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New preconditioners for Laplace and Helmholtz integral equations on open curves

II. Theoretical analysis

Martin Averseng

*Centre de Mathématiques Appliquées,
Ecole Polytechnique, France*

May 31, 2019

Abstract

This paper is the second part of a work on Laplace and Helmholtz integral equations in 2 space dimensions on open curves. A new Galerkin method in weighted L^2 spaces together with new preconditioners for the weighted layer potentials are studied. This second part provides the theoretical analysis needed to establish the results announced in the first part. The main novelty is the introduction of a pseudo-differential calculus on open curves that allows to build parametrices for the weighted layer potentials. Contrarily to more classical approaches where the Mellin transform is used, this new approach is well-suited to the specific singularities that appear in the problem.

Introduction

We are concerned with the numerical resolution of the problems of Helmholtz and Laplace scattering by open curves in the plane, with Dirichlet or Neumann boundary data. In the first part of this work [1], after recasting the classical first-kind integral equations into a form involving weighted versions of the usual layer potentials, the authors have introduced a Galerkin method in weighted L^2 spaces and some new efficient preconditioners for the resulting linear systems. A discussion on existing literature and the relation of this method to other works can be found in [1]. This second part is devoted to the mathematical analysis of the method. More precisely, on the one hand, we establish the announced optimal orders of convergence for the Galerkin method, and on the other hand, we analyze the parametrices for the weighted layer potentials that underlie the construction of the preconditioners. This involves the introduction of two classes of pseudo-differential operators on open curves with an associated symbolic calculus, transported from the usual pseudo-differential calculus available on

the torus [6]. The way we deal with the singularity of the change of variable is new, to the best of our knowledge. The resulting classes of operator are particularly simple, enjoy all the usual pseudo-differential properties and could therefore present an interest in other contexts.

The outline is as follows. In the first section, we define two scales of interpolating Hilbert spaces, T^s and U^s and analyze their properties. Some of this analysis is applied in the second section to establish the optimal orders of convergence of the Galerkin method. In the third section, we introduce the new classes pseudo-differential operators on open curves, respectively in the scales T^s and U^s . This allows us analyze the parametrices for the weighted layer potentials in the fourth and last section.

Remark 1. *Throughout all this article, we use repeatedly the letter C in estimates of the form $a \leq Cb$. From line to line, the value of the constant C may change but is independent of the relevant parameters defining a and b .*

1 Spaces T^s and U^s

All the analysis of this work takes place in two interpolating scales of Hilbert spaces T^s and U^s that we define now, before establishing some of their properties.

1.1 Definitions

The Chebyshev polynomials of first and second kinds (see e.g. the book [2]) are respectively given by

$$T_n(x) = \cos(n \arccos(x)), \quad U_n(x) = \frac{\sin((n+1) \arccos(x))}{\sqrt{1-x^2}}$$

for $x \in [-1, 1]$ and $n \in \mathbb{N}$. Letting ∂_x the derivation operator and ω the operator $u(x) \mapsto \omega(x)u(x)$ with $\omega(x) = \sqrt{1-x^2}$, T_n and U_n satisfy the following identities:

$$-(\omega \partial_x)^2 T_n = n^2 T_n, \tag{1}$$

$$-(\partial_x \omega)^2 U_n = (n+1)^2 U_n. \tag{2}$$

Notice that here and in the following, $\partial_x \omega$ denotes the composition of operators ∂_x and ω and not the function $x \mapsto \partial_x \omega(x)$. One can also check the identities

$$\partial_x T_n = n U_{n-1}, \tag{3}$$

$$-\omega \partial_x \omega U_n = (n+1) T_{n+1}. \tag{4}$$

The first one is obtained for example from the definition of T_n , from which we deduce the second one after using $-(\omega \partial_x)^2 T_{n+1} = (n+1)^2 T_{n+1}$.

Both T_n and U_n are polynomials of degree n , and provide respectively a basis of the following Hilbert spaces

$$L_{\frac{1}{\omega}}^2 := \left\{ u \in L_{\text{loc}}^1(-1, 1) \mid \int_{-1}^1 \frac{|u(x)|^2}{\sqrt{1-x^2}} dx < +\infty \right\},$$

$$L_\omega^2 := \left\{ u \in L_{\text{loc}}^1(-1, 1) \mid \int_{-1}^1 |u(x)|^2 \sqrt{1-x^2} dx < +\infty \right\}.$$

Following the notations of [3], we denote the Banach duality products of $L_{\frac{1}{\omega}}^2$ and L_ω^2 respectively by $\langle \cdot, \cdot \rangle_{\frac{1}{\omega}}$ and $\langle \cdot, \cdot \rangle_\omega$ and the inner products respectively by $(\cdot, \cdot)_{\frac{1}{\omega}}$ and $(\cdot, \cdot)_\omega$, with the following normalization:

$$(u, v)_{\frac{1}{\omega}} = \langle u, \bar{v} \rangle_{\frac{1}{\omega}} := \frac{1}{\pi} \int_{-1}^1 \frac{u(x) \overline{v(x)}}{\omega(x)} dx,$$

$$(u, v)_\omega = \langle u, \bar{v} \rangle_\omega := \frac{1}{\pi} \int_{-1}^1 u(x) \overline{v(x)} \omega(x) dx.$$

The Chebyshev polynomials satisfy

$$(T_n, T_m)_{\frac{1}{\omega}} = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } m = n = 0, \\ 1/2 & \text{otherwise,} \end{cases} \quad (5)$$

and

$$(U_n, U_m)_\omega = \begin{cases} 0 & \text{if } n \neq m, \\ 1/2 & \text{otherwise,} \end{cases} \quad (6)$$

from which we obtain the so-called Fourier-Chebyshev decomposition: any $u \in L_{\frac{1}{\omega}}^2$ can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x), \quad (7)$$

where the Fourier-Chebyshev coefficients of the first kind are given by

$$\hat{u}_n = \frac{(u, T_n)_{\frac{1}{\omega}}}{(T_n, T_n)_{\frac{1}{\omega}}}$$

and satisfy the Parseval equality

$$\forall (u, v) \in L_{\frac{1}{\omega}}^2 \quad (u, v)_{\frac{1}{\omega}} = \hat{u}_0 \bar{v}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} \hat{u}_n \bar{v}_n.$$

When u is furthermore a smooth function, one can check that the series (7) converges uniformly to u . Similarly, any function $v \in L_\omega^2$ can be decomposed along the $(U_n)_n$ as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the Fourier-Chebyshev coefficients of the second kind \check{v}_n are given by

$$\check{v}_n := \frac{(v, U_n)_\omega}{(U_n, U_n)_\omega}$$

with the Parseval identity

$$(u, v)_\omega = \frac{1}{2} \sum_{n=0}^{+\infty} \check{u}_n \bar{v}_n.$$

The preceding analysis can be used to define Sobolev-like spaces.

Definition 1. We define T^s as the set of (formal) series

$$u = \sum_{n \in \mathbb{N}} \hat{u}_n T_n$$

where the coefficients \hat{u}_n satisfy

$$\sum_{n \in \mathbb{N}} (1 + n^2)^s |\hat{u}_n|^2 < +\infty.$$

Let $T^\infty = \bigcap_{s \geq 0} T^s$ and $T^{-\infty} = \bigcup_{s \in \mathbb{R}} T^s$. For $u \in T^s$ when $s \geq 0$, the series defining u converges in $L^2_{\frac{1}{\omega}}$ and the Fourier-Chebyshev coefficients of the first kind of u coincide with \hat{u}_n , allowing to identify T^s to a subspace of $L^2_{\frac{1}{\omega}}$ with $T^0 = L^2_{\frac{1}{\omega}}$. For all $u \in T^s$, we define the linear form $\langle u, \cdot \rangle_{\frac{1}{\omega}}$ by

$$\forall \varphi \in T^\infty, \langle u, \varphi \rangle_{\frac{1}{\omega}} = \frac{1}{2} \hat{u}_0 \hat{\varphi}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} \hat{u}_n \hat{\varphi}_n. \quad (8)$$

This linear form has a unique continuous extension on T^{-s} , and the dual of T^s is the set of linear forms $\langle u, \cdot \rangle_{\frac{1}{\omega}}$ where $u \in T^{-s}$. Endowed with the scalar product

$$(u, v)_{T^s} := \hat{u}_0 \bar{\hat{v}}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} (1 + n^2)^s \hat{u}_n \bar{\hat{v}}_n,$$

T^s is a Hilbert space for all s . A homogeneous semi-norm on T^s can be defined as

$$|u|_{T^s}^2 := \frac{1}{2} \sum_{n=1}^{+\infty} n^{2s} |\hat{u}_n|^2.$$

Definition 2. In a similar fashion, we define U^s as the set of formal series

$$u = \sum_{n \in \mathbb{N}} \check{u}_n U_n$$

where the coefficients \check{u}_n satisfy

$$\sum_{n \in \mathbb{N}} (1 + n^2)^s |\check{u}_n|^2 < +\infty.$$

Let $U^\infty = \bigcap_{s \in \mathbb{R}} U^s$ and $U^{-\infty} = \bigcup_{s \in \mathbb{R}} U^s$. For $u \in U^s$ when $s \geq 0$, the series defining u converges in L^2_ω and the Fourier-Chebyshev coefficients of the second kind of u coincide with \check{u}_n , allowing to identify U^s to a subspace of L^2_ω with $U^0 = L^2_\omega$. For all $u \in U^s$, we define the linear form $\langle u, \cdot \rangle_\omega$ by

$$\forall \varphi \in U^\infty, \langle u, \varphi \rangle_\omega := \frac{1}{2} \sum_{n=0}^{+\infty} \check{\varphi}_n \check{u}_n. \quad (9)$$

This linear form has a unique continuous extension on U^{-s} , and the dual of U^s may be identified to U^{-s} with respect to the bilinear form $\langle \cdot, \cdot \rangle_\omega$. Endowed with the scalar product

$$(u, v)_{U^s} := \frac{1}{2} \sum_{n \in \mathbb{N}} (1 + (n+1)^2)^s \check{u}_n \bar{\check{v}}_n,$$

U^s is a Hilbert space for all $s \in \mathbb{R}$.

Let $s_1, s_2 \in \mathbb{R}$, $\theta \in (0, 1)$ and let $s = \theta s_1 + (1 - \theta)s_2$. It is easy to check that

$$\forall u \in T^\infty, \|u\|_{T^s} \leq \|u\|_{T^{s_1}}^\theta \|u\|_{T^{s_2}}^{1-\theta}$$

and

$$\forall u \in U^\infty, \|u\|_{U^s} \leq \|u\|_{U^{s_1}}^\theta \|u\|_{U^{s_2}}^{1-\theta}.$$

Therefore, $(T^s)_{s \in \mathbb{R}}$ and $(U^s)_{s \in \mathbb{R}}$ are exact interpolation scales.

1.2 Basic properties

Links with smooth functions and continuous inclusions For any real s , if $u \in T^s$, the sequence of polynomials

$$u_N(x) = \sum_{n=0}^N \hat{u}_n T_n(x)$$

converges to u in T^s . The same assertion holds for $u \in U^s$ when T_n is replaced by U_n . Therefore:

Lemma 1. $C^\infty([-1, 1])$ is dense in T^s and U^s for all $s \in \mathbb{R}$.

The polynomials T_n and U_n are connected by the following formulas:

$$T_0 = U_0, \quad T_1 = \frac{U_1}{2}, \quad \text{and} \quad \forall n \geq 2, \quad T_n = \frac{1}{2}(U_n - U_{n-2}), \quad (10)$$

$$\forall n \in \mathbb{N}, \quad U_{2n} = 2 \sum_{j=0}^n T_{2j} - 1, \quad U_{2n+1} = 2 \sum_{j=0}^n T_{2j+1}. \quad (11)$$

This leads us to introduce the map

$$I : T^\infty \rightarrow U^\infty$$

defined by

$$\widetilde{I}\varphi_0 = \hat{\varphi}_0 - \frac{\hat{\varphi}_2}{2}, \quad \widetilde{I}\varphi_j = \frac{\hat{\varphi}_j - \hat{\varphi}_{j+2}}{2} \text{ for } j \geq 1.$$

I is bijective has the explicit inverse

$$\widehat{I^{-1}\varphi_0} = \sum_{n=0}^{+\infty} \check{\varphi}_{2n}, \quad \widehat{I^{-1}\varphi_j} = 2 \sum_{n=0}^{+\infty} \check{\varphi}_{j+2n} \text{ for } j \geq 1.$$

Lemma 2. For all real s , I has a unique continuous extension from T^s to U^s and for $s > \frac{1}{2}$, I^{-1} has a continuous extension from U^s to T^{s-1} .

Before starting the proof, we introduce the Cesàro operator C defined on $l^2(\mathbb{N}^*)$ by

$$(Cu)_n = \frac{1}{n} \sum_{k=1}^n u_k.$$

As is well-known, this is a linear continuous operator on the Hilbert space $l^2(\mathbb{N}^*)$. Its adjoint

$$(C^*u)_n = \sum_{k=n}^{+\infty} \frac{u_k}{k},$$

is therefore also continuous on $l^2(\mathbb{N}^*)$. In other words, for all $(u_n)_n \in l^2(\mathbb{N})$,

$$\sum_{n=1}^{+\infty} \left(\sum_{k=n}^{+\infty} \frac{u_k}{k} \right)^2 \leq C \sum_{k=1}^{+\infty} u_k^2.$$

Proof. The first result is immediate from the definition of T^s , U^s and I . When $u \in U^s$ for $s > 1/2$, the series $\sum |\check{u}_n|$ is converging thus $I^{-1}u$ is well defined. By definition, $u \in U^s$ means that the sequence $((1+n^2)^{s/2} |\check{u}_n|)_{n \geq 1}$ is in $l^2(\mathbb{N}^*)$. Thus, using the continuity of the adjoint of the Cesàro operator mentioned previously, the sequence $(r_n)_n$ defined by

$$\forall n \geq 0, \quad r_n := \sum_{k=n}^{+\infty} (1+k^2)^{\frac{s-1}{2}} |\check{u}_k|$$

is in $l^2(\mathbb{N})$ with a l^2 norm bounded by $\|u\|_{U^s}$. We now write

$$\begin{aligned} \|I^{-1}u\|_{T^{s-1}}^2 &= \sum_{n=0}^{+\infty} (1+n^2)^{s-1} \left| \widehat{I^{-1}u}_n \right|^2 \\ &\leq 4 \sum_{n=0}^{+\infty} (1+n^2)^{s-1} \left(\sum_{k=n}^{+\infty} |\check{u}_k| \right)^2 \\ &\leq 4 \sum_{n=0}^{+\infty} \left(\sum_{k=n}^{+\infty} (1+k^2)^{\frac{s-1}{2}} |\check{u}_k| \right)^2 \\ &= 4 \|(r_n)_n\|_{l^2}^2. \end{aligned}$$

We saw that the last quantity is controlled by $\|u\|_{U^s}^2$ so the result is proved. \square

Let

$$u = \sum_{n \in \mathbb{N}} \hat{u}_n T_n, \quad v = \sum_{n \in \mathbb{N}} \check{v}_n U_n.$$

When $Iu = v$, we identify u and v . The previous results have shown that this identification is compatible with the equality of functions in $L^2_{\frac{1}{\omega}}$ or L^2_{ω} . The mapping I is then the identity mapping and its properties just established can be rephrased as follows.

Corollary 1. *For all $s \in \mathbb{R}$, $T^s \subset U^s$ and for all $s > \frac{1}{2}$, $U^s \subset T^{s-1}$ with continuous inclusions.*

One immediate consequence of the previous result is that $T^\infty = U^\infty$. Moreover, there holds

Lemma 3.

$$T^\infty = C^\infty([-1, 1]).$$

Proof. If $u \in C^\infty([-1, 1])$, then we can obtain by induction using integration by parts and (1), that for any $k \in \mathbb{N}$

$$\hat{u}_n = \frac{(-1)^k}{n^{2k}} \int_{-1}^1 \frac{(\omega \partial_x)^{2k} u(x) T_n(x)}{\omega(x)} dx.$$

Noting that $(\omega \partial_x)^2 = (1 - x^2) \partial_x^2 - x \partial_x$, the function $(\omega \partial_x)^{2k} u$ is C^∞ , and since $\|T_n\|_\infty = 1$, the integral is bounded independently of n . Thus, the coefficients \hat{u}_n have a fast decay, proving that $C^\infty([-1, 1]) \subset T^\infty$.

