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Scott Armstrong, Jean-Christophe Mourrat

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SCOTT ARMSTRONG AND JEAN-CHRISTOPHE MOURRAT

Abstract. We develop a functional analytic approach to the study of the Kramers and kinetic Fokker-Planck equations which parallels the classical H^1 theory of uniformly elliptic equations. In particular, we identify a function space analogous to H^1 and develop a well-posedness theory for weak solutions of the Dirichlet problem in this space. In the case of a conservative force, we identify the weak solution as the minimizer of a uniformly convex functional. We prove new functional inequalities of Poincaré and Hörmander type and combine them with basic energy estimates (analogous to the Caccioppoli inequality) in an iteration procedure to obtain the C^∞ regularity of weak solutions. We also use the Poincaré-type inequality to give an elementary proof of the exponential convergence to equilibrium for solutions of the kinetic Fokker-Planck equation which mirrors the classic dissipative estimate for the heat equation.

Contents

1. Introduction

1.1. **Motivation and informal summary of results.** In this paper, we develop a well-posedness and regularity theory for weak solutions of the hypoelliptic equation

$$-\Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f = f^* \quad \text{in } U\times\mathbb{R}^d, \quad U\subseteq\mathbb{R}^d. \tag{1.1}$$

The unknown function $f(x,v)$ is a function of the position variable $x\in U$ and the velocity variable $v\in\mathbb{R}^d$. The PDE (1.1) is sometimes called the *Kramers equation*. We also consider the time-dependent version of this equation, namely

$$\partial_t f - \Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f = f^* \quad \text{in } (0,\infty)\times U\times\mathbb{R}^d, \tag{1.2}$$

which is often called the *kinetic Fokker-Planck equation*.

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These equations were first studied by Kolmogorov [25], and were the main motivating examples for the general theory of Hörmander [22] of hypoelliptic equations. They are of physical interest due to their relation with the *Langevin diffusion process* formally defined by

$$\ddot{X} = -\mathbf{b}(X) - \dot{X} + \dot{B}, \tag{1.3}$$

where $\dot{X}$, $\ddot{X}$ stand respectively for the first and second time derivatives of X, a stochastic process taking values in $\mathbb{R}^d$, and $\dot{B}$ denotes a white noise process. Equation (1.3) can be interpreted as Newton's law of motion for a particle subject to the force field $-\mathbf{b}(X)$, friction and thermal noise. This process can be recast as a Markovian evolution for the pair (X, V) evolving according to

$$\begin{cases} \dot{X} = V, \\ \dot{V} = -\mathbf{b}(X) - V + \dot{B}. \end{cases}$$

The infinitesimal generator of this Markov process is the differential operator appearing on the left side of (1.1).

Kolmogorov [25] gave an explicit formula for the fundamental solution of (1.2) in the case $\mathbf{b} = 0$ and $U = \mathbb{R}^d$, which gives the existence of smooth solutions of (1.1) and (1.2) and implies that the operators on the left sides of (1.1) and (1.2) are *hypoelliptic*—that is, if f is a distributional solution of either of these equations and f^* is smooth, then f is also smooth. This result is extended to more general equations in Hörmander's celebrated paper [22], where he gave an essentially complete classification of hypoelliptic operators. In the case of the particular equations (1.1) and (1.2), his arguments yield a more systematic proof of Kolmogorov's results and, in particular, interior regularity estimates.

Hörmander's analysis of hypoelliptic equations relies on the theory of pseudodifferential operators, and is somewhat distinct from the variational theory of elliptic equations which begins with the construction of weak solutions in the function space H^1. The purpose of this paper is to present a functional analytic and variational theory for (1.1) and (1.2) which has strong analogies to the familiar theory of uniformly elliptic equations. In particular, in this paper we:

- identify a function space H^1_{hyp} based on the natural energy estimates and develop a notion of weak solutions in this space;
- prove functional inequalities for H^1_{hyp}, for instance a Poincaré-type inequality, which implies uniform coercivity of our equations;
- develop a well-posedness theory of weak solutions of boundary-value problems based on the minimization of a uniformly convex functional;
- develop a regularity theory for weak solutions, based on an iteration of energy estimates, which implies that weak solutions are smooth;
- prove dissipative estimates for solutions of (1.2), using the coercivity of the variational structure, which imply an exponential decay to equilibrium.

Such a theory has until now remained undeveloped, despite the attention these equations have received in the last half century. The definition of the space H^1_{hyp} is not new: it and variants of it have been previously studied previously in the works [4, 32, 10]. However, the functional inequalities and other key properties which are required to work with this space are established here. The question of well-posedness for the natural analogue of the Dirichlet boundary value problem for (1.1) is also settled for the first time in this paper. A robust notion of weak solutions and corresponding well-posedness theory—besides allowing one to prove classical results for (1.1) and (1.2) in a different way— is important because it provides a natural framework for studying the stability of solutions (i.e., proving

that a sequence of approximate solutions converges to a solution). In fact, it is just such an application—namely, developing a theory of homogenization for (1.2)—which motivated the present work. Furthermore, we expect that the theory developed here will provide a closer link between the hypoelliptic equations (1.1) and (1.2) and the classical theory of uniformly elliptic and parabolic equations, allowing, for example, for a more systematic development of regularity estimates for solutions of the former by analogy to the latter. For instance, it would be interesting to investigate a possible connection between the functional analytic framework proposed in this paper and the recent works [38, 39, 19, 29] which develop De Giorgi-Nash-type Hölder estimates for generalizations of the kinetic Fokker-Planck equations with measurable coefficients.

In the first part of the paper, we establish the well-posedness of Dirichlet-type boundary-value problems in $C^{1,1}$ domains $U \subseteq \mathbb{R}^d$ for the equation (1.1) under a weak formulation based on the Sobolev-type space $H^1_{\text{hyp}}(U)$, defined below in (1.7). In the case in which $\mathbf{b}$ is a potential field, we identify a *uniformly convex* functional that has the sought-after weak solution as its unique minimizer. The identification of the correct convex functional is inspired by previous work of Brézis and Ekeland [8, 9] on variational formulations of parabolic equations (see also the more recent works [18, 2] and the references therein). The proof that our functional is indeed uniformly convex relies on a new Poincaré-type inequality for H^1_{hyp}, see Theorem 1.2 below. The well-posedness of weak solutions for general (i.e., non-potential) bounded vector fields $\mathbf{b}$ is then obtained by a fixed point argument, using a maximum principle and a compact embedding of the space H^1_{hyp} into $L^2_x L^2_v$, in close analogy to the uniformly elliptic setting. Our convex-analytic arguments for well-posedness can be immediately adapted to cover non-linear equations such as those obtained by replacing $\Delta_v f$ in (1.1) with $\nabla_v \cdot (\mathbf{a}(x, v, \nabla_v f))$, for $p \mapsto \mathbf{a}(x, v, p)$ a Lipschitz and uniformly maximal monotone operator (uniformly over $x \in U$ and $v \in \mathbb{R}^d$).

There is a subtle point in the analysis of the energy functional which is due to the fact that we should prescribe the boundary condition only on part of the boundary, namely $\partial_{\text{hyp}} U := \{(x, v) \in \partial U \times \mathbb{R}^d : v \cdot \mathbf{n}_U(x) > 0\}$, where $\mathbf{n}_U$ denotes the outer normal to U. This requires a careful study of the boundary behavior of H^1_{hyp} functions, since there is a difficulty coming from the possibly wild behavior of the trace of an H^1_{hyp} function near the *singular set* $\{(x, v) \in \partial U \times \mathbb{R}^d : v \cdot \mathbf{n}_U(x) = 0\}$. In the case of one spatial dimension ($d = 1$), this difficulty has been previously overcome and the well-posedness result was already proved in [4]. A generalization to higher dimensions was announced in [10], but we think that the argument given there is incomplete because the difficulty concerning the boundary behavior was not satisfactorily treated. This is explained in more detail in Appendix A.

Roughly speaking, the norm $\|\cdot\|_{H^1_{\text{hyp}}(U)}$ is a measure of the size of the vector fields $\nabla_v f$ and $v \cdot \nabla_x f$, but crucially, the former is measured in a strong $L^2_x L^2_v$-type norm and the latter in a weaker $L^2_x H^{-1}_v$-type norm (see (1.7) below). Once we have proved the existence of weak solutions to (1.1) in H^1_{hyp}, we are interested in showing that these solutions are in fact smooth. It is elementary to verify that the differential operators ∇_v and $v \cdot \nabla_x$ satisfy Hörmander's bracket condition and therefore, as exposed in [22], a control of both $\nabla_v f$ and $v \cdot \nabla_x f$ in $L^2_x L^2_v$ yields control of the seminorm of the function f in a fractional Sobolev space of positive regularity, namely $H^{1/2}_x L^2_v$. However, since the natural definition of the function space $H^1_{\text{hyp}}(U)$ provides us only with control of $v \cdot \nabla_x f$ in a space of *negative* regularity in v, we are forced to revisit the arguments of [22]. What we prove is a functional inequality (see Theorem 1.3 below) which asserts that the $H^1_{\text{hyp}}(U)$ norm

controls (almost) one-third of a derivative in arbitrary x-directions, again with negative regularity in v (i.e., after "velocity averaging"). The importance of measuring the vector fields $\nabla_v f$ and $v \cdot \nabla_x f$ using different norms also features prominently in other works including [7], but only spaces of positive regularity are considered there. Measuring the term $v \cdot \nabla_x f$ in a space of negative regularity in the v-variable is related to the idea of *velocity averaging*, the idea that one should expect better control of the spatial regularity of a solution of (1.1) or (1.2) after averaging in the velocity variable. This concept is therefore wired into the definition of the H^1_{hyp} norm, allowing us to perform velocity averaging in a systematic way.

Once we have proved that an arbitrary H^1_{hyp} function possesses at least a fractional derivative in the x variable, we are in a position to iterate the estimate by repeatedly differentiating the equation a fractional number of times to obtain higher regularity (and eventually smoothness, under appropriate assumptions on $\mathbf{b}$ and f^*) of weak solutions. In order to perform this iteration, we again depart from the original arguments of [22] and subsequent treatments and rely on an appropriate version of the Caccioppoli inequality (i.e., the basic L^2 energy estimate) for the equation (1.1). This avoids any recourse to the notion of pseudodifferential operators and once again mimics the classical functional analytic arguments in the uniformly elliptic setting.

The developments described above and even the variational structure identified for the equation (1.1) are not restricted to the time-independent setting. Indeed, we show that they can be adapted in a very straightforward way to the kinetic Fokker-Planck equation (1.2), the main difference being that the first-order part in a "sum-of-squares" representation of the differential operator is now $\partial_t - v \cdot \nabla_x$ instead of just $-v \cdot \nabla_x$. The adaptation thus consists in replacing the latter by the former throughout; the natural function space associated with equation (1.2), denoted by H^1_{kin}, is defined in (7.2)–(7.3). We also prove a Poincaré inequality for functions in H^1_{kin} which implies the uniform coercivity of the variational structure with respect to the H^1_{kin} norm. This allows us to give a rather direct and natural proof of exponential long-time decay to equilibrium for solutions of (1.2) with constant-in-time right-hand sides and null Dirichlet boundary conditions. This result (stated in Theorem 1.5 below) can be compared with the celebrated results of exponential convergence to equilibrium for kinetic Fokker-Planck equations on $\mathbb{R}^d$ with confining potentials, see in particular [13, 21, 20, 15, 14, 37, 6], as well as [34, 35] and references therein for a probabilistic approach. Compared to the previous approaches, our proof of exponential convergence is once again closer to the classical dissipative argument for the heat equation based on differentiating the square of the spatial L^2 norm of the solution. Informally, our method is based on the idea that hypocoercivity is simply coercivity with respect to the correct norm.

1.2. **Statements of the main results.** We begin by introducing the Sobolev-type function space H^1_{hyp} associated with the equation (1.1). We denote by γ the standard Gaussian measure on $\mathbb{R}^d$, defined by

$$d\gamma(v) := (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{1}{2}|v|^2\right) dv. \tag{1.4}$$

For each $p \in [1, \infty)$, we denote by $L^p_\gamma := L^p(\mathbb{R}^d, d\gamma)$ the Lebesgue space with norm

$$\|f\|_{L^p_\gamma} := \left(\int_{\mathbb{R}^d} |f(v)|^p \, d\gamma(v) \right)^{\frac{1}{p}},$$

and by H^1_γ the Banach space with norm

$$\|f\|_{H^1_\gamma} := \|f\|_{L^2_\gamma} + \|\nabla f\|_{L^2_\gamma}.$$

The dual space of H^1_γ is denoted by H^{-1}_γ. By abuse of notation, we typically denote the canonical pairing $\langle\cdot,\cdot\rangle_{H^1_\gamma,H^{-1}_\gamma}$ between $f \in H^1_\gamma$ and $f^* \in H^{-1}_\gamma$ by

$$\int_{\mathbb{R}^d} f f^*\, d\gamma := \langle f, f^*\rangle_{H^1_\gamma,H^{-1}_\gamma}. \tag{1.5}$$

Given $p \in [1,\infty)$, an open set $U \subseteq \mathbb{R}^d$ and an arbitrary Banach space X, we denote by $L^p(U;X)$ the Banach space consisting of measurable functions $f : U \to X$ with norm

$$\|f\|_{L^p(U;X)} := \left(\int_U \|f(x,\cdot)\|_X^p\, dx\right)^{\frac{1}{p}}.$$

We define the space $H^1_{\mathrm{hyp}}(U)$ by

$$H^1_{\mathrm{hyp}}(U) := \left\{f \in L^2\left(U;H^1_\gamma\right) \,:\, v\cdot\nabla_x f \in L^2(U;H^{-1}_\gamma)\right\} \tag{1.6}$$

and equip it with the norm

$$\|f\|_{H^1_{\mathrm{hyp}}(U)} := \|f\|_{L^2(U;H^1_\gamma)} + \|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}. \tag{1.7}$$

Given a bounded domain $U \subseteq \mathbb{R}^d$ and a vector field $\mathbf{b} \in L^\infty(U\times\mathbb{R}^d)^d$, we say that a function $f \in H^1_{\mathrm{hyp}}(U)$ is a *weak solution of* (1.1) *in* $U\times\mathbb{R}^d$ if

$$\forall h \in L^2(U;H^1_\gamma), \qquad \int_{U\times\mathbb{R}^d} \nabla_v h\cdot\nabla_v f\, dx\, d\gamma = \int_{U\times\mathbb{R}^d} h\left(f^* + v\cdot\nabla_x f - \mathbf{b}\cdot\nabla_v f\right) dx\, d\gamma.$$

As in (1.5), the precise interpretation of the right side is

$$\int_U \left\langle h(x,\cdot), \left(f^* + v\cdot\nabla_x f - \mathbf{b}\cdot\nabla_v f\right)(x,\cdot)\right\rangle_{H^1_\gamma,H^{-1}_\gamma} dx. \tag{1.8}$$

We assume throughout that the domain $U \subseteq \mathbb{R}^d$ is bounded and has a $C^{1,1}$ boundary. We denote by $\mathbf{n}_U$ the outward-pointing unit normal to ∂U, and define the *hypoelliptic boundary* of U by

$$\partial_{\mathrm{hyp}} U := \left\{(x,v) \in \partial U\times\mathbb{R}^d \,:\, v\cdot\mathbf{n}_U(x) > 0\right\}. \tag{1.9}$$

We denote by $H^1_{\mathrm{hyp},0}(U)$ the closure in $H^1_{\mathrm{hyp}}(U)$ of the set of smooth functions with compact support in $\overline{U}\times\mathbb{R}^d$ which vanish on $\partial_{\mathrm{hyp}}U$.

We give a first demonstration that $H^1_{\mathrm{hyp}}(U)$ is indeed the natural function space on which to build a theory of weak solutions of (1.1) by presenting a very general well-posedness result for the Dirichlet problem. To the best of our knowledge, this result is new in dimensions larger than one, even in the special case in which the vector field $\mathbf{b}$ and right-hand side f^* vanish, the domain U is a ball and the boundary data f_0 is smooth. As mentioned above, in the case $d = 1$, a weak solution theory for a PDE analogous to (1.1) was previously developed in [4]. There have also been previous results for Dirichlet problems associated with certain hypoelliptic equations, including the works of Oleĭnik and Radkevič [30, 31] and of Kohn and Nirenberg [23, 24], but their assumptions do not cover the case of equations such as (1.1), since they would require that the singular set be disjoint from the closure of the set $\{(x,v) \in \partial U\times\mathbb{R}^d \,:\, v\cdot\mathbf{n}_U(x) < 0\}$.

Theorem 1.1 (Well-posedness of the Dirichlet problem)**.** *Let U be a bounded $C^{1,1}$ domain, $\mathbf{b} \in L^\infty(U; C^{0,1}(\mathbb{R}^d))^d$, $f^* \in L^2(U; H^{-1}_\gamma)$ and $f_0 \in H^1_{\mathrm{hyp}}(U)$. There exists a unique weak solution f to the Dirichlet problem*

$$\begin{cases} -\Delta_v f + v \cdot \nabla_v f - v \cdot \nabla_x f + \mathbf{b} \cdot \nabla_v f = f^* & \text{in } U \times \mathbb{R}^d, \\ f = f_0 & \text{on } \partial_{\mathrm{hyp}} U. \end{cases} \tag{1.10}$$

That is, $f \in f_0 + H^1_{\mathrm{hyp},0}(U)$ and f is a weak solution of (1.1) in $U \times \mathbb{R}^d$. Furthermore, there exists a constant $C(\mathbf{b}, U, d) < \infty$ such that f satisfies the estimate

$$\|f\|_{H^1_{\mathrm{hyp}}(U)} \leq C\left(\|f_0\|_{H^1_{\mathrm{hyp}}(U)} + \|f^*\|_{L^2(U;H^{-1}_\gamma)} \right). \tag{1.11}$$

We next give an informal discussion regarding how one could naively guess that H^1_{hyp} is the "correct" space for solving (1.1), and how our proof of Theorem 1.1 will work. We take the simpler case of matrix inversion in finite dimensions as a starting point. Given two matrices A and B with B skew-symmetric and a vector f^*, consider the problem of finding f such that

$$(A^*A + B)f = f^*, \tag{1.12}$$

where A^* denotes the transpose of A. We propose to approach this problem by looking for a minimizer of the functional

$$f \mapsto \inf\left\{ \frac{1}{2}(Af - \mathbf{g}, Af - \mathbf{g}) \ : \ \mathbf{g} \ \text{ such that } \ A^*\mathbf{g} = f^* - Bf \right\},$$

where $(\cdot,\cdot)$ denotes the underlying scalar product. It is clear that the infimum is non-negative, and if f is a solution to (1.12), then choosing $\mathbf{g} = Af$ shows that this infimum is actually zero (null). Moreover, since B is skew-symmetric, whenever $(f, \mathbf{g})$ satisfy the constraint in the infimum above, we have

$$\frac{1}{2}(Af - \mathbf{g}, Af - \mathbf{g}) = \frac{1}{2}(Af, Af) + \frac{1}{2}(\mathbf{g}, \mathbf{g}) - (f, f^*). \tag{1.13}$$

The latter quantity is clearly a convex function of the pair $(f, \mathbf{g})$. The point is that under very mild assumptions on A and B, it will in fact be *uniformly* convex on the set of pairs $(f, \mathbf{g})$ satisfying the (linear) constraint $A^*\mathbf{g} = f^* - Bf$. Informally, the functional in (1.13) is coercive with respect to the seminorm $(f, \mathbf{g}) \mapsto |Af| + |\mathbf{g}| + |A(A^*A)^{-1}Bf|$.

With this analogy in mind, and assuming that $\mathbf{b}$ vanishes for simplicity, we rewrite the problem of finding a solution to (1.1) (with $\mathbf{b} \equiv 0$) as that of finding a null minimizer of the functional

$$f \mapsto \inf\left\{ \int_{U \times \mathbb{R}^d} \frac{1}{2} |\nabla_v f - \mathbf{g}|^2 \, dx \, d\gamma \ : \ \nabla_v^* \mathbf{g} = f^* - v \cdot \nabla_x f \right\}, \tag{1.14}$$

where $\nabla_v^* F := -\nabla_v \cdot F + v \cdot F$ is the formal adjoint of ∇_v in L^2_γ. It is clear that the infimum above is non-negative, and if we are provided with a solution f to (1.1) (with $\mathbf{b} \equiv 0$), then choosing $\mathbf{g} = \nabla_v f$ reveals that this infimum vanishes at f. This functional gives strong credence to the definition of the space $H^1_{\mathrm{hyp}}(U)$ given in (1.6). Using convex-analytic arguments, we show that the mapping in (1.14), where f varies in $f_0 + H^1_{\mathrm{hyp},0}(U)$ for some fixed $f_0 \in H^1_{\mathrm{hyp}}(U)$, is uniformly convex, and that its infimum is null. This implies the well-posedness of the problem (1.1) with $\mathbf{b} \equiv 0$. The proof of coercivity relies on the following Poincaré-type inequality for $H^1_{\mathrm{hyp}}(U)$, while the proof that the infimum is null relies on a careful study of the behavior of $H^1_{\mathrm{hyp}}(U)$ functions near the singular set of the boundary of U which we carry in Section 4.

For every $f \in L^1(U; L^1_\gamma)$, we denote $(f)_U := |U|^{-1} \int_{U \times \mathbb{R}^d} f(x, v) \, dx \, d\gamma(v)$.

Theorem 1.2 (Poincaré inequality for H^1_{hyp})**.** *For every bounded $C^{1,1}$ domain $U \subseteq \mathbb{R}^d$, there exists a constant $C(U,d) < \infty$ such that for every $f \in H^1_{\mathrm{hyp}}(U)$, we have*

$$\|f - (f)_U\|_{L^2(U;L^2_\gamma)} \leqslant C\left(\|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\right). \tag{1.15}$$

Moreover, if in addition $f \in H^1_{\mathrm{hyp},0}(U)$, then we have

$$\|f\|_{L^2(U;L^2_\gamma)} \leqslant C\left(\|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\right). \tag{1.16}$$

The inequality (1.15) asserts that, up to an additive constant, the full $H^1_{\mathrm{hyp}}(U)$ norm of a function f is controlled by the seminorm

$$[\![f]\!]_{H^1_{\mathrm{hyp}}(U)} := \|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}.$$

In particular, any distribution f with $[\![f]\!]_{H^1_{\mathrm{hyp}}(U)} < \infty$ is actually a function, which moreover belongs to $L^2_x L^2_\gamma$. The proof of Theorem 1.2 thus necessarily uses the Hörmander bracket condition, although in this case the way it is used is rather implicit. If we follow Hörmander's ideas more explicitly, then we obtain more information, namely some positive (fractional) regularity in the x variable. This is encoded in the following functional inequality, which we call the *Hörmander inequality*. The definitions of the fractional Sobolev spaces H^α used in the statement are given in Section 3.2, see (3.10).

Theorem 1.3 (Hörmander inequality for H^1_{hyp})**.** *Let $\alpha \in \left[0, \frac{1}{3}\right)$ and $v_0 \in [1,\infty)$. There exists a constant $C(\alpha, v_0, d) < \infty$ such that, for every $f \in H^1_{\mathrm{hyp}}(\mathbb{R}^d)$ satisfying*

$$f = 0 \quad \text{in } \mathbb{R}^d \times \left(\mathbb{R}^d \setminus B_{v_0}\right), \tag{1.17}$$

we have the estimate

$$\|f\|_{H^\alpha(\mathbb{R}^d;H^{-1}_\gamma)} \leqslant C\,\|f\|_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)}. \tag{1.18}$$

The inequality (1.18) gives control over a norm with positive regularity in x and negative regularity in v. By interpolation, we also obtain control over norms with positive regularity in both variables: see Corollary 3.4. The estimate should be considered as an interior estimate in both variables; in other words, for any $f \in H^1_{\mathrm{hyp}}(U)$, we can apply the inequality (1.18) after multiplying f by a smooth cutoff function which vanishes for large v and for x near ∂U. Therefore, compared to the Poincaré-type inequalities given in Theorem 1.2, which are global, the inequality (1.18) gives more regularity in x, but gives less control for x near ∂U or for v large.

Our next main result asserts that weak solutions of (1.1) are actually smooth. This is accomplished by an argument which closely parallels the one for obtaining H^k regularity for solutions of uniformly elliptic equations. We first obtain a version of the Caccioppoli inequality, that is, a reverse Poincaré inequality, which states that the H^1_{hyp} seminorm of a solution of (1.1) can be controlled by its L^2 oscillation (see Lemma 6.1 for the precise statement). Combined with Theorem 1.3, this tells us that a fractional spatial derivative of a solution of (1.1) can be controlled by the L^2 oscillation of the function itself. This estimate can then be iterated: we repeatedly differentiate the equation a fractional amount to obtain estimates of the higher derivatives of the solution in the x variable; we then obtain estimates for derivatives in the v variable relatively easily.

Notice that the following statement implies that solutions of (1.1) are C^∞ in both variables (x,v) provided that the vector field $\mathbf{b}$ is assumed to be smooth. For convenience, in the statement below we use the convention $C^{-1,1} = L^\infty$.

Theorem 1.4 (Interior $H^k(H^l_\gamma)$-type regularity for (1.1))**.** *Let $k,l \in \mathbb{N}$, $r, v_0 \in [1,\infty)$ and $\mathbf{b} \in C^{k+l-1,1}(B_r; C^{l-1,1}(\mathbb{R}^d;\mathbb{R}^d))$. There exists a constant $C < \infty$ depending on*

$$\left(d,k,l,r,v_0,\left\{\|\mathbf{b}\|_{C^{k+j-1,1}(B_r;C^{l-j-1,1}(\mathbb{R}^d))}\right\}_{j\in\{0,\dots,l\}}\right)$$

such that, for every $f \in H^1_{\mathrm{hyp}}(B_r)$ and $f^* \in H^{k+l}(B_r; H^{l-1}_\gamma)$ satisfying

$$-\Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f = f^* \quad \text{in } B_r\times\mathbb{R}^d, \tag{1.19}$$

we have that $f \in H^k\left(B_{r/2}; H^l_\gamma(B_{v_0})\right)$ and the estimate

$$\left\|\left|\nabla_v^l\nabla_x^k f\right|\right\|_{L^2\left(B_{r/2};L^2_\gamma(B_{v_0})\right)} \leqslant C\left(\left\|f-(f)_{B_r}\right\|_{L^2(B_r;L^2_\gamma)} + \sum_{j=0}^{l}\left\|\widetilde{f}^*\right\|_{H^{k+j}(B_r;H^{l-j-1}_\gamma)}\right).$$

The results stated above are for the time-independent Kramers equation (1.1). In Section 7, we develop an analogous theory for the time-dependent kinetic Fokker-Planck equation (1.2) with an associated function space H^1_{kin} (defined in (7.2)–(7.3)) in place of H^1_{hyp}. In particular, we obtain analogues of the results above for (1.2) which are stated in Section 7.

The long-time behavior of solutions of (1.2) has been studied by many authors in the last two decades: see the works of Desvillettes and Villani [13], Hérau and Nier [21], Helffer and Nier [20], Eckmann and Hairer [15], Desvillettes and Villani [14] and Villani [37] as well as the references in [37]. Most of these papers consider the case in which $\mathbf{b}(x) = -\nabla V(x)$ for a potential V which has sufficient growth at infinity. In this case there is an explicit invariant measure, namely $\exp(-V(x))dx\,d\gamma(v)$, and solutions of (1.2) can be expected to converge exponentially fast to the constant which is the integral of the initial data with respect to the invariant measure. This setting is in a certain sense easier than the Dirichlet problem, since one does not have to worry about the boundary. While our methods could also handle this setting, we formulate a result for the expontential convergence of a solution of the Cauchy-Dirichlet problem with constant-in-time right-hand side (the well-posedness of which is given below in Proposition 7.12) to the solution of the time-independent problem.

Theorem 1.5 (Convergence to equilibrium)**.** *Let $U \subseteq \mathbb{R}^d$ be a $C^{1,1}$ domain and $\mathbf{b} \in L^\infty(U; C^{0,1}(\mathbb{R}^d))^d$. There exists $\lambda\left(\|\mathbf{b}\|_{L^\infty(U\times\mathbb{R}^d)}, U, d\right) > 0$ such that, for every $T \in (0,\infty)$, $f \in H^1_{\mathrm{kin}}((0,T)\times U)$ and $f^* \in L^2(U; H^{-1}_\gamma)$ satisfying*

$$\begin{cases} \partial_t f - \Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f = f^* & \text{in } (0,T)\times U\times\mathbb{R}^d, \\ f = 0 & \text{on } (0,T)\times\partial_{\mathrm{hyp}}U\times\mathbb{R}^d, \end{cases} \tag{1.20}$$

if we denote the solution of (1.10) with $f_0 = 0$ by $f_\infty \in H^1_{\mathrm{hyp}}(U)$, then we have

$$\sup_{t\in(0,T)}\left(\exp(\lambda t)\left\|f(t,\cdot)-f_\infty\right\|_{L^2(U;L^2_\gamma)}\right) \leqslant 2\left\|f(0,\cdot)-f_\infty\right\|_{L^2(U;L^2_\gamma)}. \tag{1.21}$$

The precise sense of the boundary condition in (1.20) is given below in Section 7: see the statement of Proposition 7.15. Notice that interior regularity estimates immediately upgrade the L^2 convergence in (1.21) to convergence in spaces of higher regularity (at least in the interior) with the same exponential rate.

Unlike previous arguments establishing the exponential decay to equilibrium of solutions of (1.2) which are based on differentiation of rather ad-hoc quantities involving the solution and several (possibly mixed) derivatives in both x and v, the proof of Theorem 1.5 we give here is elementary and close to the classical dissipative estimate for uniformly parabolic

equations. The essential idea is to differentiate the square of the L^2 norm of the solution and then apply the Poincaré inequality. We cannot quite perform the computation exactly like this, and so we use a finite difference instead of the time derivative and apply a version of the Poincaré inequality adapted to the kinetic equation in a thin cylinder (see Proposition 7.2). Unlike previous approaches, our method therefore relates the positive constant λ in (1.21) to the optimal constant in a Poincaré-type inequality.

1.3. **Outline of the paper.** In the next section we present the function space $H^1_{\mathrm{hyp}}(U)$ and prove the density of smooth functions. In Section 3 we prove the functional inequalities stated in Theorems 1.2 and 1.3 and establish the compactness of the embedding of $H^1_{\mathrm{hyp}}(U)$ into $L^2(U;L^2_\gamma)$. We study the behavior of $H^1_{\mathrm{hyp}}(U)$ functions near the boundary of U in Section 4, establishing a trace theorem and an integration by parts formula. In Section 5 we prove Theorem 1.1 on the well-posedness of the Dirichlet problem for the Kramers equation. The interior regularity of solutions, and in particular Theorem 1.4, is obtained in Section 6. Finally, in Section 7 we prove the analogous results for the kinetic Fokker-Planck equation (1.2) as well as the exponential decay to equilibrium (Theorem 1.5). A brief discussion on the trace theorem appears in Appendix A.

2. The function space H^1_{hyp}

In this section, we define the function space $H^1_{\mathrm{hyp}}(U)$ and establish some basic properties. We start by setting up some notation that will be used throughout the paper. We denote the formal adjoint of the operator ∇_v by ∇_v^*; that is, for every $F \in (H^1_\gamma)^d$, we denote

$$\nabla_v^* F := -\nabla_v \cdot F + v \cdot F. \tag{2.1}$$

This definition can be extended to any $F \in (L^2_\gamma)^d$, in which case $\nabla_v^* F \in H^{-1}_\gamma$ and we have, for every $f \in H^1_\gamma$,

$$\int_{\mathbb{R}^d} f\, \nabla_v^* F \, d\gamma = \int_{\mathbb{R}^d} \nabla_v f \cdot F \, d\gamma.$$

Recall that the left side above is shorthand notation for the duality pairing between H^1_γ and H^{-1}_γ. We denote the average of a function $f \in L^1_\gamma$ by

$$\langle f \rangle_\gamma := \int_{\mathbb{R}^d} f \, d\gamma. \tag{2.2}$$

Since $1 \in H^1_\gamma$, the definition of $\langle f \rangle_\gamma$ can be extended to arbitrary $f \in H^{-1}_\gamma$. The Gaussian Poincaré inequality states that, for every $f \in H^1_\gamma$,

$$\| f - \langle f \rangle_\gamma \|_{L^2_\gamma} \leqslant \| \nabla_v f \|_{L^2_\gamma}.$$

We can thus replace $\|f\|_{L^2_\gamma}$ by $\left|\langle f \rangle_\gamma\right|$ in the definition of H^1_γ and have an equivalent norm:

$$\left|\langle f \rangle_\gamma\right| + \|\nabla f\|_{L^2_\gamma} \leqslant \|f\|_{H^1_\gamma} \leqslant \left|\langle f \rangle_\gamma\right| + 2\,\|\nabla f\|_{L^2_\gamma}.$$

This comparison of norms has the following counterpart for the dual space H^{-1}_γ.

Lemma 2.1 (Identification of H^{-1}_γ)**.** *There exists a universal constant $C < \infty$ such that for every $f^* \in H^{-1}_\gamma$,*

$$C^{-1}\|f^*\|_{H^{-1}_\gamma} \leqslant \left|\langle f^* \rangle_\gamma\right| + \inf\left\{ \|\mathbf{h}\|_{L^2_\gamma} \,:\, \nabla_v^* \mathbf{h} = f^* - \langle f^* \rangle_\gamma \right\} \leqslant C \|f^*\|_{H^{-1}_\gamma}. \tag{2.3}$$

Proof. The bilinear form

$$(f,g) \mapsto \langle f \rangle_\gamma \langle g \rangle_\gamma + \int_{\mathbb{R}^d} \nabla_v f \cdot \nabla_v g \, d\gamma$$

is a scalar product for the Hilbert space H^1_γ. By the Riesz representation theorem, for every $f^* \in H^{-1}_\gamma$, there exists $g \in H^1_\gamma$ such that

$$\forall f \in H^1_\gamma \qquad \int_{\mathbb{R}^d} f f^* \, d\gamma = \langle f \rangle_\gamma \langle g \rangle_\gamma + \int_{\mathbb{R}^d} \nabla_v f \cdot \nabla_v g \, d\gamma.$$

(Recall that the integral on the left side is convenient notation for the canonical pairing between H^1_γ and H^{-1}_γ.) We clearly have $\langle g \rangle_\gamma = \langle f^* \rangle_\gamma$, and thus

$$|\langle g \rangle_\gamma|^2 + \int_{\mathbb{R}^d} |\nabla_v g|^2 \, d\gamma \leqslant \|g\|_{H^1_\gamma} \|f^*\|_{H^{-1}_\gamma}.$$

This implies that $\|\nabla_v g\|_{L^2_\gamma} \leqslant C \|f^*\|_{H^{-1}_\gamma}$, and since $\nabla_v^* \nabla_v g = f^* - \langle f^* \rangle_\gamma$, this proves the rightmost inequality in (2.3). Conversely, for any $\mathbf{h} \in L^2_\gamma$, if

$$f^* = \langle f^* \rangle_\gamma + \nabla_v^* \mathbf{h},$$

then for every $f \in H^1_\gamma$,

$$\left| \int_{\mathbb{R}^d} f f^* \, d\gamma \right| \leqslant |\langle f \rangle_\gamma| \, |\langle f^* \rangle_\gamma| + \|\nabla f\|_{L^2_\gamma} \|\mathbf{h}\|_{L^2_\gamma},$$

and thus the leftmost inequality in (2.3) holds. □

We often work with the dual pair of Banach spaces $L^2(U; H^1_\gamma)$ and $L^2(U; H^{-1}_\gamma)$. With the identification given by Lemma 2.1, we have

$$\|f^*\|_{L^2(U;H^{-1}_\gamma)} \simeq \left\| \langle f^* \rangle_\gamma \right\|_{L^2(U)} + \inf \left\{ \|\mathbf{g}\|_{L^2(U;L^2_\gamma)} \,:\, \nabla_v^* \mathbf{g} = f^* - \langle f^* \rangle_\gamma \right\}, \tag{2.4}$$

in the sense that the norms on each side are equivalent.

For convenience, for every $f \in L^1(U; L^1_\gamma)$, we use the shorthand notation

$$(f)_U := |U|^{-1} \int_{U \times \mathbb{R}^d} f(x,v) \, dx \, d\gamma(v). \tag{2.5}$$

We will occasionally also use this notation in the case when f depends only on the space variable x, in which case we simply have $(f)_U = |U|^{-1} \int_U f$.

2.1. Density of smooth functions in H^1_{hyp}.

We show that the set of smooth functions is dense in H^1_{hyp}.

Proposition 2.2. *The set $C^\infty_c(\overline{U} \times \mathbb{R}^d)$ of smooth functions with compact support in $\overline{U} \times \mathbb{R}^d$ is dense in $H^1_{\mathrm{hyp}}(U)$.*

Proof. We decompose the proof into three steps.

