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# New preconditioners for Laplace and Helmholtz integral equations on open curves

## I. Theoretical framework and numerical results

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### **Abstract**

The numerical resolution of wave scattering problems by open curves leads to ill-conditioned linear systems which are difficult to precondition due to the geometrical singularities at the edges. We introduce two new preconditioners to tackle this problem respectively for Dirichlet or Neumann boundary data, that take the form of square roots of local operators. We describe an adapted analytical setting to analyze them and demonstrate the efficiency of this method on several numerical examples. A complete new pseudo-differential calculus suited to the study of such operators is postponed to the second part of this work.

## **1 Introduction**

For the resolution of wave scattering problems in the framework of boundary element methods with first-kind integral equations, two desirable features are:

- (i) Fast convergence in terms of the mesh size,
- (ii) Availability of a preconditioner for the resulting linear system.

The situation is very different depending on the regularity of the geometry of the domain under consideration. In the case of smooth geometries, the standard Galerkin method with a uniform mesh and piecewise polynomial functions of degree  $p$  enjoys optimal rates of convergence in terms of the mesh-size  $h$ . Namely, the error in energy norm converges in  $O(h^{p-\sigma})$ ,  $\sigma$  being the pseudo-differential order of the underlying operator [36]. On the other hand, the construction of efficient preconditioners for such geometries has received considerable attention since two decades. Among the possible strategies are the so-called pseudo-differential preconditioners [3–5, 11, 18, 39]. Roughly speaking, if the original problem is written in the abstract form

$$\mathcal{L}u = f, \tag{1}$$

where  $\mathcal{L}$  is a linear operator, the strategy consists in left multiplying equation (1) by an operator  $\mathcal{K}$  and solve

$$\mathcal{K}\mathcal{L}u = \mathcal{K}f \quad (2)$$

where  $\mathcal{K}$  is chosen so that the product  $\mathcal{K}\mathcal{L}$  is a compact perturbation of the identity. Following [38], a Galerkin approximation of  $\mathcal{K}$  provides an efficient preconditioner for the linear system, whose condition number becomes independent of the mesh size. Several strategies, depending on the problem to solve (e.g. Helmholtz or Maxwell equations) have been studied in the literature to propose such operators  $\mathcal{K}$ , that often turn out to be very effective in practice, when numerical applications are considered. Among those, we would like to emphasize the viewpoint first proposed in [3,4], where, for Helmholtz equation, the authors consider integral formulations of the problem that involve the Dirichlet-Neumann map  $\Lambda$  which leads to well-conditioned systems after discretization. Combined with the approximation of  $\Lambda$  suggested in [5] under the form of the square root operator

$$\Lambda \sim \sqrt{-\Delta_\Gamma - k^2}, \quad (3)$$

where  $\Delta_\Gamma$  stands for the Laplace-Beltrami operator on the surface  $\Gamma$ , the method (called GCSIE) yields a very impressive reduction of the number of iterations in the iterative resolution of the system.

In contrast, when the domain contains geometrical singularities, the two requirements (i) and (ii) are not easily fulfilled for two reasons. First, the exact solutions of the integral equations are themselves singular (see for example [14] and references therein), leading to poor rates of convergence in the classical Galerkin method. Second, the preconditioning strategies mentioned above become inapplicable due to the singularities, and inefficient in practice. Existing theories for pseudo-differential calculus on singular geometries (see e.g. [27, 33, 34]) do not seem to be able to provide simple replacements for this context. In the literature, those two issues are mostly tackled separately. Regarding the order of convergence, possible approaches include the addition of special singular functions to the trial space [12, 40], mesh refinement [16], and a combination of the latter with an increase of the polynomial order near the singularity [31]. On the other hand, many recent contributions have tackled the question of preconditioning for some particular singular geometries, in 2D or 3D such as [10, 18–20, 32]. In most of these works, weighted versions of the usual integral operators are introduced, which enjoy better mapping properties than the standard ones, and the analysis is obtained “by hand” without any explicit use of pseudo-differential calculus, lacking genericity.

The aim of this work is to address all those questions and provide the reader with a complete and generic theory that solves the problems mentioned above.

Namely, we consider here the numerical resolution of Laplace and Helmholtz scattering problems by open curves in the plane. We introduce a piecewise affine Galerkin method for the weighted layer potentials, with optimal orders of convergence and suitable preconditioners for the resulting linear systems, thus providing a method with both features (i) and (ii).

Our contribution is split in two parts, which have been thought with the following complementary roles:

- This first part provides a self-contained description of the method and its implementation, together with an exposition of the numerical results,

- The second part [7] contains the whole theoretical analysis of the method. In particular the suitable pseudo-differential calculus on open curves is introduced there.

Preliminary analytical results, that show the optimality of the approach for Laplace problems, are nevertheless included in the first part of the work.

The paper is organized as follows: in the first section, we introduce some notation and set up the problem. In the second section, we treat the case of the Laplace equation, where it is possible to derive explicit inverses of the weighted layer potentials in terms of square roots of local operators. We then state generalizations of those results to non-zero frequencies and non-flat arcs in the third section. In the last section, we show the efficiency of this approach on several numerical examples.

## 2 Scattering by an open curve

Let  $\Gamma$  be a smooth non-intersecting bounded open curve in  $\mathbb{R}^2$ , and let  $k \geq 0$  the wave number. We seek a solution of the Helmholtz equation

$$-\Delta u - k^2 u = 0, \text{ in } \mathbb{R}^2 \setminus \Gamma \quad (4)$$

when one considers furthermore Dirichlet or Neumann boundary conditions, namely

$$u = u_D, \text{ on } \Gamma \quad (5)$$

or

$$\frac{\partial u}{\partial n} = u_N \text{ on } \Gamma \quad (6)$$

respectively. In order for the problem to be well posed, we also need to impose suitable decay at infinity, given by the Sommerfeld condition

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right) \quad (7)$$

with  $r = |x|$  for  $x \in \mathbb{R}^2$ . Notice that in equation (6),  $n$  stands for a smooth unit vector normal to  $\Gamma$ .

Existence and uniqueness of solutions to the previous problems are guaranteed by the following proposition.

**Proposition 1.** [28, 40, 41]. *Assume  $u_D \in H^{1/2}(\Gamma)$ , and  $u_N \in H^{-1/2}(\Gamma)$ . Then problems (4,5,7) and (4,6,7) both possess a unique solution  $u \in H_{\text{loc}}^1(\mathbb{R}^2 \setminus \Gamma)$ , which is of class  $C^\infty$  outside  $\Gamma$ .*

For the definition of Sobolev spaces on smooth open curves, we follow [25] by considering any smooth closed curve  $\tilde{\Gamma}$  containing  $\Gamma$ , and defining

$$H^s(\Gamma) = \{U|_\Gamma \mid U \in H^s(\tilde{\Gamma})\}.$$

This definition does not depend on the particular choice of the closed curve  $\tilde{\Gamma}$  containing  $\Gamma$ . Moreover, we also define

$$\tilde{H}^s(\Gamma) = \{u \in H^s(\Gamma) \mid \tilde{u} \in H^s(\tilde{\Gamma})\}$$

where  $\tilde{u}$  denotes the extension by zero of  $u$  on  $\tilde{\Gamma}$ .