For the converse inclusion, if $u \in T^\infty$, the series

$$u(x) = \sum_{n=0}^{\infty} \hat{u}_n T_n(x)$$

is normally converging since $\|T_n\|_\infty = 1$, so that u is a continuous function. This proves $T^\infty \subset C^0([-1, 1])$. It now suffices to show that $\partial_x u \in T^\infty$ and apply an induction argument. Applying term by term differentiation, since $\partial_x T_n = n U_{n-1}$ for all n (with the convention $U_{-1} = 0$),

$$\partial_x u(x) = \sum_{n=1}^{+\infty} n \hat{u}_n U_{n-1}(x).$$

Therefore, $\partial_x u$ is in $U^\infty = T^\infty$ which proves the result. \square

The next result shows that nothing that can be done about the case $s \leq \frac{1}{2}$:

Lemma 4. *For $s \leq \frac{1}{2}$, the functions of U^s cannot be identified to functions in $T^{-\infty}$.*

Proof. Let $s \leq \frac{1}{2}$, and let us assume by contradiction that the functions of U^s can be identified to elements of $T^{-\infty}$. Then, there must exist a continuous map I from U^s to $T^{-\infty}$ with the property

$$\forall u \in C^\infty([-1, 1]), \quad Iu = u.$$

We introduce the function u defined by $\check{u}_n = \frac{1}{n \ln(n)}$. One can check that $u \in U^{\frac{1}{2}} \subset U^s$, thus Iu must be element of $T^{-\infty}$. For all N , the function

$$u_N = \sum_{n=0}^N \check{u}_n U_n$$

is in U^∞ and $(u_N)_{N \in \mathbb{N}}$ converges to u in U^s . By continuity of I , the sequence $(\langle Iu_N, T_0 \rangle_{\frac{1}{\omega}})_{N \in \mathbb{N}}$ must converge with limit $\langle Iu, T_0 \rangle_{\frac{1}{\omega}}$. But since $Iu_N = u_N$,

$$\langle Iu_N, T_0 \rangle_{\frac{1}{\omega}} = \langle u_N, T_0 \rangle_{\frac{1}{\omega}} = \sum_{n=0}^N \check{u}_n \langle U_n, T_0 \rangle_{\frac{1}{\omega}} = \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2k \ln(2k)},$$

where, in the last equality, we have used the identity

$$\langle U_n, T_0 \rangle_{\frac{1}{\omega}} = \begin{cases} 1 & \text{if } n \text{ is even ,} \\ 0 & \text{otherwise .} \end{cases}$$

which can be checked for example using eq. (11). The last sum diverges to $+\infty$ when N goes to infinity, giving the contradiction. \square

Lemma 5. *For all $\varepsilon > 0$, if $u \in T^{\frac{1}{2}+\varepsilon}$, then u is continuous and*

$$\exists C : \forall x \in [-1, 1], \quad |u(x)| \leq C \|u\|_{T^{1/2+\varepsilon}} .$$

Similarly, if $u \in U^{3/2+\varepsilon}$, then u is continuous and

$$\exists C : \forall x \in [-1, 1], \quad |u(x)| \leq C \|u\|_{U^{3/2+\varepsilon}} .$$

Proof. Let $x \in [-1, 1]$. Using triangular inequality,

$$|u(x)| \leq \sum_{n=0}^{+\infty} |\hat{u}_n|$$

since for all n , $\|T_n\|_{L^\infty} = 1$. Applying Cauchy-Schwarz's inequality, one gets

$$|u(x)| \leq \sqrt{\sum_{n=0}^{+\infty} \frac{1}{(1+n^2)^{\frac{1}{2}+\varepsilon}}} \|u\|_{T^{\frac{1}{2}+\varepsilon}} .$$

The second statement is deduced from the first and the continuous inclusion $U^s \subset T^{s-1}$ stated in Corollary 1. \square

Derivation operators We now extend the definition of the derivation operators ∂_x and $\omega\partial_x\omega$ appearing in eqs (3) and (4).

Lemma 6. *For all real s , the operator ∂_x can be extended into a continuous map from T^{s+1} to U^s defined by*

$$\forall v \in C^\infty([-1, 1]), \quad \langle \partial_x u, v \rangle_{\omega} := - \langle u, \omega \partial_x \omega v \rangle_{\frac{1}{\omega}} .$$

In a similar fashion, the operator $\omega\partial_x\omega$ can be extended into a continuous map from U^{s+1} to T^s defined by

$$\forall v \in C^\infty([-1, 1]), \quad \langle \omega \partial_x \omega u, v \rangle_{\frac{1}{\omega}} := - \langle u, \partial_x v \rangle_{\omega} .$$

Proof. Using eqs (3) and (4), one can check that the formulas indeed extend the usual definition of both operators for smooth functions. We now show that the map ∂_x extended this way is continuous from T^{s+1} to U^s . The definition

$$\forall v \in U^\infty, \quad \langle \partial_x u, v \rangle_{\omega} := - \langle u, \omega \partial_x \omega v \rangle_{\frac{1}{\omega}}$$

gives a sense to $\partial_x u$ for all u in $T^{-\infty}$, as a duality $T^{-\infty} \times T^\infty$ product, because if $v \in U^\infty (= C^\infty([-1, 1]))$, then $\omega \partial_x \omega v = (1-x^2)v' - xv$ also lies in $C^\infty([-1, 1]) (= T^\infty)$. Letting $w = \partial_x u$, we have by definition for all n

$$\check{w}_n = \langle w, U_n \rangle_{\omega} = - \langle u, \omega \partial_x \omega U_n \rangle_{\frac{1}{\omega}} = n \langle u, T_{n+1} \rangle_{\frac{1}{\omega}} = n \hat{u}_{n+1} .$$

This implies the announced continuity with

$$\|w\|_{U^s} \leq \|u\|_{T^{s+1}} .$$

The properties of $\omega\partial_x\omega$ on T^s are established similarly. \square

Corollary 2. *The operator ∂_x is continuous from T^{s+2} to T^s for all $s > -1/2$ and from U^{s+2} to U^s for all $s > -3/2$. On the other hand, $\omega\partial_x\omega$ is continuous from T^{s+1} to T^s and from U^{s+1} to U^s for all $s \in \mathbb{R}$.*

Proof. For the continuity of ∂_x from T^{s+2} to T^s , we use the continuity of ∂_x from T^{s+2} to U^{s+1} and then of the identity from U^{s+1} to T^s . For the continuity of ∂_x from U^{s+2} to U^s , we use the same arguments in reverse order.

On the other hand, we have, for $n \geq 2$,

$$\omega\partial_x\omega T_n = \omega\partial_x\omega \frac{U_n - U_{n-2}}{2} = \frac{(n+1)T_{n+1} - (n-1)T_{n-1}}{2} .$$

Therefore $\omega\partial_x\omega$ is continuous from T^{s+1} to T^s . Finally, $\omega\partial_x\omega$ is continuous from U^{s+1} to T^s and the inclusion $T^s \subset U^s$ is continuous thus $\omega\partial_x\omega$ is continuous from U^{s+1} to U^s . \square

1.3 Equivalent norms on T^n and U^n

We now provide a characterization of the spaces T^n and U^n in terms of weighted L^2 norms of the derivatives and give equivalent norms on those spaces when n is an integer. This is done by studying the properties of the operators $(\omega\partial_x)^n$ and $(\partial_x\omega)^n$. The main results are propositions 1 and 2.

Lemma 7. *The operator ω is a bijective isometry from U^0 to T^0 with inverse $\frac{1}{\omega}$.*

Proof. This result follows from

$$\|\omega u\|_{\frac{1}{\omega}}^2 = \frac{1}{\pi} \int_{-1}^1 \frac{|(\omega u)|^2}{\omega} = \frac{1}{\pi} \int_{-1}^1 \omega |u|^2 = \|u\|_{\omega}^2 ,$$

valid for all $u \in L_{\omega}^2$. \square

Combining this with the continuity of ∂_x from T^1 to U^0 and of $\omega\partial_x\omega$ from U^1 to T^0 (see Lemma 6) we obtain:

Corollary 3. *The operators $\omega\partial_x : T^1 \rightarrow T^0$ and $\partial_x\omega : U^1 \rightarrow U^0$ are continuous.*

Note however that those two operators do not send smooth functions to smooth functions so we cannot expect a continuity of the form

$$\forall s \in \mathbb{R}, \quad \omega\partial_x : T^s \rightarrow T^{s-1} .$$

For this reason, in the analysis of $(\omega\partial_x)^n$ and $(\partial_x\omega)^n$, we must treat separately the cases of an even or odd integer n .

Definition 3. For an even integer n , the operator $(\omega\partial_x)^n : T^{-\infty} \rightarrow T^{-\infty}$ is defined by

$$(\omega\partial_x)^0 = I_d, \quad \forall k > 0, \quad (\omega\partial_x)^{2k} := (\omega\partial_x\omega)\partial_x(\omega\partial_x)^{2k-2}.$$

The operator $(\partial_x\omega)^n : U^{-\infty} \rightarrow U^{-\infty}$ is defined in an analogous way.

Lemma 8. Let n an even integer. For all $s \in \mathbb{R}$, $(\omega\partial_x)^n$ is continuous from T^s to T^{s-n} and $(\partial_x\omega)^n$ is continuous from U^s to U^{s-n} .

Proof. Those results follow from the definition of the operators and by induction using the mapping properties of ∂_x and $\omega\partial_x\omega$ established in Lemma 6. \square

Definition 4. For an odd integer n , the operator $(\omega\partial_x)^n : T^n \rightarrow T^0$ is defined by

$$(\omega\partial_x)^n := \omega\partial_x(\omega\partial_x)^{n-1}.$$

The operator $(\partial_x\omega)^n : U^n \rightarrow U^0$ is defined in an analogous way.

Using Corollary 3, one can show:

Corollary 4. The operators $(\omega\partial_x)^n$ and $(\partial_x\omega)^n$ are well defined and continuous respectively from T^n to T^0 and from U^n to U^0 .

Proposition 1. Let $n \in \mathbb{N}$. If n is even,

$$T^n = \left\{ u \in L^2_{\frac{1}{\omega}} \mid (\omega\partial_x)^n u \in L^2_{\frac{1}{\omega}} \right\}.$$

If n is odd,

$$T^n = \left\{ u \in L^2_{\frac{1}{\omega}} \mid \partial_x(\omega\partial_x)^{n-1} u \in L^2_{\omega} \right\}.$$

Moreover $u \mapsto \sqrt{\|u\|_{\frac{1}{\omega}}^2 + \|(\omega\partial_x)^n u\|_{\frac{1}{\omega}}^2}$ defines an equivalent norm on T^n , and for all $u \in T^n$,

$$|u|_{T^n} = \|(\omega\partial_x)^n u\|_{L^2_{\frac{1}{\omega}}}.$$

Proof. The direct inclusions follow from the mapping properties established in Lemma 6, Lemma 8 and Corollary 4. For the converse inclusions, let u in $L^2_{\frac{1}{\omega}}$. If n is even, say $n = 2k$, the assumption is that $(\omega\partial_x)^n u \in L^2_{\frac{1}{\omega}}$. The Fourier-Chebyshev coefficients of $a = (\omega\partial_x)^n u$ are given for $j > 0$ by

$$\hat{a}_j = \frac{((\omega\partial_x)^{2k} u(x), T_j)_{\frac{1}{\omega}}}{(T_n, T_n)_{\frac{1}{\omega}}} = \frac{(u(x), (\omega\partial_x)^{2k} T_j)_{\frac{1}{\omega}}}{(T_n, T_n)_{\frac{1}{\omega}}} = (-1)^k j^{2k} \hat{u}_j.$$

while for $j = 0$, $\hat{a}_j = 0$. Applying Parseval's equality to the function a , this gives

$$\frac{1}{2} \sum_{j>0} j^{2n} |\hat{u}_j|^2 = \|(\omega\partial_x)^n u\|_{\frac{1}{\omega}}^2. \quad (12)$$

On the other hand, if n is odd, say $n = 2k + 1$, let $b := \partial_x(\omega\partial_x)^{2k} u$. The assumption is now that $b \in L^2_{\omega}$, and by Lemma 7, $\omega b (= (\omega\partial_x)^n u) \in T^0$ with

$$\|\omega b\|_{\frac{1}{\omega}} = \|(\omega\partial_x)^n u\|_{\frac{1}{\omega}} = \|b\|_{\omega}.$$

One can write

$$\check{b}_j = 2 (\partial_x (\omega \partial_x)^{2k} u, U_j) = -2 (u, (\omega \partial_x)^{2k} (\omega \partial_x \omega) U_j) .$$

Using $-\omega \partial_x \omega U_j = (j+1)T_{j+1}$, we obtain

$$\check{b}_j = (-1)^k (j+1)^{2k+1} \hat{u}_{j+1} .$$

Parseval's equality then implies that (12) also holds for odd n . This establishes that $u \in T^n$ and $|u|_{T^n} = \|(\omega \partial_x)^n u\|_{\frac{1}{\omega}}$. For the norm equivalence, adding the Parseval equality for $u \in L_{\frac{1}{\omega}}^2$ to (12), we get

$$|\hat{u}_0|^2 + \frac{1}{2} \sum_{j>0} (1+j^{2n}) |\hat{u}_j|^2 = \|u\|_{\frac{1}{\omega}}^2 + \|(\omega \partial_x)^n u\|_{\frac{1}{\omega}}^2 . \quad (13)$$

There are two constants c and C such that $c(1+j^2)^n \leq (1+j^{2n}) \leq C(1+j^2)^n$. Injecting this in (13), we obtain

$$\frac{c}{2} \|u\|_{T^n}^2 \leq \|u\|_{\frac{1}{\omega}}^2 + \|(\omega \partial_x)^n u\|_{\frac{1}{\omega}}^2 \leq C \|u\|_{T^n}^2 ,$$

and the equivalence of the norms follows. \square

Proposition 2. *Let $n \in \mathbb{N}$. If n is even, then*

$$U^n = \{u \in L_{\omega}^2 \mid (\partial_x \omega)^n u \in L_{\omega}^2\} .$$

If n is odd, then

$$U^n = \left\{ u \in L_{\omega}^2 \mid \omega \partial_x \omega (\partial_x \omega)^{n-1} u \in L_{\frac{1}{\omega}}^2 \right\} .$$

Moreover, $u \mapsto \sqrt{\int_{-1}^1 \omega |(\partial_x \omega)^n u|^2}$ defines an equivalent norm on U^n .

Proof. The direct inclusions follow from the mapping properties established in Lemma 6, Lemma 8 and Corollary 4. For the converse inclusion, if n is even, let $a = (\partial_x \omega)^n u$, we assume that $a \in L_{\omega}^2$. One has

$$\check{a}_j = (-1)^k (1+j)^n \check{u}_j ,$$

so, by Parseval's equality,

$$\frac{1}{2} \sum_{j=0}^{+\infty} (j+1)^{2n} |\check{u}_j|^2 = \|(\partial_x \omega)^n u\|_{\omega}^2 . \quad (14)$$

If n is odd, the assumption is that $b = \omega \partial_x \omega (\partial_x \omega)^{n-1} u$ is in $L_{\frac{1}{\omega}}^2$. By calculations similar to those in the proof of the preceding lemma, we find that for $j > 0$,

$$\hat{b}_j = j^{2n} \check{u}_{j-1} .$$

while $\hat{b}_0 = 0$. By Lemma 7, $\frac{b}{\omega} (= (\partial_x \omega)^n u) \in U^0$. Applying Parseval's equality to b in $L_{\frac{1}{\omega}}^2$ and using $\|\frac{b}{\omega}\|_{\omega} = \|b\|_{\frac{1}{\omega}}$, we find that (14) also holds when n is odd, and thus the inclusion is proved. Finally, there exists two constants c and C such that for all $j \in \mathbb{N}$,

$$c(1+(j+1))^{2n} \leq (j+1)^{2n} \leq C(1+(j+1))^{2n} .$$

This implies the equivalence of the norms. \square

1.4 Link with Periodic Sobolev spaces

In this paragraph, we show how the spaces T^s and U^s are related to Sobolev spaces of periodic even and odd functions respectively through the change of variables $x = \cos \theta$. The main result is Proposition 3.

We briefly recall here the definition of the periodic Sobolev spaces on the torus $\mathbb{T}_{2\pi} := \mathbb{R}/2\pi\mathbb{Z}$. A smooth function u on $\mathbb{T}_{2\pi}$ can be decomposed in Fourier series

$$u(\theta) = \sum_{n \in \mathbb{Z}} \mathcal{F}u(n) e^{in\theta}$$

with the Fourier coefficients defined by

$$\mathcal{F}u(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\theta) e^{-in\theta} d\theta.$$

For $n \in \mathbb{Z}$, let $e_n : \theta \mapsto e^{in\theta}$. We define the Fourier coefficients of any periodic distribution u on $\mathbb{T}_{2\pi}$, by $\mathcal{F}u(n) := \langle u, e_{-n} \rangle$. For all s , the space H^s is the set of periodic distributions on $\mathbb{T}_{2\pi}$ for which

$$\|u\|_{H^s}^2 := \sum_{n \in \mathbb{Z}} (1 + n^2)^s |\mathcal{F}u(n)|^2 < +\infty.$$

Introducing the duality product

$$\langle u, v \rangle_{\mathbb{T}_{2\pi}} = \sum_{n \in \mathbb{Z}} \mathcal{F}u(n) \mathcal{F}v(-n), \quad (15)$$

H^{-s} is identified to the dual of H^s and $H^0 = L^2(\mathbb{T}_{2\pi})$. For $u, v \in H^0$,

$$\langle u, v \rangle_{\mathbb{T}_{2\pi}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)v(x) dx.$$

The space H^s is the direct sum $H_e^s \oplus H_o^s$ where

$$H_e^s := \{u \in H^s \mid \mathcal{F}u(n) = \mathcal{F}u(-n)\},$$

$$H_o^s := \{u \in H^s \mid \mathcal{F}u(n) = -\mathcal{F}u(-n)\}.$$

Note that when u is continuous,

$$u \in H_e^s \iff \forall \theta \in \mathbb{T}_{2\pi}, \quad u(-\theta) = u(\theta),$$

$$u \in H_o^s \iff \forall \theta \in \mathbb{T}_{2\pi}, \quad u(-\theta) = -u(\theta).$$

Definition 5. We define the operators $\mathcal{C} : T^{-\infty} \rightarrow H_e^{-\infty}$ by

$$\forall n \in \mathbb{Z}, \quad \mathcal{F}(\mathcal{C}u)(n) = \begin{cases} \hat{u}_0 & \text{if } n = 0, \\ \frac{\hat{u}_{|n|}}{2} & \text{otherwise,} \end{cases}$$

and $\mathcal{S} : U^{-\infty} \rightarrow H_o^{-\infty}$ by

$$\forall n \in \mathbb{Z}, \quad \mathcal{F}(\mathcal{S}u)(n) = \begin{cases} 0 & \text{if } n = 0, \\ \text{sign}(n) \frac{\hat{u}_{|n|-1}}{2} & \text{otherwise.} \end{cases}$$

