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# Null controllability of the heat equation in pseudo-cylinders by an internal control.

Jon Asier Bárcena-Petisco\*<sup>†</sup>

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## 0 Abstract and basic information

In this paper we prove the null controllability of the heat equation in domains which have a cylindrical part and which are limited by a Lipschitz graph. The proof consists mainly on getting a Carleman estimate, which presents the usual absorptions properties. In order to face the difficulty that the domain is just Lipschitz, we use its cylindrical structure and approximate the system by the same system stated in regular domains.

**Key words:** Carleman inequalities, heat equation, Lipschitz domains, null controllability

**AMS subject classification:** 35D30, 93B05, 93C20

**Abbreviated title:** Null controllability in pseudo-cylinders

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# 1 Introduction

The null controllability of the heat equation has been an interesting research topic in the last 40 years. Throughout this paper we use the term “domain” to refer to a bounded connected non-empty open set. The most general proof is given in [9], where the authors introduce a Carleman estimate to prove the null controllability of the heat equation in a  $C^2$  domain. There are other proofs, which require additional assumptions on the domain or on the data, but in which the error in estimating the cost of the control is reduced (see for example [16], [21] and [5]).

As a continuation of the work made in [9], Carleman estimates of the heat equation have been established when the system is not regular, which prove the null controllability of some related linear and non-linear systems. For instance, in [13] and [7] the authors consider non-regular source terms. Moreover, in [6] and [12] the authors study some boundary conditions which imply that the solution is not in  $L^2(0, T; H^2(\Omega))$ . Additional examples are given in [4], [1] and [15], where the authors analyse parabolic equations in which the diffusion coefficient is not continuous.

To the best of our knowledge, a lack of regularity that has not been studied is whenever the domain is not  $C^2$ . It is well-known that the method in [9] can be easily generalized to cubes or, more generally, to any cartesian product of  $C^2$  domains. Nonetheless, as far as we know, the general case of a Lipschitz domain has not been studied. As a first step on that direction, in this paper we prove the null controllability of the heat equation by an internal control in domains that are pseudo-cylinders. By pseudo-cylinder we mean the following:

*Definition 1.1.* We say that  $\Omega \subset \mathbb{R}^{d+1}$  ( $d \geq 1$ ) is a *pseudo-cylinder* if there is  $\mathfrak{B} \subset \mathbb{R}^d$  a  $C^2$  domain and if there is a Lipschitz function  $H : \overline{\mathfrak{B}} \mapsto \mathbb{R}^+$  satisfying  $\inf_{\mathfrak{B}} H > 0$  such that:

$$\Omega = \{(x, z) : x \in \mathfrak{B}, z \in (0, H(x))\}. \quad (1.1)$$

Moreover, we denote the cylindrical part by:

$$\mathfrak{C} := \mathfrak{B} \times \left(0, \inf_{\mathfrak{B}} H\right). \quad (1.2)$$

*Remark 1.2.* If  $\Omega$  is a pseudo-cylinder, we can split  $\partial\Omega$  into three parts:

$$\mathfrak{B} \times \{0\}; \quad \mathfrak{L} := \{(x, z) : x \in \partial\mathfrak{B}, z \in (0, H(x))\}; \quad \mathfrak{T} := \{(x, H(x)) : x \in \mathfrak{B}\}.$$

Before presenting the main result, we recall the usual notation:

*Definition 1.3.* Let  $\Omega$  be any domain and let  $\omega \subset \Omega$  be a subdomain. We define as usual  $Q := (0, T) \times \Omega$ ,  $Q_\omega := (0, T) \times \omega$ ,  $\Sigma := (0, T) \times \partial\Omega$  and  $n$  the outwards unit normal vector on  $\partial\Omega$ . Similarly, if the domain is denoted by  $\tilde{\Omega}$ , we define  $\tilde{Q} := (0, T) \times \tilde{\Omega}$ ,  $\tilde{\Sigma} := (0, T) \times \partial\tilde{\Omega}$  and  $\tilde{n}$  the outwards unit normal vector on  $\partial\tilde{\Omega}$ .

Now we may state the main result:

**Theorem 1.4.** *Let  $\Omega$  be a pseudo-cylinder. Then, for any subdomain  $\omega \subset \Omega$  there is  $C > 0$  such that for all  $T > 0$  and  $y^0 \in L^2(\Omega)$  there is a force  $f \in L^2(Q_\omega)$  such that the solution of the system:*

$$\begin{cases} y_t - \Delta y = f1_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y^0 & \text{on } \Omega, \end{cases}$$

*satisfies  $y(T, \cdot) = 0$ , and such that the force satisfies the estimate:*

$$\|f\|_{L^2(Q_\omega)} \leq Ce^{CT^{-1}} \|y^0\|_{L^2(\Omega)}.$$

Note that Theorem 1.4 extends the result given by Furkisso and Imanuvilov in [9] to some Lipschitz domains which are relevant, for example, in fluid mechanics (see for instance [20] and [11]). Indeed, the authors in [9] prove that for any  $C^2$  domain the heat equation is null controllable with an internal control. To this end, they prove a Carleman inequality with the help of an auxiliary function  $\eta$  which satisfies:

$$\eta \in C^2(\overline{\Omega}); \quad \eta = 0 \text{ on } \partial\Omega; \quad \eta > 0 \text{ in } \Omega; \quad \inf_{\Omega \setminus \omega} |\nabla\eta| > 0. \quad (1.3)$$

Since in a Lipschitz domain the heat equation has a unique energy solution, the existence of some Carleman estimate seems reasonable. However, two difficulties arise:

- The first one is that the construction of the function  $\eta$  given in [9] does not work when  $\Omega$  is not  $C^2$ . Indeed, if  $\Omega$  has corners and if  $\omega$  is compactly included in  $\Omega$ , the three conditions in (1.3) are incompatible. The only solution known so far for other domains (like cylinders) is to construct manually an auxiliary function which satisfies the following assumptions (which are verified by any  $\eta$  satisfying (1.3)):

$$\eta \in C^2(\overline{\Omega}); \quad \partial_n \eta \leq 0 \text{ on } \partial\Omega; \quad \inf_{\Omega \setminus \omega} |\nabla\eta| > 0. \quad (1.4)$$

The construction of  $\eta$  is done for instance as a sum of functions of different variables, which is our approach when  $\omega \subset \mathfrak{C}$ . As for the case  $\omega \subset \Omega \setminus \mathfrak{C}$ , we “transmit” the estimate from one control domain to another.

- The second difficulty is that in [9] it is used that the Laplacian behaves well in  $\Omega$  in the sense that if  $u \in H_0^1(\Omega)$  is such that  $-\Delta u \in L^2(\Omega)$ , then  $u \in H^2(\Omega)$ . However, when  $\Omega$  is Lipschitz this is not always true, since in a general Lipschitz domain the most we can ensure is that  $u \in H^{3/2}(\Omega)$ . Indeed, the fact that  $u \in H^{3/2}(\Omega)$  is proved in [14], and, for all  $\varepsilon > 0$  a counter-example of a domain  $\Omega_\varepsilon$  for which there is  $u \in H_0^1(\Omega_\varepsilon)$  such that  $-\Delta u \in C^\infty(\overline{\Omega_\varepsilon})$ , but  $u \notin H^{3/2+\varepsilon}(\Omega)$  is given for instance in [10]. In particular, if  $\varphi$  is a solution of the backwards heat equation with Dirichlet boundary conditions, the most that we can expect is that  $\varphi \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; (H^{3/2} \cap H_0^1)(\Omega))$  with  $-\Delta \varphi \in L^2(Q)$ . Solving the difficulty that in our domain the  $L^2$ -norm of the Laplacian is not equivalent to the  $H^2$ -norm is a contribution with respect to the literature on parabolic control theory.

In this paper we solve both difficulties for pseudo-cylinders and get some Carleman estimates (see Proposition 2.1 and Proposition 2.14 below) which present the usual absorption properties. However, as we explain in Section 2.3, our method does not work in any Lipschitz domain, so the general case remains as an interesting open problem.

In this document we consider the following weights:

$$\alpha_\eta(t, x, z) := \frac{e^{\lambda \frac{m+1}{m} k} - e^{\lambda(k+\eta(x,z))}}{t^m(T-t)^m}, \quad \xi_\eta(t, x, z) := \frac{e^{\lambda(k+\eta(x,z))}}{t^m(T-t)^m}, \quad (1.5)$$

for  $m \geq 1$ ,  $\eta$  some auxiliary  $C^2(\overline{\Omega})$  function such that  $\inf_{\Omega \setminus \omega} |\nabla \eta| > 0$  and  $k > m\|\eta\|_{L^\infty(\Omega)}$ , all of them to be fixed later. In (1.5) the subindex  $\eta$  might be omitted if it is clear. Moreover, if we denote some function by  $\tilde{\eta}$ , we use the notation  $\tilde{\alpha}$  and  $\tilde{\xi}$  to refer to  $\alpha_{\tilde{\eta}}$  and  $\xi_{\tilde{\eta}}$  respectively. We recall that we have the following estimates for the weights given in (1.5), for some constants  $C > 0$  that are universal (independent of any variable like  $\eta$ ,  $m$ ,  $\Omega$ , ...):

$$|\xi_t| + |\alpha_t| \leq C m T \xi^{(m+1)/m}; \quad |\alpha_{tt}| \leq C(m \xi^{(m+1)/m} + m(m+1) T^2 \xi^{(m+2)/m}). \quad (1.6)$$

Weights similar to (1.5) were first introduced in [9]. However, our version is equivalent to the one presented in [8], where the authors consider  $k\|\eta\|_{L^\infty(\Omega)}$  instead of  $k$  in the definition of (1.5), and their  $k$  must satisfy  $k > m$ .

