# Integral representation for functionals defined on SBDp in dimension two

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We prove an integral representation result for functionals with growth conditions which give coercivity on the space $SBD^p(\Omega)$, for $\Omega \subset \mathbb{R}^2$ a bounded open Lipschitz set, $p \in (1, \infty)$. The space $SBD^p$ of functions whose distributional strain is the sum of an $L^p$ part and a bounded measure supported on a set of finite $\mathcal{H}^1$-dimensional measure appears naturally in the study of fracture and damage models. Our result is based on the construction of a local approximation by $W^{1,p}$ functions. We also obtain a generalization of Korn's inequality in the $SBD^p$ setting.

## 1 Introduction

The direct methods of $\Gamma$-convergence are of paramount importance in studying variational limits and relaxation problems since their introduction in the seminal paper by Dal Maso and Modica [30]. They focus on the study of abstract limiting functionals $F(u, A)$, obtained for instance using $\overline{\Gamma}$-convergence arguments; one key ingredient is the proof of an integral representation for $F(u, A)$. Here $u : \Omega \to \mathbb{R}^N$ is an element of a suitable function space $\mathscr{X}(\Omega)$, and $A$ runs in the class $\mathcal{A}(\Omega)$ of open subsets of a given open set $\Omega \subset \mathbb{R}^n$. The notion of *variational functional* is at the heart of the matter: $F$, regarded as depending on the couple $(u, A) \in \mathscr{X}(\Omega) \times \mathcal{A}(\Omega)$, has to satisfy suitable lower semicontinuity, locality and measure theoretic properties (for more details see properties (i)-(iii) in Theorem 1.1). The specific growth conditions of the functional determine the natural functional space in which the function $u$ lies. Under these assumptions $F(u, A)$ can be written as an integral over the domain of integration $A$ with respect to a suitable measure. The integrands may depend on $x$, $u(x)$ and $\nabla u(x)$, and possibly on other local quantities of $u$, such as higher order or distributional derivatives. Furthermore, as first shown in some cases in [31] and then generalized in [12], the corresponding energy densities can be characterized in terms of cell formulas, i.e. asymptotic Dirichlet problems on small cubes or balls involving $F$ itself, with boundary data depending on the local properties of $u$.

Integral representation results have been obtained in several contexts with increasing generality: starting with the pioneering contribution by De Giorgi for limits of area-type integrals [32], it has been extended to functionals defined first on Sobolev spaces in [44, 17, 14, 16, 15] and on the space of functions with Bounded Variation in [27, 10], and then to energies defined on partitions in [2] and on the subspace $SBV$ in [13] (we refer to [15, 28, 12, 11] for a more exhaustive list of references). The global method for relaxation introduced and developed in [12, 11] provides a general approach that unifies and extends the quoted results.

In this paper we address the integral representation of functionals defined on the subspace $SBD^p(\Omega)$ of the space $BD(\Omega)$ in the two dimensional setting. The space of functions of bounded deformation $BD(\Omega)$ is characterized by the fact that the symmetric part of the distributional gradient $Eu := (Du + Du^T)/2$ of $u \in L^1(\Omega, \mathbb{R}^n)$ is a bounded Radon measure, namely

$$BD(\Omega) := \{u \in L^1(\Omega; \mathbb{R}^n) : Eu \in \mathcal{M}(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})\},$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, see [45, 46, 4]. $BD$ and its subspaces $SBD$ and $SBD^p$ constitute the natural setting for the study of plasticity, damage and fracture models in a geometrically linear framework [45, 46, 48, 6, 38, 24]. In particular, $SBD^p$, $p \in [1, \infty)$, is the set of $BD$ functions such that the strain $Eu$ can be written as the sum of an absolutely continuous measure with respect to $\mathcal{L}^n \llcorner \Omega$, with density $e(u)$ in $L^p(\Omega, \mathbb{R}^{n\times n})$, and a singular measure concentrated on the set $J_u$ of jump points of $u$, that is $(n-1)$-rectifiable and with finite $\mathcal{H}^{n-1}$-measure, see [9, 18, 19, 21]. We recall that each function $u \in BD(\Omega)$ has an approximate gradient $\nabla u(x)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$, and that the density $e(u)$ is exactly its symmetrized part (see [4, Theorem 7.4]).

For functionals with linear growth defined on $SBD$ an integral representation result was obtained by Ebobisse and Toader [34]. These functionals, however, lack coercivity on the relevant space. Integral representation for functionals defined on $BD$ was studied in [8], lower semicontinuity and relaxation in [43]. The situation of functionals defined on $SBD^p$ and with corresponding growth properties is open. We give here a solution in two dimensions.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded open Lipschitz set, $p \in (1, \infty)$, $F : SBD^p(\Omega) \times \mathcal{B}(\Omega) \to [0, \infty)$ be such that*

*(i) $F(u, \cdot)$ is a Borel measure for any $u \in SBD^p(\Omega)$;*

*(ii)* $F(\cdot, A)$ *is lower semicontinuous with respect to the strong* $L^1(\Omega, \mathbb{R}^2)$*-convergence for any open set* $A \subset \Omega$*;*

*(iii)* $F(\cdot, A)$ *is local for any open set* $A \subset \Omega$*, in the sense that if* $u, v \in SBD^p(\Omega)$ *obey* $u = v$ $\mathcal{L}^2$*-a.e. in* $A$*, then* $F(u, A) = F(v, A)$*;*

*(iv)* *There are* $\alpha, \beta > 0$ *such that for any* $u \in SBD^p(\Omega)$*, any* $B \in \mathcal{B}(\Omega)$*,*

$$\begin{aligned}\alpha\Big(\int_B |e(u)|^p dx + \int_{J_u\cap B} (1 + |[u]|) d\mathcal{H}^1\Big) \le F(u, B)\\ \le \beta\Big(\int_B (|e(u)|^p + 1) dx + \int_{J_u\cap B} (1 + |[u]|) d\mathcal{H}^1\Big).\end{aligned} \tag{1.1}$$

*Then there are two Borel functions* $f : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \to [0, \infty)$ *and* $g : \Omega \times \mathbb{R}^2 \times \mathbb{R}^2 \times S^1 \to [0, \infty)$ *such that*

$$F(u, B) = \int_B f(x, u(x), \nabla u(x)) dx + \int_{B\cap J_u} g(x, u^-(x), u^+(x), \nu_u(x)) d\mathcal{H}^1\,. \tag{1.2}$$

Above and throughout the paper we will refer to the book [5] and to the papers [4, 9] for the notation and results about $BV$ and $BD$ spaces, respectively. In particular, $\mathcal{B}(\Omega)$ is the family of Borel subsets of $\Omega$.

The proof of Theorem 1.1, which is given in Section 4, follows the general strategy introduced in [12, 11]. Their approach was based on a Poincaré-type inequality in $SBV$ by De Giorgi, Carriero and Leaci, which is not known in $SBD^p$ (see [5]). Our main new ingredient is the construction of an approximation by $W^{1,p}$ functions, discussed in Section 3, which permits to bypass the De Giorgi-Carriero-Leaci inequality. The approximation is done so that the function is only modified outside a countable set of balls with small area and perimeter. In each ball, we give a construction of a $W^{1,p}$ extension for the $SBD^p$ function by constructing a finite-element approximation on a countable mesh, which is chosen depending on the function $u$, see Section 2.

Our $W^{1,p}$ approximation result also leads naturally to the proof of the following variant of Korn's inequality for $SBD^p$ functions.

**Theorem 1.2.** *Let* $\Omega \subset \mathbb{R}^2$ *be a connected, bounded, open Lipschitz set and let* $p \in (1, \infty)$*. Then there exists a constant* $c$*, depending on* $p$ *and* $\Omega$*, with the following property: for every* $u \in SBD^p(\Omega)$ *there exist a set* $\omega \subset \Omega$ *of finite perimeter, with* $\mathcal{H}^1(\partial\omega) \le c\mathcal{H}^1(J_u)$*, and an affine function* $a(x) = Ax + b$*, with* $A \in \mathbb{R}^{2\times 2}$ *skew-symmetric and* $b \in \mathbb{R}^2$*, such that*

$$\begin{aligned}\|u - a\|_{L^p(\Omega\setminus\omega,\mathbb{R}^2)} \le c\|e(u)\|_{L^p(\Omega,\mathbb{R}^{2\times 2})},\\ \|\nabla u - A\|_{L^p(\Omega\setminus\omega,\mathbb{R}^{2\times 2})} \le c\|e(u)\|_{L^p(\Omega,\mathbb{R}^{2\times 2})}.\end{aligned}$$

This improves a result of [36] to the sharp exponent. Variants of the first inequality were first obtained in [20, 35]. Another consequence of the approximation discussed in Section 3 is the possibility to prove existence of minimizers for Griffith's fracture functional in two dimensions, which will be discussed elsewhere [23].

# 2 Approximation of $SBD^p$ functions with small jump set

In this Section we prove the following approximation result.

**Theorem 2.1.** *Let $n = 2$, $p \in [1, \infty)$. There exist $\eta > 0$ and $\tilde{c} > 0$ such that if $J \in \mathcal{B}(B_{2r})$, for some $r > 0$, satisfies*

$$\mathcal{H}^1(J) < 2r\eta, \tag{2.1}$$

*then there exists $R \in (r, 2r)$ for which the following holds: for every $u \in SBD^p(B_{2r})$ with $\mathcal{H}^1(J_u \cap B_{2r} \setminus J) = 0$ there exist $\phi(u) \in SBD^p(B_{2r}) \cap W^{1,p}(B_R, \mathbb{R}^2)$ such that*

*(i)* $\mathcal{H}^1(J_u \cap \partial B_R) = 0$;

*(ii)* $\displaystyle\int_{B_R} |e(\phi(u))|^q dx \leq \tilde{c} \int_{B_R} |e(u)|^q dx$, *for every* $q \in [1, p]$;

*(iii)* $\|u - \phi(u)\|_{L^1(B_R,\mathbb{R}^2)} \leq \tilde{c}R|Eu|(B_R)$;

*(iv)* $u = \phi(u)$ *on* $B_{2r} \setminus B_R$, $\mathcal{H}^1(J_{\phi(u)} \cap \partial B_R) = 0$;

*(v)* *if* $u \in L^\infty(B_{2r}, \mathbb{R}^2)$, *then* $\|\phi(u)\|_{L^\infty(B_{2r},\mathbb{R}^2)} \leq \|u\|_{L^\infty(B_{2r},\mathbb{R}^2)}$.

*Proof.* We follow an idea of [26, 25], which we first summarize. The basic strategy is to construct a triangular grid, see Figure 1, which refines towards the boundary of $B_R$, and to define $\phi(u)$ as the piecewise linear interpolation of the values of $u$ at the grid nodes. If the grid nodes are chosen appropriately, in the sense of having values of $u$ close to the local average, $\phi(u)$ turns out to be close to $u$. Further, the choice of the grid nodes can be done in such a way that all grid segments do not intersect the jump set of $u$, and that the average of $|e(u)|$ along the segment is not significantly larger than its average in a neighborhood of the segment. In turn, this implies that $e(\phi(u))$, which is constant in each triangle, can be estimated by the average of $|e(u)|$ in a neighborhood of the triangle itself, leading – after summing over all triangles

– to the desired $W^{1,p}$ estimate for $\phi(u)$. The critical point is the construction of the grid, which is obtained by iteratively choosing the vertices, after having globally selected the radius $R$ in such a way that the density of the jump set around $\partial B_R$ is controlled on any scale.

For some $\eta > 0$ chosen below, given $r > 0$ and a Borel set $J \in \mathcal{B}(B_{2r})$ with $\mathcal{H}^1(J) < 2r\eta$, we first claim, following [26, Lemma 4.3], that there exists $R \in (r, 2r)$ such that for $\delta_k := R\,2^{-k}$ we have

$$\mathcal{H}^1(J \cap \partial B_R) = 0, \tag{2.2}$$

$$\mathcal{H}^1(J \cap (B_R \setminus B_{R-\delta_k})) < 10\eta\delta_k, \quad \text{for every } k \in \mathbb{N}. \tag{2.3}$$

To prove this, we first observe that (2.2) holds for all but countably many $R$, therefore it suffices to show that the set of $R \in (r, 2r)$ for which (2.3) holds has positive measure. We consider the family of intervals

$$\{[R - \delta_k, R] : \mathcal{H}^1(J \cap (B_R \setminus B_{R-\delta_k})) \geq 10\eta\delta_k\}$$

and we define $I$ as the union of all intervals of the family, with $R \in (r, 2r)$, $k \in \mathbb{N}$. By Vitali's covering theorem, there exists a countable set $(R_i, k_i)_{i\in\mathbb{N}}$ such that the corresponding intervals $[R_i - \delta_{k_i}, R_i]$ are pairwise disjoint and five times their total measure is greater than or equal to the measure of $I$, $5\sum_{i\in\mathbb{N}} \delta_{k_i} \geq \mathcal{L}^1(I)$. Therefore by (2.1) we obtain

$$2r\eta > \mathcal{H}^1(J \cap B_{2r}) \geq \sum_{i\in\mathbb{N}} \mathcal{H}^1(J \cap (B_{R_i} \setminus B_{R_i - \delta_{k_i}})) \geq \sum_{i\in\mathbb{N}} 10\eta\delta_{k_i} \geq 2\eta\mathcal{L}^1(I).$$

Since the first inequality is strict, we conclude that $\mathcal{L}^1(I) < r$ and therefore there is $R \in (r, 2r) \setminus I$ such that (2.2) holds. Since $R \notin I$, by the definition of $I$ we obtain $\mathcal{H}^1(J \cap (B_R \setminus B_{R-\delta_k})) < 10\eta\delta_k\}$ for all $k \in \mathbb{N}$. Therefore (2.3) holds as well. The value of $R$ is fixed for the rest of the proof.

Let $u \in SBD^p(B_{2r})$ be such that $\mathcal{H}^1(B_{2r} \cap J_u \setminus J) = 0$. From (2.2) we deduce that (i) holds. We define $R_k := R - \delta_k$ and $\overline{x}_{k,j} := R_k(\cos \frac{2\pi j}{2^k}, \sin \frac{2\pi j}{2^k})$, $j = 1, \ldots, 2^k$. We say that $\overline{x}_{k,j}$ and $\overline{x}_{k',j'}$ are neighbors if either $k = k'$ and $j = j' \pm 1$, working modulo $2^k$, or (up to a permutation) $k = k' + 1$ and $j \in \{2j' - 1, 2j', 2j' + 1\}$, again modulo $2^k$. Connecting all neighbors we obtain a decomposition of $B_R$ into countably many triangles, whose angles are uniformly bounded away from 0 and $\pi$, see Figure 1.

We will construct $\phi(u)$ as a linear interpolation on a triangulation whose vertices are slight modifications of $\overline{x}_{k,j}$. Following the idea of [25, Proposition 2.2], we next show how to construct the modified triangulation. We start off considering two neighboring points $\overline{x}$ and $\overline{y}$ in $\{\overline{x}_{k,j}\}_{k,j}$, connected by the segment $S_{\overline{x},\overline{y}} \subset \overline{B}_{R_{k+1}} \setminus B_{R_{k-1}}$ for some $k$, and notice that $c_1\delta_k \leq |\overline{x} - \overline{y}| \leq c_2\delta_k$

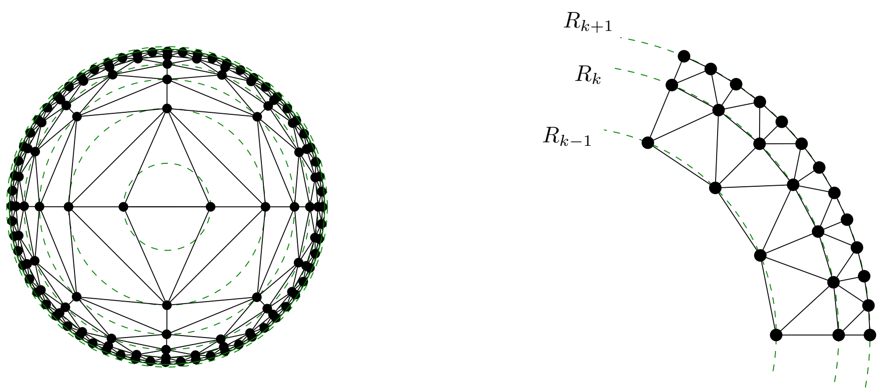


Figure 1: Sketch of the construction of the grid in the proof of Theorem 2.1.

for some $c_1 \in (0,1)$, $c_2 > 1$ independent from $k$. Let $\alpha := c_1/(8c_2)$ and consider the convex envelope

$$O_{\overline{x},\overline{y}} := \operatorname{conv}\left(B(\overline{x},\alpha\delta_k) \cup B(\overline{y},\alpha\delta_k)\right). \tag{2.4}$$

Let $a_{\overline{x},\overline{y}}$ denote the infinitesimal rigid movement appearing in the Poincaré's inequality for $u$ on the set

$$Q_{\overline{x},\overline{y}} := \{\xi \in B_R : \operatorname{dist}(\xi, S_{\overline{x},\overline{y}}) < |\overline{x}-\overline{y}|/(8c_2)\},$$

so that $\|u - a_{\overline{x},\overline{y}}\|_{L^1(Q_{\overline{x},\overline{y}};\mathbb{R}^2)} \le c|Eu|(Q_{\overline{x},\overline{y}})$; since the sets $Q_{\overline{x},\overline{y}}$ all have the same shape the constant is universal.