Lemma 9. *The operators \mathcal{C} and \mathcal{S} map smooth functions to smooth functions. For all $(u, v) \in T^{-\infty} \times T^\infty$,*

$$\langle u, v \rangle_{\frac{1}{\omega}} = \langle \mathcal{C}u, \mathcal{C}v \rangle_{\mathbb{T}_{2\pi}} .$$

For all $(u, v) \in U^{-\infty} \times U^\infty$,

$$\langle u, v \rangle_{\omega} = \langle \mathcal{S}u, \mathcal{S}v \rangle_{\mathbb{T}_{2\pi}} .$$

Proof. The first assertion is obvious from the definition of \mathcal{C} and \mathcal{S} . Let $(u, v) \in T^{-\infty} \times T^\infty$. By definition of $\langle \cdot, \cdot \rangle_{\frac{1}{\omega}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{T}_{2\pi}}$ eqs. (8) and (15),

$$\begin{aligned} \langle \mathcal{C}u, \mathcal{C}v \rangle_{\mathbb{T}_{2\pi}} &= \sum_{n \in \mathbb{Z}} \mathcal{F}(\mathcal{C}u)(n) \mathcal{F}(\mathcal{C}v)(-n) \\ &= \hat{u}_0 \hat{v}_0 + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{\hat{u}_{|n|}}{2} \frac{\hat{v}_{|n|}}{2} \\ &= \hat{u}_0 \hat{v}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} \hat{u}_n \hat{v}_n \\ &= \langle u, v \rangle_{\frac{1}{\omega}} . \end{aligned}$$

The second identity is proved similarly. \square

Proposition 3. *For all $s \in \mathbb{R}$, the operators \mathcal{C} and \mathcal{S} induce bijective isometries respectively from T^s to H_e^s and from U^s to H_o^s . Additionally, for $u \in T^s$,*

$$|u|_{T^s} = |\mathcal{C}u|_{H^s} . \quad (16)$$

For $u \in C^\infty([-1, 1])$,

$$\mathcal{C}u(\theta) = u(\cos \theta) \quad \text{and} \quad \mathcal{S}u(\theta) = \sin \theta u(\cos \theta) . \quad (17)$$

Let $v_e, v_o \in C^\infty(\mathbb{T}_{2\pi})$, an even and an odd function respectively. Then

$$\mathcal{C}^{-1}v_e(x) = v_e(\arccos x) \quad \text{and} \quad \mathcal{S}^{-1}v_o(x) = \frac{v_o(\arccos x)}{\omega(x)} . \quad (18)$$

Proof. Let J_s^T, J_s^U and J_s^H the linear continuous mappings defined respectively on $T^{-\infty}, U^{-\infty}$ and $H^{-\infty}$ by

$$J_s^T T_n = (1 + n^2)^{\frac{s}{2}} T_n, \quad J_s^U U_{n-1} = (1 + n^2)^{\frac{s}{2}} U_{n-1}, \quad J_s^H e_n = (1 + n^2)^{\frac{s}{2}} e_n .$$

By definition, for $u \in T^s$ and $v \in U^s$,

$$\|u\|_{T^s}^2 = \left\langle J_s^T u, \overline{J_s^T u} \right\rangle_{\frac{1}{\omega}} \quad \text{and} \quad \|v\|_{U^s}^2 = \left\langle J_s^U v, \overline{J_s^U v} \right\rangle_{\omega} ,$$

while for $w \in H^s$,

$$\|w\|_{H^s}^2 = \left\langle J_s^H w, \overline{J_s^H w} \right\rangle_{\mathbb{T}_{2\pi}} .$$

Moreover, notice that

$$\mathcal{C}J_s^T = J_s^H \mathcal{C}, \quad \mathcal{S}J_s^U = J_s^H \mathcal{S} .$$

The isometric property of \mathcal{C} may now be deduced from Lemma 9 as follows. Let

$$u_N = \sum_{n=0}^N u_n T_n. \text{ There holds}$$

$$\begin{aligned} \left\langle J_s^T u, \overline{J_s^T u_N} \right\rangle_{\frac{1}{\omega}} &= \left\langle \mathcal{C} J_s^T u, \overline{\mathcal{C} J_s^T u_N} \right\rangle_{\mathbb{T}_{2\pi}} \\ &= \left\langle J_s^H \mathcal{C} u, \overline{J_s^H \mathcal{C} u_N} \right\rangle_{\mathbb{T}_{2\pi}}. \end{aligned}$$

Sending N to infinity, by continuity of J_s^T , J_s^H and \mathcal{C} , this yields

$$\|u\|_{T^s}^2 = \|\mathcal{C}u\|_{H^s}^2.$$

The property (16) and the isometric property of \mathcal{S} are established in a similar manner. Let us now prove eq. (17). For the first identity, consider some smooth function u on $[-1, 1]$. Since $\mathcal{C}u$ is smooth, the Fourier series of $\mathcal{C}u$ converges pointwise to $\mathcal{C}u$. Thus, for all $\theta \in \mathbb{T}_{2\pi}$,

$$\begin{aligned} \mathcal{C}u(\theta) &= \sum_{n \in \mathbb{Z}} \mathcal{F}(\mathcal{C}u)(n) e^{in\theta} \\ &= \hat{u}_0 + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{\hat{u}_{|n|}}{2} e^{in\theta} \\ &= \hat{u}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} \hat{u}_n (e^{in\theta} + e^{-in\theta}) \\ &= \sum_{n=0}^{+\infty} \hat{u}_n T_n(\cos \theta). \end{aligned}$$

The last sum also converges pointwise to $u(\cos \theta)$ since $u \in T^\infty$. Similar calculations show that $\mathcal{S}u(\theta) = \sin \theta u(\cos \theta)$. To prove the bijectivity of \mathcal{S} and \mathcal{C} , one can check that they have the explicit inverses \mathcal{C}^{-1} and \mathcal{S}^{-1} respectively defined on H_e^s and H_o^s by

$$\forall n \in \mathbb{N}, \quad (\widehat{\mathcal{C}^{-1}u})_n = \begin{cases} \mathcal{F}u(0) & \text{if } n = 0, \\ 2\mathcal{F}u(n) & \text{otherwise,} \end{cases}$$

and

$$\forall n \in \mathbb{N}, \quad (\widehat{\mathcal{S}^{-1}u})_n = 2\mathcal{F}u(n+1).$$

Finally, identities (18) are simply deduced by inverting (17). \square

1.5 Generalization to a curve

All of the previous analysis can be generalized to define two families of spaces $T^s(\Gamma)$ and $U^s(\Gamma)$ of functions defined on a smooth curve Γ by means of a C^∞ diffeomorphism.

Parametrization of the curve

We now introduce some notation that will be used at several points in the remainder of this work. Let Γ a smooth open curve in \mathbb{R}^2 parametrized by a C^∞ diffeomorphism $r : [-1, 1] \rightarrow \Gamma$. We assume that $|r'(x)| = \frac{|\Gamma|}{2}$ for all $x \in [-1, 1]$, where $|\Gamma|$ is the length of Γ . This parametrization is related to the curvilinear abscissa $M(s)$ through

$$r(x) = M\left(\frac{|\Gamma|}{2}(1+x)\right).$$

Let $R : C^\infty(\bar{\Gamma}) \rightarrow C^\infty([-1, 1])$ defined by

$$Ru(x) = u(r(x)).$$

The tangent and normal vectors on the curve, τ and n , are respectively defined by

$$\tau(x) = \frac{\partial_x r(x)}{|\partial_x r(x)|}, \quad n(x) = \frac{\partial_x \tau(x)}{|\partial_x \tau(x)|}.$$

Let $N : \Gamma \rightarrow \mathbb{R}^2$ such that $N(r(x)) = n(x)$, that is, $N = R^{-1}n$. Let $\kappa(x)$ the signed curvature of Γ at the point $r(x)$. Frenet-Serret's formulas give

$$\begin{aligned} r(y) &= r(x) + (y-x)\frac{|\Gamma|}{2}\tau(x) + \frac{(y-x)^2}{2}\frac{|\Gamma|^2}{4}\kappa(x)n(x) \\ &\quad + \frac{(x-y)^3}{6}\frac{|\Gamma|^3}{8}(\kappa'(x)n(x) - \kappa(x)^2\tau(x)) + O((x-y)^4), \end{aligned}$$

so that

$$|r(x) - r(y)|^2 = \frac{|\Gamma|^2}{4}(y-x)^2 - \frac{(y-x)^4}{192}|\Gamma|^4\kappa(x)^2 + O(x-y)^5. \quad (19)$$

For $u, v \in L^2(\Gamma)$, we have by change of variables in the integral

$$\langle u, v \rangle_{L^2(\Gamma)} = \frac{|\Gamma|}{2} \langle Ru, Rv \rangle_{L^2(-1,1)}.$$

The tangential derivative ∂_τ on Γ satisfies

$$\partial_\tau = \frac{2}{|\Gamma|} R^{-1} \partial_x R. \quad (20)$$

We also define a "weight" on the curve as

$$\omega_\Gamma := \frac{|\Gamma|}{2} R^{-1} \omega R. \quad (21)$$

Finally, the uniform measure on Γ is denoted by $d\sigma$.

Spaces $T^s(\Gamma)$ and $U^s(\Gamma)$

The definition of the spaces T^s can be transported on the curve Γ , replacing the basis $(T_n)_n$ and $(U_n)_n$ by $(R^{-1}T_n)_n$ and $(R^{-1}U_n)_n$. The spaces $T^s(\Gamma)$ and $U^s(\Gamma)$ are thus defined as the sets of formal series respectively of the form

$$u = \sum_{n \in \mathbb{N}} \hat{u}_n R^{-1} T_n, \quad v = \sum_{n \in \mathbb{N}} \check{v}_n R^{-1} U_n,$$

where $Ru = \sum \hat{u}_n T_n \in T^s$ and $Rv = \sum \check{v}_n U_n \in U^s$. To u and v are associated the linear forms

$$\forall \varphi \in C^\infty(\bar{\Gamma}), \quad \langle u, \varphi \rangle_{\frac{1}{\omega_\Gamma}} := \langle Ru, R\varphi \rangle_{\frac{1}{\omega}} ,$$

$$\forall \varphi \in C^\infty(\bar{\Gamma}), \quad \langle v, \varphi \rangle_{\omega_\Gamma} := \frac{|\Gamma|^2}{4} \langle Rv, R\varphi \rangle_{\omega} .$$

The results of the previous section are easily extended to this new setting:

Lemma 10. *For all $s \in \mathbb{R}$, $T^s(\Gamma)$ and $U^s(\Gamma)$ are Hilbert spaces for the scalar products*

$$(u, v)_{T^s(\Gamma)} = (Ru, Rv)_{T^s} ,$$

$$(u, v)_{U^s(\Gamma)} = \frac{|\Gamma|^2}{2} (Ru, Rv)_{U^s} .$$

With these definitions,

$$(u, v)_{T^0(\Gamma)} = \langle u, \bar{v} \rangle_{\frac{1}{\omega_\Gamma}} = \int_{\Gamma} \frac{u(x)\overline{v(x)}}{\omega_\Gamma(x)} dx ,$$

$$(u, v)_{U^0(\Gamma)} = \langle u, \bar{v} \rangle_{\omega_\Gamma} = \int_{\Gamma} \omega_\Gamma(x) u(x) \overline{v(x)} dx .$$

In particular $T^0(\Gamma) = L^2_{\frac{1}{\omega_\Gamma}}$ and $U^0(\Gamma) = L^2_{\omega_\Gamma}$. For $s \in \mathbb{R}$, the dual of $T^s(\Gamma)$ is the set of linear forms $\langle u, \cdot \rangle_{\frac{1}{\omega_\Gamma}}$ where $u \in T^{-s}(\Gamma)$, and the dual of $U^s(\Gamma)$ is the set of linear forms $\langle u, \cdot \rangle_{\omega_\Gamma}$ where $u \in U^{-s}(\Gamma)$. For $s < t$, the injections $T^t(\Gamma) \subset T^s(\Gamma)$ and $U^t(\Gamma) \subset U^s(\Gamma)$ are compact. $(T^s(\Gamma))_{s \in \mathbb{R}}$ and $(U^s(\Gamma))_{s \in \mathbb{R}}$ are two Hilbert interpolation scales. For an integer n , equivalent scalar products on $T^n(\Gamma)$ and $U^n(\Gamma)$ are given respectively by

$$(u, v) \mapsto \int_{\Gamma} \frac{u(x)\overline{v(x)} + (\omega_\Gamma \partial_\tau)^n u(x) (\omega_\Gamma \partial_\tau)^n \overline{v(x)}}{\omega_\Gamma(x)} d\sigma(x) ,$$

$$(u, v) \mapsto \int_{\Gamma} (\partial_\tau \omega_\Gamma)^n u(x) (\partial_\tau \omega_\Gamma)^n \overline{v(x)} \omega_\Gamma(x) d\sigma(x) .$$

For all $s \in \mathbb{R}$, $T^s(\Gamma) \subset U^s(\Gamma)$ and for all $s > \frac{1}{2}$, $U^s(\Gamma) \subset T^{s-1}(\Gamma)$ with continuous inclusions. For $\varepsilon > 0$, $T^{1/2+\varepsilon}(\Gamma) \subset C^0(\Gamma)$ and $U^{3/2+\varepsilon} \subset C^0(\Gamma)$ with continuous inclusions. Finally, $T^\infty(\Gamma) = U^\infty(\Gamma) = C^\infty(\bar{\Gamma})$.

2 Weighted Galerkin method for the problem of scattering by open curves

In this section, we present the first-kind integral equations arising in the study of wave scattering by open curves with Dirichlet or Neumann conditions and introduce the weighted layer potentials. We then apply the theory of the first section to the analysis of the Galerkin method described in [1] in the case of a zero wavenumber and flat geometry (this can be extended to the general case by standard perturbation arguments).

2.1 First-kind integral equations

Recall the definition and parametrization of the curve Γ detailed in section 1.5. We consider the following boundary integral equations (BIEs)

$$S_k \lambda = u_D, \quad N_k \mu = u_N, \quad (22)$$

where S_k and N_k are respectively the single-layer and hypersingular operators. We refer the reader to [1] and references therein for more details on the connection between eqs. (22) and the problem of wave scattering by the curve Γ . The operators S_k and N_k admit the integral representations

$$\begin{aligned} (S_k \lambda)(x) &= \int_{\Gamma} G_k(x-y) \lambda(y) d\sigma(y), \\ (N_k \mu)(x) &= \lim_{\varepsilon \rightarrow 0^+} \int_{\Gamma} N(y) \cdot \nabla G_k(x + \varepsilon N(x) - y) \mu(y) d\sigma_y. \end{aligned} \quad (23)$$

for $x \in \Gamma$, with the Green function G_k defined by

$$\begin{cases} G_0(z) = -\frac{1}{2\pi} \ln |z|, & \text{if } k = 0, \\ G_k(z) = \frac{i}{4} H_0(k|z|), & \text{if } k > 0, \end{cases} \quad (24)$$

where H_0 is the Hankel function of the first kind. It is known that S_k maps continuously $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ [7, Theorem 1.8] and N_k maps continuously $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ [7, Theorem 1.4]. In the case $k = 0$, the Helmholtz scattering reduces to the Laplace problem. The kernel of the hypersingular operator has a non-integrable singularity, but computations are facilitated by the following formula, valid for smooth functions μ and ν that vanish at the extremities of Γ :

$$\begin{aligned} \langle N_k \mu, \nu \rangle &= \int_{\Gamma \times \Gamma} G_k(x-y) \partial_{\tau} \mu(x) \partial_{\tau} \nu(y) \\ &\quad - k^2 G_k(x, y) \mu(x) \partial_{\nu}(y) n(x) \cdot n(y) d\sigma(x) d\sigma(y). \end{aligned} \quad (25)$$

For the geometry under consideration, the solutions λ and μ of the BIEs (22) have singularities (even for C^∞ data u_D and u_N) due to the edges of the scatterer. It is now classical to introduce weighted versions of the usual layer potentials, known to enjoy better mapping properties than S_k and N_k . Namely, we define

$$S_{k, \omega_{\Gamma}} := S_k \frac{1}{\omega_{\Gamma}}, \quad N_{k, \omega_{\Gamma}} := N_k \omega_{\Gamma}, \quad (26)$$

and recast the BIEs (22) as

$$S_{k, \omega_{\Gamma}} \alpha = u_D, \quad N_{k, \omega_{\Gamma}} \beta = u_N. \quad (27)$$

where the unknowns α and β are related to λ and μ by

$$\lambda = \frac{\alpha}{\omega_{\Gamma}}, \quad \mu = \omega_{\Gamma} \beta.$$

Equation (25) give the following relation between $N_{k, \omega_{\Gamma}}$ and $S_{k, \omega_{\Gamma}}$.

Lemma 11. *There holds*

$$N_{k,\omega_\Gamma} = -\partial_\tau S_{k,\omega_\Gamma} \omega_\Gamma \partial_\tau \omega_\Gamma - k^2 V_k \omega_\Gamma^2$$

where V_k is the integral operator defined by

$$V_k u = \int_\Gamma \frac{G_k(x-y) N(x) \cdot N(y) u(y)}{\omega_\Gamma(y)} d\sigma(y).$$

Proof. Eq. (25) can be rewritten equivalently as

$$N_k u = -\partial_\tau S_k \partial_\tau u - k^2 \int_\Gamma G_k(x-y) N(x) \cdot N(y) u(y) d\sigma(y).$$

Using the definitions of N_{k,ω_Γ} and S_{k,ω_Γ} , the results follow from simple manipulations on this expression. \square

2.2 Weighted layer potentials on the flat segment

In this section, we consider the case where the wavenumber k is equal to 0 and $\Gamma = [-1, 1] \times \{0\}$. The parametrization r is then the constant function equal to 1, $\partial_\tau = \partial_x$ and $\omega_\Gamma = \omega$. In this context, the weighted potentials are thus denoted by $S_{0,\omega}$ and $N_{0,\omega}$. Some explicit properties of those operators allow us to characterize T^s and U^s for $s = \pm \frac{1}{2}$.

