \mathcal{F}')$ are foliated homotopic if there exists a continuous map $H : [0, 1] \times M \to M'$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ and for all $t \in [0, 1]$ the map $H(t, \cdot)$ is foliated and smooth. A foliated map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is a foliated homotopy equivalence if there exists a foliated map $h : (M', \mathcal{F}') \to (M, \mathcal{F})$ such that $f \circ h$ and $h \circ f$ are foliated homotopic to the identity on the two foliations.

It is proved in [2] (also in [3] for the case of foliated homeomorphisms) that foliated homotopic maps induce the same map on basic cohomology and that basic cohomology is a foliated homotopy invariant. We now examine whether or not antibasic cohomology satisfies the same property.
Note that since in general the codifferential \( \delta \) does not commute with pullback \( f^* \) by a smooth map \( f : (M, \mathcal{F}) \to (M', \mathcal{F}') \), we do not expect that pullback induces a linear map on antibasic cohomology. However, since it is true that on differential forms \( d \circ f^* = f^* \circ d \), we also have that

\[
(f^*)^\dagger \circ \delta = \delta \circ (f^*)^\dagger,
\]

where \( ^\dagger \) denotes the formal \( L^2 \)-adjoint. Note that \( f^* \) is not necessarily bounded on \( L^2 \). If we restrict to the case of closed manifolds, \( f^* \) does map smooth forms to smooth forms in \( L^2 \), so it is a densely defined operator on \( L^2 \). Here \((f^*)^\dagger\) is the formal adjoint defined on its domain. From unbounded operator theory, the domain of \((f^*)^\dagger\) is

\[
\text{Dom} \left( (f^*)^\dagger \right) = \{ \alpha \in L^2(\Omega(M)) : \exists \gamma \in L^2(\Omega(M')) \text{ such that } \langle f^* \beta, \alpha \rangle_M = \langle \beta, \gamma \rangle_{M'}, \forall \beta \in \Omega(M') \}.
\]

But we know that if \( \alpha \) is smooth, and if the linear map \( \Phi_f(\beta) := \langle f^* \beta, \alpha \rangle_M \) is bounded, then \( \Phi_f(\beta) = \langle \beta, \gamma \rangle_{M'} \) for some \( \gamma \) by the Riesz representation theorem. However, it turns out that \( \Phi_f(\cdot) \) is unbounded for almost all choices of \( f \) (the rank of its differential must be constant, for instance). However, in the cases where \( \Phi_f \) is bounded, \((f^*)^\dagger\) induces a linear map on antibasic cohomology. The usual proof applies in this case to show that maps \((f^*)^\dagger\) are invariant over the homotopy class of such \( f \).

Another possible approach is to use the Hodge star operator \( * \) and \( *' \) on \( M \) and \( M' \), respectively, and to consider \( *f^* *' \) as a map that commutes with \( \delta \) up to a sign. However, this would not apply in our case since \( *f^* *' \) does not necessarily preserve the antibasic forms.

Thus, we still have the following open problem:

**Problem 1.** If the foliations \((M, \mathcal{F})\) and \((M', \mathcal{F}')\) with Riemannian metrics are foliated homotopy equivalent, does that mean that their antibasic cohomology groups are isomorphic?

**Remark 2.7.** This problem is solved in the case of Riemannian foliations, as we see in Theorem 6.7 and Corollary 6.4. In this case, \( P_a \) preserves the smooth forms, so we show that the operator \( P_a f^* P_a' \) induces a linear map on antibasic cohomology, which is an isomorphism when \( f \) is a foliated homotopy equivalence.

### 3. Riemannian foliation setting

In the Riemannian foliation setting, we often restrict to basic forms. Let \((M, \mathcal{F})\) be a foliation of codimension \( q \) and dimension \( p \), endowed with a bundle-like metric. From [12], the orthogonal projection \( P_b : L^2(\Omega(M)) \to L^2(\Omega_b(M)) \) maps smooth forms to smooth basic forms; this was also stated and used in [I]. Because of this, it is also true that

\[
P_a = (I - P_b) : L^2(\Omega(M)) \to L^2(\Omega_b(M)^{-})
\]

maps smooth forms to smooth “antibasic forms”. As described in [12], we have

\[
d_a = P_a d P_a = d P_a,
\]

(i.e. \( d \) restricts to the basic forms). Letting \( \delta = \delta_k : \Omega^k(M) \to \Omega^{k-1}(M) \) be the \( L^2 \) adjoint of \( d_{k-1} \), we then have

\[
\delta_b = P_b \delta P_b = P_b \delta,
\]

and note that the basic adjoint is \( \delta_b = P_b \delta = P_b \delta P_b \). Note also that the formulas above imply that

\[
\begin{align*}
(I - P_b) d (I - P_b) &= (I - P_b) d \\
(I - P_b) \delta (I - P_b) &= \delta (I - P_b),
\end{align*}
\]

or

\[
\begin{align*}
d_a &= P_a d P_a = P_a d \\
\delta_a &= P_a \delta P_a = \delta P_a.
\end{align*}
\]

We see

\[
\delta_a^2 = P_a \delta P_a P_a \delta P_a = \delta^2 P_a = 0.
\]

The adjoint of \( \delta_a \) restricted to antibasic forms is \( d_a = P_a d P_a = P_a d \), and again

\[
d_a^2 = P_a d P_a d P_a = P_a d^2 = 0.
\]
Also in [12], it is shown that
\[ P_b \delta - \delta P_b = \varepsilon \circ P_b = [- (P_a \kappa)_\gamma + \varepsilon (\varphi_{\partial \gamma}) (\chi_{\partial^\gamma \gamma})] \circ P_b, \tag{3.2} \]
\[ dP_b - P_b d = P_b \circ \varepsilon^* = P_b \circ [- (P_a \kappa) \wedge + \varepsilon^* (\varphi_{\partial \gamma}) (\chi_{\partial^\gamma \gamma})] \]
on \Omega^*(M). We observe that the only information about the foliation needed to obtain the formulas above in [12] is the fact that the orthogonal projection \( P_b \) maps smooth forms to smooth forms, that \( P_b \) commutes with \( \varepsilon \), the transversal Hodge star-operator, and that \( P_b (\alpha \wedge P_b \beta) = (P_b \alpha) \wedge (P_b \beta) \) for all smooth forms \( \alpha, \beta \). These facts are true for Riemannian foliations. From the formulas above and the notation \( \kappa_a = P_a \kappa \), we obtain the following.

**Proposition 3.1.** On a Riemannian foliation \((M, \mathcal{F})\) on a closed manifold with bundle-like metric,
\[ P_b \delta - \delta P_b = \varepsilon \circ P_b = [- \kappa_a \gamma + \varepsilon (\varphi_{\partial \gamma}) (\chi_{\partial^\gamma \gamma})] \circ P_b, \tag{3.3} \]
\[ P_a d - dP_a = P_b \circ \varepsilon^* = P_b \circ [- \kappa_a \wedge + \varepsilon^* (\varphi_{\partial \gamma}) (\chi_{\partial^\gamma \gamma})] \tag{3.4} \]
on \Omega^{k-1}(M). The operation \( \varepsilon \) maps \( \Omega_b(M, \mathcal{F}) \) to \( \Omega_b(M, \mathcal{F})^\perp \), and it follows that
\[ P_b \varepsilon P_b = P_b \varepsilon^* P_b = 0, \]
\[ \varepsilon P_b = P_a \varepsilon P_b, \quad P_a \varepsilon P_a = P_b \varepsilon^* P_b, \]
\[ P_b \varepsilon P_a = P_b \varepsilon, \quad P_b \varepsilon^* P_a = P_b \varepsilon^*. \tag{3.5} \]

### 4. The antibasic Laplacian

Again we assume that \((M, \mathcal{F})\) is a foliation of codimension \( q \) and dimension \( p \), endowed with a bundle-like metric. Recall that the basic Laplacian is \( \Delta_b = \delta d_b + d_b \delta_b = \text{restriction of } P_b \delta d + d \delta P_b \) to \( \Omega_b(M, \mathcal{F}) \). We wish to do a similar restriction to antibasic forms. Let the subscript \( a \) denote the restriction to \( \Omega_b(M, \mathcal{F})^\perp \), the antibasic forms. Then
\[ \Delta_a = \delta a d_a + d_a \delta_a = (d_a + \delta_a)^2 \]
restriction of \( \delta P_a d + d \delta P_a \) to \( \Omega_b(M, \mathcal{F})^\perp \).

From the formulas (3.4) and (3.5),
\[ \Delta_a = \delta P_a d + d \delta P_a \mid_{\Omega_b(M, \mathcal{F})^\perp} \]
\[ = \delta d P_a + d P_b \varepsilon^* P_a + d P_a \delta + P_b \varepsilon^* P_a \delta \mid_{\Omega_b(M, \mathcal{F})^\perp} \]
\[ = \delta d + \delta P_b \varepsilon^* + d \delta + P_b \varepsilon^* \delta \mid_{\Omega_b(M, \mathcal{F})^\perp} \]
\[ = \Delta_a + \delta P_b \varepsilon^* + P_b \varepsilon^* \delta \mid_{\Omega_b(M, \mathcal{F})^\perp}. \]
Thus \( \Delta_a \) is the restriction of an elliptic operator on the space of all differential forms. Note that it is not clear whether this operator is differential or pseudodifferential or not, since \( P_b \) is not pseudodifferential in general.

We summarize the results below.

**Theorem 4.1.** The antibasic Laplacian \( \Delta_a \) satisfies the following.
\[ \Delta_a P_a = \Delta P_a = P_a \Delta^* = P_a \nabla P_a, \]
where \( \nabla = \Delta_a + \delta P_b \varepsilon^* + P_b \varepsilon^* \delta \), \( \Delta^* = \Delta_a + \varepsilon P_b d + d \varepsilon P_b \) is its adjoint, and \( \nabla = \Delta_a - \varepsilon P_b \varepsilon^* \).

**Proof.** The first equality was shown above. To prove that \( \Delta P_a = P_a \nabla P_a \), we compute
\[ \Delta P_a = P_a \Delta P_a = P_a (\Delta_a + \delta P_b \varepsilon^* + P_b \varepsilon^* \delta) P_a \]
\[ = P_a \Delta P_a + P_a \delta P_b \varepsilon^* P_a \]
\[ = P_a \Delta P_a + P_a (P_b \delta) P_b \varepsilon^* P_a \]
\[ = P_a \Delta P_a + P_a (\delta P_b - \varepsilon P_b) P_b \varepsilon^* P_a \]
\[ = P_a \Delta P_a - P_a \varepsilon P_b \varepsilon^* P_a \]
\[ = P_a (\Delta_a - \varepsilon P_b \varepsilon^*) P_a. \]
Here we have used the fact that \( P_b \varepsilon^* P_b = 0 \) and \( P_b^2 = P_b \) and formula (3.3). Note that since \( P_a \Delta P_a \) is formally self-adjoint, \( \Delta P_a = \left( \Delta P_a \right)^* = P_a \Delta^* \).

**Corollary 4.2.** The antibasic Laplacian is the restriction of the ordinary Laplacian if the mean curvature is basic and the normal bundle of the foliation is involutive.

*Proof.* If the mean curvature is basic and the normal bundle of the foliation is involutive, then \( P_a \kappa = 0 \) and \( \varphi_0 = 0 \), so that \( \varepsilon = 0 \). Then \( \Delta = \Delta^* = \overline{\Delta} \) in this case, so by the theorem above \( \Delta_a P_a = \Delta P_a \).

Also we show a few more facts about the projections and the operators \( d, \delta, \varepsilon \).

**Proposition 4.3.** With notation as above,

\[
P_a (d + \varepsilon^*) P_a = (d + \varepsilon^*) P_a.
\]

*Proof.* By (3.4), we have

\[
P_a (d + \varepsilon^*) P_a = P_a d P_a + P_a \varepsilon^* P_a
\]

\[
= (d P_a + P_b \varepsilon^*) P_a + P_a \varepsilon^* P_a
\]

\[
= d P_a + (P_b + P_a) \varepsilon^* P_a
\]

\[
= (d + \varepsilon^*) P_a.
\]

Now, we let the first order operator \( D^\varepsilon \) be defined as

\[
D^\varepsilon = \delta + d + \varepsilon^*,
\]

and the antibasic operator \( D_a^\varepsilon \) by

\[
D_a^\varepsilon = P_a (d + \delta + \varepsilon^*) P_a.
\]

**Corollary 4.4.** We have

\[
D_a^\varepsilon = P_a D^\varepsilon P_a = D^\varepsilon P_a,
\]

so that \( D_a^\varepsilon \) is the restriction of the elliptic operator \( D^\varepsilon \).