The solution to (4,5,7) and (4,6,7) can be furthermore expressed in terms of the Green function associated to the problem

$$\begin{cases} G_0(z) = -\frac{1}{2\pi} \ln |z|, & \text{if } k = 0, \\ G_k(z) = \frac{i}{4} H_0(k|z|), & \text{if } k > 0, \end{cases} \quad (8)$$

where  $H_0$  is the Hankel function of the first kind. The expressions involve the so-called single-layer, double-layer and hypersingular potentials that we recall now. The single-layer potential is defined by

$$\forall x \notin \Gamma, \quad \mathcal{S}_k \lambda(x) = \int_{\Gamma} G_k(x-y) \lambda(y) d\sigma(y) \quad (9)$$

and the solution  $u$  of the Dirichlet problem (4,5,7) can be expressed as

$$u = \mathcal{S}_k \lambda \quad (10)$$

where  $\lambda \in \tilde{H}^{-1/2}(\Gamma)$  is the unique solution to

$$S_k \lambda = u_D. \quad (11)$$

Here,  $S_k := \gamma \mathcal{S}_k$ , where  $\gamma$  is the trace operator on  $\Gamma$ . The operator  $S_k$  maps continuously  $\tilde{H}^{-1/2}(\Gamma)$  to  $H^{1/2}(\Gamma)$  [41, Theorem 1.8].

Similarly, we introduce the double layer potential  $\mathcal{D}_k$  defined by

$$\mathcal{D}_k \mu(x) = \int_{\Gamma} n(y) \cdot \nabla G_k(x-y) \mu(y) d\sigma(y)$$

for any smooth function  $\mu$  defined on  $\Gamma$ . The normal derivative of  $\mathcal{D}_k \mu$  is continuous across  $\Gamma$ , allowing us to define the hypersingular operator  $N_k = \partial_n \mathcal{D}_k$ . This latter operator admits the representation for  $x \in \Gamma$

$$(N_k \mu)(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{\Gamma} n(y) \cdot \nabla G_k(x + \varepsilon n(x) - y) \mu(y) d\sigma(y). \quad (12)$$

The kernel of this operator has a non-integrable singularity, but numerical calculations are made possible by the following formula, valid for smooth functions  $\mu$  and  $\nu$  that vanish at the extremities of  $\Gamma$ :

$$\begin{aligned} \langle N_k \mu, \nu \rangle &= \int_{\Gamma \times \Gamma} G_k(x-y) \mu'(x) \nu'(y) \\ &\quad - k^2 G_k(x, y) \mu(x) \nu(y) n(x) \cdot n(y) d\sigma(x) d\sigma(y). \end{aligned} \quad (13)$$

The operator  $N_k$  maps  $\tilde{H}^{1/2}(\Gamma)$  to  $H^{-1/2}(\Gamma)$  continuously [41, Theorem 1.4] and the solution  $u$  to the Neumann problem (4,6,7) can be written as

$$u = \mathcal{D}_k \mu \quad (14)$$

where  $\mu \in \tilde{H}^{1/2}(\Gamma)$  solves

$$N_k \mu = u_N. \quad (15)$$

It is known that the jumps  $\lambda$  and  $\mu$  are singular near the edges of the arc  $\Gamma$ . Their respective asymptotics are given by the following proposition.

**Proposition 2.** [28, 40, 41]. *Near the edges of the arc  $\Gamma$ ,  $\lambda$  is unbounded:*

$$\lambda(x) = O\left(\frac{1}{\sqrt{d(x, \partial\Gamma)}}\right),$$

while  $\mu$  is locally given by

$$\mu(x) = C\sqrt{d(x, \partial\Gamma)} + \psi.$$

where  $\psi \in \tilde{H}^{3/2}(\Gamma)$  and  $C$  is a constant.

In both cases (Dirichlet or Neumann problems) numerical approximations of the solution can be sought by a suitable discretization of the equations (11) and (15) respectively. This amounts to solve linear systems which turn out to be ill-conditioned in practice, and the aim of the paper is to provide the reader with a formalism that enables to give preconditioned version of them.

### 3 Laplace equation on a flat segment

We first restrict our attention to the case where  $\Gamma$  is the segment

$$\Gamma = [-1, 1] \times \{0\},$$

and the wavenumber  $k = 0$ , by considering the equations

$$S_0\lambda = u_D \quad \text{and} \quad N_0\mu = u_N.$$

These problems have already been considered thoroughly in the literature, both in terms of analytical and numerical properties (see for instance [10, 21]). However, to the best of our knowledge, the natural inverses in terms of square roots of local operators have remained unnoticed. In order to derive suitable preconditioners for such equations and their corresponding expressions, we need to introduce the following analytical setting. As it is well known, Chebyshev polynomials of first and second kind play an important role.

#### 3.1 Analytical setting

We introduce the Chebyshev polynomials of first and second kinds [24], respectively given by

$$T_n(x) = \cos(n \arccos(x)), \quad U_n(x) = \frac{\sin((n+1) \arccos(x))}{\sqrt{1-x^2}}$$

and we call  $\omega$  the operator  $\omega : u(x) \mapsto \omega(x)u(x)$  with  $\omega(x) = \sqrt{1-x^2}$ . We also denote by  $\partial_x$  the derivation operator. The Chebyshev polynomials satisfy the ordinary differential equations

$$(1-x^2)\partial_{xx}T_n - x\partial_xT_n + n^2T_n = 0$$

and

$$(1-x^2)\partial_{xx}U_n - 3x\partial_xU_n + n(n+2)U_n = 0$$

which can be rewritten under the form

$$(\omega \partial_x)^2 T_n = -n^2 T_n, \quad (16)$$

$$(\partial_x \omega)^2 U_n = -(n+1)^2 U_n. \quad (17)$$

(Notice that by  $(\partial_x \omega)f$  we mean  $\partial_x(\omega f)$ .) Both  $T_n$  and  $U_n$  are polynomials of degree  $n$ , and form orthogonal families respectively of the Hilbert spaces

$$L_{\frac{1}{\omega}}^2 := \left\{ u \in L_{\text{loc}}^1(-1, 1) \mid \int_{-1}^1 \frac{f^2(x)}{\sqrt{1-x^2}} dx < +\infty \right\}$$

and

$$L_{\omega}^2 := \left\{ u \in L_{\text{loc}}^1(-1, 1) \mid \int_{-1}^1 f^2(x) \sqrt{1-x^2} dx < +\infty \right\}.$$

We denote by  $(\cdot, \cdot)_{\frac{1}{\omega}}$  and  $(\cdot, \cdot)_{\omega}$  the inner products of  $L_{\frac{1}{\omega}}^2$  and  $L_{\omega}^2$ ,

$$(u, v)_{\frac{1}{\omega}} := \frac{1}{\pi} \int_{-1}^1 \frac{u(x) \overline{v(x)}}{\omega(x)} dx, \quad (u, v)_{\omega} := \frac{1}{\pi} \int_{-1}^1 u(x) \overline{v(x)} \omega(x) dx.$$

The Chebyshev polynomials satisfy

$$(T_n, T_m)_{\frac{1}{\omega}} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } m = n = 0 \\ 1/2 & \text{otherwise} \end{cases} \quad (18)$$

and

$$(U_n, U_m)_{\omega} = \begin{cases} 0 & \text{if } n \neq m \\ 1/2 & \text{otherwise} \end{cases} \quad (19)$$

which provides us with the so-called Fourier-Chebyshev decomposition. Any  $u \in L_{\frac{1}{\omega}}^2$  can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x) \quad (20)$$

where the Fourier-Chebyshev coefficients  $\hat{u}_n$  are given by  $\hat{u}_n := \frac{(u, T_n)_{\frac{1}{\omega}}}{(T_n, T_n)_{\frac{1}{\omega}}}$ .