*Remark 1.5.* We need  $k > m\|\eta\|_{L^\infty(\Omega)}$  to prove (1.6) and to assure that the numerator of  $\alpha$  is strictly positive, so that we can integrate by parts in the time variable with a null boundary term when proving the Carleman inequality.

*Remark 1.6.* These weights, stated as in (1.5), have the advantage that can be compared very easily when using two different auxiliary functions. Indeed, if we have  $\eta_1$  and  $\eta_2$ , for any  $m \geq 1$ , for any  $k > m \max\{\|\eta_1\|_{L^\infty(\Omega)}, \|\eta_2\|_{L^\infty(\Omega)}\}$ , for any  $t \in (0, T)$  and for any  $(x, z) \in \overline{\Omega}$ , we find:

$$\xi_{\eta_1}(t, x, z) \leq \xi_{\eta_2}(t, x, z) \text{ if and only if } \eta_1(x, z) \leq \eta_2(x, z),$$

and:

$$-\alpha_{\eta_1}(t, x, z) \leq -\alpha_{\eta_2}(t, x, z) \text{ if and only if } \eta_1(x, z) \leq \eta_2(x, z).$$

The rest of the paper is organised as follows: first, in Section 2 we prove Theorem 1.4; secondly, in Section 3 and in Appendix A we prove some technical results which have been postponed.

## 2 Proof of Theorem 1.4

This section is split in three parts: first, in Subsection 2.1 we prove Theorem 1.4 in the case  $\omega \subset \mathfrak{C}$ ; second, in Subsection 2.2 we prove Theorem 1.4 in the case  $\omega \subset \Omega \setminus \mathfrak{C}$ ; finally, in Subsection 2.3 we make some comments about the proof.

### 2.1 The case $\omega \subset \mathfrak{C}$

In this subsection we first construct the auxiliary function  $\eta$  and then prove the corresponding Carleman estimate.

#### 2.1.1 Construction of the auxiliary function

We recall that  $\Omega$  is a pseudo-cylinder and that  $\mathfrak{C}$  is given by (1.2). Let us construct an auxiliary function  $\eta$  satisfying (1.4). By taking a smaller control domain if necessary, we can suppose that  $\omega$  is compactly included in  $\Omega$  and that:

$$\omega = B(\bar{x}, r) \times (\bar{z} - r, \bar{z} + r), \quad (2.1)$$

for some  $(\bar{x}, \bar{z}) \in \Omega$  and for some  $r > 0$ . Since  $\omega \subset \mathfrak{C}$ , we have that:

$$\bar{z} < \inf_{\mathfrak{B}} H. \quad (2.2)$$

Considering that  $\mathfrak{B}$  is  $C^2$  and (1.3), there is  $\eta_{\mathfrak{B}} \in C^2(\overline{\mathfrak{B}})$  satisfying:

$$\eta_{\mathfrak{B}} = 0 \text{ on } \partial\mathfrak{B}; \quad \eta_{\mathfrak{B}} > 0 \text{ in } \mathfrak{B}; \quad \inf_{\mathfrak{B} \setminus B(\bar{x}, r_\omega)} |\nabla \eta_{\mathfrak{B}}| > 0. \quad (2.3)$$

One important consequence of (2.3) is the inequality:

$$\sup_{\partial\mathfrak{B}} \partial_{n_{\mathfrak{B}}} \eta_{\mathfrak{B}} < 0. \quad (2.4)$$

So, we define:

$$\eta(x, z) := \eta_{\mathfrak{B}}(x) - \mathfrak{c}(z - \bar{z})^2, \quad (2.5)$$

for  $\mathfrak{c}$  a positive constant to be fixed later on. It is evident that  $\eta \in C^2(\bar{\Omega})$ . As  $\nabla\eta = (\nabla\eta_{\mathfrak{B}}, -2\mathfrak{c}(z - \bar{z}))$ , it follows from (2.3)<sub>3</sub> the inequality:

$$\inf_{\Omega \setminus \omega} |\nabla\eta| > 0.$$

Finally, it can be checked that for  $\mathfrak{c}$  large enough we have  $\partial_n\eta \leq 0$  on  $\partial\Omega$ , though this last property is not used directly (see Section 2.3 for a more detailed explanation).

With the above auxiliary function we obtain the following Carleman inequality:

**Proposition 2.1.** *Let  $\Omega$  be a pseudo-cylinder, let  $\omega \subset \mathfrak{C}$ , and let  $m \geq 1$ . Then, there exists  $\mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}}) > 0$  such that if  $\eta$  is defined as in (2.5) for  $\mathfrak{c} \geq \mathfrak{c}_0$ , there is  $C > 0$  such that for all  $T > 0$ , for all  $k > m\|\eta\|_{L^\infty(\Omega)}$ , for all  $u^T \in L^2(\Omega)$ , for all  $g \in L^2(Q)$ , and for the weights defined in (1.5), we have that:*

$$s^3\lambda^4 \iint_Q e^{-2s\alpha\xi^3}|u|^2 + s\lambda^2 \iint_Q e^{-2s\alpha\xi}|\nabla u|^2 \leq C \left( s^3\lambda^4 \iint_{Q_\omega} e^{-2s\alpha\xi^3}|u|^2 + \iint_Q e^{-2s\alpha}|g|^2 \right), \quad (2.6)$$

for any  $s \geq C(T^m + T^{2m})$ , for any  $\lambda \geq C$  and for  $u$  the solution of:

$$\begin{cases} -u_t - \Delta u = g & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(T, \cdot) = u^T & \text{on } \Omega. \end{cases} \quad (2.7)$$

*Remark 2.2.* It is a classical result (see for instance [22], [18] and [9]) that Proposition 2.1 (taking  $m = 1$ ) implies that Theorem 1.4 is true for any  $\omega \subset \mathfrak{C}$ .

Proposition 2.1 is proved in the next subsection. The proof consists on approximating  $\Omega$  by some domains  $\Omega_\varepsilon$ , on applying known Carleman estimates on the solution of the heat equation stated in  $\Omega_\varepsilon$ , and on taking limits in those estimates.

### 2.1.2 Proof of Proposition 2.1

Let us now prove Proposition 2.1. We use the notation:

$$\mathcal{W}_\Omega := L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)),$$

omitting the subindex  $\Omega$  if it is clear. Moreover, we define:

*Definition 2.3.* Let  $\Omega$  be a Lipschitz domain, let  $u^0 \in L^2(\Omega)$  and let  $g \in L^2(Q)$ . Then, we define:

$$\mathbb{S}(\Omega, u^0, g)(t, x),$$

as the only solution in  $\mathcal{W}$  of:

$$\begin{cases} u_t - \Delta u = g & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0, \cdot) = u^0 & \text{on } \Omega. \end{cases} \quad (2.8)$$

Finally, we define the convergence of domains in the following way:

*Definition 2.4.* Let  $\Omega$  and  $(\Omega_\varepsilon)_{\varepsilon>0}$  be Lipschitz open sets such that  $\Omega_\varepsilon \subset \Omega$  for all  $\varepsilon > 0$ . We say that  $\Omega_\varepsilon \rightarrow_d \Omega$  if:

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{x \in \Omega \setminus \Omega_\varepsilon} d(x, \partial\Omega) = 0. \quad (2.9)$$

*Remark 2.5.* If  $\Omega_\varepsilon \rightarrow_d \Omega$ , then  $\Omega_\varepsilon$  converges to  $\Omega$  in measure; that is,  $\|1_{\Omega \setminus \Omega_\varepsilon}\|_{L^1(\mathbb{R}^d)} \rightarrow 0$ , because of the Dominated Convergence Theorem.

We have the following technical result:

**Lemma 2.6.** *Let  $\Omega$  be a Lipschitz domain, let  $\Omega_\varepsilon \subset \Omega$  be some Lipschitz domains such that  $\Omega_\varepsilon \rightarrow_d \Omega$ , let  $u^0 \in L^2(\Omega)$  and let  $g \in L^2(Q)$ . Then,*

$$\mathbb{S}(\Omega_\varepsilon, u^0 1_{\Omega_\varepsilon}, g 1_{\Omega_\varepsilon}) 1_{\Omega_\varepsilon} \rightarrow_{L^2(Q)} \mathbb{S}(\Omega, u^0, g). \quad (2.10)$$

*Remark 2.7.* The corresponding result for the backwards heat equation holds.