**Corollary 4.5.** Let \( \Delta^\varepsilon = \Delta + \varepsilon^* \delta + \delta \varepsilon^* \). Then \( P_a \Delta^\varepsilon P_a = \Delta^\varepsilon P_a \), so that the operator \( P_a \Delta^\varepsilon P_a \) on antibasic forms is the restriction of an elliptic operator.

*Proof.* By Proposition 4.3 and (3.4),

\[
P_a \Delta^\varepsilon P_a = P_a (\Delta + \varepsilon^* \delta + \delta \varepsilon^*) P_a
\]

\[
= P_a (\delta (d + \varepsilon^*) + (d + \varepsilon^*) \delta) P_a
\]

\[
= P_a \delta P_a (d + \varepsilon^*) P_a + (d + \varepsilon^*) P_a \delta P_a
\]

\[
= \delta P_a (d + \varepsilon^*) P_a + (d + \varepsilon^*) \delta P_a
\]

\[
= \delta (d + \varepsilon^*) P_a + (d + \varepsilon^*) \delta P_a = \Delta^\varepsilon P_a.
\]

5. Functional analysis of the antibasic de Rham and Laplace operators

In this section, we show that the antibasic de Rham and Laplace operators on a Riemannian foliation have properties similar to the ordinary de Rham and Laplace operators on closed manifolds, namely that they have discrete spectrum consisting of eigenvalues corresponding to finite-dimensional eigenspaces. Note that we must work out the standard Sobolev and elliptic theory for these operators, because in fact they are not pseudodifferential and are not even restrictions of pseudodifferential operators to antibasic forms.

Throughout this section, we assume that \((M, \mathcal{F})\) is a foliation with bundlelike metric \( g \), and as in the previous section, we denote the antibasic Hodge-de Rham and Laplace operators as

\[
D_a = P_a d P_a + P_a \delta P_a = d_a + \delta_a,
\]

\[
\Delta_a = d_a \delta_a + \delta_a d_a = D_a^2.
\]
First, let $H^k_a$ denote the Sobolev space $H^k(\Omega_a(M))$, defined as the closure of $\Omega_a(M) \subseteq \Omega(M)$ with respect to a choice of the $k^{th}$ Sobolev norm $\|\cdot\|_k$. We note the ordinary $L^2$ norm as the 0th Sobolev norm $\|\cdot\|_0 = L^2(\Omega^k_a(M))$. Note that Rellich’s Theorem still holds on this subspace, i.e. the inclusion of $H^k_a \rightarrow H^\ell_a$ is compact for $k > \ell$. The proof is the same as in the standard case.

Also, note that the Sobolev embedding theorem holds for the antibasic forms, so that for any integer $m > \dim M/2$, the space $H^{k+m}_a \subseteq C^k(M)$. This follows from the fact that $H^{k+m}_a \subseteq H^{k+m}(\Omega(M))$.

**Lemma 5.1.** There exists a constant $c > 0$ such that $\|D_a \psi\|_0 \leq c \|\psi\|_1$ for all $\psi \in \Omega_a(M)$.

**Proof.** By (3.4), for any $\psi \in \Omega_a(M),$

$$D_a \psi = (d_a + \delta_a) \psi = (P_a d + \delta) \psi = (dP_a + P_b \varepsilon^* + \delta) \psi = (d + \delta) \psi + P_b \varepsilon^* \psi.$$

Then, since $d + \delta$ is a first order differential operator and $\varepsilon^*$ is a bounded operator,

$$\|D_a \psi\|_0 \leq \|d+\delta\| \|\psi\|_0 + \|P_b \varepsilon^* \psi\|_0 \leq c_1 \|\psi\|_1 + \|\varepsilon^* \psi\|_0 \leq c_1 \|\psi\|_1 + c_2 \|\psi\|_0$$

for some positive constants $c_1$ and $c_2$, so that there exists $c > 0$ independent of $\psi$ such that $\|D_a \psi\|_0 \leq c \|\psi\|_1$. □

**Lemma 5.2.** (Gårding’s Inequality) There exists a positive constant $c$ such that $\|\psi\|_1 \leq c (\|\psi\|_0 + \|D_a \psi\|_0)$ for all $\psi \in \Omega_a(M)$.

**Proof.** By the ordinary Gårding’s Inequality, since $d + \delta$ is an elliptic, first order operator on $\Omega(M)$, there exists a constant $c_1$ such that for all $\psi \in \Omega_a(M) \subseteq \Omega(M),$

$$\|\psi\|_1 \leq c_1 (\|\psi\|_0 + \|(d+\delta) \psi\|_0).$$

Then, again by (3.4),

$$\|\psi\|_1 \leq c_1 (\|\psi\|_0 + \|(d_a + \delta - P_b \varepsilon^*) \psi\|_0) \leq c_1 (\|\psi\|_0 + \|P_b \varepsilon^* \psi\|_0 + \|(d_a + \delta) \psi\|_0) \leq c_1 (\|\psi\|_0 + \|\varepsilon^* \psi\|_0 + \|(d_a + \delta) \psi\|_0),$$

so since $\varepsilon^*$ is bounded, the result follows. □

**Lemma 5.3.** For all nonnegative integers $k$, there exists a positive constant $c_k$ such that

$$\|P_a \phi\|_k \leq c_k \|\phi\|_k$$

and $\|P_b \phi\|_k \leq c_k \|\phi\|_k$

for any differential form $\phi$.

**Proof.** We use induction on $k$. Let $\phi$ be any differential form. Observe first that $\|P_a \phi\|_0 \leq \|\phi\|_0$, $\|P_b \phi\|_0 \leq \|\phi\|_0$. Next, suppose that the results have been shown for some nonnegative integer $k$. Since $D^\varepsilon$ is elliptic on all forms, it satisfies the ordinary elliptic estimates: there exist constants $b_1$ and $b_2$ such that

$$\|P_a \phi\|_{k+1} + b_2 \|P_a \phi\|_k.$$

Using the fact that $\varepsilon$ is a zeroth order differential operator and the induction hypothesis,

$$\|P_a \phi\|_{k+1} \leq (\text{constant}) \|(d + \delta) \phi\|_k + \|\phi\|_k \leq \text{constant} \|\phi\|_{k+1} + \text{constant} \|\phi\|_k \leq \text{constant} \|\phi\|_{k+1},$$

since $d + \delta$ is a first order operator. Also,

$$\|P_a \phi\|_{k+1} = \|\phi - P_a \phi\|_{k+1} \leq \|\phi\|_{k+1} + \|P_a \phi\|_{k+1} \leq \text{constant} \|\phi\|_{k+1}.$$
Lemma 5.4. Let $D^c = d + \delta + \varepsilon^*$ as an operator on all differential forms, and let $D_a = d_a + \delta_a$ be the antibasic de Rham operator. For all nonnegative integers $k$, there exists a positive constant $c_k$ such that

$$\|D^c \psi\|_k - c_k \|\psi\|_k \leq \|D_a \psi\|_k \leq \|D^c \psi\|_k + c_k \|\psi\|_k,$$

for any antibasic form $\psi$.

Proof. For any antibasic $\psi$,

$$\|D_a \psi\|_k = \|D^c \psi + (D^c - D^a) \psi\|_k = \|D^c \psi + P_a \varepsilon^* \psi\|_k,$$

It suffices to bound $\|P_a \varepsilon^* \psi\|_{k+1}$. This follows from a bound on $\|P_a \phi\|_{k+1}$ from Lemma 5.3 since $\varepsilon^*$ is a zeroth order operator.

Lemma 5.5. (Elliptic Estimates for $D_a$) For every integer $k \geq 0$, there exists a positive constant $C_k$ such that $\|\psi\|_{k+1} \leq C_k (\|\psi\|_k + \|D_a \psi\|_k)$ for all $\psi \in \Omega_a(M)$.

Proof. Let $k$ be a nonnegative integer. From the elliptic estimates for the operator $D^c$ on all forms, there exists a positive constant $b_k$ such that for any $\psi \in \Omega_a(M)$,

$$\|\psi\|_{k+1} \leq b_k (\|\psi\|_k + \|D^c \psi\|_k) \leq b_k (\|\psi\|_k + \|D_a \psi\|_k + c_k \|\psi\|_k)$$

for a positive constant $c_k$, by Lemma 5.4. The inequality follows by letting $C_k = \max (b_k, b_k (1 + c_k))$.

Remark 5.6. The case $k = 0$ is Gårding’s Inequality, which we have shown independently in Lemma 5.2.

Lemma 5.7. The operator $D_a$ on $\Omega_a(M)$ is formally self-adjoint.

Proof. For any antibasic forms $\alpha$ and $\beta$,

$$\langle D_a \alpha, \beta \rangle = \langle (P_a (d + \delta) P_a) \alpha, \beta \rangle = \langle \alpha, (P_a (d + \delta) P_a) \beta \rangle = \langle \alpha, D_a \beta \rangle.$$

Lemma 5.8. The domain of the closure of $D_a$ is $H^1_a$.

Proof. The graph of $D_a$ is $G_a = \{ (\omega, D_a \omega) : \omega \in \Omega_a(M) \} \subseteq H^0_a \times H^1_a$. The closure of $G_a$ is also a graph, by the following argument. We must show that for any $(0, \eta) \in \overline{G_a}$, $\eta = 0$. For any $(0, \eta) \in \overline{G_a}$, there is a sequence $(\omega_j)$ of smooth antibasic forms with $\omega_j \to 0$ and $D_a \omega_j \to \eta$ in $H^0_a \subseteq L^2$. But then for any smooth antibasic form $\gamma$,

$$\langle D_a \omega_j, \gamma \rangle \to \langle \eta, \gamma \rangle, \text{ and } \langle \omega_j, D_a \gamma \rangle \to 0$$

as $j \to \infty$. But $\langle \omega_j, D_a \gamma \rangle = \langle D_a \omega_j, \gamma \rangle$ by Lemma 5.7. so $\langle \eta, \gamma \rangle = 0$ for all smooth $\gamma$, so $\eta = 0$. Thus $\overline{G_a} = \{ (\omega, A\omega) : \omega \in \text{dom}(A) \}$ for some operator $A$, which is defined to be the closure of $D_a$. Thus the domain is the set of all $\omega \in H^0_a$ such that there exists a sequence $(\omega_j)$ of smooth antibasic forms such that $\omega_j \to \omega$ in $H^0_a$ and $(D_a \omega_j)$ converges in $H^0_a$. By Gårding’s Inequality (Lemma 5.2) and Lemma 5.1 dom $(A) = H^1_a$.

Lemma 5.9. (Existence of Friedrichs’ mollifiers) There exists a family of self-adjoint smoothing operators $\{F_\rho\}_{\rho \in (0,1)}$ on $H^0_a$ such that $(F_\rho)$ is bounded in $H^0_a$, $F_\rho \to 1$ uniformly weakly in $H^0_a$ as $\rho \to 0$, and $[F_\rho, D_a]$ extends to a uniformly bounded family of operators on $H^0_a$.