Step 1. In this step, we show that it suffices to consider the case when U satisfies a convenient quantitative form of the star-shape property. For every $z \in \partial U$, there exist a radius $r > 0$ and a Lipschitz function $\Psi \in C^{0,1}(\mathbb{R}^{d-1}; \mathbb{R})$ such that, up to a relabelling of the axes, we have

$$U \cap B(z,r) = \{x = (x_1, \ldots, x_d) \in B(z,r) \,:\, x_d > \Psi(x_1, \ldots, x_{d-1})\}.$$

Since Ψ is a Lipschitz function, there exists $\delta > 0$ such that for every $x \in U \cap B(z,r)$, we have the cone containment property

$$\left\{ x + y \,:\, \frac{y_d}{|y|} \geqslant 1 - \delta \right\} \cap B(z,r) \subseteq U. \tag{2.6}$$

Setting

$$z' = z + \left(0, \dots, 0, \frac{r}{2}\right) \in \mathbb{R}^d,$$

and reducing $\delta > 0$ if necessary, we claim that for every $x \in U \cap B(z, \delta^2)$ and $\varepsilon \in (0,1]$, we have

$$B\left(x - \varepsilon(x - z'), \delta^2 \varepsilon\right) \subseteq U. \tag{2.7}$$

Assuming the contrary, let $y \in \mathbb{R}^d$ be such that

$$x + y \in B\left(x - \varepsilon(x - z'), \delta^2 \varepsilon\right) \setminus U.$$

Then

$$\left|y + \varepsilon(x - z')\right| \leqslant \delta^2 \varepsilon,$$

and therefore

$$\begin{aligned}\left|y - \varepsilon\left(0, \dots, 0, \frac{r}{2}\right)\right| &\leqslant \left|y + \varepsilon(x - z) - \varepsilon\left(0, \dots, 0, \frac{r}{2}\right)\right| + \varepsilon|x - z| \\ &\leqslant \left|y + \varepsilon(x - z')\right| + \varepsilon|x - z| \\ &\leqslant 2\delta^2 \varepsilon.\end{aligned}$$

Taking $\delta > 0$ sufficiently small, we arrive at a contradiction with the cone property (2.6). Now that (2.7) is proved for every x in a relative neighborhood of z, and up to a further reduction of the value of $\delta > 0$ if necessary, it is not difficult to show that one can find an open set U' containing z and z' and such that (2.7) holds for every $x \in U \cap U'$.

Summarizing, and using the fact that U is a bounded set, we have shown that there exist families of bounded open sets $U_1, \dots, U_M \subseteq \mathbb{R}^d$, of points $x_1, \dots, x_M \in \mathbb{R}^d$ and a parameter $r > 0$ such that

$$U = \bigcup_{k=1}^{M} U_i$$

and for every $k \in \{1, \dots, M\}$, $x \in U_k$ and $\varepsilon \in (0,1]$,

$$B\left(x - \varepsilon(x - x_k), r\varepsilon\right) \subseteq U_k.$$

By using a partition of unity, we can reduce our study to the case when this property is satisfied for the domain U itself (in place of each of the U_k's). By translation, we may assume that the reference point x_k is at the origin, and by scaling, we may also assume that this property holds with $r = 1$. That is, from now on, we assume that for every $x \in U$ and $\varepsilon \in (0,1]$, we have

$$B\left((1 - \varepsilon)x, \varepsilon\right) \subseteq U. \tag{2.8}$$

Step 2. Let $f \in H^1_{\mathrm{hyp}}(U)$. We aim to show that f belongs to the closure of the set $C^\infty_c(\overline{U} \times \mathbb{R}^d)$ in $H^1_{\mathrm{hyp}}(U)$. Without loss of generality, we may assume that f is compactly supported in $\overline{U} \times \mathbb{R}^d$. Indeed, if $\chi \in C^\infty_c(\mathbb{R}^d; \mathbb{R})$ is a smooth function with compact support and such that $\chi \equiv 1$ in a neighborhood of the origin, then the function $(x, v) \mapsto f(x, v)\chi(v/M)$ belongs to $H^1_{\mathrm{hyp}}(U)$ and converges to f in $H^1_{\mathrm{hyp}}(U)$ as M tends to infinity.

Let $\zeta \in C^\infty_c(\mathbb{R}^d; \mathbb{R})$ be a smooth function with compact support in $B(0,1)$ and such that $\int_{\mathbb{R}^d} \zeta = 1$. For each $\varepsilon > 0$ and $x \in \mathbb{R}^d$, we write

$$\zeta_\varepsilon(x) := \varepsilon^{-d} \zeta(\varepsilon^{-1} x), \tag{2.9}$$

and we define, for each $\varepsilon \in \left(0, \frac{1}{2}\right]$, $x \in U$ and $v \in \mathbb{R}^d$,

$$f_\varepsilon(x,v) := \int_{\mathbb{R}^d} f((1-\varepsilon)x + y, v)\zeta_\varepsilon(y)\,dy.$$

Note that this definition makes sense by the assumption of (2.8). The goal of this step is to show that f belongs to the closure in $H^1_{\mathrm{hyp}}(U)$ of the convex hull of the set $\left\{f_\varepsilon \,:\, \varepsilon \in \left(0, \frac{1}{2}\right]\right\}$. By Mazur's lemma (see [16, page 6]), it suffices to show that f_ε converges weakly to f in $H^1_{\mathrm{hyp}}(U)$. Since it is elementary to show that f_ε converges to f in the sense of distributions, this boils down to checking that f_ε is bounded in $H^1_{\mathrm{hyp}}(U)$. By Jensen's inequality,

$$\begin{aligned}\|\nabla_v f_\varepsilon\|^2_{L^2(U;L^2_\gamma)} &\leqslant \int_{U\times\mathbb{R}^d}\int_{\mathbb{R}^d} |\nabla_v f|^2\left((1-\varepsilon)x+y,v\right)\zeta_\varepsilon(y)\,dy\,dx\,d\gamma(v)\\ &\leqslant (1-\varepsilon)^{-1}\|\nabla_v f\|^2_{L^2(U;L^2_\gamma)}.\end{aligned}$$

In order to evaluate $\|v\cdot\nabla_x f_\varepsilon\|_{L^2(U;H^{-1}_\gamma)}$, we compute, for every $\varphi \in L^2(U;H^1_\gamma)$,

$$\begin{aligned}&\int_{U\times\mathbb{R}^d} v\cdot\nabla_x f_\varepsilon\,\varphi\,dx\,d\gamma\\ &\quad= (1-\varepsilon)\int_{U\times\mathbb{R}^d}\int_{\mathbb{R}^d} v\cdot\nabla_x f\left((1-\varepsilon)x+y,v\right)\zeta_\varepsilon(y)\varphi(x,v)\,dy\,dx\,d\gamma(v)\\ &\quad= \int_{U\times\mathbb{R}^d}\int_{\mathbb{R}^d} v\cdot\nabla_x f\left(x+y,v\right)\zeta_\varepsilon(y)\,\varphi\left(\frac{x}{1-\varepsilon},v\right)dy\,dx\,d\gamma(v)\\ &\quad= \int_{U\times\mathbb{R}^d}\int_{\mathbb{R}^d} v\cdot\nabla_x f\left(y,v\right)\zeta_\varepsilon(y-x)\,\varphi\left(\frac{x}{1-\varepsilon},v\right)dy\,dx\,d\gamma(v).\end{aligned}$$

Since, by Jensen's inequality,

$$\int_{U\times\mathbb{R}^d}\left|\int_U \zeta_\varepsilon(y-x)\varphi\left(\frac{x}{1-\varepsilon},v\right)dx\right|^2 dy\,d\gamma(v) \leqslant (1-\varepsilon)^{-1}\|\varphi\|^2_{L^2(U;L^2_\gamma)}$$

as well as

$$\int_{U\times\mathbb{R}^d}\left|\int_U \zeta_\varepsilon(y-x)\nabla_v\varphi\left(\frac{x}{1-\varepsilon},v\right)dx\right|^2 dy\,d\gamma(v) \leqslant (1-\varepsilon)^{-1}\|\nabla_v\varphi\|^2_{L^2(U;L^2_\gamma)},$$

we deduce that

$$\int_{U\times\mathbb{R}^d} v\cdot\nabla_x f_\varepsilon\,\varphi\,dx\,d\gamma \leqslant (1-\varepsilon)^{-\frac{1}{2}}\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\,\|\varphi\|_{L^2(U;H^1_\gamma)},$$

and therefore

$$\|v\cdot\nabla_x f_\varepsilon\|_{L^2(U;H^{-1}_\gamma)} \leqslant (1-\varepsilon)^{-\frac{1}{2}}\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}.$$

This completes the proof that the set $\left\{f_\varepsilon \,:\, \varepsilon \in \left(0, \frac{1}{2}\right]\right\}$ is bounded in $H^1_{\mathrm{hyp}}(U)$, and thus that f belongs to the closed convex hull of this set.

Step 3. It remains to be shown that for each fixed $\varepsilon \in \left(0, \frac{1}{2}\right]$, the function f_ε belongs to the closure in $H^1_{\mathrm{hyp}}(U)$ of the set $C^\infty_c(\overline{U}\times\mathbb{R}^d)$. For every $\eta \in (0,1]$, we define

$$\begin{aligned}f_{\varepsilon,\eta}(x,v) &:= \int_{\mathbb{R}^d} f_\varepsilon(x,w)\zeta_\eta(v-w)\,dw\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y,w)\zeta_\varepsilon(y-(1-\varepsilon)x)\zeta_\eta(v-w)\,dy\,dw.\end{aligned}$$

From the last expression, we see that $f_{\varepsilon,\eta}$ belongs to $C^\infty_c(\overline{U}\times\mathbb{R}^d)$ (recall that f itself has compact support in $\overline{U}\times\mathbb{R}^d$). Moreover, since $\nabla_v f_\varepsilon \in L^2(U;L^2_\gamma)$ and

$$\nabla_v f_{\varepsilon,\eta}(x,v) = \int_{\mathbb{R}^d} \nabla_v f_\varepsilon(x,v-w)\zeta_\eta(w)\,dw,$$

it is classical to verify that $\nabla_v f_{\varepsilon,\eta}$ converges to $\nabla_v f_\varepsilon$ in $L^2(U;L^2_\gamma)$ as η tends to 0. By the definition of f_ε and the fact that f_ε is compactly supported, we have that $v\cdot\nabla_x f_\varepsilon \in L^2(U;L^2_\gamma)$. The same reasoning as above thus gives that $v\cdot\nabla_x f_{\varepsilon,\eta}$ converges to $v\cdot\nabla_x f_\varepsilon$ in $L^2(U;L^2_\gamma)$, and thus a fortiori in $L^2(U;H^{-1}_\gamma)$, as η tends to 0. This shows that

$$\lim_{\eta\to 0}\|f_{\varepsilon,\eta}-f_\varepsilon\|_{H^1_{\mathrm{hyp}}(U)}=0$$

and thus completes the proof of the proposition. □

3. Functional inequalities for H^1_{hyp}

In this section we present the proofs of Theorems 1.2 and 1.3.

3.1. The Poincaré inequality for H^1_{hyp}.

We begin with the proof of Theorem 1.2, the Poincaré-type inequality for the space $H^1_{\mathrm{hyp}}(U)$. The proof requires the following fact regarding the equivalence (up to additive constants) of the norms $\|h\|_{L^2(U)}$ and $\|\nabla h\|_{H^{-1}(U)}$.

Lemma 3.1. *There exists $C(U,d)<\infty$ such that for every $h\in L^2(U)$,*

$$\|h-(h)_U\|_{L^2(U)}\leqslant C\|\nabla h\|_{H^{-1}(U)}.$$

Proof. Without loss of generality, we assume that $(h)_U=0$. We consider the problem

$$\begin{cases}\nabla\cdot\mathbf{f}=h & \text{in } U,\\ \mathbf{f}=0 & \text{on } \partial U.\end{cases}\tag{3.1}$$

According to [11, Theorem 9.2 and Remark 9.3(iii)], this problem has a solution $\mathbf{f}$ with components in $H^1_0(U)$ satisfying the estimate

$$\|\mathbf{f}\|_{H^1(U)}\leqslant C\,\|h\|_{L^2(U)}\,.\tag{3.2}$$

Then we have

$$\|h\|^2_{L^2(U)}=\int_U h\,\nabla\cdot\mathbf{f}=-\int_U\nabla h\cdot\mathbf{f}\leqslant\|\nabla h\|_{H^{-1}(U)}\,\|\mathbf{f}\|_{H^1(U)}.$$

The conclusion then follows by (3.2). □

Proof of Theorem 1.2. Let $f\in H^1_{\mathrm{hyp}}(U)$. In view of Proposition 2.2, we can without loss of generality assume that f is a smooth function. We decompose the proof into five steps.

Step 1. We show that

$$\|f-\langle f\rangle_\gamma\|_{L^2(U;L^2_\gamma)}\leqslant\|\nabla_v f\|_{L^2(U;L^2_\gamma)}\,.\tag{3.3}$$

By the Gaussian Poincaré inequality, we have for every $x\in U$ that

$$\|f(x,\cdot)-\langle f\rangle_\gamma(x)\|_{L^2_\gamma}\leqslant\|\nabla_v f(x,\cdot)\|_{L^2_\gamma}\,.$$

This yields (3.3) after integration over $x\in U$.

Step 2. We show that

$$\|\nabla\langle f\rangle_\gamma\|_{H^{-1}(U)}\leqslant C\left(\|\nabla_v f\|_{L^2(U;L^2_\gamma)}+\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\right).\tag{3.4}$$

We select $\xi_1,\dots,\xi_d\in C^\infty_c(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d}v\xi_i(v)\,d\gamma(v)=e_i,\tag{3.5}$$

and for each test function $\phi \in H_0^1(U)$ and $i \in \{1,\dots,d\}$, we compute

$$\begin{aligned}\int_U \partial_{x_i}\phi(x)\langle f\rangle_\gamma(x)\,dx &= \int_{U\times\mathbb{R}^d} v\cdot\nabla_x\phi(x)\langle f\rangle_\gamma(x)\xi_i(v)\,dx\,d\gamma(v)\\ &= \int_{U\times\mathbb{R}^d} v\cdot\nabla_x\phi(x) f(x,v)\xi_i(v)\,dx\,d\gamma(v)\\ &\quad + \int_{U\times\mathbb{R}^d} v\cdot\nabla_x\phi(x)\left(f(x,v)-\langle f\rangle_\gamma(x)\right)\xi_i(v)\,dx\,d\gamma(v).\end{aligned}$$

To control the first term on the right side, we perform an integration by parts to obtain

$$\begin{aligned}\left|\int_{U\times\mathbb{R}^d} v\cdot\nabla_x\phi(x) f(x,v)\xi_i(v)\,dx\,d\gamma(v)\right| &= \left|\int_{U\times\mathbb{R}^d} \phi(x)\xi_i(v)\,v\cdot\nabla_x f(x,v)\,dx\,d\gamma(v)\right|\\ &\leqslant C\,\|\phi\xi_i\|_{L^2(U;H^1_\gamma)}\,\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\\ &\leqslant C\,\|\phi\|_{L^2(U)}\,\|\xi_i\|_{H^1_\gamma}\,\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\\ &\leqslant C\,\|\phi\|_{L^2(U)}\,\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}.\end{aligned}$$

To control the second term, we use (3.3) and the fact that ξ_i has compact support:

$$\begin{aligned}&\left|\int_{U\times\mathbb{R}^d} v\cdot\nabla_x\phi(x)\left(f(x,v)-\langle f\rangle_\gamma(x)\right)\xi_i(v)\,dx\,d\gamma(v)\right|\\ &\quad\leqslant C\int_{U\times\mathbb{R}^d}|v||\xi_i(v)|\,|\nabla_x\phi(x)|\,|f(x,v)-\langle f\rangle_\gamma(x)|\,dx\,d\gamma(v)\\ &\quad\leqslant C\,\|\phi\|_{H^1(U)}\,\|\nabla_v f\|_{L^2(U;L^2_\gamma)}.\end{aligned}$$

Combining the above displays and taking the supremum over $\phi \in H_0^1(U)$ with $\|\phi\|_{H^1(U)} \leqslant 1$ yields (3.4).

Step 3. We deduce from Lemma 3.1, (3.3) and (3.4) that

$$\begin{aligned}\|f-(f)_U\|_{L^2(U;L^2_\gamma)} &\leqslant \left\|f-\langle f\rangle_\gamma\right\|_{L^2(U;L^2_\gamma)} + \left\|\langle f\rangle_\gamma-(f)_U\right\|_{L^2(U)}\\ &\leqslant \left\|f-\langle f\rangle_\gamma\right\|_{L^2(U;L^2_\gamma)} + C\left\|\nabla\langle f\rangle_\gamma\right\|_{H^{-1}(U)}\\ &\leqslant C\left(\|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\right).\end{aligned}$$

This completes the proof of (1.15).

Step 4. To complete the proof of (1.16), we must show that, under the additional assumption that $f \in H^1_{\mathrm{hyp},0}(U)$, we have

$$|(f)_U| \leqslant C\left(\|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\right). \tag{3.6}$$

Let f_1 be a test function belonging to $C_c^\infty\left(\overline{U}\times\mathbb{R}^d\right)$, to be constructed below, which satisfies the following:

$$f_1 = 0 \quad \text{on } (\partial U\times\mathbb{R}^d)\setminus\partial_{\mathrm{hyp}}(U), \tag{3.7}$$

$$⨍_U\int_{\mathbb{R}^d} v\cdot\nabla_x f_1\,d\gamma\,dx = 1 \tag{3.8}$$

and, for some constant $C(U,d)<\infty$,

$$\|v\cdot\nabla_x f_1\|_{L^2(U;L^2_\gamma)} \leqslant C. \tag{3.9}$$

The test function f_1 is constructed in Step 5 below. We first use it to obtain (3.6). We proceed by using (3.8) to split the mean of f as

$$(f)_U = ⨍_U\int_{\mathbb{R}^d} f\,v\cdot\nabla_x f_1\,d\gamma\,dx - ⨍_U\int_{\mathbb{R}^d}(f-(f)_U)\,v\cdot\nabla_x f_1\,d\gamma\,dx$$

and estimate the two terms on the right side separately. For the first term, we have

$$\begin{aligned}
&\left| \fint_U \int_{\mathbb{R}^d} f\, v\cdot\nabla_x f_1 \, d\gamma\, dx \right| \\
&\quad = \left| -\fint_U \int_{\mathbb{R}^d} f_1\, v\cdot\nabla_x f \, d\gamma\, dx + \frac{1}{|U|}\int_{\partial U}\int_{\mathbb{R}^d} (v\cdot \mathbf{n}_U) f f_1 \, d\gamma\, dx \right| \\
&\quad = \left| \fint_U \int_{\mathbb{R}^d} f_1 v\cdot\nabla_x f \, d\gamma\, dx \right|,
\end{aligned}$$

where we used that $(v\cdot\mathbf{n}_U)ff_1$ vanishes on $\partial U\times\mathbb{R}^d$ to remove the boundary integral. (Recall that by the definition of $H^1_{\mathrm{hyp},0}(U)$, we can assume without loss of generality that the function f is smooth, so the justification of the integration by parts above is classical.) We thus obtain that

$$\left| \fint_U \int_{\mathbb{R}^d} f_1 v\cdot\nabla_x f \, d\gamma\, dx \right| \leqslant \frac{1}{|U|}\, \|f_1\|_{L^2(U;H^1_\gamma)}\, \|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\,.$$

This completes the estimate for the first term. For the second term, we use (3.9) to get

$$\begin{aligned}
\left| \fint_U \int_{\mathbb{R}^d} (f-(f)_U)\; v\cdot\nabla_x f_1 \, d\gamma\, dx \right| &\leqslant \|f-(f)_U\|_{L^2(U;L^2_\gamma)}\, \|v\cdot\nabla_x f_1\|_{L^2(U;L^2_\gamma)} \\
&\leqslant C\, \|f-(f)_U\|_{L^2(U;L^2_\gamma)}\,,
\end{aligned}$$

which is estimated using the result of Step 3. Putting these together yields (3.6).

Step 5. We construct the test function $f_1 \in C^\infty_c\left(\overline{U}\times\mathbb{R}^d\right)$ satisfying (3.7), (3.8) and (3.9). Fix $x_0\in\partial U$. Since the unit normal $\mathbf{n}_U$ is continuous at x_0, there exist $v_0\in\mathbb{R}^d$ and $r>0$ such that for every $x,v\in\mathbb{R}^d$ satisfying $(x,v)\in(B_r(x_0)\cap\partial U)\times B_r(v_0)$, we have $v\cdot\mathbf{n}_U(x)>0$. In other words, every $(x,v)\in(B_r(x_0)\cap\partial U)\times B_r(v_0)$ is such that $(x,v)\in\partial_{\mathrm{hyp}}U$. Observe that, for every $f_1\in C^\infty_c(\mathbb{R}^d\times\mathbb{R}^d)$, we have

$$\fint_U \int_{\mathbb{R}^d} v\cdot\nabla_x f_1 \, d\gamma\, dx = \frac{1}{|U|}\int_{\partial U}\int_{\mathbb{R}^d} (v\cdot\mathbf{n}_U) f_1 \, d\gamma\, dx.$$

We select a function $f_1\in C^\infty_c(\mathbb{R}^d\times\mathbb{R}^d)$ with compact support in $B_r(x_0)\times B_r(v_0)$ and such that $f_1\geqslant 0$ and $f_1(x_0,v_0)=1$. In this case, the integral on the right side above is nonnegative, since f_1 vanishes whenever $v\cdot\mathbf{n}_U\leqslant 0$. In fact, since f_1 is positive on a set of positive measure on $\partial U\times\mathbb{R}^d$ (in the sense of the product of the $(d-1)$-dimensional Hausdorff and Lebesgue measures), the integral above is positive. Up to multiplying f_1 by a positive scalar if necessary, we can thus ensure that (3.8) holds. It is clear that this construction also ensures that (3.7) and (3.9) hold. □

Remark 3.2. As the argument above reveals, for the inequality (1.16) to hold, the assumption of $f\in H^1_{\mathrm{hyp},0}(U)$ can be weakened: it suffices that f vanishes on a relatively open piece of the boundary $\partial U\times\mathbb{R}^d$. The constant C in (1.16) then depends additionally on the identity of this piece of the boundary where f is assumed to vanish.

3.2. **The Hörmander inequality for H^1_{hyp}.** In this subsection, we use the Hörmander bracket condition to obtain a functional inequality which provides some interior spatial regularity for general H^1_{hyp} functions. Both the statement and proof of the inequality follow closely the ideas of Hörmander [22]. The main difference compared to [22] is that for a function f in H^1_{hyp}, we control $v\cdot\nabla_x f$ only weakly in the v variable (as opposed to an L^2 control in [22]). Other variants of Hörmander's inequality have been previously obtained, see in particular [7].

Since we must work in spaces of fractional differentiability, we introduce the Banach space-valued fractional Sobolev spaces, defined as follows: for every domain $U \subseteq \mathbb{R}^d$, $\alpha \in (0,1)$, Banach space X with norm $\|\cdot\|_X$ and $u \in L^2(U;X)$, we define the seminorm

$$[\![u]\!]_{H^\alpha(U;X)} := \left(\int_U \int_U \frac{\|u(x)-u(y)\|_X^2}{|x-y|^{d+2\alpha}} \, dx \, dy \right)^{\frac{1}{2}} \tag{3.10}$$

and the norm

$$\|u\|_{H^\alpha(U;X)} := \|u\|_{L^2(U;X)} + [\![u]\!]_{H^\alpha(U;X)} .$$

We then define the fractional Sobolev space

$$H^\alpha(U;X) := \left\{ u \in L^2(U;X) \,:\, \|u\|_{H^\alpha(U;X)} < \infty \right\}. \tag{3.11}$$

The space $H^\alpha(U;X)$ is a Banach space under the norm $\|\cdot\|_{H^\alpha(U;X)}$. We understand that $H^0(U;X) = L^2(U;X)$. We also set

$$\|u\|_{H^{1+\alpha}(U;X)} := \|u\|_{L^2(U;X)} + \|\nabla u\|_{H^\alpha(U;X)},$$

and define the Banach space $H^{1+\alpha}(U;X)$ as in (3.11).

We will use the following heat kernel representation of H^α spaces. We define the standard heat kernel by setting, for every $t > 0$ and $x \in \mathbb{R}^d$,

$$\Phi(t,x) := (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{4t} \right). \tag{3.12}$$

For every $\alpha \in [0,1)$, we have that

$$\|f\|_{H^\alpha(\mathbb{R}^d;X)}^2 \simeq \|f\|_{L^2(\mathbb{R}^d;X)}^2 + \int_0^1 \int_{\mathbb{R}^d} t^{-\alpha} \|(\nabla \Phi(t,\cdot) * f)(x)\|_X^2 \, dx \, dt, \tag{3.13}$$

and for every $\alpha \in [0,2)$, that

$$\|f\|_{H^\alpha(\mathbb{R}^d;X)}^2 \simeq \|f\|_{L^2(\mathbb{R}^d;X)}^2 + \int_0^1 \int_{\mathbb{R}^d} t^{1-\alpha} \left\|(\nabla^2 \Phi(t,\cdot) * f)(x)\right\|_X^2 \, dx \, dt, \tag{3.14}$$

in the sense that in (3.13) and (3.14), the squared norms on each side of the symbol $\simeq$ are equivalent, with implicit constants that may depend on α and d. A proof of these facts can be found in [36, Theorems 1.8.1] or in [3, Appendix D] in the case of real-valued functions. The argument for functions valued in a Banach space is the same (or alternatively, this more general statement can be obtained by duality from the real-valued case).

We also need to consider fractional Sobolev spaces in the velocity variable. In this case, the relevant spaces are weighted by the measure γ, which is strongly inhomogeneous. For instance, while the translation of the measure γ by a fixed vector $y \in \mathbb{R}^d$ is absolutely continuous with respect to γ, the associated Radon-Nikodym derivative is unbounded (unless $y = 0$). This renders a definition based on finite differences as in (3.10) inappropriate for spaces weighted by the measure γ. We choose instead to use the following definition. For each $f \in L^2_\gamma$ and $t > 0$, we set

$$K(t,f) := \inf \left\{ \|f_0\|_{L^2_\gamma} + t\|f_1\|_{H^1_\gamma} \,:\, f = f_0 + f_1,\ f_0 \in L^2_\gamma,\ f_1 \in H^1_\gamma \right\},$$

and, for every $\alpha \in (0,1)$, we define

$$\|f\|_{H^\alpha_\gamma} := \left(\int_0^\infty \left(t^{-\alpha} K(f,t)\right)^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \tag{3.15}$$

We also define $H^{-\alpha}_\gamma$ to be the space dual to H^α_γ.

As it turns out, we typically refer to such fractional spaces only on the subspace of functions (or distributions) f that have a uniform cutoff in the velocity variable. That is, we typically consider, for a fixed $v_0 \in [1,\infty)$, spaces of the form

$$H^\alpha_\gamma \cap \{f \in L^2_\gamma \,:\, f\mathbb{1}_{\{|v|>v_0\}} = 0\}. \tag{3.16}$$

In this case, we could as well work with a definition of fractional spaces in the velocity variable constructed in the same way as the fractional spaces in the spatial variable, based on finite differences as in (3.10). Indeed, with the additional restriction on the velocity variable in (3.16), the problem of the heterogeneity of the measure γ disappears, and the norm based on finite differences is indeed equivalent to the norm defined in (3.15). The proof of this classical fact can be found for instance in [26, Example 1.8], using also that, for every $f \in L^2_\gamma$ satisfying $f\mathbb{1}_{\{|v|>v_0\}} = 0$ and $w \in \mathbb{R}^d$, we have

$$\begin{aligned} \|f(\cdot + w)\|^2_{L^2_\gamma} &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f^2(v+w) \exp\left(-\frac{|v|^2}{2}\right) dv \\ &\leqslant (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f^2(v)\, dv \leqslant \exp\left(\frac{{v_0}^2}{2}\right) \|f\|^2_{L^2_\gamma}. \end{aligned} \tag{3.17}$$

We will prove in Theorem 1.3 below that, for every $\alpha < \frac{1}{3}$ and $v_0 \in [1,\infty)$, we have the embedding

$$H^1_{\mathrm{hyp}}(\mathbb{R}^d) \cap \left\{f \,:\, f\mathbb{1}_{\{|v|>v_0\}} = 0\right\} \hookrightarrow H^\alpha(\mathbb{R}^d; H^{-1}_\gamma). \tag{3.18}$$

Since $H^1_{\mathrm{hyp}}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d; H^1_\gamma)$ by definition, an interpolation argument also gives, for each $\beta < \frac{1}{6}$ and $v_0 \in [1,\infty)$, the embedding

$$H^1_{\mathrm{hyp}}(\mathbb{R}^d) \cap \left\{f \,:\, f\mathbb{1}_{\{|v|>v_0\}} = 0\right\} \hookrightarrow H^\beta(\mathbb{R}^d; L^2_\gamma), \tag{3.19}$$

as well as embeddings into other similar spaces of positive regularity in both variables: see Corollary 3.4 below. As already discussed above, the reason for the cutoff in the v variable has to do with technical difficulties arising due to the "tails" of the Gaussian and does not arise, for instance, if we adapt our arguments to the "flat" case in which we delete the friction term $v \cdot \nabla_v$ from the equation. Observe that, by introducing a cutoff function in the spatial variable, we also obtain analogous embeddings for bounded domains, such as

$$H^1_{\mathrm{hyp}}(U) \cap \left\{f \,:\, f\mathbb{1}_{\{|v|>v_0\}} = 0\right\} \hookrightarrow H^\alpha(U_\delta; H^{-1}_\gamma),$$

valid for every $\alpha < \frac{1}{3}$, $v_0 \geqslant 1$ and $\delta > 0$, where $U_\delta := \{x \in U \,:\, \mathrm{dist}(x, \partial U) > \delta\}$.

Proposition 3.3 (Hörmander inequality for H^1_{hyp})**.** *For every $\alpha \in \left[0, \frac{1}{3}\right)$ and $v_0 \in [1,\infty)$, there exists a constant $C(\alpha, v_0, d) < \infty$ such that, for every $f \in H^1_{\mathrm{hyp}}(\mathbb{R}^d)$ satisfying*

$$f = 0 \quad \text{in } \mathbb{R}^d \times \left(\mathbb{R}^d \setminus B_{v_0}\right),$$

we have the estimate

$$\|f\|_{H^\alpha(\mathbb{R}^d; H^{-1}_\gamma)} \leqslant C \|f\|_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)}.$$

In view of the interpolation inequality, for each $\theta \in [0,1]$,

$$\|f\|_{H^{\theta\beta}(\mathbb{R}^d; H^{1-2\theta}_\gamma)} \leqslant \|f\|^\theta_{H^\beta(\mathbb{R}^d; H^{-1}_\gamma)} \|f\|^{1-\theta}_{L^2(\mathbb{R}^d; H^1_\gamma)},$$

Theorem 1.3 immediately implies the following estimate.

Corollary 3.4 (Hörmander inequality for H^1_{hyp})**.** *Let $\alpha \in \left[0, \frac{1}{3}\right)$ and $v_0 \in [1, \infty)$. There exists a constant $C(\alpha, v_0, d) < \infty$ such that, for every $\theta \in [0,1]$ and $f \in H^1_{\mathrm{hyp}}(\mathbb{R}^d)$ satisfying*

$$f = 0 \quad \text{in } \mathbb{R}^d \times \left(\mathbb{R}^d \setminus B_{v_0}\right),$$

we have the estimate

$$\|f\|_{H^{\theta\alpha}(\mathbb{R}^d; H^{1-2\theta}_\gamma)} \leqslant C \|f\|_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)}.$$

Following [22], the proof of Theorem 1.3 is based on the splitting of a first-order finite difference in the x variable into finite differences which are either in the v variable, or in the x variable in the direction of v. Explicitly, we have

$$\begin{aligned} f(x + t^2 y, v) - f(x, v) = {} & f(x + t^2 y, v) - f(x + t^2 y, v - ty) \\ & + f(x + t^2 y, v - ty) - f(x + t^2 y + t(v - ty), v - ty) \\ & + f(x + tv, v - ty) - f(x + tv, v) \\ & + f(x + tv, v) - f(x, v). \end{aligned} \tag{3.20}$$

Notice that the right side consists of four finite differences, two for each of the derivatives ∇_v and $v \cdot \nabla_x$ which we can expect to control by the H^1_{hyp} norm. The fact that the increment on the left is of size t^2 and those on the right side are of size t suggests that we may expect to have one-half derivative in the statement of Theorem 1.3. In fact, we are only able to obtain (almost) one-third of a derivative, due to the fact that we control the quantity $v \cdot \nabla_x f$ in $L^2(U; H^{-1}_\gamma)$ only, but not in $L^2(U; L^2_\gamma)$. We do not know if the exponent $1/3$ is optimal.

The relation (3.20) is a special case of Hörmander's bracket condition introduced in [22], which for the particular equation we consider here is quite simple to check. Indeed, let $X_1, \ldots, X_d$, $V_1, \ldots, V_d$ denote the canonical vector fields and X_0 be the vector field $(x, v) \mapsto (v, 0)$. Then the Hörmander bracket condition is implied by the identity

$$[V_i, X_0] = X_i. \tag{3.21}$$

This is a local version of the identity (3.20). More precisely, for every vector field Z, if we denote by $t \mapsto \exp(tZ)$ the flow induced by the vector field Z on $\mathbb{R}^d \times \mathbb{R}^d$, then

$$\begin{aligned} & \exp(-tV_i) \exp(-tX_0) \exp(tV_i) \exp(tX_0)(x, v) \\ & \qquad = (x, v) + t^2 [V_i, X_0](x, v) + o(t^2) \qquad (t \to 0). \end{aligned} \tag{3.22}$$

For the vector fields of interest, $Z \in \{X_0, X_1, \ldots, X_d, V_1, \ldots, V_d\}$, the flows take the very simple form

$$\exp(tZ)(x, v) = (x, v) + tZ(x, v),$$

the relation (3.22) becomes an identity (that is, the term $o(t^2)$ is actually zero), and this identity can be reprased in the form of (3.20).

The proof of Theorem 1.3 relies on the representation of the H^α norm provided by (3.13). We use this characterization because we find it convenient to "do harmonic analysis" with the heat kernel; we expect that a similar proof can be made using Littlewood-Paley or other similar tools. On the other hand, a more direct approach based on the definition (3.10) would be more problematic since, as already mentioned, we only control $v \cdot \nabla_x f$ in a weak norm.

Proof of Theorem 1.3. Throughout, we select $\alpha \in \left[0, \frac{1}{3}\right)$, $v_0 \in [1, \infty)$ and a function

$$f \in C^\infty_c(\mathbb{R}^d \times B_{v_0}).$$

The goal is to show that, for a constant $C(\alpha, v_0, d) < \infty$, we have the estimate

$$\|f\|_{H^\alpha(\mathbb{R}^d;H^{-1}_\gamma)} \leqslant C\,\|f\|_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)}\,. \tag{3.23}$$

Step 1. We set up the argument. Applying (3.13), we have, for a constant $C(\alpha, d) < \infty$, the bound

$$\|f\|^2_{H^\alpha(\mathbb{R}^d;X)} \leqslant C\,\|f\|^2_{L^2(\mathbb{R}^d;X)} + C\int_0^1 \int_{\mathbb{R}^d} t^{-\alpha}\,\|(\nabla\Phi(t,\cdot) * f)(x)\|^2_X\,dx\,dt. \tag{3.24}$$

We take the convolution above with respect to the spatial variable only, in other words, the function $\nabla\Phi(t,\cdot) * f \in H^1_{\mathrm{hyp}}(\mathbb{R}^d)$ is given by

$$(\nabla\Phi(t,\cdot) * f)\,(x,v) = \int_{\mathbb{R}^d} \nabla\Phi(t, y-x) f(y,v)\,dy.$$

Obviously, we have $\|f\|_{L^2(\mathbb{R}^d;H^{-1}_\gamma)} \leqslant C\,\|f\|_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)}$. The rest of the argument is focused on estimating $\|\nabla\Phi(t,\cdot) * f\|_{L^2(\mathbb{R}^d;H^{-1}_\gamma)}$ for every $t \in (0,1]$.