We will now choose points $(x,y) \in B(\overline{x},\alpha\delta_k)\times B(\overline{y},\alpha\delta_k)$ such that $u$ does not jump on the segment $S_{x,y}$ joining them, and such that the longitudinal component has a controlled derivative. To make this precise, we define by $u_z^\nu(t) := u(z+t\nu)\cdot\nu$ the slice of $u$ along the line of direction

$$\nu := \frac{x-y}{|x-y|}, \tag{2.5}$$

and passing through

$$z := (\mathrm{Id} - \nu\otimes\nu)x \in \mathbb{R}\nu^\perp \cap (x+\mathbb{R}\nu) \tag{2.6}$$

where $\mathbb{R}\nu^\perp$ is the linear space orthogonal to $\nu$, see Figure 2. We denote by $s_{x,y} \subset \mathbb{R}$ the segment defined by $z + s_{x,y}\nu = S_{x,y}$. Given $\vartheta \in (0,1)$, we will prove that for $\eta$ sufficiently small and $\tilde{c}$ sufficiently large, depending only on $\vartheta$, there exists a subset $F \subset B(\overline{x},\alpha\delta_k)\times B(\overline{y},\alpha\delta_k)$ with $\frac{(\mathcal{L}^2\times\mathcal{L}^2)(F)}{\mathcal{L}^2(B_{\alpha\delta_k})^2} < \vartheta$, such that for every $(x,y) \notin F$ the one-dimensional section $u_z^\nu$ has the following properties:

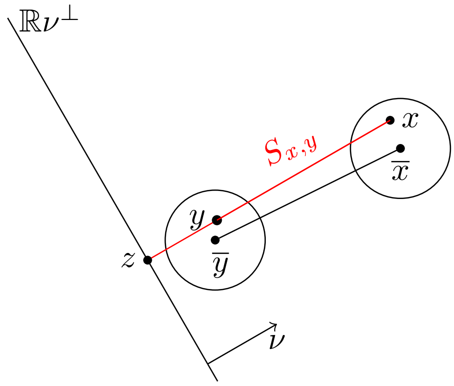


Figure 2: Slice along the line of direction $\nu = (x-y)/|x-y|$ passing through $z$ in the proof of Theorem 2.1.

(P1) $u_z^\nu \in SBV(s_{x,y})$;

(P2) $\mathcal{H}^0(J_{u_z^\nu}) = 0$, so that $u_z^\nu \in W^{1,1}(s_{x,y})$;

(P3) $\displaystyle\int_{s_{x,y}} |(u_z^\nu)'|dt \le \frac{\tilde{c}}{\delta_k}\int_{O_{\overline{x},\overline{y}}} |e(u)|dx'$;

(P4) $|u(\xi) - a_{\overline{x},\overline{y}}(\xi)| \le \dfrac{\tilde{c}}{\delta_k}|Eu|(Q_{\overline{x},\overline{y}})$, for $\xi = x, y$;

(P5) $x$ and $y$ are Lebesgue points of $u$.

The proof of (P1)–(P4) is based on the properties of slicing of $SBD$ functions; (P5) obviously holds for almost all choices. We recall that by [4, Theorem 4.5] for any fixed $\nu \in S^1$ the following holds: there is an $\mathcal{H}^1$-null set $N_\nu \subset \mathbb{R}\nu^\perp$ such that for all $z \in \mathbb{R}\nu^\perp \setminus N_\nu$ the section $u_z^\nu$ is in $SBV(s)$, where $s \subset \mathbb{R}$ is the set of $t \in \mathbb{R}$ such that $z + t\nu \in O_{\overline{x},\overline{y}}$, the jump set of $u_z^\nu$ coincides almost everywhere with the section of the jump set of $u$ (intersected with the set of points where $[u] \cdot \nu \neq 0$), and its distributional derivative obeys $\nabla u_z^\nu(t) = \nu \cdot e(u)(z + t\nu)\nu$ for almost every $t \in s$.

We now show that for almost every pair $(x, y) \in B^* := B(\overline{x}, \alpha\delta_k) \times B(\overline{y}, \alpha\delta_k)$ property (P1) holds. To see this, consider the change of variables given by $x = z+t\nu$, $y = z+t'\nu$, corresponding to the map $\psi(z, \nu, t, t') := (z+t\nu, z+t'\nu)$. This is a locally Lipschitz map from $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}^4$. Let $M := \{(z, \nu, t, t') \in \mathbb{R}^6 : \nu \in S^1, z \in N_\nu\}$ denote the exceptional set. By Fubini's theorem and $\mathcal{H}^1(N_\nu) = 0$ for all $\nu$ one obtains $\mathcal{H}^4(M) = 0$, and since $\psi$ is locally Lipschitz the set $F_1 := \psi(M)$ is also a $\mathcal{H}^4$-null set. Therefore for almost every choice of $(x,y) \in B^*$ the above slicing properties, including in particular $u^\nu_z \in SBV(s)$, hold and (P1) is proven.

In order to obtain property (P2) we first define the measure $\mu_{\nu,z} := \mathcal{H}^0 \llcorner (J_{u^\nu_z} \cap s_{x,y})$ and we observe that, by the change of variables $y = x + t\nu$,

$$\begin{aligned}&\int_{B(\overline{x},\alpha\delta_k)\times B(\overline{y},\alpha\delta_k)} \mu_{\nu,z}(s_{x,y})dx\,dy \\ &= \int_{B(\overline{x},\alpha\delta_k)} \int_{S^1} \int_0^\infty \chi_{B(\overline{y},\alpha\delta_k)}(x+t\nu)\mu_{\nu,z}(s_{x,x+t\nu})t\,dt\,d\mathcal{H}^1(\nu)\,dx\,.\end{aligned}$$

We observe that, since $\alpha \le 1$ and $|\overline{x}-\overline{y}| \le c_2\delta_k$, the characteristic function vanishes for $t \ge 3c_2\delta_k$. For any fixed $x \in B(\overline{x},\alpha\delta_k)$ and $\nu \in S^1$, we define $z$ as in (2.6) and $(O_{\overline{x},\overline{y}})^\nu_z \subset \mathbb{R}$ as the set of $t$ such that $z+t\nu \in O_{\overline{x},\overline{y}}$, so that, by convexity of the latter set, $s_{x,x+t\nu} \subset (O_{\overline{x},\overline{y}})^\nu_z$ for all $t$ for which $\chi_{B(\overline{y},\alpha\delta_k)}(x+t\nu) \ne 0$. Therefore the integral in $t$ can be estimated by $c\delta_k^2\mu((O_{\overline{x},\overline{y}})^\nu_z)$, and

$$\int_{B(\overline{x},\alpha\delta_k)\times B(\overline{y},\alpha\delta_k)} \mu_{\nu,z}(s_{x,y})dx\,dy \le c\delta_k^2 \int_{S^1}\int_{B(\overline{x},\alpha\delta_k)} \mu((O_{\overline{x},\overline{y}})^\nu_z)\,dx\,d\mathcal{H}^1(\nu)\,. \tag{2.7}$$

By Fubini's theorem the last term in the previous inequality is less than or equal to

$$c\delta_k^3 \int_{S^1} d\mathcal{H}^1(\nu) \int_{\mathbb{R}\nu^\perp} \mu_{\nu,z}((O_{\overline{x},\overline{y}})^\nu_z) d\mathcal{H}^1(z) \le c\delta_k^3\mu(O_{\overline{x},\overline{y}}), \tag{2.8}$$

where $\mu := \mathcal{H}^1 \llcorner (J_u \cap O_{\overline{x},\overline{y}})$. Now (2.3) implies

$$\int_{B(\overline{x},\alpha\delta_k)\times B(\overline{y},\alpha\delta_k)} \mathcal{H}^0(J_{u^\nu_z} \cap s_{x,y})dx\,dy \le c\delta_k^4\eta, \tag{2.9}$$

and hence the set $F_2$ of points $(x,y)$ for which $\mathcal{H}^0(J_{u^\nu_z} \cap s_{x,y}) > 1/2$ satisfies $\frac{(\mathcal{L}^2\times\mathcal{L}^2)(F_1)}{\mathcal{L}^2(B_{\alpha\delta_k})^2} < \vartheta/16$, for $\eta$ small enough. This proves property (P2). Note that this is the only step which requires the hypothesis on the dimension $n = 2$.

In order to prove (P3), for $(x,y) \in B(\overline{x},\alpha\delta_k)\times B(\overline{y},\alpha\delta_k) \setminus (F_1 \cup F_2)$ we repeat the argument in (2.7) and (2.8) above redefining

$$\mu_{\nu,z} := |D(u^\nu_z)|.$$

By the previous steps the function $u^\nu_z$ belongs to $W^{1,1}(s)$, and by the properties of slicing its weak derivative obeys $\nabla u^\nu_z(t) = \nu \cdot e(u)(z+t\nu)\nu$ for almost every $t \in s$. Repeating the same procedure as above we find that for $(x, y)$ out of a small (in the previous sense) set $F_3$ one has

$$|D(u_z^\nu)|(s_{x,y}) \leq \frac{\tilde{c}}{\delta_k} \int_{O_{\overline{x},\overline{y}}} |e(u)| dx', \tag{2.10}$$

for $\tilde{c}$ large enough.

Analogously property (P4) can be derived. From the argument above it is straightforward that for many points $x \in B(\overline{x}, \alpha\delta_k)$, still in the sense of a large $\vartheta$-fraction of $B(\overline{x}, \alpha\delta_k)$, there are many points $y \in B(\overline{y}, \alpha\delta_k)$ for which $(x, y) \notin F$.

Let us construct now the modified grid with an iterative process (see also [25, Proposition 3.4]). We will use the notation $B_i$ to indicate the balls $B(\overline{x}_{k,j}, \alpha\delta_k)$, lexicographically ordered.

We start by fixing a point $x_0 \in B_0$ for which there are many good choices in each neighboring ball. This means that for any neighbor $\overline{y}$ of $x_0$, the set of $y \in B(\overline{y}, \alpha\delta_0)$ such that $(x_0, y)$ does not have properties (P1)–(P4) has measure smaller than $\vartheta\mathcal{L}^2(B_{\alpha\delta_0})$. We next select $x_1 \in B_1$ among the points which are good choices for $x_0$ and which have many good choices in each neighboring subsequent ball $B_i$, $i \geq 2$. Iterating the process, the point $x_m \in B_m$ will be taken among the good choices for the neighboring previously fixed $x_i$, $i < m$, and with the property that have many good choices in the neighboring subsequent $B_i$, $i > m$. Since each ball can have at most seven neighbors, at each step we select $x_m$ avoiding just a small subset of $B_m$.

We call $S$ the set of points obtained by this process and we construct a new triangulation, with $x, y$ neighbors if and only if $\bar{x}$, $\bar{y}$ are neighbors. Notice that again

$$c_1\delta_k \leq |x - y| \leq c_2\delta_k, \tag{2.11}$$

for every couple of neighboring points $x, y$, with the same $k$ as for the corresponding reference points $\overline{x}$ and $\overline{y}$, and suitable $c_1, c_2 > 0$ independent from $k$. We finally define $\phi(u)$ as the linear interpolation between the values of $u(x)$, $x \in S$ on each triangle of the triangulation.

Fixed a triangle $T$ and any couple of its vertices $x, y$, we compute a component of the constant matrix $e(\phi(u))$ on $T$ by

$$e(\phi(u))\nu \cdot \nu = \frac{(\phi(u)(x) - \phi(u)(y)) \cdot \nu}{|x - y|} = \fint_{s_{x,y}} (u_z^\nu)' dt, \tag{2.12}$$

where $\nu$ and $z$ are defined in (2.5) and (2.6). We used the fact that $u$ and $\phi(u)$ agree on $x$ and $y$ and that $u$ is $W^{1,1}(s_{x,y})$ by the choice of $x$ and $y$. By

(2.12), (2.11), and property (P3) above it follows

$$|e(\phi(u))\nu \cdot \nu| \leq \frac{\tilde{c}}{\delta_k^2} \int_{O_{\overline{x},\overline{y}}} |e(u)| dx',$$

where $O_{\overline{x},\overline{y}}$ is defined in (2.4). We recall that here and henceforth $\tilde{c}$ can possibly change. Letting $\nu$ vary among the directions of the sides of $T$, we obtain a control on the whole $|e(\phi(u))|$ thanks to (2.11)

$$|e(\phi(u))| \leq \frac{\tilde{c}}{\delta_k^2} \int_{C_T} |e(u)| dx', \tag{2.13}$$

where $C_T$ denotes the convex envelope

$$C_T := \text{conv}\left(\cup B(\overline{x}, \alpha\delta_k)\right)$$

and the union is taken over the three vertices $\overline{x}$ in the old triangulation corresponding to the three vertices of $T$. We remark that $B(\overline{x}, \alpha\delta_k) \subset B_{R_{k+1}} \setminus B_{R_{k-1}}$ for all $\overline{x} \in \partial B_{R_k}$, therefore there is a universal $\tilde{c} > 0$ such that any $x \in B_R$ is contained in at most $\tilde{c}$ of the $C_T$.

We are ready to prove property (ii). By Jensen's inequality and (2.13) we have for $1 \leq q \leq p$ (changing again the value of $\tilde{c}$)

$$\begin{aligned} \int_T |e(\phi(u))|^q dx' = \mathcal{L}^2(T)|e(\phi(u))|^q &\leq \tilde{c}\mathcal{L}^2(C_T)\Big(\fint_{C_T} |e(u)| dx'\Big)^q \\ &\leq \tilde{c} \int_{C_T} |e(u)|^q dx', \end{aligned} \tag{2.14}$$

and finally summing up on all triangles $T$ we get the conclusion.

In order to prove properties (iii) and (iv) we estimate

$$\int_T |u - \phi(u)| dx' \leq \int_T |u - a_{\overline{x},\overline{y}}| dx' + \int_T |a_{\overline{x},\overline{y}} - \phi(u)| dx', \tag{2.15}$$

where $T$ is again a triangle of the modified triangulation with vertices $x, y, z$, while $\overline{x}, \overline{y}, \overline{z}$ denote the three corresponding vertices of the old triangulation, $a_{\overline{x},\overline{y}}$ is the infinitesimal rigid motion appearing in the Poincaré's inequality for $u$ on $Q_{\overline{x},\overline{y}}$ (see item (P4) above).

Let us study first the second term in (2.15). Since $a_{\overline{x},\overline{y}} - \phi(u)$ is affine, it achieves its maximum on a vertex $\xi$ of $T$, therefore

$$\int_T |a_{\overline{x},\overline{y}} - \phi(u)| dx' \leq c\delta_k^2 |a_{\overline{x},\overline{y}}(\xi) - \phi(u)(\xi)| = c\delta_k^2 |a_{\overline{x},\overline{y}}(\xi) - u(\xi)|.$$

Notice that if $\xi = z$ then by taking into account that $a_{\overline{x},\overline{z}}, a_{\overline{x},\overline{y}}$ are affine and item (P4) above we find

$$\begin{aligned}\delta_k^2|a_{\overline{x},\overline{y}}(\xi) - u(\xi)| &\leq \delta_k^2|a_{\overline{x},\overline{y}}(\xi) - a_{\overline{x},\xi}(\xi)| + \delta_k^2|a_{\overline{x},\xi}(\xi) - u(\xi)| \\ &\leq \int_{B(\overline{x},\alpha\delta_k)} |a_{\overline{x},\overline{y}} - a_{\overline{x},\xi}|dx' + c\delta_k|Eu|(Q_{\overline{x},\xi}) \\ &\leq c\int_{Q_{\overline{x},\overline{y}}} |u - a_{\overline{x},\overline{y}}(\xi)|dx' + c\int_{Q_{\overline{x},\xi}} |u - a_{\overline{x},\xi}(\xi)|dx' + c\delta_k|Eu|(Q_{\overline{x},\xi}) \\ &\leq c\delta_k|Eu|(Q_{\overline{x},\overline{y}}) + c\delta_k|Eu|(Q_{\overline{x},\xi}) \leq c\delta_k|Eu|(Q_T), \end{aligned} \tag{2.16}$$

where $Q_T := Q_{\overline{x},\overline{y}} \cup Q_{\overline{y},\overline{z}} \cup Q_{\overline{z},\overline{x}}$. Instead, if $\xi \in \{x, y\}$ we may directly apply item (P4).

For what the first term in (2.15) is concerned we first use Poincaré's inequality on $T$, obtaining for a rigid motion $a_T$ that $\|u - a_T\|_{L^1(T;\mathbb{R}^2)} \leq c\delta_k|Eu|(T)$. Since the angles of $T$ are uniformly controlled the constant is universal. We then estimate as follows:

$$\int_T |u - a_{\overline{x},\overline{y}}|dx' \leq \int_T |u - a_T|dx' + \int_T |a_{\overline{x},\overline{y}} - a_T|dx' \leq c\delta_k|Eu|(T). \tag{2.17}$$

By (2.15)-(2.17), we conclude

$$\int_T |u - \phi(u)|dx' \leq c\delta_k|Eu|(T), \tag{2.18}$$

Finally summing up over $T$ we obtain property (iii).