As far as we know, the proof of Lemma 2.6 is not available in the literature, so a proof, whose originality we do not claim, is given in Appendix A. Indeed, the proof of Lemma 2.6 consists on taking limits in the variational solution, which is done with the following interpolation result, proved for instance in [19] and [18]:

**Lemma 2.8.** *Let  $\Omega$  be a Lipschitz domain. Then,*

1. *The injection  $\mathcal{W} \subset L^2(Q)$  is compact.*
2. *The injection  $\mathcal{W} \subset C^0([0, T]; L^2(\Omega))$  is continuous.*
3. *For all  $u, v \in \mathcal{W}$ , we have  $\langle u(t, \cdot), v(t, \cdot) \rangle_{L^2(\Omega)} \in W^{1,1}(0, T)$  and:*

$$\partial_t (\langle u(t, \cdot), v(t, \cdot) \rangle_{L^2(\Omega)}) = \langle \partial_t u(t, \cdot), v(t, \cdot) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \langle \partial_t v(t, \cdot), u(t, \cdot) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}.$$



In order to prove Proposition 2.1 the approximation of  $\Omega$  must be made not only by  $C^2$  domains, but also these domains must be compatible with the function  $\eta$  in the following way:

*Definition 2.9.* Let  $\Omega$  be a domain, let  $\omega \subset \Omega$  be a subdomain, and let  $\eta$  be a real value function satisfying:

$$\eta \in C^2(\overline{\Omega}); \quad \inf_{\Omega \setminus \omega} |\nabla \eta| > 0. \quad (2.11)$$

We say that  $\tilde{\Omega}$  is a compatible open set with respect to  $\Omega$ ,  $\omega$  and  $\eta$  if we have that:

$$\tilde{\Omega} \text{ is a Lipschitz open set such that } \omega \subset \tilde{\Omega} \subset \Omega, \quad (2.12)$$

$$\text{for all } u \in H_0^1(\tilde{\Omega}) \text{ such that } \Delta u \in L^2(\tilde{\Omega}) \text{ we have that } u \in H^2(\tilde{\Omega}) \text{ continuously,} \quad (2.13)$$

$$\partial_{\tilde{n}} \eta \leq 0 \text{ on } \partial \tilde{\Omega}. \quad (2.14)$$

*Remark 2.10.* One consequence of Definition 2.9 is that  $\tilde{\Omega}$  is connected. Indeed, if  $\tilde{\Omega}_i$  is a connected component, then  $\omega \cap \tilde{\Omega}_i \neq \emptyset$ . Otherwise, because of (2.14) and (2.11)<sub>2</sub> the maximum of  $\eta$  in  $\tilde{\Omega}_i$  is not reached on the boundary (the tangential derivative of  $\eta$  is non-null), but again because of (2.11)<sub>2</sub> the maximum is not reached in the interior, getting an absurd. Consequently, all the connected components of  $\tilde{\Omega}$  intersect  $\omega$ . Since  $\omega$  is connected,  $\tilde{\Omega}$  has only one connected component; so  $\tilde{\Omega}$  is connected.

*Remark 2.11.* Every  $C^2$  domain satisfies (2.13). Moreover, there are some Lipschitz domains which also satisfy it, like  $(0, 1)^d$  or, in general, like the cartesian product of  $C^2$  domains.

Let us now state the Carleman inequality for domains which satisfy (2.12)-(2.14):

**Proposition 2.12.** *Let  $\Omega$  be a Lipschitz domain, let  $\omega \subset \Omega$  be a subdomain, let  $\eta$  be a function satisfying (2.11) and let  $m \geq 1$ . Then, there is  $C > 0$  such that for all  $T > 0$ , for all  $k > m \|\eta\|_{L^\infty(\Omega)}$ , for all  $\tilde{\Omega}$  satisfying (2.12)-(2.14), for all  $u^T \in L^2(\tilde{\Omega})$ , for all  $g \in L^2(\tilde{Q})$ , and for the weights defined in (1.5), we have that:*

$$\begin{aligned} s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\alpha \xi^3} |u|^2 + s \lambda^2 \iint_{\tilde{Q}} e^{-2s\alpha \xi} |\nabla u|^2 + s^{-1} \iint_{\tilde{Q}} e^{-2s\alpha \xi^{-1}} (|\Delta u|^2 + |u_t|^2) \\ \leq C \left( s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha \xi^3} |u|^2 + \iint_{\tilde{Q}} e^{-2s\alpha} |g|^2 \right), \end{aligned} \quad (2.15)$$

for any  $s \geq C(T^m + T^{2m})$ , for any  $\lambda \geq C$  and for  $u$  the solution of:

$$\begin{cases} -u_t - \Delta u = g & \text{in } \tilde{Q}, \\ u = 0 & \text{on } \tilde{\Sigma}, \\ u(T, \cdot) = u^T & \text{on } \tilde{\Omega}. \end{cases}$$

*Proof.* Proposition 2.12 is a direct consequence of the proof of the usual Carleman estimate, which can be found for instance in [9] and [8]. We remark that since  $\inf_{\Omega \setminus \omega} |\nabla \eta| > 0$ , there exists and we fix a domain  $\omega_0$  which is compactly included in  $\omega$  and satisfies  $\inf_{\Omega \setminus \omega_0} |\nabla \eta| > 0$ . The only additional difficulty is to prove that the constant  $C$  is independent of  $\tilde{\Omega}$ . However, their proof shows that  $C$  depends continuously only on:

- $\|\eta\|_{C^2(\tilde{\Omega})}$  (which can be bounded by  $\|\eta\|_{C^2(\bar{\Omega})}$ ),
- $m$  (see (1.6)),
- $\inf_{\tilde{\Omega} \setminus \omega_0} |\nabla \eta|$  (which can be bounded inferiorly by  $\inf_{\Omega \setminus \omega_0} |\nabla \eta| > 0$ ),
- $\omega$  and  $\omega_0$ .

□

Finally, before proving Proposition 2.1 we state the following technical result whose proof is postponed to Section 3:

**Proposition 2.13.** *Let  $\Omega$  be a pseudo-cylinder and let  $\omega$  be given by (2.1). There exists  $c_0(\Omega, \omega, \eta_{\mathfrak{B}}) > 0$  such that if  $\eta$  is given by (2.5) for  $\mathfrak{c} \geq \mathfrak{c}_0$ , there exist some  $\varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathfrak{c}) > 0$  and some domains  $(\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$  satisfying (2.12)-(2.14), and such that  $\Omega_\varepsilon \rightarrow_d \Omega$ .*

*Proof of Proposition 2.1.* We consider  $\Omega$ ,  $\omega$ ,  $m$  and  $\eta$  as in the hypothesis of Proposition 2.1, and  $(\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$  given by Proposition 2.13. Then, by Proposition 2.12 we have that for all  $m \geq 1$  there is  $C > 0$  such that for all  $T > 0$ , for all  $u^T \in L^2(\Omega)$ , for all  $g \in L^2(Q)$ , for the weights defined in (1.5) and for all  $\varepsilon \in (0, \varepsilon_0)$ :

$$s^3 \lambda^4 \iint_Q e^{-2s\alpha \xi^3} |u^\varepsilon 1_{\Omega_\varepsilon}|^2 \leq C \left( s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha \xi^3} |u^\varepsilon|^2 + \iint_Q e^{-2s\alpha} |g 1_{\Omega_\varepsilon}|^2 \right), \quad (2.16)$$

for any  $s \geq C(T^m + T^{2m})$ , for any  $\lambda \geq C$  and for:

$$u^\varepsilon(t, x) := \mathbb{S}(\Omega_\varepsilon, u^T 1_{\Omega_\varepsilon}, g(T - \cdot, \cdot) 1_{\Omega_\varepsilon})(T - t, x).$$

Considering also Lemma 2.6, we can take the limit in (2.16) and get the estimate:

$$s^3 \lambda^4 \iint_Q e^{-2s\alpha \xi^3} |u|^2 \leq C \left( s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha \xi^3} |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 \right). \quad (2.17)$$

Finally, multiplying (2.7)<sub>1</sub> by  $s\lambda^2 e^{-2s\alpha} \xi u$  we get the term of  $\nabla u$  as in [6]. Indeed, integrating by parts we get that:

$$\begin{aligned} s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 &= s\lambda^2 \iint_Q e^{-2s\alpha} g u - s\lambda^3 \iint_Q e^{-2s\alpha} \xi \nabla \eta \cdot \nabla u u + 2s^2 \lambda^3 \iint_Q e^{-2s\alpha} \xi^2 \nabla \eta \cdot \nabla u u \\ &\quad - \frac{s\lambda^2}{2} \iint_Q e^{-2s\alpha} \xi_t |u|^2 + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \alpha_t \xi |u|^2. \end{aligned} \quad (2.18)$$

So, by doing weighted Cauchy-Schwarz estimates in (2.18), and considering also (1.6) and (2.17), we get (2.6).  $\square$

## 2.2 The case $\omega \subset \Omega \setminus \mathfrak{C}$

We recall that the definition of  $\mathfrak{C}$  is given in (1.2). For this case we suppose, by making again the control domain smaller if necessary, that:

$$\omega = B(\tilde{x}, r) \times (\tilde{z} - r, \tilde{z} + r),$$

for some:

$$\tilde{z} > \inf_{\mathfrak{B}} H,$$

and some:

$$r \in \left(0, \inf_{\mathfrak{B}} \frac{H}{4}\right), \quad (2.19)$$

such that:

$$\tilde{\Omega} := B(\tilde{x}, 3r) \times (0, \tilde{z} + 3r) \subset \Omega.$$

In this case we cannot consider an auxiliary function similar to (2.5) because now we have that  $\mathfrak{T} \cap \{(x, z) : z = \tilde{z}\} \neq \emptyset$ . However, thanks to Proposition 2.1 our problem can be viewed as a problem of transmitting the estimate from one control domain to another.