We will show that there exists $C(v_0, d) < \infty$ such that, for every $t \in (0,1]$,

$$\|\nabla\Phi(t,\cdot) * f\|_{L^2(\mathbb{R}^d;H^{-1}_\gamma)} \leqslant Ct^{-\frac{1}{3}}\,\|f\|_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)}\,. \tag{3.25}$$

Observe that (3.24) and (3.25) imply the statement of the proposition: since $\alpha < \frac{1}{3}$,

$$\begin{aligned}
\|f\|^2_{H^\alpha(\mathbb{R}^d;H^{-1}_\gamma)} &\leqslant C\|f\|^2_{L^2(\mathbb{R}^d;H^{-1}_\gamma)} + C\int_0^1 \int_{\mathbb{R}^d} t^{-\alpha}\,\|(\nabla\Phi(t,\cdot) * f)(x)\|^2_{H^{-1}_\gamma}\,dx\,dt \\
&\leqslant C\|f\|^2_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)} + C\,\|f\|^2_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)} \int_0^1 t^{-\alpha-\frac{2}{3}}\,dt \\
&\leqslant C(1-3\alpha)^{-1}\,\|f\|^2_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)}\,.
\end{aligned}$$

To prove (3.25), it suffices by duality to show that, for every test function $g \in L^2(\mathbb{R}^d; H^1_\gamma)$, we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla\Phi(t,\cdot) * f)\,g\,dx\,d\gamma \leqslant Ct^{-\frac{1}{3}}\,\|f\|_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)}\,\|g\|_{L^2(\mathbb{R}^d;H^1_\gamma)}\,. \tag{3.26}$$

Since $\nabla\Phi(t,\cdot) * f$ has zero mean and is supported in the set $\{(x,v) : |v| \leqslant v_0\}$, it suffices to show (3.26) for test functions $g \in L^2(\mathbb{R}^d; H^1_\gamma)$ satisfying $g(x,v) = 0$ for $|v| \geqslant v_0 + 1$.

Using the semigroup property $\nabla\Phi(t+s,\cdot) = \nabla\Phi(t,\cdot) * \Phi(s,\cdot)$, we may express the left side of (3.26), for $t \in (0,1)$, as

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\nabla\Phi(t,\cdot) * f)\,(x,v) g(x,v)\,dx\,d\gamma(v) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\nabla\Phi\left(\tfrac{t}{2},\cdot\right) * f\right)(x,v)\left(\Phi\left(\tfrac{t}{2},\cdot\right) * g\right)(x,v)\,dx\,d\gamma(v) \\
&= \int_{\mathbb{R}^d} \nabla\Phi\left(\tfrac{t}{2},y\right) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x+y,v) - f(x,v))\left(\Phi\left(\tfrac{t}{2},\cdot\right) * g(\cdot,v)\right)(x)\,dx\,d\gamma(v)\,dy.
\end{aligned} \tag{3.27}$$

Therefore we are interested in estimating the quantity

$$\int_{\mathbb{R}^d} \nabla\Phi\left(\tfrac{t}{2},y\right)\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x,v)\,(f(x+y,v) - f(x,v))\,dx\,d\gamma(v)\right)dy \tag{3.28}$$

for an appropriate test function ψ, which is a smooth, H^1_γ-valued function on $\mathbb{R}^d$ such that $\psi\mathbb{1}_{\{|v|\geqslant v_0+1\}} = 0$. We split the inner integral in parentheses in (3.28) into four pieces,

using a variation of (3.20):

$$
\begin{aligned}
(3.29)\qquad & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(f(x+y,v)-f(x,v)\right)dx\,d\gamma(v)\\
&\quad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(f(x+y,v)-f(x+y,v-t^{-\frac13}y)\right)dx\,d\gamma(v)\\
&\qquad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(f(x+y,v-t^{-\frac13}y)-f(x+t^{\frac13}v,v-t^{-\frac13}y)\right)dx\,d\gamma(v)\\
&\qquad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(f(x+t^{\frac13}v,v-t^{-\frac13}y)-f(x+t^{\frac13}v,v)\right)dx\,d\gamma(v)\\
&\qquad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(f(x+t^{\frac13}v,v)-f(x,v)\right)dx\,d\gamma(v).
\end{aligned}
$$

The fourth integral we can ignore, because it does not depend on y and so its contribution in (3.28) vanishes in view of the fact that $\int_{\mathbb{R}^d}\nabla\Phi(\frac{t}{2},y)\,dy=0$.

As we will see, the first and third integrals are estimated relatively straightforwardly as they involve ∇_v derivatives, while the second is more tricky since it involves $v\cdot\nabla_x$. We assume throughout that ψ is a smooth test function which vanishes for $|v|\geqslant v_0+1$.

Step 2. We estimate the contribution of the first and third integrals on the right side of (3.29). The claim is that there exists $C(v_0,d)<\infty$ such that, for every $t\in(0,1]$,

$$
\begin{aligned}
(3.30)\qquad & \int_{\mathbb{R}^d}\nabla\Phi\left(\tfrac{t}{2},y\right)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(f(x+y,v)-f(x+y,v-t^{-\frac13}y)\right)dx\,d\gamma(v)\,dy\\
&\qquad\leqslant Ct^{-\frac13}\left\|\nabla_v f\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}\left\|\psi\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}.
\end{aligned}
$$

Notice that this estimate takes care of both the first and third integrals on the right side of (3.29), since the third integral is, after up to a minus sign, a translation of the first integral in the x variable.

We write

$$
\begin{aligned}
\left|f(x+y,v)-f(x+y,v-t^{-\frac13}y)\right| &= t^{-\frac13}\left|\int_0^1 y\cdot\nabla_v f(x+y,v-t^{-\frac13}sy)\,ds\right|\\
&\leqslant t^{-\frac13}|y|\int_0^1\left|\nabla_v f(x+y,v-t^{-\frac13}sy)\right|ds.
\end{aligned}
$$

By Hölder's inequality, we deduce

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(f(x+y,v)-f(x+y,v-t^{-\frac13}y)\right)dx\,d\gamma(v)\right|\\
&\qquad\leqslant t^{-\frac13}|y|\left\|\psi\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}\int_0^1\left\|\nabla_v f(\cdot,\cdot-t^{-\frac13}sy)\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}ds.
\end{aligned}
$$

Since $f\mathbb{1}_{\{|v|\geqslant v_0+1\}}=0$, we deduce from (3.17) that

$$
(3.31)\qquad \left\|\nabla_v f(\cdot,\cdot-t^{-\frac13}sy)\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}\leqslant\exp\left(\frac{(v_0+1)^2}{2}\right)\left\|\nabla_v f\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)},
$$

and thus

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\nabla\Phi\left(\tfrac{t}{2},y\right)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(f(x+y,v)-f(x+y,v-t^{-\frac13}y)\right)dx\,d\gamma(v)\,dy\\
&\quad\leqslant Ct^{-\frac13}\left\|\nabla_v f\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}\left\|\psi\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}\int_{\mathbb{R}^d}|y|\left|\nabla\Phi\left(\tfrac{t}{2},y\right)\right|dy\\
&\quad\leqslant Ct^{-\frac13}\left\|\nabla_v f\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}\left\|\psi\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}.
\end{aligned}
$$

This is (3.30).

Step 3. We estimate the contribution of the second integral on the right side of (3.29). The claim is that, for a constant $C(v_0, d) < \infty$,

$$\int_{\mathbb{R}^d} \nabla\Phi\left(\tfrac{t}{2}, y\right) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x,v)\left(f(x+y, v-t^{-\frac{1}{3}}y) - f(x+t^{\frac{1}{3}}v, v-t^{-\frac{1}{3}}y)\right) dx\, d\gamma(v) \tag{3.32}$$
$$\leqslant C\left\|v\cdot\nabla_x f\right\|_{L^2(\mathbb{R}^d;H^{-1}_\gamma)}\left(t^{-\frac{1}{6}}\left\|\psi\right\|_{L^2(\mathbb{R}^d;H^1_\gamma)} + t^{\frac{1}{6}}\left\|\nabla_x\psi\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}\right).$$

Here we use

$$f(x+y, v-t^{-\frac{1}{3}}y) - f(x+t^{\frac{1}{3}}v, v-t^{-\frac{1}{3}}y)$$
$$= -t^{\frac{1}{3}}\int_0^1 \left(v-t^{-\frac{1}{3}}y\right)\cdot\nabla_x f(x+y+st^{\frac{1}{3}}(v-t^{-\frac{1}{3}}y), v-t^{-\frac{1}{3}}y)\, ds.$$

Using this identity, performing a change of variables, and recalling (3.17), we find that

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(f(x+y,v-t^{-\frac{1}{3}}y) - f(x+t^{\frac{1}{3}}v, v-t^{-\frac{1}{3}}y)\right)dx\, d\gamma(v)\right|$$
$$= t^{\frac{1}{3}}\left|\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(v-t^{-\frac{1}{3}}y\right)\cdot\nabla_x f(x+y+st^{\frac{1}{3}}(v-t^{-\frac{1}{3}}y), v-t^{-\frac{1}{3}}y)\, dx\, d\gamma(v)\, ds\right|$$
$$\leqslant Ct^{\frac{1}{3}}\int_0^1\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x-y-st^{\frac{1}{3}}v, v+t^{-\frac{1}{3}}y)v\cdot\nabla_x f(x,v)\, dx\, d\gamma(v)\right| ds$$
$$\leqslant Ct^{\frac{1}{3}}\sup_{s\in[0,1]}\left\|\psi_{y,s,t}\right\|_{L^2(\mathbb{R}^d;H^1_\gamma)}\left\|v\cdot\nabla_x f\right\|_{L^2(\mathbb{R}^d;H^{-1}_\gamma)},$$

where we denote, for each $y\in\mathbb{R}^d$ and $s,t\in[0,1]$,

$$\psi_{y,s,t}(x,v) := \psi(x-y-st^{\frac{1}{3}}v, v+t^{-\frac{1}{3}}y).$$

Using again (3.17), we see that

$$\left\|\psi_{y,s,t}\right\|_{L^2(\mathbb{R}^d;H^1_\gamma)} \leqslant C\|\psi\|_{L^2(\mathbb{R}^d;H^1_\gamma)} + Ct^{\frac{1}{3}}\|\nabla_x\psi\|_{L^2(\mathbb{R}^d;L^2_\gamma)}.$$

Combining the inequalities above, we obtain

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,v)\left(f(x+y,v-t^{-\frac{1}{3}}y) - f(x+t^{\frac{1}{3}}v, v-t^{-\frac{1}{3}}y)\right)dx\, d\gamma(v)\right|$$
$$\leqslant C\left(t^{\frac{1}{3}}\|\psi\|_{L^2(\mathbb{R}^d;H^1_\gamma)} + t^{\frac{2}{3}}\|\nabla_x\psi\|_{L^2(\mathbb{R}^d;L^2_\gamma)}\right)\left\|v\cdot\nabla_x f\right\|_{L^2(\mathbb{R}^d;H^{-1}_\gamma)}.$$

Multiplying by $\left|\nabla\Phi\left(\frac{t}{2},y\right)\right|$ and integrating in y, using that $\int_{\mathbb{R}^d}\left|\nabla\Phi(\frac{t}{2},y)\right| dy \leqslant Ct^{-\frac{1}{2}}$, we obtain (3.32).

Step 4. The conclusion. Combining (3.27) and (3.29) with the results of Steps 2 and 3, we obtain

$$\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}(\nabla\Phi(t,\cdot)*f)\, g\, dx\, d\gamma\right|$$
$$\leqslant C\left\|f\right\|_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)}\left(t^{-\frac{1}{3}}\left\|\Phi\left(\tfrac{t}{2},\cdot\right)*g\right\|_{L^2(\mathbb{R}^d;H^1_\gamma)} + t^{\frac{1}{6}}\left\|\nabla\Phi\left(\tfrac{t}{2},\cdot\right)*g\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}\right).$$

We now obtain (3.26) after combining the previous inequality with the estimates

$$\left\|\Phi\left(\tfrac{t}{2},\cdot\right)*g\right\|_{L^2(\mathbb{R}^d;H^1_\gamma)} \leqslant \|g\|_{L^2(\mathbb{R}^d;H^1_\gamma)}$$

and

$$\left\|\nabla\Phi\left(\tfrac{t}{2},\cdot\right)*g\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)} \leqslant \left\|\nabla\Phi\left(\tfrac{t}{2},\cdot\right)\right\|_{L^1(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d;L^2_\gamma)} \leqslant Ct^{-\frac{1}{2}}\left\|g\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}.$$

The proof is complete. □

3.3. **Compact embedding of H^1_{hyp} into $L^2(U;L^2_\gamma)$.** Using the results of the previous subsection, we show that the embedding $H^1_{\mathrm{hyp}}(U) \hookrightarrow L^2(U;L^2_\gamma)$ is compact. We recall that $U \subseteq \mathbb{R}^d$ is assumed to be a bounded $C^{1,1}$ domain.

Proposition 3.5 (Compact embedding of $H^1_{\mathrm{hyp}}(U)$ into $L^2(U;L^2_\gamma)$)**.** *The inclusion map $H^1_{\mathrm{hyp}}(U) \hookrightarrow L^2(U;L^2_\gamma)$ is compact.*

Before we give the proof of Proposition 3.5, we need to review some basic facts concerning the logarithmic Sobolev inequality and a generalized Hölder inequality for Orlicz norms. The logarithmic Sobolev inequality states that, for some $C < \infty$,

$$\int_{\mathbb{R}^d} f^2(v) \log\left(1 + f^2(v)\right) d\gamma(v) \leq C \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma(v), \quad \forall f \in H^1_\gamma \text{ with } \|f\|_{L^2_\gamma} = 1. \tag{3.33}$$

Let $F : \mathbb{R} \to [0,\infty)$ denote the (strictly) convex function

$$F(t) := |t| \log\left(1 + |t|\right).$$

Let F^* denote its dual convex conjugate function, defined by

$$F^*(s) := \sup_{t \in \mathbb{R}} \left(st - F(t)\right).$$

Then (F, F^*) is a *Young pair* (see [33]), that is, both F and F^* are nonnegative, even, convex, satisfy $F(0) = F^*(0) = 0$ as well as

$$\lim_{|t| \to \infty} |t|^{-1} F(t) = \lim_{|s| \to \infty} |s|^{-1} F^*(s) = \infty.$$

Moreover, both F and F^* are strictly increasing on $[0,\infty)$ and in particular vanish only at $t = 0$. Given any measure space (X,ω), the *Orcliz spaces* $L_F(X,\omega)$ and $L_{F^*}(X,\omega)$, which are defined by the norms

$$\begin{cases} \|g\|_{L_F(X,\omega)} := \inf\left\{t > 0 : \displaystyle\int_X F\left(t^{-1} g\right) d\omega \leq F(1)\right\}, \quad \text{and} \\ \|g\|_{L_{F^*}(X,\omega)} := \inf\left\{t > 0 : \displaystyle\int_X F^*\left(t^{-1} g\right) d\omega \leq F^*(1)\right\} \end{cases}$$

are dual Banach spaces and the following generalized version of the Hölder inequality is valid (see [33, Proposition 3.3.1]):

$$\int_X |g g^*| \, d\omega \leq \|g\|_{L_F(X,\omega)} \|g^*\|_{L_{F^*}(X,\omega)}, \quad \forall g \in L_F(X,\omega),\, g^* \in L_{F^*}(X,\omega).$$

The logarithmic Sobolev inequality (3.33) may be written in terms of the Orcliz norm as

$$\left\|f^2\right\|_{L_F(\mathbb{R}^d,\gamma)} \leq C\left(|\langle f \rangle_\gamma|^2 + \|\nabla f\|^2_{L^2_\gamma}\right), \quad \forall f \in H^1_\gamma.$$

The previous two displays imply that

$$\left(\int_{U \times \mathbb{R}^d} g\, |f|^2 \, dx\, d\gamma(v)\right)^{\frac{1}{2}} \leq C \|g\|^{\frac{1}{2}}_{L_{F^*}(U \times \mathbb{R}^d, dx d\gamma)} \|f\|_{L^2(U;H^1_\gamma)}. \tag{3.34}$$

We do not identify F^* with an explicit formula, although we notice that the inequality

$$s(t+1) \leq \exp(s) + t \log(1+t), \quad \forall s,t \in (0,\infty)$$

implies that

$$F^*(s) \leq \exp(s) - s.$$

This allows us in particular to obtain from (3.34) that

$$\left(\int_{U \times \mathbb{R}^d} |v|^2 |f|^2 \, dx\, d\gamma(v)\right)^{\frac{1}{2}} \leq C \|f\|_{L^2(U;H^1_\gamma)}. \tag{3.35}$$

We also point out that (3.35) also implies the existence of $C(d,U)<\infty$ such that, for every $f\in L^2(U;L^2_\gamma)$,

$$\|\nabla_v f\|_{L^2(U;H^{-1}_\gamma)}\leqslant C\,\|f\|_{L^2(U;L^2_\gamma)}\,. \tag{3.36}$$

We now turn to the proof of Proposition 3.5.

Proof of Proposition 3.5. For each $\theta>0$, we denote

$$U_\theta:=\{x\,:\,\operatorname{dist}(x,\partial U)<\theta\}\,. \tag{3.37}$$

Since U is a $C^{1,1}$ domain, we can extend the outer normal $\mathbf{n}_U$ to a globally Lipschitz function on $\overline{U}$. We can moreover assume that, for some $\theta_0(U)>0$, this extension $\mathbf{n}_U$ coincides with the gradient of the mapping $x\mapsto-\operatorname{dist}(x,\partial U)$ in U_{θ_0}.

By Proposition 2.2, we may work under the qualitative assumption that all of our $H^1_{\mathrm{hyp}}(U)$ functions belong to $C^\infty_c(\overline{U}\times\mathbb{R}^d)$. Select $\varepsilon>0$ and a sequence $\{f_n\}_{n\in\mathbb{N}}\subseteq H^1_{\mathrm{hyp}}(U)$ satisfying

$$\sup_{n\in\mathbb{N}}\|f_n\|_{H^1_{\mathrm{hyp}}(U)}\leqslant 1.$$

We will argue that there exists a subsequence $\{f_{n_k}\}$ such that

$$\limsup_{k\to\infty}\sup_{i,j\geqslant k}\left\|f_{n_i}-f_{n_j}\right\|_{L^2(U;L^2_\gamma)}\leqslant\varepsilon. \tag{3.38}$$

By a diagonalization argument, this suffices to obtain the proposition.

Step 1. We claim that there exists $v_0\in[1,\infty)$ such that, for every $f\in H^1_{\mathrm{hyp}}(U)$,

$$\left(\int_U\int_{\mathbb{R}^d\smallsetminus B_{v_0}}|f(x,v)|^2\,dx\,d\gamma(v)\right)^{\frac12}\leqslant\frac{\varepsilon}{3}\,\|f\|_{H^1_{\mathrm{hyp}}(U)}\,.$$

Indeed, applying (3.34), we find that

$$\left(\int_U\int_{\mathbb{R}^d\smallsetminus B_{v_0}}|f(x,v)|^2\,dx\,d\gamma(v)\right)^{\frac12}\leqslant C\left\|\mathbb{1}_{U\times(\mathbb{R}^d\smallsetminus v_0)}\right\|^{\frac12}_{L_{F^*}(U\times\mathbb{R}^d,dxd\gamma)}\|f\|_{H^1_{\mathrm{hyp}}(U)}\,.$$

Taking v_0 sufficiently large, depending on ε, ensures that

$$C\left\|\mathbb{1}_{U\times(\mathbb{R}^d\smallsetminus v_0)}\right\|^{\frac12}_{L_{F^*}(U\times\mathbb{R}^d,dxd\gamma)}\leqslant\frac{\varepsilon}{3}.$$

Step 2. We next claim that there exists $\delta\in\left(0,\frac12\right]$ such that, for every $f\in H^1_{\mathrm{hyp}}(U)$,

$$\left(\int_U\int_{\mathbb{R}^d}|f(x,v)|^2\,\mathbb{1}_{\{|\mathbf{n}_U\cdot v|<\delta\}}\,dx\,d\gamma(v)\right)^{\frac12}\leqslant\frac{\varepsilon}{3}\,\|f\|_{H^1_{\mathrm{hyp}}(U)}\,.$$

The argument here is similar to the estimate in Step 1, above. We simply apply (3.34) after choosing δ small enough that

$$C\left\|\mathbb{1}_{\{|\mathbf{n}_U\cdot v|<\delta\}}\right\|^{\frac12}_{L_{F^*}(U\times\mathbb{R}^d,dxd\gamma)}\leqslant\frac{\varepsilon}{3}.$$

Step 3. We next show that, for every $\delta>0$, there exists $\theta>0$ such that, for every function $f\in H^1_{\mathrm{hyp}}(U)$,

$$\left(\int_U\int_{\mathbb{R}^d}|f(x,v)|^2\,\mathbb{1}_{\{|\mathbf{n}_U\cdot v|\geqslant\delta\}}\mathbb{1}_{\{\operatorname{dist}(x,\partial U)<\theta\}}\,dx\,d\gamma(v)\right)^{\frac12}\leqslant\frac{\varepsilon}{3}\,\|f\|_{H^1_{\mathrm{hyp}}(U)}\,. \tag{3.39}$$

For $\theta\in\left(0,\frac{\theta_0}{2}\right]$ to be taken sufficiently small in terms of $\delta>0$ in the course of the argument, we let $\varphi\in C^{1,1}(\overline{U})$ be defined by

$$\varphi(x):=-\eta\left(\operatorname{dist}(x,\partial U)\right),$$

where $\eta \in C_c^\infty([0,\infty))$ satisfies

$$0 \leq \eta \leq 2\theta, \quad 0 \leq \eta' \leq 1, \quad \eta(x) = x \text{ on } [0,\theta], \quad \eta' = 0, \text{ on } [2\theta,\infty).$$

We have $-2\theta \leq \varphi \leq 0$. Moreover, by the definition of θ_0 below (3.37), its gradient $\nabla\varphi$ is proportional to $\mathbf{n}_U$ in U, it vanishes outside of $U_{2\theta}$, and $\nabla\varphi = \mathbf{n}_U$ in U_θ. We next select another test function $\chi \in C_c^\infty([0,\infty))$ satisfying

$$0 \leq \chi \leq 1, \quad \chi \equiv 0 \text{ on } \left[0, \tfrac{1}{2}\delta\right], \quad \chi \equiv 1 \text{ on } [\delta,\infty), \quad |\chi'| \leq \delta^{-1},$$

and define

$$\psi_\pm(x,v) := \chi\left((v\cdot\mathbf{n}_U(x))_\pm\right),$$

where for $r \in \mathbb{R}$, we use the notation $r_- := \max(0,-r)$ and $r_+ := \max(0,r)$. Observe that

$$|\nabla_v \psi_\pm(x,v)| = \left|\chi'\left((v\cdot\mathbf{n}_U(x))_\pm\right)\right| |\mathbf{n}_U(x)| \leq C\delta^{-1}.$$

Therefore

$$\begin{aligned}
\|\varphi f\psi_\pm\|_{L^2(U;H^1_\gamma)} &\leq C\left(\|\varphi f\psi_\pm\|_{L^2(U;L^2_\gamma)} + \|\varphi\nabla_v(f\psi_\pm)\|_{L^2(U;L^2_\gamma)}\right)\\
&\leq C\theta\left(\|f\|_{L^2(U;L^2_\gamma)} + \|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|f\nabla_v\psi_\pm\|_{L^2(U;L^2_\gamma)}\right)\\
&\leq C\theta\delta^{-1}\|f\|_{L^2(U;H^1_\gamma)},
\end{aligned}$$

and hence

$$\left|\int_{U\times\mathbb{R}^d} \varphi f\psi_\pm v\cdot\nabla_x f\,dx\,d\gamma(v)\right| \leq C\theta\delta^{-1}\|f\|^2_{H^1_{\mathrm{hyp}}(U)}.$$

On the other hand,

$$\begin{aligned}
&\int_{U\times\mathbb{R}^d} \varphi f\psi_\pm v\cdot\nabla_x f\,dx\,d\gamma(v)\\
&\quad = -\frac{1}{2}\int_{U\times\mathbb{R}^d} f^2 v\cdot\nabla_x(\varphi\psi_\pm)\,dx\,d\gamma(v)\\
&\quad = -\frac{1}{2}\int_{U\times\mathbb{R}^d} \varphi f^2 v\cdot\nabla_x\psi_\pm\,dx\,d\gamma(v) - \frac{1}{2}\int_{U\times\mathbb{R}^d}\psi_\pm f^2 v\cdot\nabla\varphi\,dx\,d\gamma(v).
\end{aligned}$$

Since $|v\cdot\nabla_x\psi_\pm(x,v)| \leq C\delta^{-1}|v|^2$, we have, by (3.35),

$$\left|\int_{U\times\mathbb{R}^d}\varphi f^2 v\cdot\nabla_x\psi_\pm\,dx\,d\gamma(v)\right| \leq C\theta\delta^{-1}\int_{U\times\mathbb{R}^d}|v|^2 f^2\,dx\,d\gamma(v) \leq C\theta\delta^{-1}\|f\|^2_{H^1_{\mathrm{hyp}}(U)}.$$

We deduce that

$$\left|\int_{U\times\mathbb{R}^d}\psi_\pm f^2 v\cdot\nabla\varphi\,dx\,d\gamma(v)\right| \leq C\theta\delta^{-1}\|f\|^2_{H^1_{\mathrm{hyp}}(U)}.$$

Finally, we observe from the properties of φ and $\psi_\pm$ that

$$\begin{aligned}
&\int_U\int_{\mathbb{R}^d}|f(x,v)|^2\,\mathbb{1}_{\{|\mathbf{n}_U\cdot v|\geq\delta\}}\mathbb{1}_{\{\mathrm{dist}(x,\partial U)<\theta\}}\,dx\,d\gamma(v)\\
&\quad\leq \delta^{-1}\left(\left|\int_{U\times\mathbb{R}^d}\psi_+ f^2 v\cdot\nabla\varphi\,dx\,d\gamma(v)\right| + \left|\int_{U\times\mathbb{R}^d}\psi_- f^2 v\cdot\nabla\varphi\,dx\,d\gamma(v)\right|\right)\\
&\quad\leq C\theta\delta^{-2}\|f\|^2_{H^1_{\mathrm{hyp}}(U)}.
\end{aligned}$$

Taking $\theta = c\varepsilon^2\delta^2$ for a sufficiently small constant $c > 0$ yields the claimed inequality (3.39).

Step 4. By the results of the previous three steps, to obtain (3.38) it suffices to exhibit a subsequence $\{f_{n_k}\}$ satisfying

$$\limsup_{k\to\infty}\sup_{i,j\geq k}\int_{U_\theta\times B_{v_0}}\left|f_{n_i}-f_{n_j}\right|^2\,dx\,d\gamma(v) = 0.$$

This is an immediate consequence of Corollary 3.4 and the compactness of the embedding $H^{\frac{1}{10}}\left(U_\theta; H_\gamma^{\frac{1}{3}}\right) \hookrightarrow L^2(U_\theta; L^2_\gamma(B_{v_0}))$ (see for instance [1, Theorem 2.32]). □

4. Trace theorem and integration by parts

In this section, we study the boundary behavior of functions in $H^1_{\mathrm{hyp}}(U)$. In particular, we prove a trace theorem and an integration by parts formula. These results are an important ingredient in the well-posedness theory presented in Section 5.

In the case when the vector field $\mathbf{b}(x)$ only depends on the space variable x and is a potential field, we can prove more general integration by parts formulas that include the term $\mathbf{b}\cdot\nabla_v f$. We therefore introduce the following assumption.

Assumption 4.1 (Conservative force)**.** *There exists a Lipschitz function $H \in C^{0,1}(\mathbb{R}^d;\mathbb{R})$ such that $\mathbf{b}(x) = \nabla H(x)$ for almost every $x \in U$.*

Under Assumption 4.1, we denote by σ the measure on $\mathbb{R}^d$ defined by

$$d\sigma(x) := \exp(-H(x))\,dx, \tag{4.1}$$

and by m the measure on $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$dm(x,v) := d\sigma(x)\,d\gamma(v) = \exp\left(-H(x) - \frac{1}{2}|v|^2\right) dx\,dv. \tag{4.2}$$

Since the domain U is bounded, the measure σ and the Lebesgue measure are equivalent on U. As a consequence, the Lebesgue space $L^2(U\times\mathbb{R}^d, m)$ is identical to the space $L^2(U; L^2_\gamma)$. We will make use of the shorthand notation

$$Bf := -v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f. \tag{4.3}$$

Recalling the definition of ∇_v^* in (2.1), we see that the left side of (1.1) can be rewritten in the form

$$\nabla_v^*\nabla_v f + Bf.$$

Moreover, the formal adjoint of B in $L^2(\mathbb{R}^d\times\mathbb{R}^d, m)$ is $B^* = -B$. Note also that the norm

$$f \mapsto \|f\|_{L^2(U;H^1_\gamma)} + \|Bf\|_{L^2(U;H^{-1}_\gamma)}$$

is equivalent to the $H^1_{\mathrm{hyp}}(U)$ norm, since $\|\mathbf{b}\|_{L^\infty(U)} < \infty$. Abusing notation, we denote, for every $f \in C_c^\infty(\overline{U}\times\mathbb{R}^d)$,

$$\int_{\partial U\times\mathbb{R}^d} f\,dm = \int_{\partial U\times\mathbb{R}^d} f(x,v)\,dm(x,v) := \int_{\partial U\times\mathbb{R}^d} f(x,v)\,\exp\left(-H(x)\right)\,d\mathcal{H}_{d-1}(x)\,d\gamma(v),$$

where $\mathcal{H}_{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure on ∂U. We also write

$$\int_{\partial U\times\mathbb{R}^d} f\,dx\,d\gamma = \int_{\partial U\times\mathbb{R}^d} f(x,v)\,dx\,d\gamma(v) := \int_{\partial U\times\mathbb{R}^d} f(x,v)\,d\mathcal{H}_{d-1}(x)\,d\gamma(v).$$

An integration by parts reveals that for every $f, g \in C_c^\infty(\overline{U}\times\mathbb{R}^d)$, we have

$$\int_{U\times\mathbb{R}^d} (fBg + gBf)\,dm = -\int_{\partial U\times\mathbb{R}^d} fg\,v\cdot\mathbf{n}_U\,dm, \tag{4.4}$$

where the term on the right side is shorthand notation for

$$-\int_{\partial U\times\mathbb{R}^d} f(x,v)g(x,v)\,v\cdot\mathbf{n}_U(x)\,dm(x,v).$$

In particular, we have for every $f \in C_c^\infty(\overline{U}\times\mathbb{R}^d)$ that

$$\int_{U\times\mathbb{R}^d} fBf\,dm = -\frac{1}{2}\int_{\partial U\times\mathbb{R}^d} f^2\,v\cdot\mathbf{n}_U\,dm. \tag{4.5}$$

The functional on the left side of (4.5) is continuous in f for the topology of $H^1_{\mathrm{hyp}}(U)$. However, since the measure $v \cdot \mathbf{n}_U \, dm$ has a varying sign, we cannot immediately deduce that functions in $H^1_{\mathrm{hyp}}(U)$ have a trace in $L^2(\partial U \times \mathbb{R}^d; |v \cdot \mathbf{n}_U| \, dm)$. Note that the measures dm and $dx \, d\gamma$ are equivalent on $\partial U \times \mathbb{R}^d$. Also, if we specify to $\mathbf{b} \equiv 0$ in (4.5), then the identity becomes

$$\int_{U \times \mathbb{R}^d} f \, v \cdot \nabla_x f \, dx \, d\gamma = \frac{1}{2} \int_{\partial U \times \mathbb{R}^d} f^2 \, v \cdot \mathbf{n}_U \, dx \, d\gamma, \tag{4.6}$$

and again, the left side of (4.6) depends continuously on $f \in H^1_{\mathrm{hyp}}(U)$. We are thus naturally led to the following question.

Question 4.2. Does there exist $C(U,d) < \infty$ such that for every $f \in C_c^\infty(\overline{U} \times \mathbb{R}^d)$,

$$\int_{\partial U \times \mathbb{R}^d} f^2 \, |v \cdot \mathbf{n}_U| \, dx \, d\gamma \leqslant C \|f\|^2_{H^1_{\mathrm{hyp}}(U)} \quad ?$$

When the spatial dimension is $d = 1$, it was shown in [4] that the answer to this question is positive. A generalization to higher dimensions is stated in [10, Lemma 2.3], but we think that the argument is incomplete, as discussed below in Appendix A. We were not able to resolve Question 4.2 and so we leave it as an open problem. What we will prove instead is the weaker statement that the boundary value of f is well-defined locally in L^2 away from the singular set $\{v \cdot \mathbf{n}_U = 0\}$. We introduce the notation

$$\partial^{\pm}_{\mathrm{hyp}} U := \{(x,v) \in \partial U \times \mathbb{R}^d \ : \ v \cdot \mathbf{n}_U(x) \neq 0\}, \tag{4.7}$$

and

$$\partial^{=}_{\mathrm{hyp}} U := \{(x,v) \in \partial U \times \mathbb{R}^d \ : \ v \cdot \mathbf{n}_U(x) = 0\}. \tag{4.8}$$

Lemma 4.3 (trace lemma)**.** *For every compact set $K \subseteq \partial^{\pm}_{\mathrm{hyp}} U$, the trace operator from $C_c^\infty(\overline{U} \times \mathbb{R}^d)$ to $L^2(K, |v \cdot \mathbf{n}_U| \, dx \, d\gamma)$ extends to a continuous linear operator on $H^1_{\mathrm{hyp}}(U)$.*

Proof. For $r \in \mathbb{R}$, we use the notation $(r)_- := \max(0, -r)$ and $(r)_+ := \max(0, r)$. Since U is a $C^{1,1}$ domain, we can extend the outer normal $\mathbf{n}_U$ to a globally Lipschitz function on U. We still denote the extension by $\mathbf{n}_U$. Let $\chi \in C_c^\infty(\mathbb{R}_+)$ be a smooth function satisfying

$$0 \leqslant \chi \leqslant 1, \quad \chi = 1 \text{ on } [0,1], \quad \chi = 0 \text{ on } [2, +\infty), \quad |\chi'| \leqslant 2. \tag{4.9}$$

For every $r_-, r_+ \in [0, \infty)$ and $x, v \in \mathbb{R}^d$, we define

$$\chi_{r_-,r_+}(x,v) := \chi\left(r_- \frac{(v \cdot \mathbf{n}_U(x))_-}{1 + |v|^2} + r_+ \frac{(v \cdot \mathbf{n}_U(x))_+}{1 + |v|^2} \right). \tag{4.10}$$

Step 1. We show that there exists a constant $C(\chi, d) < \infty$ such that, for every $f \in H^1_{\mathrm{hyp}}(U)$ and $r_-, r_+ \in [0, \infty)$, we have that $f \chi_{r_-,r_+} \in H^1_{\mathrm{hyp}}(U)$ and

$$\|f \chi_{r_-,r_+}\|_{H^1_{\mathrm{hyp}}(U)} \leqslant C(1 + r_- + r_+) \|f\|_{H^1_{\mathrm{hyp}}(U)}. \tag{4.11}$$

We have that $|f \chi_{r_-,r_+}| \leqslant 2|f|$ and

$$|\nabla_v (f \chi_{r_-,r_+})| \leqslant |\nabla_v f| + C(r_- + r_+)|f|. \tag{4.12}$$

Therefore

$$\|f \chi_{r_-,r_+}\|_{L^2(U; H^1_\gamma)} \leqslant C(1 + r_- + r_+) \, \|f\|_{L^2(U; H^1_\gamma)} .$$

We next observe that

$$v \cdot \nabla_x (f \chi_{r_-,r_+}) = \chi_{r_-,r_+} \, v \cdot \nabla_x f + f v \cdot \nabla_x \chi_{r_-,r_+}.$$

We clearly have

$$\|f v \cdot \nabla_x \chi_{r_-,r_+}\|_{L^2(U; L^2_\gamma)} \leqslant C \, \|f\|_{L^2(U; H^1_\gamma)} ,$$

and also

$$\begin{aligned}\|\chi_{r_-,r_+}\, v\cdot\nabla_x f\|_{L^2(U;H_\gamma^{-1})} &\leqslant C\,\|\chi_{r_-,r_+}\|_{L^\infty(U;C^{0,1}(\mathbb{R}^d))}\,\|v\cdot\nabla_x f\|_{L^2(U;H_\gamma^{-1})}\\ &\leqslant C\,\|v\cdot\nabla_x f\|_{L^2(U;H_\gamma^{-1})}\,.\end{aligned}$$

Combining the above inequalities yields (4.11).