We prove now property (iv), property (v) holding true by (P5) and convexity of the Euclidean norm. We define $\phi(u) := u$ outside $\overline{B}_R$ and know that $\phi(u) \in W^{1,p}(B_R, \mathbb{R}^2) \cap SBD(B_{2r})$. It remains to prove that the traces on $\partial B_R$ coincide, or, equivalently, that $\mathcal{H}^1(J_{\phi(u)} \cap \partial B_R) = 0$. Let $\psi_k \in C^\infty(B_R, [0, 1])$ be such that $\psi_k = 0$ on $B_{R_k}$, $\psi_k = 1$ in a neighborhood of $\partial B_R$, and $|\nabla\psi_k| \leq c/\delta_k$. We define $v_k := (u - \phi(u))\psi_k \in SBD(B_R)$ and we prove that $v_k \to 0$ strongly in $BD$, this implying in turn $v_k|_{\partial B_R} \to 0$ in $L^1(\partial B_R, \mathbb{R}^2)$ in the sense of traces and therefore property (iv). Clearly

$$\int_{B_R} |v_k|dx \leq \int_{B_R \setminus B_{R_k}} |u - \phi(u)|dx \to 0$$

by the dominated convergence theorem. Finally, using (2.18) and the fact that the triangles have finite overlap,

$$\begin{aligned}|Ev_k|(B_R) &\leq |E(u - \phi(u))|(B_R \setminus B_{R_k}) + \frac{c}{\delta_k}\int_{B_R \setminus B_{R_k}} |u - \phi(u)|dx \\ &\leq \tilde{c}|E(u - \phi(u))|(B_R \setminus B_{R_k}).\end{aligned}$$

Since $B_R$ is open, by monotonicity the last term tends to 0 and this concludes the proof of property (iv).

The thesis follows choosing $\eta$ and $\tilde{c}$ such that (2.9), (2.10), and (2.14) hold.

□

# 3 Regularity of $SBD^p$ functions with small jump set

We first discuss how $SBD^p$ functions can be approximated by $W^{1,p}$ functions locally away from the jump set (Section 3.1), and then how they can be approximated by piecewise $W^{1,p}$ functions around the jump set (Section 3.3). Our approximation result also leads to the Korn inequality stated in Theorem 1.2. The key ingredient for all these results is the construction of Theorem 2.1. Throughout the section $\eta \in (0,1)$ will be the constant from Theorem 2.1 and $n = 2$.

## 3.1 Approximation of $SBD^p$ functions with $W^{1,p}$ functions

We will show that the construction of Theorem 2.1, using a suitable covering argument, permits to approximate $SBD^p$ functions by $W^{1,p}$ functions which coincide away from a small neighborhood of the jump set. The neighborhood is the union of countably many balls, such that each of them contains an amount of jump set proportional to the radius. Before discussing the covering argument in Proposition 3.2, we show that (away from the boundary) almost any point of the jump set is the center of a ball with the appropriate density.

**Lemma 3.1.** *Let $s \in (0,1)$. Let $J \in \mathcal{B}(B_{2\rho})$, for some $\rho > 0$, be such that $\mathcal{H}^1(J) < \eta(1-s)\rho$. Then for $\mathcal{H}^1$-a.e. $x \in J \cap B_{2s\rho}$ there exists a radius $r_x \in (0, (1-s)\rho)$ such that*

$$\mathcal{H}^1\big(J \cap \partial B_{r_x}(x)\big) = 0, \tag{3.1}$$

$$\eta\, r_x \le \mathcal{H}^1\big(J \cap B_{r_x}(x)\big) \le \mathcal{H}^1\big(J \cap B_{2r_x}(x)\big) < 2\,\eta\, r_x. \tag{3.2}$$

*Proof.* We fix $x \in J \cap B_{2s\rho}$, choose $\lambda_x \in (\rho, 2\rho)$ such that $\mathcal{H}^1\big(J \cap \partial B_{\lambda_x/2^k}(x)\big) = 0$ for all $k \in \mathbb{N}$, and define

$$r_x := \max\{{}^{\lambda_x}\!/_{2^k} : k \in \mathbb{N},\, \mathcal{H}^1\big(J \cap B_{\lambda_x/2^k}(x)\big) \ge \eta\, \lambda_x 2^{-k}\}.$$

The set is nonempty for $\mathcal{H}^1$-almost every $x$ because $\eta < 1$. The estimates (3.2) hold by definition. To conclude that $r_x < (1-s)\rho$ it is enough to notice that the opposite inequality would give the ensuing contradiction

$$\mathcal{H}^1(J) \geq \mathcal{H}^1\big(J \cap B_{r_x}(x)\big) \geq \eta\, r_x \geq (1-s)\eta\,\rho > \mathcal{H}^1(J).$$

□

We are now ready to prove the main result of the section via a covering argument, Lemma 3.1, and Theorem 2.1.

**Proposition 3.2.** *Let $p \in (1,\infty)$, $n = 2$. There exists a universal constant $c > 0$ such that if $u \in SBD^p(B_{2\rho})$, $\rho > 0$, satisfies*

$$\mathcal{H}^1(J_u \cap B_{2\rho}) < \eta\,(1-s)\rho$$

*for $\eta \in (0,1)$ as in Theorem 2.1 and some $s \in (0,1)$, then there is a countable family $\mathcal{F} = \{B\}$ of closed balls of radius $r_B < 2(1-s)\rho$, such that their union is compactly contained in $B_{2\rho}$, and a field $w \in SBD^p(B_{2\rho})$ such that*

*(i)* $\rho^{-1}\sum_{\mathcal{F}}\mathcal{L}^2(B) + \sum_{\mathcal{F}}\mathcal{H}^1(\partial B) \leq {}^{c}\!/\!_{\eta}\,\mathcal{H}^1(J_u \cap B_{2\rho})$;

*(ii)* $\mathcal{H}^1\big(J_u \cap \cup_{\mathcal{F}}\partial B\big) = \mathcal{H}^1\big((J_u \cap B_{2s\rho}) \setminus \cup_{\mathcal{F}} B\big) = 0$;

*(iii)* $w = u$ *$\mathcal{L}^2$-a.e. on* $B_{2\rho} \setminus \cup_{\mathcal{F}} B$;

*(iv)* $w \in W^{1,p}(B_{2s\rho},\mathbb{R}^2)$ *and* $\mathcal{H}^1(J_w \setminus J_u) = 0$;

*(v)*

$$\int_{\cup_{\mathcal{F}} B} |e(w)|^p dx \leq c \int_{\cup_{\mathcal{F}} B} |e(u)|^p dx; \tag{3.3}$$

*(vi)* $\|u - w\|_{L^1(B,\mathbb{R}^2)} \leq c\, r_B\, |Eu|(B)$, *for every* $B \in \mathcal{F}$;

*(vii) if, additionally,* $u \in L^\infty(B_{2\rho},\mathbb{R}^2)$ *then* $w \in L^\infty(B_{2\rho},\mathbb{R}^2)$ *with*

$$\|w\|_{L^\infty(B_{2\rho},\mathbb{R}^2)} \leq \|u\|_{L^\infty(B_{2\rho},\mathbb{R}^2)}.$$

Before giving the proof we show an immediate consequence of this result.

**Corollary 3.3.** *Under the same assumptions and notation of Proposition 3.2, there is an affine map $a : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\int_{B_{2s\rho}\setminus\omega} |\nabla u - \nabla a|^p dx \leq c(p) \int_{B_{2\rho}} |e(u)|^p dx; \tag{3.4}$$

*and*

$$\int_{B_{2s\rho}\setminus\omega} |u - a|^p dx \leq c(p)\rho^p \int_{B_{2\rho}} |e(u)|^p dx\,, \tag{3.5}$$

*where $\omega := \cup_{\mathcal{F}} B$.*

*Proof.* By Korn's inequality applied to $w$ on the ball $B_{2s\rho}$ there is a skew-symmetric matrix $A$ such that

$$\int_{B_{2s\rho}} |\nabla w - A|^p dx \leq c(p) \int_{B_{2s\rho}} |e(w)|^p dx,$$

and by Poincaré's inequality applied to $x \mapsto w(x) - Ax$ on the same domain there is $d \in \mathbb{R}^2$ such that

$$\int_{B_{2s\rho}} |w(x) - Ax - d|^p dx \leq c(p)\rho^p \int_{B_{2s\rho}} |\nabla w - A|^p dx.$$

To conclude the proof we define $a(x) := Ax + d$ and for the left-hand side observe that $w = u$, $\nabla w = \nabla u$ on $B_{2s\rho} \setminus \cup_{\mathcal{F}} B$ (more precisely $\nabla u = \nabla w$ $\mathcal{L}^n$ a.e. on $\{u = w\}$ by [5, Proposition 3.73]), and instead use (v) from Proposition 3.2 to estimate the right-hand side. □

*Proof of Proposition 3.2.* By Lemma 3.1 we find a family $\mathcal{F}'$ of open balls covering $\mathcal{H}^1$-a.e. $J_u \cap B_{2s\rho}$ that satisfies (3.1) and (3.2). Since the inequality $\mathcal{H}^1\big(J \cap \partial B_{r_x}(x)\big) = 0$ is strict, we can further assume that they are all contained in $B_{2\rho'}$ for some $\rho' < \rho$. Setting $J = J_u$, to every $B \in \mathcal{F}'$ we associate a new ball $B^* \subset B$ with the properties (i)-(v) of Theorem 2.1. Let $\mathcal{F}^*$ be the family of the new balls $B^*$, this is still a cover of $J$. Further, the balls $B^*$ can be taken to be closed. By the Besicovitch covering theorem [5, Theorem 2.17] there are $\xi$ countable subfamilies $\mathcal{F}'_j = \{B^i_j\}_{i\in\mathbb{N}}$ of disjoint balls. Therefore, setting $\mathcal{F} := \cup_{j=1}^{\xi} \mathcal{F}'_j$ we have $\mathcal{H}^1\big((J_u \cap B_{2s\rho}) \setminus \cup_{\mathcal{F}} B\big) = 0$. In addition, by (3.1) the first condition in item (ii) is satisfied as well, so that (ii) is established. Furthermore,

$$\begin{aligned}\sum_{B\in\mathcal{F}} \mathcal{H}^1(\partial B) =& 2\pi \sum_{B\in\mathcal{F}} r_B \overset{(3.2)}{\leq} \frac{2\pi}{\eta} \sum_{B\in\mathcal{F}} \mathcal{H}^1\big(J_u \cap B\big) \\ \leq& \xi\, \frac{2\pi}{\eta} \mathcal{H}^1\big(J_u \cap \cup_{B\in\mathcal{F}} B\big) \leq \xi\, \frac{2\pi}{\eta} \mathcal{H}^1(J_u \cap B_{2\rho}).\end{aligned}$$

The volume estimate follows since $r_B \leq \rho$ implies $\sum r_B^2 \leq \rho \sum r_B$. We remark that a quadratic volume estimate also follows by $\sum r_B^2 \leq (\sum r_B)^2$.

Let $\phi(u)$ be the function given by Theorem 2.1 on the balls of the first family $\mathcal{F}'_1$ and define for every $h \in \mathbb{N}$ a function

$$w_1^h := \begin{cases} \phi(u) & B_1^i,\ i \leq h \\ u & \text{otherwise} \end{cases}$$

such that $w_1^h \in SBD^p(B_{2\rho})$, $w_1^h \in W^{1,p}(\cup_{i\leq h} B_1^i;\mathbb{R}^2)$ with $w_1^h = u$ $\mathcal{L}^2$-a.e. on $B_{2\rho} \setminus \cup_{i\leq h} B_1^i$ and $\mathcal{H}^1(J_{w_1^h} \setminus J_u) = 0$. In addition by item (ii) in Theorem 2.1

$$\int_{B_{2\rho}} |e(w_1^h)|^p\,dx = \int_{\cup_{i\leq h} B_1^i} |e(\phi(u))|^p\,dx + \int_{B_{2\rho}\setminus\cup_{i\leq h} B_1^i} |e(u)|^p\,dx$$

$$\leq \tilde{c} \int_{\cup_{i\leq h} B_1^i} |e(u)|^p\,dx + \int_{B_{2\rho}\setminus\cup_{i\leq h} B_1^i} |e(u)|^p\,dx \leq (1+\tilde{c}) \int_{B_{2\rho}} |e(u)|^p\,dx, \quad (3.6)$$

and

$$|Ew_1^h|(B_{2\rho}) \leq |Eu|\big(B_{2\rho} \setminus \cup_{i\leq h} B_1^i\big) + \tilde{c} \int_{\cup_{i\leq h} B_1^i} |e(u)|\,dx.$$

Moreover, recalling that the $B_1^i$'s are disjoint and that $w_1^{h-1} = u$ on $B_1^h$, item (iii) in Theorem 2.1 gives

$$\|w_1^h - w_1^{h-1}\|_{L^1(B_{2\rho};\mathbb{R}^2)} = \|w_1^h - u\|_{L^1(B_1^h;\mathbb{R}^2)} \leq c\,\rho\,|Eu|(B_1^h),$$

in turn implying that for all $h \geq k \geq 1$

$$\|w_1^h - w_1^k\|_{L^1(B_{2\rho};\mathbb{R}^2)} \leq \sum_{i=k+1}^{h} \|w_1^i - w_1^{i-1}\|_{L^1(B_1^h;\mathbb{R}^2)} \leq c\,\rho\,|Eu|\big(\cup_{k+1\leq i\leq h} B_1^i\big).$$

Thus, $w_1^h \to w_1$ in $L^1(B_{2\rho};\mathbb{R}^2)$ with

$$w_1 := \begin{cases} \phi(u) & \cup_{\mathcal{F}_1'} B \\ u & \text{otherwise.} \end{cases}$$

The $BD$ compactness theorem then yields that $w_1 \in BD(B_{2\rho})$. In turn, by (3.6) and since $\mathcal{H}^1(J_{w_1^h} \setminus J_u) = 0$, the $SBD$ compactness theorem implies that actually $w_1 \in SBD^p(B_{2\rho})$ (see also [29, Theorem 11.3]). Furthermore, since

$$\mathcal{H}^1\big(J_{w_1^h} \cap \cup_{\mathcal{F}_1'} B\big) = \mathcal{H}^1(J_u \cap \cup_{i\geq h+1} B_1^i),$$

we may conclude that

$$\mathcal{H}^1\big(J_{w_1} \cap \cup_{\mathcal{F}_1'} B\big) \leq \liminf_h \mathcal{H}^1\big(J_{w_1^h} \cap \cup_{\mathcal{F}_1'} B\big) = 0,$$

and therefore $w_1 \in W^{1,p}(\cup_{\mathcal{F}_1'} B, \mathbb{R}^2)$. Finally, by construction $w_1 = u$ $\mathcal{L}^2$-a.e. on $B_{2\rho} \setminus \cup_{\mathcal{F}_1'} B$ and $\mathcal{H}^1(J_{w_1} \setminus J_u) = 0$.

By iterating the latter construction, for all $1 < k \leq \xi$ and for every $h \in \mathbb{N}$ we find

$$w_k^h := \begin{cases} \phi(w_{k-1}) & B_k^i,\ i \leq h \\ w_{k-1} & \text{otherwise} \end{cases}$$

such that $w_k^h \in SBD^p(B_{2\rho})$, $w_k^h \in W^{1,p}(\cup_{i\le h} B_k^i;\mathbb{R}^2)$, $w_k^h = w_{k-1}$ $\mathcal{L}^2$-a.e. on $B_{2\rho}\setminus \cup_{i\le h} B_k^i$ , $\mathcal{H}^1(J_{w_k^h}\setminus J_{w_{k-1}})=0$. In addition, arguing as above, $w_k^h \to w_k$ in $L^1(B_{2\rho},\mathbb{R}^2)$ with

$$w_k := \begin{cases} \phi(w_{k-1}) & \cup_{\mathcal{F}_k'} B \\ w_{k-1} & \text{otherwise,} \end{cases}$$

$w_k \in SBD^p(B_{2\rho})$, $w_k \in W^{1,p}(\cup_{j\le k}\cup_{\mathcal{F}_j'} B;\mathbb{R}^2)$, $w_k = w_{k-1}$ $\mathcal{L}^2$-a.e. on $B_{2\rho}\setminus \cup_{\mathcal{F}_k'} B$ and $\mathcal{H}^1(J_{w_k}\setminus J_{w_{k-1}})=0$.

Set $w := w_\xi$, then $w\in SBD^p(B_{2\rho})$, $w\in W^{1,p}(\cup_{\mathcal{F}}B;\mathbb{R}^2)$, $w=u$ $\mathcal{L}^2$-a.e. on $B_{2\rho}\setminus\cup_{\mathcal{F}}B$, $\mathcal{H}^1(J_w\setminus J_u)=0$. Iterating estimate (3.6), inequality (3.3) follows at once with $c:=\max\{1+\tilde c, 2\pi\}\,\xi$, with $\tilde c$ the constant in Theorem 2.1.

Finally, it is clear that also items (vi) and (vii) are satisfied in view of properties (iii) and (v) in Theorem 2.1. □

### 3.2 Korn's inequality in $SBD^p$

*Proof of Theorem 1.2.* By standard scaling and covering arguments it suffices to prove the assertion for a special Lipschitz domain. Precisely, let $\varphi:\mathbb{R}\to\mathbb{R}$ Lipschitz with $\min\varphi[(-1,1)]=2$, and set $U:=\{x: x_1\in(-2,2), x_2\in(-2,\varphi(x_1))\}$, and $U^{\rm int}:=\{x: x_1\in(-1,1), x_2\in(-1,\varphi(x_1))\}$. It suffices to show that for any $u\in SBD^p(U)$ there are $\omega$ with $\mathcal{H}^1(\partial\omega)+\mathcal{L}^2(\omega)\le c\mathcal{H}^1(J_u)$ and an affine function $a:\mathbb{R}^2\to\mathbb{R}^2$ such that $\|u-a\|_{L^p(U^{\rm int}\setminus\omega,\mathbb{R}^2)}+\|\nabla u-\nabla a\|_{L^p(U^{\rm int}\setminus\omega,\mathbb{R}^2)}\le c_{L,p}\|e(u)\|_{L^p(U,\mathbb{R}^{2\times 2})}$, with $c$ depending on $p$ and the Lipschitz constant $L$ of $\varphi$. Obviously we can assume $\mathcal{H}^1(J_u)$ to be small.