Indeed, we define:

$$\omega^* := B(\tilde{x}, r) \times (2r, 4r),$$

which because of (2.19) satisfies that  $\omega^* \subset \tilde{\Omega} \cap \mathfrak{C}$ . We fix  $m \geq 1$ , and define  $\eta$  as in (2.5) for some  $\mathfrak{c} \geq \mathfrak{c}_0(\Omega, \omega^*, \eta_{\mathfrak{B}})$  (and for  $\bar{x} := \tilde{x}$  and  $\bar{z} := 3r$ ). We obtain by Proposition 2.1 a constant  $C > 0$  such that for all  $T > 0$ ,  $u^T \in L^2(\Omega)$ ,  $g \in L^2(Q)$  and  $k > m \|\eta\|_{L^\infty(\Omega)}$ , we have that:

$$s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 + s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 \leq C \left( s^3 \lambda^4 \iint_{Q_{\omega^*}} e^{-2s\alpha} \xi^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 \right), \quad (2.20)$$

for  $s \geq C(T^m + T^{2m})$ ,  $\lambda \geq C$  and  $u$  the solution of (2.7).

Next, we estimate the weighted  $L^2(Q_{\omega^*})$ -norm by a weighted  $L^2(Q_{\omega})$ -norm. To do this, we consider a function whose maximum is in  $\omega$  but which is large in  $\omega^*$ . In particular, we consider:

$$\tilde{\eta}(x, z) := -\tilde{\eta}_1(|x - \tilde{x}|) - \tilde{\eta}_2(z) + 2 \sup_{\Omega} |\eta|,$$

for  $\tilde{\eta}_1$  a positive  $C^\infty$  function supported in  $(r/2, +\infty)$  such that:

$$\begin{cases} \inf_{(r, +\infty)} \tilde{\eta}'_1 > 0, \\ \tilde{\eta}_1 < \sup_{\Omega} |\eta| / 2 \text{ in } (0, r), \\ \tilde{\eta}_1 > 3 \sup_{\Omega} |\eta| \text{ in } (2r, 3r), \end{cases} \quad (2.21)$$

and for  $\tilde{\eta}_2$  a positive  $C^\infty$  function such that:

$$\begin{cases} \inf_{\mathbb{R} \setminus (\tilde{z}-r, \tilde{z}+r)} |\tilde{\eta}'_2| > 0, \\ \tilde{\eta}_2 < \sup_{\Omega} |\eta| / 2 \text{ in } (2r, 4r), \\ \tilde{\eta}_2 > 3 \sup_{\Omega} |\eta| \text{ in } (0, r) \cup (\tilde{z} + 2r, \tilde{z} + 3r). \end{cases} \quad (2.22)$$

We remark that  $\tilde{\eta} \in C^2(\tilde{\Omega})$  and satisfies:

$$\partial_n \tilde{\eta} < 0 \text{ on } \partial \tilde{\Omega}, \quad \inf_{\mathcal{E} \setminus \omega} |\nabla_{x,z} \tilde{\eta}| > 0. \quad (2.23)$$

Moreover, from (2.21) and (2.22) we get that:

$$\begin{cases} \eta \leq \tilde{\eta} \text{ in } \omega^*, \\ \tilde{\eta} \leq \eta \text{ in } \tilde{\Omega} \setminus (B(\tilde{x}, 2r) \times (r, \tilde{z} + 2r)). \end{cases} \quad (2.24)$$

With the function  $\tilde{\eta}$ , we can define the weights  $\tilde{\alpha}$  and  $\tilde{\xi}$  as in (1.5). Let us fix:

$$k > m \max\{\|\tilde{\eta}\|_{L^\infty(\Omega)}, \|\eta\|_{L^\infty(\Omega)}\}.$$

Thanks to (2.24) and Remark 1.6 we have that:

$$\begin{cases} \xi \leq \tilde{\xi}; & -\alpha \leq -\tilde{\alpha} \text{ in } Q_{\omega^*}, \\ \tilde{\xi} \leq \xi; & -\tilde{\alpha} \leq -\alpha \text{ in } (0, T) \times \left( \tilde{\Omega} \setminus (B(\tilde{x}, 2r) \times (r, \tilde{z} + 2r)) \right). \end{cases} \quad (2.25)$$

Next, let us consider  $\chi(x, z)$  a positive regular cut-off function such that  $\text{supp}(\chi) \subset \tilde{\Omega}$  and  $\chi = 1$  in  $B(\tilde{x}, 2r) \times (r, \tilde{z} + 2r)$ . Then,  $v := \chi u$  is the solution of:

$$\begin{cases} -v_t - \Delta v = -2\nabla \chi \cdot \nabla u - \Delta \chi u + \chi g & \text{in } \tilde{Q}, \\ v = 0 & \text{on } (0, T) \times \partial \tilde{\Omega}, \\ v(T, \cdot) = \chi u^T & \text{on } \tilde{\Omega}. \end{cases} \quad (2.26)$$

By using Proposition 2.12 ( $\tilde{\Omega}$  satisfies (2.13) by Remark 2.11), we have that there is  $C(\tilde{\eta}, m) > 0$  such that if  $u^T \in L^2(\Omega)$ ,  $g \in L^2(Q)$ ,  $s \geq C(T^m + T^{2m})$  and  $\lambda \geq C$ :

$$s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |\chi u|^2 \leq C \left( s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |\chi u|^2 + \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} | -2\nabla\chi \cdot \nabla u - \Delta\chi u + \chi g|^2 \right). \quad (2.27)$$

To continue with, as  $\chi = 1$  in  $\omega$ ,  $\chi \in \mathcal{D}(\tilde{\Omega})$ , and (2.25)<sub>1</sub> we may combine (2.20) and (2.27) and get that:

$$\begin{aligned} s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 + s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |\chi u|^2 + s \lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 \\ \leq C \left( s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |u|^2 + \iint_Q (e^{-2s\alpha} |g|^2 + e^{-2s\tilde{\alpha}} |\chi g|^2) \right. \\ \left. + \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} | -2\nabla\chi \cdot \nabla u - \Delta\chi u|^2 \right). \quad (2.28) \end{aligned}$$

Next, we recall that  $\chi \in C^2(\overline{\Omega})$ , that:

$$\text{supp}(\Delta\chi), \text{supp}(\nabla\chi) \subset \tilde{\Omega} \setminus (B(\tilde{x}, 2r) \times (r, \tilde{z} + 2r)),$$

and (2.25)<sub>2</sub>. Thus, by taking  $s$  and  $\lambda$  large enough the last term in the right-hand side of (2.28) is absorbed, so we get:

$$\begin{aligned} s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 + s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |\chi u|^2 + s \lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 \\ \leq C \left( s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |u|^2 + \iint_Q (e^{-2s\alpha} |g|^2 + e^{-2s\tilde{\alpha}} |\chi g|^2) \right). \quad (2.29) \end{aligned}$$

Finally, let us state what we have proved:

**Proposition 2.14.** *Let  $\Omega$  be a pseudo-cylinder,  $\omega \subset \Omega \setminus \mathfrak{C}$  be a subdomain and  $m \geq 1$ . Then, there exists a constant  $C(\Omega, \omega, m) > 0$  and a constant  $k_0(\Omega, \omega) > 0$  such that, if  $u^T \in L^2(\Omega)$ ,  $g \in L^2(Q)$ ,  $T > 0$ ,  $k \geq k_0 m$ ,  $s \geq C(T^m + T^{2m})$  and  $\lambda \geq C$ , we have (2.29) for  $u$  the solution of (2.7).*

*Remark 2.15.* It is a classical result (see for instance [22], [18] and [9]) that Proposition 2.14 (taking  $m = 1$ ) implies that Theorem 1.4 is true for any  $\omega \subset \Omega \setminus \mathfrak{C}$ . Together with Remark 2.2, it ends the proof of Theorem 1.4 (up to proving Proposition 2.13).