Similarly, any function  $v \in L_{\omega}^2$  can be decomposed along the  $(U_n)_{n \geq 0}$  as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the coefficients  $\check{v}_n$  are given by  $\check{v}_n := \frac{(v, U_n)_{\omega}}{(U_n, U_n)_{\omega}}$ . Those properties can be used to define Sobolev-like spaces.

**Definition 1.** For all  $s \geq 0$ , we define

$$T^s = \left\{ u \in L_{\frac{1}{\omega}}^2 \mid \sum_{n=0}^{+\infty} (1+n^2)^s |\hat{u}_n|^2 < +\infty \right\}.$$

Endowed with the scalar product

$$(u, v)_{T^s} = \hat{u}_0 \overline{\hat{v}_0} + \frac{1}{2} \sum_{n=1}^{+\infty} (1+n^2)^s \hat{u}_n \overline{\hat{v}_n},$$

$T^s$  is a Hilbert space for all  $s \geq 0$ . Similarly, we set

$$U^s = \left\{ u \in L^2_\omega \left| \sum_{n=0}^{+\infty} (1+n^2)^s |\check{u}_n|^2 \right. \right\},$$

which is a Hilbert space for the scalar product

$$(u, v)_{U^s} = \frac{1}{2} \sum_{n=0}^{+\infty} (1+(n+1)^2)^s \check{u}_n \overline{\check{v}_n}.$$

One can extend the definition of  $T^s$  and  $U^s$  for  $s \in \mathbb{R}$ , in which case they form interpolating scales of Hilbert space, see [7] for details. For  $s = \pm \frac{1}{2}$ , those spaces have been analyzed (with different notation) e.g. in [20] and satisfy

$$T^{-1/2} = \omega \tilde{H}^{-1/2}(-1, 1), \quad T^{1/2} = H^{1/2}(-1, 1), \quad (21)$$

$$U^{-1/2} = H^{-1/2}(-1, 1), \quad U^{1/2} = \frac{1}{\omega} \tilde{H}^{1/2}(-1, 1). \quad (22)$$

Denoting by  $T^\infty = \bigcap_{s \geq 0} T^s$  and similarly for  $U^\infty$ , it is shown in [7] that

**Lemma 1.**

$$T^\infty = U^\infty = C^\infty([-1, 1]).$$

### 3.2 Single layer equation

In the case of Dirichlet condition, we seek a solution to the equation  $S_0 \lambda = g$ , with  $\lambda \in \tilde{H}^{-1/2}(\Gamma)$ , that is

$$-\frac{1}{2\pi} \int_{-1}^1 \log|x-y| \lambda(y) dy = g(x), \quad \forall x \in (-1, 1). \quad (23)$$

This equation is sometimes called ‘‘Symm’s integral equation’’ and its resolution has received a lot of attention in the 1990’s. Numerical methods, using both the collocation and Galerkin paradigms have been presented and analyzed [6, 37, 44–46].

Our analysis lies on the following formula. (For a proof, see for example [24] Theorem 9.2. Note that this is also the main ingredient in several connected works, such as [10] and [21].)

**Lemma 2.** For all  $n \in \mathbb{N}$ , we have

$$-\frac{1}{2\pi} \int_{-1}^1 \frac{\ln|x-y|}{\sqrt{1-y^2}} T_n(y) dy = \sigma_n T_n(x)$$



where

$$\sigma_n = \begin{cases} \frac{\ln(2)}{2} & \text{if } n = 0, \\ \frac{1}{2n} & \text{otherwise.} \end{cases}$$

Using the decomposition of  $g$  and of the logarithmic kernel on the basis  $(T_n)_n$ , we see that the solution  $\lambda$  to equation (23) admits the following expansion

$$\lambda(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{+\infty} \frac{\hat{g}_n}{\sigma_n} T_n(x). \quad (24)$$

We deduce the following well-known fact (e.g. [14])

**Corollary 1.** *If the data  $g$  is in  $C^\infty([-1, 1])$ , the solution  $\lambda$  to the equation*

$$S_0\lambda = g$$

*is of the form*

$$\lambda(x) = \frac{\alpha(x)}{\sqrt{1-x^2}}$$

*with  $\alpha \in C^\infty([-1, 1])$ .*

*Proof.* Let  $\alpha(x) = \sqrt{1-x^2}\lambda(x)$  where  $\lambda$  is the solution of  $S_0\lambda = g$ . By Lemma 1, if  $g \in C^\infty([-1, 1])$ , then  $g \in T^\infty$ , and by equation (24),

$$\hat{\alpha}_n = \frac{\hat{g}_n}{\sigma_n},$$

from which we deduce that  $\alpha$  also belongs to  $T^\infty = C^\infty([-1, 1])$ . □

Following [10], we introduce the weighted single layer operator as the operator that appears in Lemma 2.

**Definition 2.** *Let  $S_{0,\omega}$  be the weighted single layer operator defined by*

$$S_{0,\omega} : \alpha \in C^\infty([-1, 1]) \longrightarrow -\frac{1}{2\pi} \int_{-1}^1 \frac{\ln|x-y|}{\omega(y)} \alpha(y) dy.$$

To obtain the solution of (23), we thus solve

$$S_{0,\omega}\alpha = u_D, \quad (25)$$

and let  $\lambda = \frac{\alpha}{\omega}$ , which indeed belongs to  $\tilde{H}^{-1/2}$  by (21). From the previous considerations, the inverse for  $S_{0,\omega}$  can be derived by building an operator  $R$  such that  $RT_n = \frac{1}{\sigma_n} T_n$ . It turns out that such an operator  $R$  has been characterized in [20, 42], through explicit variational forms in closed forms (see also the recent paper [19] for the case of the unit disk in  $\mathbb{R}^3$ ).

Here, we give an alternative form for the operator  $R$ , that we will then generalize for a non-zero frequency.

**Theorem 1.** For any  $s$ , the operator  $-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0$  is bicontinuous from  $T^{s+2}$  to  $T^s$  and

$$S_{0,\omega}^2 = \frac{1}{4} \left( -(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0 \right)^{-1}.$$

**Corollary 2.** The operator  $S_{0,\omega}$  being non negative, its inverse can thus be equivalently expressed as

$$S_{0,\omega}^{-1} = R := 2\sqrt{-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0}$$

where the square root of the operator is defined using standard spectral theory.

*Proof.* We simply notice that for all  $n \in \mathbb{N}$ ,  $(\omega\partial_x)^2 T_n = -n^2 T_n$ . Therefore

$$\begin{aligned} \left( -(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0 \right) T_n &= \begin{cases} n^2 T_n & \text{if } n \neq 0 \\ \frac{1}{\ln(2)^2} & \text{otherwise} \end{cases} \\ &= \frac{1}{4\sigma_n^2} T_n. \end{aligned}$$

□

Since the operator  $\sqrt{-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0}$  is the inverse of  $S_{0,\omega}$ , in particular, it can be used as an efficient preconditioner for the weighted integral equation (25).

### 3.3 Hypersingular equation

We now turn our attention to the equation

$$N_0\mu = g. \tag{26}$$

Similarly to the previous section and following [10], we consider the weighted version of the hypersingular operator  $N_{0,\omega} := N_0\omega$  defined by

$$N_{0,\omega}\mu = \lim_{\varepsilon \rightarrow 0} \int_{-1}^1 n(y) \cdot \nabla G(x + \varepsilon n(x) - y) \sqrt{1 - y^2} dy.$$

We can get the solution to equation (26) by solving

$$N_{0,\omega}\beta = u_N \tag{27}$$

and letting  $\mu = \omega\beta$ . We now show that  $N_{0,\omega}$  can be analyzed using this time the spaces  $U^s$ .