Proof. Let $F_\rho^0$ be defined as the usual Friedrichs’ mollifiers; c.f. [16, Definition 5.21, Exercise 5.34]. Thus, these operators satisfy the properties above, except with $H^0_a$ replaced by $L^2(\Omega(M))$ and $D_a$ replaced by any first order differential operator. Now let $F_\rho = P_a F_\rho^0$. Note that $F_\rho$ is smoothing because $F_\rho^0$ is smoothing and since $P_a$ maps smooth forms to smooth forms; its kernel is the kernel of $F_\rho^0$ followed by $P_a$. We now check the three properties. First, for any $\alpha \in H^0_a \subseteq L^2(\Omega(M))$,

$$\|F_\rho \alpha\|_0 = \|P_a F_\rho^0 \alpha\|_0 \leq \|F_\rho^0 \alpha\|_0 \leq c \|\alpha\|_0$$
for some $c > 0$, by the first property of $F_0^\rho$. Next, for any $\alpha \in H^0_a$, for all smooth antibasic forms $\beta$,
\[
\langle (F_\rho - 1) \alpha, \beta \rangle = \langle (P_\rho F_\rho^0 - 1) \alpha, \beta \rangle = \langle (F_\rho^0 - 1) \alpha, \beta \rangle,
\]
which approaches 0 uniformly as $\rho \to 0$ by the corresponding property of $F_\rho^0$. Lastly, for any smooth antibasic forms $\omega, \eta$,
\[
\langle [F_\rho, D_\alpha] \omega, \eta \rangle = \langle P_\rho F_\rho^0 D_\alpha \omega, \eta \rangle - \langle D_\alpha P_\rho F_\rho^0 \omega, \eta \rangle
\]
\[
= \langle F_\rho^0 (\delta + d + P_\rho \varepsilon^* \omega), \eta \rangle - \langle P_\rho (\delta + d + \varepsilon P_\rho) F_\rho^0 \omega, \eta \rangle
\]
\[
= \langle F_\rho^0 (\delta + d) \omega, \eta \rangle + \langle F_\rho F_\rho^0 \varepsilon^* \omega, \eta \rangle - \langle (\delta + d + \varepsilon P_\rho) F_\rho^0 \omega, \eta \rangle
\]
\[
= \langle [F_\rho^0, (\delta + d)] \omega, \eta \rangle + \langle F_\rho^0 P_\rho \varepsilon^* \omega, \eta \rangle - \langle \varepsilon P_\rho F_\rho^0 \omega, \eta \rangle .
\]
The operator $[F_\rho^0, (\delta + d)]$ is bounded by the corresponding property of $F_\rho^0$, and $F_\rho^0 P_\rho \varepsilon^*$ and $\varepsilon P_\rho F_\rho^0$ are both zeroth order operators that are uniformly bounded in $\rho$ on $L^2$, so we conclude that $[F_\rho, D_\alpha]$ is uniformly bounded on $H^k_a$.

**Corollary 5.10.** Let $\{F_\rho\}$ be a family of Friedrichs’ mollifiers. Then $F_\rho$ and $[D_\alpha, F_\rho]$ are uniformly bounded families of operators on $H^k_a$ for any $k \geq 0$.

**Proof.** We proceed by induction using the elliptic estimates in Lemma 5.9.

**Proposition 5.11.** Suppose that $\alpha, \beta \in H^0_a$ and $D_\alpha \alpha = \beta$ weakly. Then $\alpha \in H^1_a = \mathrm{dom} \overline{D_\alpha}$, and $D_\alpha \alpha = \beta$.

**Proof.** For any $\alpha, \beta \in H^0_a$, suppose $D_\alpha \alpha = \beta$ weakly. Then for any smooth antibasic form $\gamma$ and any $\rho \in (0, 1)$,
\[
\langle D_\alpha F_\rho \alpha, \gamma \rangle = \langle F_\rho \alpha, D_\alpha \gamma \rangle
\]
\[
= \langle \alpha, F_\rho D_\alpha \gamma \rangle + \langle \alpha, D_\alpha F_\rho \alpha \rangle + \langle [F_\rho, D_\alpha] \alpha, \gamma \rangle
\]
\[
= \langle \beta, F_\rho \gamma \rangle + \langle \alpha, [F_\rho, D_\alpha] \alpha \rangle + \langle F_\rho \beta, \gamma \rangle + \langle \alpha, [F_\rho, D_\alpha] \alpha \rangle
\]
by Lemma 5.9. Thus, for a constant $C > 0$ independent of $\rho$ and $\gamma$,
\[
|\langle D_\alpha F_\rho \alpha, \gamma \rangle| \leq C \| \gamma \|_0
\]
independent of $\rho$ and of $\gamma$. Then $\| D_\alpha F_\rho \alpha \|_0 \leq C$. By Gårding’s Inequality (Lemma 5.2) and the fact that $F_\rho$ is a bounded operator in $H^0_a$, $\{F_\rho \alpha\}_{\rho \in (0, 1)}$ is a bounded set in $H^1_a$. By the weak compactness of a ball in the Hilbert space $H^1_a$ (with equivalent metric $(\xi, \theta)_1 = \langle D_\alpha \xi, D_\alpha \theta \rangle + \langle \xi, \theta \rangle$), there is a sequence $\rho_j \to 0$ and $\alpha' \in H^1_a$ such that $F_{\rho_j} \alpha \to \alpha'$ weakly in $H^1_a$. By Rellich’s Theorem, the subsequence converges strongly in $H^0_a$, so $F_{\rho_j} \alpha \to \alpha'$ in $H^0_a$. But we know already that $F_{\rho_j} \alpha \to \alpha$ in $H^0_a$, so $\alpha = \alpha' \in H^1_a$.

**Corollary 5.12.** The antibasic de Rham and antibasic Laplacian are essentially self-adjoint operators.

**Proof.** From the proposition above, the domain of the closure of the symmetric operator $D_\alpha$ is the domain of its $H^0_a$-adjoint, so that $\overline{D_\alpha}$ is self-adjoint. Then $\overline{\Delta_\alpha} = D_\alpha^2$ is also essentially self-adjoint.

**Proposition 5.13.** (Elliptic regularity) Suppose that $\omega \in \ker D_\alpha \subseteq H^1_a$. Then $\omega$ is smooth.

**Proof.** If $D_\alpha \omega = 0$ for some $\omega \in H^1_a$. We will show by induction that $\omega \in H^k_a$ for all $k$, and then the Sobolev embedding theorem implies that $\omega$ is smooth. Suppose that we know $\omega \in H^{k-1}_a$ for some $k \geq 2$. Let $\{F_{\rho_j}\}$ be a be a family of Friedrichs’ mollifiers. Then from the elliptic estimates (Lemma 5.9), there is a constant $C_{k-1} > 0$ such that
\[
\| F_{\rho_j} \omega \|_k \leq C_{k-1} \left( \| F_{\rho_j} \omega \|_{k-1} + \| D_\alpha F_{\rho_j} \omega \|_{k-1} \right)
\]
\[
\leq C_{k-1} \left( \| F_{\rho_j} \omega \|_{k-1} + \| F_{\rho_j} D_\alpha \omega \|_{k-1} + \| D_\alpha F_{\rho_j} \omega \|_{k-1} \right)
\]
\[
= C_{k-1} \left( \| F_{\rho_j} \omega \|_{k-1} + \| D_\alpha F_{\rho_j} \omega \|_{k-1} \right).
\]
Thus \( \|F_{\rho} \omega\|_k \) is bounded by Corollary 5.10. We now proceed as in the proof of Proposition 5.11 to say that there is a sequence \( \rho_j \to 0 \) such that \( F_{\rho_j} \omega \to \omega' \) weakly in \( H^k_a \) and strongly to \( H^0_a \). Thus, we get \( \omega = \omega' \in H^k_a \).

**Corollary 5.14.** Eigenforms of \( D_a \) are smooth.

**Proof.** The proof above also is easily modified if \( D_a \omega = \lambda \omega \) to show that the eigenforms of \( D_a \) are smooth. \( \square \)

We will now use a standard technique to derive the spectral theorem for \( D_a \) and \( \Delta_a \) from these basic facts (c.f. [16] Chapter 5)

**Lemma 5.15.** Let \( \overline{G} = (H^1_a, D_a (H^1_a)) \subseteq (H^1_a, H^0_a) \) denote the closure of the graph of \( D_a \). Let \( J : H^0_a \times H^0_a \to H^0_a \times H^0_a \) be defined by \( J(x, y) = (y, -x) \). Then there is an orthogonal direct sum decomposition

\[
H^0_a \oplus H^0_a = \overline{G} \oplus \overline{JG}
\]

**Proof.** Suppose \((x, y) \in \overline{G} \). Then, for all \( \omega \in \Omega_a (M) \),

\[
0 = \langle (x, y), (\omega, D_a \omega) \rangle = \langle x, \omega \rangle + \langle y, D_a \omega \rangle,
\]

so that \( D_a y + x = 0 \) weakly. By Proposition 5.11, \( y \in H^0_a \), so \( (y, -x) \in \overline{G} \), so \( (x, y) \in \overline{JG} \). \( \square \)

**Definition 5.16.** Let the operator \( Q_a : H^0_a \to H^1_a \) be defined by the equation for any \( \alpha \in H^0_a \), \( (Q_a \alpha, D_a Q_a \alpha) \) is the orthogonal projection of \((\alpha, 0)\) on \( \overline{G} \) in \( H^0_a \oplus H^0_a \).

As \( \|\alpha\|^2_0 \geq \|Q_a \alpha\|^2_0 + \|D_a Q_a \alpha\|^2_0 \geq c \|Q_a \alpha\|^2_1 \), then \( Q_a \) is bounded as an operator from \( H^0_a \) to \( H^1_a \). By Rellich’s Theorem, \( Q_a \) is compact as an operator from \( H^0_a \) to \( H^0_a \). It is self-adjoint, positive, and injective, and has norm \( \leq 1 \). By the spectral theorem for compact, self-adjoint operators, \( H^0_a \) can be decomposed as a direct sum of finite-dimensional eigenspaces of \( Q_a \), and the eigenvalues approach 0 as the only accumulation point. Given an eigenvector \( \alpha \) of \( Q_a \) corresponding to the eigenvalue \( \mu > 0 \), so that \( 0 < \mu \leq 1 \), by Lemma 5.14 there exists \( \eta \) such that

\[
\alpha = (Q_a \alpha, D_a Q_a \alpha) + (-D_a \eta, \eta) = \mu (\alpha, D_a \alpha) + (-D_a \eta, \eta),
\]

so that \( (\mu - 1) \alpha = D_a \eta \) and \( \eta = -\mu D_a \alpha \). Letting \( \lambda^2 = \frac{1}{\mu} \) and \( \beta = -\frac{1}{\mu} \eta \), we have

\[
D_a \alpha = \lambda \beta, \quad D_a \beta = \lambda \alpha.
\]

Thus \( \alpha \pm \beta \) are eigenforms of \( D_a \) with eigenvalues \( \pm \lambda \). Thus, \( H^0_a \) can be decomposed as an orthogonal direct sum of finite-dimensional eigenspaces of \( D_a \). We now have the following.

**Theorem 5.17.** (Spectral Theorem for the antibasic operators) The spectrum of the antibasic Laplacian and antibasic de Rham operators consists of real eigenvalues of finite multiplicity, with accumulation points at infinity. The smooth eigenforms of \( D_a \) are also the eigenforms of \( \Delta_a \) and can be chosen to form a complete orthonormal basis of \( H^0_a \).

**Proof.** Besides the above computations, observe that \( \Delta_a = D_a^2 \). \( \square \)

6. THE ANTIBASIC HODGE DECOMPOSITION AND HOMOTOPY INVARIANCE

Let \( M \) be a closed manifold of dimension \( n \) endowed with a foliation of dimension \( q \) and a bundle-like metric. The basic Hodge decomposition theorem (proved in [12]) gives

\[
\Omega^k_a (M) = \text{im} (d_{b,k-1}) \oplus \mathcal{H}^k_b \oplus \text{im} (\delta_{b,k+1}),
\]

where \( d_{b,k} = d : \Omega^k_b (M) \to \Omega^{k+1}_b (M) \) is the exterior derivative restricted to basic forms with \( L^2 \)-adjoint \( \delta_{b,k+1} = P_b \delta : \Omega^{k+1}_b (M) \to \Omega^k_b (M) \), and where \( \mathcal{H}^k_b = \ker (\Delta_{b,k}) \) is the space of basic harmonic \( k \)-forms. Also

\[
\ker (d_{b,k}) = \text{im} (d_{b,k-1}) \oplus \mathcal{H}^k_b\text{ and } \ker (\delta_{b,k}) = \mathcal{H}^k_b \oplus \text{im} (\delta_{b,k+1}),
\]

so the basic cohomology groups satisfy \( H^k_b (M, \mathcal{F}) \cong \mathcal{H}^k_b \). We now have the tools to prove the antibasic version.
First, note that there is an alternative de Rham complex that uses $\delta$ as a differential. Writing $\Omega^i = \Omega^i(M)$, the complex
\[
\Omega^n \xrightarrow{\delta_n} \Omega^{n-1} \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_1} \Omega^0 \rightarrow 0
\]
satisfies $\delta_{k-1} \delta_k = 0$ for $0 \leq k \leq n$, and the de Rham cohomology satisfies
\[
H^k(M) = \ker \delta_k : \Omega^k \rightarrow \Omega^{k-1} / \text{im} \delta_{k+1} : \Omega^{k-1} \rightarrow \Omega^k.
\]
Abbreviating $\Omega^i_\alpha = \Omega^i_\alpha(M)$, the antibasic de Rham complex is a subcomplex
\[
\Omega^n_\alpha \xrightarrow{\delta_n} \Omega^{n-1}_\alpha \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_1} \Omega^0_\alpha \rightarrow 0
\]
We adopt a standard proof of Hodge decomposition to our case (c.f. [16, Chapter 6]) and utilize the analytic results proved in the previous section.