Step 2. We complete the proof. By (4.11) and (4.17), we have, for every $r\in[0,\infty)$ and $f\in C_c^\infty(\overline{U}\times\mathbb{R}^d)$,

$$\begin{aligned}\int_{\partial U\times\mathbb{R}^d} f^2(1-\chi_{r,0})\,|v\cdot\mathbf{n}_U|\,dm &= -\int_{\partial U\times\mathbb{R}^d} f^2(1-\chi_{r,0})\,v\cdot\mathbf{n}_U\,dm\\ &\leqslant C(1+r)\|f\|^2_{H^1_{\mathrm{hyp}}(U)},\end{aligned}$$

and similarly,

$$\int_{\partial U\times\mathbb{R}^d} f^2(1-\chi_{0,r})\,|v\cdot\mathbf{n}_U|\,dm\leqslant C(1+r)\|f\|^2_{H^1_{\mathrm{hyp}}(U)}.$$

From the definition of χ_{r_-,r_+}, we see that

$$1-\chi_{r,0}+1-\chi_{0,r}=1-\chi_{r,r}.$$

We deduce that, for every $r\in[0,\infty)$ and $f\in C_c^\infty(\overline{U}\times\mathbb{R}^d)$,

$$\int_{\partial U\times\mathbb{R}^d} f^2(1-\chi_{r,r})\,|v\cdot\mathbf{n}_U|\,dm\leqslant C(1+r)\|f\|^2_{H^1_{\mathrm{hyp}}(U)}. \tag{4.13}$$

By continuity of $\mathbf{n}_U$, we have that

$$\bigcup_{r\geqslant 0}\{(x,v)\in\partial U\times\mathbb{R}^d\ :\ \chi_{r,r}(x,v)=0\}=\partial^{\neq}_{\mathrm{hyp}}U.$$

The lemma now follows from the previous two displays and Proposition 2.2. □

By Lemma 4.3, it makes sense to ask whether for a given $f\in H^1_{\mathrm{hyp}}(U)$, we have

$$\int_{\partial U\times\mathbb{R}^d} f^2\,|v\cdot\mathbf{n}_U|\,dx\,d\gamma<\infty.$$

As discussed above, we cannot rule out the possibility that this integral is infinite in general. The next lemma asserts that in case this integral is finite, the integration by parts formula is valid. We introduce the function space

$$H^1_{\mathrm{hyp},\partial}(U):=\left\{f\in H^1_{\mathrm{hyp}}(U)\ :\ \int_{\partial U\times\mathbb{R}^d} f^2\,|v\cdot\mathbf{n}_U|\,dx\,d\gamma<\infty\right\}, \tag{4.14}$$

equipped with the norm

$$\|f\|_{H^1_{\mathrm{hyp},\partial}(U)}:=\left(\|f\|^2_{H^1_{\mathrm{hyp}}(U)}+\int_{\partial U\times\mathbb{R}^d} f^2\,|v\cdot\mathbf{n}_U|\,dx\,d\gamma\right)^{\frac12}. \tag{4.15}$$

Before stating the lemma, we observe that we have the inclusion $H^1_{\mathrm{hyp},0}(U)\subseteq H^1_{\mathrm{hyp},\partial}(U)$. Indeed, for every $f\in C_c^\infty(\overline{U}\times\mathbb{R}^d)$ which vanishes on $\partial_{\mathrm{hyp}}U$, we have that

$$\int_{U\times\mathbb{R}^d} f\,v\cdot\nabla_x f\,dx\,d\gamma=-\frac12\int_{\partial U\times\mathbb{R}^d} f^2\,v\cdot\mathbf{n}_U\,dx\,d\gamma=\frac12\int_{\partial U\times\mathbb{R}^d} f^2\,|v\cdot\mathbf{n}_U|\,dx\,d\gamma.$$

By density, we deduce that for every $f\in H^1_{\mathrm{hyp},0}(U)$,

$$\int_{\partial U\times\mathbb{R}^d} f^2\,|v\cdot\mathbf{n}_U|\,dx\,d\gamma\leqslant 2\|f\|^2_{H^1_{\mathrm{hyp}}(U)}. \tag{4.16}$$

Arguing again by density, we can already assert the validity of the integration by parts formula (4.4) for every $f, g \in H^1_{\mathrm{hyp},0}(U)$. Similarly, if $f, g \in H^1_{\mathrm{hyp}}(U)$ vanish in a neighborhood of $\partial^{=}_{\mathrm{hyp}} U$, then we can readily obtain the validity of (4.4) from Proposition 2.2 and Lemma 4.3. More generally, we have the following result.

Proposition 4.4 (Integration by parts in $H^1_{\mathrm{hyp},\partial}(U)$)**.** *Under Assumption 4.1, we have, for every $f, g \in H^1_{\mathrm{hyp},\partial}(U)$,*

$$\int_{U\times\mathbb{R}^d} (fBg + gBf)\, dm = -\int_{\partial U\times\mathbb{R}^d} fg\, v\cdot \mathbf{n}_U\, dm. \tag{4.17}$$

The interpretation of the right side of (4.17) is as a Lebesgue integral, which is well-defined since we assume that $f, g \in H^1_{\mathrm{hyp},\partial}(U)$. The fundamental ingredient for the proof of Proposition 4.4 is the density result stated in part (2) of the following lemma. A comparable result can be found in [5].

Lemma 4.5 (Density of smooth functions in $H^1_{\mathrm{hyp},\partial}(U)$)**.** *(1) Under Assumption 4.1, there exists a constant $C(\|\mathbf{b}\|_{L^\infty(U)}, U, d) < \infty$ such that for every $f \in H^1_{\mathrm{hyp}}(U)$,*

$$\begin{aligned}\int_{\partial U\times\mathbb{R}^d} f^2\, |v\cdot \mathbf{n}_U|\, dm \\ \leqslant \min\left(\int_{\partial U\times\mathbb{R}^d} f^2 (v\cdot\mathbf{n}_U)_-\, dm, \int_{\partial U\times\mathbb{R}^d} f^2 (v\cdot\mathbf{n}_U)_+\, dm\right) + C\|f\|^2_{H^1_{\mathrm{hyp}}(U)}.\end{aligned} \tag{4.18}$$

(2) The set of smooth functions with compact support in $\overline{U}\times\mathbb{R}^d$ and which vanish on a neighborhood of $\partial^{=}_{\mathrm{hyp}} U$ is dense in $H^1_{\mathrm{hyp}}(U)$. It is also dense $H^1_{\mathrm{hyp},\partial}(U)$ with respect to the norm in (4.15).

Proof. We decompose the argument into four steps.

Step 1. Part (2) of the lemma can be restated as the existence of smooth functions with certain properties that approximate a given $f \in H^1_{\mathrm{hyp}}(U)$, respectively $f \in H^1_{\mathrm{hyp},\partial}(U)$. In this first step, we show that it suffices to prove part (1) and these approximation statements for functions f with compact support in $\overline{U}\times\mathbb{R}^d$ and which are also in $L^\infty(U\times\mathbb{R}^d)$. The fact that it suffices to consider functions with compact support is obtained as in the beginning of Step 2 of the proof of Proposition 2.2. From now on, we therefore restrict our attention to the case when $f \in H^1_{\mathrm{hyp}}(U)$, resp. $f \in H^1_{\mathrm{hyp},\partial}(U)$, is compactly supported in $\overline{U}\times\mathbb{R}^d$. Let $\Psi \in C^\infty(\mathbb{R};\mathbb{R})$ be a smooth function such that

$$\Psi(x) = \begin{cases} -1 & \text{if } x \leqslant -2, \\ x & \text{if } |x| \leqslant \frac{1}{2}, \\ 1 & \text{if } x \geqslant 2. \end{cases}$$

For convenience, we can further assume that the mapping $x \mapsto x^{-1}\Psi(x)$ is non-decreasing on $(-\infty, 0]$ and non-increasing on $[0,\infty)$. For each $M \geqslant 1$, we consider the truncated function $f_M := M\Psi(f/M) \in H^1_{\mathrm{hyp}}(U)$. Since the mapping $g \mapsto M\Psi(g/M)$ is continuous over $H^1_{\mathrm{hyp}}(U)$, Lemma 4.3 implies that the trace of f_M on $\partial U\times\mathbb{R}^d$ is indeed $M\Psi(f/M) \in L^2_{\mathrm{loc}}(\partial^{\neq}_{\mathrm{hyp}} U, |v\cdot\mathbf{n}_U|\, dm)$. Moreover, one can verify that f_M converges to f in $H^1_{\mathrm{hyp}}(U)$ and in $L^2_{\mathrm{loc}}(\partial^{\neq}_{\mathrm{hyp}} U, |v\cdot\mathbf{n}_U|\, dm)$, and that if f belongs to $H^1_{\mathrm{hyp},\partial}(U)$, then f_M also converges to f with respect to the norm in (4.15). By the monotone convergence theorem, it thus suffices to prove (4.18) for $f \in H^1_{\mathrm{hyp}}(U) \cap L^\infty(U\times\mathbb{R}^d)$. For part (2), it suffices to consider the approximation of functions in $H^1_{\mathrm{hyp}}(U) \cap L^\infty(U\times\mathbb{R}^d)$ and $H^1_{\mathrm{hyp},\partial}(U) \cap L^\infty(U\times\mathbb{R}^d)$ respectively.

Step 2. For every $f \in H^1_{\mathrm{hyp}}(U) \cap L^\infty(U \times \mathbb{R}^d)$ with compact support in $\overline{U} \times \mathbb{R}^d$, we introduce a family of approximations of f that vanish near $\partial^=_{\mathrm{hyp}} U$. For each $r \geqslant 1$, we select a function $\phi_r \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying $0 \leqslant \phi_r \leqslant 1$ and such that

$$\begin{aligned} \phi_r &= 0 \quad \text{on } \left\{(x,v) \in U \times \mathbb{R}^d : \operatorname{dist}((x,v), \partial^=_{\mathrm{hyp}} U) \leqslant r^{-1}\right\} \cap \operatorname{supp} f, \\ \phi_r &= 1 \quad \text{on } \left\{(x,v) \in U \times \mathbb{R}^d : \operatorname{dist}((x,v), \partial^=_{\mathrm{hyp}} U) \geqslant 2r^{-1}\right\} \cap \operatorname{supp} f, \\ |\nabla \phi_r| &\leqslant Cr \quad \text{on } \operatorname{supp} f. \end{aligned}$$

We aim to show first that $f\phi_r$ converges to f weakly in $H^1_{\mathrm{hyp}}(U)$ as r tends to infinity. Since $f\phi_r$ converges to f almost everywhere in $U \times \mathbb{R}^d$, the only possible weak limit to $f\phi_r$ is f. It thus suffices to verify that $f\phi_r$ remains bounded in $H^1_{\mathrm{hyp}}(U)$. By the chain rule,

$$\|\nabla_v (f\phi_r)\|_{L^2(U;L^2_\gamma)} \leqslant \|\phi_r \nabla_v f\|_{L^2(U;L^2_\gamma)} + \|f \nabla_v \phi_r\|_{L^2(U;L^2_\gamma)}. \tag{4.19}$$

The first term on the right side above is clearly bounded uniformly over r. This is also true of the second term, since $f\nabla_v\phi_r$ is non-zero only on a set of measure Cr^{-2} (notice that $\partial^=_{\mathrm{hyp}} U$ is a set of Hausdorff dimension $2d-2$), and is bounded by Cr there, by the assumption of $f \in L^\infty(U \times \mathbb{R}^d)$. (Here we allow the constant $C < \infty$ to also depend on $\|f\|_{L^\infty(U\times\mathbb{R}^d)}$.) Similarly,

$$\|v \cdot \nabla_x (f\phi_r)\|_{L^2(U;H^{-1}_\gamma)} \leqslant \|\phi_r v \cdot \nabla_x f\|_{L^2(U;H^{-1}_\gamma)} + \|f v \cdot \nabla_x \phi_r\|_{L^2(U;H^{-1}_\gamma)}. \tag{4.20}$$

For the first term on the right side above, we note that if $\psi \in L^2(U;H^1_\gamma)$, then

$$\|\psi\phi_r\|_{L^2(U;H^1_\gamma)} \leqslant \|\psi\phi_r\|_{L^2(U;L^2_\gamma)} + \|\phi_r \nabla_v \psi\|_{L^2(U;L^2_\gamma)} + \|\psi_r \nabla_v \phi\|_{L^2(U;L^2_\gamma)}.$$

Arguing as for the last term in (4.19), we deduce that for some constant $C(U,d) < \infty$,

$$\sup\left\{\|\psi\phi_r\|_{L^2(U;H^1_\gamma)} \ : \ \|\psi\|_{L^2(U;H^1_\gamma)} \leqslant 1\right\} \leqslant C,$$

and thus that

$$\|\phi_r v \cdot \nabla_x f\|_{L^2(U;H^{-1}_\gamma)} \leqslant C\|v \cdot \nabla_x f\|_{L^2(U;H^{-1}_\gamma)}$$

is bounded uniformly over $r \geqslant 1$. For the second term on the right side of (4.20), we simply use that

$$\|f v \cdot \nabla_x \phi_r\|_{L^2(U;H^{-1}_\gamma)} \leqslant \|f v \cdot \nabla_x \phi_r\|_{L^2(U;L^2_\gamma)}$$

and then argue as for the last term on the right side of (4.19). This completes the argument for the fact that $f\phi_r$ converges to f weakly in $H^1_{\mathrm{hyp}}(U)$. By Mazur's lemma (see [16, page 6]), we deduce the existence of a sequence $(\phi'_n)_{n\in\mathbb{N}}$ of functions in $C^\infty(U \times \mathbb{R}^d)$ such that for each n, the function ϕ'_n is a convex combination of a finite number of $(\phi_k)_{k\geqslant n}$, and $f\phi'_n$ converges strongly to f in $H^1_{\mathrm{hyp}}(U)$. In particular, each function $f\phi'_n$ vanishes in a neighborhood of $\partial^=_{\mathrm{hyp}} U$. One can check that the same construction allows to obtain strong convergence in $H^1_{\mathrm{hyp},\partial}(U)$ if $f \in H^1_{\mathrm{hyp},\partial}(U) \cap L^\infty(U \times \mathbb{R}^d)$.

Step 3. We complete the proof of part (1) of the lemma. Using the notation of the previous step, since $f\phi'_n$ vanishes in a neighborhood of $\partial^=_{\mathrm{hyp}} U$, we have

$$\int_{U\times\mathbb{R}^d} (f\phi'_n) B(f\phi'_n)\, dm = -\frac{1}{2}\int_{\partial U\times\mathbb{R}^d} (f\phi'_n)^2\, v \cdot \mathbf{n}_U\, dm.$$

If both of the boundary integrals on the right side of (4.18) are infinite, then there is nothing to prove. Otherwise, letting n tend to infinity in the previous identity, we deduce that

$$\left|\int_{\partial U\times\mathbb{R}^d} f^2 (v\cdot\mathbf{n}_U)_+\, dm - \int_{\partial U\times\mathbb{R}^d} f^2 (v\cdot\mathbf{n}_U)_-\, dm\right| \leqslant C\|f\|^2_{H^1_{\mathrm{hyp}}(U)}.$$

This implies (4.18).

Step 4. We complete the proof of part (2) of the lemma, starting with the statement of density in $H^1_{\mathrm{hyp}}(U)$. In view of the result of Step 2, it suffices to show that any $f \in H^1_{\mathrm{hyp}}(U)$ with compact support in $\overline{U} \times \mathbb{R}^d$ and vanishing in a neighborhood of $\partial^{=}_{\mathrm{hyp}} U$ can be approximated by a sequence of smooth functions with the same properties. This is immediate from the proof of Proposition 2.2. Similarly, for the statement of density in $H^1_{\mathrm{hyp},\partial}(U)$, we only need to consider the approximation of functions in $H^1_{\mathrm{hyp},\partial}(U)$ with compact support in $\overline{U} \times \mathbb{R}^d$ and which vanish in a neighborhood of $\partial^{=}_{\mathrm{hyp}} U$. In this case, the approximations used in the proof of Proposition 2.2 can be chosen so that they all vanish in a fixed neighborhood of $\partial^{=}_{\mathrm{hyp}} U$. The convergence of these approximations in $H^1_{\mathrm{hyp},\partial}(U)$ is then a consequence of Lemma 4.3. □

Proof of Proposition 4.4. The integration by parts formula (4.17) is valid for smooth functions with compact support in $\overline{U} \times \mathbb{R}^d$. Proposition 4.4 is thus a consequence of the density statement in $H^1_{\mathrm{hyp},\partial}(U)$ contained in part (2) of Lemma 4.5. □

Abusing notation, if $f, g \in H^1_{\mathrm{hyp}}(U)$ but the assumption $f, g \in H^1_{\mathrm{hyp},\partial}(U)$ of Proposition 4.4 fails to hold, then we *define* the boundary integral on the right side of (4.17) to be the left side of (4.17). We stress however that in this case, we cannot interpret the boundary integral as a Lebesgue integral.

One fundamental ingredient in the proof of well-posedness of the Dirichlet problem presented in the next section is a boundary approximation result. This result addresses the possibility of taking a function $f \in H^1_{\mathrm{hyp}}(U)$ and approximating it by a function b which ideally would be taken to vanish on $\partial_{\mathrm{hyp}} U$ and be equal to f elsewhere on $\partial U \times \mathbb{R}^d$. The precise statement is the following.

Lemma 4.6 (boundary approximation)**.** *Under Assumption 4.1, we have, for every* $f \in H^1_{\mathrm{hyp}}(U)$,

$$\sup_{b \in H^1_{\mathrm{hyp},0}(U)} \int_{\partial U \times \mathbb{R}^d} (b - f)^2 \, v \cdot \mathbf{n}_U \, dm \geqslant 0.$$

Proof. We use some notation introduced in the proof of Lemma 4.3, in particular the functions χ and χ_{r_-,r_+} defined in (4.9) and (4.10). Without loss of generality, we may further assume that $\sqrt{1-\chi^2} \in C^\infty(\mathbb{R})$. We decompose the proof into two steps.

Step 1. We first treat the case $f \in H^1_{\mathrm{hyp},\partial}(U)$. For each $r \geqslant 0$, we set

$$b_r := f(1 - \chi_{r,0}). \tag{4.21}$$

By (4.11), we have $b_r \in H^1_{\mathrm{hyp}}(U)$. Since b_r vanishes in a neighborhood of $\partial_{\mathrm{hyp}} U$, we also have that $b_r \in H^1_{\mathrm{hyp},0}(U)$, and in particular $b_r \in H^1_{\mathrm{hyp},\partial}(U)$. This allows us to interpret the following boundary integrals as Lebesgue integrals and conclude:

$$\begin{aligned} \int_{\partial U \times \mathbb{R}^d} (b_r - f)^2 \, v \cdot \mathbf{n}_U \, dm &= \int_{\partial U \times \mathbb{R}^d} \chi_{r,0}^2 f^2 \, v \cdot \mathbf{n}_U \, dm \\ &\xrightarrow[r \to \infty]{} \int_{\partial U \times \mathbb{R}^d} f^2 \, (v \cdot \mathbf{n}_U)_+ \, dm \geqslant 0. \end{aligned}$$

Step 2. We now consider the case when $f \notin H^1_{\mathrm{hyp},\partial}(U)$. By part (1) of Lemma 4.5, this implies that

$$\int_{\partial U \times \mathbb{R}^d} f^2 (v \cdot \mathbf{n}_U)_- \, dm = +\infty. \tag{4.22}$$

We recall that the integral on the left side of (4.22) is interpreted as a Lebesgue integral using Lemma 4.3 and the fact that the integrand is non-negative. For each $r \geqslant 0$, we define $b_r \in H^1_{\mathrm{hyp},0}(U) \subseteq H^1_{\mathrm{hyp},\partial}(U)$ as in (4.21). For each $r \in [1,\infty)$, the identity

$$\int_{\partial U\times\mathbb{R}^d} \left(g^2 - (\chi_{r,0}g)^2\right) v\cdot \mathbf{n}_U \, dm = \int_{\partial U\times\mathbb{R}^d} (1-\chi_{r,0}^2) g^2 \, v\cdot \mathbf{n}_U \, dm \tag{4.23}$$

holds for every $g \in C^\infty_c(\overline{U}\times\mathbb{R}^d)$. Morever, arguing as for (4.11), we can show that

$$\| g\left(1-\chi_{r,0}^2\right)^{\frac{1}{2}} \|_{H^1_{\mathrm{hyp}}(U)} \leqslant C(1+r)\|g\|_{H^1_{\mathrm{hyp}}(U)}.$$

Hence, we can extend the identity (4.23) to every $g \in H^1_{\mathrm{hyp}}(U)$ by density using Proposition 2.2, and in particular,

$$\int_{\partial U\times\mathbb{R}^d} f^2 \, v\cdot \mathbf{n}_U \, dm - \int_{\partial U\times\mathbb{R}^d} (b_r - f)^2 \, v\cdot \mathbf{n}_U \, dm = \int_{\partial U\times\mathbb{R}^d} (1-\chi_{r,0}^2) f^2 \, v\cdot \mathbf{n}_U \, dm. \tag{4.24}$$

Since $f\left(1-\chi_{r,0}^2\right)^{\frac{1}{2}}$ vanishes in a neighborhood of $\partial_{\mathrm{hyp}}U$, it follows that this function belongs to $H^1_{\mathrm{hyp},0}(U) \subseteq H^1_{\mathrm{hyp},\partial}(U)$, and we thus deduce from Proposition 4.4 that the boundary integral on the right side of (4.24) is a Lebesgue integral. By (4.22), this quantity diverges to $-\infty$ as r tends to infinity. We have thus shown that

$$\lim_{r\to\infty} \int_{\partial U\times\mathbb{R}^d} (b_r - f)^2 \, v\cdot \mathbf{n}_U \, dm = +\infty,$$

which clearly implies the statement of the lemma. □

We next record the simple but useful observation that the $L^2(U;H^1_\gamma)$ norm is not sufficient to control the trace of a function.

Lemma 4.7 (No-trace result for $L^2(U;H^1_\gamma)$)**.** *For every $f \in H^1_{\mathrm{hyp}}(U)$, $b \in H^1_{\mathrm{hyp},0}(U)$ and $\varepsilon > 0$, there exists $f' \in H^1_{\mathrm{hyp},0}(U)$ such that $f' = b$ on $\partial U \times \mathbb{R}^d$ and*

$$\|f - f'\|_{L^2(U;H^1_\gamma)} \leqslant \varepsilon.$$

Remark 4.8. In Lemma 4.7, we understand the statement that $f' = b$ on $\partial U \times \mathbb{R}^d$ to mean that the equality holds almost everywhere on $\partial U \times \mathbb{R}^d$, which is a meaningful statement by Lemma 4.3. The proof in fact allows to construct a function f' such that $f' = b$ in a full neighborhood of the boundary $\partial U \times \mathbb{R}^d$.

Proof of Lemma 4.7. For each $\delta > 0$, denote

$$U_\delta := \{x \in U \ : \ \mathrm{dist}(x,\partial U) \geqslant \delta\}.$$

For every $\delta > 0$ sufficiently small that $U_{2\delta} \neq \varnothing$, , let $\psi_\delta \in C^\infty_c(U)$ denote a smooth function such that

$$\begin{aligned} \psi_\delta &\equiv 1 \quad \text{on } U_{2\delta}, \\ \psi_\delta &\equiv 0 \quad \text{on } \mathbb{R}^d \setminus U_\delta. \end{aligned}$$

We think of ψ_δ as a function on $U\times\mathbb{R}^d$ that does not depend on the second variable. For every $f \in H^1_{\mathrm{hyp}}(U)$, consider the function

$$f_\delta := f\psi_\delta + b(1-\psi_\delta). \tag{4.25}$$

This function clearly satisfies $f_\delta \in H^1_{\mathrm{hyp},0}(U)$ and is such that $f' = b$ on $\partial U\times\mathbb{R}^d$. Moreover, it converges to f in $L^2(U;L^2_\gamma)$ as δ tends to 0. Since

$$\nabla_v f_\delta = \psi_\delta \nabla_v f + (1-\psi_\delta)\nabla_v b,$$

we deduce that $\|\nabla_v(f_\delta - f)\|_{L^2(U;L^2_\gamma)} \to 0$ as δ tends to 0. □

Remark 4.9. Recall that for $f, g \in H^1_{\mathrm{hyp}}(U)$, we have defined the boundary integral on the right side of (4.17) as being equal to the left side of (4.17), but we were not able to interpret this integral as a Lebesgue integral in general. However, the notion is not misleading in the sense that the boundary integral only depends on "what f and g do on the boundary $\partial U \times \mathbb{R}^d$", in the following sense. Using the assumptions of Lemma 4.7 for convenience, and recalling that $H^1_{\mathrm{hyp},0}(U) \subseteq H^1_{\mathrm{hyp},\partial}(U)$, we have that for every $h \in H^1_{\mathrm{hyp},\partial}(U)$,

$$\int_{\partial U \times \mathbb{R}^d} f' h \, v \cdot \mathbf{n}_U \, dm = \int_{\partial U \times \mathbb{R}^d} b h \, v \cdot \mathbf{n}_U \, dm, \tag{4.26}$$

simply because these integrals do make sense as Lebesgue integrals, by Proposition 4.4. By (4.17), both sides of this identity depend continuously on $h \in H^1_{\mathrm{hyp}}(U)$, so we can then use part (2) of Lemma 4.5 to infer that the identity (4.26) holds for every $h \in H^1_{\mathrm{hyp}}(U)$ (although for $h \in H^1_{\mathrm{hyp}}(U)$ we may loose the interpretation of each integral as a Lebesgue integral in general).

To conclude this section, we identify elements of $H^1_{\mathrm{hyp},0}(U)$ with elements of $H^1_{\mathrm{hyp}}(U)$ with vanishing trace on $\partial_{\mathrm{hyp}} U$. This result will not be used in the paper, but we find it interesting nonetheless. Also, we will appeal to a time-dependent version of this result in Section 7.

Proposition 4.10 ($H^1_{\mathrm{hyp},0}(U)$ as a trace space)**.** *For every $f \in H^1_{\mathrm{hyp}}(U)$, we have the equivalence*

$$f \in H^1_{\mathrm{hyp},0}(U) \quad \Longleftrightarrow \quad f = 0 \ \text{ on } \partial_{\mathrm{hyp}} U.$$

Proof. The direct implication follows from Lemma 4.3. For the converse implication, we start by arguing as in Step 1 of the proof of Proposition 2.2 to see that, without loss of generality, we can assume that for every $x \in \mathbb{R}^d \setminus U$ and $\varepsilon \in (0,1]$, we have

$$B((1+\varepsilon)x, \varepsilon) \subseteq \mathbb{R}^d \setminus U. \tag{4.27}$$

Arguing as in the proof of Lemma 4.5, we can also assume without loss of generality that the function $f \in H^1_{\mathrm{hyp}}(U)$, in addition to having vanishing trace on $\partial_{\mathrm{hyp}} U$, is also of compact support in $\overline{U} \times \mathbb{R}^d$ and vanishes in a neighborhood of $\partial^=_{\mathrm{hyp}} U$. We aim to show that there exists a sequence of smooth functions which vanish on $\partial_{\mathrm{hyp}} U$ and converge to f in $H^1_{\mathrm{hyp}}(U)$. We write

$$\partial^*_{\mathrm{hyp}} U := \left\{ (x,v) \in \partial U \times \mathbb{R}^d \ : \ v \cdot \mathbf{n}_U(x) < 0 \right\} = (\partial U \times \mathbb{R}^d) \setminus \left(\partial_{\mathrm{hyp}} U \cup \partial^=_{\mathrm{hyp}} U \right).$$

Using a partition of unity and the fact that f vanishes in a neighborhood of $\partial^=_{\mathrm{hyp}} U$, we can further assume that the support of f is either contained in $U \times \mathbb{R}^d$, or intersects $\partial^*_{\mathrm{hyp}} U$ but not $(\partial U \times \mathbb{R}^d) \setminus \partial^*_{\mathrm{hyp}} U$, or intersects $\partial_{\mathrm{hyp}} U$ but not $(\partial U \times \mathbb{R}^d) \setminus \partial_{\mathrm{hyp}} U$. In the first two cases, it is clear using Proposition 2.2 that the function can be approximated by a smooth function which vanishes on $\partial_{\mathrm{hyp}} U$. There remains to consider the case when the support of f intersects $\partial_{\mathrm{hyp}} U$ but not $(\partial U \times \mathbb{R}^d) \setminus \partial_{\mathrm{hyp}} U$. Under this circumstance, we have in particular that $f = 0$ on $\partial U \times \mathbb{R}^d$. For every $x, v \in \mathbb{R}^d$, we set

$$\widetilde{f}(x,v) := f(x,v) \mathbb{1}_{x \in U},$$

and claim that $\widetilde{f} \in H^1_{\mathrm{hyp}}(\mathbb{R}^d)$. We clearly have

$$\| \widetilde{f} \|_{L^2(\mathbb{R}^d; H^1_\gamma)} = \| f \|_{L^2(U; H^1_\gamma)}.$$

We have $f \in H^1_{\mathrm{hyp},\partial}(U)$, and thus, by Proposition 4.4, that for every $\phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\begin{aligned}\int_{U\times\mathbb{R}^d} \phi\, v\cdot\nabla_x f\, dx\, d\gamma &= -\int_{U\times\mathbb{R}^d} f\, v\cdot\nabla_x\phi\, dx\, d\gamma \\ &= -\int_{\mathbb{R}^d\times\mathbb{R}^d} \widetilde{f}\, v\cdot\nabla_x\phi\, dx\, d\gamma.\end{aligned}$$

It thus follows that $v\cdot\nabla_x\widetilde{f} \in L^2(\mathbb{R}^d; H^{-1}_\gamma)$,

$$\|v\cdot\nabla_x\widetilde{f}\|_{L^2(\mathbb{R}^d;H^{-1}_\gamma)} = \|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)},$$

and in particular, $\widetilde{f} \in H^1_{\mathrm{hyp}}(\mathbb{R}^d)$. We can now adapt the proof of Proposition 2.2 to produce a sequence of smooth approximations of $\widetilde{f}$ that vanish on $(\mathbb{R}^d\setminus U)\times\mathbb{R}^d$. Indeed, with ζ_ε defined in (2.9), we set

$$f_\varepsilon(x,v) := \int_{\mathbb{R}^d} \widetilde{f}((1+\varepsilon)x+y,v)\zeta_\varepsilon(y)\, dy.$$

By (4.27), we have that $f_\varepsilon(x,v) = 0$ whenever $x \in \mathbb{R}^d\setminus U$. We then follows Steps 2 and 3 of the proof of Proposition 2.2 with this definition of f_ε. This produces the desired sequence of approximations of f that vanish on $(\mathbb{R}^d\setminus U)\times\mathbb{R}^d$, and thus completes the proof. □

5. Well-posedness of the Dirichlet problem

In this section, we prove Theorem 1.1 on the well-posedness of the Dirichlet problem. We begin in the first subsection by considering the case when the vector field is conservative, that is, with the additional Assumption 4.1. This is less difficult than the general case, due to the explicit invariant measure, and can be solved by a variational method. We show that solutions exist and are unique by demonstrating the equivalence of the boundary-value problem to the minimization of a functional which is uniformly convex with respect to the $H^1_{\mathrm{hyp}}(U)$ norm. We next extend the well-posedness result to more general vector fields $\mathbf{b}$, as stated in Theorem 1.1. The proof is based on a fixed point argument and a weak maximum principle proved below in Proposition 5.3.

5.1. Well-posedness of the Dirichlet problem for a conservative field.

Throughout this subsection, we will assume that the force field $\mathbf{b}$ depends only on x and is conservative, in the precise sense given by Assumption 4.1. We let H be as in that assumption and B and m be as defined in (4.3) and (4.2). We define the function space

$$Z(U) := \left\{(f,f^*) \,:\, f\in L^2(U;H^1_\gamma) \text{ and } f^* - Bf \in L^2(U;H^{-1}_\gamma)\right\}, \tag{5.1}$$

and for every pair $(f,f^*)\in Z(U)$, we set

$$J[f,f^*] := \inf\left\{\int_{U\times\mathbb{R}^d}\frac{1}{2}|\nabla_v f-\mathbf{g}|^2\, dm \,:\, \mathbf{g}\in L^2(U;L^2_\gamma)^d \text{ s.t. } \nabla_v^*\mathbf{g} = f^*-Bf\right\}. \tag{5.2}$$

We keep the dependence of J on the domain U implicit in the notation. The condition

$$\nabla_v^*\mathbf{g} = f^* - Bf$$

appearing in (5.2) is interpreted as

$$\forall \phi\in L^2(U;H^1_\gamma) \qquad \int_{U\times\mathbb{R}^d}\mathbf{g}\cdot\nabla_v\phi\, dm = \int_{U\times\mathbb{R}^d}(f^*-Bf)\phi\, dm. \tag{5.3}$$

As in (1.5), the interpretation of the right side is through the duality pairing

$$\int_U \langle\phi(x,\cdot),(f^*-Bf)(x,\cdot)\rangle_{H^1_\gamma,H^{-1}_\gamma}\, d\sigma(x),$$

where we recall that the measure σ is defined in (4.1). Note that replacing the measure dm by the measure $dx\, d\gamma$ in (5.3) would yield an equivalent statement.

Proposition 5.1 (Solvability of the boundary value problem)**.** *Under Assumption 4.1, for each $(g, g^*) \in Z(U)$, the mapping*

$$\left\{ \begin{array}{ccc} g + H^1_{\mathrm{hyp},0}(U) & \to & [0, +\infty] \\ f & \mapsto & J[f, g^*] \end{array} \right. \tag{5.4}$$

is uniformly convex. Moreover, its minimum is zero, and the associated minimizer is the unique $f \in g + H^1_{\mathrm{hyp},0}(U)$ solution of the equation

$$\nabla_v^* \nabla_v f + Bf = g^*. \tag{5.5}$$

Equation (5.5) is interpreted in the sense that

$$\forall \phi \in L^2(U; H^1_\gamma) \qquad \int_{U \times \mathbb{R}^d} \nabla_v f \cdot \nabla_v \phi \, dm = \int_{U \times \mathbb{R}^d} (g^* - Bf) \phi \, dm,$$

or equivalently, the same property with the measure dm replaced by $dx\, d\gamma$ on both sides. Recall the notation $\langle \cdot \rangle_\gamma$ introduced in (2.2). By testing the condition (5.3) with functions ϕ that depend only on the position variable, we see that this condition cannot be fulfilled unless for almost every $x \in U$,

$$\langle (f^* - Bf)(x, \cdot) \rangle_\gamma = 0. \tag{5.6}$$

In order to verify that the mapping in (5.4) is not constantly equal to $+\infty$, the following lemma will be useful. It asserts that any function in $L^2(U)$ can be written in the form $\langle Bf \rangle_\gamma$ for some $f \in H^1_{\mathrm{hyp},0}(U)$. Note that, by integration by parts, we have for every $f \in C_c^\infty(\overline{U} \times \mathbb{R}^d)$ that

$$\langle Bf \rangle_\gamma = \langle -v \cdot \nabla_x f \rangle_\gamma. \tag{5.7}$$

This property extends to every $f \in H^1_{\mathrm{hyp}}(U)$ by density.

Lemma 5.2. *There exist $C(U, d) < \infty$ and, for each $h \in L^2(U)$, a function $f \in H^1_{\mathrm{hyp},0}(U)$ satisfying*

$$\langle v \cdot \nabla_x f \rangle_\gamma = h \tag{5.8}$$

and the estimate

$$\| f \|_{H^1_{\mathrm{hyp}}(U)} \leqslant C \, \| h \|_{L^2(U)}. \tag{5.9}$$

Proof. Step 1. We prove the proposition under the additional assumption that $\int_U h = 0$. We consider the problem (3.1) which, as recalled there, has a solution $\mathbf{f}$ with components in $H^1_0(U)$ satisfying the estimate (3.2). We next select smooth functions $\xi_1, \ldots, \xi_d \in C_c^\infty(\mathbb{R}^d)$ which satisfy

$$\int_{\mathbb{R}^d} v \xi_i(v) \, d\gamma(v) = e_i. \tag{5.10}$$

Notice that by integration by parts against the Gaussian measure, we may rewrite this as

$$\int_{\mathbb{R}^d} \nabla \xi_i(v) \, d\gamma(v) = e_i. \tag{5.11}$$

We define

$$f(x, v) := \sum_{j=1}^d \mathbf{f}_i(x) \xi_i(v).$$

Since each component of $\mathbf{f}$ belongs to $H^1_0(U)$, they each admit a sequence of approximating functions in $H^1(U)$ that are smooth and vanish on ∂U. We thus deduce that $f \in H^1_{\mathrm{hyp},0}(U)$. (We could also have appealed to Proposition 4.10 to obtain this.) Moreover, we have

$$\|f\|_{H^1_{\mathrm{hyp}}(U)} \leqslant C \sum_{i=1}^{d} \|\mathbf{f}_i\|_{H^1(U)} \leqslant C \, \|h\|_{L^2(U)} .$$

This is (5.9). To verify (5.8), we use (5.10) to notice that

$$\langle v \cdot \nabla_x f(x,v) \rangle_\gamma = \int_{\mathbb{R}^d} \sum_{i=1}^{d} v \cdot \nabla \mathbf{f}_i(x) \xi_i(v) \, d\gamma(v) = \nabla \cdot \mathbf{f}(x) = h(x).$$

This completes the proof in the case $\int_U h = 0$.