Consider first two squares $q_j := y_j+(-r_j/2,r_j/2)^2$ and $Q_j := y_j+(-r_j,r_j)^2$ contained in $U$, and let $\eta$ be the constant from Proposition 3.2. If $u\in SBD^p(Q_j)$ obeys $\mathcal{H}^1(J_u\cap Q_j)\le \eta r_j/8$, then by Proposition 3.2 and Corollary 3.3 with $\rho:=r_j/2$ and $s:=1/\sqrt{2}$ there are a measurable set $\omega_j\subset Q_j$ and an affine function $a_j:\mathbb{R}^2\to\mathbb{R}^2$ such that $\mathcal{H}^1(\partial\omega_j)+r_j^{-1}\mathcal{L}^2(\omega_j)\le c\mathcal{H}^1(J_u\cap Q_j)$ and $r_j^{-1}\|u_j-a_j\|_{L^p(q_j\setminus\omega_j,\mathbb{R}^2)}+\|\nabla u_j-\nabla a_j\|_{L^p(q_j\setminus\omega_j,\mathbb{R}^{2\times2})}\le c_p\|e(u)\|_{L^p(Q_j,\mathbb{R}^{2\times 2})}$, with a constant which depends only on the exponent $p$. If instead $\mathcal{H}^1(J_u\cap Q_j)>\eta r_j/8$ we define $\omega_j=q_j$, $a_j=0$, and trivially obtain the same estimates.

To pass to the estimate on $U^{\rm int}$ one uses a Whitney cover with pairs of open cubes $q_j\subset Q_j\subset U$ such that the exterior ones have finite overlap and the interior ones cover $U^{\rm int}$, as done for example in proving the nonlinear Korn's inequality in [37, Theorem 3.1]. We can additionally require that $Q_0=(-2,2)^2$, $q_0=(-1,1)^2$, and that if $q_i\cap q_j\neq\emptyset$ then $c\mathcal{L}^2(q_i\cap q_j)\ge\mathcal{L}^2(q_i)+\mathcal{L}^2(q_j)$. Following [36], if $\mathcal{H}^1(J_u\cap Q_j)\ge\eta r_j/8$ we define $P_j := (y_j+(-r_j,r_j)\times(-r_j,\infty))\cap U$, otherwise $P_j=\emptyset$ and $\omega_j$, $a_j$ are obtained as above. Notice that $\mathcal{H}^1(\partial P_j) \le c_L r_j$ by the properties of Lipschitz functions and of the Whitney covering.

By a triangular inequality for any pair of overlapping squares $q_i$ and $q_j$ such that $P_i = P_j = \emptyset$ we obtain that $c^{-1} r_i \le r_j \le c r_i$ and

$$r_j^{-1}\|a_j - a_i\|_{L^p(q_j\cap q_i\setminus(\omega_j\cup\omega_i),\mathbb{R}^2)} + \|\nabla a_j - \nabla a_i\|_{L^p(q_j\cap q_i\setminus(\omega_j\cup\omega_i),\mathbb{R}^{2\times 2})} \le c_p\|e(u)\|_{L^p(Q_j\cup Q_i,\mathbb{R}^{2\times 2})}.$$

By the properties of the covering, for any $i$ and $j$ such that $q_i \cap q_j \neq \emptyset$ one has $\mathcal{L}^2(q_j \cap q_i) \ge c r_i^2$. If $\eta$ is sufficiently small, the bounds on $\omega_i$ and $\omega_j$ give $\mathcal{L}^2(q_j \cap q_i \setminus (\omega_j \cup \omega_i)) \ge c r_i^2$ and, since $a_i - a_j$ is affine,

$$r_j^{-1}\|a_j - a_i\|_{L^p(q_j\cap q_i,\mathbb{R}^2)} + \|\nabla a_j - \nabla a_i\|_{L^p(q_j\cap q_i,\mathbb{R}^{2\times 2})} \le c_p\|e(u)\|_{L^p(Q_j\cup Q_i,\mathbb{R}^{2\times 2})}.$$

Fix now a partition of unity $\theta_j \in C^2_c(q_j)$ with $|D\theta_j| \le c/r_j$, and define $a^* \in C^\infty(U,\mathbb{R}^2)$ by $a^* := \sum_j \theta_j a_j$. Since $\sum_i D\theta_i = 0$, for $x \in q_j$ we obtain $Da^*(x) = \sum_{i:q_i\cap q_j\neq\emptyset}(\theta_i Da_i(x) + (a_i - a_j)\otimes D\theta_i(x))$, and correspondingly

$$r_j|D^2 a^*|(x) \le c \sum_{i:q_i\cap q_j\neq\emptyset} (r_j^{-1}|a_i - a_j|(x) + |Da_i - Da_j|(x)).$$

At the same time, by the properties of the covering $r_j$ can be estimated with the distance from the boundary, which in turn, since $U$ is a Lipschitz set, behaves as $\varphi(x_1) - x_2$. Taking the $L^p$ norm we conclude

$$\int_{U^{\mathrm{int}}\setminus\cup_j P_j} (\varphi(x_1) - x_2)^p |D^2 a^*|^p(x)dx \le c_{L,p}\|e(u)\|^p_{L^p(U,\mathbb{R}^2)}.$$

At this point we apply a weighted Poincaré inequality, as was done in [37, Theorem 3.1]. The inequality we use states that for any bounded, connected open Lipschitz set $\Omega$, any $p \in [1,\infty)$ and any $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ there is $f_* \in \mathbb{R}$ such that

$$\int_\Omega |f - f_*|^p(x)dx \le c\int_\Omega \mathrm{dist}^p(x,\partial\Omega)|Df|^p(x)dx$$

where the constant may depend on $p$ and $\Omega$, see [41, Theorem 1.5] or [39, Theorem 8.8] for a proof. In one dimension, this corresponds to the fact that for any function $f \in C^0([s,t)) \cap W^{1,p}_{\mathrm{loc}}((s,t))$ one has

$$\int_s^t |f(x) - f(s)|^p(x)dx \le c\int_s^t |x - t|^p|f'|^p(x)dx.$$

Since the cube $Q_0 = (-2,2)^2$ was not removed one has $a^* = a_0$ in $q_0 = (-1,1)^2$ and application of the one-dimensional weighted Poincaré inequality to $Da^*(x_1,\cdot)$ on the segment $(-2,\varphi(x_1))$ leads to the assertion, with $\omega := \cup_j(P_j\cup\omega_j)$ and $a := a_0$. Equivalently, in the last step one may use a Poincaré or Korn inequality on John domains, as done in [36, Theorem 4.2]. □

We remark that the nonoptimality of the exponent in [36, Theorem 4.2] is only consequence of the nonoptimal local estimate employed there (see [36, Theorem 3.1]).

## 3.3 Reflection

In this subsection we establish a technical result instrumental for the identification of the surface energy density in Section 4.3. To this aim, given $u \in SBD^p(\Omega)$ and a point $x_0 \in J_u$ we set

$$u_{x_0}(x) := \begin{cases} u^+(x_0) & \text{if } \langle x - x_0, \nu_{x_0} \rangle > 0, \\ u^-(x_0) & \text{if } \langle x - x_0, \nu_{x_0} \rangle < 0. \end{cases} \tag{3.7}$$

**Lemma 3.4.** *Let $p \in (1, \infty)$, $u \in SBD^p(\Omega)$, $\Omega \subset \mathbb{R}^2$ open. For $\mathcal{H}^1$-a.e. $x_0 \in J_u$ and any $\rho > 0$ sufficiently small there is $v_\rho \in SBD^p(B_{2\rho}(x_0)) \cap SBV^p(B_\rho(x_0), \mathbb{R}^2)$ such that:*

*(i)* $\displaystyle\lim_{\rho\to 0} \frac{1}{\rho} \mathcal{H}^1(B_\rho(x_0) \cap J_{v_\rho} \setminus J_u) = 0;$

*(ii)* $\displaystyle\lim_{\rho\to 0} \frac{1}{\rho} \int_{B_\rho(x_0)} |\nabla v_\rho|^p dx = 0;$

*(iii)* $\displaystyle\lim_{\rho\to 0} \frac{1}{\rho^2} \mathcal{L}^2(\{x \in B_\rho(x_0) : u \neq v_\rho\}) = 0;$

*(iv)* $\displaystyle\lim_{\rho\to 0} \frac{1}{\rho^2} \int_{B_\rho(x_0)} |v_\rho - u| dx = 0;$

*(v)* $\displaystyle\lim_{\rho\to 0} \frac{1}{\rho^{p+1}} \int_{B_\rho(x_0)} |v_\rho - u_{x_0}|^p dx = 0;$

*(vi)* $\displaystyle\lim_{\rho\to 0} \frac{1}{\rho} \int_{B_\rho(x_0) \cap J_{v_\rho}} |[v_\rho] - [u]| d\mathcal{H}^1 = 0.$

*Proof.* Since $J_u$ is $(\mathcal{H}^1, 1)$ rectifiable, there exists a sequence $(\Gamma_i)_{i=1}^\infty$ of $C^1$ curves such that $\mathcal{H}^1(J_u \setminus \cup_{i=1}^\infty \Gamma_i) = 0$. For $\mathcal{H}^1$-a.e. $x_0 \in J_u$ we have

$$\lim_{\rho\to 0} \frac{1}{2\rho} \int_{J_u \cap B_\rho(x_0)} (|[u]| + 1) d\mathcal{H}^1 = |[u](x_0)| + 1,$$
$$\lim_{\rho\to 0} \frac{1}{2\rho} \int_{J_u \cap \Gamma \cap B_\rho(x_0)} (|[u]| + 1) d\mathcal{H}^1 = |[u](x_0)| + 1,$$

for one of the aforementioned curves $\Gamma$. Therefore

$$\lim_{\rho\to 0}\frac{1}{2\rho}\int_{(J_u\triangle\Gamma)\cap B_\rho(x_0)}(|[u]|+1)d\mathcal{H}^1=0 \tag{3.8}$$

and for $\rho$ small $\Gamma$ separates $B_{6\rho}(x_0)$ into two connected components. It is not restrictive to assume that $\Gamma\cap B_{6\rho}(x_0)$ is the graph of a function $h\in C^1(\mathbb{R})$. Moreover the following properties hold for $\mathcal{H}^1$-a.e. $x_0\in J_u$

$$\lim_{\rho\to 0}\frac{1}{\rho}\int_{B_\rho(x_0)}|e(u)|^p dx=0, \tag{3.9}$$

$$\lim_{\rho\to 0}\frac{1}{2\rho}|Eu|(B_\rho(x_0))=|[u]\odot\nu_u|(x_0), \tag{3.10}$$

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\int_{B_\rho(x_0)\cap\{\pm(x_2-h(x_1))>0\}}|u-u^\pm(x_0)|dx=0. \tag{3.11}$$

Indeed, the first one follows from $|e(u)|^p\in L^1(\Omega)$ and $\mathcal{L}^2\llcorner\Omega\perp\mathcal{H}^1\llcorner J_u$, the second one from [4, Eq. (4.2)], and the third one from [7, Prop. 4.1, Eq. (4.2)].

For simplicity we next assume that the point $x_0=0$ satisfies all the previous properties (3.8)-(3.11), with $h(0)=h'(0)=0$. We also set $\tau_\rho:=\|h\|_{L^\infty(B_{6\rho})}$ and note that $\tau_\rho/\rho\to 0$ as $\rho\to 0$. We now define the reflections of $u$ with respect to the lines $\{x_2=\pm\tau_\rho\}$, in the sense of [42, Lemma 1]. More precisely, define $\tilde{u}^+_\rho$ on the set $B_{2\rho}\cap\{x_2<\tau_\rho\}$ by

$$\begin{cases}(\tilde{u}^+_\rho)_1(x_1,x_2):=-2u_1(x_1,3\tau_\rho-2x_2)+3u_1(x_1,2\tau_\rho-x_2)\\ (\tilde{u}^+_\rho)_2(x_1,x_2):=4u_2(x_1,3\tau_\rho-2x_2)-3u_2(x_1,2\tau_\rho-x_2)\end{cases}$$

and by $u$ otherwise in $B_{2\rho}$. Note that $\tilde{u}^+_\rho\in SBD^p(B_{2\rho})$ and that

$$\lim_{\rho\to 0}\frac{1}{2\rho}\mathcal{H}^1(J_{\tilde{u}^+_\rho}\cap B_{2\rho})=0, \tag{3.12}$$

$$\|e(\tilde{u}^+_\rho)\|_{L^p(B_{2\rho},\mathbb{R}^{2\times 2})}\le c\|e(u)\|_{L^p(B_{6\rho},\mathbb{R}^{2\times 2})}, \tag{3.13}$$

for a universal constant $c$. Using a similar reflection we define $\tilde{u}^-_\rho$ in $B_{2\rho}\cap\{(x_1,x_2):x_2>-\tau_\rho\}$ and we set $\tilde{u}^-_\rho:=u$ otherwise in $B_{2\rho}$.

By (3.8) and (3.12) for $\rho$ small we have that $\tilde{u}^\pm_\rho$ satisfy the hypotheses of Proposition 3.2 on $B_{2\rho}$ with $s=1/2$. Thus, there exist $w^\pm_\rho\in SBD^p(B_{2\rho})\cap W^{1,p}(B_\rho,\mathbb{R}^2)$, for which properties (i)-(vii) hold true. Finally let us define $v_\rho\in SBD^p(B_{2\rho})$ by

$$v_\rho:=\begin{cases}w^+_\rho & \text{in } B_{2\rho}\cap\{x_2>h(x_1)\},\\ w^-_\rho & \text{in } B_{2\rho}\cap\{x_2<h(x_1)\}.\end{cases}$$

Since $w_\rho^\pm \in W^{1,p}(B_\rho,\mathbb{R}^2)$ we obtain $v_\rho \in SBV^p(B_\rho,\mathbb{R}^2)$ with

$$\begin{aligned}Dv_\rho \llcorner B_\rho =& \nabla w_\rho^+ \, \mathcal{L}^2 \llcorner B_\rho \cap \{x_2 > h(x_1)\} + (w_\rho^+ - w_\rho^-) \otimes \nu_\Gamma \mathcal{H}^1 \llcorner \Gamma \cap B_\rho \\ &+ \nabla w_\rho^- \, \mathcal{L}^2 \llcorner B_\rho \cap \{x_2 < h(x_1)\}.\end{aligned}$$

We next check that $v_\rho$ satisfies the properties in the statement in the ball $B_\rho$. Property (i) comes straightforwardly from (3.8) and from the fact that $J_{v_\rho} \subset \Gamma$. Moreover (3.13), (3.3), and (3.9) yield

$$\lim_{\rho\to 0} \frac{1}{\rho} \int_{B_\rho} |e(w_\rho^\pm)|^p dx = 0. \tag{3.14}$$

As for property (iii), we observe that

$$\lim_{\rho\to 0} \frac{1}{\rho^2} \mathcal{L}^2(\{x \in B_\rho : \, u(x) \neq v_\rho(x)\}) \leq \lim_{\rho\to 0} (c\frac{\tau_\rho}{\rho} + \frac{c}{\rho}\mathcal{H}^1((J_u \setminus \Gamma) \cap B_{6\rho})) = 0,$$

where we have used Proposition 3.2 (i) and (3.8).

Let us now prove property (iv). By the definition of $v_\rho$ and $\tilde{u}_\rho^\pm$ and by triangular inequality we obtain

$$\begin{aligned}\frac{1}{\rho^2}\int_{B_\rho} |v_\rho - u| dx \leq& \\ &\frac{1}{\rho^2}\int_{B_\rho \cap \{h(x_1)<x_2\}} |w_\rho^+ - \tilde{u}_\rho^+| dx + \frac{1}{\rho^2}\int_{B_\rho \cap \{h(x_1)<x_2<\tau_\rho\}} |\tilde{u}_\rho^+ - u| dx + \\ &\frac{1}{\rho^2}\int_{B_\rho \cap \{x_2<h(x_1)\}} |w_\rho^- - \tilde{u}_\rho^-| dx + \frac{1}{\rho^2}\int_{B_\rho \cap \{-\tau_\rho<x_2<h(x_1)\}} |\tilde{u}_\rho^- - u| dx.\end{aligned} \tag{3.15}$$

By the definition of $w_\rho^+$ and Proposition 3.2 (vi) we can estimate

$$\frac{1}{\rho^2}\int_{B_\rho} |w_\rho^+ - \tilde{u}_\rho^+| dx \leq \frac{c}{\rho}|E\tilde{u}_\rho^+|(B_{2\rho}) \leq \frac{c}{\rho}|Eu|(B_{6\rho} \setminus \Gamma).$$

By (3.8) and (3.9) we conclude that the first term of (3.15) tends to 0. Clearly, the same argument can be applied to the third term there. So, it remains to treat the second term in (3.15), being the fourth one similar. By triangular inequality and a change of variable we infer

$$\begin{aligned}\frac{1}{\rho^2}\int_{B_\rho \cap \{h(x_1)<x_2<\tau_\rho\}} |\tilde{u}_\rho^+ - u| dx \leq& \\ &\frac{1}{\rho^2}\int_{B_\rho} |\tilde{u}_\rho^+ - u^+(x_0)| dx + \frac{1}{\rho^2}\int_{B_\rho \cap \{h(x_1)<x_2\}} |u^+(x_0) - u| dx \leq \\ &\qquad \frac{c}{\rho^2}\int_{B_{6\rho} \cap \{h(x_1)<x_2\}} |u^+(x_0) - u| dx,\end{aligned}$$

and the last term tends to 0 by (3.11), hence property (iv) follows.