**Theorem 6.1. (Antibasic Hodge Theorem)** Suppose that $(M,\mathcal{F})$ is a Riemannian foliation with bundle-like metric. Then for $0 \leq k \leq n$ the antibasic cohomology groups satisfy
\[
H^k_a(M,\mathcal{F}) \cong \mathcal{H}^k_a.
\]

**Proof.** From Theorem 5.17 $\mathcal{H}^k_a = \ker \left( \Delta_a |_{\Omega^k_a} \right)$ is finite dimensional for all $k$. Consider the following subcomplex of the antibasic de Rham complex, with $0$ being the codifferential, and the inclusion maps:
\[
\cdots \xrightarrow{0} \mathcal{H}^{j+1}_a \xrightarrow{0} \mathcal{H}^j_a \xrightarrow{0} \mathcal{H}^{j-1}_a \xrightarrow{0} \cdots
\]

We will show that the inclusion $i$ is a chain equivalence. We define the map $P : \Omega^j_a \rightarrow \mathcal{H}^j_a$ to be the restriction of the orthogonal projection $L^2(\Omega^j_a) \rightarrow \mathcal{H}^j_a$ to smooth antibasic forms. Then $Pf = 1$ and $iP = 1 - f(D_a)$, where
\[
f(\lambda) = \begin{cases} 
1 & \text{if } \lambda \neq 0 \\
0 & \text{if } \lambda = 0 
\end{cases}
\]
and where we have used Theorem 5.17 and the functional calculus. Let
\[
g(\lambda) = \begin{cases} 
\frac{1}{\lambda} & \text{if } \lambda \neq 0 \\
0 & \text{if } \lambda = 0 
\end{cases}.
\]

Then $g$ is bounded on $\sigma(D_a)$, so the Green’s operator $G_a = g(D_a)$ extends to a bounded operator on $H^0_a$. We see that $D^2_a G_a = f(D_a) = 1 - iP$, and also
\[
D^2_a G_a = (\delta d_a + d_a \delta) G_a = \delta d_a G_a + G_a d_a \delta.
\]
Since $\Delta_a$ commutes with $d_a$, we have $H_a = d_a G_a = G_a d_a$. Thus, $H_a$ satisfies
\[
1 - iP = \delta H_a + H_a \delta
\]
and is thus a chain homotopy between $iP$ and $1$, so $i$ is a chain equivalence.

**Corollary 6.2.** On a Riemannian foliation on a closed manifold, the antibasic cohomology groups are finite dimensional.

The following corollary follows in the standard way.

**Corollary 6.3.** We have the following $L^2$-orthogonal decomposition:
\[
\Omega^i_a = \mathcal{H}^i_a \oplus \text{im} \left( \delta |_{\Omega^i_a} \right) \oplus \text{im} \left( d_a |_{\Omega^{i+1} a} \right).
\]

**Proof.** We utilize the spectral theorem again, noting the eigenform decomposition $\Delta_a \geq 0$. For any smooth antibasic $k$-form $\alpha$, $\Delta_a \alpha = (d_a + \delta_a)^2 \alpha = 0$ if and only if
\[
0 = \langle (d_a + \delta_a) \alpha, (d_a + \delta_a) \alpha \rangle = \langle d_a \alpha, d_a \alpha \rangle + \langle \delta \alpha, \delta \alpha \rangle.
\]
Then and let Lemma 6.5.

\[ \Omega^a_{k,+} \cong d_a (\Omega^a_{k,+}) \oplus \delta (\Omega^a_{k,+}) \subseteq \Omega^a_{k+1,+} \oplus \Omega^a_{k-1,+.} \]

Then

\[ \Omega^*_a = \mathcal{H}^*_a + \text{im} \left( \delta|_{\Omega^a_{k,+}} \right) + \text{im} \left( d_a|_{\Omega^a_{k,-}} \right) \]

Also \( \text{im} \left( \delta|_{\Omega^a_{k,+}} \right) \) and \( \text{im} \left( d|_{\Omega^a_{k,-}} \right) \) are orthogonal:

\[ \langle d_a \alpha, \delta \beta \rangle = \langle d^2_a \alpha, \beta \rangle = 0 \]

for all \( \alpha, \beta \) and likewise if \( \gamma \in \mathcal{H}^a_k \), then for all antibasic forms \( \eta, \theta \) we have

\[ \langle d_a \eta, \gamma \rangle = \langle \eta, \delta \gamma \rangle = 0, \]
\[ \langle \delta \theta, \gamma \rangle = \langle \theta, d_a \gamma \rangle = 0. \]

Therefore, the result follows.

Remark 6.4. For the same reason that \( \delta \) can be used in place of \( d \) in computing de Rham cohomology, the same reasoning shows from the Hodge theorem (in the Riemannian foliation case) that

\[ H^k_a(M, \mathcal{F}) \cong \frac{\ker d_a : \Omega^k_a \to \Omega^{k+1}_a}{\text{im} d_a : \Omega^{k-1}_a \to \Omega^k_a}. \]

We now use the above formula for antibasic cohomology to prove the foliated homotopy invariance of antibasic cohomology in the case of Riemannian foliations.

Lemma 6.5. Let \((M, \mathcal{F})\) and \((M', \mathcal{F}')\) be Riemannian foliations of closed manifolds with bundle-like metrics, and let \( f \) be a foliated map from \((M, \mathcal{F}) \to (M', \mathcal{F}')\). Then \( P_a f^* P'_a \) induces a linear map on antibasic cohomology.

Proof. We consider antibasic cohomology through the isomorphism \( H^k_a(M, \mathcal{F}) \cong \frac{\ker d_a : \Omega^k_a \to \Omega^{k+1}_a}{\text{im} d_a : \Omega^{k-1}_a \to \Omega^k_a} \). Then we have the following equation on antibasic forms, using \([3.3]\) and the fact that pullbacks by foliated maps preserve the basic forms:

\[ P_a f^* P'_a d^*_a = P_a f^* P'_a \]
\[ = P_a f^* (dP'_a + P'_a \xi^*_r) \]
\[ = P_a f^* dP'_a + P_a f^* P'_b \xi^*_r \]
\[ = P_a dP'_a + P_a f^* P'_a \]
\[ = P_a dP'_a + P_a f^* P'_a = d_a P_a f^* P'_a. \]

Therefore, \( P_a f^* P'_a \) induces a linear map from \( d'_a \)-cohomology to \( d_a \)-cohomology.

Theorem 6.6. (Foliated Homotopy Axiom of Antibasic Cohomology) Let \((M, \mathcal{F})\) and \((M', \mathcal{F}')\) be Riemannian foliations of closed manifolds with bundle-like metrics, and let \( f \) and \( g \) be two foliated maps from \((M, \mathcal{F}) \to (M', \mathcal{F}')\). If \( f \) is foliated homotopic to \( g \), then \( P_a f^* P'_a \) and \( P_a g^* P'_a \) induce the same map on antibasic cohomology.

Proof. Again we view antibasic cohomology through the isomorphism \( H^k_a(M, \mathcal{F}) \cong \frac{\ker d_a : \Omega^k_a \to \Omega^{k+1}_a}{\text{im} d_a : \Omega^{k-1}_a \to \Omega^k_a} \). Let \( H : [0, 1] \times M \to M' \) be a foliated homotopy such that \( H(0, x) = f(x) \) and \( H(1, x) = g(x) \). Then, define \( j_x : M \to [0, 1] \times M \) by \( j_x(x) = (s, x) \), and let \( h : \Omega^k(M') \to \Omega^{k-1}(M) \) by

\[ h(\sigma) = \int_0^1 j^*_x (\partial_t H^* \sigma) \, ds. \]

Note that \( h \) preserves the basic forms since \( H \) is a foliated homotopy. By a standard calculation, we have that

\[ g^* - f^* = dh + hd \]
on $\Omega(M')$. Then we apply $P_a$ on the left and $P'_a$ on the right and use the equation $P_a dP_a = P_a d$ to get

$$P_a g^* P'_a - P_a f^* P'_a = P_a dhP'_a + P_a h dP'_a$$

$$= P_a dP_a h P'_a + P_a h (P'_a d - P'_a e^{*'})$$

$$= P_a dP_a h P'_a + P_a h P'_a P'_a d - P_a h P'_a e^{*'}$$

$$= d_a (P_a h P'_a) (P_a h P'_a d) - P_a h P'_a e^{*'}$$

$$= d_a (P_a h P'_a) + (P_a h P'_a d)^a.$$ 

Thus $P_a h P'_a$ is a chain homotopy between $P_a g^* P'_a$ and $P_a f^* P'_a$. \hfill $\square$

**Corollary 6.7.** (Foliated Homotopy Invariance of Antibasic Cohomology) If $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are Riemannian foliations of closed manifolds with bundle-like metrics that foliated homotopy equivalent, then their antibasic cohomology groups are isomorphic.

**Proof.** Suppose that $f : M \to M'$ is a foliated map such that there exists a foliated map $g : M' \to M$ such that $f \circ g$ and $g \circ f$ are each foliated homotopic to the identity. Then we have that

$$P'_a (f \circ g)^* P'_a = Id' : H^k_a (M') \to H^k_a (M'),$$

$$P_a (g \circ f)^* P_a = Id : H^k_a (M) \to H^k_a (M).$$

Since the pullback by $g$ preserves the basic forms,

$$Id' = P'_a g^* f^* P'_a = P'_a g^* (P_a + P_b) f^* P'_a$$

$$= P'_a g^* P_a P_a f^* P'_a + P'_a g^* P_b f^* P'_a$$

$$= P'_a g^* P_a P_a f^* P'_a + P'_a P'_b g^* P_b f^* P'_a = (P'_a g^* P_a) (P_a f^* P'_a),$$

and similarly $(P_a f^* P'_a) (P'_a g^* P_a) = Id$, so we must have that $P_a f^* P'_a$ is an isomorphism from $H^k_a (M')$ to $H^k_a (M)$. \hfill $\square$

### 7. Properties and Applications

First we consider the simple case when the operators $\varepsilon$ and $\varepsilon^*$ are zero. In this case, the antibasic Betti numbers can be computed from the ordinary Betti numbers and basic Betti numbers.

**Proposition 7.1.** Suppose that $(M, \mathcal{F})$ is a Riemannian foliation of a closed manifold of dimension $n$, such that the normal bundle $TM/T\mathcal{F}$ is involutive. Then for any metric, the antibasic cohomology and basic cohomology add to the ordinary cohomology. That is, for $0 \leq k \leq n$

$$H^k (M) \cong H^k_a (M, \mathcal{F}) \oplus H^k_b (M, \mathcal{F}).$$

**Proof.** First, we choose a bundle-like metric so that the mean curvature is basic; this can always be done \cite{7}. The operator $\varepsilon^*$ satisfies

$$\varepsilon^* = -\kappa_a \wedge + (-1)^k (\chi_{\mathcal{F}} \wedge) (\varphi_0 \wedge)$$

$$= 0$$

under the hypotheses, since $\varphi_0 = 0$ if and only if the normal bundle is involutive. Then, by Theorem \cite{12} the antibasic Laplacian is precisely a restriction of the ordinary Laplacian. Similarly, by formula \cite{12} and the results of \cite{12}, the basic Laplacian is a restriction of the ordinary Laplacian. Thus the ordinary Laplacian preserves the basic and antibasic forms, and it decomposes as a direct sum of the basic and antibasic Laplacians. The harmonic forms decompose into basic and antibasic parts, and the Hodge theorem implies the result. \hfill $\square$

**Corollary 7.2.** Suppose that $(M, \mathcal{F})$ is a Riemannian foliation of a closed manifold of dimension $n$, such that the normal bundle $TM/T\mathcal{F}$ is involutive. Then $\dim H^k_b (M, \mathcal{F}) \leq \dim H^k (M), \dim H^k_a (M, \mathcal{F}) \leq \dim H^k (M)$.