### 2.3 Further comments

- If  $g$  is just a zero-order term of  $u$ , for  $s$  and  $\lambda$  large enough it can be absorbed by the left-hand side of (2.29). Moreover, using all the information (a extended version of (2.27)),  $g$  can also be absorbed if it is a first-order term of  $u$ . Therefore, the Carleman estimate obtained by this method presents the usual absorption properties.
- The method explained in this section also works by symmetry if we consider  $H < 0$  such that  $\sup H < 0$ , a structure which appears naturally in fluid mechanics.
- It seems impossible to approximate some domain  $\Omega$  by some subdomains  $\Omega_\varepsilon$  satisfying (2.12)-(2.14) without having  $\partial_n \eta < 0$  on  $\partial\Omega$ . This implies that if  $\Omega$  is a pseudo-cylinder, if  $\omega$  is given by (2.1) and if  $\eta$  is the auxiliary function defined in (2.5), no matter who  $\Omega_\varepsilon$  are, we need to take  $\mathfrak{c}$  large enough with respect to  $\eta_{\mathfrak{B}}$ . Indeed, let us compute  $\partial_n \eta$  and verify that it is strictly negative on all the subsets of the boundary given by Remark 1.2. On  $\mathfrak{B} \times \{0\}$  we have that  $\partial_n \eta = -\partial_z \eta = -2\mathfrak{c}\bar{z} < 0$ . Moreover, on  $\mathfrak{L}$  we have by (2.4) that  $\partial_n \eta = \partial_{n_{\mathfrak{B}}} \eta_{\mathfrak{B}} < 0$ . However, on  $\mathfrak{T}$  we find that:

$$n(x, H(x)) = \frac{1}{\sqrt{1 + |\nabla H(x)|^2}} (-\nabla H(x), 1).$$

Thus, on  $\mathfrak{T}$  the normal derivative is given by:

$$\partial_n \eta(x, H(x)) = \frac{-\nabla H(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - 2\mathfrak{c}(H(x) - \bar{z})}{\sqrt{1 + |\nabla H(x)|^2}}.$$

This implies that the condition:

$$\mathfrak{c} > \frac{\|\nabla H\|_{L^\infty(\mathfrak{B})} \|\nabla \eta_{\mathfrak{B}}\|_{C^0(\overline{\mathfrak{B}})}}{2(\inf_{\mathfrak{B}} H(x) - \bar{z})}, \quad (2.30)$$

is necessary to ensure that  $\partial_n \eta < 0$  on  $\mathfrak{T}$  (the denominator in (2.30) is not null by (2.2)).

- A problem that remains open is the null controllability of the heat equation in any Lipschitz domain. A natural attempt is first, to split the domain into a finite amount of pseudo-cylinders and a single compactly included  $C^2$  subdomain; and then, to apply a Carleman estimate in each subdomain (with the help of some cut-off functions). To that end, we need to absorb what is next to  $\mathfrak{L}$  (see Remark 1.2) with estimates in some other pseudo-cylinders and then use the technique of Section 2.2 to pass from a finite number of control domains that are compactly contained in  $\Omega$  to a single control domain. This idea could work if we have a function  $\eta$  satisfying (2.11) which is much bigger in the interior of the cylinders than close to  $\mathfrak{L}$ . However, it seems incompatible with picking in (2.5)  $\mathfrak{c}$  large with respect to  $\eta_{\mathfrak{B}}$ .

### 3 Proof of Proposition 2.13

In this section we first construct the open sets  $\Omega_\varepsilon$ , and then prove that for  $\mathfrak{c} \gg \gg \mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$  and  $\varepsilon \ll \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathfrak{c})$  the open sets  $\Omega_\varepsilon$  satisfy (2.12)-(2.14). We recall that  $\Omega$ ,  $\omega$  and  $\eta$  are defined in the statement of Proposition 2.13 and that, once the open sets  $\Omega_\varepsilon$  satisfy (2.12)-(2.14), Remark 2.10 implies that they are connected. Moreover,  $\varepsilon$  is always taken as a strictly positive parameter. Similarly, throughout this proof we use  $c$  and  $C$  to denote respectively small and large strictly positive constants that may be different each time and only depend on  $\Omega$ ,  $\omega$  and  $\eta_{\mathfrak{B}}$ .

#### 3.1 Introducing the open sets $\Omega_\varepsilon$

In order to construct these domains, we need the following basic result from differential geometry, whose proof can be found for instance in [3]:

**Lemma 3.1.** *Let  $\tilde{\Omega}$  be a domain and let  $\Phi : \tilde{\Omega} \mapsto \mathbb{R}$  be a positive function that belongs to  $C^2(\tilde{\Omega})$  and such that  $\Phi = 0$  on  $\partial\tilde{\Omega}$ . Then, if  $\theta > 0$  such that  $\Phi^{-1}(\theta, +\infty) \neq \emptyset$  and such that:*

$$\Phi^{-1}(\{\theta\}) \cap (\nabla\Phi)^{-1}(\{0\}) = \emptyset, \quad (3.1)$$

*we have that  $\Phi^{-1}(\theta, +\infty)$  is a  $C^2$  open subset whose boundary is given by  $\Phi^{-1}(\{\theta\})$ . Moreover, the outwards normal unit vector is given, for all  $x \in \Phi^{-1}(\{\theta\})$ , by:*

$$n_\theta(x) = -\frac{\nabla\Phi(x)}{|\nabla\Phi(x)|}. \quad (3.2)$$

The first natural approach is to use Lemma 3.1 where  $\Phi(x, z)$  is the distance from  $(x, z)$  to  $\partial\Omega$ . Nonetheless, this is not possible because the distance is not differentiable. On the one hand, near  $\mathfrak{T}$  the distance is not  $C^1$  when  $\mathfrak{T}$  is just Lipschitz. On the other hand, even for regular domains the distance may not be  $C^1$  in the interior (for instance, in the disk the gradient of the distance to the boundary is not differentiable in its center). However, it is possible to construct a function which behaves similarly to the distance. In particular, to face the first problem we approximate  $H$  by regular functions  $H_\varepsilon$ , and to face the second one we regularize the distance with the tools introduced in [17].

We define the function:

$$\mathfrak{d} : x \in \overline{\mathfrak{B}} \mapsto d(x, \partial\mathfrak{B}),$$

and the domain:

$$\mathfrak{B}_\varepsilon := \mathfrak{d}^{-1}(\varepsilon, +\infty).$$

Since  $\mathfrak{B}$  is  $C^2$ , we have for some  $\varepsilon_0(\Omega) > 0$  the following properties:

- $\mathfrak{d}$  is a  $C^2(\mathfrak{d}^{-1}([0, \varepsilon_0]))$  function such that  $|\nabla \mathfrak{d}| = 1$  in  $\mathfrak{d}^{-1}([0, \varepsilon_0])$ .
- For all  $x \in \mathfrak{d}^{-1}([0, \varepsilon_0])$  there is a unique point  $P(x)$  such that:

$$d(x, P(x)) = \mathfrak{d}(x). \quad (3.3)$$

Moreover, the function  $x \mapsto P(x)$  is continuous in  $\mathfrak{d}^{-1}([0, \varepsilon_0])$ .

- The open set  $\mathfrak{B}_\varepsilon$  is a non-empty connected  $C^2$  manifold and its boundary is given by:

$$\partial \mathfrak{B}_\varepsilon = \mathfrak{d}^{-1}(\varepsilon).$$

The proof of the first two assertions can be found for instance in [2, Section III.3]. The third one is a classical result that follows from Lemma 3.1 and the previous two assertions.

In order to regularize  $H$  we consider  $\varsigma$  a mollifier in  $\mathbb{R}^d$ ; that is, a positive function belonging to  $\mathcal{D}(B(0, 1))$  such that  $\int_{B(0, 1)} \varsigma = 1$ . As usual, we define  $\varsigma_\varepsilon := \varepsilon^{-d} \varsigma(\frac{\cdot}{\varepsilon})$ . In addition, we define in  $\mathfrak{B}_\varepsilon$  the function  $H_\varepsilon := (1 - \sqrt{\varepsilon}) H * \varsigma_\varepsilon$ , which is well defined because  $x \in \mathfrak{B}_\varepsilon$  implies that  $B(x, \varepsilon) \subset \mathfrak{B}$ . Furthermore, if  $x \in \mathfrak{B}_\varepsilon$ :

$$\begin{aligned} H_\varepsilon(x) &= (1 - \sqrt{\varepsilon}) \int_{B(0, \varepsilon)} H(x - \tilde{x}) \varsigma_\varepsilon(\tilde{x}) d\tilde{x} \\ &\leq (1 - \sqrt{\varepsilon}) (H(x) + \varepsilon \|H\|_{W^{1, \infty}(\mathfrak{B})}) \int_{B(0, \varepsilon)} \varsigma_\varepsilon(\tilde{x}) d\tilde{x} \\ &= H(x) - \sqrt{\varepsilon} H(x) + \varepsilon \|H\|_{W^{1, \infty}(\mathfrak{B})} - \varepsilon^{3/2} \|H\|_{W^{1, \infty}(\mathfrak{B})}. \end{aligned} \quad (3.4)$$

So, since  $\inf_{\mathfrak{B}} H > 0$ , we have for  $\varepsilon \ll \varepsilon_0(\Omega)$  that:

$$H_\varepsilon \leq H \text{ in } \mathfrak{B}_\varepsilon. \quad (3.5)$$

Similarly, we have for  $x \in \mathfrak{B}_\varepsilon$  the lower bound:

$$H_\varepsilon(x) \geq (1 - \sqrt{\varepsilon}) (H(x) - \varepsilon \|H\|_{W^{1, \infty}(\mathfrak{B})}). \quad (3.6)$$

One important consequence of (3.4) and (3.6) is the limit:

$$\inf_{\mathfrak{B}_\varepsilon} H_\varepsilon \rightarrow \inf_{\mathfrak{B}} H. \quad (3.7)$$

A last important property of  $H_\varepsilon$  is that it is uniformly Lipschitz. Indeed:

$$\|\nabla H_\varepsilon\|_{L^\infty(\mathfrak{B}_\varepsilon)} = (1 - \sqrt{\varepsilon}) \|\varsigma_\varepsilon * \nabla H\|_{L^\infty(\mathfrak{B}_\varepsilon)} \leq \|\nabla H\|_{L^\infty(\mathfrak{B})}. \quad (3.8)$$