Single layer potential. The operator $S_{0,\omega}$ takes the form

$$S_{0,\omega} \alpha(x) = \int_{-1}^1 \frac{\ln|x-y| \alpha(y)}{\sqrt{1-y^2}} dy.$$

There holds

$$S_{0,\omega} T_n = \sigma_n T_n \tag{28}$$

where

$$\sigma_n = \begin{cases} \frac{\ln(2)}{1^2} & \text{if } n = 0, \\ \frac{1}{2n} & \text{otherwise.} \end{cases}$$

Those identities are fundamental in our analysis. A proof can be found in [2, Theorem 9.2]. We deduce easily

Lemma 12. *The operator $S_{0,\omega}$ is a positive bicontinuous bijection from T^s to T^{s+1} for all $s \in \mathbb{R}$.*

In particular, $S_{0,\omega}$ maps T^∞ to itself, so the image of a smooth function by $S_{0,\omega}$ is a smooth function. We now proceed to show the following characterization of $T^{-1/2}$ and $T^{1/2}$. The next result, and Lemma 15 stated below are equivalent to results formulated in [4] (see equations (4.77-4.86), and Propositions 3.1 and 3.3 therein).

Lemma 13. *We have $T^{-1/2} = \omega \tilde{H}^{-1/2}(-1, 1)$ and for all $u \in \tilde{H}^{-1/2}(-1, 1)$,*

$$\|u\|_{\tilde{H}^{-1/2}} \sim \|\omega u\|_{T^{-1/2}}.$$

Moreover, $T^{1/2} = H^{1/2}(-1, 1)$ and

$$\|u\|_{H^{1/2}} = \|u\|_{T^{1/2}}.$$

Proof. Since the logarithmic capacity of the segment is $\frac{1}{4}$, the (unweighted) single-layer operator S_0 is positive and bounded from below on $\tilde{H}^{-1/2}(-1, 1)$, (see [3] chap. 8). Therefore the norm on $\tilde{H}^{-1/2}(-1, 1)$ is equivalent to

$$\|u\|_{\tilde{H}^{-1/2}} \sim \sqrt{\langle S_0 u, u \rangle}.$$

On the other hand, the explicit expression (28) implies that if $\alpha \in T^{-1/2}$, then

$$\|\alpha\|_{T^{-1/2}} \sim \sqrt{\langle S_{0,\omega} \alpha, \alpha \rangle_{\frac{1}{\omega}}}.$$

It remains to notice that, since $\alpha = \omega u$, $\langle S_{0,\omega} \alpha, \alpha \rangle_{\frac{1}{\omega}} = \langle S_0 u, u \rangle$. This proves the first result. For the second result, we know that,

$$(H^{1/2}(-1, 1))' = \tilde{H}^{-1/2}(-1, 1)$$

([3, Chap. 3] taking the identification with respect to the usual L^2 duality denoted by $\langle \cdot, \cdot \rangle$) and therefore

$$\|u\|_{H^{\frac{1}{2}}} = \sup_{v \neq 0} \frac{\langle u, v \rangle}{\|v\|_{\tilde{H}^{-\frac{1}{2}}}}.$$

According to the previous result, for all $v \in \tilde{H}^{-\frac{1}{2}}$, the function $\alpha = \omega v$ is in $T^{-1/2}$, and $\|v\|_{\tilde{H}^{-1/2}} \sim \|\alpha\|_{T^{-1/2}}$, while $\langle u, v \rangle = \langle u, \alpha \rangle_{\frac{1}{\omega}}$. Thus

$$\|u\|_{H^{1/2}} \sim \sup_{\alpha \neq 0} \frac{\langle u, \alpha \rangle_{\frac{1}{\omega}}}{\|\alpha\|_{T^{-1/2}}}.$$

The last quantity is the $T^{1/2}$ norm of u since $T^{1/2}$ is identified to the dual of $T^{-1/2}$ for $\langle \cdot, \cdot \rangle_{\frac{1}{\omega}}$, concluding the proof. \square

Hypersingular operator. In the present context, the identity (11) takes the form

$$\langle N_{0,\omega} \beta, \beta' \rangle_{\omega} = \langle S_{0,\omega} (\omega \partial_x \omega) \beta, (\omega \partial_x \omega) \beta' \rangle_{\frac{1}{\omega}}.$$

Noticing that $(\omega \partial_x \omega) U_n = -(n+1) T_{n+1}$, we have, for all $n \neq m$,

$$\langle N_{0,\omega} U_n, U_m \rangle_{\omega} = 0.$$

Therefore,

$$N_{0,\omega} U_n = \nu_n U_n$$

with $\nu_n \|U_n\|_{\omega}^2 = (n+1)^2 \sigma_{n+1} \|T_{n+1}\|_{\frac{1}{\omega}}^2$, that is, $\nu_n = \frac{(n+1)}{2}$. We deduce the following:

Lemma 14. *The operator $N_{0,\omega}$ is a positive bicontinuous bijection from U^s to U^{s-1} for all $s \in \mathbb{R}$.*

In particular, $N_{0,\omega}$ maps smooth functions to smooth functions. As before, we obtain a characterization of U^s for $s = \pm \frac{1}{2}$ from the previous formula.

Lemma 15. *We have $U^{1/2} = \frac{1}{\omega} \tilde{H}^{1/2}(-1, 1)$ and for all $u \in \tilde{H}^{1/2}(-1, 1)$,*

$$\|u\|_{\tilde{H}^{1/2}} \sim \left\| \frac{u}{\omega} \right\|_{U^{1/2}} .$$

Moreover, $U^{-1/2} = H^{1/2}(-1, 1)$ and

$$\|u\|_{H^{1/2}} = \|u\|_{U^{1/2}} .$$

2.3 Optimal orders of convergence for the weighted Galerkin method

A Galerkin method based on a refined mesh and weighted L^2 scalar products is described in [1] to solve eqs. (27) and orders of convergences are announced for the Laplace problem ($k = 0$) when $\Gamma = [-1, 1] \times \{0\}$. In this section we provide the proofs for those statements. Let us consider the following discretization of the segment $[-1, 1]$

$$-1 = x_0 < x_1 < \dots < x_N = 1 ,$$

where $x_i = \cos(i \frac{\pi}{N})$.

2.3.1 Dirichlet problem

Let V_h the Galerkin space of (discontinuous) piecewise affine functions with breakpoints at $(x_i)_{0 \leq i \leq N}$. Let α_h the unique solution in V_h to the variational problem

$$(S_{0,\omega} \alpha_h, \alpha'_h)_{\frac{1}{\omega}} = (u_D, \alpha'_h)_{\frac{1}{\omega}} , \quad \forall \alpha'_h \in V_h . \quad (29)$$

Let $\lambda_h = \frac{\alpha_h}{\omega}$ and $\alpha_0 = \omega \lambda$.

Theorem 1. *If the data u_D is in T^{s+1} for some $-\frac{1}{2} \leq s \leq 2$, then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq Ch^{s+1/2} \|\omega \lambda\|_{T^s} \leq Ch^{s+1/2} \|u_D\|_{T^{s+1}} .$$

The proof of this theorem is similar in structure to the standard proof in non-weighted L^2 -spaces. What we expose below borrows many ideas from [5]. The only difference, and source of difficulty, comes from the fact that the semi-norms $|P|_{T^n}$ of polynomials P do not vanish on each segment contrary to the non-weighted L^2 case. For this reason, an additional estimate, eq. (37), is required to establish the result.

Proof. Let us denote by Π_h the operator that maps a function $\alpha \in T^{-1/2}$ to the element $\alpha_h \in V_h$ such that

$$(S_{0,\omega} \alpha_h, \alpha'_h)_{\frac{1}{\omega}} = (S_{0,\omega} \alpha, \alpha'_h)_{\frac{1}{\omega}} \quad \forall \alpha'_h \in V_h .$$

Since $S_{0,\omega}$ is coercive in $T^{-1/2}$ by Lemma 12, we have an analog of C ea's lemma in our context:

$$\forall \alpha \in T^{-1/2}, \quad \|\alpha - \Pi_h \alpha\|_{T^{-1/2}} \leq C \inf_{\alpha'_h \in V_h} \|\alpha - \alpha'_h\|_{T^{-1/2}}. \quad (30)$$

From this we deduce, taking $\alpha'_h = 0$ in the infimum:

$$\forall \alpha \in T^{-1/2} \quad \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq C \|\alpha\|_{T^{-1/2}}.$$

In addition, we are going to show

$$\forall \alpha \in T^2, \quad \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq Ch^{5/2} \|\alpha\|_{T^2}. \quad (31)$$

By interpolation, this implies for all $s \in [-\frac{1}{2}, 2]$

$$\forall \alpha \in T^s, \quad \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq Ch^{s+1/2} \|\alpha\|_{T^s}.$$

The result then follows from this, since, on the one hand

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq C \|\alpha_0 - \alpha_h\|_{T^{-1/2}}$$

by Lemma 13, with $\alpha_h = \Pi_h \alpha_0$ and, on the other hand, $\omega u = \alpha_0 = S_{0,\omega}^{-1} u_D$ which by Lemma 12, gives

$$\|\alpha_0\|_{T^s} \leq C \|u_D\|_{T^{s+1}}.$$

We prove (31) by studying the properties of two particular operators: the $L^2_{\frac{1}{\omega}}$ orthonormal projection \mathbb{P}_h on V_h and the interpolation operator I_h which maps a continuous function α to the (continuous) function of V_h that matches α at the breakpoints x_i . Because of C ea's lemma, we have

$$\forall \alpha \in T^{-1/2}, \quad \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq C \|(I_d - \mathbb{P}_h)\alpha\|_{T^{-1/2}}.$$

Therefore, it suffices to show (31) where Π_h is replaced by \mathbb{P}_h to establish the theorem. We shall first show that

$$\forall \alpha \in T^s, \quad \|(I_d - \mathbb{P}_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq Ch^s \|\alpha\|_{T^s} \quad (32)$$

for $s \in [0, 2]$. The estimate in $T^{-1/2}$ norm is then deduced by the classical duality method:

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \sup_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_h \alpha, \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}},$$

and since \mathbb{P}_h is an orthonormal projection on $L^2_{\frac{1}{\omega}}$,

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \sup_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_N \alpha, \eta - \mathbb{P}_h \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}.$$

This, by Cauchy-Schwarz equality and (32), gives

$$\forall \alpha \in T^2, \quad \|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq Ch^2 \|\alpha\|_{T^2} \sup_{\eta \in T^{1/2}, \eta \neq 0} \frac{h^{1/2} \|\eta\|_{T^{1/2}}}{\|\eta\|_{T^{1/2}}},$$

implying the desired estimate. Let us prove (32). First, since \mathbb{P}_h is an orthonormal projection on $L^2_{\frac{1}{\omega}}$

$$\forall \alpha \in L^2_{\frac{1}{\omega}}, \quad \|(I_d - \mathbb{P}_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq C \|\alpha\|_{L^2_{\frac{1}{\omega}}} .$$

Using again an interpolation argument, it is thus sufficient to show

$$\forall \alpha \in T^2, \quad \|(I_d - \mathbb{P}_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|\alpha\|_{T^2} . \quad (33)$$

To this aim, we establish the following estimate:

$$\forall \alpha \in T^2, \quad \|(I_d - I_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|\alpha\|_{T^2} , \quad (34)$$

and conclude with $\|(I_d - \mathbb{P}_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq \|(I_d - I_h)\alpha\|_{L^2_{\frac{1}{\omega}}}$ since \mathbb{P}_h minimizes the $L^2_{\frac{1}{\omega}}$ error. Any function $\alpha \in T^2$ is continuous by Lemma 5, thus $I_h\alpha$ is well-defined. To prove (34), let us fix $\alpha \in T^2$ and $i \in \{0, \dots, N-1\}$. The function $\mathcal{C}\alpha$ belongs to H^2 by Proposition 3 and thus its restriction on $[\theta_i, \theta_{i+1}]$ belongs to $H^2(\theta_i, \theta_{i+1})$. The function $I_h\alpha$ being C^∞ on $[x_i, x_{i+1}]$, the restriction of $\mathcal{C}I_h\alpha$ to $[\theta_i, \theta_{i+1}]$ also belongs to $H^2(\theta_i, \theta_{i+1})$. Moreover, $\mathcal{C}(I_h - I_d)\alpha$ vanishes on θ_i and θ_{i+1} , and it is well-known that for a function v in $H^2(a, b)$ which vanishes on a and b , there holds

$$\int_a^b |v(\theta)|^2 d\theta \leq C(b-a)^4 \int_a^b |\partial_{\theta\theta} v(\theta)|^2 d\theta . \quad (35)$$

Thus, applying this result on $[\theta_i, \theta_{i+1}]$ to $v = (\mathcal{C}(I_d - I_h)\alpha)|_{[\theta_i, \theta_{i+1}]}$ and summing those inequalities for $i = 0$ to $N-1$, we obtain

$$\frac{1}{\pi} \int_0^\pi |\mathcal{C}(I_d - I_h)\alpha(\theta)|^2 d\theta \leq \frac{h^4}{\pi} \sum_{i=0}^{N-1} \int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta} \mathcal{C}(I_d - I_h)\alpha(\theta)|^2 d\theta ,$$

where on the left hand side, the integral is equal to $\|(I_d - I_h)\alpha\|_{L^2_{\frac{1}{\omega}}}^2$ by Proposition 3. If $\mathcal{C}I_h\alpha$ were affine on each segment $[\theta_i, \theta_{i+1}]$, this inequality would simplify to

$$\|(I_d - I_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|\alpha\|_{T^2} . \quad (36)$$

This does not occur in our, which is the main difference with the standard proof. Nevertheless, if we show that

$$\sum_{i=0}^{N-1} \int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta} \mathcal{C}I_h\alpha(\theta)|^2 d\theta \leq C \|\alpha\|_{T^2}^2 , \quad (37)$$

then eq. (34) follows by triangular inequality. The expression of $\mathcal{C}I_h\alpha$ on $[\theta_i, \theta_{i+1}]$ is

$$\mathcal{C}I_h\alpha(\theta) = \alpha(x_i) + \frac{\alpha(x_i) - \alpha(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)} (\cos(\theta) - \cos(\theta_i)),$$

thus

$$\int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta} \mathcal{C}I_h\alpha(\theta)|^2 d\theta = \left| \frac{\alpha(x_i) - \alpha(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)} \right|^2 \int_{\theta_i}^{\theta_{i+1}} \cos^2(\theta) d\theta .$$

We can rewrite

$$|\alpha(x_{i+1}) - \alpha(x_i)|^2 = \left| \int_{x_i}^{x_{i+1}} \partial_x \alpha(x) dx \right|^2,$$

and apply Cauchy-Schwarz's inequality and the change of variables $x = \cos(\theta)$ to find

$$|\alpha(x_{i+1}) - \alpha(x_i)|^2 \leq \int_{x_i}^{x_{i+1}} \frac{|\partial_x \alpha(x)|^2}{\omega(x)} dx \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2 d\theta.$$

Notice that the quantity

$$\frac{\int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2}{(\cos(\theta_{i+1}) - \cos(\theta_i))^2}$$

is bounded uniformly in (θ_i, θ_{i+1}) . Indeed, since \cos is injective on $[0, \pi]$, the only problematic case is the limit when $\theta_i = \theta_{i+1}$. It is easy to check that the fraction then takes the limit $\cos(\theta_i)^2$, which is indeed uniformly bounded in θ_i . We deduce

$$\sum_{i=0}^{N-1} \int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta} CI_h \alpha(\theta)|^2 \leq C \|\partial_x \alpha\|_{\frac{1}{\omega}}^2.$$

By Corollary 2, one has $\|\partial_x \alpha\|_{\frac{1}{\omega}} \leq C \|\alpha\|_{T^2}$, thus (34) is established, concluding the proof. \square

2.3.2 Neumann problem

Let W_h the Galerkin space of continuous piecewise affine functions on the mesh $(x_i)_{0 \leq i \leq N}$ defined above, and β_h the unique solution in W_h to

$$(N_{0,\omega} \beta_h, \beta'_h)_\omega = (u_N, \beta'_h)_\omega, \quad \forall \beta'_h \in W_h. \quad (38)$$

Let $\mu_h = \beta_h \omega$ and $\beta_0 = \frac{\mu}{\omega}$.