**Remark 7.3.** It was essentially already known that $\dim H^k_b (M, \mathcal{F}) \leq \dim H^k (M)$ in this case, because using \cite{12} and \cite{7} we see that for a metric with basic mean curvature, $\delta_b = \delta$ when restricted to basic forms.
Proposition 7.4. Suppose that \((M, \mathcal{F}, g)\) is a Riemannian foliation of a closed manifold of dimension \(n\) with bundle-like metric, such that the mean curvature form is basic and the normal bundle \(\mathcal{N}\) is involutive. Then the wedge product induces a bilinear product on basic and antibasic cohomology:

\[ \wedge : H^k_b(M, \mathcal{F}) \otimes H^r_b(M, \mathcal{F}) \to H^{k+r}_a(M, \mathcal{F}). \]

Proof. The operator \(\varepsilon = 0\) under the assumptions, so that both \(d\) and \(\delta\) restrict to both basic and antibasic forms. The wedge product of a basic and antibasic form is antibasic (since \(\varepsilon = 0\)), so that the result follows from the standard result in de Rham cohomology.

In more generality, if the mean curvature is basic but without the assumption on \(\varphi_0\), the same result is true for \(k = 0\).

Proposition 7.5. Suppose that \((M, \mathcal{F}, g)\) is a Riemannian foliation of a closed manifold of dimension \(n\). Then

\[ H^0(M) \cong H^0_b(M, \mathcal{F}) \oplus H^0_a(M, \mathcal{F}). \]

In particular, if \(M\) is connected, then \(H^0_b(M, \mathcal{F}) \cong \mathbb{R}\) and \(H^0_a(M, \mathcal{F}) \cong \{0\} \).

Proof. We first choose a bundle-like metric such that the mean curvature is basic. By Theorem \[\text{[4.1]}\], \(\Delta_a\) is the restriction of \(\Delta + \delta P_a \varepsilon^* + P_a \varepsilon^* \delta\) to \(\Omega^*_a(M, \mathcal{F})\), and on functions this is \(\Delta + \delta P_a \varepsilon^*\). But also \(\varepsilon^* = -\kappa_a \wedge + (-1)^k (\chi \varepsilon) (\varphi_0 \wedge) = 0\) on functions, so that \(\Delta_a\) is the restriction of the ordinary Laplacian. Also, by the results of \[\text{[12]}\], \(\Delta_b\) is the restriction of \(\Delta + \varepsilon d + d \varepsilon\) to \(\Omega^*_b(M, \mathcal{F})\), and on functions this is \(\Delta + d \varepsilon\), but in our case \(\varepsilon = (-1)^k (\varphi_0 \wedge) (\chi \varepsilon \wedge)\) is zero on basic one-forms so that \(\Delta_b\) is the restriction of \(\Delta\). Thus \(\Delta\) is the orthogonal direct sum of the restrictions to basic and antibasic functions, and the result follows from the Hodge theorem.

Proposition 7.6. Suppose that \((M, \mathcal{F}, g)\) is a Riemannian foliation on a closed, connected manifold. Then for any metric,

\[ \dim H^1(M) \leq \dim H^1_b(M, \mathcal{F}) + \dim H^1_a(M, \mathcal{F}). \]

Proof. First, we choose a bundle-like metric with basic mean curvature. Given a \(\Delta\)-harmonic form \(\beta\), consider \(P_a \beta\). We see that

\[ P_a d (P_a \beta) = P_a (P_a (d \beta) - P_b (\varepsilon^* \beta)) = 0. \]

Also,

\[ \delta_a P_a \beta = \delta P_a \beta = P_a \delta \beta + \varepsilon (P_b \beta) = 0, \]

because \(\varepsilon = 0\) on basic one-forms. Thus the map \(\beta \mapsto P_a \beta\) maps harmonic one-forms to antibasic harmonic one-forms. The kernel of this map is the set of basic forms such that \(\delta \beta = 0\) and \(\delta \beta = (\delta_b - \varepsilon) \beta = \delta_b \beta\), since \(\varepsilon\) is zero on basic one-forms. Thus the kernel is the set of \(\Delta_b\)-harmonic forms. By the Hodge theorem, the result follows, since \(H^1(M) \cong H^1_b(M, \mathcal{F}) \oplus P_a (H^1(M)) \subseteq H^1_b(M, \mathcal{F}) \oplus H^1_b(M, \mathcal{F}). \)

Remark 7.7. Note that in general \(H^1_b(M) \hookrightarrow H^1(M)\) is an injection for all foliations, so always \(\dim H^1_b(M) \leq \dim H^1(M)\). Thus, if \(H^1_b(M, \mathcal{F}) \cong \{0\}\), then \(H^1_a(M) \cong H^1(M)\), so that every harmonic one-form is basic.

Another simple class of examples of Riemannian foliations occurs when the orbits of a compact connected Lie group action all have the same dimension. In this case, we may choose a metric such that the Lie group acts by isometries. The Lie group acts on differential forms by pullback, and this action commutes with \(d\) and \(\delta\). Thus, if we decompose the differential forms according to the irreducible representations \(\rho \in \hat{G}\) of \(G\), we have the \(L^2\)-orthogonal direct sum

\[ \Omega^*(M) = \bigoplus_{\rho \in \hat{G}} \Omega^{*, \rho}(M) \]

where \(\Omega^{*, \rho}(M)\) is the space of differential forms of type \(\rho : G \to U(V_{\rho})\). That is,

\[ \Omega^{*, \rho}(M) = \bigcup_{f \in \text{Hom}_c(V_{\rho}, \Omega^*(M))} f(V_{\rho}). \]

Because of the metric invariance, both \(d\) and \(\delta\) respect this decomposition. It is well-known that the harmonic forms are always invariant (i.e., belong to \(\Omega^{*, \rho_0}(M)\), where \(\rho_0\) is the trivial representation). Also, for the
foliation \( \mathcal{F} \) by \( G \)-orbits, we have \( \Omega^*_b(M, \mathcal{F}) \subseteq \Omega^{*,\rho_0}(M) \). We let \( d_j, \delta_j \) refer to the restrictions of \( d, \delta \) to \( \Omega^j \), and we let \( d_{a,j}, \delta_{a,j}, d_{b,j}, \delta_{b,j} \) denote the corresponding restrictions to basic and antibasic forms. We use the superscript \( \rho \) to denote further restrictions to \( \Omega^{*,\rho}(M) \). Then we have

\[
\begin{align*}
    d_j &= \bigoplus_{\rho \in \hat{G}} d^\rho_j \\
    d_{b,j} &= d^\rho_{b,j} \\
    d_{a,j} &= d^\rho_{a,j} \oplus \bigoplus_{\rho \notin \hat{G}} d^\rho_j \\
    \delta_{b,j} &= \delta^\rho_{b,j} \\
    \delta_{a,j} &= \delta^\rho_{a,j} \oplus \bigoplus_{\rho \notin \hat{G}} \delta^\rho_j.
\end{align*}
\]

Thus, in computing either the basic or antibasic cohomology, it is sufficient to restrict to invariant forms.

The result below follows.

**Proposition 7.8.** Let \( G \) be a connected, compact Lie group that acts on a connected closed manifold \( M \) by isometries. Let \((M, \mathcal{F}, g)\) be the Riemannian foliation with bundle-like metric given by the \( G \)-orbits. Then

\[
H^j_b(M, \mathcal{F}) \cong \frac{\ker d^\rho_{b,j}}{\text{im} d^\rho_{b,j}}; \quad H^j_a(M, \mathcal{F}) \cong \frac{\ker \delta^\rho_{a,j}}{\text{im} \delta^\rho_{a,j+1}}.
\]

In particular,

\[
H^0_b(M, \mathcal{F}) \cong \mathbb{R}; \quad H^0_a(M, \mathcal{F}) \cong \{0\}.
\]

**Proof.** The first part follows from the discussion above. Next, observe that all \( G \)-invariant functions are basic, so that \( \Omega^0_b(M, \mathcal{F}) = \Omega^{0,\rho_0}(M) \), so that \( \ker (d^\rho_{b,j}) = \ker (d^\rho_{b,j}) \) consists of the constant functions, and \( \ker \delta^\rho_{a,0} = \{0\} \). The second part also follows from Proposition 7.5 since the mean curvature is always basic in this case.

\[\square\]

8. The case of Riemannian flows

In this section, we study tautness and cohomology for Riemannian flows. The following result shows a relationship between basic and antibasic cohomology when the flow is taut. We will see evidence of this behavior in Example 9.1 and Example 9.2. We will use standard techniques in the study of these flows, which can also be found for example in [3, 11, 1].

**Proposition 8.1.** Suppose that \((M, \mathcal{F})\) is a Riemannian flow on a closed manifold with bundle-like metric \( g \) with basic mean curvature \( \kappa \) and characteristic form \( \chi \). If \( \kappa = 0 \), then for all \( r \), the map \( \alpha \mapsto \chi \wedge \alpha \) maps basic harmonic \( r \)-forms to antibasic harmonic \( (r+1) \)-forms injectively. Thus, \( \dim (H^r_a(M, \mathcal{F})) \geq \dim (H^r_b(M, \mathcal{F})) \) whenever \( [\alpha] = 0 \in H^1_b(M, \mathcal{F}) \).

**Proof.** Suppose that \( [\alpha] = 0 \). We then choose a bundle-like metric such that \( \kappa = 0 \). With this metric, for any basic harmonic \( r \)-form \( \alpha \),

\[
\begin{align*}
    d(\chi \wedge \alpha) &= d\chi \wedge \alpha - \chi \wedge d\alpha \\
                       &= \varphi_0 \wedge \alpha,
\end{align*}
\]

which is basic, so that \( d_a(\chi \wedge \alpha) = 0 \). Let \( \chi^\# = \xi \), and choose the usual adapted orthonormal frame \( \{b_i\} = \{e_i\} \cup \{\xi\} \) near a point, where the \( e_i \) are basic and \( \nabla^Q \)-parallel at the point in question. Note that
\[ \delta \chi = 0 \] because the metric is bundle-like, and then
\[
\delta (\chi \wedge \alpha) = - \sum_i b_i \nabla^M_{b_i} (\chi \wedge \alpha) \\
= - \sum_i b_i (\nabla^M_{b_i} \chi \wedge \alpha + \chi \wedge \nabla^M_{b_i} \alpha) \\
= (\delta \chi) \alpha + \sum_i \nabla^M_{b_i} \chi \wedge (b_i \alpha) - \nabla^M_{\xi} \alpha + \chi \wedge \sum_i (b_i \nabla^M_{b_i} \alpha) \\
= \sum_i \nabla^M_{b_i} \chi \wedge (b_i \alpha) - \nabla^M_{\xi} \alpha - \chi \wedge \delta \alpha.
\]

But since \( \alpha \) is basic harmonic, \( \delta \alpha = \delta b \alpha = \varepsilon \alpha = 0 - \varphi_{0 \alpha} (\chi \wedge \alpha) = \chi \wedge (\varphi_{0 \alpha} \alpha) \). Thus, \( \chi \wedge \delta \alpha = 0 \), so that
\[
\delta (\chi \wedge \alpha) = \sum_i \nabla^M_{b_i} \chi \wedge (f_i \alpha) - \nabla^M_{\xi} \alpha \\
= \sum_k \nabla^M_{e_k} \chi \wedge (e_k \alpha) - \nabla^M_{\xi} \alpha \\
= \sum_k (he_k \gamma) \wedge (e_k \alpha) - \nabla^M_{\xi} \alpha \\
= \sum_{k,j} (he_k, e_j) e^j \wedge (e_k \alpha) - \nabla^M_{\xi} \alpha \\
= \sum_j -e^j \wedge ((he_j) \alpha) - \nabla^M_{\xi} \alpha,
\]

where the skew-adjoint O’Neill tensor \( h \) satisfies \( hX = \nabla^M_{X} \xi \) for \( X \in \Gamma (\mathcal{NF}) \). Now observe from one hand that \( \left( \nabla^M_{\xi} \alpha \right) (\xi, e_i, \ldots, e_{i-1}) = 0 \) since \( \alpha \) is basic. On the other hand, we use
\[
\nabla^M_{\xi} Z = \nabla^Q_{\xi} Z + h (Z) - \kappa (Z) \xi = \nabla^Q_{\xi} Z + h (Z)
\]

for \( Z \in \Gamma (\mathcal{NF}) \) to compute
\[
\left( \nabla^M_{\xi} \alpha \right) (e_i, \ldots, e_{i_r}) = \xi (\alpha (e_i, \ldots, e_{i_r})) - \sum_k \alpha (e_i, \ldots, \nabla^M_{\xi} e_k, \ldots, e_{i_r}) \\
= \xi (\alpha (e_i, \ldots, e_{i_r})) - \sum_k \alpha (e_i, \ldots, \nabla^M_{\xi} e_k, \ldots, e_{i_r}) \\
= \xi (\alpha (e_i, \ldots, e_{i_r})) - \sum_k \alpha (e_i, \ldots, he_k, \ldots, e_{i_r}) \\
= \left( \nabla^Q_{\xi} \alpha \right) (e_i, \ldots, e_{i_r}) - \sum_k \alpha (e_i, \ldots, he_k, \ldots, e_{i_r}) \\
= - \sum_k \alpha (e_i, \ldots, he_k, \ldots, e_{i_r}).
\]