Step 2. We construct a function $f_0 \in H^1_{\mathrm{hyp},0}(U)$ satisfying

$$\int_{U \times \mathbb{R}^d} v \cdot \nabla_x f_0 \, dx \, d\gamma = 1 \qquad \text{and} \qquad \|f_0\|_{H^1_{\mathrm{hyp}}(U)} \leqslant C.$$

We take $\psi_0 \in C^\infty_c(\overline{U} \times \mathbb{R}^d)$ satisfying $\psi_0 \geqslant 0$, $\psi_0 \equiv 0$ on $\partial_{\mathrm{hyp}} U$ and $\psi_0(x_0, v_0) > 0$ at some point $(x_0, v_0) \in \partial U \times \mathbb{R}^d$. Integrating by parts and recalling the definition of $\partial_{\mathrm{hyp}} U$ in (1.9), we find that

$$\begin{aligned} \int_{U \times \mathbb{R}^d} v \cdot \nabla_x \psi_0(x,v) \, dx \, d\gamma(v) &= \int_{\partial U \times \mathbb{R}^d} (\mathbf{n}_U(x) \cdot v) \psi_0(x,v) \, dx \, d\gamma(v) \\ &= - \int_{\partial U \times \mathbb{R}^d} |\mathbf{n}_U(x) \cdot v| \, \psi_0(x,v) \, dx \, d\gamma(v) < 0. \end{aligned}$$

We can clearly choose ψ_0 to depend only on (U, d). It thus suffices to take

$$f_0 := - \left(\int_{U \times \mathbb{R}^d} v \cdot \nabla_x \psi_0 \, dx \, d\gamma \right)^{-1} \psi_0.$$

Step 3. We combine the previous steps to conclude. For a general $h \in L^2(U)$, we set

$$\widetilde{h} := h - (h)_U \, \langle v \cdot \nabla_x f_0 \rangle_\gamma .$$

Since this function has zero mean, we can apply the result of the first step to obtain the existence of some $\widetilde{f} \in H^1_{\mathrm{hyp},0}(U)$ satisfying the desired properties with respect to $\widetilde{h}$. We then obtain the conclusion by setting $f := \widetilde{f} + (h)_U f_0$. □

Note that the function $f \in H^1_{\mathrm{hyp},0}(U)$ we constructed in the proof of Lemma 5.2 has in fact vanishing trace everywhere on $\partial U \times \mathbb{R}^d$, not only on $\partial_{\mathrm{hyp}} U$.

Proof of Proposition 5.1. We decompose the proof into five steps.

Step 1. We show that the functional in (5.4) is not constantly equal to $+\infty$ and that it is uniformly convex. For every $f \in H^1_{\mathrm{hyp}}(U)$ and $\mathbf{j} \in L^2(U; L^2_\gamma)^d$, we define

$$\mathcal{J}[f, \mathbf{j}] := \int_{U \times \mathbb{R}^d} \frac{1}{2} |\nabla_v f - \mathbf{j}|^2 \, dm.$$

We also fix $(g, g^*) \in Z(U)$ and define

$$\mathcal{A}(g, g^*) := \left\{ (f, \mathbf{j}) \in (g + H^1_{\mathrm{hyp},0}(U)) \times L^2(U; L^2_\gamma)^d \ : \ \nabla_v^* \mathbf{j} = g^* - Bf \right\}.$$

By Lemma 5.2 and (5.7), there exists $f_0 \in H^1_{\mathrm{hyp},0}(U)$ such that

$$\langle g^* - Bg \rangle_\gamma = \langle B f_0 \rangle_\gamma. \tag{5.12}$$

It follows that the set $\mathcal{A}(g, g^*)$ is non-empty, since it contains

$$\left(g + f_0, \nabla_v (\nabla_v^* \nabla_v)^{-1} (g^* - B(g + f_0)) \right). \tag{5.13}$$

We now show that the functional $\mathcal{J}$ is uniformly convex on $\mathcal{A}(g,g^*)$. Since for every $(f',\mathbf{j}') \in \mathcal{A}(g,g^*)$ and $(f,\mathbf{j}) \in \mathcal{A}(0,0)$,

$$\frac{1}{2}\mathcal{J}[f'+f,\mathbf{j}'+\mathbf{j}] + \frac{1}{2}\mathcal{J}[f'-f,\mathbf{j}'-\mathbf{j}] - \mathcal{J}[f',\mathbf{j}'] = \mathcal{J}[f,\mathbf{j}],$$

it suffices to show that there exists $C(d,U) < \infty$ such that for every $(f,\mathbf{j}) \in \mathcal{A}(0,0)$,

$$\mathcal{J}[f,\mathbf{j}] \geqslant C^{-1}\left(\|f\|^2_{H^1_{\mathrm{hyp}}(U)} + \|\mathbf{j}\|^2_{L^2(U;L^2_\gamma)}\right). \tag{5.14}$$

Expanding the square and using that $\nabla_v^*\mathbf{j} = -Bf$, we find

$$\mathcal{J}[f,\mathbf{j}] = \int_{U\times\mathbb{R}^d}\left(\frac{1}{2}|\nabla_v f|^2 + \frac{1}{2}|\mathbf{j}|^2 + fBf\right)dm.$$

Moreover, by (4.5) and the fact that $f \in H^1_{\mathrm{hyp},0}(U)$, the term $\int fBf\,dm$ is nonnegative. Finally, we have $\langle Bf\rangle_\gamma = 0$ and thus (2.4) gives us that

$$\begin{aligned}\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)} &\leqslant \|Bf\|_{L^2(U;H^{-1}_\gamma)} + \|\mathbf{b}(x)\cdot\nabla_v f\|_{L^2(U;H^{-1}_\gamma)}\\ &\leqslant C\|\mathbf{j}\|_{L^2(U;L^2_\gamma)} + C\|\nabla_v f\|_{L^2(U;L^2_\gamma)}.\end{aligned}$$

Combining the last displays yields (5.14), and thus also the uniform convexity of the functional in (5.4).

Step 2. Denote by $(f_1,\mathbf{j}_1)$ the unique minimizing pair of the functional $\mathcal{J}$ over $\mathcal{A}(g,g^*)$. In particular, f_1 is the unique minimizer of the functional in (5.4), and we obviously have

$$J[f_1,g^*] \geqslant 0.$$

Observe that there is a one-to-one correspondence between solutions of (5.5) and null minimizers of J: for every $f \in g + H^1_{\mathrm{hyp},0}(U)$, we have

$$f \text{ solves (5.5)} \iff J[f,g^*] = 0.$$

Indeed, the implication $\Longrightarrow$ is clear since if f solves (5.5), then

$$(f,\nabla_v f) \in \mathcal{A}(g,g^*) \quad\text{and}\quad \mathcal{J}[f,\nabla_v f] = 0.$$

Conversely, if $J[f,g^*] = 0$, then $f = f_1$ and $\mathcal{J}[f_1,\mathbf{j}_1] = 0$. This implies that

$$\nabla_v f_1 = \mathbf{j}_1 \qquad \text{a.e. in } U\times\mathbb{R}^d,$$

and since $\nabla_v^*\mathbf{j}_1 = g^* - Bf_1$, we recover that $f = f_1$ is indeed a solution of (5.5). In particular, the fact that there is at most one solution to (5.5) is clear.

To complete the proof of Proposition 5.1, it thus remains to show that

$$J[f_1,g^*] \leqslant 0. \tag{5.15}$$

In order to do so, we introduce the perturbed convex minimization problem defined, for every $f^* \in L^2(U;H^{-1}_\gamma)$, by

$$G(f^*) := \inf_{f\in H^1_{\mathrm{hyp},0}(U)}\left(J[f+g,f^*+g^*] + \int_{U\times\mathbb{R}^d} ff^*\,dm\right).$$

The inequality (5.15) we intend to prove can be rephrased as $G(0) \leqslant 0$. We decompose the proof of this fact into the next three steps.

Step 3. In this step, we verify that the function G is convex and reduce the problem of showing (5.15) to that of showing that the convex dual of G is nonnegative. For every pair $(f,\mathbf{j})$ satisfying $(f+g,\mathbf{j}) \in \mathcal{A}(g,f^*+g^*)$, we have

$$\nabla_v^*\mathbf{j} = f^* + g^* - B(f+g), \tag{5.16}$$

and thus

$$\begin{aligned}\mathcal{J}[f+g,\mathbf{j}] &= \int_{U\times\mathbb{R}^d} \frac{1}{2}|\nabla_v(f+g)-\mathbf{j}|^2\,dm \\ &= \int_{U\times\mathbb{R}^d}\left(\frac{1}{2}|\nabla_v(f+g)|^2+\frac{1}{2}|\mathbf{j}|^2+(f+g)B(f+g)-(f+g)(f^*+g^*)\right)dm.\end{aligned}$$

By adding $\int ff^*\,dm$ to the expression above and then taking the infimum over all $(f,\mathbf{j})$ satisfying the affine constraint $(f+g,\mathbf{j})\in\mathcal{A}(g,f^*+g^*)$, we obtain the quantity $G(f^*)$. Using (4.5) and that $f\in H^1_{\mathrm{hyp},0}(U)$, we thus infer that G is convex. By Lemma 5.2, (5.7) and the construction in (5.13), the function G is also locally bounded above. These two properties imply that G is lower semi-continuous, see [16, Lemma I.2.1 and Corollary I.2.2]. We denote by G^* the convex dual of G, defined for every $h\in L^2(U;H^1_\gamma)$ by

$$G^*(h) := \sup_{f^*\in L^2(U;H^{-1}_\gamma)}\left(-G(f^*)+\int_{U\times\mathbb{R}^d} hf^*\,dm\right),$$

and by G^{**} the bidual of G. Since G is lower semi-continuous, we have that $G^{**}=G$ (see [16, Proposition I.4.1]), and in particular,

$$G(0)=G^{**}(0)=\sup_{h\in L^2(U;H^1_\gamma)}\left(-G^*(h)\right).$$

In order to prove that $G(0)\leqslant 0$, it therefore suffices to show that

$$\forall h\in L^2(U;H^1_\gamma),\qquad G^*(h)\geqslant 0. \tag{5.17}$$

Step 4. Note that for every $h\in L^2(U;H^1_\gamma)$, we have $G^*(h)\in\mathbb{R}\cup\{+\infty\}$. In this step, we show that

$$G^*(h)<+\infty \quad\Longrightarrow\quad h\in H^1_{\mathrm{hyp}}(U). \tag{5.18}$$

We rewrite $G^*(h)$ in the form

$$G^*(h)=\sup\left\{\int_{U\times\mathbb{R}^d}\left(-\frac{1}{2}|\nabla_v(f+g)-\mathbf{j}|^2-ff^*+hf^*\right)dm\right\}, \tag{5.19}$$

where the supremum is over every $f\in H^1_{\mathrm{hyp},0}(U)$, $\mathbf{j}\in L^2(U;L^2_\gamma)^d$ and $f^*\in L^2(U;H^{-1}_\gamma)$ satisfying the constraint (5.16). Recall that we can construct $f_0\in H^1_{\mathrm{hyp},0}(U)$ such that (5.12) holds. We choose to restrict the supremum above to $f^*:=B(f-f_0)$ and $\mathbf{j}=\mathbf{j}_0$ solution of $\nabla_v^*\mathbf{j}_0=g^*-B(g+f_0)$. Recall from (5.13) that such a $\mathbf{j}_0\in L^2(U;L^2_\gamma)^d$ exists since $\langle g^*-B(g+f_0)\rangle_\gamma\equiv 0$. With such choices of f^* and $\mathbf{j}$, the constraint (5.16) is satisfied, and we obtain that

$$\begin{aligned}&G^*(h)\\ &\geqslant\sup\left\{\int_{U\times\mathbb{R}^d}\left(-\frac{1}{2}|\nabla_v(f+g)-\mathbf{j}_0|^2-fB(f-f_0)+hB(f-f_0)\right)dm\ :\ f\in H^1_{\mathrm{hyp},0}(U)\right\}.\end{aligned}$$

For smooth functions f with compact support in $U\times\mathbb{R}^d$, we can use (4.5) to infer that $\int fBf\,dm=0$. Hence,

$$G^*(h)\geqslant\sup\left\{\int_{U\times\mathbb{R}^d}\left(-\frac{1}{2}|\nabla_v(f+g)-\mathbf{j}_0|^2+fBf_0+hB(f-f_0)\right)dm\ :\ f\in C^\infty_c(U\times\mathbb{R}^d)\right\},$$

and therefore, since $Bf_0\in L^2(U;H^{-1}_\gamma)$, the assumption of $G^*(h)<\infty$ implies that

$$\sup\left\{\int_{U\times\mathbb{R}^d} hBf\,dm\ :\ f\in C^\infty_c(U\times\mathbb{R}^d),\ \|f\|_{L^2(U;H^1_\gamma)}\leqslant 1\right\}<\infty.$$

This shows that the distribution Bh belongs to the dual of $L^2(U;H^1_\gamma)$, which is $L^2(U;H^{-1}_\gamma)$. Since

$$v\cdot\nabla_x h = -Bh - \mathbf{b}\cdot\nabla_v h,$$

the proof of (5.18) is complete.

Step 5. In place of (5.17), we have left to show that

$$\forall h \in H^1_{\mathrm{hyp}}(U), \qquad G^*(h) \geqslant 0. \tag{5.20}$$

Since $Bf \in L^2(U;H^{-1}_\gamma)$, we may replace f^* by $f^* + Bf$ in the variational formula (5.19) for G^* to get that

$$G^*(h) = \sup\left\{\int_{U\times\mathbb{R}^d}\left(-\frac{1}{2}|\nabla_v(f+g)-\mathbf{j}|^2 + (h-f)(f^*+Bf)\right)dm\right\}, \tag{5.21}$$

where the supremum is over every $f \in H^1_{\mathrm{hyp},0}(U)$, $\mathbf{j}\in L^2(U;L^2_\gamma)^d$ and $f^*\in L^2(U;H^{-1}_\gamma)$ satisfying the constraint

$$\nabla_v^*\mathbf{j} = f^* + g^* - Bg. \tag{5.22}$$

We would like to choose $f = h$ in (5.21), but the class of allowed choices for f is restricted by the boundary condition $f\in H^1_{\mathrm{hyp},0}(U)$. We will now show that this restriction can be lifted using Lemma 4.7. By definition of the boundary integral, see (4.4)–(4.5), we have

$$\int_{U\times\mathbb{R}^d}(h-f)Bf\,dm = -\int_{U\times\mathbb{R}^d} fBh\,dm + \int_{\partial U\times\mathbb{R}^d}\left(\frac{1}{2}f^2 - fh\right)v\cdot\mathbf{n}_U\,dm,$$

and thus

$$G^*(h) = \sup\left\{\int_{U\times\mathbb{R}^d}\left(-\frac{1}{2}|\nabla_v(f+g)-\mathbf{j}|^2+(h-f)f^*-fBh\right)dm + \int_{\partial U\times\mathbb{R}^d}\left(\frac{1}{2}f^2-fh\right)v\cdot\mathbf{n}_U\,dm\right\},$$

where the supremum is taken over f, $\mathbf{j}$,f^* as in (5.21). Note that for each fixed $\mathbf{j}$ and f^*, the functional

$$f\mapsto \int_{U\times\mathbb{R}^d}\left(-\frac{1}{2}|\nabla_v(f+g)-\mathbf{j}|^2+(h-f)f^*-fBh\right)dm$$

is continuous with respect to the topology of $L^2(U;H^1_\gamma)$, and the function f does not appear in the constraint (5.22). In view of Lemma 4.7 and Remark 4.9, we deduce that

$$G^*(h) = \sup\left\{\int_{U\times\mathbb{R}^d}\left(-\frac{1}{2}|\nabla_v(f+g)-\mathbf{j}|^2+(h-f)f^*-fBh\right)dm + \int_{\partial U\times\mathbb{R}^d}\left(\frac{1}{2}b^2-bh\right)v\cdot\mathbf{n}_U\,dm\right\},$$

where now the supremum is over every $f\in H^1_{\mathrm{hyp}}(U)$, $b\in H^1_{\mathrm{hyp},0}(U)$, $\mathbf{j}\in L^2(U;L^2_\gamma)^d$, and $f^*\in L^2(U;H^1_\gamma)$ satisfying the constraint (5.22). Let us stress that the boundary constraint on f has been removed. We are thus allowed to select $f = h$ and obtain that

$$G^*(h)\geqslant \sup\left\{\int_{U\times\mathbb{R}^d}\left(-\frac{1}{2}|\nabla_v(h+g)-\mathbf{j}|^2-hBh\right)dm + \int_{\partial U\times\mathbb{R}^d}\left(\frac{1}{2}b^2-bh\right)v\cdot\mathbf{n}_U\,dm\right\},$$

where the supremum is now over every b, $\mathbf{j}$ and f^* as above. Using the integration by parts formula (4.17) once more yields

$$\int_{\partial U\times\mathbb{R}^d}\left(\frac{1}{2}b^2-bh\right)v\cdot\mathbf{n}_U\,dm - \int_{U\times\mathbb{R}^d}hBh\,dm = \int_{\partial U\times\mathbb{R}^d}\frac{1}{2}(b-h)^2\,v\cdot\mathbf{n}_U\,dm.$$

We thus deduce from Lemma 4.6 that

$$G^*(h)\geqslant \sup\left\{\int_{U\times\mathbb{R}^d}-\frac{1}{2}|\nabla_v(h+g)-\mathbf{j}|^2\,dm\right\},$$

with the supremum ranging over all $f^* \in L^2(U;H_\gamma^{-1})$ and $\mathbf{j} \in L^2(U;L_\gamma^2)^d$ satisfying the constraint (5.22). We now simply select $\mathbf{j} = \nabla_v(h+g) \in L^2(U;L_\gamma^2)^d$ and then $f^* = \nabla_v^*\mathbf{j} - g^* + Bg \in L^2(U;H_\gamma^{-1})$ to conclude that $G^*(h) \geqslant 0$. □

5.2. **Well-posedness for a general bounded vector field.** In this subsection, we complete the proof of Theorem 1.1. In the previous section, we proved the result for conservative vector fields, and thus in particular in the case $\mathbf{b} = 0$. Here we will use this result and an abstract fixed point argument (based on Schaeffer's fixed point theorem) to obtain solvability for general forcing fields $\mathbf{b}$. We begin by formulating a version of the weak maximum principle for weak subsolutions.

We say that a function $f \in H^1_{\mathrm{hyp}}(U)$ is a *weak subsolution of* (1.1) *in* $U \times \mathbb{R}^d$ or, equivalently, is a weak solution of the differential inequality

$$-\Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f \leqslant f^* \quad \text{in } U\times\mathbb{R}^d,$$

if, for every $h \in L^2(U;H^1_\gamma)$ with $h \geqslant 0$ in $U\times\mathbb{R}^d$,

$$\int_{U\times\mathbb{R}^d} \nabla_v h\cdot\nabla_v f\,dx\,d\gamma \leqslant \int_{U\times\mathbb{R}^d} h\left(f^* + v\cdot\nabla_x f - \mathbf{b}\cdot\nabla_v f\right)dx\,d\gamma.$$

Proposition 5.3 (Weak maximum principle)**.** *Suppose that* $\mathbf{b} \in L^\infty(U\times\mathbb{R}^d;\mathbb{R}^d)$ *and* $f \in H^1_{\mathrm{hyp},0}(U)$ *satisfy*

$$-\Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f \leqslant 0 \quad \text{in } U\times\mathbb{R}^d. \tag{5.23}$$

Then $f \leqslant 0$ *in* $U\times\mathbb{R}^d$.

Proof. Arguing by contradiction, we suppose that $k := \sup_{U\times\mathbb{R}^d} f > 0$. Let $h \in (0,k)$ be a constant which will be selected below. Testing (5.23) with $g_h := (f-h)_+ \in L^2(U;H^1_\gamma)$, we obtain

$$\int_{U\times\mathbb{R}^d} \left(\nabla_v f\cdot\nabla_v g_h - g_h v\cdot\nabla_x f + g_h\mathbf{b}\cdot\nabla_v f\right)dx\,d\gamma \leqslant 0. \tag{5.24}$$

Denote

$$\Gamma_h := \left\{(x,v)\in U\times\mathbb{R}^d \,:\, \nabla_v g_h(x,v) \neq 0\right\}.$$

Observe that

$$\nabla_v f = \nabla_v g_h \quad \text{a.e. on } \{g_h \neq 0\} \supseteq \Gamma_h.$$

Therefore, the inequality (5.24) may be written as

$$\int_{U\times\mathbb{R}^d} \left(|\nabla_v g_h|^2 - g_h v\cdot\nabla_x f + g_h\mathbf{b}\cdot\nabla_v g_h\right)dx\,d\gamma \leqslant 0.$$

For every $f' \in C_c^\infty(\overline{U}\times\mathbb{R}^d)$ which vanish on $\partial_{\mathrm{hyp}}(U)$ and $h>0$, if we set $g_h' := (f'-h)_+$, then we have $\nabla_x f' = \nabla_x g_h'$ on the set $\{g_h' \neq 0\}$, and therefore by (4.6) and (1.9),

$$\int_{U\times\mathbb{R}^d} g_h' v\cdot\nabla_x f'\,dx\,d\gamma = \int_{U\times\mathbb{R}^d} g_h' v\cdot\nabla_x g_h'\,dx\,d\gamma = \frac{1}{2}\int_{\partial U\times\mathbb{R}^d} (g_h')^2 v\cdot\mathbf{n}_U\,dx\,d\gamma \leqslant 0.$$

Since the mapping $f' \mapsto (f'-h)_+$ is continuous from $L^2(U;H^1_\gamma)$ to $L^2(U;H^1_\gamma)$, we therefore obtain by density that

$$\int_{U\times\mathbb{R}^d} g_h v\cdot\nabla_x f\,dx\,d\gamma \leqslant 0.$$

We next estimate, using Young's inequality,

$$\begin{aligned}
&\left| \int_{U\times\mathbb{R}^d} g_h \mathbf{b}\cdot\nabla_v g_h \, dx\, d\gamma \right| \\
&\quad = \left| \int_{\Gamma_h} g_h \mathbf{b}\cdot\nabla_v g_h \, dx\, d\gamma \right| \\
&\quad \leqslant \|\mathbf{b}\|_{L^\infty(U\times\mathbb{R}^d)} \left(\int_{U\times\mathbb{R}^d} |\nabla_v g_h|^2 \, dx\, d\gamma \right)^{\frac12} \left(\int_{\Gamma_h} |g_h|^2 \, dx\, d\gamma \right)^{\frac12} \\
&\quad \leqslant \frac12 \int_{U\times\mathbb{R}^d} |\nabla_v g_h|^2 \, dx\, d\gamma + \frac12 \|\mathbf{b}\|^2_{L^\infty(U\times\mathbb{R}^d)} \left(\int_{\Gamma_h} |g_h|^2 \, dx\, d\gamma \right).
\end{aligned}$$

Combining the previous displays, we obtain

$$\frac12 \int_{U\times\mathbb{R}^d} |\nabla_v g_h|^2 \, dx\, d\gamma \leqslant \frac12 \|\mathbf{b}\|^2_{L^\infty(U\times\mathbb{R}^d)} \left(\int_{\Gamma_h} |g_h|^2 \, dx\, d\gamma \right).$$

Applying (3.34) and using the Gaussian Poincaré inequality, we find that

$$\|\nabla g_h\|_{L^2(U;L^2_\gamma)} \leqslant C \|\mathbf{b}\|_{L^\infty(U\times\mathbb{R}^d)} \|\mathbb{1}_{\Gamma_h}\|^{\frac12}_{L_{F^*}(U\times\mathbb{R}^d,dxd\gamma)} \|\nabla g_h\|_{L^2(U;L^2_\gamma)}. \tag{5.25}$$

Now, since $h < k := \sup_{U\times\mathbb{R}^d} f$, the function g_h does not vanish a.e. in $U\times\mathbb{R}^d$ and therefore neither does ∇g_h. We deduce that there exists a positive constant $c\left(\|\mathbf{b}\|_{L^\infty(U\times\mathbb{R}^d)}\right) > 0$ such that, for every $0 < h < k$,

$$\|\mathbb{1}_{\Gamma_h}\|_{L_{F^*}(U\times\mathbb{R}^d,dxd\gamma)} \geqslant c. \tag{5.26}$$

It is clear that $\Gamma_h \supseteq \Gamma_{h'}$ for $h < h'$, and by the fact that $\Gamma_h \subseteq \{(x,v) : g(x,v) \neq 0\}$, we have

$$\bigcap_{0<h<k} \Gamma_h \subseteq \left\{ (x,v) : f(x,v) \equiv \sup_{U\times\mathbb{R}^d} f \right\}. \tag{5.27}$$

We also have that ∇f and hence ∇g_h vanish almost everywhere on the set on the right side of (5.27). This implies that the set on the left side of (5.27) has null measure, and therefore that

$$\lim_{h\nearrow k} \|\mathbb{1}_{\Gamma_h}\|_{L_{F^*}(U\times\mathbb{R}^d,dxd\gamma)} = 0.$$

This stands in violation of (5.26), and thus completes the proof. □

We next complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We may assume without loss of generality that $f_0 = 0$.

Step 1. We set up the fixed point argument for obtaining existence. Define an operator

$$\mathsf{S} : H^1_{\mathrm{hyp}}(U) \to H^1_{\mathrm{hyp}}(U)$$

by taking $\mathsf{S}f := h$ to be the solution of the Dirichlet problem

$$\begin{cases} -\Delta_v h + v\cdot\nabla_v h - v\cdot\nabla_x h = f^* - \mathbf{b}\cdot\nabla_v f & \text{in } U, \\ h = 0 & \text{on } \partial_{\mathrm{hyp}} U. \end{cases} \tag{5.28}$$

By (3.36), for every $f \in L^2(U;L^2_\gamma)$, we have that $\mathbf{b}\cdot\nabla_v f \in L^2(U;H^{-1}_\gamma)$ and

$$\begin{aligned}
\|\mathbf{b}\cdot\nabla_v f\|_{L^2(U;H^{-1}_\gamma)} &\leqslant C\|\mathbf{b}\|_{L^\infty(U;C^{0,1}(\mathbb{R}^d))} \|\nabla_v f\|_{L^2(U;H^{-1}_\gamma)} \\
&\leqslant C\|\mathbf{b}\|_{L^\infty(U;C^{0,1}(\mathbb{R}^d))} \|f - (f)_U\|_{L^2(U;L^2_\gamma)}.
\end{aligned} \tag{5.29}$$

Therefore, by Proposition 5.1, there exists a unique solution to the problem (5.28) and, for a constant $C(d,U)<\infty$, we have the estimate

$$\|\mathsf{S}f\|_{H^1_{\mathrm{hyp}}(U)} \leqslant C\left(\|f^*\|_{L^2(U;H^{-1}_\gamma)} + \|\mathbf{b}\|_{L^\infty(U;C^{0,1}(\mathbb{R}^d))}\,\|f-(f)_U\|_{L^2(U;L^2_\gamma)}\right). \tag{5.30}$$

In order to argue that there exists $f\in H^1_{\mathrm{hyp}}(U)$ satisfying $f=\mathsf{S}f$, we check the hypotheses of Schaeffer's fixed point theorem (see for instance [17]). That is, we will show

$$\mathsf{S} \text{ is a continuous and compact operator on } H^1_{\mathrm{hyp}}(U) \tag{5.31}$$

and

$$\left\{f\in H^1_{\mathrm{hyp}}(U) : \exists \alpha\in[0,1] \text{ such that } f=\alpha\mathsf{S}f\right\} \quad \text{is bounded in } H^1_{\mathrm{hyp}}(U). \tag{5.32}$$

Step 2. The proof of (5.31). If $f_1,f_2\in H^1_{\mathrm{hyp}}(U)$, then $\widetilde{h}:=\mathsf{S}f_1-\mathsf{S}f_2$ is the solution of

$$\begin{cases} -\Delta_v\widetilde{h}+v\cdot\nabla_v\widetilde{h}-v\cdot\nabla_x\widetilde{h}=-\mathbf{b}\cdot\nabla_v(f_1-f_2) & \text{in } U,\\ \widetilde{h}=0 & \text{on } \partial_{\mathrm{hyp}}U. \end{cases}$$

Since, as in (5.29), $\|\mathbf{b}\cdot\nabla_v(f_1-f_2)\|_{L^2(U;H^{-1}_\gamma)}\leqslant C\|\mathbf{b}\|_{L^\infty(U;C^{0,1}(\mathbb{R}^d))}\|f_1-f_2\|_{L^2(U;L^2_\gamma)}$, an application of Proposition 5.1 yields the estimate

$$\|\mathsf{S}f_1-\mathsf{S}f_2\|_{H^1_{\mathrm{hyp}}(U)}\leqslant C\|\mathbf{b}\|_{L^\infty(U;C^{0,1}(\mathbb{R}^d))}\|f_1-f_2\|_{L^2(U;L^2_\gamma)}. \tag{5.33}$$

This yields the continuity of S. The compactness of S also follows from (5.33) and Proposition 3.5, which asserts that $H^1_{\mathrm{hyp}}(U)$ is compactly embedded in $L^2(U;L^2_\gamma)$.

Step 3. The proof of (5.32). Suppose on the contrary that (5.32) is false. Then for each $k\in\mathbb{N}$, there exists $f_k\in H^1_{\mathrm{hyp}}(U)$ and $\alpha_k\in[0,1]$ satisfying

$$f_k=\alpha_k\mathsf{S}f_k, \quad \text{and} \quad \|f_k\|_{H^1_{\mathrm{hyp}}(U)}\geqslant k.$$

In particular, f_k is the solution of the problem

$$\begin{cases} -\Delta_v f_k+v\cdot\nabla_v f_k-v\cdot\nabla_x f_k=\alpha_k(f^*-\mathbf{b}\cdot\nabla_v f_k) & \text{in } U,\\ f_k=0 & \text{on } \partial_{\mathrm{hyp}}U. \end{cases} \tag{5.34}$$

Observe that $\|f_k\|_{H^1_{\mathrm{hyp}}(U)}\to\infty$ as $k\to\infty$. Define $\widetilde{f}_k:=\|f_k\|^{-1}_{H^1_{\mathrm{hyp}}(U)}f_k$ and observe that $\widetilde{f}_k$ is a solution of the problem

$$\begin{cases} -\Delta_v\widetilde{f}_k+v\cdot\nabla_v\widetilde{f}_k-v\cdot\nabla_x\widetilde{f}_k=\alpha_k\left(\|f_k\|^{-1}_{H^1_{\mathrm{hyp}}(U)}f^*-\mathbf{b}\cdot\nabla_v\widetilde{f}_k\right) & \text{in } U,\\ f_k=0 & \text{on } \partial_{\mathrm{hyp}}U. \end{cases} \tag{5.35}$$

In view of Proposition 3.5, by extracting a subsequence we may suppose that there exists $\alpha\in[0,1]$ and $\widetilde{f}\in H^1_{\mathrm{hyp}}(U)$ such that $\alpha_k\to\alpha$ and

$$\widetilde{f}_k\to\widetilde{f} \quad \text{in} \quad L^2(U;L^2_\gamma) \quad \text{as} \quad k\to\infty.$$

By (5.35) and the continuity of the solution operator from $L^2(U;H^{-1}_\gamma)$ to $H^1_{\mathrm{hyp}}(U)$, see Proposition 5.1, we see that $\widetilde{f}$ belongs to $H^1_{\mathrm{hyp}}(U)$, is the limit of $\widetilde{f}_k$ in $H^1_{\mathrm{hyp}}(U)$, and is the solution of

$$\begin{cases} -\Delta_v\widetilde{f}+v\cdot\nabla_v\widetilde{f}-v\cdot\nabla_x\widetilde{f}+\alpha\mathbf{b}\cdot\nabla_v\widetilde{f}=0 & \text{in } U,\\ \widetilde{f}=0 & \text{on } \partial_{\mathrm{hyp}}U. \end{cases} \tag{5.36}$$

By Proposition 5.3, we conclude that $\widetilde{f}=0$. However, this is not possible since

$$\|\widetilde{f}\|_{H^1_{\mathrm{hyp}}(U)}=\lim_{k\to\infty}\|\widetilde{f}_k\|_{H^1_{\mathrm{hyp}}(U)}=1.$$

This contradiction completes the proof of (5.32). We may now apply Schaeffer's fixed point theorem to deduce the existence of a solution f of the boundary value problem (1.10). The bound (1.11) follows from (5.32) and the linearity of the problem in the data (f^*, f_0). □

6. Interior regularity of solutions

In this subsection, we use energy methods to obtain interior regularity estimates for solutions of the equation

$$-\Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f = f^*. \tag{6.1}$$

In analogy to the classical theory for uniformly elliptic equations (such as the Laplace or Poisson equations), we obtain an appropriate version of the Caccioppoli inequality, apply it iteratively to obtain H^1_{hyp} estimates on all spatial derivatives of the solution and then apply the Hörmander and Sobolev inequalities to obtain pointwise estimates. In particular, we obtain higher regularity estimates—strong enough to implying that our weak solutions are C^∞—without resorting to the theory of pseudodifferential operators.

We begin with a version of the Caccioppoli inequality for the equation (1.1).

Lemma 6.1 (Caccioppoli inequality)**.** *Suppose that $r>0$, $\mathbf{b}\in L^\infty(B_r\times\mathbb{R}^d;\mathbb{R}^d)$ and the pair $(f,f^*)\in H^1_{\mathrm{hyp}}(B_r)\times L^2(B_r;H^{-1}_\gamma)$ satisfies the equation*

$$-\Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f = f^* \quad \text{in } B_r\times\mathbb{R}^d. \tag{6.2}$$

There exists $C\left(d,r,\|\mathbf{b}\|_{L^\infty(B_r\times\mathbb{R}^d)}\right)<\infty$ such that

$$\|\nabla_v f\|_{L^2(B_{r/2};L^2_\gamma)} + \|v\cdot\nabla_x f\|_{L^2(B_{r/2};H^{-1}_\gamma)} \leqslant C\left\|f-(f)_{B_r}\right\|_{L^2(B_r;L^2_\gamma)} + C\|f^*\|_{L^2(B_r;H^{-1}_\gamma)}. \tag{6.3}$$

Proof. We may assume without loss of generality that $(f)_{B_r}=0$.

Step 1. We show that there exists $C(d)<\infty$ such that

$$\|\nabla_v f\|_{L^2(B_{r/2};L^2_\gamma)} \leqslant C\left(\frac{1}{r}+\|\mathbf{b}\|_{L^\infty(B_r\times\mathbb{R}^d)}\right)\|f\|_{L^2(B_r;L^2_\gamma)} + C(1+r)\|f^*\|_{L^2(B_r;H^{-1}_\gamma)}. \tag{6.4}$$

Select a smooth cutoff function $\phi\in C^\infty_c(B_r)$ which is compactly supported in B_r and satisfies $0\leqslant\phi\leqslant 1$ in B_r, $\phi\equiv 1$ on $B_{r/2}$ and $\|\nabla\phi\|_{L^\infty(B_r)}\leqslant 4r^{-1}$. Testing the equation (6.2) with $(x,v)\mapsto\phi^2(x)f(x,v)$ yields

$$\begin{aligned}\int_{B_r\times\mathbb{R}^d}\phi^2|\nabla_v f|^2\,dx\,d\gamma &= \int_{B_r\times\mathbb{R}^d}\phi^2 f\,f^*\,dx\,d\gamma + \int_{B_r\times\mathbb{R}^d}\phi^2 f v\cdot\nabla_x f\,dx\,d\gamma - \int_{B_r\times\mathbb{R}^d}\phi^2 f\mathbf{b}\cdot\nabla_v f\,dx\,d\gamma.\end{aligned} \tag{6.5}$$

Recall that the first two integrals on the right side above are interpreted as in (1.8). We estimate each of the three terms on the right side of (6.5) separately.