In order to regularize  $\mathfrak{d}$  in the interior we use the function introduced in [17], whose properties can be consulted for instance in [2, Section III.3.3]. Indeed, we consider  $\rho(x)$ , which is defined as the only fixed point of  $\tau \mapsto G(x, \tau)$ , for:

$$G(x, \tau) := \int_{B(0,1)} \mathfrak{d} \left( x + \frac{\tau}{2} \tilde{x} \right) \varsigma(\tilde{x}) d\tilde{x},$$

for  $\varsigma$  again a mollifier. We have that  $\rho$  satisfies the following properties, which are proved for instance in [2, Section III.3.3]:

$$\rho \in C^2(\overline{\mathfrak{B}}), \quad (3.9)$$

$$\rho \geq 0 \text{ in } \overline{\mathfrak{B}} \text{ and } \rho = 0 \text{ if and only if } x \in \partial\mathfrak{B}, \quad (3.10)$$

$$\frac{1}{2} \leq \frac{\mathfrak{d}(x)}{\rho(x)} \leq \frac{3}{2} \quad \forall x \in \mathfrak{B}, \quad (3.11)$$

$$\nabla \rho(x) = -n_{\mathfrak{B}}(x) \quad \forall x \in \partial\mathfrak{B}. \quad (3.12)$$

So, bearing the regularized distance in mind we introduce the functions:

$$\mathfrak{D}_\varepsilon(x, z) := \chi_\varepsilon(x) \rho(x) (H_\varepsilon(x) - z)z,$$

for  $\varepsilon \ll \varepsilon_0(\Omega)$  such that  $\mathfrak{B}_{2\varepsilon_0} \neq \emptyset$ , and for  $\chi_\varepsilon$  a positive  $C^\infty$  cut-off function which takes values in  $[0, 1]$  such that  $\chi_\varepsilon = 1$  in  $\mathfrak{B}_{2\varepsilon}$  and supported in  $\mathfrak{B}_\varepsilon$  (for instance  $1_{\mathfrak{B}_{(3/2)\varepsilon}} * \varsigma_{\varepsilon/4}$ ). The function  $\mathfrak{D}_\varepsilon$  is clearly a positive function which belongs to  $C^2(\overline{\Omega})$ . So, we approximate  $\Omega$  by:

$$\Omega_\varepsilon := \mathfrak{D}_\varepsilon^{-1} \left( \varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2, +\infty \right),$$

for  $\varepsilon \ll \varepsilon_0(\Omega)$ . In order to see that the open sets  $\Omega_\varepsilon$  are  $C^2$  domains we will use Lemma 3.1 for:

$$\tilde{\Omega}_\varepsilon := \{(x, z) : x \in \mathfrak{B}_\varepsilon, z \in (0, H_\varepsilon(x))\}.$$

The fact that  $\mathfrak{D}_\varepsilon = 0$  on  $\partial\Omega_\varepsilon$  is trivial. Moreover, it follows from (3.11) that  $\Omega_\varepsilon \rightarrow_d \Omega$ . Thus, to apply Lemma 3.1 we just have to check that  $\varepsilon \ll \varepsilon_0(\Omega)$ ,  $(x, z) \in \tilde{\Omega}_\varepsilon$ , and  $\mathfrak{D}_\varepsilon(x, z) = \varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2$  imply that  $\nabla \mathfrak{D}_\varepsilon \neq 0$ .

We first remark that if  $\mathfrak{D}_\varepsilon(x, z) = \varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2$ , we have that  $x \in \overline{\mathfrak{B}_{2\varepsilon}}$ . Indeed, from (see (3.5)):

$$\chi_\varepsilon(x) (H_\varepsilon(x) - z)z \leq \frac{\|H\|_{L^\infty(\mathfrak{B})}^2}{4},$$

we obtain that:

$$\rho(x) \geq 4\varepsilon,$$

which implies using (3.11) that:

$$\mathfrak{d}(x) \geq 2\varepsilon;$$

that is:

$$x \in \overline{\mathfrak{B}_{2\varepsilon}}. \quad (3.13)$$

In particular, from  $\chi_\varepsilon = 1$  in  $\overline{\mathfrak{B}_{2\varepsilon}}$ , we deduce that if  $\mathfrak{D}_\varepsilon(x, z) = \varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2$ , then:

$$\nabla \mathfrak{D}_\varepsilon(x, z) = (z(\rho(x)\nabla H_\varepsilon(x) + (H_\varepsilon(x) - z)\nabla \rho(x)), \rho(x)(H_\varepsilon(x) - 2z)). \quad (3.14)$$

We now prove that  $\mathfrak{D}_\varepsilon(x, z) = \varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2$  and  $\partial_z \mathfrak{D}_\varepsilon(x, z) = 0$  imply that  $\nabla_x \mathfrak{D}_\varepsilon(x, z) \neq 0$ . From (3.14) and  $\partial_z \mathfrak{D}_\varepsilon(x, z) = 0$  we get that  $z = H_\varepsilon(x)/2$ . Hence, we have the equality:

$$\rho(x) = \frac{4\varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2}{H_\varepsilon^2(x)},$$

which implies the equality:

$$\nabla \mathfrak{D}_\varepsilon(x, z) = \left( \frac{H_\varepsilon(x)}{2} \left( \frac{4\varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2}{H_\varepsilon^2(x)} \nabla H_\varepsilon(x) + \frac{H_\varepsilon(x)}{2} \nabla \rho(x) \right), 0 \right).$$

So, because of (3.7)-(3.12), we have that for  $\varepsilon \ll \varepsilon_0(\Omega)$  the equalities  $\mathfrak{D}_\varepsilon(x, z) = \varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2$  and  $\partial_z \mathfrak{D}_\varepsilon(x, z) = 0$  imply that  $\nabla_x \mathfrak{D}_\varepsilon(x, z) \neq 0$ .

Summing up, for  $\varepsilon \ll \varepsilon_0(\Omega)$ , we have that  $\Omega_\varepsilon$  is  $C^2$ . Moreover, using (3.13), (3.2) and (3.14), we obtain for all  $(x, z) \in \partial\Omega_\varepsilon$  that:

$$n_\varepsilon(x, z) = -\frac{\nabla \mathfrak{D}_\varepsilon(x, z)}{|\nabla \mathfrak{D}_\varepsilon(x, z)|} = \frac{(-z(\rho(x)\nabla H_\varepsilon(x) + (H_\varepsilon(x) - z)\nabla \rho(x)), \rho(x)(2z - H_\varepsilon(x)))}{|(-z(\rho(x)\nabla H_\varepsilon(x) + (H_\varepsilon(x) - z)\nabla \rho(x)), \rho(x)(2z - H_\varepsilon(x)))|}, \quad (3.15)$$

where  $n_\varepsilon$  denotes the outwards unit normal vector on  $\partial\Omega_\varepsilon$ .

### 3.2 The open sets $\Omega_\varepsilon$ satisfy (2.12)-(2.14)

The last thing to prove in Proposition 2.13 is that for  $\mathfrak{c} \gg \mathfrak{c}_0(\Omega, \omega, \eta_\mathfrak{B})$  and for  $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_\mathfrak{B}, \mathfrak{c})$  we have that  $\partial_{n_\varepsilon} \eta \leq 0$  on  $\partial\Omega_\varepsilon$ . Thanks to (3.15) and (2.5), we have to prove that on  $\partial\Omega_\varepsilon$ :

$$\begin{aligned} |\nabla \mathfrak{D}_\varepsilon(x, z)| \partial_{n_\varepsilon} \eta(x, z) &= -z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_\mathfrak{B}(x) - z(H_\varepsilon(x) - z)\nabla \rho(x) \cdot \nabla \eta_\mathfrak{B}(x) \\ &\quad - 2\rho(x)(2z - H_\varepsilon(x))\mathfrak{c}(z - \bar{z}) < 0. \end{aligned} \quad (3.16)$$

Intuitively, the idea is that  $\partial\Omega_\varepsilon$  is near  $\partial\Omega$  and that  $n_\varepsilon$  somehow approximates  $n$ . Doing this rigorously implies splitting  $\partial\Omega_\varepsilon$  into several subsets depending if we are close to  $\mathfrak{B} \times \{0\}$ ,  $\mathfrak{L}$  or  $\mathfrak{T}$  (see Remark 1.2 for the notation). In particular, we split the boundary into:

$$\begin{cases} \partial\Omega_{\varepsilon,b} := \partial\Omega_\varepsilon \cap \{(x, z) : z \in (0, H_\varepsilon(x)/2]\}, \\ \partial\Omega_{\varepsilon,t} := \partial\Omega_\varepsilon \cap \{(x, z) : z \in (H_\varepsilon(x)/2, H_\varepsilon(x))\}. \end{cases} \quad (3.17)$$

*Case 1:*  $(x, z) \in \partial\Omega_{\varepsilon,b}$ . The fact that  $\mathfrak{D}_\varepsilon(x, z) = \varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2$  (recall (3.13) and (3.5)) implies for  $\varepsilon \ll \varepsilon_0(\Omega)$  that:

$$\varepsilon \|H\|_{L^\infty(\mathfrak{B})} \leq z\rho(x) \leq \frac{2\varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2}{\inf_{\mathfrak{B}_\varepsilon} H_\varepsilon}. \quad (3.18)$$

Thanks to (3.7) we have for  $\varepsilon \ll \varepsilon_0(\Omega)$  the inequality:

$$\frac{1}{\inf_{\mathfrak{B}_\varepsilon} H_\varepsilon} \leq \frac{2}{\inf_{\mathfrak{B}} H};$$

so if we define:

$$c_1 := \|H\|_{L^\infty(\mathfrak{B})}, \quad c_2 := \frac{4\|H\|_{L^\infty(\mathfrak{B})}^2}{\inf_{\mathfrak{B}} H}, \quad (3.19)$$

we have that (3.18) turns into:

$$c_1\varepsilon \leq z\rho(x) \leq c_2\varepsilon. \quad (3.20)$$

So, either  $z \geq \sqrt{c_1\varepsilon}$  or  $\rho(x) \geq \sqrt{c_1\varepsilon}$  is obtained. We study both cases separately, since in the first one we are near  $\mathfrak{L}$ ; whereas in the second one we are near  $\mathfrak{B} \times \{0\}$ .