Now we write
\[
\nabla^M_{\xi} \alpha = \frac{1}{r!} \sum_{(i_1, \ldots, i_r)} \left( \nabla^M_{\xi} \alpha \right) (e_{i_1}, \ldots, e_{i_r}) e^{i_1} \wedge \ldots \wedge e^{i_r} \\
= - \frac{1}{r!} \sum \alpha (e_{i_1}, \ldots, he_k, \ldots, e_{i_r}) e^{i_1} \wedge \ldots \wedge e^{i_r} \\
= - \frac{1}{r!} \sum (-1)^{k-1} (he_k \alpha) (e_{i_1}, \ldots, e_{i_r}) e^{i_1} \wedge \ldots \wedge e^{i_r} \\
= - \frac{r}{r!} \sum (he_{\ell \alpha}) (e_{i_1}, \ldots, e_{i_{r-1}}) e^{\ell} \wedge e^{i_1} \wedge \ldots \wedge e^{i_{r-1}} \\
= - \sum e^{\ell} \wedge (he_{\ell \alpha}).
\]
Thus, substituting we have
\[
\delta (\chi \wedge \alpha) = - \sum e^j \wedge ((h e_j) \alpha) - \nabla^M \alpha
\]
\[
= - \sum e^j \wedge ((h e_j) \alpha) + \sum \varepsilon^\ell \wedge (h e_\ell) \alpha
\]
\[
= 0,
\]
so that \(\chi \wedge \alpha \in \mathcal{H}^1_{c+1}(M,F)\). If \(\alpha\) is nonzero, then \(\chi \wedge \alpha\) is nonzero, so the class \([\chi \wedge \alpha]\) is nontrivial. \(\square\)

**Remark 8.2.** In particular, if \([\kappa] = 0 \in H^1_b(M,F)\), then \(\dim H^1_b(M,F) \geq 1\).

**Lemma 8.3.** Suppose that \((M,F)\) is a Riemannian flow on a closed, connected manifold, with a bundle-like metric chosen so that the mean curvature is basic. Then for any antibasic one-forms \(\alpha\) and \(\beta\),
\[
\langle \Delta_{\alpha} \alpha, \beta \rangle = \langle \Delta_{\alpha} \beta, \alpha \rangle - \int P_b(\chi,\alpha) P_b(\chi,\beta) |\varphi_0|^2.
\]

**Proof.** From Theorem 6.1,
\[
\Delta_{\alpha} \alpha = \Delta \alpha + \delta P_b \varepsilon^* \alpha + P_b \varepsilon^* \delta \alpha.
\]
Since \(\varepsilon^* \delta \alpha = 0\) and
\[
\varepsilon^* \alpha = - \chi \, (\varphi_0 \wedge \alpha) = (\chi, \alpha) \varphi_0,
\]
we have
\[
\delta P_b \varepsilon^* \alpha = \delta P_b ((\chi, \alpha) \varphi_0) = \delta (P_b (\chi, \alpha) \varphi_0)
\]
\[
= - df \varphi_0 + f \delta \varphi_0,
\]
where \(f = P_b (\chi, \alpha)\). Now,
\[
\langle \Delta_{\alpha} \alpha, \beta \rangle = \langle \Delta \alpha, \beta \rangle - \int (P_b(\chi,\alpha) P_b(\chi,\beta) |\varphi_0|^2.
\]
This completes the proof. \(\square\)

**Theorem 8.4.** Suppose that \((M,F)\) is a Riemannian flow on a closed, connected manifold. If \(H^1(M) = \{0\}\), then
\[
\dim H^1_b(M,F) = 1.
\]
If \([\kappa]\) is a nonzero class in \(H^1_b(M,F) \subseteq H^1(M)\), then
\[
\dim H^1_b(M,F) = 0.
\]

**Proof.** We choose the bundle-like metric so that \(\kappa\) is basic-harmonic 1-form (as in \([10]\)). We write any antibasic one-form \(\alpha\) as
\[
\alpha = f \chi + \beta = (f_a \chi + \beta) + (f_b \chi) = \alpha_1 + \alpha_2,
\]
where \(f_a = P_a f, f_b = P_b f\), \(\alpha_1 = f_a \chi + \beta, \alpha_2 = f_b \chi\) and \(\beta\) is an antibasic section of \(N^* F\). The \(L^2\) inner product gives
\[
\langle \Delta_{\alpha} \alpha, \alpha \rangle = \langle \Delta_{\alpha} \alpha_1, \alpha_1 \rangle + \langle \Delta_{\alpha} \alpha_2, \alpha_2 \rangle + 2 \langle \Delta_{\alpha} \alpha_1, \alpha_2 \rangle
\]
\[
= \langle \Delta_{\alpha} \alpha_1, \alpha_1 \rangle + \langle \Delta_{\alpha} \alpha_2, \alpha_2 \rangle + 2 \langle \Delta_{\alpha} (f_a \chi), f_b \chi \rangle + 2 \langle \Delta_{\alpha} \beta, f_b \chi \rangle
\]
\[
= \langle \Delta \alpha_1, \alpha_1 \rangle + \langle \Delta (f_a \chi), f_b \chi \rangle - \int_M f_b^2 |\varphi_0|^2 + 2 \langle \Delta (f_b \chi), f_a \chi \rangle + 2 \langle \Delta (f_b \chi), \beta \rangle.
\]
(8.1)
In the last equality, we use the formula in Lemma 8.1. In order to express each of the above inner product, we will compute $\Delta (f_b \chi)$, for any basic function $f_b$. To simplify the notation, we will omit the subscript "b" in $f_b$ in the following computations. First we have (keep in mind that $f = f_b$ is basic)

$$\delta (f \chi) = -df \chi + f \delta \chi = 0$$

since $\chi$ is divergence free. Therefore, $d(\delta(f \chi)) = 0$. Next, using Rummler’s formula $d\chi = -\kappa \wedge \chi + \varphi_0$, we write

$$d(f \chi) = f \delta \chi + (f_\alpha \wedge \varphi_0) + \delta (f_\alpha \wedge \chi),$$

$$\delta d(f \chi) = \delta(f \delta \chi + f \delta \varphi_0 + df \wedge \chi),$$

$$= \delta\left( -f \kappa \wedge \chi + f \varphi_0 + df \wedge \chi \right),$$

$$= df \left( (\kappa \wedge \chi - \varphi_0) - f \delta (\kappa \wedge \chi - \varphi_0) + \delta (df \wedge \chi) \right).$$

(8.2)

To express the divergence terms in the above equality, we consider an orthonormal frame $\{b_i\}$ of $TM$ and we compute for any basic 1-form $\theta$,

$$\delta (\theta \wedge \chi) = -\sum_i b_i \wedge \nabla_{b_i} (\theta \wedge \chi)$$

$$= -\sum_i b_i \wedge (\nabla_{b_i} \theta \wedge \chi + \theta \wedge \nabla_{b_i} \chi)$$

$$= (\delta \theta) \chi + \nabla_x \theta - \nabla_{\theta x} \chi - \theta \wedge \delta \chi$$

$$= (\delta \theta) \chi + \nabla_x \theta - \nabla_{\theta x} \chi$$

$$= (\delta \theta) \chi + [\xi, \theta]^b$$

$$= (\delta \theta - (\kappa, \theta)) \chi.$$ 

Therefore, we deduce for either $\theta = \kappa$ or $\theta = df$ that

$$\delta (\kappa \wedge \chi) = -|\kappa|^2 \chi$$

and $\delta (df \wedge \chi) = (\Delta_b f - (df, \kappa)) \chi$

since $\kappa$ is basic harmonic. Then we substitute into (8.2) to get

$$\delta d(f \chi) = df \left( (\kappa \wedge \chi - \varphi_0) - f \delta (\kappa \wedge \chi) + f \delta \varphi_0 + \delta (df \wedge \chi) \right)$$

$$= df \left( f \delta \varphi_0 + f |\kappa|^2 \chi + f \delta \varphi_0 + (\Delta_b f - (df, \kappa)) \chi \right)$$

$$= -df \delta \varphi_0 + f |\kappa|^2 \chi + f \delta \varphi_0 + (\Delta_b f - (df, \kappa)) \chi$$

$$= -df \delta \varphi_0 + f |\kappa|^2 \chi + f (\delta_b - \varepsilon) \varphi_0 + (\Delta_b f) \chi$$

$$\delta P_b = P_\theta - \varepsilon P_b$$ from (8.3). As $\varepsilon \varphi_0 = -\varphi_0 \wedge (\chi \wedge \varphi_0) = -|\varphi_0|^2 \chi$, we arrive at (replace $f$ by $f_b$)

$$\Delta (f_b \chi) = -df_b \wedge \varphi_0 + f_b |\kappa|^2 \chi + f_b \delta_b \varphi_0 + f_b |\varphi_0|^2 \chi + (\Delta_b f_b) \chi.$$

In particular, one can easily get that

$$\langle \Delta (f_b \chi), f_\alpha \chi \rangle = 0$$

and $\langle \Delta (f_b \chi), \beta \rangle = 0$, (8.3)

since $\beta$ is antibasic and orthogonal to $\xi$. Also, we have that

$$\langle \Delta (f_b \chi), f_b \chi \rangle = \int f_b^2 |\kappa|^2 + f_b^2 |\varphi_0|^2 + |df_b|^2.$$

(8.4)

Now substituting Equations (8.3) and (8.4) into Equation (8.1), we find that

$$\langle \Delta_a \alpha, \alpha \rangle = \langle \Delta_{a_1} \alpha_1, \alpha_1 \rangle + \int f_a^2 |\kappa|^2 + |df_a|^2,$$

which is non-negative. Then $\langle \Delta_a \alpha, \alpha \rangle = 0$ if and only if $a_1$ is harmonic (i.e. $\alpha_1 \in H^1(M)$), $f_b \rightarrow cte$ and $f_b \kappa = 0$. Recall that $\alpha = \alpha_1 + \alpha_2$ with $\alpha_2 = f_b \chi$. In the case where $H^1(M) = \{0\}$ and $\alpha$ is $\Delta_{a}$-harmonic 1-form, then $\alpha_1 = 0$ and $\alpha = f_b \chi = cte \chi$. Hence, $dim H^1(M, F) = 1$. This proves the first part of the theorem. To prove the second part, we use the exact Gysin sequence for non-taut Riemannian flows established in 17

$$0 \rightarrow H^1_b(M) \rightarrow H^1(M) \rightarrow H^0_{\kappa, b}(M) \rightarrow \ldots$$

where $H^0_{\kappa, b}(M) \cong H^0_b(M)$, which is zero because the foliation is not taut. Thus, $H^1_b(M) \cong H^1(M)$. By the proof of Proposition 7.3, we get that $P_a (H^1_a(M)) = 0$. That means for every harmonic one-form $\omega$, we
have $P_αω = 0$, and thus is basic. Hence $(Δ_αω, α) = 0$ implies that $α_1$ is basic-harmonic and $f_b = 0$. Thus both $α_1$ and $α_2$ are zero and then $H^j_b (M, F) = \{0\}$.

**Remark 8.5.** It might seem at first glance that Proposition 8.4 may contradict Proposition 7.1. But in fact, if $H^j (M) = \{0\}$ for some compact manifold $M$, then any Riemannian flow of $M$ must have a normal bundle that is not involutive. The reason is as follows. First, the mean curvature can be chosen to be zero after a change in bundle-like metric, since the mean curvature must be exact. If the normal bundle is involutive, then $dχ = 0$ from Rummler’s formula, and $δχ = 0$ (true for any Riemannian flow), so that $χ$ is a harmonic one-form and therefore represents a nontrivial class in $H^1 (M)$, a contradiction. So Proposition 7.1 does not apply.