Theorem 2. *If $u_N \in U^{s-1}$, for some $\frac{1}{2} \leq s \leq 2$, there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq Ch^{s-\frac{1}{2}} \left\| \frac{\mu}{\omega} \right\|_{U^s} \leq Ch^{s-\frac{1}{2}} \|u_N\|_{U^{s-1}}.$$

Proof. Like before, let us denote by Π_h the operator that maps a function $\beta \in U^{1/2}$ to the element $\beta_h \in W_h$ such that

$$(N_{0,\omega} \beta_h, \beta'_h) = (N_{0,\omega} \beta, \beta'_h) \quad \forall \beta'_h \in W_h.$$

The operator $N_{0,\omega}$ being coercive on $U^{1/2}$ by Lemma 15, we have a Céa's lemma:

$$\forall \beta \in U^{\frac{1}{2}}, \quad \|\beta - \Pi_h \beta\|_{U^{1/2}} \leq C \inf_{\beta'_h \in W_h} \|\beta - \beta'_h\|_{U^{1/2}}.$$

In particular taking $\beta'_h = 0$ in the infimum,

$$\forall \beta \in U^{1/2}, \quad \|(I_d - \Pi_h)\beta\|_{U^{1/2}} \leq C \|\beta\|_{U^{1/2}}.$$

Once we prove

$$\forall \beta \in U^2, \quad \|(I_d - \Pi_h)\beta\|_{U^{1/2}} \leq Ch^{\frac{3}{2}} \|\beta\|_{U^2}, \quad (39)$$

we get by interpolation

$$\forall \beta \in U^s, \quad \|(I_d - \Pi_h)\beta\|_{U^{1/2}} \leq Ch^{s-\frac{1}{2}} \|\beta\|_{U^s}$$

for all $s \in [\frac{1}{2}, 2]$. The result follows since on the one hand,

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} = \|\beta_0 - \beta_h\|_{U^{1/2}},$$

by Lemma 15, with $\Pi_h\beta_0 = \beta_h$ and, on the other hand, $\frac{\mu}{\omega} = \beta_0 = N_{0,\omega}^{-1}u_N$ which, by Lemma 14, implies

$$\|\beta_0\|_{U^s} \leq C \|u_N\|_{U^{s-1}}.$$

Like before, the proof of (39) involves the study of the interpolation operator I_h . Namely, if we have

$$\forall \beta \in U^2, \quad \|(I_d - I_h)\beta\|_{\omega} \leq Ch^2 \|\beta\|_{U^2}, \quad \|(I_d - I_h)\beta\|_{U^1} \leq Ch \|\beta\|_{U^2}, \quad (40)$$

then, by interpolation, we obtain

$$\|(I_d - I_h)\beta\|_{U^{1/2}} \leq Ch^{3/2} \|\beta\|_{U^2},$$

which gives (39) after applying Céa's lemma. Let us show the first estimate in (40). Applying Proposition 3 and using again the property of H^2 functions vanishing at the boundary (35) one can write

$$\begin{aligned} \int_{x_i}^{x_{i+1}} \omega |(I_d - I_h)\beta|^2 &\leq C(\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}(\mathcal{S}\beta - \mathcal{S}I_h\beta)|^2 \\ &\leq Ch^4 \left(2 \int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}\mathcal{S}\beta|^2 + 2 \int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}\mathcal{S}I_h\beta|^2 \right). \end{aligned}$$

Summing for $i = 0, \dots, N-1$, by Proposition 3, we get

$$\|(I_d - I_h)\beta\|_{\omega} \leq Ch^2 \left(\|\beta\|_{U^2} + \sqrt{\sum_{i=0}^{N-1} \int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}\mathcal{S}I_h\beta(\theta)|^2 d\theta} \right). \quad (41)$$

Just like for the Dirichlet problem, the proof would end here if $\mathcal{S}I_h u$ were affine on each segment, because the second term in the right hand side would be 0. Here, we need to show that this term is controlled by $\|\beta\|_{U^2}$. Using the expression of I_h , one can write

$$\begin{aligned} \int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}\mathcal{S}I_h\beta|^2 &\leq C \left(|\beta(x_i)|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 \theta d\theta \right. \\ &\quad \left. + \left| \frac{\beta(x_{i+1}) - \beta(x_i)}{\cos \theta_{i+1} - \cos \theta_i} \right|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 \theta (1 + \cos^2 \theta) d\theta \right). \quad (42) \end{aligned}$$

We can estimate the first term, thanks to Lemma 5:

$$|\beta(x_i)| \leq C \|\beta\|_{U^2},$$

while for the second term, the numerator of the fraction is estimated as follows:

$$\begin{aligned} |\beta(x_{i+1}) - \beta(x_i)|^2 &= \left| \int_{x_i}^{x_{i+1}} \partial_x \beta \right|^2 \\ &\leq \int_{x_i}^{x_{i+1}} \omega |\partial_x \beta|^2 \int_{x_i}^{x_{i+1}} \frac{1}{\omega} \\ &= |\theta_{i+1} - \theta_i| \int_{x_i}^{x_{i+1}} \omega |\partial_x \beta|^2. \end{aligned}$$

Observe that the quantity

$$\frac{|\theta_{i+1} - \theta_i| \int_{\theta_i}^{\theta_{i+1}} \sin^2 \theta (1 + \cos^2 \theta) d\theta}{(\cos \theta_i - \cos \theta_{i+1})^2}$$

is bounded by a constant independent of θ_i and θ_{i+1} . Indeed, in the limit $\theta_{i+1} \rightarrow \theta_i$, the fraction takes the value $1 + \cos^2 \theta_i$. Therefore, (41) leads to

$$\|(I_d - I_h)\beta\|_{\omega} \leq Ch^2 (\|\beta\|_{U^2} + \|\partial_x \beta\|_{\omega}).$$

Recalling Corollary 2, the second term in this estimate is controlled by $\|\beta\|_{U^2}$ and the first estimate of (40) is established. The second estimate of (40) can be shown in a similar manner, concluding the proof. \square

2.3.3 Estimates in the weighted L^2 norms

We have just established the optimal orders of convergence of the Galerkin method in the energy norms. It is also possible to prove optimal order of convergence in the weighted L^2 norm of the problem, using inverse estimates and Aubin-Nitsche's duality method. The proofs do not differ in any essential way from the standard theory, so they are omitted.

Proposition 4. *For all $s \in [0, 2]$,*

$$\|\alpha_0 - \alpha_h\|_{\frac{1}{\omega}} \leq Ch^s \|\alpha_0\|_{T^s},$$

$$\|\beta_0 - \beta_h\|_{\omega} \leq Ch^s \|\beta_0\|_{U^s}.$$

and for all $s \in [1, 2]$,

$$\|\beta_0 - \beta_h\|_{U^1} \leq Ch^{s-1} \|\beta_0\|_{U^s}.$$

3 Pseudo-differential operators on open curves

This section is devoted to the introduction of two classes of pseudo-differential operators on open curves, which, to the best of our knowledge, have not been studied elsewhere in the literature. Our approach can be summarized as follows.

Through the change of variables $x = \cos \theta$, an operator on the segment can be viewed as an operator on the torus $\mathbb{T}_{2\pi}$ (where $\mathbb{T}_{2\pi} = \mathbb{R}/2\pi\mathbb{Z}$ as in section 1.4). On this geometry, a simple algebra of pseudo-differential operators exists [6]. However, the inverse change of variables, $\theta = \arccos(x)$ has singularities at $x = -1$ and $x = 1$, preventing the transfer of the properties of this algebra back to the segment. We solve this difficulty by using some parity arguments. Loosely speaking, we first remove the problematic part of the operators before applying the pullback change of variables. As a result, two classes of operators emerge, related to pseudo-differential operators on the torus that stabilize even and odd functions respectively.

We start by collecting some facts on periodic pseudo-differential operators in section 3.1. We then introduce a first class of pseudo-differential operators on the segment (and more generally on smooth open curves) in section 3.2, which is based on the scales of Hilbert spaces $(T^s)_{s \in \mathbb{R}}$ presented in section 1. We show that the usual properties of pseudo-differential operators hold in this class. The pseudo-differential operators based on $(U^s)_{s \in \mathbb{R}}$ are introduced in section 3.3.

3.1 Periodic pseudo-differential operators

On the family of periodic Sobolev spaces H^s , a class of periodic pseudo differential operators (PPDO) is studied in [6]. We briefly reproduce here the material needed for our purposes. A PPDO of order α on H^s is an operator of the form

$$Au(\theta) = \sum_{n \in \mathbb{Z}} \sigma_A(\theta, n) \mathcal{F}u(n) e^{in\theta}$$

for a ‘‘prolongated symbol’’ $\sigma_A \in C^\infty(\mathbb{T}_{2\pi} \times \mathbb{R})$ satisfying

$$\forall j, k \in \mathbb{N}, \quad \exists C_{j,k} > 0 : \quad \left| D_\theta^j D_\xi^k \sigma_A(\theta, \xi) \right| \leq C_{j,k} (1 + |\xi|)^{\alpha-k}. \quad (43)$$

Here, as in section 1.4, $\mathcal{F}u(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t) e^{-in\theta} d\theta$ are the usual Fourier coefficients of u and

$$D_\theta := \frac{1}{i} \frac{\partial}{\partial \theta}, \quad D_\xi := \frac{1}{i} \frac{\partial}{\partial \xi}.$$

The class of symbols that satisfy (43) is denoted by Σ^α , and $\Sigma^\infty := \cup_{\alpha \in \mathbb{Z}} \Sigma^\alpha$. The operator defined by a symbol σ is denoted by $Op(\sigma)$ and the set of PPDOs of order α is denoted by $Op(\Sigma^\alpha)$.

The prolonged symbol is not unique but determined uniquely at integer values of ξ by (see [6]):

$$\sigma_A(\theta, n) = e_{-n}(\theta) A e_n(\theta), \quad (44)$$

where we recall the notation $e_n(\theta) = e^{in\theta}$. This justifies the terminology of ‘‘prolongated symbol’’. The operator A is in $Op(\Sigma^\alpha)$ if and only if

$$\forall j, k \in \mathbb{N}, \quad \exists C_{j,k} > 0 : \quad \left| D_\theta^j \Delta_n^k \sigma_A(\theta, n) \right| \leq C_{j,k} (1 + |n|)^{\alpha-k}, \quad (45)$$

where $\Delta_n \phi(\theta, n) = \phi(\theta, n+1) - \phi(\theta, n)$. That is, if the symbol defined in (44) satisfies (45), then there exists a prolonged symbol satisfying (43). Because of

this, we write $\sigma \in \Sigma^p$ for a symbol $\sigma(\theta, n)$ that can be prolonged to a symbol $\sigma(\theta, \xi) \in \Sigma^p$. An operator in $Op(\Sigma^\alpha)$ maps H^s to $H^{s-\alpha}$ continuously for all $s \in \mathbb{R}$. The composition of two operators in $Op(\Sigma^\alpha)$ and $Op(\Sigma^\beta)$ gives rise to an operator in $Op(\Sigma^{\alpha+\beta})$. If two symbols a and b in Σ^∞ satisfy $a - b \in \Sigma^\alpha$, we write $a = b + \Sigma^\alpha$.

Definition 6. Let $a \in \Sigma^\infty$. If there exists a sequence of reals $(p_j)_{j \in \mathbb{N}}$ such that $p_{j+1} < p_j$ and a sequence of symbols $a_j \in \Sigma^{p_j}$ such that for all N , $a = \sum_{i=0}^N a_i + \Sigma^{p_{N+1}}$, we write

$$a = \sum_{i=0}^{+\infty} a_i.$$

This is called an asymptotic expansion of the symbol a .

The symbol of the composition of two PPDOs A and B is denoted by $\sigma_A \# \sigma_B$ and satisfies the asymptotic expansion [6]

$$\sigma_A \# \sigma_B(t, \xi) = \sum_{j=0}^{+\infty} \frac{1}{j!} \left(\frac{\partial}{\partial \xi} \right)^j \sigma_A(\theta, \xi) D_\theta^j \sigma_B(\theta, \xi). \quad (46)$$

We also need the following result.

Proposition 5. (see [6]) Consider an integral operator K of the form

$$K : u \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} a(\theta, \theta') h(\theta - \theta') u(\theta') d\theta',$$

where a is 2π -periodic and C^∞ in both arguments and h is a 2π -periodic distribution. Assume that the Fourier coefficients $\mathcal{F}h(n)$ of h can be prolonged to a function $\hat{h}(\xi)$ on \mathbb{R} such that

$$\forall k \in \mathbb{N}, \quad \exists C_k > 0 : \quad \left| \partial_\xi^k \hat{h}(\xi) \right| \leq C_k (1 + |\xi|)^{\alpha-k}$$

for some α . Then K is in $Op(\Sigma^\alpha)$ with a symbol satisfying the asymptotic expansion

$$\sigma_K(\theta, \xi) = \sum_{j=0}^{+\infty} \frac{1}{j!} \left(\frac{\partial}{\partial \xi} \right)^j \hat{h}(\xi) D_t^j a(t, \theta)|_{t=\theta}. \quad (47)$$

In particular, taking $h \equiv 1$, we see that for any functions $a \in C^\infty(\mathbb{T}_{2\pi}^2)$, the operator

$$K : u \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} a(\theta, \theta') u(\theta') d\theta'$$

is in $Op(\Sigma^{-\infty})$.

3.2 Pseudo-differential operators on $T^s(\Gamma)$

We now introduce a new class of pseudo-differential operators on the segment $[-1, 1]$, which is based on the scale of Hilbert spaces $(T^s)_{s \in \mathbb{R}}$. Those operators are characterized in Theorem 3 in terms of PPDOs, and from this we deduce the main properties needed for our purposes in Corollary 5. Those definitions and results are then generalized to smooth open curves.

3.2.1 Definitions and main results

Definition 7. Let A an operator on $T^{-\infty}$ and assume that there exists a couple of functions a_1 and a_2 defined on $[-1, 1] \times \mathbb{N}$, that are C^∞ in the first variable and such that for all $n \in \mathbb{N}$,

$$AT_n = a_1(x, n)T_n - \omega^2 a_2(x, n)U_{n-1}, \quad (48)$$

with, by convention, $U_{-1} = 0$. The operator defined by the previous formula is denoted by $Op_T(a_1, a_2)$. Define the symbol $\tilde{\sigma}(a_1, a_2)$ on $\mathbb{T}_{2\pi} \times \mathbb{Z}$ by

$$\tilde{\sigma}(a_1, a_2)(\theta, n) := a_1(\cos \theta, |n|) + i \sin \theta \operatorname{sign}(n) a_2(\cos \theta, |n|).$$

We say that $(a_1, a_2) \in S_T^\alpha$ if $\tilde{\sigma}(a_1, a_2) \in \Sigma^\alpha$. In this case, we say that A is a pseudo-differential operator on T^s and that the couple of functions (a_1, a_2) is a pair of symbols for A . We denote $S_T^\infty := \cup_{\alpha \in \mathbb{R}} S_T^\alpha$ and $S_T^{-\infty} = \cap_{\alpha \in \mathbb{R}} S_T^\alpha$. The set of pseudo-differential operators (of order α) in $T^{-\infty}$ is denoted by $Op(S_T^\alpha)$ (by $Op(S_T^\alpha)$).

Remark 2. It is easy to construct non-trivial symbols in $S_T^{-\infty}$ for the null operator. For example, for some $m \in \mathbb{N}$, take $a(x, n) = \delta_{n=m} \omega^2(x) U_{m-1}(x)$ and $b(x, n) = \delta_{n=m} T_m(x)$. Because of this, a pair of symbol for a pseudo-differential operator on T^s is not unique.

Definition 8. For a PPDO \tilde{A} of symbol $\sigma_{\tilde{A}}$, we define $a_1(\tilde{A})$ and $a_2(\tilde{A})$ by

$$a_1(\tilde{A})(x, n) = \frac{\tilde{\sigma}(\arccos(x), n) + \tilde{\sigma}(\arccos(x), -n)}{2},$$

$$a_2(\tilde{A})(x, n) = \frac{\tilde{\sigma}(\arccos(x), n) - \tilde{\sigma}(\arccos(x), -n)}{2i\sqrt{1-x^2}},$$

where

$$\tilde{\sigma}(\theta, n) = \frac{\sigma_{\tilde{A}}(\theta, n) + \sigma_{\tilde{A}}(-\theta, -n)}{2}.$$

We can now state the main results of this section. Recall the definition of the operator \mathcal{C} from Proposition 3.

Theorem 3. Let $A : T^\infty \rightarrow T^{-\infty}$. Assume that for some PPDO $\tilde{A} \in Op(\Sigma^\alpha)$, there holds

$$\mathcal{C}A = \tilde{A}\mathcal{C} \quad (49)$$

as an equality of operators on T^∞ . Then A has a unique continuous extension as an element of $Op(S_T^\alpha)$, and $(a_1(\tilde{A}), a_2(\tilde{A}))$ is a pair of symbols for A . Reciprocally let $A = Op_T(a_1, a_2) \in Op(S_T^\alpha)$. Then (49) holds, taking for \tilde{A} the PPDO of order α given by the symbol

$$\sigma_{\tilde{A}} = \tilde{\sigma}(a_1, a_2).$$

We immediately derive the following properties of those new pseudo-differential operators as an easy consequence of the previous result.

Corollary 5. *If $A \in Op(S_T^\alpha)$, then for all $s \in \mathbb{R}$,*

$$A : T^s \rightarrow T^{s-\alpha}$$

is continuous. Moreover,

$$A \in Op(S_T^\alpha) \text{ and } B \in Op(S_T^\beta) \implies AB \in Op(S_T^{\alpha+\beta}).$$

Finally, if A and B admit the pairs of symbols (a_1, a_2) and (b_1, b_2) respectively, then AB admits the pair of symbol $(a_1(\tilde{C}), a_2(\tilde{C}))$ where

$$\tilde{C} = Op(\tilde{\sigma}(a_1, a_2))Op(\tilde{\sigma}(b_1, b_2)).$$

Proof of Corollary 5. Let $A \in Op(S_T^\alpha)$, and let \tilde{A} a PPDO of order α such that (49) holds. Let $u \in T^s$ for some $s \in \mathbb{R}$. Applying the isometric property of \mathcal{C} (cf. Proposition 3) and the continuity of \tilde{A} from H^s to $H^{s-\alpha}$,

$$\|Au\|_{T^s} = \|CAu\|_{H^s} = \|\tilde{A}\mathcal{C}u\|_{H^s} \leq C\|\mathcal{C}u\|_{H^{s-\alpha}} = C\|u\|_{T^{s-\alpha}}.$$

This establishes the first claim. Let $A = Op_T(a_1, a_2) \in S_T^\alpha$, $B = Op_T(b_1, b_2) \in S_T^\beta$ and let $\tilde{A} = Op(\tilde{\sigma}(a_1, a_2))$, $\tilde{B} = Op(\tilde{\sigma}(b_1, b_2))$. We have

$$CAB = \tilde{A}CB = \tilde{A}\tilde{B}C.$$

Applying the properties of the PPDOs, one has $\tilde{A}\tilde{B} \in Op(\Sigma^{\alpha+\beta})$ therefore, by Theorem 3, $AB \in Op(S_T^{\alpha+\beta})$ establishing the second claim. In addition, by Theorem 3, AB admits the pair of symbols $(a_1(\tilde{A}\tilde{B}), a_2(\tilde{A}\tilde{B}))$, concluding the proof. \square

Remark 3. *The previous results provide a symbolic calculus on the class $S_T^\alpha(\Gamma)$ as follows. If B and C respectively admit the pair of symbols (b_1, b_2) and (c_1, c_2) , then BC admits the pair of symbols*

$$(b_1, b_2)\#_T(c_1, c_2) := (a_1(\tilde{A}), a_2(\tilde{A}))$$

where $\tilde{A} = Op(\tilde{\sigma}(b_1, b_2)\#\tilde{\sigma}(c_1, c_2))$. One can use (46) to compute an asymptotic expansion of $\tilde{\sigma}(b_1, b_2)\#\tilde{\sigma}(c_1, c_2)$ which, in turn, gives an asymptotic expansion of $(b_1, b_2)\#_T(c_1, c_2)$.