Let us prove now property (v). By Korn's inequality and Poincaré's inequality in $W^{1,p}$, there exists an affine function $a_\rho(x) := d_\rho + \beta_\rho x$ such that

$$\frac{1}{\rho^{p+1}} \int_{B_\rho} |w_\rho^+ - a_\rho|^p dx \le \frac{c}{\rho} \int_{B_\rho} |e(w_\rho^+)|^p dx. \tag{3.16}$$

We first claim that

$$\lim_{\rho \to 0} d_\rho = u^+(x_0). \tag{3.17}$$

Let $\omega_\rho^+ := B_\rho \cap \{u = w_\rho^+\} \cap \{x_2 > h(x_1)\}$. Since $|\omega_\rho^+|/\rho^2 \to \pi/2$, and $a_\rho$ is affine, by [22, Lemma 4.3] we obtain, for $\rho$ small,

$$\|a_\rho - u^+(x_0)\|_{L^\infty(B_\rho^+,\mathbb{R}^2)} \le \frac{c}{\rho^2} \int_{\omega_\rho^+} |w_\rho^+ - a_\rho| dx + \frac{c}{\rho^2} \int_{\omega_\rho^+} |u - u^+(x_0)| dx.$$

The right hand side above is infinitesimal by (3.16), (3.14) and (3.11), thus we conclude

$$\limsup_{\rho \to 0} |d_\rho - u^+(x_0)| \le \lim_{\rho \to 0} \|a_\rho - u^+(x_0)\|_{L^\infty(B_\rho^+,\mathbb{R}^2)} = 0,$$

which proves (3.17).

Next we prove that

$$\lim_{\rho \to 0} \rho |\beta_\rho|^p = 0, \tag{3.18}$$

$$\lim_{\rho \to 0} \rho^{\frac{1-p}{p}} |d_\rho - u^+(x_0)| = 0. \tag{3.19}$$

To establish (3.18), we fix $\delta > 0$ small and we consider $\hat\rho$ such that

$$\Big(\frac{1}{\rho} \int_{B_\rho} |e(w_\rho^+)|^p dx\Big)^{\frac{1}{p}} < \delta, \qquad \text{for } \rho \le \hat\rho, \tag{3.20}$$

note that this is possible by (3.14). For $\rho < \hat\rho$ we define $\rho_k := (2^k \rho) \wedge \hat\rho$ and we adopt the notation $k$ in place of $\rho_k$ for the subscriptions. As above, using [22, Lemma 4.3] and the triangular inequality we infer

$$\|a_k - a_{k+1}\|_{L^\infty(B_{\rho_k}^+,\mathbb{R}^2)} \le \\ \frac{c}{{\rho_k}^2} \int_{\{u = w_k^+\}} |w_k^+ - a_k| dx + \frac{c}{\rho_{k+1}^2} \int_{\{u = w_{k+1}^+\}} |w_{k+1}^+ - a_{k+1}| dx \le c\delta \rho_k^{\frac{p-1}{p}},$$

where the last estimate follows by Hölder's inequality, (3.16), and (3.20). Therefore

$$|d_k - d_{k+1}| \le \|a_k - a_{k+1}\|_{L^\infty(B^+_{\rho_k},\mathbb{R}^2)} \le c\delta\rho_k^{\frac{p-1}{p}}, \tag{3.21}$$

and hence once more by [22, Lemma 4.3] and by the triangular inequality we conclude

$$|\beta_k - \beta_{k+1}| \le c\delta\rho_k^{-\frac{1}{p}}.$$

Collecting these estimates as $k$ varies we obtain

$$\rho|\beta_\rho|^p \le \rho\Big(|\hat\beta| + \sum_{k=0}^{\hat k-1}|\beta_k - \beta_{k+1}|\Big)^p \le c\delta^p + c\rho|\hat\beta|^p,$$

where $\hat k$ is the first index such that $\rho_{\hat k} = \hat\rho$ and $\hat\beta := \beta_{\hat k} = \beta_{\hat\rho}$. This proves (3.18) as $\rho \to 0$ and $\delta \to 0$.

We next prove (3.19). Similarly to the previous estimate, summing (3.21) gives

$$|d_\rho - d_{\hat\rho}| \le c\delta\,\hat\rho^{(p-1)/p}$$

for all $0 < \rho < \hat\rho \le \rho_\delta$, with $\delta$ arbitrary and $\rho_\delta$ depending only on $\delta$. Taking $\rho \to 0$ and using (3.17) yields

$$\hat\rho^{(1-p)/p}|u^+(x_0) - d_{\hat\rho}| \le c\delta$$

which, since $\delta$ was arbitrary, proves (3.19) and therefore (v).

At this point we turn to property (ii). Korn's inequality implies that

$$\begin{aligned}\|\nabla w^+_\rho\|_{L^p(B_\rho,\mathbb{R}^{2\times2})} &\le \|\nabla w^+_\rho - \beta_\rho\|_{L^p(B_\rho,\mathbb{R}^{2\times2})} + c\,\rho^{2/p}|\beta_\rho| \\ &\le c\,\|e(w^+_\rho)\|_{L^p(B_\rho,\mathbb{R}^{2\times2})} + c\,\rho^{2/p}|\beta_\rho|,\end{aligned}$$

where $c > 0$ is a universal constant. This, together with (3.14) and (3.18) and the corresponding estimates for $w^-_\rho$, implies property (ii).

We finally show property (vi). Note that by the trace theorem we have

$$\begin{aligned}&\frac{1}{\rho}\int_{\Gamma\cap B_\rho}|v^\pm_\rho - u^\pm|d\mathcal{H}^1 \le \\ &\qquad \frac{c}{\rho^2}\int_{B_\rho}|v_\rho - u|dx + \frac{c}{\rho}|E(v_\rho - u)|(B_\rho\setminus\Gamma) \le \\ &\frac{c}{\rho^2}\int_{B_\rho}|v_\rho - u|dx + \frac{c}{\rho}\int_{B_\rho}|e(v_\rho)|dx + \frac{c}{\rho}\int_{B_\rho}|e(u)|dx + \frac{c}{\rho}\int_{J_u\setminus\Gamma}|[u]|d\mathcal{H}^1\end{aligned}$$

and all terms in the last expression approach 0 respectively by (iv), (3.9), (3.14) and (3.8).

□

# 4 Integral representation

## 4.1 Preliminaries

In this Section we prove Theorem 1.1, along the lines of [11, Section 2.2].

Before starting we specify that property (ii) means that if $u_j, u \in SBD^p(\Omega)$ obey $u_j \to u$ in $L^1(\Omega, \mathbb{R}^2)$, then $F(u, A) \le \liminf_{j\to\infty} F(u_j, A)$ for any open set $A$. By property (iii), if $u, v \in SBD^p(\Omega)$ obey $u = v$ $\mathcal{L}^2$-a.e. in $A$, then $F(u, A) = F(v, A)$. The functions $f$ and $g$ are defined in (4.1) and (4.2) below. We recall that any $u \in SBD^p(\Omega)$ (actually, also any function in $BD(\Omega)$) for $\mathcal{L}^2$-a.e. $x \in \Omega$ has an approximate gradient $\nabla u(x) \in \mathbb{R}^{2\times 2}$, defined by the fact that

$$\lim_{\rho\to 0} \frac{1}{\rho^2}\mathcal{L}^2\Big(\Big\{y \in B_\rho(x) : \frac{|u(y) - u(x) - \nabla u(x)(y - x)|}{|y - x|} > \varepsilon\Big\}\Big) = 0$$

for every $\varepsilon > 0$ (see [4, Theorem 7.4]). It is easy to see that this definition implies $e(u) = (\nabla u + \nabla u)^T/2$.

The family of balls contained in $\Omega$ is denoted by

$$\mathcal{A}^*(\Omega) := \{B_\varepsilon(x) : x \in \Omega, \varepsilon > 0, B_\varepsilon(x) \subset \Omega\} .$$

Let $B \in \mathcal{A}^*(\Omega)$. We can identify any $u \in SBD^p(B)$ with its zero extension $u\chi_B \in SBD^p(\Omega)$, and correspondingly write $F(u, B)$ for $F(u\chi_B, B)$. By locality, for any other extension the value of the functional is the same.

For $B \in \mathcal{A}^*(\Omega)$ we define

$$m(u, B) := \inf\{F(w, B) : w \in SBD^p(B),\ w = u \text{ around } \partial B\}$$

where the condition $w = u$ around $\partial B$ means that a ball $B' \subset\subset B$ exists, so that $w = u$ on $B \setminus B'$. For $\delta > 0$, $A \in \mathcal{A}(\Omega)$, we set

$$m^\delta(u, A) := \inf\{\sum_{i=1}^{\infty} m(u, B_i) : B_i \in \mathcal{A}^*, B_i \cap B_j = \emptyset, B_i \subset A,$$
$$\operatorname{diam}(B_i) < \delta,\ \mu(A \setminus \bigcup_{i=1}^{\infty} B_i) = 0\} ,$$

where $\mu := \mathcal{L}^2 \llcorner \Omega + (1 + |[u]|)\mathcal{H}^1 \llcorner (J_u \cap \Omega)$.

Since $\delta \mapsto m^\delta(u, A)$ is decreasing, we can define

$$m^*(u, A) := \lim_{\delta\to 0} m^\delta(u, A).$$

Moreover, we set

$$f(x_0, u_0, \xi) := \limsup_{\varepsilon \to 0} \frac{m(u_0 + \xi(\cdot - x_0), B_\varepsilon(x_0))}{\mathcal{L}^2(B_\varepsilon)} \tag{4.1}$$

$$g(x_0, a, b, \nu) := \limsup_{\varepsilon \to 0} \frac{m(u_{x_0,a,b,\nu}, B_\varepsilon(x_0))}{2\varepsilon}, \tag{4.2}$$

where $u_{x_0,a,b,\nu}$ is defined as

$$u_{x_0,a,b,\nu}(x) := \begin{cases} a & \text{if } \langle x - x_0, \nu \rangle > 0, \\ b & \text{if } \langle x - x_0, \nu \rangle < 0. \end{cases}$$

In the next Lemmas we will see that $F$ is equivalent to $m$, in the sense that the two quantities have the same Radon-Nykodym derivative with respect to $\mu$, see Lemma 4.3 below. This will then be used in the next Section to determine the structure of $F$, separately for the diffuse part, which is absolutely continuos with respect to $\mathcal{L}^2$, and the jump part, which is orthogonal to it. We start by showing that $F = m^*$ on open sets (Lemma 4.1) and determining continuity of $m$ in the radius of the ball (Lemma 4.2).

**Lemma 4.1.** *For all $u \in SBD^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$, $F(u, A) = m^*(u, A)$.*

*Proof.* By definition, $m(u, B) \leq F(u, B)$ for any ball $B$. Since $F(u, \cdot)$ is a measure, we obtain $m^\delta(u, A) \leq F(u, A)$ for any $\delta > 0$. Therefore $m^*(u, A) \leq F(u, A)$.

To prove the converse inequality, let $\delta > 0$, pick countably many balls $B_i^\delta$ as in the definition of $m^\delta(u, A)$, such that

$$\sum_{i=1}^\infty m(u, B_i^\delta) < m^\delta(u, A) + \delta\,.$$

By the definition of $m$ there are functions $v_i^\delta \in SBD^p(B_i^\delta)$ such that $v_i^\delta = u$ around $\partial B_i^\delta$ and $F(v_i^\delta, B_i^\delta) \leq m(u, B_i^\delta) + \delta \mathcal{L}^2(B_i^\delta)$. We define

$$v^\delta := \sum_{i=1}^\infty v_i^\delta \chi_{B_i^\delta} + u \chi_{N_0^\delta}$$

where $N_0^\delta := \Omega \setminus \cup_i B_i^\delta$.

By the $BD$ compactness theorem $v^\delta \in BD(\Omega)$ and by the $SBD$ closure theorem (see also [29, Theorem 11.3]) we conclude that $v^\delta \in SBD^p(\Omega)$ and

$$Ev^\delta = \sum_{i=1}^\infty Ev_i^\delta \llcorner B_i^\delta + Eu \llcorner N_0^\delta\,,$$

with

$$|Ev^\delta|\llcorner N^\delta = 0, \quad \mu(N^\delta) = 0\,, \qquad F(v^\delta, N^\delta) = 0$$

where $N^\delta := A \cap N_0^\delta$. Further,

$$F(v^\delta, A) = \sum_{i=1}^{\infty} F(v_i^\delta, B_i^\delta) + F(v^\delta, N^\delta) \le m^\delta(u, A) + \delta + \delta\mathcal{L}^2(A)\,.$$

We claim that $v^\delta \to u$ in $L^1(\Omega, \mathbb{R}^2)$. Since $F(\cdot, A)$ is lower semicontinuous, this will imply

$$F(u, A) \le \liminf_{\delta\to 0} F(v^\delta, A) \le \liminf_{\delta\to 0} m^\delta(u, A) = m^*(u, A)\,.$$

To prove $v^\delta \to u$, we observe that by Poincaré's inequality (see for example [33, Proposition 1.7.6]), or [47, Theorem 2.2]), since diam $B_i^\delta \le \delta$ and $v^\delta = u$ on $\partial B_i^\delta$ we obtain

$$\|v^\delta - u\|_{L^1(B_i^\delta, \mathbb{R}^2)} \le c\delta|Ev^\delta - Eu|(B_i^\delta)\,.$$

Therefore

$$\begin{aligned}\|v^\delta - u\|_{L^1(\Omega, \mathbb{R}^2)} \le \sum_i \|v^\delta - u\|_{L^1(B_i^\delta, \mathbb{R}^2)} &\le c\delta(|Ev^\delta|(A) + |Eu|(A)) \\ &\le c\delta(F(v^\delta, A) + F(u, A))\,.\end{aligned}$$

Since $F(v^\delta, A)$ has a finite limit as $\delta \to 0$, this proves $v^\delta \to u$ in $L^1(\Omega, \mathbb{R}^2)$. □

**Lemma 4.2.** *For any ball $B_r(x_0) \subset \Omega$ and $\delta > 0$ sufficiently small we have*

*(i)* $\lim_{\delta\to 0} m(u, B_{r-\delta}(x_0)) = m(u, B_r(x_0))$*;*

*(ii)* $m(u, B_{r+\delta}(x_0))) \le m(u, B_r(x_0)) + \beta \int_{B_{r+\delta}(x_0)\setminus B_r(x_0)} (1 + |e(u)|^p)dx + \beta \int_{J_u \cap B_{r+\delta}(x_0)\setminus B_r(x_0)} (1 + |[u]|)d\mathcal{H}^1.$

*Proof.* We drop $x_0$ from the notation. Choose $v_\delta \in SBD^p(B_{r-\delta})$ with $v_\delta = u$ around $\partial B_{r-\delta}$ and $F(v_\delta, B_{r-\delta}) \le m(u, B_{r-\delta}) + \delta$. We define

$$w_\delta(x) := \begin{cases} v_\delta(x) & \text{if } x \in B_{r-\delta}\,, \\ u(x) & \text{if } x \in \Omega \setminus B_{r-\delta}\,. \end{cases}$$

We have

$$
\begin{aligned}
m(u, B_r) \leq & F(w_\delta, B_r) \leq F(v_\delta, B_{r-\delta}) + F(w_\delta, B_r \setminus B_{r-\delta}) \\
\leq & m(u, B_{r-\delta}) + \delta \\
& + \beta \int_{B_r \setminus B_{r-\delta}} (|e(u)|^p + 1) dx + \beta \int_{J_u \cap B_r \setminus B_{r-\delta}} (1 + |[u]|) d\mathcal{H}^{n-1} .
\end{aligned}
$$

Since $(1+|e(u)|^p)\mathcal{L}^2 \llcorner \Omega + (1+|[u]|)\mathcal{H}^1 \llcorner J_u$ is a bounded measure, we conclude that

$$
m(u, B_r) \leq \liminf_{\delta \to 0} m(u, B_{r-\delta}) .
$$

Conversely, for any $\varepsilon > 0$ there is $v_\varepsilon \in SBD^p(B_r)$ with $v_\varepsilon = u$ around $\partial B_r$ and $F(v_\varepsilon, B_r) \leq m(u, B_r) + \varepsilon$. For $\delta > 0$ sufficiently small one has $v_\varepsilon = u$ on $B_r \setminus B_{r-2\delta}$ and therefore $m(u, B_{r-\delta}) \leq F(v_\varepsilon, B_{r-\delta}) \leq m(u, B_r) + \varepsilon$. Taking first $\delta \to 0$ and then $\varepsilon \to 0$ concludes the proof of (i). The proof of (ii) is analogous. □

**Lemma 4.3.** *For $\mu$-a.e. $x_0 \in \Omega$,*

$$
\lim_{\varepsilon \to 0} \frac{F(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon \to 0} \frac{m(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} ,
$$

*where, as above, $\mu := \mathcal{L}^2 \llcorner \Omega + (1 + |[u]|)\mathcal{H}^1 \llcorner (J_u \cap \Omega)$.*

*Proof.* From $m(u, B_\varepsilon(x_0)) \leq F(u, B_\varepsilon(x_0))$ one immediately obtains

$$
\limsup_{\varepsilon \to 0} \frac{m(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} \leq \limsup_{\varepsilon \to 0} \frac{F(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))}
$$

for any $x_0 \in \Omega$. To prove the converse inequality, we define for $t > 0$ the set

$$
\begin{aligned}
E_t := \{ x \in \Omega : & \text{ there is } \varepsilon_h \to 0 \text{ such that} \\
& F(u, B_{\varepsilon_h}(x)) > m(u, B_{\varepsilon_h}(x)) + t\mu(B_{\varepsilon_h}(x)) \text{ for all } h \} .
\end{aligned}
$$

From this definition one immediately has

$$
\liminf_{\varepsilon \to 0} \frac{F(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} \leq \liminf_{\varepsilon \to 0} \frac{m(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} + t \quad \text{ for all } x_0 \in \Omega \setminus E_t .
$$

If we can prove that

$$
\mu(E_t) = 0 \text{ for all } t > 0 \tag{4.3}
$$

then, recalling that $\lim_{\varepsilon \to 0} \frac{F(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))}$ exists $\mu$-almost everywhere, the proof is concluded.