9. Examples

We illustrate the antibasic cohomology and our theorems in some low-dimensional examples of foliations.

To simplify the exposition, we denote the Betti numbers for each example foliation $(M, F)$ as follows:

$$h^3 = \dim H^3 (M) , \ h^j_b = \dim H^j_b (M, F) , \ h^j_a = \dim H^j_a (M, F).$$

We start with the Hopf fibration, which is a taut Riemannian flow.

**Example 9.1.** Using Proposition 8.4 above, we consider the Hopf fibration of $S^3 \subset \mathbb{C}^2 \to \mathbb{C}P^1$ via $(z_0, z_1) \to [z_0, z_1]$. The leaves of the foliation $F$ are the orbits of the $S^1$ action $e^{it} \mapsto (e^{it}z_0, e^{-it}z_1)$. This is a Riemannian flow, but the normal bundle is not involutive. The lengths of the circular leaves are constant, so the mean curvature is zero. By Theorem 5.3, $h^3_3 = 1$, and from Proposition 7.3, $h^2_a \equiv 0$. Also $H^2_a (S^3, F) \subset H^2 (S^3)$ because of Lemma 2.4, so that $h^2_a = 0$, and $h^3_a = h^3 = 1$. In summary, we have

$$\begin{align*}
(h^0, h^1, h^2, h^3) &= (1, 0, 0, 1) \\
(h^0_b, h^1_b, h^2_b) &= (1, 0, 1) \\
(h^0_a, h^1_a, h^2_a, h^3_a) &= (0, 1, 0, 1).
\end{align*}$$

The following example is a Riemannian flow of a 3-manifold that is not taut.

**Example 9.2.** We will compute the antibasic cohomology groups of the Carrière example from [3] in the 3-dimensional case. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. We denote respectively by $V_1$ and $V_2$ the eigenvectors associated with the eigenvalues $\lambda$ and $\lambda^2$ of $A$ with $\lambda > 1$ irrational. Let the hyperbolic torus $T^3$ be the quotient of $T^2 \times \mathbb{R}$ by the equivalence relation which identifies $(m, t)$ to $(A(m), t + 1)$. The flow generated by the vector field $\mathbf{V}_2$ is a transversally Lie foliation of the affine group. The Betti numbers of this closed manifold are $h^0 = 1 = h^3$, $h^1 = h^2 = 3$. We choose the bundle-like metric (letting $(x, s, t)$ denote the local coordinates in the $V_2$ direction, $V_1$ direction, and $\mathbb{R}$ direction, respectively) as $g = \lambda^{-2t}dx^2 + \lambda^{2t}ds^2 + dt^2$.

The mean curvature of the flow is $κ = κ_0 = \log (λ) dt$, since $χ_F = λ^{-t}dx$ is the characteristic form and $dχ_F = -\log (λ) λ^{-t}dt ∧ dx = -κ ∧ χ_F$. It is easily seen that the basic cohomology satisfies $h^j_b = 1$ for $j = 0, 1$ and $h^k_a = 0$ (class of the mean curvature class being nonzero implies this; see [1]). The foliation has an involutive normal bundle, so that Proposition 7.1 applies, so that $h^2_a = 3$, $h^3_a = 1$ and $h^k_a = 0$ for $k = 0, 1$. In summary,

$$\begin{align*}
(h^0, h^1, h^2, h^3) &= (1, 3, 3, 1) \\
(h^0_b, h^1_b, h^2_b) &= (1, 1, 0) \\
(h^0_a, h^1_a, h^2_a, h^3_a) &= (0, 0, 3, 1).
\end{align*}$$

We now consider an example of a foliation that is not Riemannian (for any metric). This is a standard example of a foliation on a connected, compact manifold with infinite-dimensional basic cohomology (similar to [3]).

**Example 9.3.** Let $M$ be the closed 3-manifold defined as $\mathbb{R} \times T^2 / \mathbb{Z}$, where $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ and $m \in \mathbb{Z}$ acts on $\mathbb{R} \times T^2$ by $m(t, x) = (t + m, A^m x)$, where $A$ is the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We define the leaves of the foliation...
to be the $t$-parameter curves. Then observe that that leaf closures intersect each torus with a set of the form $S \times \{x_2\}$, where $x_2 \in \mathbb{R}/\mathbb{Z}$ and $S$ is a finite number of points for rational $x_2$ and is $\mathbb{R}/\mathbb{Z}$ for irrational $x_2$. Thus, the basic forms in the “coordinates” $(t, x_1, x_2)$ have the form

$$\Omega^0_b (M) = \{ \text{functions of } x_2 \}$$

$$\Omega^1_b (M) = \{ (\text{functions of } x_2) \, dx_2 \}$$

$$\Omega^2_b (M) = \{ (\text{functions of } x_2) \, dx_1 \wedge dx_2 \}.$$

From this we can easily calculate that $h^0_b = 1$, $h^1_b = 1$, and $H^2_b (M, \mathcal{F}) \cong \Omega^2_b (M)$, which is infinite dimensional. One may also check with a cell complex that the ordinary homology satisfies $H^j (M, \mathbb{Z}) \cong \mathbb{Z}$ for $j = 0, 3$ and $H^j (M, \mathbb{Z}) \cong \mathbb{Z}^2$ for $j = 1, 2$, so that the ordinary de Rham cohomology satisfies $h^j = 1$ for $j = 0, 3$ and $h^j = 2$ for $j = 1, 2$. We choose the metric so that $\partial_t$ is a unit vector and perpendicular to each torus and such that on the metric on $T^2$ at each $t$ is

$$(g_{ij}) = \begin{pmatrix} 1 & -t \\ -t & 1 + t^2 \end{pmatrix}.$$ 

One can check the invariance with respect to the action of $m \in \mathbb{Z}$; it is chosen so that $\{ \partial_{x_1}, t \partial_{x_1} + \partial_{x_2} \}$ forms an orthonormal basis at each $t$. Then

$$(g^{ij}) = \begin{pmatrix} 1 + t^2 & t \\ t & 1 \end{pmatrix},$$

so the dual orthonormal basis is $\{ dx_1 - t dx_2, dx_2 \}$. We now compute the antibasic forms with respect to this metric. For any $t, x_1$, let $C_{t,x_1}$ denote the circle $\{(t, x_1, x_2) : x_2 \in \mathbb{R}/\mathbb{Z}\}$

$$\Omega^0_a (M) = \left\{ \text{functions } f \text{ of } t, x_1, x_2 : \int_{C_{t,x_1}} f (t, x_1, x_2) \, dx_2 = 0 \text{ for all } t, x_1 \right\}$$

$$\Omega^1_a (M) = \{ \text{antibasic function} \, dx_2 + (\text{any function}) \, (dx_1 - t dx_2) + (\text{any function}) \, dt \}$$

$$\Omega^2_a (M) = \{ \text{antibasic function} \, dx_1 \wedge dx_2 + (\text{any function}) dt \wedge (dx_1 - t dx_2) + (\text{any function}) dt \wedge dx_2 \}$$

$$\Omega^3_a (M) = \Omega^3 (M).$$

Immediately we have $h^3_a = 1$. We compute the divergence on one-forms with respect to the nice basis:

$$\langle df, adt + c_1 (dx_1 - t dx_2) + c_2 dx_2 \rangle = \int f \, a(t, x_1, x_2) + (t f(x_1) + f(x_2)) c_2$$

$$= \int f \left( -a_1 - (c_1)_{x_1} + t (c_2)_{x_1} - (c_2)_{x_2} \right)$$

$$\delta (adt + c_1 (dx_1 - t dx_2) + c_2 dx_2) = -a_1 - \partial_{x_1} (c_1) - (t \partial_{x_1} + \partial_{x_2}) (c_2).$$

**Divergence of 2-forms:** writing in terms of the nice basis. Note that in terms of an orthonormal frame, $\{e_i\}$ with Christoffel symbols defined by $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$ or $d(e^k) = -\Gamma^k_{ij} e_i \wedge e_j$ we have $\delta (b_{ij} e^i \wedge e^j) = (e^*_i (b_{ij}) - \Gamma^j_{ik} b_{ik}) e^j$. In our case with $e^1 = dt$, $e^2 = dx_1 - t dx_2$, $e^3 = dx_2$, we have $d(e^2) = -dt \wedge dx_2 = -e^1 \wedge e^3$, so only $\Gamma^2_{13} - \Gamma^2_{31} = 1$, all other combinations zero. Also $e^*_1 = -e_1 = -\partial_{x_1}$, $e^*_2 = -\partial_{x_1}$, $e^*_3 = -e_3 = -t \partial_{x_1} - \partial_{x_2}$. So we have, assuming $b_{ij}$ is antisymmetrized, $\delta (b_{ij} e^i \wedge e^j) = -e_1 (b_{ij}) e^j - (b_{13} - b_{31}) e^2$. Hence in our situation (antisymmetrizing $b'_{ij} = \frac{1}{2} b_{ij}$ if $i < j$ and $b'_{ij} = -\frac{1}{2} b_{ij}$ if $j < i$).

$$\delta (b_{12} dt \wedge (dx_1 - t dx_2) + b_{13} dt \wedge dx_2 + b_{23} dx_1 \wedge dx_2)$$

$$= \left( -(b_{12})_{x_1} + (b_{23})_{x_2} - (b_{13}) \right) (dx_1 - t dx_2) + \left( -(b_{13})_{x_1} - (b_{23}) \right) (dx_2)$$

$$+ ((b_{12})_{x_1} + (b_{13})_{x_1} + (b_{13})_{x_2}) (dt).$$
Finally we calculate divergence of 3-forms:

\[
\langle d (b_{12} dt \wedge (dx_1 - tdx_2) + b_{13} dt \wedge dx_2 + b_{23} dx_1 \wedge dx_2), f dt \wedge dx_1 \wedge dx_2 \rangle \\
= \langle d (b_{12} dt \wedge dx_1 - b_{13} dt \wedge dx_2 + b_{13} dt \wedge dx_2 + b_{23} dx_1 \wedge dx_2), f dt \wedge dx_1 \wedge dx_2 \rangle \\
= \langle ((b_{12})_{x_2} + t (b_{12})_{x_1} - (b_{13})_{x_1} + (b_{23})_{t}), f \rangle \\
= \int b_{12} (-f_{x_2} - tf_{x_1}) + b_{13} (f_{x_1}) + b_{23} (-f_t),
\]

so that

\[
\delta (f dt \wedge dx_1 \wedge dx_2) = (-f_{x_2} - tf_{x_1}) dt \wedge (dx_1 - tdx_2) + f_{x_1} dt \wedge dx_2 - f_t dx_1 \wedge dx_2.
\]

From these formulas, we note that the divergence of a basic 1-form (one of the type $c_2(x_2) dx_2$) is always a basic function $(-\partial_2 c_2)$, so that $\delta$ maps basic one-forms to basic functions and antibasic one-forms to antibasic functions. Then $h^0_a = 1$, since in this case

\[
H^0(M) = \frac{\Omega^0}{\text{im} \delta|_{\Omega^0}} = \frac{\Omega^0_a \oplus \Omega^0_b}{\left(\text{im} \delta|_{\Omega^0_a}\right) \oplus \left(\text{im} \delta|_{\Omega^0_b}\right)} = \frac{\Omega^0_a}{\left(\text{im} \delta|_{\Omega^0_a}\right)} \oplus \frac{\Omega^0_b}{\left(\text{im} \delta|_{\Omega^0_b}\right)} = H^0_a(M,F) \oplus H^0_b(M,F) = H^0_a(M,F) \oplus H^0(M).
\]

(Note that the first and second step fail for foliations in general).