**Lemma 3.** For any  $\beta, \beta'$ , one has

$$\langle N_{0,\omega}\beta, \beta' \rangle_\omega = \langle S_{0,\omega}\omega\partial_x\omega\beta, \omega\partial_x\omega\beta' \rangle_{\frac{1}{\omega}}.$$

*Proof.* We use the well-known integration by part formula

$$\langle N_0 u, v \rangle = \langle S_0 \partial_x u, \partial_x v \rangle,$$

valid when  $u$  and  $v$  are regular enough and vanish at the extremities of the segment. For a smooth  $\beta$ , we thus have

$$\langle N_0(\omega\beta), (\omega\beta') \rangle = \langle S_0 \partial_x(\omega\beta), \partial_x(\omega\beta') \rangle$$

which obviously implies the claimed identity. □

**Lemma 4.** For all  $n \in \mathbb{N}$ , we have

$$N_{0,\omega}U_n = \frac{n+1}{2}U_n.$$

*Proof.* From the identity  $\partial_x T_{n+1} = (n+1)U_n$  and Equation (16) we obtain

$$\omega \partial_x \omega U_n = -(n+1)T_{n+1}.$$

Therefore, by Lemma 3

$$\begin{aligned} \langle N_{0,\omega}U_m, U_n \rangle_\omega &= (n+1)(m+1) \langle S_{0,\omega}T_{m+1}, T_{n+1} \rangle_{\frac{1}{\omega}} \\ &= \delta_{m=n} \frac{n+1}{2}. \end{aligned} \quad (28)$$

□

Moreover, the identity  $-(\partial_x \omega)^2 U_n = (n+1)^2 U_n$  leads to the following result, that expresses again the operator  $N_\omega$  as a square root operator.

**Theorem 2.**

$$N_{0,\omega} = \frac{1}{2} \sqrt{-(\partial_x \omega)^2}. \quad (29)$$

## 4 Helmholtz equation

In this section, we aim at generalizing the preceding analysis to the case of Helmholtz equation on  $\mathbb{R}^2 \setminus \Gamma$  with  $\Gamma = [-1, 1] \times \{0\}$ , based on the explicit formulas presented in the previous section. Recall the definition of the single layer and hypersingular operators,  $S_k$  and  $N_k$ , given in (9) and (12), and the integral equations (11) and (15) for the Dirichlet and Neumann problems respectively. As before, let  $S_{k,\omega} := S_k \frac{1}{\omega}$  and  $N_{k,\omega} := N_k \omega$ . The following commutation holds:

**Theorem 3.**

$$S_{k,\omega} [-(\omega \partial_x)^2 - k^2 \omega^2] = [-(\omega \partial_x)^2 - k^2 \omega^2] S_{k,\omega},$$

*Proof.* Since  $(\omega \partial_x)^2$  is self adjoint and symmetric (with respect to the bilinear form  $(\cdot, \cdot)_{\frac{1}{\omega}}$ ), we have

$$(S_{k,\omega}(\omega \partial_x)^2 u)(x) = \int_{-1}^1 \frac{(\omega_y \partial_y)^2 [G_k(x-y)] u(y)}{\omega(y)} dy,$$

where we use the notation  $\omega_y$  and  $\partial_y$  to emphasize the dependence in the variable  $y$ . Thus,

$$((S_{k,\omega}(\omega \partial_x)^2 - (\omega \partial_x)^2 S_{k,\omega}) u)(x) = \int_{-1}^1 \frac{D_k(x,y) u(y)}{\omega(y)} dy,$$

where  $D_k(x,y) := [(\omega_y \partial_y)^2 - (\omega_x \partial_x)^2] [G_k(x-y)]$ . A simple computation leads to

$$D_k(x,y) = \partial_{xx} G_k(x-y)(\omega_y^2 - \omega_x^2) + \partial_x G_k(x-y)(y+x).$$

Since  $G_k$  is a solution of the Helmholtz equation, we have for all  $(x \neq y) \in \mathbb{R}^2$

$$\partial_x G_k(x-y) = (y-x)(\partial_{xx} G_k(x-y) + k^2 G(x-y)),$$

thus

$$D_k(x, y) = \partial_{xx} G_k(x-y) (\omega_y^2 - \omega_x^2 + y^2 - x^2) + k^2(y^2 - x^2)G_k(x-y).$$

A careful analysis shows that no Dirac mass appears in the previous formula. Note that  $y^2 - x^2 = \omega_x^2 - \omega_y^2$  so the first term vanishes and we find

$$S_{k,\omega}(\omega\partial_x)^2 - (\omega\partial_x)^2 S_{k,\omega} = k^2 (\omega^2 S_{k,\omega} - S_{k,\omega}\omega^2)$$

as claimed.  $\square$

There also holds the following identity:

$$N_{k,\omega} [-(\partial_x \omega)^2 - k^2 \omega^2] = [-(\partial_x \omega)^2 - k^2 \omega^2] N_{k,\omega},$$

however, the proof of this commutation is quite heavy. We chose to not include it in the present work for the sake of conciseness.

Those commutations imply that the operators  $S_{k,\omega}$  and  $N_{k,\omega}$  share the same eigenvectors as, respectively,  $[-(\omega\partial_x)^2 - k^2\omega^2]$  and  $[-(\partial_x\omega)^2 - k^2\omega^2]$ . The eigenfunctions of the operator  $[-(\omega\partial_x)^2 - k^2\omega^2]$  thus provide us with a diagonal basis for  $S_{k,\omega}$ . They are the solutions to the differential equation

$$(1-x^2)\partial_{xx}y - x\partial_x y - k^2\omega^2 y = \lambda y.$$

Once we set  $x = \cos\theta$ ,  $\tilde{y}(\theta) = y(x)$ ,  $q = \frac{k^2}{4}$ ,  $a = \lambda + 2q$ ,  $\tilde{y}$  is a solution of the standard Mathieu equation

$$\tilde{y}'' + (a - 2q \cos(2\theta))\tilde{y} = 0. \quad (30)$$

There exists a discrete set of values  $a_{2n}(q)$  for which this equation possesses even and  $2\pi$  periodic solutions, which are known as the Mathieu cosine functions, and usually denoted by  $ce_n$ . Here, we use the notation  $ce_n^k$  to emphasize the dependency in the parameter  $k = \sqrt{2q}$  of those functions. The normalization is taken as

$$\int_{-\pi}^{\pi} ce_n^k(\theta)^2 d\theta = \pi.$$

The Mathieu cosine functions also satisfy

$$\int_{-\pi}^{\pi} ce_n^k(\theta) ce_m^k(\theta) d\theta = \pi \delta_{m,n}$$

so that any even  $2\pi$  periodic function in  $L^2(-\pi, \pi)$  can be expanded along the functions  $ce_n^k$ , with the coefficients obtained by orthonormal projection. Setting

$$T_n^k := ce_n^k(\arccos(x)),$$

in analogy to the zero-frequency case, we have

$$[-(\omega\partial_x)^2 - k^2\omega^2] T_n^k = \lambda_{n,k}^2 T_n^k.$$

For large  $n$ , using the general results from the theory of Hill's equations (see e.g. [43, eqs. (21), (28) and (29)]), we have the following asymptotic formula for  $\lambda_{n,k}$ :

$$\lambda_{n,k}^2 = n^2 - \frac{k^4}{16n^2} + o(n^{-2}).$$

The first commutation established in Theorem 3 implies that Mathieu cosine functions are also the eigenfunctions of the single-layer operator. (An equivalent statement is given in [9, Thm 4.2], if we allow the degenerate case  $\mu = 0$ .)