It remains to prove (4.3) for an arbitrary $t > 0$. For $\delta > 0$ we define

$$X^\delta := \{B_\varepsilon(x) : \varepsilon < \delta,\ \ \overline{B}_\varepsilon(x) \subset \Omega,\ \ \mu(\partial B_\varepsilon(x)) = 0,\\ F(u, B_\varepsilon(x)) > m(u, B_\varepsilon(x)) + t\mu(B_\varepsilon(x))\}$$

and

$$U^* := \bigcap_{\delta>0} \{x : \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \in X^\delta\}\,.$$

We first show that $E_t \subset U^*$. Let $x \in E_t$. Then for any $\delta > 0$ there is $\varepsilon \in (0, \delta)$ such that $F(u, B_\varepsilon(x)) > m(u, B_\varepsilon(x)) + t\mu(B_\varepsilon(x))$. By Lemma 4.2 the function $\varepsilon \to m(u, B_\varepsilon(x))$ is left-continuous; $F(u, B_\varepsilon(x))$ is left-continuous because $F(u, \cdot)$ is a measure, therefore the same inequality holds for all $\varepsilon' \in (\varepsilon'', \varepsilon)$. In particular, there is one which additionally obeys $\mu(\partial B_\varepsilon(x)) = 0$, so that $x \in U^*$.

It remains to show that $\mu(U^*) = 0$. We fix a compact set $K \subset U^*$ and $0 < \delta < \eta$. Let $U^\eta := \bigcup\{B_\varepsilon(x) : B_\varepsilon(x) \in X^\eta\}$ and

$$Y^\delta := \{B_\varepsilon(x) : \varepsilon < \delta, \overline{B_\varepsilon(x)} \subset U^\eta \setminus K, \mu(\partial B_\varepsilon(x)) = 0\}\,.$$

By definition, $X^\delta$ is a fine cover of $K$ and $Y^\delta$ of $U^\eta \setminus K$. Therefore there are countably many pairwise disjoint balls $B_i \in X^\delta$ and $\hat{B}_j \in Y^\delta$ and a set $N$ with $\mu(N) = 0$ such that

$$U^\eta = \left(\bigcup_{i\in\mathbb{N}} B_i\right) \cup \left(\bigcup_{j\in\mathbb{N}} \hat{B}_j\right) \cup N\,.$$

Then

$$\begin{aligned}
F(u, U^\eta) &= \sum_i F(u, B_i) + \sum_j F(u, \hat{B}_j) + F(u, N)\\
&\geq \sum_i (m(u, B_i) + t\mu(B_i)) + \sum_j m(u, \hat{B}_j)\\
&= \sum_i m(u, B_i) + \sum_j m(u, \hat{B}_j) + t\mu(\cup_i B_i)\\
&\geq m^\delta(u, U^\eta) + t\mu(K)
\end{aligned}$$

where in the last step we used the definition of $m^\delta$. For $\delta \to 0$, the definition of $m^*$ and Lemma 4.1 give

$$F(u, U^\eta) \geq m^*(u, U^\eta) + t\mu(K) = F(u, U^\eta) + t\mu(K)\,.$$

Therefore $\mu(K) = 0$, and by the regularity of $\mu$ we conclude $\mu(U^*) = 0$.

□

## 4.2 Bounds on the volume term

In this subsection we identify the volume energy density in the integral representation for $F$ to be the function $f$ defined in (4.1). Throughout the whole subsection we consider a fixed map $u \in SBD^p(\Omega)$. Our first result shows that the local volume energy density can be computed with a $W^{1,p}$-approximation to the blow-ups of $u$ (see (4.7–4.8) below), in the sense that

$$\frac{dF(u,\cdot)}{d\mathcal{L}^2}(x_0) = \lim_{\varepsilon\to 0} \frac{m\big(w_\varepsilon, B_\varepsilon(x_0)\big)}{\mathcal{L}^2(B_\varepsilon)}. \tag{4.4}$$

We will however not need (4.4), but only the apparently more complex version in (4.5)-(4.6). Taking a diagonal subsequence they imply (4.4).

**Lemma 4.4.** *For $\mathcal{L}^2$-almost any $x_0 \in \Omega$, any $\varepsilon > 0$, and any $s \in (0,1)$ there are functions $w_\varepsilon^s \in W^{1,p}(B_{s\varepsilon}(x_0);\mathbb{R}^2)$ which obey*

$$\frac{dF(u,\cdot)}{d\mathcal{L}^2}(x_0) \le \liminf_{s\to 1} \liminf_{\varepsilon\to 0} \frac{m\big(w_\varepsilon^s, B_{s\varepsilon}(x_0)\big)}{\mathcal{L}^2(B_{s\varepsilon})} \tag{4.5}$$

*and*

$$\limsup_{s\to 1} \limsup_{\varepsilon\to 0} \frac{m\big(w_\varepsilon^s, B_{s^2\varepsilon}(x_0)\big)}{\mathcal{L}^2(B_{s\varepsilon})} \le \frac{dF(u,\cdot)}{d\mathcal{L}^2}(x_0) \tag{4.6}$$

*and which approximate the affine function $y \mapsto \nabla u(x_0)(y-x_0) + u(x_0)$ in the sense that*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{B_\varepsilon(x_0)} |e(w_\varepsilon^s) - e(u)(x_0)|^p dx = 0 \tag{4.7}$$

*and*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{2+p}} \int_{B_\varepsilon(x_0)} |w_\varepsilon^s(x) - u(x_0) - \nabla u(x_0)(x-x_0)|^p dx = 0\,. \tag{4.8}$$

We remark that the ball in (4.6) has radius $s^2\varepsilon$ instead of $s\varepsilon$. The estimate would also hold on $B_{s\varepsilon}$, the variant we chose is more convenient in the proof of Lemma 4.7 (cp. (4.12)).

*Proof.* Let $x_0 \in \Omega$ be such that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{B_\varepsilon(x_0)} |e(u)(x) - e(u)(x_0)|^p dx = 0\,, \tag{4.9}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{B_\varepsilon(x_0)\cap J_u}(1+|[u]|)d\mathcal{H}^1=0\,, \tag{4.10}$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{B_\varepsilon(x_0)}|u(x)-u(x_0)-\nabla u(x_0)(x-x_0)|dx=0\,. \tag{4.11}$$

By [4, Th. 7.4], $\mathcal{L}^2$-almost every $x_0$ obeys (4.11), the other two are standard.

By (4.10), for sufficiently small $\varepsilon$ one has $\mathcal{H}^1(J_u\cap B_\varepsilon(x_0))\le\eta(1-s)\varepsilon/2$, where $\eta$ is the constant from Theorem 2.1. By Proposition 3.2 applied to $u-u(x_0)-\nabla u(x_0)(\cdot-x_0)$ there is $\tilde w^s_\varepsilon\in SBD^p(B_\varepsilon(x_0))\cap W^{1,p}(B_{s\varepsilon}(x_0);\mathbb{R}^2)$ with properties (i)-(vii), then we set $w^s_\varepsilon:=\tilde w^s_\varepsilon+u(x_0)+\nabla u(x_0)(\cdot-x_0)$. In particular, (4.7) follows from (3.3) and (4.9), while (4.8) follows from Lemma 4.5 below applied to $\tilde w^s_\varepsilon$, estimating the right-hand side with (4.7), (vi), and (4.9)-(4.11).

We first prove (4.6). By the very definition of $m$ and the fact that $F(w^s_\varepsilon,\cdot)$ is a positive measure, it follows

$$m(w^s_\varepsilon,B_{s^2\varepsilon}(x_0))\le F(w^s_\varepsilon,B_{s^2\varepsilon}(x_0))\le F(w^s_\varepsilon,B_\varepsilon(x_0))\,. \tag{4.12}$$

Let $(B_i)_{i\in\mathbb{N}}$ be the balls from Proposition 3.2. For $M\in\mathbb{N}$ we define

$$w^{s,M}_\varepsilon:=u+\chi_{\cup_{i=1}^M\overline{B}_i}(w^s_\varepsilon-u)\,.$$

Then $w^{s,M}_\varepsilon\in SBD^p(B_\varepsilon(x_0))$ and $w^{s,M}_\varepsilon\to w^s_\varepsilon$ in $L^1$ as $M\to\infty$. Further,

$$\begin{aligned}F(w^{s,M}_\varepsilon,B_\varepsilon(x_0))\le&F(w^{s,M}_\varepsilon,B_\varepsilon(x_0)\setminus\cup_{i=1}^M\overline{B}_i)+\sum_{i=1}^M F(w^{s,M}_\varepsilon,\overline{B}_i)\\ \le&F(u,B_\varepsilon(x_0)\setminus\cup_{i=1}^M\overline{B}_i)+\beta\sum_{i=1}^M\int_{\overline{B}_i}(1+|e(w^s_\varepsilon)|^p)dx\end{aligned}$$

since $w^{s,M}_\varepsilon=w^s_\varepsilon$ is a $W^{1,p}$ function on each $\overline{B}_i$. By monotonicity and lower semicontinuity of $F$ we obtain

$$\begin{aligned}F(w^s_\varepsilon,B_\varepsilon(x_0))\le&F(u,B_\varepsilon(x_0))+\beta\sum_{i=1}^\infty\int_{\overline{B}_i}(1+|e(w^s_\varepsilon)|^p)dx\\ \le&F(u,B_\varepsilon(x_0))+c\mathcal{L}^2(\cup_iB_i)(1+|e(u)|^p(x_0))\\ &+c\int_{B_\varepsilon(x_0)}|e(w^s_\varepsilon)-e(u)(x_0)|^pdx\end{aligned}$$

and, recalling Proposition 3.2 (i), conclude the proof of (4.6) by (4.7) and (4.9).

It remains to prove (4.5). Let $v_\varepsilon \in SBD^p(B_{s\varepsilon}(x_0))$ be such that $v_\varepsilon = w^s_\varepsilon$ around $\partial B_{s\varepsilon}(x_0)$ and $F(v_\varepsilon, B_{s\varepsilon}(x_0)) \le m(w^s_\varepsilon, B_{s\varepsilon}(x_0)) + \varepsilon^3$. We define

$$\tilde v_\varepsilon(x) := \begin{cases} v_\varepsilon(x) & \text{if } x \in B_{s\varepsilon}(x_0) \\ w^s_\varepsilon(x) & \text{if } x \in B_\varepsilon(x_0) \setminus B_{s\varepsilon}(x_0)\,. \end{cases}$$

By definition of $m$ and additivity of $F$ we obtain

$$m(u, B_\varepsilon(x_0)) \le F(\tilde v_\varepsilon, B_\varepsilon(x_0)) = F(\tilde v_\varepsilon, B_{s\varepsilon}(x_0)) + F(\tilde v_\varepsilon, B_\varepsilon(x_0) \setminus B_{s\varepsilon}(x_0))$$

where by locality of $F$ and definition of $v_\varepsilon$

$$F(\tilde v_\varepsilon, B_{s\varepsilon}(x_0)) = F(v_\varepsilon, B_{s\varepsilon}(x_0)) \le m(w^s_\varepsilon, B_{s\varepsilon}(x_0)) + \varepsilon^3$$

and, since $\tilde v_\varepsilon = w^s_\varepsilon$ outside $B_{s\varepsilon}(x_0)$ and $\mathcal{H}^1(J_{\tilde v_\varepsilon} \cap \partial B_{s\varepsilon}(x_0)) = 0$, recalling (3.3) we obtain

$$\begin{aligned} F(\tilde v_\varepsilon, B_\varepsilon(x_0) \setminus B_{s\varepsilon}(x_0)) \le & \beta \int_{B_\varepsilon(x_0)\setminus B_{s\varepsilon}(x_0)} (1 + |e(w^s_\varepsilon)|^p) dx \\ & + \beta \int_{J_u \cap B_\varepsilon(x_0) \setminus B_{s^2\varepsilon}(x_0)} (1 + |[u]|) d\mathcal{H}^1 \\ \le & c\beta \mathcal{L}^2(B_\varepsilon)(1 - s^2)(1 + |e(u)|^p(x_0)) \\ & + c\beta \int_{B_\varepsilon(x_0)} |e(w^s_\varepsilon)(x) - e(u)(x_0)|^p dx \\ & + \beta \int_{J_u \cap B_\varepsilon(x_0)} (1 + |[u]|) d\mathcal{H}^1\,. \end{aligned}$$

Dividing by $\mathcal{L}^2(B_\varepsilon)$ and taking the limit as $\varepsilon \to 0$ gives

$$\lim_{\varepsilon\to0} \frac{m(u, B_\varepsilon(x_0))}{\mathcal{L}^2(B_\varepsilon)} \le \liminf_{\varepsilon\to0} \frac{m(w^s_\varepsilon, B_{s\varepsilon}(x_0))}{\mathcal{L}^2(B_\varepsilon)} + c\beta(1 - s^2)(1 + |e(u)|^p(x_0))\,,$$

where we used (4.7) and (4.10). Recalling Lemma 4.3 we obtain

$$\frac{dF(u, \cdot)}{d\mathcal{L}^2}(x_0) = \lim_{\varepsilon\to0} \frac{m(u, B_\varepsilon(x_0))}{\mathcal{L}^2(B_\varepsilon)} \le \liminf_{s\to1} \liminf_{\varepsilon\to0} \frac{m(w^s_\varepsilon, B_{s\varepsilon}(x_0))}{\mathcal{L}^2(B_{s\varepsilon})}\,.$$

This concludes the proof of (4.5). □

The next Lemma is a reverse-Hölder estimate for functions with small strain, of the form $\|v\|_p \le r\|e(v)\|_p + r^{-n/p'}\|v\|_1$.