For future use, we compute the Hodge star operator. Observe that

\[
\ast 1 = dt \wedge dx_1 \wedge dx_2 \\
\ast (adt + c_1(dx_1 - tdx_2) + c_2 dx_2) = adx_1 \wedge dx_2 - c_1 dt \wedge dx_2 + c_2 dt \wedge (dx_1 - tdx_2) \\
\ast (b_{12} dt \wedge (dx_1 - tdx_2) + b_{13} dt \wedge dx_2 + b_{23} dx_1 \wedge dx_2) = b_{12} dx_2 - b_{13} (dx_1 - tdx_2) + b_{23} dt \\
\ast dt \wedge dx_1 \wedge dx_2 = 1.
\]

We check that since $\delta = (-1)^{n+k+1} \ast d \ast = (-1)^k \ast d \ast$ on k-forms,

\[
\delta (adt + c_1(dx_1 - tdx_2) + c_2 dx_2) = -\ast d \ast (adt + c_1(dx_1 - tdx_2) + c_2 dx_2) \\
= -\ast d((a_1) dx_1 - dx_2 - c_1 dt \wedge dx_2 + c_2 dt \wedge (dx_1 - tdx_2)) \\
= -\ast ((a_1) dV + (c_1)_{x_1} dV + (t (c_2)_{x_1} + (c_2)_{x_2}) dV) \\
= -a_t - (c_1)_{x_1} - t(c_2)_{x_1} - (c_2)_{x_2}.
\]
\[ \delta(b_{12} dt \wedge (dx_1 - tdx_2) + b_{13} dt \wedge dx_2 + b_{23} dx_1 \wedge dx_2) \]
\[ = *d * (b_{12} dt \wedge (dx_1 - tdx_2) + b_{13} dt \wedge dx_2 + b_{23} dx_1 \wedge dx_2) \]
\[ = *d (b_{12} dx_2 - b_{13} (dx_1 - tdx_2) + b_{23} dt) \]
\[ = * (b_{12}) dt \wedge dx_2 + * (b_{12}) x_1 (dx_1 - tdx_2) \wedge dx_2 \]
\[ - * (b_{13}) dt \wedge (dx_1 - tdx_2) + * b_{13} dt \wedge dx_2 - * (t (b_{13}) x_1 + (b_{13})^3) dx_2 \wedge (dx_1 - tdx_2) \]
\[ + * (b_{23}) x_1 (dx_1 - tdx_2) \wedge dt + * (t (b_{23}) x_1 + (b_{23})^3) dx_2 \wedge dt \]
\[ = - (b_{12}) (dx_1 - tdx_2) + (b_{12}) x_1 dt - b_{13} (dx_1 - tdx_2) - (b_{13}) x_2 dx_2 + (t (b_{13}) x_1 + (b_{13})^3) dt \]
\[ - (b_{23}) x_1 dx_2 + (t (b_{23}) x_1 + (b_{23})^3) (dx_1 - tdx_2). \]

Finally,
\[ \delta(f dt \wedge dx_1 \wedge dx_2) = - *d * f dt \wedge (dx_1 - tdx_2) \wedge dx_2 \]
\[ = *df \]
\[ = - f (f dt + f x_1 (dx_1 - tdx_2) + (tf x_1 + f x_2) dx_2) \]
\[ = - f (dx_1 - tdx_2) \wedge dx_2 + f x_1 dt \wedge dx_2 - (tf x_1 + f x_2) dt \wedge (dx_1 - tdx_2). \]

We now compute \( H^2_a(M, \mathcal{F}) \). We have
\[ H^2_a(M, \mathcal{F}) = \frac{\ker \delta|_{\Omega^2}}{\text{im} \delta|_{\Omega^3}} \subset \frac{\ker \delta|_{\Omega^2}}{\text{im} \delta|_{\Omega^2}} = H^2(M) \cong \mathbb{R}^2. \]

In the ordinary \( \delta \)-cohomology, the generators of \( H^2(M) \) are \([dt \wedge (dx_1 - tdx_2)]\) and \([dx_1 \wedge dx_2]\). But \( b_{12} dt \wedge (dx_1 - tdx_2) \) is basic and \( b_{23} dx_1 \wedge dx_2 \) is basic for any choice of constants \( b_{12}, b_{23} \). Thus, \( h^2_a = 1 \).

We now consider 1-forms. Observe that basic 2 forms have the form \( b_{12} (x_2) dx_1 \wedge dx_2 \) and from the formulas above \( \delta(b_{12} (x_2) dx_1 \wedge dx_2) = 0 \) for all functions \( b_{12} \). Thus, the image of \( \delta_a \) and the image of \( \delta \) coincide for 2-forms. Thus, we have
\[ H^1_a(M, \mathcal{F}) = \frac{\ker \delta|_{\Omega^1}}{\text{im} \delta|_{\Omega^2}} \subset \frac{\ker \delta|_{\Omega^1}}{\text{im} \delta|_{\Omega^2}} = H^1(M) \cong \mathbb{R}^2. \]

The generators of \( H^1(M) \) are \([dx_2]\) and \([dt]\). But note that \( c_2 dx_2 \) is basic and \( adt \) is basic for any constants \( c_2 \) and \( a \). Thus \( h^1_a = 1 \). This contrasts to \( h_a^2 = \infty \) and \( h^2 = 2 \). In summary,
\[ (h^0, h^1, h^2, h^3) = (1, 2, 2, 1) \]
\[ (h^0_a, h^1_a, h^2_a) = (1, 1, \infty) \]
\[ (h^0_a, h^1_a, h^2_a, h^3_a) = (0, 1, 1, 1). \]

The following non-Riemannian flow is a simple example where the basic projection \( P_b \) and antibasic projection \( P_a \) do not preserve smoothness. In spite of that, the basic and antibasic cohomology can be calculated. From the calculations, we also see that the Hodge theorem is false for this foliation.

**Example 9.4.** Let \( M \) be the flat torus \((0, 2) \times [0, 1)\) with opposite sides of the boundary identified. Let \( \phi(x) \) be a smooth function on the circle \([0, 2] \mod 2\) such that \( \phi \) is positive on \((0, 1)\) and identically zero on \([1, 2]\). Consider the foliation that whose tangent space at each point is spanned by the vector field \( V(x, y) = (\phi(x), \sqrt{1 - \phi(x)^2}) \). All the leaves in the region \( 0 < x < 1 \) are noncompact and have \( x = 0 \) and \( x = 1 \) in their closure, and the leaves in the region \( 1 \leq x \leq 2 \) are vertical circles. The set of basic functions is
\[ \Omega^0_b(M) = \left\{ f : [0, 2] \times [0, 1] \to \mathbb{R} : f(x, y) = g(x) \text{ for a smooth function } g \text{ on } \mathbb{R} / 2\mathbb{Z} \text{ such that } g(x) = \text{constant} \text{ for } x \in [0, 1] \mod 2 \right\}. \]

Since every basic normal vector field approaches 0 as \( x \to 0^+ \) and \( x \to 1^- \), there are no bounded basic one-forms for \( 0 < x < 1 \), so we have
\[ \Omega^1_b(M) = \left\{ \omega = h(x) dx : h \text{ is a smooth function on } \mathbb{R} / 2\mathbb{Z} \text{ such that } h(x) = 0 \text{ for } x \in [0, 1] \mod 2 \right\}. \]
Then, the set of antibasic forms are those smooth forms orthogonal to $\Omega^a_0(M)$ in $L^2$. We obtain

$$\Omega^a_0(M) = \left\{ \text{smooth } f : M \to \mathbb{R} : \int_0^1 f(x, y) \, dy = 0 \text{ for } x \in [1, 2] \mod 2 \text{ and } \int_0^1 \int_0^1 f(x, y) \, dx \, dy = 0 \right\}.$$  

$$\Omega^a_1(M) = \left\{ \alpha = a_1 \, dx + a_2 \, dy : \int_0^1 a_1(x, y) \, dy = 0 \text{ for } x \in [1, 2] \mod 2 \right\}.$$  

Note that in this example, the basic and antibasic projections are not smooth maps to differential forms. Observe that on functions,

$$P_a(f)(x, y) = \left\{ \begin{array}{ll} \text{average of } f \text{ over } [0, 1]^2 & x \in (0, 1) \\ \text{average of } f(x, \cdot) \text{ over } [0, 1] & x \in (1, 2) \end{array} \right.$$  

so for instance

$$P_a(\sin(\pi x)) = \left\{ \begin{array}{ll} \frac{2}{\pi} x \sin(\pi x) & x \in (0, 1) \\ x \sin(\pi x) & x \in (1, 2) \end{array} \right.$$  

which is not a continuous function. Likewise, $P_a(\sin(\pi x)) = \sin(\pi x) - P_b(\sin(\pi x))$ is not smooth. In any case, we may calculate the basic and antibasic cohomology groups.

$$H^a_k(M, \mathcal{F}) = \ker (d : \Omega^a_k(M) \to \Omega^a_{k+1}(M)) \cong \mathbb{R},$$

$$H^b_k(M, \mathcal{F}) = \frac{\Omega^b_k(M)}{\text{im} (d : \Omega^b_k(M) \to \Omega^b_{k+1}(M))} = \frac{\Omega^b_k(M)}{\Omega^a_0(M) \cup \left\{ f_1^2 h(x) \, dx = 0 \text{ and } h(x) = 0 \text{ for } x \in [0, 1] \right\}}.$$  

Note that $H^b_1$ is not represented by a basic harmonic form. Also,

$$H^a_0(M, \mathcal{F}) = \frac{\Omega^a_0(M)}{\text{im} (\delta : \Omega^a_0(M) \to \Omega^a_1(M))} = \frac{\Omega^a_0(M)}{\left\{(a_1)_x + (a_2)_y : \int_0^1 a_1(x, y) \, dy = 0 \text{ for } x \in [1, 2] \mod 2 \right\}} = \{0\}.$$  

$$H^a_1(M, \mathcal{F}) = \frac{\ker (\delta : \Omega^a_1(M) \to \Omega^a_0(M))}{\text{im} (\delta : \Omega^b(M) \to \Omega^a_1(M))} = \frac{\left\{ a_1 \, dx + a_2 \, dy : \int_0^1 a_1(x, y) \, dy = 0 \text{ for } x \in [1, 2] \mod 2 \text{ and } (a_1)_x + (a_2)_y = 0 \right\}}{\{f_g \, dx - f_x \, dy\}} = \{[(c \cdot \text{bump on } [0, 1])] \cong \mathbb{R}.\}$$  

$$H^a_2(M, \mathcal{F}) = H^2(M) \cong \mathbb{R}.$$  

And we also have $h^0 = h^2 = 1$, $h^1 = 2$. In summary,

$$(h^0, h^1, h^2) = (1, 2, 1)$$

$$(h^0_a, h^1_a, h^2_a) = (1, 1)$$

$$(h^0_b, h^1_b, h^2_b) = (0, 1, 1).$$  

The next example is a non-Riemannian foliation of the torus $T^3$. This example, along with Example 9.2, demonstrates that the antibasic cohomology truely gives additional topological information. All of the ordinary Betti numbers and basic Betti numbers are the same for these two examples, but the antibasic Betti numbers distinguish them.

**Example 9.5.** We consider the following non-Riemannian flow on the flat 3-torus $T^3 = \mathbb{R}^3 / \mathbb{Z}^3$. Let $\psi(x)$ be a smooth function on the circle $\mathbb{R}/\mathbb{Z}$ such that $\phi$ is positive on $(0, 1)$ and zero to infinite order at $0$ (and 1). Consider the foliation $\mathcal{F}$ that whose tangent space at each point is spanned by the vector field $W(x, y, z) = \left( \phi(x), \sqrt{1 - \phi(x)^2}, 0 \right)$. This foliation is a circle cross the subset of the foliation in Example
for \( x \in (0, 1] \). Since all basic functions are constant in \( x \) and \( y \) and all basic one-forms must have no \( dx \) or \( dy \) components, the basic forms are as follows:

\[
\Omega^0_0 (T^3, \mathcal{F}) = \{ h(z) \} \cong \Omega^0 (S^1), \\
\Omega^1_0 (T^3, \mathcal{F}) = \{ f(z) dz \} \cong \Omega^1 (S^1), \\
\Omega^j_0 (T^3, \mathcal{F}) = \{ 0 \}, \quad j \geq 2.
\]

Then the antisymmetric forms are as follows:

\[
\Omega^0_0 (T^3, \mathcal{F}) = \{ \alpha \in \Omega^0 : \int_0^1 \alpha(x, y, z) \, dz = 0 \}, \\
\Omega^1_0 (T^3, \mathcal{F}) = \{ \beta_1 dx + \beta_2 dy + \beta_3 dz : \int_{[0,1]^2} \beta_3 (x, y, z) \, dx \, dy = 0 \}, \\
\Omega^j_0 (T^3, \mathcal{F}) = \Omega^j (T^3), \quad j \geq 2.
\]

Then we calculate that

\[
(h^0_0, h^1_0, h^2_0, h^3_0) = (1, 3, 3, 1), \\
(h^0_1, h^1_1, h^2_1) = (1, 1, 0), \\
(h^0_2, h^1_2, h^2_2) = (0, 2, 3, 1).
\]

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