Remark 4. *We have already stated that a pseudo-differential operator on T^s always admits several distinct symbols. Theorem 3 gives another way to view this fact, by observing that, when $A \in Op(S_T^\alpha)$, there is an infinite number of operators \tilde{A} satisfying $CA = \tilde{A}C$. Indeed, if \tilde{B} is any PPDO that vanishes on the set of even functions, one has*

$$CA = (\tilde{A} + \tilde{B})C.$$

In light of this, a natural idea to define uniquely the symbol of A would be to set

$$a_1 = a_1(\tilde{A}^*), \quad a_2 = a_2(\tilde{A}^*),$$

where \tilde{A}^ is the operator defined by*

$$\begin{aligned} \tilde{A}^*\mathcal{C}u &= CAu, \\ \tilde{A}^*Su &= 0. \end{aligned}$$

However, though $A \in Op(S_T^\alpha)$, \tilde{A}^* may fail to be a PPDO of order α . To see why, one can check that if \tilde{A} is a PPDO of order α such that $CA = \tilde{A}C$, then the symbol of \tilde{A}^* must be given by

$$\sigma_{\tilde{A}^*}(\theta, n) = \sigma_{\tilde{A}}(\theta, n) + e^{-2in\theta} \sigma_{\tilde{A}}(\theta, -n).$$

In general, this symbol is not in Σ^α because of the oscillatory term $e^{-2in\theta}$. In conclusion, there is no clear way how to fix a natural representative in the class of pairs (a_1, a_2) that define the same operator A . However, although unusual, this is not an obstacle for the theory.

3.2.2 Proof of Theorem 3

The proof of Theorem 3 is decomposed through the following lemmas.

Lemma 16. *Let $A = Op_T(a_1, a_2)$ and let $\tilde{A} = Op(\tilde{\sigma}(a_1, a_2))$. Then for all $n \in \mathbb{N}$,*

$$CAT_n = \tilde{A}CT_n.$$

Proof. Let $A = Op_T(a_1, a_2)$. Let $\sigma = \tilde{\sigma}(a_1, a_2)$, and $\tilde{A} = Op(\sigma)$. Fix $n \in \mathbb{N}$. On the one hand, we can write

$$\begin{aligned} AT_n(\cos(\theta)) &= a_1(\cos \theta, n)T_n(\cos(\theta)) - \omega^2(\cos \theta)a_2(\cos \theta, n)U_{n-1}(\cos \theta) \\ &= a_1(\cos \theta, n) \cos(n\theta) - \sin \theta a_2(\cos \theta, n) \sin(n\theta). \end{aligned}$$

On the other hand,

$$\begin{aligned} \tilde{A}(CT_n)(\theta) &= \frac{\tilde{A}(e_n)(\theta) + \tilde{A}(e_{-n})(\theta)}{2} \\ &= \frac{\sigma(\theta, n)e^{in\theta} + \sigma(\theta, -n)e^{-in\theta}}{2} \\ &= \left(\frac{\sigma(\theta, n) + \sigma(\theta, -n)}{2} \right) \cos(n\theta) + i \left(\frac{\sigma(\theta, n) - \sigma(\theta, -n)}{2} \right) \sin(n\theta). \end{aligned}$$

From the definition of $\tilde{\sigma}(a_1, a_2)$, we see that the first term in parenthesis is equal to $a_1(\cos \theta, n)$, and the second is $i \sin \theta a_2(\cos \theta, n)$. Therefore,

$$CAT_n = \tilde{A}CT_n$$

for all $n \in \mathbb{N}$ and the result is proved. \square

Lemma 17. *If $\tilde{A} = Op(\sigma_{\tilde{A}})$ is a PPDO, then the functions $a_1(\tilde{A})$ and $a_2(\tilde{A})$ are C^∞ in the variable x and there holds the identity*

$$\tilde{\sigma}(a_1(\tilde{A}), a_2(\tilde{A}))(\theta, n) = \frac{\sigma_{\tilde{A}}(\theta, n) + \sigma_{\tilde{A}}(-\theta, -n)}{2}.$$

Proof. Let $\tilde{\sigma}(\theta, n) = \frac{\sigma_{\tilde{A}}(\theta, n) + \sigma_{\tilde{A}}(-\theta, -n)}{2}$. We decompose $\tilde{\sigma}$ as

$$\tilde{\sigma}(\theta, n) = f(\theta, n) + g(\theta, n)$$

where $f(\theta, n) = \frac{\tilde{\sigma}(\theta, n) + \tilde{\sigma}(\theta, -n)}{2}$ and $g(\theta, n) = \frac{\tilde{\sigma}(\theta, n) - \tilde{\sigma}(\theta, -n)}{2}$. By construction, f (resp. g) is even (resp. odd) in both θ and n . For $n > 0$, there holds

$$a_1(\tilde{A})(x, n) = f(\arccos(x), n), \quad a_2(\tilde{A})(x, n) = \frac{g(\arccos(x), n)}{i\sqrt{1-x^2}}.$$

Recalling Proposition 3, this is equivalently expressed as

$$a_1(\tilde{A})(\cdot, n) = \mathcal{C}^{-1}f(\cdot, n), \quad a_2(\tilde{A})(\cdot, n) = -i\mathcal{S}^{-1}g(\cdot, n).$$

Therefore, $a_1(\tilde{A})$ and $a_2(\tilde{A})$ are C^∞ in x since f (resp. g) is a smooth even (resp. odd) function. By definition, we have

$$\begin{aligned} \tilde{\sigma}_T(a_1(\tilde{A}), a_2(\tilde{A}))(\theta, n) &= \mathcal{C}a_1(\tilde{A})(\theta, |n|) + i\text{sign}(n)\mathcal{S}a_2(\tilde{A})(\theta, |n|) \\ &= f(\theta, |n|) + \text{sign}(n)g(\theta, |n|) \\ &= f(\theta, n) + g(\theta, n) \\ &= \tilde{\sigma}(\theta, n), \end{aligned}$$

recalling that f (resp. g) is even (resp. odd) in n . □

Lemma 18. *If $\tilde{A} = \text{Op}(\sigma_{\tilde{A}})$ is a PPDO that stabilizes the set of smooth even functions, then \tilde{A} coincides on this set with $\tilde{B} = \text{Op}(\sigma_{\tilde{B}})$ where*

$$\sigma_{\tilde{B}}(\theta, n) = \frac{\sigma_{\tilde{A}}(\theta, n) + \sigma_{\tilde{A}}(-\theta, -n)}{2}.$$

Proof. Let u a smooth even function. Since Au is even, we have

$$Au(\theta) = \frac{Au(\theta) + Au(-\theta)}{2}.$$

Thus

$$\begin{aligned} Au(\theta) &= \frac{1}{2} \left(\sum_{n \in \mathbb{Z}} \sigma_{\tilde{A}}(\theta, n) \mathcal{F}u(n) e^{in\theta} + \sigma_{\tilde{A}}(-\theta, n) \mathcal{F}u(n) e^{-in\theta} \right) \\ &= \frac{1}{2} \left(\sum_{n \in \mathbb{Z}} \sigma_{\tilde{A}}(\theta, n) \mathcal{F}u(n) e^{in\theta} + \sigma_{\tilde{A}}(-\theta, -n) \mathcal{F}u(-n) e^{in\theta} \right) \\ &= \sum_{n \in \mathbb{Z}} \frac{\sigma_{\tilde{A}}(\theta, n) + \sigma_{\tilde{A}}(-\theta, -n)}{2} \mathcal{F}u(n) e^{in\theta} \end{aligned}$$

since $\mathcal{F}u(n) = \mathcal{F}u(-n)$. This proves the claim. □

Lemma 19. *If $A : T^\infty \rightarrow T^{-\infty}$ is such that there exists a PPDO \tilde{A} satisfying*

$$\forall n \in \mathbb{N}, \quad \mathcal{C}AT_n = \tilde{A}\mathcal{C}T_n,$$

then $A = \text{Op}_T(a_1(\tilde{A}), a_2(\tilde{A}))$.

Proof. Notice that the assumption implies that \tilde{A} stabilizes the set of smooth even functions. If $\sigma_{\tilde{A}}$ is the symbol of \tilde{A} and

$$\sigma(\theta, n) = \frac{\sigma_{\tilde{A}}(\theta, n) + \sigma_{\tilde{A}}(-\theta, -n)}{2}$$

then, by Lemma 18, we have $\tilde{A}\mathcal{C} = Op(\sigma)\mathcal{C}$. Moreover, by Lemma 17, we know that

$$\sigma = \tilde{\sigma}_T(a_1(\tilde{A}), a_2(\tilde{A}))$$

and thus, letting $B = Op_T(a_1(\tilde{A}), a_2(\tilde{A}))$, Lemma 16 tells us that

$$\forall n \in \mathbb{N}, \quad \mathcal{C}BT_n = Op(\sigma)\mathcal{C}T_n.$$

Summing up, we have

$$\forall n \in \mathbb{N}, \quad \mathcal{C}AT_n = \tilde{A}\mathcal{C}T_n = Op(\sigma)\mathcal{C}T_n = \mathcal{C}BT_n.$$

This ensures $A = B$. □

The proof of Theorem 3 is concluded as follows. Assume that for any $u \in T^\infty$, $\mathcal{C}Au = \tilde{A}\mathcal{C}u$ where \tilde{A} is some PPDO of order α with a symbol $\sigma_{\tilde{A}}$. The linear continuous extension of A is uniquely defined for $u \in T^{-\infty}$ by

$$\forall v \in T^\infty, \quad \langle Au, v \rangle_{\frac{1}{\omega}} = \langle \mathcal{C}Au, \mathcal{C}v \rangle_{\mathbb{T}_{2\pi}} = \langle \tilde{A}\mathcal{C}u, \mathcal{C}v \rangle_{\mathbb{T}_{2\pi}}.$$

where the last quantity makes sense since $\mathcal{C}u \in H^{-\infty}$. By Lemma 19, we have $A = Op_T(a_1(\tilde{A}), a_2(\tilde{A}))$. It remains to show that $\sigma := \tilde{\sigma}(a_1(\tilde{A}), a_2(\tilde{A})) \in \Sigma^\alpha$. By Lemma 17, if \tilde{A} has a symbol $\sigma_{\tilde{A}}$, then

$$\sigma(\theta, n) = \frac{\sigma_{\tilde{A}}(\theta, n) + \sigma_{\tilde{A}}(-\theta, -n)}{2},$$

and $\sigma_{\tilde{A}} \in \Sigma^\alpha$ immediately implies $\sigma \in \Sigma^\alpha$. This proves the first assertion. The second assertion is an immediate consequence of Lemma 16.

3.2.3 Extension to open curves

Definition 9. Let $A : T^{-\infty}(\Gamma) \rightarrow T^{-\infty}(\Gamma)$. We say that A is a pseudo-differential operator (of order α) on $T^{-\infty}(\Gamma)$ if $RAR^{-1} \in Op(S_T^\alpha)$ ($\in Op(S_T^\alpha)$). The set of pseudo-differential operators of order α on $T^{-\infty}(\Gamma)$ is denoted by $Op(S_T^\alpha(\Gamma))$. We say that (a_1, a_2) is a pair of symbols of A if it is a pair of symbols of RAR^{-1} .

The properties of $Op(S_T^\alpha)$ are transferred to $Op(S_T^\alpha(\Gamma))$ without difficulty.

Proposition 6. Let $A \in Op(S_T^\alpha(\Gamma))$. Then for all s , A is continuous from $T^s(\Gamma)$ to $T^{s-\alpha}(\Gamma)$. If A and B respectively belong to $Op(S_T^\alpha(\Gamma))$ and $Op(S_T^\beta(\Gamma))$, with pairs of symbols (a_1, a_2) and (b_1, b_2) , then AB is in $Op(S_T^{\alpha+\beta}(\Gamma))$ and admits the pair of symbols $(a_1(\tilde{C}), a_2(\tilde{C}))$ where

$$\tilde{C} = Op(\tilde{\sigma}(a_1, a_2))Op(\tilde{\sigma}(b_1, b_2)) = Op(\tilde{\sigma}(a_1, a_2)\#\tilde{\sigma}(b_1, b_2)).$$

3.3 Pseudo-differential operators on $U^s(\Gamma)$

We proceed to introduce an analogous family of pseudo-differential operators defined this time on the spaces U^s . Similar properties hold for this new family

of pseudo-differential operators. They are stated here but the proofs do not differ in any significant way from the previous, and are thus omitted. At the end of this section, we also establish two results linking pseudo-differential operators on $T^s(\Gamma)$ to pseudo-differential operators on $U^s(\Gamma)$.

Definition 10. Let A an operator on $U^{-\infty}$ and assume that there exists a couple of smooth functions a_1 and a_2 defined on $[-1, 1] \times \mathbb{N}$, that are C^∞ in the first argument and such that for all $n \in \mathbb{N}^*$,

$$AU_{n-1} = a_1(x, n)U_{n-1} + a_2(x, n)T_n. \quad (50)$$

The operator defined by the previous formula is denoted by $Op_U(a_1, a_2)$. For $n \in \mathbb{Z}$ and $\theta \in [0, 2\pi]$, define the symbol $\tilde{\sigma}(a_1, a_2)$ as before by

$$\tilde{\sigma}(a_1, a_2)(\theta, n) = a_1(\cos \theta, |n|) + i \sin \theta \operatorname{sign}(n) a_2(\cos \theta, |n|)$$

with the convention $a_1(x, 0) = a_2(x, 0) = 0$. We say that $(a_1, a_2) \in S_U^\alpha$ if $\tilde{\sigma}(a_1, a_2) \in \Sigma^\alpha$. In this case, we say that A is a pseudo-differential operator on U^s and the (non-unique) couple of functions (a_1, a_2) is called a pair of symbols of A . We also take the notation $S_U^\infty := \cup_{\alpha \in \mathbb{R}} S_U^\alpha$ and $S_U^{-\infty} = \cap_{\alpha \in \mathbb{R}} S_U^\alpha$. and the set of pseudo-differential operators (of order α) in $T^{-\infty}$ by $Op(S_U^\alpha)$ (by $Op(S_U^\alpha)$).

Recall the definition of the isometric mapping \mathcal{S} from Proposition 3.

Theorem 4. Let $A : U^\infty \rightarrow U^{-\infty}$. Assume that for some PPDO $\tilde{A} \in Op(\Sigma^\alpha)$, there holds

$$SA = \tilde{A}\mathcal{S} \quad (51)$$

as an equality of operators on U^∞ . Then A has a unique continuous extension as an element of $Op(S_U^\alpha)$, and $(a_1(\tilde{A}), a_2(\tilde{A}))$ is a pair of symbols for A . Reciprocally let $A = Op_U(a_1, a_2) \in Op(S_U^\alpha)$. Then (51) holds, taking for \tilde{A} the PPDO of order α given by the symbol

$$\sigma_{\tilde{A}} = \tilde{\sigma}(a_1, a_2).$$

Definition 11. Let $A : U^{-\infty}(\Gamma) \rightarrow U^{-\infty}(\Gamma)$. We say that A is a pseudo-differential operator (of order α) on $U^{-\infty}(\Gamma)$ if RAR^{-1} belongs to $Op(S_U^\infty)$ (to $Op(S_U^\alpha)$). The set of pseudo-differential operators of order α on $U^{-\infty}(\Gamma)$ is denoted by $Op(S_U^\alpha(\Gamma))$. We say that (a_1, a_2) is a pair of symbols of A if it is a pair of symbols of RAR^{-1} .