For the first term on the right side of (6.5), we use

$$\begin{aligned}\left|\int_{B_r\times\mathbb{R}^d}\phi^2 f\,f^*\,dx\,d\gamma\right| &\leqslant \left\|\phi^2 f\right\|_{L^2(B_r;H^1_\gamma)}\|f^*\|_{L^2(B_r;H^{-1}_\gamma)}\\ &\leqslant\left(\left\|\phi^2\nabla_v f\right\|_{L^2(B_r;L^2_\gamma)}+\|f\|_{L^2(B_r;L^2_\gamma)}\right)\|f^*\|_{L^2(B_r;H^{-1}_\gamma)}\end{aligned}$$

and then apply Young's inequality to obtain

$$\begin{aligned}&\left|\int_{B_r\times\mathbb{R}^d}\phi^2 f\,f^*\,dx\,d\gamma\right|\\ &\qquad\leqslant\frac{1}{6}\int_{B_r\times\mathbb{R}^d}\phi^2|\nabla_v f|^2\,dx\,d\gamma+\frac{C}{r^2}\int_{B_r\times\mathbb{R}^d}f^2\,dx\,d\gamma+C(1+r^2)\|f^*\|^2_{L^2(B_r;H^{-1}_\gamma)}.\end{aligned} \tag{6.6}$$

For the second term on the right side of (6.5), we integrate by parts to find

$$\begin{aligned}\int_{B_r\times\mathbb{R}^d}\phi^2 f v\cdot\nabla_x f\,dx\,d\gamma &= \int_{B_r\times\mathbb{R}^d}\phi^2 v\cdot\nabla_x\left(\frac{1}{2}f^2\right)dx\,d\gamma\\ &= -\int_{B_r\times\mathbb{R}^d}\phi\nabla_x\phi\cdot v f^2\,dx\,d\gamma\\ &= -\int_{B_r\times\mathbb{R}^d}\phi(x)\nabla_x\phi(x)\cdot v\exp\left(-\frac{|v|^2}{2}\right)f^2(x,v)\,dx\,dv\\ &= \int_{B_r\times\mathbb{R}^d}2f\phi\nabla_x\phi\cdot\nabla_v f\,dx\,d\gamma.\end{aligned}$$

Thus, by Young's inequality,

$$\begin{aligned}\left|\int_{B_r\times\mathbb{R}^d}\phi^2 f v\cdot\nabla_x f\,dx\,d\gamma\right| &\leqslant \frac{1}{6}\int_{B_r\times\mathbb{R}^d}\phi^2\left|\nabla_v f\right|^2\,dx\,d\gamma + C\int_{B_r\times\mathbb{R}^d}f^2\left|\nabla_x\phi\right|^2\,dx\,d\gamma\\ &\leqslant \frac{1}{6}\int_{B_r\times\mathbb{R}^d}\phi^2\left|\nabla_v f\right|^2\,dx\,d\gamma + \frac{C}{r^2}\int_{B_r\times\mathbb{R}^d}f^2\,dx\,d\gamma.\end{aligned}\tag{6.7}$$

For the last term on the right side of (6.5), we use Young's inequality to obtain

$$\begin{aligned}\left|\int_{B_r\times\mathbb{R}^d}\phi^2 f\mathbf{b}\cdot\nabla_v f\right| &\leqslant \frac{1}{6}\int_{B_r\times\mathbb{R}^d}\phi^2\left|\nabla_v f\right|^2\,dx\,d\gamma + C\int_{B_r\times\mathbb{R}^d}\phi^2 f^2\left|\mathbf{b}\right|^2\,dx\,d\gamma\\ &\leqslant \frac{1}{6}\int_{B_r\times\mathbb{R}^d}\phi^2\left|\nabla_v f\right|^2\,dx\,d\gamma + C\left\|\mathbf{b}\right\|^2_{L^\infty(B_r\times\mathbb{R}^d)}\int_{B_r\times\mathbb{R}^d}f^2\,dx\,d\gamma.\end{aligned}\tag{6.8}$$

To conclude, we combine (6.5), (6.6), (6.7) and (6.8) to obtain

$$\begin{aligned}\int_{B_r\times\mathbb{R}^d}\phi^2\left|\nabla_v f\right|^2\,dx\,d\gamma \leqslant \frac{1}{2}\int_{B_r\times\mathbb{R}^d}\phi^2\left|\nabla_v f\right|^2\,dx\,d\gamma + \frac{C}{r^2}\int_{B_r\times\mathbb{R}^d}f^2\,dx\,d\gamma\\ + C(1+r^2)\left\|f^*\right\|^2_{L^2(B_r;H^{-1}_\gamma)} + C\left\|\mathbf{b}\right\|^2_{L^\infty(B_r\times\mathbb{R}^d)}\int_{B_r\times\mathbb{R}^d}f^2\,dx\,d\gamma.\end{aligned}$$

The first term on the right may now be reabsorbed on the left. Using that $\phi = 1$ on $B_{r/2}$, we thus obtain (6.4).

Step 2. We show that there exists $C(d) < \infty$ such that

$$\begin{aligned}&\left\|v\cdot\nabla_x f\right\|_{L^2(B_{r/2};H^{-1}_\gamma)}\\ &\qquad\leqslant C\left(1+\left\|\mathbf{b}\right\|_{L^\infty(B_{r/2}\times\mathbb{R}^d)}\right)\left\|\nabla_v f\right\|_{L^2(B_{r/2};L^2_\gamma)} + C\left\|f^*\right\|_{L^2(B_{r/2};H^{-1}_\gamma)}.\end{aligned}\tag{6.9}$$

This estimate may be combined with (6.4) to obtain the bound for the second term in (6.3), which competes the proof of the lemma.

To obtain (6.9), we test the equation (6.2) with $w \in L^2(B_{r/2};H^1_\gamma)$ to find that

$$\int_{B_r\times\mathbb{R}^d} w\left(v\cdot\nabla_x f\right)dx\,d\gamma = \int_{B_r\times\mathbb{R}^d}\nabla_v f\cdot\left(\nabla_v w + w\mathbf{b}\right) - \int_{B_r\times\mathbb{R}^d} w f^*\,dx\,d\gamma.$$

We deduce that

$$\begin{aligned}&\left|\int_{B_r\times\mathbb{R}^d} w\left(v\cdot\nabla_x f\right)dx\,d\gamma\right|\\ &\quad\leqslant\left\|\nabla_v f\right\|_{L^2(B_{r/2};L^2_\gamma)}\left(\left\|\nabla_v w\right\|_{L^2(B_{r/2};L^2_\gamma)} + \left\|\mathbf{b}\right\|_{L^\infty(B_{r/2}\times\mathbb{R}^d)}\left\|w\right\|_{L^2(B_{r/2};L^2_\gamma)}\right)\\ &\qquad+\left\|w\right\|_{L^2(B_{r/2};H^1_\gamma)}\left\|f^*\right\|_{L^2(B_{r/2};H^{-1}_\gamma)}.\end{aligned}$$

Taking the supremum over $w \in L^2(B_{r/2};H^1_\gamma)$ with $\left\|w\right\|_{L^2(B_{r/2};H^1_\gamma)} \leqslant 1$ yields (6.9).

The combination of (6.4) and (6.9) yields (6.3). □

In the next lemma, under appropriate regularity conditions on the coefficients, we differentiate the equation (1.1) with respect to x_i to obtain an equation for $\partial_{x_i} f$, and then apply the previous lemma to obtain an interior H^1_{hyp} estimate for $\partial_{x_i} f$. The proof also uses the identification of H^α spaces recalled in (3.13)–(3.14), because we need to essentially differentiate the equation a fractional number of times (cf. [27, 28]).

Lemma 6.2. *Fix $r \in (0,\infty)$, $v_0 \in [1,\infty)$ and coefficients $\mathbf{b} \in C^{0,1}(B_r; L^\infty(\mathbb{R}^d;\mathbb{R}^d))$ and $c \in C^{0,1}(B_r; L^\infty(\mathbb{R}^d))$. Suppose that $f^* \in H^1(B_r; H^{-1}_\gamma)$ and $f \in H^1_{\mathrm{hyp}}(B_r)$ satisfy*

$$-\Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f + cf = f^* \quad \text{in } B_r\times\mathbb{R}^d \tag{6.10}$$

as well as

$$f \equiv 0 \quad \text{in } B_r \times \left(\mathbb{R}^d \setminus B_{v_0}\right). \tag{6.11}$$

Then, for each $i \in \{1,\dots,d\}$, the function $h := \partial_{x_i} f$ belongs to $H^1_{\mathrm{hyp}}(B_{r/2})$ and satisfies

$$-\Delta_v h + v\cdot\nabla_v h - v\cdot\nabla_x h + \mathbf{b}\cdot\nabla_v h + ch = \partial_{x_i} f^* - \partial_{x_i}\mathbf{b}\cdot\nabla_v f - \partial_{x_i} c\, f \quad \text{in } B_r\times\mathbb{R}^d. \tag{6.12}$$

Moreover, there exists $C\left(d, r, v_0, \|\mathbf{b}\|_{C^{0,1}(B_r;L^\infty(\mathbb{R}^d))}, \|c\|_{C^{0,1}(B_r;L^\infty(\mathbb{R}^d))}\right) < \infty$ such that

$$\|\partial_{x_i} f\|_{H^1_{\mathrm{hyp}}(B_{r/2})} \leq C\left\|f - (f)_{B_r}\right\|_{L^2(B_r;L^2_\gamma)} + C\|f^*\|_{H^1(B_r;H^{-1}_\gamma)}. \tag{6.13}$$

Proof. Step 1. We first prove that, for every $(f,f^*) \in H^1_{\mathrm{hyp}}(B_r)\times H^1(B_r,H^{-1}_\gamma)$ satisfying (6.10) and (6.11), there exists $C\left(d,r,v_0,\|\mathbf{b}\|_{C^{0,1}(B_r;L^\infty(\mathbb{R}^d))},\|c\|_{C^{0,1}(B_r;L^\infty(\mathbb{R}^d))}\right) < \infty$ such that f belongs to $H^1(B_{r/2};L^2_\gamma)$ and satisfies the estimate

$$\|\nabla_x f\|_{L^2(B_{r/2};L^2_\gamma)} \leqslant C\left\|f - (f)_{B_r}\right\|_{L^2(B_r;L^2_\gamma)} + C\|f^*\|_{H^1(B_r;H^{-1}_\gamma)}. \tag{6.14}$$

The argument is by induction on the fractional exponent of differentiability of f in the spatial variable x. Essentially, we want to differentiate the equation a fractional amount (almost $\frac{1}{6}$ times in fact), apply the Caccioppoli inequality to the fractional derivative, and then iterate this a few times until we have one full spatial derivative.

Let us suppose that $\alpha_0 \in (0,1)$ is such that the following statement is valid: for every $\alpha \in (0,\alpha_0)$, $r>0$, pair $(f,f^*) \in H^1_{\mathrm{hyp}}(B_r)\times H^{\alpha-\frac{1}{8}}(B_r,H^{-1}_\gamma)$ satisfying (6.10) and (6.11), we have $f \in H^\alpha(B_{r/2};L^2_\gamma)$ and, for $C\left(d,r,v_0,\|\mathbf{b}\|_{C^{0,1}(B_r;L^\infty(\mathbb{R}^d))},\|c\|_{C^{0,1}(B_r;L^\infty(\mathbb{R}^d))},\alpha\right) < \infty$, the estimate

$$\|f\|_{H^\alpha(B_{r/2};L^2_\gamma)} \leqslant C\left\|f - (f)_{B_r}\right\|_{L^2(B_r;L^2_\gamma)} + C\|f^*\|_{H^{\alpha-\frac{1}{8}}(B_r;H^{-1}_\gamma)}. \tag{6.15}$$

We will then argue that the statement is also valid for $\alpha_0 + \frac{1}{6}$ in place of α_0. Since this statement is clearly valid for $\alpha_0 < \frac{1}{6}$ by the Caccioppoli inequality (Lemma 6.1) and the Hörmander inequality (Corollary 3.4), this suffices by induction to obtain the statement for any $\alpha < \frac{7}{6}$ and therefore for $\alpha = 1$, yielding the desired bound (6.14).

We therefore assume that $\alpha_0 \in (0,1)$ is such that (6.15) is valid for every $\alpha \in (0,\alpha_0)$, $r>0$ and pair $(f,f^*) \in H^1_{\mathrm{hyp}}(B_r)\times H^{\alpha-\frac{1}{8}}(B_r,H^{-1}_\gamma)$ satisfying (6.10) and (6.11). We fix $\alpha \in (0,\alpha_0)$ and another pair

$$(f,f^*) \in H^1_{\mathrm{hyp}}(B_r)\times H^\alpha(B_r,H^{-1}_\gamma)$$

satisfying (6.10) and (6.11), an index $i \in \{1,\dots,d\}$, a cutoff function $\phi \in C^\infty_c(B_{r/2})$ with $0\leqslant\phi\leqslant 1$ and $\phi \equiv 1$ on $B_{r/4}$. Recall from (3.12) that Φ denotes the standard heat kernel.

In preparation for an application of (3.14), we define the functions

$$\begin{cases} \widetilde{f} := \phi^2 f, \\ \widetilde{f}^* := \phi^2 f^* + 2f\phi\, v\cdot\nabla_x\phi \end{cases}$$

and, for each $t\in(0,1)$, the functions

$$\begin{cases} f_t := \partial_{x_i}\Phi(t,\cdot) * \widetilde{f}, \\ f_t^* := \partial_{x_i}\Phi(t,\cdot) * \widetilde{f}^*, \\ h_t(x,v) := \displaystyle\int_{\mathbb{R}^d} \phi^2(y)\nabla_v f(y,v)\cdot(\mathbf{b}(y,v)-\mathbf{b}(x,v))\,\partial_{x_i}\Phi(t,y-x)\,dy \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^d} \phi^2(y) f(y,v)\cdot(c(y,v)-c(x,v))\,\partial_{x_i}\Phi(t,y-x)\,dy. \end{cases}$$

Observe that $\widetilde{f}\in H^1_{\mathrm{hyp}}(\mathbb{R}^d)$ and $\widetilde{f}^*\in H^\alpha(\mathbb{R}^d;H^{-1}_\gamma)$ are compactly supported in B_r, and we check directly that they satisfy the equations

$$-\Delta_v\widetilde{f} + v\cdot\nabla_v\widetilde{f} - v\cdot\nabla_x\widetilde{f} + \mathbf{b}\cdot\nabla_v\widetilde{f} + c\widetilde{f} = \widetilde{f}^* \quad \text{in } \mathbb{R}^d\times\mathbb{R}^d.$$

Likewise, we have that $f_t\in H^1_{\mathrm{hyp}}(B_r)$ and $h_t, f_t^*\in H^\alpha(B_r;H^{-1}_\gamma)$ and they satisfy

$$-\Delta_v f_t + v\cdot\nabla_v f_t - v\cdot\nabla_x f_t + \mathbf{b}\cdot\nabla_v f_t + cf_t = f_t^* - h_t \quad \text{in } \mathbb{R}^d\times\mathbb{R}^d. \tag{6.16}$$

The condition (6.11) is also satisfied for both $\widetilde{f}$ and f_t. Observe that, by the induction hypothesis that (6.15) is valid for α, we have

$$\|\widetilde{f}\|_{H^\alpha(\mathbb{R}^d;L^2_\gamma)} \leqslant C\|f\|_{H^\alpha(B_{r/2};L^2_\gamma)} \leqslant C\left\|f-(f)_{B_r}\right\|_{L^2(B_r;L^2_\gamma)} + C\|f^*\|_{H^{\alpha-\frac18}(B_r;H^{-1}_\gamma)} \tag{6.17}$$

and

$$\begin{aligned} \|\widetilde{f}^*\|_{H^\alpha(\mathbb{R}^d;L^2_\gamma)} &\leqslant C\|f\|_{H^\alpha(B_{r/2};L^2_\gamma)} + C\|f^*\|_{H^\alpha(B_{r/2};H^{-1}_\gamma)} \\ &\leqslant C\left\|f-(f)_{B_r}\right\|_{L^2(B_r;L^2_\gamma)} + C\|f^*\|_{H^\alpha(B_r;H^{-1}_\gamma)}. \end{aligned} \tag{6.18}$$

By the Caccioppoli inequality (Lemma 6.1),

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\phi^2(y)\nabla_v f(y,v)\cdot(\mathbf{b}(y,v)-\mathbf{b}(x,v))\,\partial_{x_i}\Phi(t,y-x)\,dy\right|^2 dx\,d\gamma(v) \\ &\leqslant C\|\mathbf{b}\|^2_{C^{0,1}(B_r;L^\infty(\mathbb{R}^d))}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\phi^2(y)\,|\nabla_v f(y,v)|\,t^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{Ct}\right)dy\right)^2 dx\,d\gamma(v) \\ &\leqslant C\|\mathbf{b}\|^2_{C^{0,1}(B_r;L^\infty(\mathbb{R}^d))}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi^4(y)\,|\nabla_v f(y,v)|^2\,t^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{Ct}\right)dy\,dx\,d\gamma(v) \\ &\leqslant C\|\mathbf{b}\|^2_{C^{0,1}(B_r;L^\infty(\mathbb{R}^d))}\|\nabla_v f\|^2_{L^2(B_{r/2};L^2_\gamma)} \\ &\leqslant C\left\|f-(f)_{B_r}\right\|^2_{L^2(B_r;L^2_\gamma)} + C\|f^*\|^2_{L^2(B_r;H^{-1}_\gamma)}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\phi^2(y)f(y,v)\cdot(c(y,v)-c(x,v))\,\partial_{x_i}\Phi(t,y-x)\,dy\right|^2 dx\,d\gamma(v) \\ &\qquad\leqslant C\left\|f-(f)_{B_r}\right\|^2_{L^2(B_r;L^2_\gamma)} + C\|f^*\|^2_{L^2(B_r;H^{-1}_\gamma)}. \end{aligned}$$

Combining these, we obtain

$$\|h_t\|^2_{L^2(B_r;L^2_\gamma)} \leqslant C\left\|f-(f)_{B_r}\right\|^2_{L^2(B_r;L^2_\gamma)} + C\|f^*\|^2_{L^2(B_r;H^{-1}_\gamma)}.$$

The Caccioppoli inequality (Lemma 6.1) applied to (6.16) yields

$$\begin{aligned}\|f_t\|_{H^1_{\mathrm{hyp}}(B_{r/2})} &\leqslant C\|f_t\|_{L^2(B_r;L^2_\gamma)} + C\|f_t^* - h_t\|_{L^2(B_r;H^{-1}_\gamma)} \\ &\leqslant C\|f_t\|_{L^2(B_r;L^2_\gamma)} + C\left(\|f_t^*\|_{L^2(B_r;H^{-1}_\gamma)} + \|h_t\|_{L^2(B_r;L^2_\gamma)}\right).\end{aligned}$$

Therefore, by (3.13) and Corollary 3.4 with $\theta = \frac{1}{2}$, for every $\beta \in \left(0, \frac{1}{6}\right)$,

$$\begin{aligned}&\int_0^1 \int_{\mathbb{R}^d} s^{-\beta} \|(\nabla\Phi(s,\cdot) * f_t)(x,\cdot)\|_{L^2_\gamma}^2 \,dx\,ds \\ &\quad\leqslant C\|f_t\|^2_{H^\beta(\mathbb{R}^d;L^2_\gamma)} \\ &\quad\leqslant C\|f_t\|^2_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)} \\ &\quad\leqslant C\|f_t\|^2_{L^2(B_r;L^2_\gamma)} + C\|f_t^*\|^2_{L^2(B_r;H^{-1}_\gamma)} + C\left\|f - (f)_{B_r}\right\|^2_{L^2(B_r;L^2_\gamma)} + C\|f^*\|^2_{L^2(B_r;H^{-1}_\gamma)}.\end{aligned}$$

Multiplying by $t^{-\alpha}$, integrating the resulting inequality over $(0,1)$ with respect t and using (6.17) and (6.18), we obtain

$$\begin{aligned}&\int_0^1 \int_0^1 \int_{\mathbb{R}^d} t^{-\alpha} s^{-\beta} \|(\nabla\Phi(s,\cdot) * f_t)(x,\cdot)\|_{L^2_\gamma}^2 \,dx\,ds\,dt \\ &\quad\leqslant C\int_0^1 t^{-\alpha}\|f_t\|^2_{L^2(\mathbb{R}^d;L^2_\gamma)}\,dt + C\int_0^1 t^{-\alpha}\|f_t^*\|^2_{L^2(\mathbb{R}^d;H^{-1}_\gamma)}\,dt \\ &\qquad + C\left\|f - (f)_{B_r}\right\|^2_{L^2(B_r;L^2_\gamma)} + C\|f^*\|^2_{L^2(B_r;H^{-1}_\gamma)} \\ &\quad\leqslant C\left\|\widetilde{f}\right\|^2_{H^\alpha(\mathbb{R}^d;L^2_\gamma)} + C\left\|\widetilde{f}^*\right\|^2_{H^\alpha(\mathbb{R}^d;H^{-1}_\gamma)} + C\left\|f - (f)_{B_r}\right\|^2_{L^2(B_r;L^2_\gamma)} + C\|f^*\|^2_{L^2(B_r;H^{-1}_\gamma)} \\ &\quad\leqslant C\left\|f - (f)_{B_r}\right\|^2_{L^2(B_r;L^2_\gamma)} + C\|f^*\|^2_{H^\alpha(B_r;H^{-1}_\gamma)}.\end{aligned}$$

Summing over the index i and using that

$$\nabla\Phi(s,\cdot) * f_t = \nabla\partial_{x_i}\Phi(s+t,\cdot) * (\phi^2 f),$$

we obtain, for any $\beta \in \left(0, \frac{1}{6}\right)$,

$$\begin{aligned}&\int_0^1 \int_0^1 \int_{\mathbb{R}^d} t^{-\alpha} s^{-\beta} \left\|\nabla^2\Phi(t+s,\cdot) * (\phi^2 f)\right\|_{L^2_\gamma}^2 \,dx\,ds\,dt \\ &\qquad\qquad\leqslant C\left\|f - (f)_{B_r}\right\|^2_{L^2(B_r;L^2_\gamma)} + C\|f^*\|^2_{H^\alpha(B_r;H^{-1}_\gamma)}.\end{aligned}$$

We bound the left side of the previous display from below by

$$\begin{aligned}&\int_0^1 \int_0^1 \int_{\mathbb{R}^d} t^{-\alpha} s^{-\beta} \left\|\nabla^2\Phi(t+s,\cdot) * (\phi^2 f)\right\|_{L^2_\gamma}^2 \,dx\,ds\,dt \\ &\qquad\qquad\geqslant C^{-1}\int_0^1 \int_{\mathbb{R}^d} t^{1-\beta-\alpha} \left\|\nabla^2\Phi(t,\cdot) * (\phi^2 f)\right\|_{L^2_\gamma}^2 \,dx\,dt.\end{aligned}$$

By (3.14) and the properties of ϕ, we thus obtain, for any $\alpha \in (0,\alpha_0)$ and $\beta \in \left(0, \frac{1}{6}\right)$,

$$\|f\|_{H^{\alpha+\beta}(B_{r/4};L^2_\gamma)} \leqslant C\left\|f - (f)_{B_r}\right\|_{L^2(B_r;L^2_\gamma)} + C\|f^*\|_{H^\alpha(B_r;H^{-1}_\gamma)}.$$

Restricting β to lie in the interval $\left[\frac{1}{8}, \frac{1}{6}\right)$, we obtain

$$\|f\|_{H^{\alpha+\beta}(B_{r/4};L^2_\gamma)} \leqslant C\left\|f - (f)_{B_r}\right\|_{L^2(B_r;L^2_\gamma)} + C\|f^*\|_{H^{\alpha+\beta-\frac{1}{8}}(B_r;H^{-1}_\gamma)}.$$

This is almost the desired statement (6.15) for the exponent $\alpha+\beta$ in place of α. The difference is that we have $B_{r/4}$ on the left side instead of $B_{r/2}$. Of course, this is easy to fix by a covering argument. Since $\alpha+\beta$ can be any exponent in $\left(0, \alpha_0 + \frac{1}{6}\right)$, we

therefore obtain (6.15) for all $\alpha \in \left(0, \alpha_0 + \frac{1}{6}\right)$. By induction, we therefore obtain (6.15) for every $\alpha \in \left(0, \frac{7}{6}\right)$. This completes the proof of (6.14).

Step 2. We complete the proof of the lemma. Fix an index $i \in \{1, \dots, d\}$. By formally differentiating the equation (6.10) with respect to x_i and applying the Caccioppoli inequality (Lemma 6.1) and (6.14), we expect that the function $h := \partial_{x_i} f$ satisfies (6.12) as well as the bound

$$\begin{aligned}
&\|\partial_{x_i} f\|_{H^1_{\mathrm{hyp}}(B_{r/2})} \\
&\quad \leqslant C \Big(\|\partial_{x_i} f\|_{L^2(B_{3r/4}; L^2_\gamma)} + \|\partial_{x_i} f^*\|_{L^2(B_r; H^{-1}_\gamma)} + \|\partial_{x_i} \mathbf{b}\|_{L^\infty(B_r \times \mathbb{R}^d)} \|\nabla_v f\|_{L^2(B_r; L^2_\gamma)} \\
&\qquad + \|\partial_{x_i} c\|_{L^\infty(B_r \times \mathbb{R}^d)} \|f\|_{L^2(B_r; L^2_\gamma)} \Big) \\
&\quad \leqslant C \left\| f - (f)_{B_r} \right\|_{L^2(B_r; L^2_\gamma)} + C \|f^*\|_{H^1(B_r; H^{-1}_\gamma)} .
\end{aligned}$$

This is (6.13). To make this argument rigorous, what is missing is the qualitative statement that $\partial_{x_i} f \in H^1_{\mathrm{hyp}}(B_{3r/4})$. We obtain this by using finite difference quotients, in the standard way. For $\delta > 0$ small, we set

$$\partial^\delta_{x_i} f(x, v) := \frac{1}{\delta} \left(f(x + \delta e_i, v) - f(x, v) \right).$$

We argue that $\partial^\delta_{x_i} f$ satisfies a similar equation as (in fact, a finite difference version of) equation (6.12). We then obtain the same estimate for $\partial^\delta_{x_i} f$ as in (6.13), independently of δ, using the bound (6.14) and the fact that, for every $\delta > 0$ sufficiently small,

$$\left\| \partial^\delta_{x_i} f \right\|_{L^2(B_{5r/8}; L^2_\gamma)} \leqslant C \|\partial_{x_i} f\|_{L^2(B_{3r/4}; L^2_\gamma)} .$$

We can then pass to the limit $\delta \to 0$ to obtain the result for $\partial_{x_i} f$. □

Iterating the previous lemma yields interior higher regularity estimates for solutions of (1.1) in the spatial variable, under appropriate regularity assumptions on $\mathbf{b}$, c and f^*.

Proposition 6.3 (Interior $H^k(L^2_\gamma)$-type regularity)**.** *Suppose that $k \in \mathbb{N}$, $r, v_0 \in [1, \infty)$, $\mathbf{b} \in C^{k,1}(B_r; L^\infty(\mathbb{R}^d; \mathbb{R}^d))$, $c \in C^{k,1}(B_r; L^\infty(\mathbb{R}^d))$ and $f^* \in H^{k+1}(B_r; H^{-1}_\gamma)$. Let $f \in H^1_{\mathrm{hyp}}(B_r)$ be a solution of*

$$-\Delta_v f + v \cdot \nabla_v f - v \cdot \nabla_x f + \mathbf{b} \cdot \nabla_v f + cf = f^* \quad \text{in } B_r \times \mathbb{R}^d. \tag{6.19}$$

Then there exists $C\left(d, k, r, v_0, \|\mathbf{b}\|_{C^{k,1}(B_r; L^\infty(\mathbb{R}^d))}, \|c\|_{C^{k,1}(B_r; L^\infty(\mathbb{R}^d))}\right) < \infty$ such that $f \in H^k\left(B_{r/2}; L^2_\gamma(B_{v_0})\right)$ and

$$\left\| |\nabla^k_x f| \right\|_{L^2(B_{r/2}; L^2_\gamma(B_{v_0}))} \leqslant C \left(\left\| f - (f)_{B_r} \right\|_{L^2(B_r; L^2_\gamma)} + \|f^*\|_{H^k(B_r; H^{-1}_\gamma)} \right).$$

Proof. Step 1. We first prove the proposition under the additional assumption that

$$f \equiv 0 \quad \text{in } B_r \times \left(\mathbb{R}^d \setminus B_{2v_0}\right). \tag{6.20}$$

The case $k = 0$ of the proposition, under the extra hypothesis (6.20), is implied by Lemma 6.2. Arguing by induction, let us assume the validity of the proposition, under the extra hypothesis (6.20), for some $k \in \mathbb{N}$. Fixing $i \in \{1, \dots, d\}$ and applying this assumption in the ball $B_{r/2}$ to the function $\partial_{x_i} f \in H^1_{\mathrm{hyp}}(B_{r/2})$, in combination with the result of

Lemma 6.2, we obtain that $\partial_{x_i} f \in H^k\left(B_{r/4}; L^2_\gamma\right)$ and the estimate

$$\begin{aligned}\left\|\nabla_x^k \partial_{x_i} f\right\|_{L^2\left(B_{r/4};L^2_\gamma\right)} &\leqslant C\left(\left\|\partial_{x_i} f\right\|_{L^2(B_{r/2};L^2_\gamma)} + \left\|\partial_{x_i} f^* - \partial_{x_i}\mathbf{b}\cdot\nabla_v f - \partial_{x_i} c f\right\|_{H^k(B_{r/2};H^{-1}_\gamma)}\right)\\ &\leqslant C\left(\|f\|_{H^1(B_{r/2};L^2_\gamma)} + \|\mathbf{b}\|_{C^{k+1,1}(B_r;L^\infty(\mathbb{R}^d))}\|f\|_{H^k(B_{r/2};L^2_\gamma)}\right.\\ &\qquad \left. + \|f^*\|_{H^{k+1}(B_r;H^{-1}_\gamma)} + \|c\|_{C^{k+1,1}(B_r;L^\infty(\mathbb{R}^d))}\|f\|_{H^k(B_{r/2};L^2_\gamma)}\right),\end{aligned}$$

where the constants C above depend on $\left(d,k,r,v_0,\|\mathbf{b}\|_{C^{k,1}(B_r;L^\infty(\mathbb{R}^d))},\|c\|_{C^{k,1}(B_r;L^\infty(\mathbb{R}^d))}\right)$. Applying the induction hypothesis again, we obtain that the right side of the previous display is bounded above by

$$C\left(\left\|f-(f)_{B_r}\right\|_{L^2(B_r;L^2_\gamma)} + \|f^*\|_{H^{k+1}(B_r;H^{-1}_\gamma)}\right),$$

where C depends now on $\left(d,k,r,v_0,\|\mathbf{b}\|_{C^{k+1,1}(B_r;L^\infty(\mathbb{R}^d))},\|c\|_{C^{k+1,1}(B_r;L^\infty(\mathbb{R}^d))}\right)$. We can replace $r/4$ in the statements and estimates above by $r/2$ after performing a routine covering argument. This yields the statement of the proposition, under the extra hypothesis (6.20), for $k+1$. The statement of the proposition under the extra hypothesis (6.20) now follows by induction.

Step 2. We remove the additional hypothesis (6.20). If $f \in H^1_{\mathrm{hyp}}(U)$ is a solution of (6.19) and $\varphi \in C^\infty_c(\mathbb{R}^d)$ is a function only of v, then

$$\widetilde{f}(x,v) := \varphi(v) f(x,v)$$

satisfies

$$-\Delta_v \widetilde{f} + v\cdot\nabla_v\widetilde{f} - v\cdot\nabla_x\widetilde{f} + \widetilde{\mathbf{b}}\cdot\nabla_v\widetilde{f} + \widetilde{c}\widetilde{f} = \widetilde{f}^* \quad \text{in } B_r\times\mathbb{R}^d,$$

where

$$\begin{cases}\widetilde{f}^* := \varphi f^*,\\ \widetilde{\mathbf{b}} := \mathbf{b} + 2\nabla_v\varphi,\\ \widetilde{c} := c + \left(-\Delta_v\varphi + v\cdot\nabla_v\varphi + \mathbf{b}\cdot\nabla_v\varphi\right).\end{cases}$$

We select φ to satisfy $0\leqslant\varphi\leqslant 1$, $\varphi\equiv 1$ in B_{v_0}, $|\nabla\varphi| + \left|\nabla^2\varphi\right| \leqslant C$, and $\varphi\equiv 0$ in $\mathbb{R}^d\setminus B_{2v_0}$. It is easy to check that

$$\begin{aligned}\|\widetilde{\mathbf{b}}\|_{C^{k,1}(B_r;L^\infty(\mathbb{R}^d))} &\leqslant \|\mathbf{b}\|_{C^{k,1}(B_r;L^\infty(\mathbb{R}^d))} + C,\\ \|\widetilde{c}\|_{C^{k,1}(B_r;L^\infty(\mathbb{R}^d))} &\leqslant \|c\|_{C^{k,1}(B_r;L^\infty(\mathbb{R}^d))} + C\left(1 + \|\mathbf{b}\|_{C^{k,1}(B_r;L^\infty(\mathbb{R}^d))}\right).\end{aligned}$$

We may then apply the result of Step 1 above to $\widetilde{f}$ to obtain the proposition. □

We turn to the proof of Theorem 1.4, concerning the interior regularity, jointly in the variables x and v. We actually prove the slightly more general statement, allowing for a zeroth-order term in the equation.

Proposition 6.4 (Interior $H^k(H^l_\gamma)$-type regularity)**.** *Let $k,l\in\mathbb{N}$, $r,v_0\in[1,\infty)$, $\mathbf{b}\in C^{k+l-1,1}(B_r;C^{l-1,1}(\mathbb{R}^d;\mathbb{R}^d))$ and $c\in C^{k+l-1,1}(B_r;C^{l-1,1}(\mathbb{R}^d))$. There exists a constant $C<\infty$ depending on*

$$\left(d,k,l,r,v_0,\left\{\|\mathbf{b}\|_{C^{k+j-1,1}(B_r;C^{l-j-1,1}(\mathbb{R}^d))}\right\}_{j\in\{0,\dots,l\}},\left\{\|c\|_{C^{k+j-1,1}(B_r;C^{l-j-1,1}(\mathbb{R}^d))}\right\}_{j\in\{0,\dots,l\}}\right)$$

such that, for every $f\in H^1_{\mathrm{hyp}}(B_r)$ and $f^*\in H^{k+l}(B_r;H^{l-1}_\gamma)$ satisfying

$$-\Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f + cf = f^* \quad \text{in } B_r\times\mathbb{R}^d, \tag{6.21}$$

we have that $f \in H^k\left(B_{2^{-k}r}; H^l_\gamma(B_{v_0})\right)$ *and the estimate*

$$\left\| \left| \nabla_v^l \nabla_x^k f \right| \right\|_{L^2\left(B_{r/2}; L^2_\gamma(B_{v_0})\right)} \leqslant C \left(\left\| f - (f)_{B_r} \right\|_{L^2(B_r; L^2_\gamma)} + \sum_{j=0}^{l} \left\| \widetilde{f}^* \right\|_{H^{k+j}(B_r; H^{l-j-1}_\gamma)} \right).$$

Proof. Observe that the case $l = 0$ of the proposition is a consequence of Proposition 6.3. By the same argument as in the proof of Proposition 6.3, it suffices to prove the proposition under the additional restriction that f satisfies (6.20). Arguing by induction, we assume there exists $l \in \mathbb{N}$ such that the proposition, subject to the additional hypothesis (6.20), is valid (for every $k \in \mathbb{N}$). We select $i \in \{1, \dots, d\}$ and apply this assumption to the function $h := \partial_{v_i} f$, which we observe satisfies the equation

$$-\Delta h + v \cdot \nabla_v h - v \cdot \nabla_x g + \mathbf{b} \cdot \nabla_v h + (c+1)h = \widetilde{f}^* \quad \text{in } B_r \times \mathbb{R}^d,$$

where

$$\widetilde{f}^* := \partial_{v_i} f^* + \partial_{x_i} f - (\partial_{v_i} \mathbf{b}) \cdot \nabla_v f + (\partial_{v_i} c) f.$$

As a result we obtain

$$\begin{aligned} \left\| \left| \partial_{v_i} \nabla_v^l \nabla_x^k f \right| \right\|_{L^2(B_{r/4}; L^2_\gamma)} &= \left\| \left| \nabla_v^l \nabla_x^k h \right| \right\|_{L^2(B_{r/4}; L^2_\gamma)} \\ &\leqslant C \left(\left\| h - (h)_{B_{r/2}} \right\|_{L^2(B_{r/2}; L^2_\gamma)} + \sum_{j=0}^{l} \left\| \widetilde{f}^* \right\|_{H^{k+j}(B_{r/2}; H^{l-j-1}_\gamma)} \right). \end{aligned}$$

By the Caccioppoli inequality (Lemma 6.1) we have

$$\left\| h - (h)_{B_{r/2}} \right\|_{L^2(B_{r/2}; L^2_\gamma)} \leqslant \left\| \nabla_v f \right\|_{L^2(B_r; L^2_\gamma)} \leqslant C \left(\left\| f - (f)_{B_r} \right\|_{L^2(B_r; L^2_\gamma)} + \left\| f^* \right\|_{L^2(B_r; H^{-1}_\gamma)} \right).$$

By a direct computation, for each $j \in \{0, \dots, l\}$,

$$\begin{aligned} &\left\| \widetilde{f}^* \right\|_{H^{k+j}(B_{r/2}; H^{l-j-1}_\gamma)} \\ &= \left\| \partial_{v_i} f^* + \partial_{x_i} f - (\partial_{v_i} \mathbf{b}) \cdot \nabla_v f + (\partial_{v_i} c) f \right\|_{H^{k+j}(B_{r/2}; H^{l-j-1}_\gamma)} \\ &\leqslant C \left\| f^* \right\|_{H^{k+j}(B_r; H^{l-j}_\gamma)} + C \left\| f \right\|_{H^{k+j+1}(B_{r/2}; H^{l-j-1}_\gamma)} \\ &\quad + C \left\| \mathbf{b} \right\|_{C^{k+j-1,1}(B_r; C^{l-j-1,1})} \left\| f \right\|_{H^{k+j}(B_{r/2}; H^{l-j}_\gamma)} \\ &\quad + C \left\| c \right\|_{C^{k+j-1,1}(B_r; C^{l-j-1,1})} \left\| f \right\|_{H^{k+j}(B_{r/2}; H^{l-j-1}_\gamma)}. \end{aligned}$$

Using again the induction hypothesis and absorbing the factors depending on $\mathbf{b}$ and c, we deduce that the right side of the previous display is bounded by

$$C \left(\left\| f - (f)_{B_r} \right\|_{L^2(B_r; L^2_\gamma)} + \sum_{j'=0}^{l+1} \left\| f^* \right\|_{H^{k+j'}(B_r; H^{l-j'}_\gamma)} \right),$$

where the constant C depends on

$$\left(d, k, l, r, v_0, \left\{ \left\| \mathbf{b} \right\|_{C^{k+j-1,1}(B_r; C^{l-j-1,1}(\mathbb{R}^d))} \right\}_{j \in \{0,\dots,l\}}, \left\{ \left\| c \right\|_{C^{k+j-1,1}(B_r; C^{l-j-1,1}(\mathbb{R}^d))} \right\}_{j \in \{0,\dots,l\}} \right).$$

Combining the above inequalities and summing over $j \in \{0, \dots, l\}$ and $i \in \{1, \dots, d\}$ yields

$$\left\| \left| \nabla_v^{l+1} \nabla_x^k f \right| \right\|_{L^2(B_{r/4}; L^2_\gamma)} \leqslant C \left(\left\| f - (f)_{B_r} \right\|_{L^2(B_r; L^2_\gamma)} + \sum_{j=0}^{l+1} \left\| f^* \right\|_{H^{k+j}(B_r; H^{l-j}_\gamma)} \right).$$

By a routine covering argument, we can replace $r/4$ in the norm on the left side by $r/2$. This is the statement of the proposition for $l+1$, under the additional hypothesis (6.20). The proposition now follows by induction. □

7. The kinetic Fokker-Planck equation

In this last section, we study the time-dependent kinetic Fokker-Planck equation

$$\partial_t f - \Delta_v f + v \cdot \nabla_v f - v \cdot \nabla_x f + \mathbf{b} \cdot \nabla_v f = f^*. \tag{7.1}$$

We aim to solve this problem on a bounded domain $V \subseteq \mathbb{R} \times \mathbb{R}^d$ which is either $C^{1,1}$ or of the form $I \times U$ with $I \subseteq \mathbb{R}$ and $U \subseteq \mathbb{R}^d$ a $C^{1,1}$ domain.