*Case 1.1:*  $(x, z) \in \partial\Omega_{\varepsilon,b}$  and  $z \geq \sqrt{c_1\varepsilon}$ . Due to (3.20) we have the bound:

$$\rho(x) \leq c_2\sqrt{c_1^{-1}\varepsilon}. \quad (3.21)$$

We first focus on the sign of:

$$(-z\rho(x)\nabla H_\varepsilon(x) - z(H_\varepsilon(x) - z)\nabla\rho(x)) \cdot \nabla\eta_{\mathfrak{B}}(x).$$

From (3.20) we obtain the estimate:

$$|z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla\eta_{\mathfrak{B}}(x)| \leq C\varepsilon. \quad (3.22)$$

Moreover, we have the equality (see (3.3)):

$$-z(H_\varepsilon(x) - z)\nabla\rho(x) = z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) + z(H_\varepsilon(x) - z)(-n_{\mathfrak{B}}(P(x)) - \nabla\rho(x)). \quad (3.23)$$

Recalling (3.9)-(3.12) and (3.3) we obtain for  $\varepsilon \ll \varepsilon_0(\mathfrak{B})$  the estimate:

$$\begin{aligned} | -n_{\mathfrak{B}}(P(x)) - \nabla\rho(x) | &= | \nabla\rho(P(x)) - \nabla\rho(x) | \\ &\leq \|\rho\|_{W^{2,\infty}(\Omega)} |P(x) - x| = \|\rho\|_{W^{2,\infty}(\Omega)} \mathfrak{d}(x) \leq \frac{3}{2} \|\rho\|_{W^{2,\infty}(\Omega)} \rho(x). \end{aligned} \quad (3.24)$$

So, combining (3.24) and (3.20) we get the bound:

$$|z(H_\varepsilon(x) - z)(-n_{\mathfrak{B}}(P(x)) - \nabla\rho(x)) \cdot \nabla\eta_{\mathfrak{B}}(x)| \leq C|z\rho(x)| \leq C\varepsilon. \quad (3.25)$$

Finally, we have the equality:

$$\begin{aligned} z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) \cdot \nabla\eta_{\mathfrak{B}}(x) &= z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) \cdot \nabla\eta_{\mathfrak{B}}(P(x)) \\ &\quad + z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) \cdot (-\nabla\eta_{\mathfrak{B}}(P(x)) + \nabla_{\mathfrak{B}}\eta(x)). \end{aligned} \quad (3.26)$$

Since  $\eta_{\mathfrak{B}} \in C^2(\overline{\mathfrak{B}})$ , arguing similarly to (3.24) we obtain the estimate:

$$|z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) \cdot (-\nabla\eta_{\mathfrak{B}}(P(x)) + \nabla_{\mathfrak{B}}\eta(x))| \leq C\varepsilon. \quad (3.27)$$

Moreover, if we take into account (2.4), since  $z \geq \sqrt{c_1\varepsilon}$  we have the bound:

$$z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) \cdot \nabla\eta_{\mathfrak{B}}(P(x)) \leq -c\sqrt{\varepsilon}. \quad (3.28)$$

Consequently, if we combine (3.22)-(3.28), we find for  $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}})$  that:

$$(-z\rho\nabla H_\varepsilon - z(H_\varepsilon - z)\nabla\rho) \cdot \nabla\eta_{\mathfrak{B}} \leq C\varepsilon - c\sqrt{\varepsilon} < 0. \quad (3.29)$$

Let us now analyse the term:

$$2\rho(x)(H_\varepsilon(x) - 2z)\mathfrak{c}(z - \bar{z}).$$

On the one hand, if  $z \leq \bar{z}$  (see (2.1) for the definition of  $\bar{z}$ ), we have that (see (3.17)<sub>1</sub>):

$$2\rho(x)(H_\varepsilon(x) - 2z)\mathfrak{c}(z - \bar{z}) \leq 0,$$

which, together with (3.29) implies (3.16). On the other hand, if  $z \geq \bar{z}$ , we get from (3.20) the inequality:

$$\rho(x) \leq \frac{c_2\varepsilon}{\bar{z}}. \quad (3.30)$$

Thus, we obtain the estimate:

$$|2\rho(x)(H_\varepsilon(x) - 2z)\mathfrak{c}(z - \bar{z})| \leq C\mathfrak{c}\varepsilon. \quad (3.31)$$

Thus, if we combine (3.29) and (3.31), we get for  $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathfrak{c})$  the bound:

$$\partial_{n_\varepsilon}\eta(x, z) \leq \frac{C(1 + \mathfrak{c})\varepsilon - c\sqrt{\varepsilon}}{|\nabla\mathfrak{D}_\varepsilon(x, z)|} < 0;$$

that is, we get (3.16).

*Case 1.2:*  $(x, z) \in \partial\Omega_{\varepsilon,b}$  and  $\rho(x) \geq \sqrt{c_1\varepsilon}$ . We first remark from (3.20) that we have for  $\varepsilon \ll \varepsilon_0(\omega, \Omega)$  that:

$$z \leq c_2\sqrt{c_1^{-1}\varepsilon} \leq \frac{\bar{z}}{4} \leq \frac{\inf_{\mathfrak{B}} H}{4}. \quad (3.32)$$

If we use (3.20), (3.8), (3.5), (3.32)<sub>1</sub>, (3.9), that  $\eta_{\mathfrak{B}} \in C^2(\overline{\mathfrak{B}})$ , (3.32)<sub>2</sub> and (3.32)<sub>3</sub> and (3.7) in (3.16), we obtain for  $\varepsilon \ll \varepsilon(\Omega)$  the inequality:

$$|\nabla \mathfrak{D}_\varepsilon(x, z)| \partial_{n_\varepsilon} \eta(x, z) \leq C\varepsilon + C\varepsilon^{1/2} - c\varepsilon^{1/2}.$$

Consequently, for  $\mathfrak{c} \gg \mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$  and for  $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathfrak{c})$  we have (3.16).

*Case 2:*  $(x, z) \in \partial\Omega_{\varepsilon,t}$ . As for  $\partial\Omega_{\varepsilon,t}$  the situation is very similar. Indeed, from  $\mathfrak{D}_\varepsilon(x, z) = \varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2$ , we find the bounds:

$$c_1\varepsilon \leq (H_\varepsilon(x) - z)\rho(x) \leq c_2\varepsilon, \quad (3.33)$$

for  $c_1$  and  $c_2$  defined in (3.19). We again distinguish the cases  $H_\varepsilon(x) - z \geq \sqrt{c_1\varepsilon}$  and  $\rho(x) \geq \sqrt{c_1\varepsilon}$ . The first one concerns the points near  $\mathfrak{L}$ , whereas the second one concerns the points near  $\mathfrak{T}$ .

*Case 2.1:*  $(x, z) \in \partial\Omega_{\varepsilon,t}$  and  $H_\varepsilon(x) - z \geq \sqrt{c_1\varepsilon}$ . Due to (3.33) we have the bound:

$$\rho(x) \leq c_2\sqrt{c_1^{-1}\varepsilon}. \quad (3.34)$$

Arguing as before, we get the estimate:

$$-z(H_\varepsilon(x) - z)\nabla\rho(x) \cdot \nabla\eta_{\mathfrak{B}}(x) \leq C\varepsilon - c\varepsilon^{1/2}. \quad (3.35)$$

In order to continue, as before, we make a distinction depending on how close  $z$  is to  $H_\varepsilon(x)$  and taking into account (2.2).