Corollary 6. Let Γ a smooth open curve and let $A \in Op(S_U^\alpha(\Gamma))$ Then for all s , A is continuous from $U^s(\Gamma)$ to $U^{s-\alpha}(\Gamma)$. If B and C respectively belong to $Op(S_U^{\alpha_1}(\Gamma))$ and $Op(S_U^{\alpha_2}(\Gamma))$, with pairs of symbols (b_1, b_2) and (c_1, c_2) , then BC is in $Op(S_U^{\alpha_1+\alpha_2}(\Gamma))$ and admits the pair of symbols $(a_1(\tilde{A}), a_2(\tilde{A}))$ where

$$\tilde{A} = Op(\tilde{\sigma}(b_1, b_2))Op(\tilde{\sigma}(c_1, c_2)) = Op(\tilde{\sigma}(b_1, b_2) \# \tilde{\sigma}(c_1, c_2)).$$

Lemma 20. Let $A \in Op(S_T^\alpha(\Gamma))$ and

$$B = -\partial_\tau A \omega_\Gamma \partial_\tau \omega_\Gamma.$$

Then $B \in Op(S_U^{\alpha+2}(\Gamma))$ and if \tilde{A} is a PPDO such that $CRAR^{-1} = \tilde{A}C$, then

$$SRBR^{-1} = -\partial_\theta \tilde{A} \partial_\theta \mathcal{S}.$$

Proof. One can check the following identities:

$$\begin{aligned}\partial_\theta \mathcal{S} &= -\mathcal{C}\omega\partial_x\omega, \\ \partial_\theta \mathcal{C} &= -\mathcal{S}\partial_x.\end{aligned}$$

Let $A' = RAR^{-1}$ and $B' = RBR^{-1}$. Assuming that $CA' = \tilde{A}\mathcal{C}$, there holds

$$\begin{aligned}\mathcal{S}B' &= -\mathcal{S}R\partial_\Gamma A\omega_\Gamma\partial_\Gamma\omega_\Gamma R^{-1} \\ &= -\mathcal{S}\partial_x A'\omega\partial_x\omega \\ &= \partial_\theta \mathcal{C}A'\omega\partial_x\omega \\ &= \partial_\theta \tilde{A}\mathcal{C}\omega\partial_x\omega \\ &= -\partial_\theta \tilde{A}\partial_\theta \mathcal{S}.\end{aligned}$$

Since \tilde{A} can be chosen as a PPDO of order α by Corollary 5, $\partial_\theta \tilde{A}\partial_\theta$ is then a PPDO of order $\alpha+2$ and by Theorem 4, we conclude that $B \in Op(S_U^{\alpha+2}(\Gamma))$. \square

Lemma 21. *Let $A \in Op(S_T^\alpha(\Gamma))$ and let*

$$B = A\omega_\Gamma^2.$$

Then $B \in Op(S_U^\alpha(\Gamma))$ and if \tilde{A} is a PPDO such that $\mathcal{C}RAR^{-1} = \tilde{A}\mathcal{C}$, then

$$\mathcal{S}RBR^{-1} = \sin \tilde{A} \sin \mathcal{S}$$

where \sin denotes the operator $f(\theta) \mapsto \sin(\theta)f(\theta)$.

Proof. This follows from the identities

$$\mathcal{S} = \sin \mathcal{C}, \quad \mathcal{C}\omega^2 = \sin \mathcal{S}$$

and the same arguments as in the proof of Lemma 20. \square

Definition 12. *Let A and B in $Op(S_T^\infty(\Gamma))$ (resp. $Op(S_U^\infty(\Gamma))$). If $A - B \in Op(S_T^\alpha(\Gamma))$ (resp. $Op(S_U^\alpha(\Gamma))$), we write $A = B + T_\alpha$ (resp. $A = B + U_\alpha$).*

4 Parametrices for the weighted layer potentials

Here, we apply the pseudo-differential theory on $T^s(\Gamma)$ and $U^s(\Gamma)$ to build parametrices for the weighted layer potentials introduced at the beginning of the second section. Asymptotic expansions are performed with the help of the symbolic calculus software Maple. The proofs of the next results are accompanied by commented Maple worksheets in Appendix A.

4.1 Dirichlet problem

Lemma 22. *The operator S_{k,ω_Γ} belongs to $Op(S_T^{-1}(\Gamma))$ and satisfies*

$$\mathcal{C}RS_{k,\omega_\Gamma}R^{-1} = \tilde{S}_k\mathcal{C}$$

where the symbol of $\tilde{S}_k \in Op(\Sigma^{-1})$ has the asymptotic expansion

$$\begin{aligned} \sigma_{\tilde{S}_k}(\theta, \xi) &= \frac{1}{2|\xi|} + \frac{k^2 |\Gamma|^2 \sin(\theta)^2}{16 |\xi|^3} + \frac{3ik^2 |\Gamma|^2 \sin \theta \cos \theta}{16 \xi^4 \text{sign}(\xi)} \\ &+ k^2 |\Gamma|^2 \frac{-768\kappa(\theta)^2 |\Gamma|^2 \sin^4 \theta + 112 \sin^2 \theta + 3k^2 |\Gamma|^2 \sin^4 \theta - 48}{128 |\xi|^5} \quad (52) \\ &+ \Sigma^{-6}. \end{aligned}$$

Proof. The Hankel function admits the following expansion

$$H_0(z) = \frac{-1}{2\pi} \ln |z| J_0(z) + F_1(z^2) \quad (53)$$

where J_0 is the Bessel function of first kind and order 0 and where F_1 is analytic. Let us define

$$S_{k,\omega} := RS_{k,\omega_\Gamma} R^{-1}.$$

We fix a smooth function $u \in T^\infty$. One has

$$(S_{k,\omega} u)(x) = \int_{-1}^1 H_0(k|r(x) - r(y)|) \frac{u(y)}{\omega(y)} dy.$$

Using the change of variables $x = \cos \theta$, $y = \cos \theta'$, we get

$$S_{k,\omega} u(\cos \theta) = \int_0^\pi H_0(k|r(\cos \theta) - r(\cos \theta')|) u(\cos(\theta)) d\theta,$$

which, in view of (53), can be rewritten as

$$\begin{aligned} S_{k,\omega} u(\cos \theta) &= \frac{-1}{2\pi} \int_0^\pi \ln |\cos \theta - \cos \theta'| J_0(k|r(\cos \theta) - r(\cos \theta')|) \mathcal{C}u(\theta) d\theta \\ &+ \int_0^\pi F_2(\cos \theta, \cos \theta') \mathcal{C}u(\theta) d\theta' \end{aligned}$$

where

$$F_2(x, y) = \ln \frac{|r(x) - r(y)|}{|x - y|} + F_1(k^2(x - y)^2)$$

is a C^∞ function. By parity, the second integral defines an operator

$$Ku(\theta) = \frac{1}{2} \int_{-\pi}^\pi F_2(\cos \theta, \cos \theta') \mathcal{C}u(\theta) d\theta.$$

There holds $K = \tilde{R}_1 \mathcal{C}$ where, by Proposition 5, $R_1 \in Op(\Sigma^{-\infty})$. For the first integral, we make the following classical manipulations. We first write $\cos \theta - \cos \theta' = -2 \sin \frac{\theta + \theta'}{2} \sin \frac{\theta - \theta'}{2}$. Thus

$$\ln |\cos \theta - \cos \theta'| = \ln \left| \sqrt{2} \sin \frac{\theta + \theta'}{2} \right| + \ln \left| \sqrt{2} \sin \frac{\theta - \theta'}{2} \right|.$$

We then integrate and apply the change of variables $\theta \rightarrow -\theta$ for the second term, yielding

$$S_{k,\omega} u(\cos \theta) = (\tilde{S}_{k,1} + \tilde{R}_1) \mathcal{C}u(\theta)$$

where

$$\tilde{S}_{k,1}u(\theta) = \frac{-1}{2\pi} \int_{-\pi}^{\pi} \ln \left| \sqrt{2} \sin \frac{\theta - \theta'}{2} \right| J_0(k|r(\cos \theta) - r(\cos \theta')|) u(\theta') d\theta'.$$

Let $g : \theta \mapsto -\frac{1}{2\pi} \ln \left| \sqrt{2} \sin \frac{\theta}{2} \right|$. It is well-known that $\hat{g}(n) = \frac{1}{2|n|}$ for $n \neq 0$. We may prolong this by $\hat{g}(\xi) = \frac{1}{2|\xi|}$ away from $\xi = 0$. Let

$$a(\theta, \theta') = J_0(k|r(\cos \theta) - r(\cos \theta')|),$$

which is a smooth function. By Proposition 5, the operator

$$\tilde{S}_{k,1}u(\theta) := \int_{-\pi}^{\pi} g(\theta - \theta') a(\theta, \theta') u(\theta') d\theta'$$

is in $Op(\Sigma^{-1})$. In particular, $\tilde{S}_{k,1}u$ is a smooth function, from which we deduce that $\theta \mapsto S_{k,\omega}u(\cos \theta)$ is a smooth (even) function. Proposition 3 then ensures

$$S_{k,\omega}u(\cos \theta) = \mathcal{C}S_{k,\omega}u(\theta).$$

This establishes that $\mathcal{C}S_{k,\omega}u = \tilde{S}_k \mathcal{C}u$ for any smooth function u . By Corollary 5, this implies that $S_{k,\omega} \in Op(S_T^{-1})$. We can compute the symbol of $\tilde{S}_{k,1}$ using the asymptotic expansion (47). The terms $\partial_s^j a(t, s)|_{t=s}$, can be related to the geometric characteristics of Γ through expansion (19). The asymptotic expansion (52) for $\tilde{S}_{k,1}$ is obtained the help of Maple, and we refer the reader to appendix A.1, eq. (1). To simplify the expressions, the computations were only performed for $\xi > 0$, where $\hat{g}(\xi) = \frac{1}{2\xi}$. In the general case, the $\frac{1}{\xi^4}$ term in the asymptotic expansion must be multiplied by $\text{sign}(\xi)$ to account for the case $\xi < 0$. Obviously since $\tilde{R}_1 \in Op(\Sigma^{-\infty})$, the asymptotic expansion also holds for $\tilde{S}_k := \tilde{S}_{k,1} + \tilde{R}_1$, concluding the proof. \square

In particular, by Proposition 6,

Corollary 7. S_{k,ω_Γ} is continuous from $T^s(\Gamma)$ to $T^{s+1}(\Gamma)$ for all $s \in \mathbb{R}$ and thus maps $C^\infty(\Gamma)$ to itself. A pair of symbols of S_{k,ω_Γ} is given by

$$\begin{aligned} a_1(x, n) &= \frac{1}{2n} + \frac{k^2 |\Gamma|^2 \omega(x)^2}{16n^3} + \tilde{a}_1, \\ a_2(x, n) &= \frac{3xk^2 |\Gamma|^2}{16n^4} + \tilde{a}_2, \end{aligned}$$

where $(\tilde{a}_1, \tilde{a}_2) \in S_T^{-5}$.

Lemma 23. The operator $-(\omega_\tau \partial_\tau)^2 - k^2 \omega_\Gamma^2$ is in $Op(S_T^2(\Gamma))$ with the pair of symbols

$$\begin{aligned} a_1(x, n) &= n^2 - k^2 \frac{|\Gamma|^2 \omega^2(x)}{4}, \\ a_2(x, n) &= 0. \end{aligned}$$

It satisfies

$$\mathcal{C}R [-(\omega_\tau \partial_\tau)^2 - k^2 \omega_\Gamma^2] R^{-1} = \tilde{D}_k \mathcal{C}$$

where $\tilde{D}_k \in Op(\Sigma^2)$ has the symbol

$$\sigma_{\tilde{D}_k}(\theta, \xi) = |\xi|^2 - k^2 |\Gamma|^2 \sin^2(\theta). \quad (54)$$

Proof. Recalling eqs. (20),(21), one has

$$-(\omega_\Gamma)^2 - k^2\omega_\Gamma^2 = R^{-1} [-(\omega\partial_x)^2 - k^2\omega^2] R.$$

Letting $D_k = -(\omega\partial_x)^2 - k^2\omega^2$,

$$D_k T_n = (n^2 - k^2 |\Gamma|^2 \omega^2) T_n.$$

The result is then a consequence of Corollary 5. \square

Theorem 5. *The operators $[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2]$ and S_{k,ω_Γ} are respectively in $Op(S_T^2(\Gamma))$ and $Op(S_T^{-1}(\Gamma))$ and satisfy*

$$[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2] S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + T_{-4}.$$

Proof. We have shown that $[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2]$ and S_{k,ω_Γ} are respectively in $Op(S_T^2(\Gamma))$ and $Op(S_T^{-1}(\Gamma))$ in the previous two lemmas. Using the symbolic calculus described in section 3, we can compute an asymptotic expansion of the symbol of the pseudo-differential operator

$$[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2] S_{k,\omega_\Gamma}^2 - \frac{I_d}{4}.$$

The symbol of this operator is found to be in $S_T^{-4}(\Gamma)$, from which the result follows. Details of the computations can be found in Appendix A.1. \square

Remark 5. *The previous theorem implies the following fact*

$$-(\omega_\Gamma\partial_\tau)^2 S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + R$$

where R is in $Op(S_T^{-2}(\Gamma))$. This is also a compact perturbation of the identity. Nevertheless, since $R = k^2\omega_\Gamma^2 S_{k,\omega_\Gamma}^2 + T_{-4}$ the term $k^2\omega^2 S_{k,\omega}^2 \in Op(S_T^{-2})$ can be viewed as the leading first order correction accounting for the wavenumber. In numerical applications, the inclusion of this term in the preconditioners leads to very significant reductions of the number of GMRES iterations to solve the linear systems arising from the Galerkin discretization, as demonstrated in [1].

4.2 Neumann problem

We saw in Lemma 11 that $N_{k,\omega_\Gamma} = N_1 - k^2 N_2$ where

$$N_1 = -\partial_\tau S_{k,\omega_\Gamma} \partial_\tau \omega_\Gamma$$

and $N_2 = V_k \omega_\Gamma^2$ with

$$V_k u(x) = \int_\Gamma \frac{G_k(x-y) N(x) \cdot N(y) u(y)}{\omega_\Gamma(y)} d\sigma(y).$$

Lemma 24. *The operator N_1 is in $Op(S_T^2(\Gamma))$ and*

$$\mathcal{S} R N_1 R^{-1} = \tilde{N}_1 \mathcal{S}$$

where \tilde{N}_1 is a PPDO with a symbol $\sigma_{\tilde{N}_1}$ satisfying

$$\sigma_{\tilde{N}_1}(\theta, \xi) = \frac{|\xi|}{2} + \frac{1}{16} \frac{k^2 |\Gamma|^2 \sin^2(\theta)}{|\xi|} + i \frac{k^2 |\Gamma|^2 \sin \theta \cos \theta}{16 \xi^2 \text{sign}(\xi)} + \Sigma^{-3}. \quad (55)$$

As a consequence, N_1 admits the pair of symbols

$$a_1(x, n) = \frac{n}{2} + \frac{k^2 |\Gamma|^2 \omega(x)^2}{16n} + \tilde{a}_1, \quad (56)$$

$$a_2(x, n) = \frac{k^2 |\Gamma|^2 x}{16n^2} + \tilde{a}_2, \quad (57)$$

where $(\tilde{a}_1, \tilde{a}_2) \in S_U^{-3}$

Proof. This result is obtained by symbolic calculus combining Lemma 22 and Lemma 20. See appendix A.2 eq. (1). \square

A small adaptation of the proof of Lemma 22 yields the following result (see Appendix A.2 eq. (2))

Lemma 25. *The operator V_k is in $Op(S_T^{-1}(\Gamma))$ and*

$$CRV_k R^{-1} = \tilde{V}_k \mathcal{C}$$

where \tilde{V}_k is a PPDO with a symbol $\sigma_{\tilde{V}_k}$ satisfying

$$\sigma_{\tilde{V}_k} = \frac{1}{2|\xi|} + \Sigma^{-3}.$$

V_k thus admits the pair of symbols

$$\begin{aligned} a_1(x, n) &= \frac{1}{2n} + \tilde{a}_1(x, n), \\ a_2(x, n) &= \tilde{a}_2(x, n), \end{aligned}$$

where $(\tilde{a}_1, \tilde{a}_2) \in S_T^{-3}$.

Applying Lemma 21, we deduce

Corollary 8. *The operator N_2 is in $Op(S_U^{-1}(\Gamma))$ and satisfies*

$$\mathcal{S}RN_2 R^{-1} = \tilde{N}_2 \mathcal{S}$$

where the symbol of \tilde{N}_2 has the asymptotic expansion

$$\sigma_{\tilde{N}_2} = \frac{\sin^2 \theta}{2|\xi|} + i \frac{\sin \theta \cos \theta}{2\xi^2 \text{sign}(\xi)} + \Sigma^{-3}. \quad (58)$$

A pair of symbols for N_2 is thus

$$\begin{aligned} a_1(x, n) &= \frac{\omega(x)^2}{2n} + \tilde{a}_1, \\ a_2(x, n) &= \frac{x}{2n^2} + \tilde{a}_2, \end{aligned}$$

where $(\tilde{a}_1, \tilde{a}_2) \in S_U^{-3}$.

It is also easy to check the following result:

Lemma 26. *The operator $[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2]$ is in $Op(S_U^2(\Gamma))$ and satisfies*

$$SR [-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2] R^{-1} = \tilde{D}_k S$$

where \tilde{D}_k is the operator defined in Lemma 23.

Theorem 6. *The operators $[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2]$ and N_{k,ω_Γ} are respectively in $Op(S_U^2(\Gamma))$ and $Op(S_U^1(\Gamma))$ and satisfy*

$$N_{k,\omega_\Gamma}^2 = \frac{1}{4} [-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2] + U_{-2}.$$

Proof. Gathering the previous lemmas, we have asymptotic expansions available for the symbols of the operators N_{k,ω_Γ} and $[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2]$. We can thus compute an asymptotic expansion of the symbol of the operator $N_{k,\omega_\Gamma}^2 - [-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2]$ which turns out to be in $S_U^{-2}(\Gamma)$, giving the result. Details of the computations can be found in Appendix A.2. \square

5 Conclusion

In this work and [1], we described and analyzed a new method for the numerical resolution of first-kind integral equations modeling wave scattering by open curves, with Dirichlet or Neumann boundary conditions. The equations are discretized with a Galerkin method in weighted L^2 space, with piecewise linear functions defined on an explicitly refined mesh. We have established optimal rates of convergence for the method. We have also introduced an adapted analytic framework with, in particular, a new pseudo-differential calculus on open curves, providing a precise understanding of the weighted layer potentials. A simple and effective symbolic calculus has enabled us to build parametrices for those operators. In turn, those parametrices can be discretized to provide preconditioners for the linear systems. Our numerical results, exposed in [1], are in close agreement with the theoretical predictions. The preconditioners have been tested on various examples, including cases at high frequencies. They indeed lead to great reductions in the number of GMRES iterations for the resolution of the linear systems.