A similar analysis can be applied to the hypersingular operator. The eigenfunctions of  $[-(\partial_x \omega)^2 - k^2 \omega^2]$  are given by

$$U_n^k := \frac{\text{se}_n^k(\arccos(x))}{\omega(x)}$$

where  $\text{se}_n^k$  are the so-called Mathieu sine functions, which also satisfy the Mathieu differential equation (30), but with the condition that they are  $2\pi$  periodic and odd functions.

All the previous considerations highly suggest the following theorem:

**Theorem 4.** *There exists a compact operator  $K$  from  $T^s$  to  $T^{s+2}$  such that*

$$[-(\omega \partial_x)^2 - k^2 \omega^2] S_{k,\omega}^2 = \frac{Id}{4} + K,$$

and a compact operator  $K'$  from  $U^s$  to  $U^{s-2}$  such that

$$N_{k,\omega}^2 = [-(\partial_x \omega)^2 - k^2 \omega^2] + K'.$$

The proof of this theorem, requires the introduction of additional analytic tools that are out of the scope of this introductory paper. It is thus omitted in the present work and given in full details in [7]. Nevertheless, Theorem 4 gives a generalization to Helmholtz equation of the result obtained precedingly for Laplace equation, namely, the operator  $[-(\omega \partial_x)^2 - k^2 \omega^2]^{1/2}$  is expected to be a good preconditioner for  $S_{k,\omega}$ , as well as  $[-(\partial_x \omega)^2 - k^2 \omega^2]^{-1/2}$  for  $N_{k,\omega}$ .

Finally, we note that in the more general case of a  $C^\infty$  non-intersecting open curve  $\Gamma$  and non-zero frequency  $k$ , the previous result extends, replacing  $\partial_x$  by  $\partial_\tau$  the tangential derivative on  $\Gamma$ ,  $S_{k,\omega}$  and  $N_{k,\omega}$  respectively by  $S_{k,\omega_\Gamma} := S_k \frac{1}{\omega_\Gamma}$ , and  $N_{k,\omega_\Gamma} := N_k \omega_\Gamma$ , and  $\omega$  by  $\omega_\Gamma$ . Here  $\omega_\Gamma$  is defined by  $\omega_\Gamma(x) = \frac{|\Gamma|}{2} \omega(r(x))$ , where  $|\Gamma|$  is the length of the curve and  $r : [-1, 1] \rightarrow \Gamma$  is a parametrization of the curve such that for all  $x$ ,  $|\partial_x r(x)| = \frac{|\Gamma|}{2}$ .

## 5 Numerical results

### 5.1 Galerkin setting

To solve numerically the integral equations presented earlier, various methods have been described and analyzed in the literature. The standard discretization on a uniform mesh with piecewise polynomial trial functions leads to very poor rates of convergence (see for example [36, Chap. 4]). Several strategies have been developed to remedy this problem. One can for example enrich the trial space

with special singular functions, refine the mesh near the segment tips (h-BEM) or increase the polynomial order in the trial space (p-BEM). The combination of the last two methods, known as h-p BEM, can achieve an exponential rate of convergence with respect to the dimension of the trial space, see [31] and references therein. Spectral methods, involving trigonometric polynomials have also been analyzed for example in [10], and some results exist for piecewise linear functions in the collocation setting [13].

Here, we describe a simple Galerkin setting, suited to the spaces  $T^s$  and  $U^s$ . We use piecewise affine functions defined on a non-uniform mesh, which is refined towards the edges of the curve. More precisely, let  $-1 = x_0 < x_1 < \dots < x_N = 1$  and let  $\theta_i := \arccos(x_i)$ . We choose the points  $x_i$  such that  $(\theta_i)_{0 \leq i \leq N}$  are equispaced, that is to say  $\theta_i = ih$  with  $h = \pi/N$ . The nodes of the mesh are then set to  $X_i = r(x_i)$ , where  $r$  is a smooth parametrization of the curve  $\Gamma$ . This turns out to be analogous to a graded mesh with a grading parameter equal to 2 which means that, near the edge, the width of the  $i - th$  interval is approximately  $(ih)^2$ . Notice that it is known that this modification alone, i.e. using the h-BEM method with a polynomial order  $p = 1$ , is not sufficient to get an optimal rate of convergence. Indeed, it can be shown that it only leads to a convergence rate in  $O(h)$  for the  $L^2$  norm (cf. [31, Theorem 1.3]) instead of the expected  $O(h^2)$  behavior.

The key ingredient, beside the graded mesh, to recover optimal convergence is to use a weighted  $L^2$  scalar product (with weight  $\frac{1}{\omega}$  or  $\omega$  depending on the considered equation), in order to assemble the operators in their natural spaces. We state here the orders of convergence that one gets with this new method, and refer again the reader to [7] for proofs. We also restrict our presentation to the case where  $\Gamma = [-1, 1] \times \{0\}$  and  $k = 0$ , the general case being then obtained by standard perturbation arguments.

**Dirichlet problem.** For the resolution of the single-layer equation (23) we use a variational formulation of (25) to compute an approximation  $\alpha_h$  of  $\alpha$ . Namely, let  $V_h$  the Galerkin space of (discontinuous) piecewise affine functions defined on the mesh  $(x_i)_{0 \leq i \leq N}$  defined above, and  $\alpha_h$  the unique solution in  $V_h$  to

$$(S_{0,\omega} \alpha_h, \alpha'_h)_{\frac{1}{\omega}} = (u_D, \alpha'_h)_{\frac{1}{\omega}}, \quad \forall \alpha'_h \in V_h.$$

We then compute  $\lambda_h = \frac{\alpha_h}{\omega}$ . Using the notation  $C$  to denote any constant that does not depend on the relevant parameters, we then have

**Theorem 5.** (see [7]). *If the data  $u_D$  is in  $T^{s+1}$  for some  $-1/2 \leq s \leq 2$ , then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq Ch^{s+1/2} \|\omega\lambda\|_{T^s} \leq Ch^{s+1/2} \|u_D\|_{T^{s+1}}.$$

In particular, when  $u_D$  is smooth, the solution  $\alpha = \omega\lambda$  belongs to  $T^\infty$ , and we get the optimal rate of convergence of the error in  $O(h^{5/2})$ .

**Neumann problem.** For the numerical resolution of (26), we use a variational form for equation (27) to compute an approximation  $\beta_h$  of  $\beta$ , and solve it using a Galerkin method with continuous piecewise affine functions. Introducing  $W_h$  the space of continuous piecewise affine functions on the mesh defined

by the points  $(x_i)_{0 \leq i \leq N}$ , we denote by  $\beta_h$  the unique solution in  $W_h$  to the variational equation:

$$(N_\omega \beta_h, \beta'_h)_\omega = (u_N, \beta'_h)_\omega, \quad \forall \beta'_h \in W_h. \quad (31)$$

Then, the proposed approximation for  $\mu$ , given by  $\mu_h = \omega \beta_h$ , satisfies the following error estimate.