We start with the subcase:

$$z \geq \max \left\{ \sqrt{\frac{\bar{z}}{\inf_{\mathfrak{B}} H}}, \frac{2}{3} \right\} H_\varepsilon(x). \quad (3.36)$$

We trivially have for  $\varepsilon \ll \varepsilon_0(\Omega)$  the inequality:

$$H_\varepsilon \geq \sqrt[4]{\frac{\bar{z}}{\inf_{\mathfrak{B}} H}} H. \quad (3.37)$$

Thanks to (3.36) and (3.37) we have for  $\varepsilon \ll \varepsilon_0(\Omega)$  the lower bound:

$$z \geq \sqrt[4]{(\bar{z})^3 \inf_{\mathfrak{B}} H} = \sqrt[4]{\frac{\inf_{\mathfrak{B}} H}{\bar{z}}} \bar{z}. \quad (3.38)$$

So, combining (3.36), (3.38) and (2.2) we obtain for all  $\varepsilon \ll \ll \varepsilon_0(\Omega)$  the inequality:

$$-2\rho(x)(2z - H_\varepsilon(x))\mathbf{c}(z - \bar{z}) \leq -c\mathbf{c}\rho(x). \quad (3.39)$$

Moreover, we clearly have the estimate:

$$-z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_{\mathfrak{B}}(x) \leq C\rho(x). \quad (3.40)$$

Thus, for  $\mathbf{c} \gg \gg \mathbf{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$ , using that  $\rho \geq 0$ , we get from (3.39) and (3.40) the upper bound:

$$-z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - 2\rho(x)(2z - H_\varepsilon(x))\mathbf{c}(z - \bar{z}) \leq 0. \quad (3.41)$$

Hence, combining (3.35) and (3.41) we get (3.16) for  $\varepsilon \ll \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}})$ .

Let us now suppose that:

$$z < \max \left\{ \sqrt{\frac{\bar{z}}{\inf_{\mathfrak{B}} H}}, \frac{2}{3} \right\} H_\varepsilon(x).$$

Considering (3.33) we have that  $\rho(x) < c\varepsilon$ . Thus, we find the estimate:

$$-z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - 2\rho(x)(2z - H_\varepsilon(x))\mathbf{c}(z - \bar{z}) < Cc\varepsilon. \quad (3.42)$$

So, combining (3.35) and (3.42), we obtain (3.16) for  $\varepsilon \ll \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathbf{c})$ .

*Case 2.2:*  $(x, z) \in \partial\Omega_{\varepsilon,t}$  and  $\rho(x) \geq \sqrt{c_1\varepsilon}$ . We have from (3.33) the inequality:

$$H_\varepsilon(x) - z \leq c_2\sqrt{c_1^{-1}\varepsilon}. \quad (3.43)$$

So, we find the upper bound:

$$(-z\rho(x)\nabla H_\varepsilon(x) - z(H_\varepsilon(x) - z)\nabla\rho(x)) \cdot \nabla \eta_{\mathfrak{B}}(x) \leq C\left(\rho(x) + \varepsilon^{1/2}\right). \quad (3.44)$$

Moreover, since (3.43) implies for  $\varepsilon \ll \ll \varepsilon_0(\Omega)$  that  $z > \sqrt{\bar{z}\inf_{\mathfrak{B}} H}$  and since:

$$2\rho(x) \geq \sqrt{c_1\varepsilon} + \rho(x),$$

we have the estimate:

$$-2\rho(x)(2z - H_\varepsilon(x))\mathbf{c}(z - \bar{z}) \leq -c\mathbf{c}\left(\rho(x) + \varepsilon^{1/2}\right). \quad (3.45)$$

Consequently, combining (3.44) and (3.45), we obtain (3.16) under the hypothesis of the case 2.2 for  $\mathbf{c} \gg \gg \mathbf{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$  and for  $\varepsilon \ll \ll \varepsilon_0(\Omega)$ .

## A Proof of Lemma 2.6

In order to prove Lemma 2.6 we recall that for any Lipschitz domain  $\Omega$ , for any  $u^0 \in L^2(\Omega)$  and for any  $g \in L^2(Q)$ ,  $\mathbb{S}(\Omega, u^0, g)$  is characterized as the only element in  $\mathcal{W}_\Omega$  such that:

$$\begin{cases} \int_0^T \langle \partial_t \mathbb{S}(\Omega, u^0, g), \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_Q \nabla \mathbb{S}(\Omega, u^0, g) \cdot \nabla \phi = \int_Q g \phi & \forall \phi \in \mathcal{D}(Q), \\ \mathbb{S}(\Omega, u^0, g)(0, \cdot) = u^0. \end{cases} \quad (\text{A.1})$$

From now on, let us fix  $\Omega, u^0, g$  and  $\Omega_\varepsilon$  as stated in Lemma 2.6. We define:

$$\mathbb{S} := S(\Omega, u^0, g), \quad \mathbb{S}_\varepsilon := \mathbb{S}(\Omega_\varepsilon, u^0 1_{\Omega_\varepsilon}, g 1_{\Omega_\varepsilon}) 1_{\Omega_\varepsilon} \quad \text{and} \quad Q_\varepsilon := (0, T) \times \Omega_\varepsilon.$$

We recall that, since  $v \mapsto v 1_{\Omega_\varepsilon}$  is an injection from  $\mathcal{W}_{\Omega_\varepsilon}$  to  $\mathcal{W}_\Omega$ , we have  $\mathbb{S}_\varepsilon \in \mathcal{W}_\Omega$  and the equality:

$$\|\mathbb{S}_\varepsilon\|_{\mathcal{W}_\Omega} = \|\mathbb{S}_\varepsilon\|_{\mathcal{W}_{\Omega_\varepsilon}}. \quad (\text{A.2})$$

By density of  $\mathcal{D}(Q_\varepsilon)$  in  $\mathcal{W}_{\Omega_\varepsilon}$  we find the equality:

$$\|\mathbb{S}_\varepsilon(T, \cdot)\|_{L^2(\Omega_\varepsilon)}^2 + 2 \iint_{Q_\varepsilon} |\nabla \mathbb{S}_\varepsilon|^2 = 2 \iint_Q g \mathbb{S}_\varepsilon + \|u^0\|_{L^2(\Omega_\varepsilon)}^2.$$

In particular, if we use the Poincaré inequality on  $\Omega$  and Cauchy-Schwarz, we obtain for a constant  $C(\text{diam}(\Omega)) > 0$ :

$$\|\mathbb{S}_\varepsilon\|_{L^2(0, T; H_0^1(\Omega_\varepsilon))} \leq C (\|g\|_{L^2(Q_\varepsilon)} + \|u^0\|_{L^2(\Omega_\varepsilon)}) \leq C (\|g\|_{L^2(Q)} + \|u^0\|_{L^2(\Omega)}).$$

In addition, looking at (2.8)<sub>1</sub> we obtain that  $\partial_t \mathbb{S}_\varepsilon \in L^2(0, T; H^{-1}(\Omega_\varepsilon))$  and the existence of a constant  $C(\text{diam}(\Omega)) > 0$  such that:

$$\|\mathbb{S}_\varepsilon\|_{\mathcal{W}_{\Omega_\varepsilon}} \leq C (\|g\|_{L^2(Q)} + \|u^0\|_{L^2(Q)}).$$

Consequently,  $\mathbb{S}_\varepsilon$  is uniformly bounded in  $\mathcal{W}_\Omega$  due to (A.2); thus,  $\mathbb{S}_\varepsilon$  has at least a weakly convergence sequence in  $\mathcal{W}_\Omega$ . Moreover, thanks to item 1 of Lemma 2.8, we have that the sequence is convergent in  $L^2(Q)$ .

In order to end the proof, it suffices to see that for all  $u \in \mathcal{W}_\Omega$  such that there is a sequence  $\mathbb{S}_{\varepsilon_i}$  such that  $\mathbb{S}_{\varepsilon_i} \rightharpoonup u$  in  $\mathcal{W}_\Omega$  (with  $\varepsilon_i \rightarrow 0$ ), then  $u = \mathbb{S}$ . To get the equality, we prove that  $u$  satisfies (A.1):

- As for the variational condition of (A.1), let us pick  $\phi \in \mathcal{D}(Q)$ . We remark that  $d(\text{supp}_x \phi, \partial\Omega) > 0$ , for:

$$\text{supp}_x \phi := \overline{\{x : \exists t \in [0, T] : \phi(t, x) \neq 0\}}.$$

In particular, if  $i$  is sufficiently large, we have that  $\text{supp}_x \phi \subset \Omega_{\varepsilon_i}$ . Consequently, by (A.1)<sub>1</sub> (applied in  $\Omega_{\varepsilon_i}$  and then taking into account the support of  $\phi$ ) we obtain for  $i$  large enough the equality:

$$\int_0^T \langle \partial_t \mathbb{S}_{\varepsilon_i}, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_Q \nabla \mathbb{S}_{\varepsilon_i} \cdot \nabla \phi = \int_Q g \phi.$$

Thus, if we take the weak limit in  $\mathcal{W}_\Omega$ , we get the equation:

$$\int_0^T \langle \partial_t u, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_Q \nabla u \cdot \nabla \phi = \int_Q g \phi.$$

Since  $\phi$  is arbitrary,  $u$  satisfies (A.1)<sub>1</sub>.

- As for the initial condition of (A.1), we recall that because of item 2 of Lemma 2.8,

$$v \in \mathcal{W} \mapsto v(0, \cdot) \in L^2(\Omega)$$

is a continuous operator. Therefore, since the weak limit is preserved by linear continuous operators between Hilbert spaces, we have that  $\mathbb{S}_{\varepsilon_i}(0, \cdot) \rightharpoonup u(0, \cdot)$  in  $L^2(\Omega)$ . Moreover,  $\mathbb{S}_{\varepsilon_i}(0, \cdot) = u^0 1_{\Omega_{\varepsilon_i}} \rightarrow u^0$  in  $L^2(\Omega)$  by Remark 2.5. Consequently, from the uniqueness of the weak limit, we obtain that  $u(0, \cdot) = u^0$ .

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