7.1. Function spaces.

We define the function space

$$H^1_{\mathrm{kin}}(V) := \left\{ f \in L^2(V; H^1_\gamma) \ : \ \partial_t f - v \cdot \nabla_x f \in L^2(V; H^{-1}_\gamma) \right\}, \tag{7.2}$$

equipped with the norm

$$\|f\|_{H^1_{\mathrm{kin}}(V)} := \|f\|_{L^2(V;H^1_\gamma)} + \|\partial_t f - v \cdot \nabla_x f\|_{L^2(V;H^{-1}_\gamma)}. \tag{7.3}$$

We denote the unit exterior normal to V by $\mathbf{n}_V \in L^\infty(\partial V; \mathbb{R}^{d+1})$. If V is a $C^{1,1}$ domain, then $\mathbf{n}_V(t,x)$ is well-defined for every $(t,x) \in \partial V$; if V is of the form $I \times U$, then $\mathbf{n}_V(t,x)$ is well-defined unless $(t,x) \in \partial I \times \partial U$, in which case we take the convention that $\mathbf{n}_V(t,x) = 0$. We define the hypoelliptic boundary of $V \subseteq \mathbb{R} \times \mathbb{R}^d$ as

$$\partial_{\mathrm{kin}}(V) := \left\{ ((t,x),v) \in \partial V \times \mathbb{R}^d \ : \ \begin{pmatrix} 1 \\ -v \end{pmatrix} \cdot \mathbf{n}_V(t,x) < 0 \right\},$$

We denote by $H^1_{\mathrm{kin},0}(V)$ the closure in $H^1_{\mathrm{kin}}(V)$ of the set of smooth functions which vanish on $\partial_{\mathrm{kin}} V$.

Proposition 7.1 (Density of smooth functions)**.** *Let $V \subseteq \mathbb{R} \times \mathbb{R}^d$ be a Lipschitz domain. The set $C^\infty_c(\overline{V} \times \mathbb{R}^d)$ of smooth functions with compact support in $\overline{V} \times \mathbb{R}^d$ is dense in $H^1_{\mathrm{kin}}(V)$.*

Proof. Mimicking the first step of the proof of Proposition 2.2, which only uses that the domain is Lipschitz, we see that we can assume without loss of generality that for every $z \in V$ and $\varepsilon \in (0,1]$, we have

$$B((1-\varepsilon)z, \varepsilon) \subseteq V.$$

Here we use z to denote a generic variable in $\mathbb{R} \times \mathbb{R}^d$; in standard notation, $z = (t,x)$. Let ζ_ε be a $(1+d)$-dimensional version of the mollifier defined in (2.9), and let $f \in H^1_{\mathrm{kin}}(V)$. We define, for every $\varepsilon \in \left(0, \frac{1}{2}\right]$, $z \in V$ and $v \in \mathbb{R}^d$,

$$f_\varepsilon(z,v) := \int_{\mathbb{R}^{1+d}} f((1-\varepsilon)z + z', v) \zeta_\varepsilon(z')\, dz'.$$

We then show as in Step 2 of the proof of Proposition 2.2 that f belongs to the closed convex hull of the set $\left\{ f_\varepsilon \ : \ \varepsilon \in \left(0, \frac{1}{2}\right] \right\}$, and then, as in Step 3 of this proof, that for each $\varepsilon > 0$, we have that f_ε belongs to the closure of the set $C^\infty_c(\overline{V} \times \mathbb{R}^d)$. □

7.2. Functional inequalities for H^1_{kin}.

We next show a Poincaré inequality for $H^1_{\mathrm{kin}}(V)$ in the case when $V \subseteq \mathbb{R} \times \mathbb{R}^d$ is a $C^{1,1}$ domain. We do not know how to prove the result for cylindrical domains, but as will be clear below, this limitation does not cause much trouble. Since this will be useful later in Subsection 7.6, we allow for more flexible boundary conditions than in Theorem 1.2, in the spirit of Remark 3.2.

Proposition 7.2 (Poincaré inequality)**.** *Let $V \subseteq \mathbb{R} \times \mathbb{R}^d$ be a $C^{1,1}$ domain.*

(1) There exists a constant $C(V,d) < \infty$ such that for every $f \in H^1_{\mathrm{kin}}(V)$, we have

$$\|f - (f)_V\|_{L^2(V;L^2_\gamma)} \leq C \left(\|\nabla_v f\|_{L^2(V;L^2_\gamma)} + \|v \cdot \nabla_x f - \partial_t f\|_{L^2(V;H^{-1}_\gamma)} \right).$$

(2) Let W be a relatively open subset of $\partial V \times \mathbb{R}^d$. There exists a constant $C(V,W,d) < \infty$ such that for every $f \in C_c^\infty(\overline{V} \times \mathbb{R}^d)$ that vanishes on W, we have

$$\|f\|_{L^2(V;L^2_\gamma)} \leqslant C\left(\|\nabla_v f\|_{L^2(V;L^2_\gamma)} + \|v\cdot\nabla_x f - \partial_t f\|_{L^2(V;H^{-1}_\gamma)}\right).$$

Proof of Proposition 7.2. The proof is similar to that of Theorem 1.2. By Proposition 7.1, we can assume that $f \in C_c^\infty(\overline{W} \times \mathbb{R}^d)$. We start by using the Gaussian Poincaré inequality to assert that

$$\|f - \langle f\rangle_\gamma\|_{L^2(V;L^2_\gamma)} \leqslant \|\nabla_v f\|_{L^2(V;L^2_\gamma)}.$$

Paralleling the second step of the proof of Theorem 1.2, we then aim to gain control on a negative Sobolev norm of the derivatives of $\langle f\rangle_\gamma$. Here we treat the time and space variables on an equal footing, and thus are interested in controlling $\partial_t\langle f\rangle_\gamma$ and $\nabla\langle f\rangle_\gamma$ in the $H^{-1}(V)$ norm. The precise claim is that there exists $C(d,V) < \infty$ such that for every test function $\phi \in C_c^\infty(V)$ satisfying

$$\|\phi\|_{L^2(V)} + \|\nabla\phi\|_{L^2(V)} + \|\partial_t\phi\|_{L^2(V)} \leqslant 1, \tag{7.4}$$

we have

$$\left|\int_V \phi\,\partial_t\langle f\rangle_\gamma\right| + \sum_{i=1}^d \left|\int_V \phi\,\partial_{x_i}\langle f\rangle_\gamma\right| \leqslant C\left(\|\nabla_v f\|_{L^2(V;L^2_\gamma)} + \|v\cdot\nabla_x f - \partial_t f\|_{L^2(V;H^{-1}_\gamma)}\right). \tag{7.5}$$

We start by showing that the first term on the left side of (7.5), which refers to the time derivative of $\langle f\rangle_\gamma$, is estimated by the right side of (7.5). We select a smooth function $\xi_0 \in C_c^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \xi_0(v)\,d\gamma(v) = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} v\xi_0(v)\,d\gamma(v) = 0, \tag{7.6}$$

and observe that, using these properties of ξ_0, we can write

$$\begin{aligned}
&\int_V \partial_t\phi(t,x)\,\langle f\rangle_\gamma(t,x)\,dt\,dx \\
&\quad= \int_{V\times\mathbb{R}^d} \xi_0(v)\,(\partial_t\phi(t,x) - v\cdot\nabla_x\phi(t,x))\,\langle f\rangle_\gamma(t,x)\,dt\,dx\,d\gamma(v) \\
&\quad= \int_{V\times\mathbb{R}^d} \xi_0(v)\,(\partial_t - v\cdot\nabla_x)\,\phi(t,x)\,f(t,x,v)\,dt\,dx\,d\gamma(v) \\
&\qquad+ \int_{V\times\mathbb{R}^d} \xi_0(v)\,(\partial_t - v\cdot\nabla_x)\,\phi(t,x)\,(\langle f\rangle_\gamma(t,x) - f(t,x,v))\,dt\,dx\,d\gamma(v).
\end{aligned}$$

Using (7.4) and the fact that ξ_0 has compact support, we can bound the second integral above by

$$C\|f - \langle f\rangle_\gamma\|_{L^2(V;L^2_\gamma)} \leqslant C\|\nabla_v f\|_{L^2(V;L^2_\gamma)}.$$

By integration by parts, the absolute value of the first integral is equal to

$$\left|\int_{V\times\mathbb{R}^d} \xi_0(v)\phi(t,x)\,(v\cdot\nabla_x - \partial_t)\,f(t,x,v)\,dt\,dx\,d\gamma(v)\right| \leqslant C\|v\cdot\nabla_x f - \partial_t f\|_{L^2(V;H^{-1}_\gamma)}.$$

This completes the proof of the estimate in (7.5) involving the time derivative. To estimate the terms involving the space derivatives, we fix $i \in \{1,\dots,d\}$ and use a smooth function $\xi_i \in C_c^\infty(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} \xi_i(v)\,d\gamma(v) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} v\xi_i(v)\,d\gamma(v) = e_i$$

to get that

$$\int_V \partial_{x_i}\phi(t,x)\,\langle f\rangle_\gamma(t,x)\,dt\,dx \\ = \int_{V\times\mathbb{R}^d} \xi_i(v)\,(v\cdot\nabla_x\phi(t,x)-\partial_t\phi(t,x))\,\langle f\rangle_\gamma(t,x)\,dt\,dx\,d\gamma(v).$$

The rest of the argument is then identical to the estimate involving the time derivative, and thus (7.5) is proved. The remainder of the proof is then identical to that for Theorem 1.2. Note that we need to invoke Lemma 3.1 for the domain V, and this is where the assumption that V is a $C^{1,1}$ domain is used. □

7.3. **The Hörmander inequality for H^1_{kin}.** The Hörmander-type inequality for H^1_{kin} is proved in an almost identical way to the one for H^1_{hyp}, with the time variable playing very little role since unlike the x variable it requires no extra smoothing. For the convenience of the reader, we give a complete proof anyway. The version of (3.20) we use here is

$$\begin{aligned} f(t,x+s^2y,v)-f(t,x,v) &= f(t,x+s^2y,v)-f(t,x+s^2y,v-sy) \\ &\quad + f(t,x+s^2y,v-sy)-f(t-s,x+sv,v-sy) \\ &\quad + f(t-s,x+sv,v-sy)-f(t-s,x+sv,v) \\ &\quad + f(t-s,x+sv,v)-f(t,x,v). \end{aligned} \tag{7.7}$$

This choice is a reflection of the fact that

$$[\nabla_v, v\cdot\nabla_x-\partial_t] = [\nabla_v, v\cdot\nabla_x] = \nabla_x.$$

Proposition 7.3 (Hörmander inequality for H^1_{kin})**.** *Let $\alpha\in\left[0,\frac{1}{3}\right)$ and $v_0\in[1,\infty)$. There exists a constant $C(\alpha,v_0,d)<\infty$ such that, for every $f\in H^1_{\mathrm{kin}}(\mathbb{R}\times\mathbb{R}^d)$ satisfying*

$$f=0 \quad \text{in } \mathbb{R}\times\mathbb{R}^d\times\left(\mathbb{R}^d\setminus B_{v_0}\right), \tag{7.8}$$

we have the estimate

$$\|f\|_{L^2\left(\mathbb{R};H^\alpha\left(\mathbb{R}^d;H^{-1}_\gamma\right)\right)} \leqslant C\,\|f\|_{H^1_{\mathrm{kin}}(\mathbb{R}^d)}. \tag{7.9}$$

Proof. As in (3.12), we let $\Phi(s,x)$ denotes the standard heat kernel in dimension d and we apply (3.13) (in the space variable only) to obtain

$$\|f\|^2_{L^2\left(\mathbb{R};H^\alpha\left(\mathbb{R}^d;H^{-1}_\gamma\right)\right)} \leqslant \int_0^1 s^{-\alpha}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left\|\left(\nabla\Phi(s,\cdot)*_x f\right)(t,x,\cdot)\right\|^2_{H^{-1}_\gamma}\,dx\,dt\,ds. \tag{7.10}$$

Here the symbol $*_x$ denotes convolution in the spatial variable x; in other words, the function inside the norm on the right side of the previous display is written explicitly as

$$\left(\nabla\Phi(s,\cdot)*_x f\right)(t,x,v) = \int_{\mathbb{R}^d}\nabla\Phi(s,y-x)f(t,y,v)\,dy.$$

We claim that, for every $s\in(0,1)$,

$$\|\nabla\Phi(s,\cdot)*_x f\|_{L^2(\mathbb{R}\times\mathbb{R}^d;H^{-1}_\gamma)} \leqslant Cs^{-\frac{1}{3}}\,\|f\|_{H^1_{\mathrm{kin}}(\mathbb{R}\times\mathbb{R}^d)}. \tag{7.11}$$

In view of (7.10), the bound (7.11) implies the proposition.

To prove (7.11) we show that, for every $s\in(0,1)$ and test function $g\in L^2(\mathbb{R}\times\mathbb{R}^d;H^1_\gamma)$ satisfying $g(t,x,v)\equiv 0$ for $|v|\geqslant v_0+1$, we have

$$\left|\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}\left(\nabla\Phi(s,\cdot)*_x f\right)(t,x,v)g(t,x,v)\,dt\,dx\,d\gamma(v)\right| \\ \leqslant Cs^{-\frac{1}{3}}\,\|f\|_{H^1_{\mathrm{kin}}(\mathbb{R}\times\mathbb{R}^d)}\,\|g\|_{L^2(\mathbb{R}\times\mathbb{R}^d;H^1_\gamma)}. \tag{7.12}$$

The proof of (7.12) is similar to the one of (3.26), since the time variable plays only a passive role in the argument. Using the semigroup property, we have

$$\begin{aligned}(7.13)\qquad &\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d} \left(\nabla\Phi(s,\cdot) *_x f\right)(t,x,v) g(t,x,v)\, dt\, dx\, d\gamma(v)\\ &\quad= \int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d} \left(\nabla\Phi\left(\tfrac{s}{2},\cdot\right) *_x f\right)(t,x,v) \left(\Phi\left(\tfrac{s}{2},\cdot\right) *_x g\right)(t,x,v)\, dt\, dx\, d\gamma(v)\\ &\quad= \int_{\mathbb{R}^d} \nabla\Phi(\tfrac{s}{2},y) \int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d} \left(f(t,x+y,v)-f(t,x,v)\right)\psi_s(t,x,v)\, dt\, dx\, d\gamma(v)\, dy\end{aligned}$$

where we denote $\psi_s := \Phi\left(\frac{s}{2},\cdot\right) *_x g$, which is an H^1_γ-valued function on $\mathbb{R}\times\mathbb{R}^d$ which is smooth in x. Similar to (3.29), we split up the inner integral on the right side of (7.13) as follows:

$$\begin{aligned}(7.14)\qquad &\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d} \psi_s(t,x,v)\left(f(t,x+y,v)-f(t,x,v)\right)\, dt\, dx\, d\gamma(v)\\ &= \int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d} \psi_s(t,x,v)\left(f(t,x+y,v)-f\left(t,x+y,v-s^{-\frac13}y\right)\right)\, dt\, dx\, d\gamma(v)\\ &\quad+\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d} \psi_s(t,x,v)\left(f\left(t,x+y,v-s^{-\frac13}y\right)-f\left(t-s^{\frac13},x+s^{\frac13}v,v-s^{-\frac13}y\right)\right)\, dt\, dx\, d\gamma(v)\\ &\quad+\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d} \psi_s(t,x,v)\left(f\left(t-s^{\frac13},x+s^{\frac13}v,v-s^{-\frac13}y\right)-f\left(t-s^{\frac13},x+s^{\frac13}v,v\right)\right)\, dt\, dx\, d\gamma(v)\\ &\quad+\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d} \psi_s(t,x,v)\left(f\left(t-s^{\frac13},x+s^{\frac13}v,v\right)-f(t,x,v)\right)\, dt\, dx\, d\gamma(v).\end{aligned}$$

The fourth integral makes zero contribution to the right side of (7.13) since it is independent of the variable y. We estimate the first and third integrals together, and the second integral separately, in the following two steps.

Step 1. The estimate of the first and third terms on the right side of (7.14). The claim is that, for a constant $C(v_0,d)<\infty$,

$$\begin{aligned}(7.15)\qquad &\int_{\mathbb{R}^d}\left|\nabla\Phi(\tfrac{s}{2},y)\right|\left|\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}\psi_s(t,x,v)\left(f(t,x+y,v)-f(t,x+y,v-s^{-\frac13}y)\right)\, dt dx d\gamma(v)\right| dy\\ &\qquad\leqslant Cs^{-\frac13}\left\|\psi_s\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;L^2_\gamma)}\left\|\nabla_v f\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;L^2_\gamma)}.\end{aligned}$$

We use the identity

$$\begin{aligned}f(t,x+y,v)-f\left(t,x+y,v-s^{-\frac13}y\right) &= -\int_0^1 \partial_u\left[f\left(t,x+y,v-us^{-\frac13}y\right)\right] du\\ &= s^{-\frac13}\int_0^1 y\cdot\nabla_v f\left(t,x+y,v-us^{-\frac13}y\right) du.\end{aligned}$$

Inserting this into the left side of (7.15), changing variables and using (3.17) (see also (3.31)), we have

$$\begin{aligned}&\left|\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}\psi_s(t,x,v)\left(f(t,x+y,v)-f\left(t,x+y,v-s^{-\frac13}y\right)\right)\, dt\, dx\, d\gamma(v)\right|\\ &\quad= s^{-\frac13}\left|\int_0^1\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}\psi_s(t,x,v)y\cdot\nabla_v f\left(t,x+y,v-us^{-\frac13}y\right)\, dt\, dx\, d\gamma(v)\, du\right|\\ &\quad\leqslant Cs^{-\frac13}|y|\int_0^1\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}|\psi_s(t,x,v)|\left|\nabla_v f\left(t,x+y,v-us^{-\frac13}y\right)\right| dt\, dx\, d\gamma(v)\, du\\ &\quad\leqslant Cs^{-\frac13}|y|\left\|\psi_s\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;L^2_\gamma)}\left\|\nabla_v f\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;L^2_\gamma)}.\end{aligned}$$

Multiplying by $\nabla\Phi(\frac{s}{2},y)$, integrating over y and using that $\int_{\mathbb{R}^d}|y|\left|\nabla\Phi(\frac{s}{2},y)\right|\,dy\leq C$, we obtain the desired inequality (7.15).

Step 2. The estimate of the second term on the right side of (7.14). The claim is that, for a constant $C(v_0,d)<\infty$,

(7.16)
$$\left|\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}\psi_s(t,x,v)\left(f\left(t,x+y,v-s^{-\frac{1}{3}}y\right)-f\left(t-s^{\frac{1}{3}},x+s^{\frac{1}{3}}v,v-s^{-\frac{1}{3}}y\right)\right)dt\,dx\,d\gamma(v)\right|$$
$$\leq C\left\|\partial_t-v\cdot\nabla_x f\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;H^{-1}_\gamma)}\left(s^{-\frac{1}{6}}\left\|\psi_s\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;H^1_\gamma)}+s^{\frac{1}{6}}\left\|\nabla_x\psi_s\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;L^2_\gamma)}\right).$$

Here we use the expression

$$\begin{aligned}&f\left(t,x+y,v-s^{-\frac{1}{3}}y\right)-f\left(t-s^{\frac{1}{3}},x+s^{\frac{1}{3}}v,v-s^{-\frac{1}{3}}y\right)\\&\quad=-\int_0^1\partial_u\left[f\left(t-us^{\frac{1}{3}},x+y+us^{\frac{1}{3}}\left(v-s^{-\frac{1}{3}}y\right),v-s^{-\frac{1}{3}}y\right)\right]\\&\quad=s^{\frac{1}{3}}\int_0^1\left(\partial_t-v\cdot\nabla_x f\right)\left(t-us^{\frac{1}{3}},x+us^{\frac{1}{3}}v,v-s^{-\frac{1}{3}}y\right)du.\end{aligned}$$

Inserting this identity, changing variables and using (3.17), we have

$$\begin{aligned}&\left|\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}\psi_s(t,x,v)\left(f\left(t,x+y,v-s^{-\frac{1}{3}}y\right)-f\left(t-s^{\frac{1}{3}},x+s^{\frac{1}{3}}v,v-s^{-\frac{1}{3}}y\right)\right)dt\,dx\,d\gamma(v)\right|\\&=s^{\frac{1}{3}}\left|\int_0^1\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}\psi_s(t,x,v)\left(\partial_t-v\cdot\nabla_x f\right)\left(t-us^{\frac{1}{3}},x+us^{\frac{1}{3}}v,v-s^{-\frac{1}{3}}y\right)dt\,dx\,d\gamma(v)\,du\right|\\&\leq Cs^{\frac{1}{3}}\int_0^1\left|\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}\psi_s(t+us^{\frac{1}{3}},x-us^{\frac{1}{3}}v,v+s^{-\frac{1}{3}}y)\left(\partial_t-v\cdot\nabla_x f\right)(t,x,v)\,dtdxd\gamma(v)\right|du\\&\leq Cs^{\frac{1}{3}}\left\|\partial_t-v\cdot\nabla_x f\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;H^{-1}_\gamma)}\sup_{u\in[0,1]}\left\|\widetilde{\psi}_{s,y,u}\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;H^1_\gamma)},\end{aligned}$$

where we set

$$\widetilde{\psi}_{s,y,u}(t,x,v):=\psi_s\left(t+us^{\frac{1}{3}},x-us^{\frac{1}{3}}v,v+s^{-\frac{1}{3}}y\right).$$

Using again (3.17), we find that

$$\left\|\widetilde{\psi}_{s,y,u}\right\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d;H^1_\gamma)}\leq C\left\|\psi_s\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;H^1_\gamma)}+Cs^{\frac{1}{3}}\left\|\nabla_x\psi_s\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;L^2_\gamma)}.$$

Combining the previous two displays, we obtain

(7.17)
$$\left|\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}\psi_s(t,x,v)\left(f\left(t,x+y,v-s^{-\frac{1}{3}}y\right)-f\left(t-s^{\frac{1}{3}},x+s^{\frac{1}{3}}v,v-s^{-\frac{1}{3}}y\right)\right)dtdxd\gamma(v)\right|$$
$$\leq C\left\|\partial_t-v\cdot\nabla_x f\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;H^{-1}_\gamma)}\left(s^{\frac{1}{3}}\left\|\psi_s\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;H^1_\gamma)}+s^{\frac{2}{3}}\left\|\nabla_x\psi_s\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;L^2_\gamma)}\right).$$

Multiplying by $\nabla\Phi(\frac{s}{2},y)$, integrating over y and using that $\int_{\mathbb{R}^d}\left|\nabla\Phi(\frac{s}{2},y)\right|\,dy\leq Cs^{-\frac{1}{2}}$, we obtain (7.16).

Step 3. The conclusion. Combining (7.13), (7.14), (7.15) and (7.16), we obtain

$$\left|\int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d}\left(\nabla\Phi(s,\cdot)\ast_x f\right)(t,x,v)g(t,x,v)\,dt\,dx\,d\gamma(v)\right|$$
$$\leq C\left\|f\right\|_{H^1_{\mathrm{kin}}(\mathbb{R}^d)}\left(s^{-\frac{1}{3}}\left\|\psi_s\right\|_{L^2(\mathbb{R}^d;H^1_\gamma)}+s^{\frac{1}{6}}\left\|\psi_s\right\|_{L^2(\mathbb{R}^d;L^2_\gamma)}\right).$$

We now obtain (7.11) after combining the previous inequality with the estimates

$$\left\|\psi_s\right\|_{L^2(\mathbb{R}^d;H^1_\gamma)}\leq C\left\|g\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d;H^1_\gamma)}$$

and

$$\|\nabla_x \psi_s\|_{L^2(\mathbb{R}\times\mathbb{R}^d;L^2_\gamma)} \leqslant \left\|\nabla\Phi\left(\tfrac{t}{2},\cdot\right)\right\|_{L^1(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}\times\mathbb{R}^d;L^2_\gamma)} \leqslant Cs^{-\frac{1}{2}} \|g\|_{L^2(\mathbb{R}\times\mathbb{R}^d;L^2_\gamma)}.$$

The proof is complete. □

Remark 7.4 (Regularity in time)**.** By an interpolation argument, the result of Proposition 7.3 implies some time regularity for a function $f \in H^1_{\mathrm{kin}}(\mathbb{R}\times\mathbb{R}^d)$ which satisfies (7.8) for $v_0 \in [1,\infty)$. Indeed, by the definition of the norm $\|\cdot\|_{H^1_{\mathrm{kin}}}$, we have that

$$\|f\|_{L^2(\mathbb{R};L^2(\mathbb{R}^d;H^1_\gamma))} \leqslant \|f\|_{H^1_{\mathrm{kin}}(\mathbb{R}\times\mathbb{R}^d)}.$$

By interpolation and (7.9), for every $\theta \in [0,1]$ and $\alpha \in \left[0,\frac{1}{3}\right)$,

$$\|f\|_{L^2(\mathbb{R};H^{\theta\alpha}(\mathbb{R}^d;H^{1-2\theta}_\gamma))} \leqslant C\|f\|_{H^1_{\mathrm{kin}}(\mathbb{R}\times\mathbb{R}^d)}.$$

We also have, by (7.9), for any $\alpha \in \left[0,\frac{1}{3}\right)$,

$$\begin{aligned}
&\|f\|_{H^1(\mathbb{R};H^{\alpha-1}(\mathbb{R}^d;H^{-1}_\gamma))}\\
&\quad\leqslant \|f\|_{L^2(\mathbb{R};H^{\alpha-1}(\mathbb{R}^d;H^{-1}_\gamma))} + \|\partial_t f\|_{L^2(\mathbb{R};H^{\alpha-1}(\mathbb{R}^d;H^{-1}_\gamma))}\\
&\quad\leqslant \|f\|_{L^2(\mathbb{R};L^2(\mathbb{R}^d;H^{-1}_\gamma))} + \|\partial_t f - v\cdot\nabla_x f\|_{L^2(\mathbb{R};L^2(\mathbb{R}^d;H^{-1}_\gamma))} + \|v\cdot\nabla_x f\|_{L^2(\mathbb{R};H^{\alpha-1}(\mathbb{R}^d;H^{-1}_\gamma))}\\
&\quad\leqslant C\|f\|_{H^1_{\mathrm{kin}}(\mathbb{R}\times\mathbb{R}^d)}.
\end{aligned}$$

By interpolation of the previous two displays, we obtain, for any $\theta,\sigma \in [0,1]$ and $\alpha \in \left[0,\frac{1}{3}\right)$,

$$\|f\|_{H^\sigma(\mathbb{R};H^{\theta\alpha-\sigma(1-\alpha+\theta\alpha)}(\mathbb{R}^d;H^{1-2(\theta+\sigma-\theta\sigma)}_\gamma))} \leqslant C\|f\|_{H^1_{\mathrm{kin}}(\mathbb{R}\times\mathbb{R}^d)}. \tag{7.18}$$

As in the statement of Proposition 7.3, each of the constants C above depends only on (α, v_0, d). Note that all three exponents can be made simultaneously positive, for example taking $\alpha = \theta = \frac{1}{4}$ and $\sigma = \frac{1}{32}$ yields

$$\|f\|_{H^{\frac{1}{32}}(\mathbb{R};H^{\frac{1}{32}}(\mathbb{R}^d;H^{\frac{7}{16}}_\gamma))} \leqslant C\|f\|_{H^1_{\mathrm{kin}}(\mathbb{R}\times\mathbb{R}^d)}. \tag{7.19}$$

By (7.19) and an argument very similar to the proof of Proposition 3.5, which we omit, we obtain the following compact embedding statement.

Proposition 7.5 (Compact embedding of H^1_{kin} into L^2)**.** *For any $C^{1,1}$ domain $V \subseteq \mathbb{R}\times\mathbb{R}^d$, the inclusion map $H^1_{\mathrm{kin}}(V) \hookrightarrow L^2(V;L^2_\gamma)$ is compact.*

7.4. **Trace theorem and integration by parts.** We now study trace theorems and integration by parts formulas. Throughout this section, we assume that $V \subseteq \mathbb{R}\times\mathbb{R}^d$ is either a $C^{1,1}$ domain, or of the form $I\times U$ with $I \subseteq \mathbb{R}$ a bounded interval and $U \subseteq \mathbb{R}^d$ a $C^{1,1}$ domain. As in Section 4, the most general integration by parts formula is obtained when $\mathbf{b}(x)$ is a conservative force field, see Assumption 4.1. Under this assumption, we recall that the measure σ on $\mathbb{R}^d$ is defined in (4.1). For notational convenience, we redefine m to be the measure on $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ given by

$$dm(t,x,v) := dt\, d\sigma(x)\, d\gamma(v).$$

Recalling the definition of the differential operator B in (4.3), we see that the operator $\partial_t + B$ is formally skew-symmetric in $L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d, dm)$, and that the operator on the left side of (7.1) can be rewritten as

$$\nabla_v^*\nabla_v f + (\partial_t + B)f.$$

In practice, most of the results of the previous sections carry over provided that we replace B by $\partial_t + B$ throughout. Under Assumption 4.1, we write, for every $f \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$,

$$\int_{\partial V\times\mathbb{R}^d} f\,dm = \int_{\partial V\times\mathbb{R}^d} f(t,x,v)\,dm(t,x,v) := \int_{\partial V\times\mathbb{R}^d} f(t,x,v)\,\exp\left(-H(x)\right)\,d\mathcal{H}_d(t,x)\,d\gamma(v),$$

where $\mathcal{H}_d$ denotes the d-dimensional Hausdorff measure on ∂V. We also write

$$\int_{\partial V\times\mathbb{R}^d} f\,dt\,dx\,d\gamma = \int_{\partial U\times\mathbb{R}^d} f(t,x,v)\,dx\,d\gamma(v) := \int_{\partial U\times\mathbb{R}^d} f(t,x,v)\,d\mathcal{H}_d(t,x)\,d\gamma(v).$$

By integration by parts, we have for every $f,g\in C_c^\infty(\overline{V}\times\mathbb{R}^d)$ that

$$\int_{V\times\mathbb{R}^d}\left(f(\partial_t+B)g + g(\partial_t+B)f\right)\,dm = \int_{\partial V\times\mathbb{R}^d} fg\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V\,dm.$$

We denote

$$\partial^\pm_{\mathrm{kin}}V := \left\{((t,x),v)\in\partial V\times\mathbb{R}^d \;:\; \begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V(t,x)\neq 0\right\}.$$

Recall that we consider either that V is a $C^{1,1}$ domain, or that $V = I\times U$ with U a $C^{1,1}$ domain. In the latter case, by convention, we have set $\mathbf{n}_V(t,x) = 0$ whenever $(t,x)\in\partial I\times\partial U$. In both cases, we have the following analogue of Lemma 4.3.