**Lemma 4.5.** *For any $p \geq 1$ there is $c > 0$ (depending on $n$ and $p$) such that for any $v \in W^{1,p}(B_r;\mathbb{R}^n)$ one has*

$$\frac{1}{r^{n+p}}\int_{B_r}|v|^p dx \leq c\frac{1}{r^n}\int_{B_r}|e(v)|^p dx + c\left(\frac{1}{r^{n+1}}\int_{B_r}|v|dx\right)^p .$$

*Proof.* By scaling it suffices to consider $r = 1$. By Korn's inequality there is an affine function $a$ such that

$$\int_{B_1}|v-a|^p dx \leq c\int_{B_1}|e(v)|^p dx .$$

Since $a$ is affine,

$$\int_{B_1}|a|^p dx \leq c\left(\int_{B_1}|a|dx\right)^p \leq c\left(\int_{B_1}|v|dx\right)^p + c\int_{B_1}|v-a|^p dx .$$

A triangular inequality concludes the proof. □

**Lemma 4.6.** *For $\mathcal{L}^2$-a.e. $x_0 \in \Omega$,*

$$\frac{dF(u,\cdot)}{d\mathcal{L}^2}(x_0) \leq f(x_0, u(x_0), \nabla u(x_0))$$

*where $f$ was defined in (4.1).*

*Proof.* Let $x_0$, $w^s_\varepsilon$ be as in Lemma 4.4, for $s \in (0,1)$. We choose $v^s_\varepsilon \in SBD^p(B_{s^2\varepsilon}(x_0))$ such that $v^s_\varepsilon(x) = u(x_0)+\nabla u(x_0)(x-x_0)$ around $\partial B_{s^2\varepsilon}(x_0)$ and $F(v^s_\varepsilon, B_{s^2\varepsilon}(x_0)) \leq m(u(x_0)+\nabla u(x_0)(\cdot - x_0), B_{s^2\varepsilon}(x_0))+\varepsilon^3$. We extend it to $\mathbb{R}^2$ setting it equal to $u(x_0)+\nabla u(x_0)(\cdot - x_0)$ outside $B_{s^2\varepsilon}(x_0)$ and choose $\varphi \in C^\infty_c(B_{s\varepsilon}(x_0))$ with $\varphi = 1$ on $B_{s^2\varepsilon}(x_0)$ and $\|D\varphi\|_\infty \leq c/(s(1-s)\varepsilon)$. We define

$$z^s_\varepsilon := \varphi v^s_\varepsilon + (1-\varphi)w^s_\varepsilon .$$

We remark that $z^s_\varepsilon = v^s_\varepsilon$ on $B_{s^2\varepsilon}(x_0)$ and $z^s_\varepsilon \in W^{1,p}(B_{s\varepsilon}(x_0)\setminus B_{s^2\varepsilon}(x_0);\mathbb{R}^2)$. Then

$$\begin{aligned} m(w^s_\varepsilon, B_{s\varepsilon}(x_0)) \leq& F(z^s_\varepsilon, B_{s\varepsilon}(x_0)) \leq F(v^s_\varepsilon, B_{s^2\varepsilon}(x_0)) + F(z^s_\varepsilon, B_{s\varepsilon}(x_0)\setminus B_{s^2\varepsilon}(x_0)) \\ \leq& m(u(x_0)+\nabla u(x_0)(\cdot - x_0), B_{s^2\varepsilon}(x_0)) + \varepsilon^3 \\ &+ \beta\int_{B_{s\varepsilon}(x_0)\setminus B_{s^2\varepsilon}(x_0)}(1+|e(z^s_\varepsilon)|^p)dx . \end{aligned}$$

In order to estimate the error term, we observe that in $B_{s\varepsilon}(x_0)\setminus B_{s^2\varepsilon}(x_0)$ one has

$$\nabla z^s_\varepsilon - \nabla u(x_0) = (u(x_0)+\nabla u(x_0)(\cdot - x_0) - w^s_\varepsilon)\otimes\nabla\varphi + (1-\varphi)(\nabla w^s_\varepsilon - \nabla u(x_0))$$

which implies

$$\begin{aligned}\int_{B_{s\varepsilon}(x_0)\setminus B_{s^2\varepsilon}(x_0)}(1+|e(z^s_\varepsilon)|^p)dx \le & c(1-s)\mathcal{L}^2(B_{s\varepsilon})(1+|e(u)|^p(x_0))\\ &+c\int_{B_{s\varepsilon}(x_0)}|e(w^s_\varepsilon)-e(u)(x_0)|^p dx\\ &+c\int_{B_{s\varepsilon}(x_0)}\frac{|u(x_0)+\nabla u(x_0)(x-x_0)-w^s_\varepsilon|^p}{\varepsilon^p s^p(1-s)^p}dx.\end{aligned}$$

Therefore, recalling (4.7) and (4.8),

$$\limsup_{\varepsilon\to0}\frac{F(z^s_\varepsilon,B_{s\varepsilon}(x_0)\setminus B_{s^2\varepsilon}(x_0))}{\mathcal{L}^2(B_{s\varepsilon})}\le c(1-s)(1+|e(u)|^p(x_0))$$

and

$$\begin{aligned}\limsup_{\varepsilon\to0}\frac{m(w^s_\varepsilon,B_{s\varepsilon}(x_0))}{\mathcal{L}^2(B_{s\varepsilon})}\le&\limsup_{\varepsilon\to0}\frac{m(u(x_0)+\nabla u(x_0)(\cdot-x_0),B_{s^2\varepsilon}(x_0))}{\mathcal{L}^2(B_{s\varepsilon})}\\ &+c(1-s)(1+|e(u)|^p(x_0))\\ =&s^2f(x_0,u_0,\nabla u(x_0))+c(1-s)(1+|e(u)|^p(x_0))\,.\end{aligned}$$

Since $s$ was arbitrary, this concludes the proof. □

**Lemma 4.7.** *For $\mathcal{L}^2$-a.e. $x_0\in\Omega$,*

$$f(x_0,u(x_0),\nabla u(x_0))\le\frac{dF(u,\cdot)}{d\mathcal{L}^2}(x_0)$$

*where $f$ was defined in (4.1).*

*Proof.* We choose $x_0$ and $w^s_\varepsilon$ as in Lemma 4.4, for $s\in(0,1)$. We let $v^s_\varepsilon\in SBD^p(B_{s^2\varepsilon}(x_0))$ be such that $v^s_\varepsilon=w^s_\varepsilon$ around $\partial B_{s^2\varepsilon}(x_0)$ and $F(v^s_\varepsilon,B_{s^2\varepsilon}(x_0))\le m(w^s_\varepsilon,B_{s^2\varepsilon}(x_0))+\varepsilon^3$, and extend it to $B_{s\varepsilon}(x_0)$ setting it equal to $w^s_\varepsilon$ outside $B_{s^2\varepsilon}(x_0)$. We choose $\varphi\in C^\infty_c(B_{s\varepsilon}(x_0))$ with $\varphi=1$ on $B_{s^2\varepsilon}(x_0)$ and $\|D\varphi\|_\infty\le c/(s(1-s)\varepsilon)$ and define

$$z^s_\varepsilon:=\varphi v^s_\varepsilon+(1-\varphi)(u(x_0)+\nabla u(x_0)(x-x_0))\,.$$

Then

$$\begin{aligned}m(u(x_0)+\nabla u(x_0)(\cdot-x_0),&B_{s\varepsilon}(x_0))\le F(z^s_\varepsilon,B_{s\varepsilon}(x_0))\\ &=F(v^s_\varepsilon,B_{s^2\varepsilon}(x_0))+F(z^s_\varepsilon,B_{s\varepsilon}(x_0)\setminus B_{s^2\varepsilon}(x_0))\\ &\le m(w^s_\varepsilon,B_{s^2\varepsilon}(x_0))+\varepsilon^3+F(z^s_\varepsilon,B_{s\varepsilon}(x_0)\setminus B_{s^2\varepsilon}(x_0))\,.\end{aligned}$$

In order to estimate the error term, we observe that in $B_{s\varepsilon}(x_0)\setminus B_{s^2\varepsilon}(x_0)$ one has

$$\nabla z^s_\varepsilon - \nabla u(x_0) = -(u(x_0)+\nabla u(x_0)(\cdot - x_0) - w^s_\varepsilon)\otimes\nabla\varphi + \varphi(\nabla w^s_\varepsilon - \nabla u(x_0))$$

which leads as in the proof of Lemma 4.6 to

$$\limsup_{\varepsilon\to 0}\frac{F(z^s_\varepsilon, B_{s\varepsilon}(x_0)\setminus B_{s^2\varepsilon}(x_0))}{\mathcal{L}^2(B_{s\varepsilon})} \le c(1-s)(1+|e(u)|^p(x_0))\,.$$

We conclude that for any $s\in(0,1)$

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\frac{m(u(x_0)+\nabla u(x_0)(\cdot - x_0), B_{s\varepsilon}(x_0))}{\mathcal{L}^2(B_{s\varepsilon})}\\ &\le \limsup_{\varepsilon\to 0}\frac{m(w^s_\varepsilon, B_{s^2\varepsilon}(x_0))}{\mathcal{L}^2(B_{s\varepsilon})} + c(1-s)(1+|e(u)|^p(x_0))\,.\end{aligned}$$

Since $s$ was arbitrary, this concludes the proof. □

### 4.3 Bounds on the surface term

In the current subsection we identify the function $g$ in (4.2) to be the surface energy density in the integral representation of $F$. As above, we work with a fixed map $u\in SBD^p(\Omega)$.

We first prove a technical result.

**Lemma 4.8.** *For $\mathcal{H}^1$-a.e. $x_0\in J_u$ there are functions $w_\varepsilon\in SBV^p(B_{2\varepsilon}(x_0),\mathbb{R}^2)$ satisfying for all $t\in(0,2)$*

$$\frac{dF(u,\cdot)}{d\mathcal{H}^1\llcorner J_u}(x_0) = \lim_{\varepsilon\to 0}\frac{m(w_\varepsilon, B_{t\varepsilon}(x_0))}{2t\varepsilon}. \tag{4.13}$$

*Proof.* It suffices to consider points $x_0$ such that the conclusions of Lemmata 3.4 and 4.3 hold true, the Radon-Nikodym derivative $\frac{dF(u,\cdot)}{d\mathcal{H}^1\llcorner J_u}(x_0)$ exists finite,

$$\lim_{\varepsilon\to 0}\frac{\mu(B_\varepsilon(x_0))}{2\varepsilon} = 1+|[u](x_0)|, \tag{4.14}$$

and

$$\lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon}\int_{B_\varepsilon(x_0)}|e(u)|^p dx + \frac{1}{\varepsilon^2}\int_{B_\varepsilon(x_0)}|u(x)-u_{x_0}|dx\Big) = 0, \tag{4.15}$$

where $u_{x_0}$ is the piecewise constant function defined in (3.7). In view of all these choices and thanks to Lemma 4.3 we may conclude that

$$\frac{dF(u,\cdot)}{d\mathcal{H}^1\llcorner J_u}(x_0) = \lim_{\varepsilon\to 0}\frac{F(u,B_\varepsilon(x_0))}{2\varepsilon} = \lim_{\varepsilon\to 0}\frac{m(u,B_\varepsilon(x_0))}{2\varepsilon}. \tag{4.16}$$

For $\varepsilon > 0$ small enough the function $v_{2\varepsilon}$ introduced in Lemma 3.4 belongs to $SBD^p(B_{4\varepsilon}(x_0)) \cap SBV^p(B_{2\varepsilon}(x_0), \mathbb{R}^2)$ and it satisfies properties (i)-(vi). We set $w_\varepsilon := v_{2\varepsilon}$, we are left with proving that for all $t \in (0, 2)$

$$\frac{dF(u,\cdot)}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq \limsup_{\varepsilon \to 0} \frac{m(w_\varepsilon, B_{t\varepsilon}(x_0))}{2t\varepsilon}, \tag{4.17}$$

$$\frac{dF(u,\cdot)}{d\mathcal{H}^1 \llcorner J_u}(x_0) \leq \liminf_{\varepsilon \to 0} \frac{m(w_\varepsilon, B_{t\varepsilon}(x_0))}{2t\varepsilon}. \tag{4.18}$$

For the sake of notational simplicity we will prove inequalities (4.17) and (4.18) only for $t = 1$.

We start off with (4.17). Let $(\varepsilon_j)_j$ be a sequence such that

$$\lim_{j \to \infty} \frac{m(w_{\varepsilon_j}, B_{\varepsilon_j}(x_0))}{2\varepsilon_j} = \limsup_{\varepsilon \to 0} \frac{m(w_\varepsilon, B_\varepsilon(x_0))}{2\varepsilon}. \tag{4.19}$$

Items (iii) and (iv) in Lemma 3.4 and the Coarea formula yield for a subsequence not relabeled for convenience that for $\mathcal{L}^1$-a.e. $s \in (0, 1)$

$$\lim_{j \to \infty} \frac{1}{\varepsilon_j} \int_{\partial B_{s\varepsilon_j}(x_0) \cap \{u \neq w_{\varepsilon_j}\}} (1 + |u - w_{\varepsilon_j}|) d\mathcal{H}^1 = 0, \tag{4.20}$$

$$\mu\big(\partial B_{s\varepsilon_j}(x_0)\big) = \mathcal{H}^1\big(\partial B_{s\varepsilon_j}(x_0) \cap J_{w_{\varepsilon_j}}\big) = 0. \tag{4.21}$$

We choose $z_j \in SBD^p(B_{s\varepsilon_j}(x_0))$ such that $z_j = u$ around $\partial B_{s\varepsilon_j}(x_0)$ and

$$F(z_j, B_{s\varepsilon_j}(x_0)) \leq m(u, B_{s\varepsilon_j}(x_0)) + \varepsilon_j^2,$$

and define

$$\zeta_j := \begin{cases} z_j & B_{s\varepsilon_j}(x_0) \\ w_{\varepsilon_j} & B_{\varepsilon_j}(x_0) \setminus B_{s\varepsilon_j}(x_0). \end{cases}$$

The definition of $z_j$, the growth conditions in (1.1), and the locality of $F$ yield

$$\begin{aligned} m(w_{\varepsilon_j}, B_{\varepsilon_j}(x_0)) &\leq F(\zeta_j, B_{\varepsilon_j}(x_0)) \\ &\leq F(z_j, B_{s\varepsilon_j}(x_0)) + \underbrace{\beta \int_{B_{\varepsilon_j}(x_0) \setminus B_{s\varepsilon_j}(x_0)} (1 + |e(w_{\varepsilon_j})|^p)\, dx}_{=: I_j^{(1)}} \\ &+ \underbrace{\beta \int_{\partial B_{s\varepsilon_j}(x_0) \cap \{u \neq w_{\varepsilon_j}\}} (1 + |u - w_{\varepsilon_j}|) d\mathcal{H}^1}_{=: I_j^{(2)}} + \underbrace{\beta \int_{(B_{\varepsilon_j}(x_0) \setminus \overline{B_{s\varepsilon_j}(x_0)}) \cap J_{w_{\varepsilon_j}}} (1 + |[w_{\varepsilon_j}]|) d\mathcal{H}^1}_{=: I_j^{(3)}} \\ &\leq m(u, B_{s\varepsilon_j}(x_0)) + \varepsilon_j^2 + I_j^{(1)} + I_j^{(2)} + I_j^{(3)}. \end{aligned}$$

We note that $I_j^{(1)}$ and $I_j^{(2)}$ are $o(\varepsilon_j)$ as $j \to \infty$ thanks to Lemma 3.4 (ii) and (4.20), respectively. Instead, employing Lemma 3.4 (vi) and (4.14) to bound $I_j^{(3)}$ we infer that

$$\limsup_{j\to\infty} \frac{I_j^{(3)}}{2\varepsilon_j} \leq \limsup_{j\to\infty} \frac{\beta}{2\varepsilon_j} \int_{((B_{\varepsilon_j}(x_0)\setminus\overline{B_{s\varepsilon_j}(x_0)})\cap J_u} (1+|[u]|)d\mathcal{H}^1$$
$$= \beta \limsup_{j\to\infty} \frac{\mu\big((B_{\varepsilon_j}(x_0)\setminus\overline{B_{s\varepsilon_j}(x_0)})\cap J_u\big)}{2\varepsilon_j} = (1-s)\beta(1+|[u](x_0)|). \quad (4.22)$$

Therefore, by (4.16) we conclude

$$\lim_{j\to\infty} \frac{m(w_{\varepsilon_j}, B_{\varepsilon_j}(x_0))}{2\varepsilon_j} \leq \liminf_{j\to\infty} \frac{m(u, B_{s\varepsilon_j}(x_0))}{2\varepsilon_j} + (1-s)\beta(1+|[u](x_0)|)$$
$$= s\frac{dF(u,\cdot)}{d\mathcal{H}^1 \llcorner J_u}(x_0) + (1-s)\beta(1+|[u](x_0)|).$$

Estimate (4.17) follows at once by (4.19) and by letting $s \uparrow 1$ in the last inequality.

Let now $(\varepsilon_j)_j$ be a sequence such that

$$\lim_{j\to\infty} \frac{m(w_{\varepsilon_j}, B_{\varepsilon_j}(x_0))}{2\varepsilon_j} = \liminf_{\varepsilon\to 0} \frac{m(w_\varepsilon, B_\varepsilon(x_0))}{2\varepsilon}. \quad (4.23)$$

Let $\lambda \in (1,2)$, arguing as for (4.20) and (4.21), up to a subsequence depending on $\lambda$ and not relabeled for convenience we may assume that for $\mathcal{L}^1$-a.e. $s \in (0,1)$

$$\lim_{j\to\infty} \frac{1}{\varepsilon_j} \int_{\partial B_{s\lambda\varepsilon_j}(x_0)\cap\{u\neq w_{\varepsilon_j}\}} \big(1+|u-w_{\varepsilon_j}|\big)d\mathcal{H}^1 = 0, \quad (4.24)$$

and

$$\mu\big(\partial B_{s\lambda\varepsilon_j}(x_0)\big) = \mathcal{H}^1\big(\partial B_{s\lambda\varepsilon_j}(x_0)\cap J_{w_{\varepsilon_j}}\big) = 0. \quad (4.25)$$

Given $z_j \in SBD^p(B_{s\lambda\varepsilon_j}(x_0))$ with $z_j = w_{\varepsilon_j}$ around $\partial B_{s\lambda\varepsilon_j}(x_0)$ and such that

$$F(z_j, B_{s\lambda\varepsilon_j}(x_0)) \leq m(w_{\varepsilon_j}, B_{s\lambda\varepsilon_j}(x_0)) + \varepsilon_j^2,$$

define

$$\zeta_j := \begin{cases} z_j & B_{s\lambda\varepsilon_j}(x_0) \\ u & B_{\lambda\varepsilon_j}(x_0)\setminus B_{s\lambda\varepsilon_j}(x_0). \end{cases}$$

Using $\zeta_j$ as a test field for $m(u, B_{\lambda\varepsilon_j}(x_0))$, by the locality of $F$ and its growth conditions in (1.1)

$$
\begin{aligned}
m(u, B_{\lambda\varepsilon_j}(x_0)) \le F(\zeta_j, B_{\lambda\varepsilon_j}(x_0)) \le m(w_{\varepsilon_j}, B_{s\lambda\varepsilon_j}(x_0)) + \varepsilon_j^2 \\
+ \underbrace{\beta \int_{B_{\lambda\varepsilon_j}(x_0)} (1 + |e(u)|^p)\, dx}_{I_j^{(4)}} + \underbrace{\beta \int_{\partial B_{s\lambda\varepsilon_j}(x_0) \cap \{u \neq w_{\varepsilon_j}\}} (1 + |u - w_{\varepsilon_j}|) d\mathcal{H}^1}_{I_j^{(5)}} \\
+ \underbrace{\beta \int_{(B_{\lambda\varepsilon_j}(x_0) \setminus \overline{B_{s\lambda\varepsilon_j}(x_0)}) \cap J_u} (1 + |[u]|) d\mathcal{H}^1}_{I_j^{(6)}} .
\end{aligned}
$$