**Theorem 6.** (see [7]). *If  $u_N \in U^{s-1}$ , for some  $\frac{1}{2} \leq s \leq 2$ , there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq Ch^{s-\frac{1}{2}} \left\| \frac{\mu}{\omega} \right\|_{U^s} \leq Ch^{s-\frac{1}{2}} \|u_N\|_{U^{s-1}}.$$

**Numerical validation.** In addition to the energy norm, estimates in the local weighted  $L^2$  norms can be derived from the previous, namely:

$$\forall s \in [0, 2], \quad \|\alpha - \alpha_h\|_{\frac{1}{\omega}} \leq Ch^s \|\alpha\|_{T^s}$$

for the Dirichlet problem and

$$\forall s \in [0, 2], \quad \|\beta - \beta_h\|_\omega \leq Ch^s \|\beta\|_{U^s}$$

$$\forall s \in [1, 2], \quad \|\beta - \beta_h\|_{U^1} \leq Ch^{s-1} \|\beta\|_{U^s}$$

for the Neumann problem. We verify those rates numerically. For the Dirichlet problem, we solve two test cases  $S_{0,\omega} \alpha_1 = u_1$  and  $S_{0,\omega} \alpha_2 = u_2$  having the explicit solutions  $\alpha_1(x) = \omega(x)$  and  $\alpha_2 = \omega(x)^3$ , for adequately chosen rhs  $u_1$  and  $u_2$ . One can check that  $\alpha_1 \in T^s$  for  $s < \frac{3}{2}$  and  $\alpha_1 \notin T^{3/2}$ , while  $\alpha_2 \in T^2$ . The  $L^2_{\frac{1}{\omega}}$  error is plotted in Figure 1 in each case as a function of the mesh size  $h$ . We find that the expected rates  $O(h^{3/2})$  and  $O(h^2)$  predicted by the theory are precisely recovered in practice.

Similarly, for the Neumann case, we solve a test case  $N_{0,\omega} \beta = u_N$  where the solution  $\beta$  is explicit. We take  $u_N = U_2$  the second Chebyshev polynomial of the second kind. The corresponding solution  $\beta$  is proportional to  $U_2$  and thus belongs to  $U^\infty$ . The theory therefore predicts a convergence rate of the error in the  $L^2_\omega$  and  $U^1$  norms respectively in  $O(h^2)$  and  $O(h)$ . This behavior is again confirmed by our numerical results, exposed in Figure 2.

## 5.2 Preconditioning the linear systems

Let  $X_h$  the considered finite element space ( $X_h = V_h$  or  $W_h$ ), and  $(\phi_i)_i$  the basis functions. For an operator  $A$ , we denote by  $[A]_p$  the Galerkin matrix of the operator for the relevant weight  $p(x) = \frac{1}{\omega(x)}$  or  $\omega(x)$ , defined by

$$[A]_{p,ij} = \int_\Gamma (A\phi_j)(x) \phi_i(x) p(x) dx.$$

When the operator  $BA$  is a compact perturbation of the identity (either in  $T^s$  or  $U^s$ ) then, following [38], we precondition the linear system  $[A]_p x = b$  by the matrix  $[I_d]_p^{-1} [B]_p [I_d]_p^{-1}$ , which amounts to solve

$$[I_d]_p^{-1} [B]_p [I_d]_p^{-1} [A]_p x = [I_d]_p^{-1} [B]_p [I_d]_p^{-1} b.$$

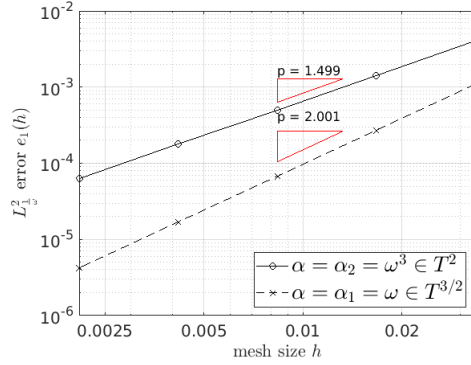


Figure 1: Effective order of convergence of the approximation of the solution  $\beta$  to (27) by the weighted Galerkin method. Two cases are considered where  $\alpha \in T^s$  for all  $s < 3/2$  but  $\alpha \notin T^{3/2}$  (solid line and circles) and  $\alpha \in T^2$  (dashed line, crosses) respectively. The approximate slope  $p$  is displayed above each curve. Theoretical convergence rates, respectively  $O(h^{3/2})$  and  $O(h^2)$  are recovered in practice

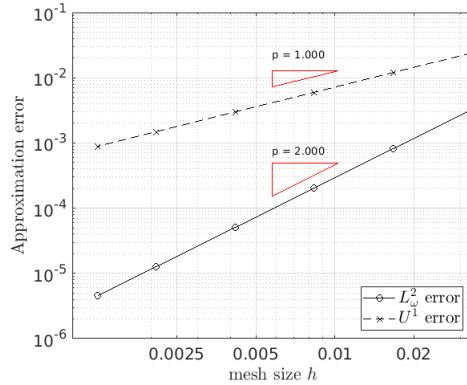


Figure 2: Effective order of convergence of the approximation of the solution  $\beta$  to (27) by the weighted Galerkin method. In this test, the solution  $\beta$  lies in  $U^2$  and we measure the error in two different norms, respectively  $L^2_\omega$  (solid line, circles) and  $U^1$  (dashed line, crosses). The approximate slope  $p$  is displayed above each curve. The theoretical order of convergence, in  $O(h^2)$  and  $O(h)$  respectively for the  $L^2_\omega$  and  $U^1$  norms, are recovered in practice



When  $B$  is the inverse of a local operator  $C$ , then it may be more convenient to compute  $[C]_p$ , and solve instead

$$[C]_p^{-1} [A]_p x = [C]_p^{-1} b.$$

The preconditioners introduced in this work are in the form of square roots of local operators. More precisely, we introduced two preconditioners  $P_1$  and  $P_2$  with

$$\begin{aligned} P_1(k) &= (-\omega \partial_x)^2 - k^2 \omega^2)^{1/2}, \\ P_2(k) &= (-\partial_x \omega)^2 - k^2 \omega^2)^{-1/2}. \end{aligned}$$

For the second equation, we rewrite

$$P_2(k) = (-\partial_x \omega)^2 - k^2 \omega^2)^{-1} (-\partial_x \omega)^2 - k^2 \omega^2)^{1/2}.$$

This brings us back to computing Galerkin matrix of the square root of a differential operator. When the frequency is 0, we use the method exposed in [17]. When the frequency is non-zero, the previous method fails since the spectrum of the matrix contains negative values. In [5], a method involving a Padé approximation of the square root, with a rotated branch cut, is used to compute the matrix of an operator of the form  $\sqrt{X - k^2 I_d}$  where  $X$  is a positive definite operator. This method gives excellent results in our context when using  $X = -(\partial_x \omega)^2 + k^2 (I_d - \omega^2)$ . More specifically, we build a rational approximation of the function  $X \mapsto \sqrt{X - k^2}$  in the form

$$\sqrt{X - k^2} = a_0 + \sum_{i=0}^{N_p} \frac{a_i}{b_i + X},$$

and then take

$$\left[ \sqrt{X - k^2 I_d} \right]_p \approx a_0 [I_d]_p + \sum_{i=0}^{N_p} a_i (b_i [I_d]_p + [X]_p)^{-1}.$$

### 5.3 Preconditioning results

All the numerical results exposed here are obtained on a personal laptop running on an eight cores intel i7 processor with a clock rate of 2.8GHz. The Galerkin method has been implemented in the language Matlab R2018. For problems such that  $\frac{|\Gamma|}{\lambda} \leq 200$ , where  $\lambda$  stands for the wavelength, the problems are solved in full BEM without compression method. For the cases with  $\frac{|\Gamma|}{\lambda} > 200$ , ( $N > 10^4$  where  $N$  is the number of unknowns) the memory of the computer is insufficient to store the full problem, and in this case the Efficient Bessel Decomposition (EBD) [8] is used to compress and accelerate the matrix-vector products.