Lemma 7.6 (trace lemma)**.** *For every compact set $K\subseteq\partial^\pm_{\mathrm{kin}}V$, the trace operator from $C_c^\infty(\overline{V}\times\mathbb{R}^d)$ to $L^2\left(K,\left|\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V\right|\,dt\,dx\,d\gamma\right)$ extends to a continuous linear operator on $H^1_{\mathrm{kin}}(V)$.*

Proof. When V is a $C^{1,1}$ domain, the proof is almost identical to that of Lemma 4.3, the only difference being that we replace $-v\cdot\mathbf{n}_U(x)$ by

$$\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V(t,x) \tag{7.20}$$

throughout. We now turn to the case when $V = I\times U$. We denote by χ the function introduced in (4.9), and for each $r_-,r_+\in[0,\infty)$, we set

$$\chi'_{r_-,r_+}(t,x,v) := \chi\left(r_-\left[\frac{(v\cdot\mathbf{n}_U(x))_-}{1+|v|}\wedge|t-I_-|\right] + r_+\left[\frac{(v\cdot\mathbf{n}_U(x))_+}{1+|v|}\wedge|t-I_+|\right]\right), \tag{7.21}$$

where we recall that we write $a\wedge b := \min(a,b)$. As for (4.11), we verify that for every $r_-,r_+\in[0,\infty)$ and $f\in H^1_{\mathrm{kin}}(V)$,

$$\|f\chi'_{r_-,r_+}\|_{H^1_{\mathrm{kin}}(V)}\leqslant C(1+r_-+r_+)\|f\|_{H^1_{\mathrm{kin}}(V)}.$$

Moreover, for every $r\in[0,\infty)$ and $f\in C_c^\infty(\overline{V}\times\mathbb{R}^d)$, we have

$$\int_{\partial V\times\mathbb{R}^d} f^2(1-\chi'_{r,0})\left|\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V\right|\,dm = \int_{\partial V\times\mathbb{R}^d} f^2(1-\chi'_{r,0})\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V\,dm \leqslant C(1+r)\|f\|_{H^1_{\mathrm{kin}}(V)},$$

and similarly,

$$\int_{\partial V\times\mathbb{R}^d} f^2(1-\chi'_{0,r})\left|\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V\right|\,dm \leqslant C(1+r)\|f\|_{H^1_{\mathrm{kin}}(V)}.$$

Using that $1-\chi'_{r,0}+1-\chi'_{0,r}=1-\chi_{r,r}$, we conclude that

$$\int_{\partial V\times\mathbb{R}^d} f^2(1-\chi'_{r,r})\left|\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V\right| dm \leqslant C(1+r)\|f\|_{H^1_{\mathrm{kin}}(V)},$$

which yields the result. □

We next introduce the function space

$$H^1_{\mathrm{kin},\partial}(V) := \left\{ f\in H^1_{\mathrm{kin}}(V) \ : \ \int_{\partial V\times\mathbb{R}^d} f^2\left|\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V\right| dm<\infty\right\},$$

equipped with the norm

$$\|f\|_{H^1_{\mathrm{kin},\partial}(V)} := \left(\|f\|^2_{H^1_{\mathrm{kin}}(V)}+\int_{\partial V\times\mathbb{R}^d} f^2\left|\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V\right| dm\right)^{\frac12}.$$

Proposition 7.7 (Integration by parts in $H^1_{\mathrm{kin},\partial}(V)$)**.** *Under Assumption 4.1, we have for every $f,g\in H^1_{\mathrm{kin},\partial}(V)$ that*

$$\int_{V\times\mathbb{R}^d}\left(f(\partial_t+B)g+g(\partial_t+B)f\right)dm=\int_{\partial V\times\mathbb{R}^d} fg\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V\,dm.$$

Proof. The proof is essentially identical to that of Proposition 4.4, with only minor changes to the notation. □

Lemma 7.8 (Boundary approximation)**.** *Under Assumption 4.1, we have for every $f\in H^1_{\mathrm{kin}}(V)$ that*

$$\inf_{b\in H^1_{\mathrm{kin},0}(V)}\int_{\partial V\times\mathbb{R}^d}(b-f)^2\begin{pmatrix}1\\-v\end{pmatrix}\cdot\mathbf{n}_V\,dm\leqslant 0.$$

Proof. If V is a $C^{1,1}$ domain, then the argument is almost identical to the proof Lemma 4.6; the only difference is that we replace $-v\cdot\mathbf{n}_U(x)$ by (7.20) throughout. In the case $V=I\times U$, we replace the function χ_{r_-,r_+} used in the proof of Lemma 4.6 by the function χ'_{r_-,r_+} introduced in (7.21). The rest of the argument follows similarly. □

Lemma 7.9 (No-trace result in $L^2(V;H^1_\gamma)$)**.** *For every $f\in H^1_{\mathrm{kin}}(V)$, $b\in H^1_{\mathrm{kin},0}(V)$ and $\varepsilon>0$, there exists $f'\in H^1_{\mathrm{kin},0}(V)$ such that $f'=b$ on $\partial V\times\mathbb{R}^d$ and*

$$\|f-f'\|_{L^2(V;H^1_\gamma)}\leqslant\varepsilon.$$

Proof. The argument is a simple adaptation of the proof of Lemma 4.7: we use a cutoff in the time-space variable (t,x) where a cutoff in x was used in the stationary setting. □

7.5. **Well-posedness of the Cauchy-Dirichlet problem.** We first show the well-posedness of the Cauchy-Dirichlet problem under Assumption 4.1 and when $V\subseteq\mathbb{R}\times\mathbb{R}^d$ is a $C^{1,1}$ domain (the case of a cylindrical domain is considered afterwards). We define

$$Z_{\mathrm{kin}}(V):=\left\{(f,f^*)\ :\ f\in L^2(V;H^1_\gamma)\ \text{ and }\ f^*-Bf-\partial_t f\in L^2(V;H^{-1}_\gamma)\right\},$$

and for every pair $(f,f^*)\in Z_{\mathrm{kin}}(V)$,

$$J_{\mathrm{kin}}[f,f^*]:=\inf\left\{\int_{V\times\mathbb{R}^d}\frac12|\nabla_v f-\mathbf{g}|^2\,dm\ :\ \mathbf{g}\in L^2(V;L^2_\gamma)^d\ \text{ s.t. }\ \nabla_v^*\mathbf{g}=f^*-Bf-\partial_t f\right\}. \tag{7.22}$$

The constraint on $\mathbf{g}$ appearing in (7.22) is interpreted as

$$\forall \phi \in L^2(V; H^1_\gamma) \qquad \int_{V\times\mathbb{R}^d} \mathbf{g}\cdot\nabla_v\phi\, dm = \int_{V\times\mathbb{R}^d} \phi\,(f^* - Bf - \partial_t f)\, dm.$$

As in previous sections, the right side above stands for

$$\int_V \langle \phi(t,x,\cdot), (f^* - Bf - \partial_t f)(t,x,\cdot)\rangle_{H^1_\gamma, H^{-1}_\gamma}\, dt\, d\sigma(x).$$

Proposition 7.10 (Solvability of Cauchy-Dirichlet problem)**.** *Let V be a $C^{1,1}$ domain. Under Assumption 4.1, for each $(g,g^*)\in Z_{\mathrm{kin}}(V)$, the mapping*

$$\left\{\begin{array}{ccc} g + H^1_{\mathrm{kin},0}(V) & \to & [0,+\infty] \\ f & \mapsto & J_{\mathrm{kin}}[f,g^*] \end{array}\right. \tag{7.23}$$

is uniformly convex. Moreover, its minimum is zero, and the associated minimizer is the unique $f\in g + H^1_{\mathrm{kin},0}(V)$ solution of the equation

$$\nabla_v^*\nabla_v f + \partial_t f + Bf = g^*. \tag{7.24}$$

We intepret (7.24) as meaning that

$$\forall \phi \in L^2(V; H^1_\gamma) \qquad \int_{V\times\mathbb{R}^d} \nabla_v\phi\cdot\nabla_v f\, dm = \int_{V\times\mathbb{R}^d} \phi\,(g^* - Bf - \partial_t f)\, dm$$

The proof is essentially identical to that of Proposition 5.1. The key step in which we use the assumption that the domain is $C^{1,1}$ is the following analogue of Lemma 5.2.

Lemma 7.11. *Let $V\subseteq\mathbb{R}\times\mathbb{R}^d$ be a $C^{1,1}$ domain. There exists a constant $C(V,d)<\infty$ and, for each $h\in L^2(V)$, a function $f\in H^1_{\mathrm{kin},0}(V)$ satisfying*

$$\langle \partial_t f - v\cdot\nabla_x f\rangle_\gamma = h \tag{7.25}$$

and the estimate

$$\|f\|_{H^1_{\mathrm{kin}}(V)} \leqslant C\|h\|_{L^2(V)}. \tag{7.26}$$

Proof. We can reproduce the proof of Lemma 5.2 with only minor modifications. Indeed, we first notice that by arguing as in Steps 2 and 3 of the proof of Lemma 5.2, it suffices to consider the case when $\int_V h = 0$. In this case, as in Step 1 of the proof of Lemma 5.2, we can obtain the existence of a vector field $\mathbf{f} = (\mathbf{f}_0,\ldots,\mathbf{f}_d)\in H^1_0(V)^{1+d}$ satisfying

$$\left\{\begin{array}{ll} \partial_t \mathbf{f}_0 + \displaystyle\sum_{i=1}^d \partial_{x_i}\mathbf{f}_i = h & \text{in } V, \\ \mathbf{f} = 0 & \text{on } \partial V. \end{array}\right. \tag{7.27}$$

and such that, for some $C(V,d)<\infty$,

$$\|\mathbf{f}\|_{H^1(V)} \leqslant C\|h\|_{L^2(V)}.$$

We stress that here we treat the time variable in just the same way as the spatial variables; a function belongs to $H^1(V)$ if its derivatives in time and space belong to $L^2(V)$. We then select smooth functions $\xi_0,\xi_1,\ldots,\xi_d\in C^\infty_c(\mathbb{R}^d)$ such that ξ_0 satisfies (7.6) and, for every $i\in\{1,\ldots,d\}$,

$$\int_{\mathbb{R}^d} \xi_i(v)\, d\gamma(v) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} v\xi_i(v)\, d\gamma(v) = e_i. \tag{7.28}$$

Finally, we set

$$f(t,x,v) := \mathbf{f}_0(t,x)\xi_0(v) - \sum_{i=1}^d \mathbf{f}_i(t,x)\xi_i(v),$$

and check that f belongs to $H^1_{\mathrm{kin},0}(V)$ and satisfies (7.25) and (7.26). □

Proof of Proposition 7.10. The proof parallels that of Proposition 5.1.

Step 1. We show that the mapping in (5.4) is uniformly convex and not constantly equal to $+\infty$. As in Step 1 of Proposition 5.1, we can use Lemma 5.2 and (5.7) to assert that the mapping in (5.4) is not constantly equal to $+\infty$. Verifying the uniform convexity boils down to showing the existence of a constant $C(V,d) < \infty$ such that for every $(f,\mathbf{j}) \in H^1_{\mathrm{kin},0}(V) \times L^2(V;L^2_\gamma)^d$ satisfying the constraint

$$\nabla_v^* \mathbf{j} = -Bf - \partial_t f \tag{7.29}$$

we have the estimate

$$\frac{1}{2}\int_{V\times\mathbb{R}^d} |\nabla_v f - \mathbf{j}|^2 \, dm \geqslant C^{-1}\left(\|f\|^2_{H^1_{\mathrm{kin}}(V)} + \|\mathbf{j}\|^2_{L^2(V;L^2_\gamma)}\right).$$

Expanding the square and using (7.29), we can rewrite the left side above as

$$\int_{V\times\mathbb{R}^d}\left(\frac{1}{2}|\nabla_v f|^2 + \frac{1}{2}|\mathbf{j}|^2 + f(Bf + \partial_t f)\right) dm.$$

We then verify that $\int_{V\times\mathbb{R}^d} f(\partial_t + B)f \, dm \geqslant 0$ and, using (7.29), that

$$\|\partial_t f + Bf\|_{L^2(V;H^{-1}_\gamma)} \leqslant C\|\mathbf{j}\|_{L^2(V;L^2_\gamma)}.$$

Step 2. We introduce the perturbed convex minimization problem defined, for every $f^* \in L^2(V;H^{-1}_\gamma)$, by

$$G(f^*) := \inf_{f\in H^1_{\mathrm{kin},0}(V)}\left(J_{\mathrm{kin}}[f+g, f^*+g^*] + \int_{V\times\mathbb{R}^d} f f^* \, dm\right),$$

and we aim to show that $G(0) = 0$. Since $G(0) \geqslant 0$ and G is convex and lower semi-continuous, this boils down to the verification that $G^* \geqslant 0$, where G^* is defined for every $h \in L^2(V;H^1_\gamma)$ by

$$G^*(h) := \sup_{f^*\in L^2(V;H^{-1}_\gamma)}\left(-G(f^*) + \int_{V\times\mathbb{R}^d} h f^* \, dm\right).$$

We decompose the proof of the fact that $G^* \geqslant 0$ into the remaining two steps.

Step 3. We show that for every $h \in L^2(V;H^{-1}_\gamma)$,

$$G^*(h) < +\infty \quad \Longrightarrow \quad \partial_t h + Bh \in L^2(V;H^{-1}_\gamma).$$

Note that

$$G^*(h) = \sup\left\{\int_{V\times\mathbb{R}^d}\left(-\frac{1}{2}|\nabla_v(f+g) - \mathbf{j}|^2 + (h-f)f^*\right) dm\right\}, \tag{7.30}$$

where the supremum ranges over every $f \in H^1_{\mathrm{kin},0}(V)$, $\mathbf{j} \in L^2(V;L^2_\gamma)^d$ and $f^* \in L^2(V;H^{-1}_\gamma)$ satisfying the constraint

$$\nabla_v^* \mathbf{j} = f^* + g^* - (\partial_t + B)(f+g).$$

We use Lemma 7.11 to select $f_0 \in H^1_{\mathrm{kin},0}(V)$ such that

$$\langle g^* - (\partial_t + B)g\rangle_\gamma = \langle(\partial_t + B)f_0\rangle_\gamma,$$

then $\mathbf{j}_0 \in L^2(V;L^2_\gamma)^d$ such that

$$\nabla_v^* \mathbf{j}_0 = g^* - (\partial_t + B)(f_0 + g),$$

and restrict the supremum in (7.30) to $f^* = (\partial_t + B)(f - f_0)$ and $\mathbf{j} = \mathbf{j}_0$, obtaining that

$$G^*(h) \geqslant \sup\left\{\int_{V\times\mathbb{R}^d}\left(-\frac{1}{2}|\nabla_v(f+g) - \mathbf{j}_0|^2 + (h-f)(\partial_t + B)(f-f_0)\right) dm\right\},$$

with the supremum now ranging over $f \in H^1_{\mathrm{kin},0}(V)$ only. For $f \in C_c^\infty(V \times \mathbb{R}^d)$, the integral $\int f(\partial_t + B)f \, dm$ vanishes, and thus

$$\sup\left\{ \int_{V\times\mathbb{R}^d} h(\partial_t + B)f \ : \ f \in C_c^\infty(V\times\mathbb{R}^d), \ \|f\|_{L^2(V;H^1_\gamma)} \leqslant 1 \right\} < +\infty.$$

This implies that $(\partial_t + B)h \in L^2(V; H^{-1}_\gamma)$, as desired.

Step 4. To conclude the proof, there remains to show that for every $h \in H^1_{\mathrm{kin}}(V)$, we have $G^*(h) \geqslant 0$. We replace f^* by $f^* + (\partial_t + B)f$ in the formula (7.30) to get that

$$G^*(h) = \sup\left\{ \int_{V\times\mathbb{R}^d} \left(-\frac{1}{2}|\nabla_v(f+g) - \mathbf{j}|^2 + (h-f)(f^* + \partial_t f + Bf) \right) dm \right\}, \tag{7.31}$$

where the supremum is now over every $f \in H^1_{\mathrm{kin},0}(V)$, $\mathbf{j} \in L^2(V; L^2_\gamma)^d$ and $f^* \in L^2(V; H^{-1}_\gamma)$ satisfying the constraint

$$\nabla_v^* \mathbf{j} = f^* + g^* - (\partial_t + B)g. \tag{7.32}$$

We can rewrite the supremum in (7.31) as

$$G^*(h) = \sup\left\{ \int_{V\times\mathbb{R}^d} \left(-\frac{1}{2}|\nabla_v(f+g) - \mathbf{j}|^2 + (h-f)f^* - f(\partial_t + B)h) \right) dm \right.$$
$$\left. - \int_{\partial V\times\mathbb{R}^d} \left(\frac{1}{2}f^2 - fh \right) \begin{pmatrix} 1 \\ -v \end{pmatrix} \cdot \mathbf{n}_V \, dm \right\},$$

with the supremum over f, $\mathbf{j}$ and f^* as above. Since the functional

$$f \mapsto \int_{V\times\mathbb{R}^d} \left(-\frac{1}{2}|\nabla_v(f+g) - \mathbf{j}|^2 + (h-f)f^* - f(\partial_t + B)h) \right) dm$$

is continuous with respect to the topology of $L^2(V; H^1_\gamma)$, we can use Lemma 7.9 to infer that

$$G^*(h) = \sup\left\{ \int_{V\times\mathbb{R}^d} \left(-\frac{1}{2}|\nabla_v(f+g) - \mathbf{j}|^2 + (h-f)f^* - f(\partial_t + B)h) \right) dm \right.$$
$$\left. - \int_{\partial V\times\mathbb{R}^d} \left(\frac{1}{2}b^2 - bh \right) \begin{pmatrix} 1 \\ -v \end{pmatrix} \cdot \mathbf{n}_V \, dm \right\},$$

where the supremum is now over every $f \in H^1_{\mathrm{kin}}(V)$, $b \in H^1_{\mathrm{kin},0}(V)$, $\mathbf{j} \in L^2(V; L^2_\gamma)^d$ and $f^* \in L^2(V; H^{-1}_\gamma)$ satisfying (7.32). We then select $f = h$ to get that

$$G^*(h) \geqslant \sup\left\{ -\int_{V\times\mathbb{R}^d} \frac{1}{2}|\nabla_v(h+g) - \mathbf{j}|^2 \, dm - \frac{1}{2}\int_{\partial V\times\mathbb{R}^d} (b-h)^2 \begin{pmatrix} 1 \\ -v \end{pmatrix} \cdot \mathbf{n}_V \, dm \right\},$$

where the supremum is now over every b, $\mathbf{j}$ and f^* as above. By Lemma 7.8, it follows that

$$G^*(h) \geqslant \sup\left\{ -\int_{V\times\mathbb{R}^d} \frac{1}{2}|\nabla_v(h+g) - \mathbf{j}|^2 \, dm \right\},$$

where the supremum ranges over every $\mathbf{j} \in L^2(V; L^2_\gamma)^d$ and $f^* \in L^2(V; H^{-1}_\gamma)$ satisfying (7.32). To conclude, we select $\mathbf{j} = \nabla_v(h+g)$ and then $f^* = \nabla_v^* \mathbf{j} - g^* + (\partial_t + B)g$ and thus obtain that $G^*(h) \geqslant 0$. □

We now extend the well-posedness result to the case of cylindrical domains.

Proposition 7.12 (Well-posedness in $I \times U$)**.** *Let $I \subseteq \mathbb{R}$ be a bounded interval, $U \subseteq \mathbb{R}^d$ be a $C^{1,1}$ domain, $V := I \times U$, $f^* \in L^2(V; H_\gamma^{-1})$ and $f_0 \in H^1_{\mathrm{kin}}(V)$. Under Assumption 4.1, there exists a unique weak solution of the Cauchy-Dirichlet problem*

$$\begin{cases} \partial_t f - \Delta_v f + v \cdot \nabla_v f - v \cdot \nabla_x f + \mathbf{b} \cdot \nabla_v f = f^* & \text{in } V \times \mathbb{R}^d, \\ f = f_0 & \text{on } \partial_{\mathrm{hyp}} V. \end{cases} \tag{7.33}$$

Proof. Step 1. We first show uniqueness of solutions. By linearity, this amounts to showing that for $f^* = 0$ and $f_0 = 0$, the only solution to (7.33) is the null solution. Testing the equation with f itself, we find that

$$\int_{V \times \mathbb{R}^d} |\nabla_v f|^2 \, dm + \int_{V \times \mathbb{R}^d} f(\partial_t + B) f \, dm = 0.$$

Since $f \in H^1_{\mathrm{kin},0}(V) \subseteq H^1_{\mathrm{kin},\partial}(V)$, the second term on the left side above is non-negative. It thus follows that $\nabla_v f = 0$, and then that $(\partial_t + B) f = 0$. By the Poincaré inequality (Proposition 7.2), for each $C^{1,1}$ domain $W \subseteq V$, we thus have that f is constant on W. Using overlapping domains, we see that this constant does not depend on W. Finally, the only constant that also belongs to $H^1_{\mathrm{kin},0}(V)$ is the null function.

Step 2. We now prove the existence of solutions. By considering the equation for $f - f_0$, we can assume without loss of generality that $f_0 = 0$. Let $(V_n)_{n \geqslant 1}$ be an increasing sequence of $C^{1,1}$ domains such that $\bigcup_{n \geqslant 1} V_n = V$. We choose these approximating domains in such a way that $\partial V_n = \partial V$ outside of smaller and smaller neighborhoods of the set $\partial I \times \partial U$ where the $C^{1,1}$ regularity of V fails to hold. For each fixed n, there exists a weak solution $f_n \in H^1_{\mathrm{kin},0}(V_n)$ of the equation

$$\begin{cases} \nabla_v^* \nabla_v f_n + (\partial_t + B) f_n = f^* & \text{in } V_n \times \mathbb{R}^d, \\ f_n = 0 & \text{on } \partial_{\mathrm{hyp}} V_n. \end{cases}$$

By considering the equation for the function $\widetilde{f}_n(t,x,v) := e^{-t} f_n(t,x,v)$, testing this equation with $\widetilde{f}_n$, and arguing as in the first step to treat the boundary term, we obtain that

$$\|f_n\|_{L^2(V_n; H^1_\gamma)} \leqslant C \|f^*\|_{L^2(V_n; H^{-1}_\gamma)}.$$

Using also that $(\partial_t + B) f_n = f^* - \nabla_v^* \nabla_v f_n$, we deduce that

$$\|f_n\|_{H^1_{\mathrm{kin}}(V_n)} \leqslant C \|f^*\|_{L^2(V_n; H^{-1}_\gamma)}. \tag{7.34}$$

Up to the extraction of a subsequence, which we keep implicit in the notation for simplicity, we can thus assume that there exists $f \in \bigcap_{n \geqslant 1} H^1_{\mathrm{kin}}(V_n)$ such that for each $k \geqslant 1$, the sequence f_n converges to f weakly in $H^1_{\mathrm{kin}}(V_k)$. Using (7.34), we have that for each $k \geqslant 1$,

$$\|f\|_{H^1_{\mathrm{kin}}(V_k)} \leqslant \liminf_{n \to \infty} \|f_n\|_{H^1_{\mathrm{kin}}(V_k)} \leqslant C \|f^*\|_{L^2(V; H^{-1}_\gamma)}.$$

Letting k go to infinity, this shows that $f \in H^1_{\mathrm{kin}}(V)$. Moreover, we have for every $k \geqslant 1$ and $\phi \in L^2(V_k; H^1_\gamma)$ that

$$\int_{V_k \times \mathbb{R}^d} \nabla_v \phi \cdot \nabla_v f \, dm + \int_{V_k \times \mathbb{R}^d} \phi \left(\partial_t f + B f - f^* \right) dm = 0.$$

Using that $f \in H^1_{\mathrm{kin}}(V)$, we can extend this relation to every $\phi \in L^2(V; H^1_\gamma)$, thus obtaining that f is a weak solution. There now remains to verify that $f \in H^1_{\mathrm{kin},0}(V)$. Let $\psi \in C_c^{0,1}(\partial V \times \mathbb{R}^d)$ be a boundary datum with compact support in $\partial_{\mathrm{kin}} V$. We will argue that

$$\int_{\partial V \times \mathbb{R}^d} \psi f \begin{pmatrix} 1 \\ -v \end{pmatrix} \cdot \mathbf{n}_V \, dm = 0. \tag{7.35}$$

If this can be done, it implies that $f = 0$ on $\partial_{\mathrm{kin}} V$, and by the time-dependent version of Proposition 4.10, that $f \in H^1_{\mathrm{kin},0}(V)$. Hence, there only remains to argue for (7.35). Since ψ has compact support in $\partial_{\mathrm{kin}} V$, for k sufficiently large and every $n \geqslant k$, we have that the support of ψ is a subset of $\partial_{\mathrm{kin}} V_n$; we can then extend ψ to $\partial_{\mathrm{kin}} V_k$ by setting $\psi = 0$ whenever ψ was not defined. We can then further extend ψ to be a function in $H^1_{\mathrm{kin}}(V)$, e.g. by solving for a Dirichlet problem in V_k and extending the function to be 0 in $V \setminus V_k$. For every $n \geqslant k$, we have

$$\int_{V_n \times \mathbb{R}^d} \left(\psi(\partial_t + B) f_n + f_n(\partial_t + B)\psi\right) dm = \int_{\partial V_n \times \mathbb{R}^d} \psi f_n \begin{pmatrix} 1 \\ -v \end{pmatrix} \cdot \mathbf{n}_V \, dm = 0.$$

By the construction of ψ, we can replace V_n by V_k in the domain of integration on the left of this identity. Since f_n converges to f weakly in $H^1_{\mathrm{kin}}(V_k)$, we deduce that

$$\int_{V_k \times \mathbb{R}^d} \left(\psi(\partial_t + B) f + f(\partial_t + B)\psi\right) dm = 0.$$

By the construction of ψ, we can then replace V_k by V in the integral above, and thus conclude that (7.35) holds, as desired. □

We do not include a proof of the following statement in this paper, since the argument is a close adaptation of the one of Theorem 1.4. We denote $V_r := (-r, r) \times B_r$ and by $\nabla_{t,x}$ the full gradient in t and x, that is, $\nabla_{t,x} = (\partial_t, \nabla_x)$.

Proposition 7.13 (Interior regularity, kinetic Fokker-Planck)**.** *Let $k, l \in \mathbb{N}$, $r, v_0 \in [1, \infty)$ and $\mathbf{b} \in C^{k+l-1,1}(V_r; C^{l-1,1}(\mathbb{R}^d; \mathbb{R}^d))$. There exists a constant $C < \infty$ depending on*

$$\left(d, k, l, r, v_0, \left\{ \|\mathbf{b}\|_{C^{k+j-1,1}(V_r; C^{l-j-1,1}(\mathbb{R}^d))} \right\}_{j \in \{0,\dots,l\}}\right)$$

such that, for every $f \in H^1_{\mathrm{kin}}(V_r)$ and $f^* \in H^{k+l}(V_r; H^{l-1}_\gamma)$ satisfying

$$\partial_t - \Delta_v f + v \cdot \nabla_v f - v \cdot \nabla_x f + \mathbf{b} \cdot \nabla_v f = f^* \quad \text{in } V_r \times \mathbb{R}^d, \tag{7.36}$$

we have that $f \in H^k\left(V_{r/2}; H^l_\gamma(B_{v_0})\right)$ and the estimate

$$\left\| \left| \nabla_v^l \nabla_{t,x}^k f \right| \right\|_{L^2\left(V_{r/2}; L^2_\gamma(B_{v_0})\right)} \leqslant C \left(\left\| f - (f)_{V_r} \right\|_{L^2(V_r; L^2_\gamma)} + \sum_{j=0}^{l} \left\| \widetilde{f}^* \right\|_{H^{k+j}(V_r; H^{l-j-1}_\gamma)} \right).$$

7.6. **Exponential decay in time.** For each bounded interval $I = (I_-, I_+) \subseteq \mathbb{R}$ and bounded Lipschitz domain U, we denote by $H^1_{\mathrm{kin},\|}(I \times U)$ the closure in $H^1_{\mathrm{kin}}(I \times U)$ of the set of smooth functions which vanish on $I \times \partial_{\mathrm{hyp}} U$. Note that in particular, we allow the trace of $f \in H^1_{\mathrm{kin},\|}(I \times U)$ on the initial time slice $\{I_-\} \times U$ to be non-zero. In this section, we show that a solution to the kinetic Fokker-Planck equation with zero right-hand side and belonging to $H^1_{\mathrm{kin},\|}(I \times U)$ decays to zero exponentially fast in time. We start with a preliminary classical lemma.

Lemma 7.14 (continuity in L^2)**.** *Every function in $H^1_{\mathrm{kin},\|}(I \times U)$ can be identified (up to a set of null measure) with an element of $C(\overline{I}; L^2(U; L^2_\gamma))$.*

Proof. If f is a smooth function which vanishes on $I \times \partial_{\mathrm{hyp}} U$, then for every $t \in I$, we have

$$\partial_t \|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)} + \int_{\partial U \times \mathbb{R}^d} f^2(t,x,v)(v \cdot \mathbf{n}_U(x))_- \, dx \, d\gamma(v) = 2 \int_{U \times \mathbb{R}^d} \left(f(\partial_t f - v \cdot \nabla_x f)\right)(t,x,v) \, dx \, d\gamma(v),$$

where we recall that $(r)_- := \max(0,-r)$. Since the second integral on the left side is nonnegative, we deduce that for every $s,t \in I$,

$$\left| \|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(s,\cdot)\|^2_{L^2(U;L^2_\gamma)} \right| \leqslant 2\|f\|_{L^2((s,t)\times U;H^1_\gamma)} \, \|\partial_t f - v\cdot\nabla_x f\|_{L^2((s,t)\times U;H^{-1}_\gamma)},$$

and thus, for a constant $C(I) < \infty$,

$$\sup_{t\in\overline{I}} \|f(t,\cdot)\|_{L^2(U;L^2_\gamma)} \leqslant C\|f\|_{H^1_{\mathrm{kin}}(I\times U)}.$$

For a general $f \in H^1_{\mathrm{kin},\|}(I\times U)$, there exists a sequence (f_n) of functions which vanish on $I\times\partial_{\mathrm{hyp}}U$ and such that f_n converges to f in $H^1_{\mathrm{kin}}(I\times U)$. It follows from the inequality above that f_n converges to f with respect to the $L^\infty(I;L^2(U;L^2_\gamma))$ norm; in particular, $f \in C(\overline{I};L^2(U;L^2_\gamma))$. □

We finally turn to the proof of Theorem 1.5, which is restated in the following proposition. Notice that, by linearity, it suffices to prove the theorem in the case $f^* = 0$ and $f_\infty = 0$.

Proposition 7.15 (Exponential decay to equilibrium)**.** *Let $U \subseteq \mathbb{R}^d$ be a $C^{1,1}$ domain and $\mathbf{b} \in L^\infty(U\times\mathbb{R}^d)^d$. There exists $\lambda(\|\mathbf{b}\|_{L^\infty(U\times\mathbb{R}^d)}, U, d) > 0$ such that, for every $T \in (0,\infty)$ and $f \in H^1_{\mathrm{kin},\|}((0,T)\times U)$ satisfying*

$$\partial_t f - \Delta_v f + v\cdot\nabla_v f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f = 0 \qquad \text{in } (0,T)\times U\times\mathbb{R}^d,$$

we have, for every $t \in (0,T)$,

$$\|f(t,\cdot)\|_{L^2(U;L^2_\gamma)} \leqslant 2\exp(-\lambda t)\,\|f(0,\cdot)\|_{L^2(U;L^2_\gamma)}.$$

Proof. For every $0 \leqslant s < t$, we compute

$$\frac{1}{2}\left(\|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(s,\cdot)\|^2_{L^2(U;L^2_\gamma)}\right) \leqslant -\|\nabla_v f\|^2_{L^2((s,t)\times U;L^2_\gamma)}.$$

In particular,

$$\text{the mapping } t \mapsto \|f(t,\cdot)\|_{L^2(U;L^2_\gamma)} \text{ is nonincreasing.} \tag{7.37}$$

Since

$$-\nabla_v^*\nabla_v f = \partial_t f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f,$$

we have

$$\begin{aligned}
&\|\partial_t f - v\cdot\nabla_x f\|_{L^2((s,t)\times U;H^{-1}_\gamma)} \\
&\quad\leqslant \|\partial_t f - v\cdot\nabla_x f + \mathbf{b}\cdot\nabla_v f\|_{L^2((s,t)\times U;H^{-1}_\gamma)} + \|\mathbf{b}\cdot\nabla_v f\|_{L^2((s,t)\times U;H^{-1}_\gamma)} \\
&\quad\leqslant C\|\nabla_v f\|_{L^2((s,t)\times U;L^2_\gamma)},
\end{aligned}$$

and thus

$$\begin{aligned}
&-\left(\|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(s,\cdot)\|^2_{L^2(U;L^2_\gamma)}\right) \\
&\qquad\geqslant \frac{1}{C}\left(\|\nabla_v f\|^2_{L^2((s,t)\times U;L^2_\gamma)} + \|\partial_t f - v\cdot\nabla_x f\|^2_{L^2((s,t)\times U;H^{-1}_\gamma)}\right).
\end{aligned} \tag{7.38}$$

We aim to appeal to Proposition 7.2 to conclude. Since we have only proved this proposition for $C^{1,1}$ domains, we find a $C^{1,1}$ domain $V \subseteq \mathbb{R}\times\mathbb{R}^d$ satisfying

$$\left[\frac{1}{4},\frac{3}{4}\right]\times U \subseteq V \subseteq [0,1]\times U. \tag{7.39}$$

For every $t \geqslant 0$, we write

$$V_t := (t,0) + V = \{(t+s,x) \in \mathbb{R}\times\mathbb{R}^d \ : \ (s,x) \in V\}.$$

Inequality (7.38) implies that, for every $t \geqslant 0$,

$$-\left(\|f(t+1,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)}\right) \geqslant \frac{1}{C}\left(\|\nabla_v f\|^2_{L^2(V_t;L^2_\gamma)} + \|\partial_t f - v\cdot\nabla_x f\|^2_{L^2(V_t;H^{-1}_\gamma)}\right).$$

Proposition 7.2 yields that

$$-\left(\|f(t+1,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)}\right) \geqslant \frac{1}{C}\|f\|^2_{L^2(V_t;L^2_\gamma)}.$$

Using (7.37) and (7.39), we deduce that

$$-\left(\|f(t+1,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)}\right) \geqslant \frac{1}{C}\|f(t+1,\cdot)\|^2_{L^2(U;L^2_\gamma)}.$$

This implies exponential decay of the mapping $t \mapsto \|f(t,\cdot)\|_{L^2(U;L^2_\gamma)}$ along integer values of t, and we then obtain the conclusion of the proposition by using (7.37) once more. □

Appendix A. Discussion on trace theorems

In this appendix, we discuss previous results in the literature related to Question 4.2. The first contribution is due to [4], where Question 4.2 is answered positively in the case $d = 1$. The argument is based on the construction of an auxiliary function defined by

$$\widetilde{f}(x,v) := \begin{cases} f(x,v) & \text{if } v > 0, \\ f(x,-v) & \text{if } v < 0. \end{cases}$$

It is shown in [4] that $|v|\partial_x \widetilde{f}$ can be estimated in $L^2_x H^{-1}_v$ in terms of the $L^2_x H^{-1}_v$ norm of $v\partial_x f$. Once an estimate on $|v|\partial_x \widetilde{f}$ is known, one can use the integration by parts in (4.6) with f replaced by $\widetilde{f}$ and $v\partial_x f$ replaced by $|v|\partial_x \widetilde{f}$ to obtain the result.

One may at first try to generalize this construction to higher dimensions. Assuming that $\mathbf{n}_U(0) = e_1$ for definiteness, we can consider the auxiliary function

$$\widetilde{f}(x,v) := \begin{cases} f(x,v) & \text{if } v_1 > 0, \\ f(x,-v_1,v_2,\ldots,v_d) & \text{if } v_1 < 0. \end{cases}$$

An attempt to mimic the one-dimensional argument above would then require an estimate of $\operatorname{sign}(v_1) v\cdot\nabla_x \widetilde{f}$ in $L^2_x H^{-1}_v$ in terms of the $L^2_x H^{-1}_v$ norm of $v\cdot\nabla_x f$, but this is no longer possible. One way to understand that the situation is easier in dimension $d = 1$ is to observe that, in this case, an estimate on $v\cdot\nabla_x f$ is an estimate on the full gradient of $\nabla_x f$ which happens to degenerate on a hyperplane, while in higher dimension it is only an estimate of one particular projection of $\nabla_x f$ at any given point.

The same one-dimensional positive result is also stated in [12, Lemma II.1]. The authors explain that they follow the approach of [4], but also provide an independent argument which seems to be incomplete. A positive answer to Question 4.2 in full generality ($d \geqslant 1$) is then asserted in [10, Lemma 2.3]. As stated there: "to define the traces we use again the ideas of [12] that we sketch in the following lemma [...]". The starting point of the argument is that any function in $H^1_{\mathrm{hyp}}(U)$ can be approximated in $H^1_{\mathrm{hyp}}(U)$ by functions which vanish in a neighborhood of the singular set $\partial^=_{\mathrm{hyp}}(U)$, as was recalled in part (2) of Lemma 4.5. One can use this result to find a sequence of functions $f_n, g_n \in H^1_{\mathrm{hyp}}(U)$ such that for each n, the function f_n vanishes on a neighborhood of $\partial_{\mathrm{hyp}}(U)$, the function g_n vanishes on a neighborhood of $(\partial U \times \mathbb{R}^d) \setminus \partial_{\mathrm{hyp}}(U)$, and

$$\lim_{n\to\infty} \|f_n + g_n - f\|_{H^1_{\mathrm{hyp}}(U)} = 0.$$

It is then clear from Lemma 4.3 above that the functions f_n and g_n have traces in the space $L^2(\partial U \times \mathbb{R}^d, |v \cdot \mathbf{n}_U| \, dx \, d\gamma)$, and that the norms of these traces can be estimated by a constant times $\|f_n\|_{H^1_{\mathrm{hyp}}(U)}$ and $\|g_n\|_{H^1_{\mathrm{hyp}}(U)}$, respectively. However, in order to conclude, one would then need to assert that

$$\limsup_{n \to \infty} \left(\|f_n\|_{H^1_{\mathrm{hyp}}(U)} + \|g_n\|_{H^1_{\mathrm{hyp}}(U)} \right) < +\infty. \tag{A.1}$$

Unfortunately, we do not know how to construct f_n and g_n in such a way that (A.1) holds. To see the difficulty, we now explain why the most natural approach to this problem does not work. As can be seen from Proposition 2.2, we may assume without loss of generality that the function $f \in H^1_{\mathrm{hyp}}(U)$ we wish to approximate belongs to $L^\infty(U \times \mathbb{R}^d)$ and has compact support in $\overline{U} \times \mathbb{R}^d$. Similarly to the construction involving the functions ϕ_r in the proof of Lemma 4.5, we would wish to identify a sequence of functions $\phi_r \in H^1(U \times \mathbb{R}^d)$ satisfying, for each $r \geqslant 1$,

$$\begin{aligned} \phi_r &\equiv 0 \quad \text{on } \{(x,v) \in \partial U \times \mathbb{R}^d \ : \ \mathrm{dist}((x,v), \partial_{\mathrm{hyp}} U) \leqslant r^{-1}\} \cap \mathrm{supp}\, f, \\ \phi_r &\equiv 1 \quad \text{on } \{(x,v) \in \partial U \times \mathbb{R}^d \ : \ \mathrm{dist}((x,v), \partial_{\mathrm{hyp}} U) \geqslant 2r^{-1}\} \cap \mathrm{supp}\, f, \end{aligned}$$

and such that

$$\sup_{r \geqslant 1} \|\nabla \phi_r\|_{L^2(U \times \mathbb{R}^d)} < +\infty.$$

Here ∇ denotes the full gradient in the variables (x,v). However, the H^1 norm of ϕ_r is bounded below by the $H^{\frac{1}{2}}$ norm of its trace on $\partial U \times \mathbb{R}^d$, and a calculation shows that the latter quantity must blow up like $\log r$ as $r \to \infty$.

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