The terms $I_j^{(4)}$ and $I_j^{(5)}$ are $o(\varepsilon_j)$ by (4.15) and (4.24), respectively. The term $I_j^{(6)}$ can be estimated thanks to (4.14). Hence, we get by (4.16)

$$
\begin{aligned}
\frac{dF(u,\cdot)}{d\mathcal{H}^1 \llcorner J_u}(x_0) &= \limsup_{j\to\infty} \frac{m(u, B_{\lambda\varepsilon_j}(x_0))}{2\lambda\varepsilon_j} \\
&\le \limsup_{j\to\infty} \frac{m(w_{\varepsilon_j}, B_{s\lambda\varepsilon_j}(x_0))}{2\lambda\varepsilon_j} + (1-s)\beta(1 + |[u]|(x_0)).
\end{aligned} \tag{4.26}
$$

Next, by choosing $s \in (0,1)$ for which (4.24) and (4.25) hold and $s\lambda > 1$, we may use Lemma 4.2(ii) to infer

$$
\begin{aligned}
m(w_{\varepsilon_j}, B_{s\lambda\varepsilon_j}(x_0)) \le & m(w_{\varepsilon_j}, B_{\varepsilon_j}(x_0)) + \beta \int_{B_{s\lambda\varepsilon_j}(x_0) \setminus B_{\varepsilon_j}(x_0)} (1 + |e(w_{\varepsilon_j})|^p) dx \\
& + \beta \int_{(B_{s\lambda\varepsilon_j}(x_0) \setminus B_{\varepsilon_j}(x_0)) \cap J_{w_{\varepsilon_j}}} (1 + |[w_{\varepsilon_j}]|) d\mathcal{H}^1.
\end{aligned} \tag{4.27}
$$

Clearly, the first integral is $o(\varepsilon_j)$ by Lemma 3.4 (ii), while the other one can be dealt with as $I_j^{(3)}$ in (4.22). Thus, (4.26) and (4.27) give

$$
\frac{dF(u,\cdot)}{d\mathcal{H}^1 \llcorner J_u}(x_0) \le \frac{1}{\lambda} \lim_{j\to\infty} \frac{m(w_{\varepsilon_j}, B_{\varepsilon_j}(x_0))}{2\varepsilon_j} + (\lambda - 1)\beta(1 + |[u](x_0)|).
$$

In conclusion, by taking into account (4.23), we deduce (4.18) by taking first the limit as $s \uparrow 1$, for $s \in (0,1)$ chosen as explained above, and then as $\lambda \downarrow 1$ in the latter inequality. □

We are now ready to show that the function $g$ in (4.2) is the surface energy density of $F$. This task will be accomplished by proving two inequalities.

**Lemma 4.9.** *For $\mathcal{H}^1$-a.e. $x_0 \in J_u$,*

$$\frac{dF(u,\cdot)}{d\mathcal{H}^1 \llcorner J_u}(x_0) \leq g(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0))$$

*where $g$ was defined in (4.2).*

*Proof.* We consider the same $x_0$ as in Lemma 4.8. In view of (4.13) and the definition of $g$ in (4.2) it suffices to show that

$$\lim_{\varepsilon\to 0} \frac{m(w_\varepsilon, B_\varepsilon(x_0))}{2\varepsilon} \leq \limsup_{\varepsilon\to 0} \frac{m(u_{x_0}, B_\varepsilon(x_0))}{2\varepsilon}, \tag{4.28}$$

where $w_\varepsilon$ is the function introduced in Lemma 4.8. To prove such a claim consider any sequence $(\varepsilon_j)_j$, we have that for $\mathcal{L}^1$-a.e. $s \in (0,1)$

$$\mu\big(\partial B_{s\varepsilon_j}(x_0)\big) = \mathcal{H}^1\big(\partial B_{s\varepsilon_j}(x_0) \cap J_{w_j}\big) = 0, \tag{4.29}$$

where we have set $w_j := w_{\varepsilon_j}$.

Fix $s \in (0,1)$ as above and a test field $z_j \in SBD^p(B_{s\varepsilon_j}(x_0))$ with $z_j = u_{x_0}$ on $\partial B_{s\varepsilon_j}(x_0)$ such that

$$F(z_j, B_{s\varepsilon_j}(x_0)) \leq m(u_{x_0}, B_{s\varepsilon_j}(x_0)) + \varepsilon_j^2.$$

Consider a cut-off function $\varphi \in C_c^\infty(B_{\varepsilon_j}(x_0), [0,1])$ such that $\varphi \equiv 1$ on $B_{s\varepsilon_j}(x_0)$ and $\|\nabla\varphi\|_{L^\infty} \leq \frac{2}{(1-s)\varepsilon_j}$. Define $\zeta_j := \varphi\, z_j + (1-\varphi)w_j$, with the convention that $z_j$ is extended equal to $u_{x_0}$ outside $B_{s\varepsilon_j}(x_0)$. Therefore, by using $\zeta_j$ as a test field for $m(w_j, B_{\varepsilon_j}(x_0))$ we infer from the growth condition in (1.1) and the locality of $F$

$$\begin{aligned}
&m(w_j, B_{\varepsilon_j}(x_0)) \leq F(\zeta_j, B_{\varepsilon_j}(x_0)) \leq F(z_j, B_{s\varepsilon_j}(x_0)) \\
&\underbrace{+C\int_{B_{\varepsilon_j}(x_0)\setminus B_{s\varepsilon_j}(x_0)} (1+|e(w_j)|^p)\,dx}_{=:I_j^{(7)}} + \underbrace{\frac{C}{((1-s)\varepsilon_j)^p}\int_{B_{\varepsilon_j}(x_0)\setminus B_{s\varepsilon_j}(x_0)} |w_j - u_{x_0}|^p dx}_{=:I_j^{(8)}} \\
&\quad + \underbrace{C\,\mathcal{H}^1\big((B_{\varepsilon_j}(x_0)\setminus B_{s\varepsilon_j}(x_0))\cap J_{\zeta_j}\big)}_{=:I_j^{(9)}} + \underbrace{C\int_{(B_{\varepsilon_j}(x_0)\setminus B_{s\varepsilon_j}(x_0))\cap J_{\zeta_j}} |[\zeta_j]|\,d\mathcal{H}^1}_{=:I_j^{(10)}} \\
&\qquad \leq m(u_{x_0}, B_{s\varepsilon_j}(x_0)) + \varepsilon_j^2 + I_j^{(7)} + I_j^{(8)} + I_j^{(9)} + I_j^{(10)},
\end{aligned} \tag{4.30}$$

with $C = C(\beta, p) > 0$.

By taking into account Lemma 3.4 (ii) and (v) we deduce that $I_j^{(7)}+I_j^{(8)} = o(\varepsilon_j)$ as $j \to \infty$. Moreover, as

$$\mathcal{H}^1((B_{\varepsilon_j}(x_0) \setminus B_{s\varepsilon_j}(x_0)) \cap J_{\zeta_j} \setminus (J_{u_{x_0}} \cup J_{w_j})) = 0,$$

item (i) in Lemma 3.4 together with (4.14) give

$$\limsup_{j\to\infty} \frac{I_j^{(9)}}{2\varepsilon_j} \leq C(1-s)(1+|[u](x_0)|).$$

Furthermore, for $\mathcal{H}^1$-a.e. $x \in J_{\zeta_j} \cap (B_{\varepsilon_j}(x_0) \setminus \overline{B_{s\varepsilon_j}(x_0)})$ it holds

$$|[\zeta_j]| \leq |[u_{x_0}]|\chi_{J_{u_{x_0}} \cap J_{\zeta_j}} + |[w_j]|\chi_{J_{w_j} \cap J_{\zeta_j}} \leq 2|[u_{x_0}]|\chi_{J_{\zeta_j}} + |[w_j] - [u_{x_0}]|\chi_{J_{w_j}}.$$

In turn the latter inequality implies by (4.14) and (4.29)

$$\begin{aligned}\limsup_{j\to\infty} \frac{I_j^{(10)}}{2\varepsilon_j} &\leq C(1-s)|[u](x_0)| \\ &\quad + C \limsup_{j\to\infty} \frac{1}{2\varepsilon_j} \int_{(B_{\varepsilon_j}(x_0)\setminus B_{s\varepsilon_j}(x_0))\cap J_{\zeta_j}} (|[w_j] - [u](x_0)|)\, d\mathcal{H}^1 \\ &\qquad\qquad \leq C(1-s)|[u](x_0)|,\end{aligned}$$

thanks to item (vi) in Lemma 3.4.

Finally, we obtain from (4.30)

$$\begin{aligned}\liminf_{j\to\infty} \frac{m(w_j, B_{\varepsilon_j}(x_0))}{2\varepsilon_j} &\leq s \limsup_{j\to\infty} \frac{m(u_{x_0}, B_{s\varepsilon_j}(x_0))}{2s\varepsilon_j} + C(1-s)(1+|[u](x_0)|) \\ &\leq s \limsup_{\varepsilon\to 0} \frac{m(u_{x_0}, B_\varepsilon(x_0))}{2\varepsilon} + C(1-s)(1+|[u](x_0)|),\end{aligned}$$

and the claim in (4.28) follows at once by letting $s \to 1$ in the inequality above. □

The reverse inequality is established arguing in an analogous fashion, therefore we provide a more concise proof.

**Lemma 4.10.** *For $\mathcal{H}^1$-a.e. $x_0 \in J_u$,*

$$\frac{dF(u,\cdot)}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq g(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0))$$

*where $g$ was defined in (4.2).*

*Proof.* We consider the same points $x_0$ as in Lemma 4.8. Take any infinitesimal sequence $(\varepsilon_j)_j$ such that

$$g(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)) = \lim_{j\to\infty} \frac{m(u_{x_0}, B_{\varepsilon_j}(x_0))}{2\varepsilon_j},$$

and recall that (4.29) is valid for $\mathcal{L}^1$-a.e. $s \in (0,1)$ (as usual $w_j = w_{\varepsilon_j}$). Having fixed such an $s$, let $z_j \in SBD^p(B_{s\,\varepsilon_j}(x_0))$ with $z_j = w_j$ on $\partial B_{s\varepsilon_j}(x_0)$ be such that

$$F(z_j, B_{s\varepsilon_j}(x_0)) \le m(w_j, B_{s\varepsilon_j}(x_0)) + \varepsilon_j^2.$$

Let $\varphi \in C_c^\infty(B_{\varepsilon_j}(x_0), [0,1])$ be a cut-off function such that $\varphi \equiv 1$ on $B_{s\varepsilon_j}(x_0)$ and $\|\nabla\varphi\|_{L^\infty} \le \frac{2}{(1-s)\varepsilon_j}$. Define $\zeta_j := \varphi\, z_j + (1-\varphi)u_{x_0}$, with the convention that $z_j$ is extended equal to $w_j$ outside $B_{s\varepsilon_j}(x_0)$. By using $\zeta_j$ as a test field for $m(u_{x_0}, B_{\varepsilon_j}(x_0))$ we infer from the growth condition in (1.1) and the locality of $F$

$$\begin{aligned} &m(u_{x_0}, B_{\varepsilon_j}(x_0)) \le F(\zeta_j, B_{\varepsilon_j}(x_0)) \le m(w_j, B_{s\varepsilon_j}(x_0)) + \varepsilon_j^2 \\ &+C\int_{B_{\varepsilon_j}(x_0)\setminus B_{s\,\varepsilon_j}(x_0)} (1+|e(w_j)|^p)\,dx + \frac{C}{((1-s)\varepsilon_j)^p}\int_{B_{\varepsilon_j}(x_0)\setminus B_{s\varepsilon_j}(x_0)} |w_j - u_{x_0}|^p dx \\ &\qquad + C\int_{(B_{\varepsilon_j}(x_0)\setminus B_{s\varepsilon_j}(x_0))\cap J_{\zeta_j}} (1+|[\zeta_j]|)d\mathcal{H}^1, \end{aligned}$$

where $C = C(\beta, p) > 0$. Arguing as in the corresponding estimate in Lemma 4.9 (cf. (4.30)), and by taking into account the choice of $(\varepsilon_j)_j$ we conclude that

$$\begin{aligned} g(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)) &\le \liminf_{j\to\infty} \frac{m(w_j, B_{s\varepsilon_j}(x_0))}{2\varepsilon_j} + C(1-s)(1+|[u](x_0)|) \\ &= s\frac{dF(u,\cdot)}{d\mathcal{H}^1 \llcorner J_u}(x_0) + C(1-s)(1+|[u](x_0)|). \end{aligned}$$

The last equality follows from (4.13). The conclusion is achieved by letting $s \uparrow 1$ in the last inequality, with $s \in (0,1)$ satisfying (4.29). □

## 4.4 Proof of Theorem 1.1

*Proof of Theorem 1.1.* The conclusion straightforwardly follows by Lemmata 4.6, 4.7, 4.9, and 4.10. □

**Proposition 4.11.** *The assertion in Theorem 1.1 holds also if property (iv) is replaced by the weaker* *(iv') There are* $\alpha, \beta > 0$ *such that for any* $u \in SBD^p(\Omega)$*, any* $B \in \mathcal{B}(\Omega)$*,*

$$\alpha\Big(\int_B |e(u)|^p dx + \mathcal{H}^1(J_u \cap B)\Big) \leq F(u,B)$$
$$\leq \beta\Big(\int_B (|e(u)|^p + 1)dx + \int_{J_u\cap B} (1 + |[u]|)d\mathcal{H}^1\Big).$$

*Proof.* Given $F$ satisfying properties (i)-(iii) and (iv'), we define for $\delta > 0$ a functional $F_\delta : SBD^p(\Omega) \times \mathcal{B}(\Omega) \to [0, \infty)$ by

$$F_\delta(u,B) := F(u,B) + \delta \int_{J_u\cap B} |[u]| d\mathcal{H}^1,$$

for $u \in SBD^p(\Omega)$ and $B \in \mathcal{B}(\Omega)$. Since $F_\delta$ satisfies properties (i)-(iv) of Theorem 1.1, there are two functions $f$ and $g_\delta$ such that $F_\delta$ can be represented as in (1.2). The family of functionals $F_\delta$ is pointwise increasing in $\delta$, therefore there exists the pointwise limit $g$ of $g_\delta$ as $\delta \to 0$. We conclude that the representation (1.2) holds for $F$ with densities $f$ and $g$. □

**Remark 4.12.** *Since* $F$ *is lower semicontinuous on* $W^{1,p}$*, the integrand* $f$ *is quasiconvex [1, 40]. Since* $F$ *is lower semicontinuous on piecewise constant functions,* $g$ *is* $BV$*-elliptic [2, 3].*

**Remark 4.13.** *If the functional* $F$ *additionally obeys*

$$F(u + a, B) = F(u,B),$$

*for every* $u \in SBD^p(\Omega)$*, every ball* $B \subset \Omega$*, and every affine function* $a : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $e(a) = 0$*, then there are two functions* $f : \Omega \times \mathbb{R}^{2\times 2} \to [0,\infty)$ *and* $g : \Omega \times \mathbb{R}^2 \times S^1 \to [0, \infty)$ *such that*

$$F(u,B) = \int_B f(x, e(u(x)))dx + \int_{B\cap J_u} g(x, [u](x), \nu_u(x))d\mathcal{H}^1 .$$

**Remark 4.14.** *The bulk density* $f$ *satisfies the growth conditions*

$$\alpha \left|\frac{\xi + \xi^T}{2}\right|^p \leq f(x,u,\xi) \leq \beta\Big(1 + \left|\frac{\xi + \xi^T}{2}\right|^p\Big) \qquad (4.31)$$

*for* $\mathcal{L}^2$ *a.e.* $x \in \Omega$ *and for all* $(u,\xi) \in \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$*.*

*In particular, with fixed* $u$ *and* $x$ *for which (4.31) holds, if* $f(x,u,\cdot)$ *turns out to be convex then the restriction of* $f(x,u,\cdot)$ *to the subspace of skew-symmetric matrices is constant. Therefore,* $f(x,u,\cdot)$ *depends only on the symmetric part of the matrix* $\xi$ *rather than on the whole matrix.*

*Instead, if $f$ is not convex in $\xi$ the growth condition in (4.31) does not prevent the dependence on the skew-symmetric part of $\xi$. As an example, the integrand $f : \mathbb{R}^{2\times 2} \to [0, \infty)$ defined by*

$$f(\xi) := (\xi_{11} + \xi_{22})^2 + \sqrt{(\xi_{12}^2 + \xi_{21}^2)^2 + 1} - 2\det(\xi) \tag{4.32}$$

*satisfies*

$$\frac{1}{4}|\xi + \xi^T|^2 \le f(\xi) \le \frac{1}{2}|\xi + \xi^T|^2 + 1$$

*for every $\xi \in \mathbb{R}^{2\times 2}$, but evidently $f(\xi)$ depends also on the skew-symmetric part $\xi - \xi^T$. In particular, $f$ is not convex; note that $f$ is actually polyconvex.*

*We do not know if there is $g$ such that the functional $F$ defined as in (1.2) with $f$ given by (4.32) satisfies the growth condition (1.1) and is lower semicontinuous.*

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