**Flat segment, Laplace-Dirichlet problem.** In Table 1, we report the number of iterations for the numerical resolution of the Laplace problem (25) by the method detailed above, in section 5. Two cases are considered, first without

$N$	with Prec.		without Prec.	
	$n_{it}$	t(s)	$n_{it}$	t(s)
50	7	0.05	30	0.05
200	8	0.05	58	0.05
800	8	0.06	93	0.10
3200	8	0.13	141	0.90

Table 1: Computing time and number of GMRES iterations for the numerical resolution of the Laplace single layer integral equation on the segment respectively with the square root preconditioner and without preconditioner

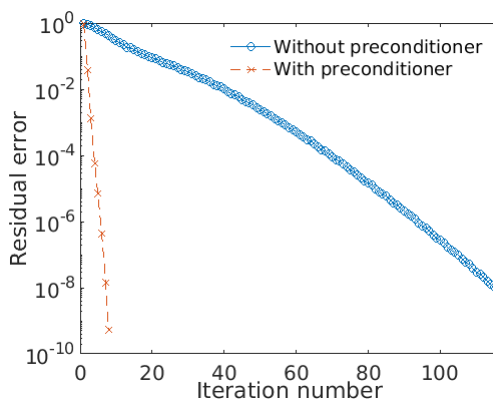


Figure 3: Number of GMRES iterations for the resolution of the Laplace single layer integral equation on the segment with a mesh of size  $N = 1600$  respectively without preconditioner (circles) and with the square root preconditioner (crosses)

any preconditioner, and then with a preconditioner given by  $[I_d]_{\frac{1}{\omega}}^{-1} [B]_{\frac{1}{\omega}} [I_d]_{\frac{1}{\omega}}^{-1}$  where  $[B]_{\frac{1}{\omega}}$  is the Galerkin matrix of the operator  $\sqrt{-(\omega \partial_x)^2 + \frac{1}{\ln(2)^2} \pi_0}$ . The right hand side in (25) is chosen as  $u_D(x) = (x^2 + 0.001)^{-1/2}, x \in [-1, 1]$ . A graph of the residual along the iterations is given in Figure 3 for a refined mesh with 1600 node points.

**Flat segment, Laplace-Neumann problem.** For the Neumann problem (27), we also report in Table 2 the number of iterations for the numerical resolution with and without the preconditioner given by  $[I_p]_{\omega}^{-1} [C]_{\omega} [I_p]_{\omega}^{-1}$  where  $[C]_{\omega}$  is the Galerkin matrix of the operator  $\sqrt{-(\partial_x \omega)^{-2}}$ . The right hand side in (27) is chosen as  $u_N(x) = (x^2 + 0.001)^{1/2}, x \in [-1, 1]$ . The decay of the residual along the iterations is shown in Fig. 4. We observe in both Dirichlet and Neumann cases a drastic decay of the number of iteration which justifies the approach. We also see that, as expected, the number of iterations obtained with the preconditioned version does not depend on the mesh size.

We now turn our attention to Helmholtz equation. In each case, in order

$N$	with Prec.		without Prec.	
	$n_{it}$	t(s)	$n_{it}$	t(s)
50	5	0.05	29	0.05
200	5	0.05	127	0.1
800	4	0.05	530	1.0
3200	4	0.07	2153	43

Table 2: Computing time and number of GMRES iterations for the numerical resolution of the Laplace hypersingular equation respectively with the square root preconditioner and without preconditioner

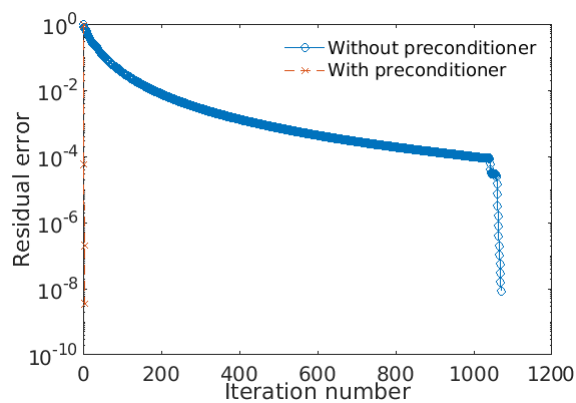


Figure 4: Comparison of the number of GMRES iterations in the resolution of the Laplace hypersingular integral equation on the segment with a mesh of size  $N = 1600$  respectively without preconditioner (circles) and with the square root preconditioner (crosses). The importance of preconditioning in this case is even more spectacular than in the case of the single-layer equation

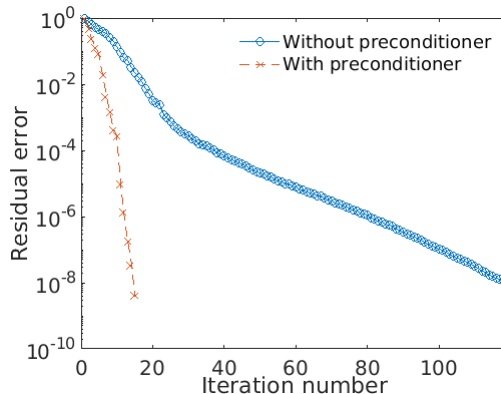


Figure 5: Comparison of the number of GMRES iterations in the resolution of the Helmholtz single layer integral equation on the segment with a mesh of size  $N \approx 2000$ ,  $L = 200\lambda$ , respectively without preconditioner (circles) and with the square root preconditioner (crosses)

to fully resolve the frequency, the number of segments in the discretization is set to  $N \approx 10k$ , where  $k = \frac{\pi}{\lambda}$  is the wavenumber. In the GMRES iteration, we require a relative residual below  $10^{-8}$ .

**Flat segment, Helmholtz-Dirichlet problem.** In Table 3 we report the number of GMRES iterations for the numerical resolution of Equation (11) on the segment  $\Gamma = [-1, 1] \times \{0\}$ , when the linear system is preconditioned by the operator

$$\sqrt{-(\omega\partial_x)^2 - k^2\omega^2},$$

as compared to the case where no preconditioner is used. We take, for the Dirichlet data, the plane wave  $u_D(x) = e^{ikx}$ . We also provide, in Figure 5, the value of the relative residual in the GMRES method along the iterations, with and without preconditioner, for a problem with  $L = 800\lambda$ . As before, we see that the number of iterations needed to reach a given precision decreases significantly but, this time, we observe a very slight increase with respect to the wavenumber.