The numerical method is straightforward to implement and is only in a small variant of the standard method. The only differences are summarized below:

Mesh refinement: The mesh has to be refined in an explicit way near the edges. In fact, it must be uniform with respect to the weight function, that is the quantity

$$\int_{S_i} \frac{1}{\omega(x)} dx$$

must be constant over each segment S_i of the mesh.

Numerical quadratures: The numerical quadratures used to discretize the integrals must be modified to take the weight $w = \frac{1}{\omega}$ into account. Explicit Gaussian quadratures are available in the case of $w(x) = \frac{1}{\sqrt{1-x^2}}$.

Singular integrals: The main difficulty is the efficient computation of singular integrals of the form

$$\int_a^b \frac{\ln|x-y|dy}{\omega(y)},$$

where a , b and x are close to 1 or -1 . An efficient method is to rewrite this integral as

$$\int_{\theta_a}^{\theta_b} \ln|\sin\theta(\theta-\theta')| d\theta + \int_{\theta_a}^{\theta_b} \ln\left(\frac{|\cos\theta-\cos\theta'|}{|\sin\theta(\theta-\theta')|}\right) d\theta,$$

where $\theta_a = \arccos(a)$, $\theta_b = \arccos(b)$ and $\theta = \arccos(x)$. The first term can be computed explicitly, while the second is the integral of a smooth function, which is approximated using the standard Gaussian quadrature.

Finally, the method easily lends itself to formal generalizations. It is not difficult to treat the case of a flat disk in \mathbb{R}^3 (this work is in progress). Future works also include the treatment of general angles in \mathbb{R}^2 . Eventually, we expect that a suitable change of the weight function could allow to extend the approach to arbitrary geometries.

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A Symbolic Calculus

A.1 Single layer potential

restart, gc();

Procedure for the (usual) symbolic calculs:

```

symbolCalc := proc(a, b, k)
local res, j, resPast,
resPast := 0;
for j from 0 to k do
res := resPast +  $\frac{1}{j!} \cdot \left(\frac{1}{l}\right)^j \text{D}[2\$j](a) \cdot \text{D}[1\$j](b)$ ;
resPast := res
end do;
return res(x, xi);
end proc;

```

Symbol of the operator $-(\omega\partial_x)^2 - k^2\omega^2$

$\Delta := (x, xi) \rightarrow \xi^2 - \frac{k^2 \cdot L^2}{4} \cdot \sin^2(x)$:

Taylor expansion of $|r(\cos t) - r(\cos s)|$ where $r : [-1, 1] \rightarrow \Gamma$ is a parametrization such that $|r'(x)| = \frac{L}{2}$ for all x . L is the length of Γ and $C(t)$ is the curvature. K denotes the unknown constant in the next order.

$r := (t, s) \rightarrow \text{sqrt}\left(\frac{L^2}{4} \cdot (\cos(t) - \cos(s))^2 - \frac{L^4}{192} \cdot C(t)^2 \cdot (\cos(t) - \cos(s))^4 + K \cdot (\cos(t) - \cos(s))^5\right)$:

Taylor expansion of $b(t, s) = J_0(k|r(\cos t) - r(\cos s)|)$.

$b := (t, s) \rightarrow 1 - k^2 \cdot \frac{r(t, s)^2}{4} + k^4 \cdot \frac{r(t, s)^4}{16} + K \cdot r(t, s)^6$:

We use the following procedure to compute an asymptotic expansion of the symbol of \tilde{S}_k . K denotes the unknown constant in the next order.

```

makeSk := proc(N)
local res, j;
res := 0;
res := res +  $\frac{\text{limit}(b(s, x), s = x)}{xi}$  ;
for j from 1 to N - 1 do;
res := res +  $\frac{1}{j!} \cdot \left(\frac{1}{l}\right)^j \cdot \text{limit}(\text{diff}(b(s, x), s\$j), s = x) \cdot (-1)^j \cdot j!$ ;
end do;
return res +  $\frac{K}{\xi^{N+1}}$ ;

```

end proc:

We obtain the following asymptotic expansion up to order 6:

$sk := unapply\left(\frac{1}{2} \cdot makeSk(6), x, xi\right);$

$$\begin{aligned}
(x, \xi) \rightarrow & \frac{1}{2\xi} + \frac{1}{16} \frac{k^2 L^2 \sin(x)^2}{\xi^3} + \frac{\frac{3}{16} I k^2 L^2 \sin(x) \cos(x)}{\xi^4} \\
& + \frac{1}{2} \frac{-\frac{3}{8} k^2 L^2 \cos(x)^2 + \frac{1}{32} k^2 L^4 C(x)^2 \sin(x)^4 + \frac{1}{2} k^2 L^2 \sin(x)^2 + \frac{3}{32} k^4 L^4 \sin(x)^4}{\xi^5} \\
& + \frac{1}{\xi^6} \left(\frac{1}{2} I \left(\frac{5}{16} k^2 L^4 C(x) \sin(x)^4 D(C)(x) + \frac{5}{16} k^2 L^4 C(x)^2 \sin(x)^3 \cos(x) + 30 k^2 K \sin(x)^5 \right. \right. \\
& \left. \left. + \frac{15}{8} k^2 L^2 \sin(x) \cos(x) + \frac{15}{16} k^4 L^4 \sin(x)^3 \cos(x) \right) \right) + \frac{1}{2} \frac{K}{\xi^7}
\end{aligned} \tag{1}$$

We can thus compute the symbol of the operator \tilde{S}_k^2 using symbolic calculus, and keep the terms up to order -6 .

$sk2tmp := collect(symbolCalc(sk, sk, 3), xi); sk2 := convert([op(1..4, sk2tmp)], `+`) + \frac{K}{\xi^7}; sk2 := unapply(sk2, x, xi)$

$$\begin{aligned}
(x, \xi) \rightarrow & \frac{1}{4\xi^2} + \frac{1}{16} \frac{k^2 L^2 \sin(x)^2}{\xi^4} + \frac{\frac{1}{4} I k^2 L^2 \sin(x) \cos(x)}{\xi^5} + \frac{1}{\xi^6} \left(\frac{13}{256} k^4 L^4 \sin(x)^4 \right. \\
& - \frac{1}{4} k^2 L^2 \cos(x)^2 + \frac{1}{64} k^2 L^4 C(x)^2 \sin(x)^4 + \frac{5}{16} k^2 L^2 \sin(x)^2 + \frac{1}{2} I \left(\frac{3}{16} I k^2 L^2 \cos(x)^2 \right. \\
& \left. \left. - \frac{3}{16} I k^2 L^2 \sin(x)^2 \right) \right) + \frac{K}{\xi^7}
\end{aligned} \tag{2}$$

We now apply symbolic calculus to compute an asymptotic expansion of the symbol of the composition $\tilde{D}_k \tilde{S}_k^2$, keeping only the first two terms

$DkSk2 := collect(symbolCalc(Delta, sk2, 4), xi); a := unapply\left(convert([op(1..2, DkSk2)], `+`) + \frac{K}{\xi^5}, x, xi \right)$

$$\begin{aligned}
(x, \xi) \rightarrow & \frac{1}{4} + \frac{1}{\xi^4} \left(-\frac{3}{8} k^2 L^2 \cos(x)^2 + \frac{9}{256} k^4 L^4 \sin(x)^4 + \frac{1}{64} k^2 L^4 C(x)^2 \sin(x)^4 \right. \\
& + \frac{7}{16} k^2 L^2 \sin(x)^2 + \frac{1}{2} I \left(\frac{3}{16} I k^2 L^2 \cos(x)^2 - \frac{3}{16} I k^2 L^2 \sin(x)^2 \right) - 2 I \left(\frac{1}{4} I k^2 L^2 \cos(x)^2 \right. \\
& \left. \left. - \frac{1}{4} I k^2 L^2 \sin(x)^2 \right) \right) + \frac{K}{\xi^5}
\end{aligned} \tag{3}$$

We see that this is of the form $\frac{1}{4} + \sigma$ where $\sigma \in \Sigma^{-4}$.

A.2 Hypersingular operator

restart; gc();

Procedure for the (usual) symbolic calculs:

```

symbolCalc := proc(a, b, k)
local res, j, resPast;
resPast := a(x, xi)·b(x, xi);
for j from 1 to k do
res := resPast +  $\frac{1}{j!} \cdot \left(\frac{1}{I}\right)^j \text{diff}(a(x, xi), xi\$j) \cdot \text{diff}(b(x, xi), x\$j)$ ;
resPast := res
end do;
return res;
end proc;

```

Symbols of the operators $(\partial_x \omega)^2 - k^2 \omega^2$, $u(x) \mapsto \sin(x)u(x)$ and ∂_θ :

Delta := (x, xi) → $\xi^2 - \frac{k^2 \cdot L^2}{4} \cdot \sin^2(x)$:

sinx := (x, xi) → sin(x) :

dtheta := (x, xi) → I·xi :

Taylor expansion of $|r(\cos t) - r(\cos s)|$ where $r : [-1, 1] \rightarrow \Gamma$ is a parametrization such that $|r'(x)| = \frac{L}{2}$ for all x . L is the length of Γ and $C(t)$ is the curvature. K denotes the unknown constant in the next order:

r := (t, s) → sqrt($\left(\frac{L^2}{4} \cdot (\cos(t) - \cos(s))^2 - \frac{L^4}{192} \cdot C(t)^2 \cdot (\cos(t) - \cos(s))^4 + K \cdot (\cos(t) - \cos(s))^5\right)$):

Taylor expansion of $b(t, s) = J_0(k|r(\cos t) - r(\cos s)|)n(\cos t) \cdot n(\cos s)$ where $n(t)$ is the normal vector at the point $r(t)$:

b := (t, s) → $\left(1 - k^2 \cdot \frac{r(t, s)^2}{4} + k^4 \cdot \frac{r(t, s)^4}{16} + K \cdot r(t, s)^6\right) \cdot \left(1 - \frac{L^2 \cdot C(t)^2}{8} \cdot (\cos(t) - \cos(s))^2 + \frac{(\cos(t) - \cos(s))^4 \cdot L^4}{16 \cdot 24} + K \cdot (\cos(t) - \cos(s))^5\right)$:

We first compute an asymptotic expansion up to order 3 of the symbol of $\tilde{N}_1 = -\partial_\theta \tilde{S}_k \partial_\theta$. We already know the symbol of \tilde{S}_k

sk := (x, xi) → $\frac{1}{2\xi} + \frac{1}{16} \frac{k^2 L^2 \sin(x)^2}{\xi^3} + \frac{\frac{3}{16} k^2 L^2 \sin(x) \cos(x)}{\xi^4}$

$$+ \frac{1}{2} \frac{-\frac{3}{8} k^2 L^2 \cos(x)^2 + \frac{1}{32} k^2 L^4 C(x)^2 \sin(x)^4 + \frac{1}{2} k^2 L^2 \sin(x)^2 + \frac{3}{32} k^4 L^4 \sin(x)^4}{\xi^5} + \frac{1}{2} \frac{K}{\xi^6};$$

so we just need to use the usual symbol calculus. We obtain the following symbol:

```
N1tmp := unapply(symbolCalc(dtheta, sk, 3), x, xi) : N1 :=
- collect(simplify(symbolCalc(N1tmp, dtheta, 3)), xi) : N1 := unapply(
convert((op(1..4,
N1))L `+`) +  $\frac{K}{\xi^4}$ , x, xi)
```

$$(x, \xi) \rightarrow \frac{1}{2} \xi - \frac{\frac{1}{64} I (4 I k^2 L^2 - 4 I k^2 L^2 \cos(x)^2)}{\xi} + \frac{\frac{1}{16} I k^2 L^2 \sin(x) \cos(x)}{\xi^2} - \frac{1}{\xi^3} \left(\frac{1}{64} I (3 I k^4 L^4 - 6 I k^4 L^4 \cos(x)^2 + 3 I k^4 L^4 \cos(x)^4 - 4 I k^2 L^2 \cos(x)^2 + I k^2 L^4 C(x)^2 - 2 I k^2 L^4 C(x)^2 \cos(x)^2 + I k^2 L^4 C(x)^2 \cos(x)^4 + 4 I k^2 L^2) \right) + \frac{K}{\xi^4} \quad (1)$$

We then turn to the computation of an asymptotic expansion of the symbol of \tilde{N}_2 . We start with a procedure to compute the symbol of the operator \tilde{V}_k :

```
makeVk := proc(N)
local res, j;
res := 0;

res := res +  $\frac{\text{limit}(b(s, x), s = x)}{xi}$ ;

for j from 1 to N - 1 do;
res := res +  $\frac{1}{j!} \cdot \left(\frac{1}{I}\right)^j \cdot \frac{\text{limit}(\text{diff}(b(s, x), s\$j), s = x) \cdot (-1)^j \cdot j!}{\xi^{j+1}}$ ;
end do;
return res +  $\frac{K}{\xi^{N+1}}$ ;
end proc;
```

The following symbol is obtained for \tilde{V}_k :

```
vk := unapply( $\frac{1}{2}$  · makeVk(3), x, xi)
```

$$(x, \xi) \rightarrow \frac{1}{2} \xi - \frac{1}{2} \frac{-\frac{1}{8} k^2 L^2 \sin(x)^2 - \frac{1}{4} L^2 C(x)^2 \sin(x)^2}{\xi^3} + \frac{1}{2} \frac{K}{\xi^4} \quad (2)$$

The operator \tilde{N}_2 is then obtained by multiplying left and right by the operator $u(x) \mapsto u(x)\sin(x)$. We obtain the following asymptotic expansion up to order 3.

```
N2tmp := unapply(collect(symbolCalc(sin(x), v(x), 2), x), x, xi) : N2 := collect(symbolCalc(N2tmp,
sin(x), 2), xi) : N2 := unapply( convert([op(1..3, N2)] , `+ `) +  $\frac{K}{\xi^4}$ , x, xi )
```

$$\begin{aligned} (x, \xi) \rightarrow & \frac{1}{2} \frac{\sin(x)^2}{\xi} + \frac{\frac{1}{2} I \sin(x) \cos(x)}{\xi^2} \\ & + \frac{\sin(x)^2 \left(\frac{1}{16} k^2 L^2 \sin(x)^2 + \frac{1}{8} L^2 C(x)^2 \sin(x)^2 \right) + \frac{1}{2} \sin(x)^2}{\xi^3} + \frac{K}{\xi^4} \end{aligned} \quad (3)$$

The symbol of $\tilde{N}_k = \tilde{N}_1 - \frac{k^2 L^2}{4} \tilde{N}_2$ is thus, retaining only the terms up to order 3:

```
tildeNk := (x, xi) -> N1(x, xi) -  $\frac{k^2 \cdot L^2}{4}$  \cdot N2(x, xi) : tildeNk := unapply( convert([op(1..4,
collect(simplify(tildeNk(x, xi)), xi)] , `+ `) +  $\frac{K}{\xi^4}$ , x, xi )
```

$$\begin{aligned} (x, \xi) \rightarrow & \frac{1}{2} \xi + \frac{-\frac{1}{16} k^2 L^2 + \frac{1}{16} k^2 L^2 \cos(x)^2}{\xi} - \frac{\frac{1}{16} I k^2 L^2 \sin(x) \cos(x)}{\xi^2} + \frac{1}{\xi^3} \left(-\frac{1}{16} k^4 L^4 \cos(x)^4 \right. \\ & + \frac{1}{32} k^4 L^4 \cos(x)^4 + \frac{1}{16} k^2 L^2 \cos(x)^2 + \frac{1}{32} k^2 L^4 C(x)^2 \cos(x)^2 - \frac{1}{64} k^2 L^4 C(x)^2 \cos(x)^4 \\ & \left. - \frac{1}{16} k^2 L^2 + \frac{1}{32} k^4 L^4 - \frac{1}{64} k^2 L^4 C(x)^2 \right) + \frac{K}{\xi^4} \end{aligned} \quad (4)$$

We can now compute the symbol of \tilde{N}_k^2 by usual symbolic calculus, retaining terms up to order 2.

```
nksquared := collect(simplify(symbolCalc(tildeNk, tildeNk, 2)), xi) : nksquared
:= unapply( convert([op(1..4, nksquared)] , `+ `) +  $\frac{K}{\xi^3}$ , x, xi )
```

$$\begin{aligned} (x, \xi) \rightarrow & \frac{1}{4} \xi^2 - \frac{1}{16} k^2 L^2 + \frac{1}{16} k^2 L^2 \cos(x)^2 + \frac{1}{\xi^2} \left(\frac{1}{32} k^2 L^4 C(x)^2 \cos(x)^2 - \frac{1}{64} k^2 L^4 C(x)^2 \right. \\ & - \frac{1}{64} k^2 L^4 C(x)^2 \cos(x)^4 - \frac{9}{128} k^4 L^4 \cos(x)^2 + \frac{9}{256} k^4 L^4 \cos(x)^4 + \frac{9}{256} k^4 L^4 - \frac{1}{32} k^2 L^2 \left. \right) \\ & + \frac{K}{\xi^3} \end{aligned} \quad (5)$$

The difference $\tilde{N}_k^2 - \frac{1}{4} \tilde{D}_k$,

```
collect( simplify( nksquared(x, xi) -  $\frac{1}{4}$  \cdot Delta(x, xi) ), xi )
```

$$\begin{aligned} & \frac{1}{\xi^2} \left(\frac{1}{32} k^2 L^4 C(x)^2 \cos(x)^2 - \frac{1}{64} k^2 L^4 C(x)^2 - \frac{1}{64} k^2 L^4 C(x)^2 \cos(x)^4 - \frac{9}{128} k^4 L^4 \cos(x)^2 \right. \\ & \left. + \frac{9}{256} k^4 L^4 \cos(x)^4 + \frac{9}{256} k^4 L^4 - \frac{1}{32} k^2 L^2 \right) + \frac{K}{\xi^3} \end{aligned} \tag{6}$$

is in Σ^{-2}