$ \Gamma /\lambda$	with Prec.		without Prec.	
	$n_{it}$	t(s)	$n_{it}$	t(s)
50	9	0.07	79	0.21
200	11	0.37	106	1.35
800	15	15	155	110

Table 3: Computing time and number of GMRES iterations for the numerical resolution of the Helmholtz single-layer integral equation on the segment respectively with the square root preconditioner and without preconditioner

**Flat segment, Helmholtz-Neumann problem.** We run the same numerical comparisons, this time solving (15) and considering the preconditioning op-

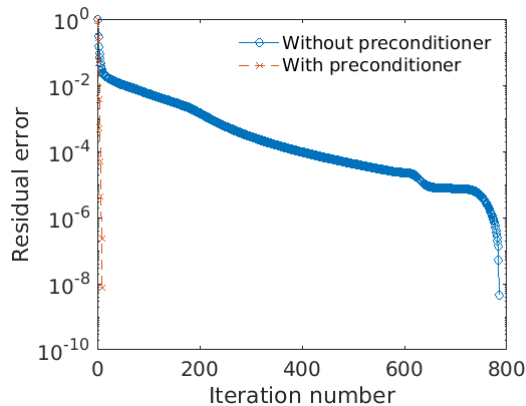


Figure 6: Comparison of the number of GMRES iterations in the resolution of the Helmholtz hypersingular integral equation on the segment with a mesh of size  $N \approx 800$ ,  $L = 50\lambda$ , respectively without preconditioner (circles) and with the square root preconditioner (crosses)

erator  $(-\partial_x \omega)^2 - k^2 \omega^2)^{-1/2}$ . Results are given in Table 4 for different meshes and in Figure 6 for the evolution of the residual along the iterations. We take, for the Neumann data, the normal derivative of a plane wave  $u_N = \frac{\partial}{\partial n} e^{ikx}$ . Huge differences, both in time and number of iterations are shown in favor of the preconditioned system.

$ \Gamma /\lambda$	with Prec.		without Prec.	
	$n_{it}$	t(s)	$n_{it}$	t(s)
50	9	0.07	785	7
200	12	0.5	3115	6 min
800	18	28	?	> 1 hour

Table 4: Computing time and number of GMRES iterations for the numerical resolution of the Helmholtz hypersingular equation on the segment respectively with the square root preconditioner and without preconditioner  
**Non-flat arc.** We also consider a spiral shaped arc of equation

$$\begin{aligned} x(t) &= e^{0.1(s+s_0)} \cos(2(s-0.2)) \\ y(t) &= e^{0.1(s+s_0)} \sin(2(s-0.2)) \end{aligned}$$

and report in Tables 5 and 6 the number of iterations and computing times respectively for the Dirichlet and Neumann problems. The results show that the preconditioning strategy is also efficient in presence of non-zero curvature, especially for the Dirichlet problem. For the Neumann problem, it might be necessary to add a curvature term in the preconditioner for a faster resolution. The scattering problem with Dirichlet condition is illustrated in Figure 7 for this geometry.

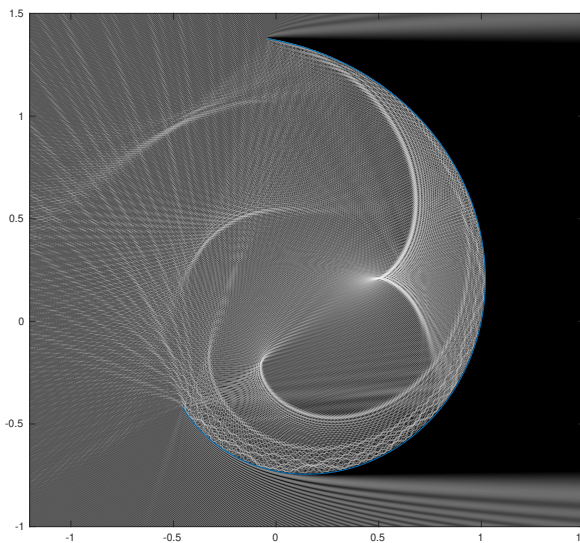


Figure 7: Sample diffraction pattern (Dirichlet boundary conditions) with left to right incidence for an arc of spiral of size  $L = 800\lambda$ . After the resolution of the integral equation, the computation of the image is accelerated by the EBD method [8].

$ \Gamma /\lambda$	With prec.		Without prec.	
	$n_{it}$	t(s)	$n_{it}$	t(s)
50	20	0.19	117	0.46
200	25	1.2	163	4.8
800	31	28	220	2min 30s

Table 5: Computing time and number of GMRES iterations for the numerical resolution of the Helmholtz single-layer integral equation on the spiral-shaped arc respectively with the square root preconditioner and without preconditioner

$ \Gamma /\lambda$	With prec.		Without prec.	
	$n_{it}$	t(s)	$n_{it}$	t(s)
50	64	0.34	800	7
200	155	4.2	3250	7 min
800	345	6 min	?	> 1 hour

Table 6: Computing time and number of GMRES iterations for the numerical resolution of the Helmholtz hypersingular integral equation on the spiral-shaped arc, respectively with the square root preconditioner and without preconditioner

**Comparison with the generalized Calderón relations.** Eventually, we test the idea presented in [10], namely to use  $S_{k,\omega}$  and  $N_{k,\omega}$  as mutual pre-

conditioners. This alternative method is also very efficient in our numerical setting (here, we use simple piecewise affine functions, whereas in [10], spectral discretization with trigonometric polynomials is used). We report the number of iterations and computing times respectively for the Dirichlet and Neumann problems on the flat segment respectively in Tables 7 and 8. The second members are respectively  $u_D = u_{inc}$  and  $u_N = \frac{\partial}{\partial n} u_{inc}$  where  $u_{inc}$  is a plane wave of wavenumber  $k$  and angle of incidence  $\frac{\pi}{4}$ . The number of iterations is comparable for both methods in each case. However, our preconditioner being a sparse matrix, our method leads to a faster resolution of the linear system particularly for the Dirichlet problem.

$ \Gamma /\lambda$	Calderón Prec.		Square root Prec.	
	$n_{it}$	t(s)	$n_{it}$	t(s)
50	15	0.1	9	0.07
200	15	0.47	9	0.35
800	16	41	16	16.2

Table 7: Computing time and number of GMRES iterations for the Helmholtz single-layer integral equation on the flat segment, respectively with the Calderón preconditioner and with the square root preconditioner

$ \Gamma /\lambda$	Calderón Prec.		Square root Prec.	
	$n_{it}$	t(s)	$n_{it}$	t(s)
50	15	0.1	9	0.07
200	16	0.5	12	0.4
800	17	44	18	32

Table 8: Computing time and number of GMRES iterations for the Helmholtz hypersingular integral equation on the flat segment, respectively with the Calderón preconditioner and with the square root preconditioner

## 6 Conclusion

We have presented a new approach for the preconditioning of integral equations coming from the discretization of wave scattering problems in 2D by open arc domains. The methodology is very effective and proven to be optimal for Laplace problems on straight segments. It has three advantages:

- It generalizes the formulas mainly proposed in [5] for regular domains, which is only modified by a suitable weight.
- We can show that one can recover optimal error estimates, provided that the mesh is suitably graded near the edges.
- Eventually, a novel pseudo-differential approach, adapted to the corner singularities that appear in the problem is proposed. It is sketched in the present paper, but given in full details in [7].

We deeply believe that the methodology opens new perspectives for such problems. First, a generalization to specific 3D scattering problem, e.g. by a flat disc seems a simple generalization. We plan to extend the results presented here to such problems in the very near future. Second, the strategy that we used here seems very likely to be extended to the half line and hopefully to 2D sectors, giving, on the one hand a new pseudo-differential analysis more suitable than classical ones (see e.g. [27, 30, 33, 34]) for handling Helmholtz-like problems on singular domains, and, on the other hand, a completely new preconditioning technique adapted to the treatment of BEM operators on domains with corners or wedges in 3D. Eventually, the weighted square root operators that appeared in the present context might well be generalized to give suitable approximation of the exterior Dirichlet to Neumann map for the Helmholtz equation which is of particular importance in e.g. domain decomposition methods. Having such approximations might therefore lead to better methods in that context too.

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