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ABSTRACT. In this paper, we define the core entropy for postcritically-finite Newton maps and study its continuity within this family. We show that the entropy function is not continuous in this family, which is different from the polynomial case studied by Thurston, Gao, Dudko-Schleicher, Tiozzo [Th+, GT, DS, Ti2], and describe completely the continuity of the entropy function at generic parameters.

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1. INTRODUCTION

A classical way to measure the topological complexity of a dynamical system is its *entropy*. In particular, to each real polynomial map f one can associate the topological entropy of f as a dynamical system on the real line.

The classical entropy theory goes back to the seminal work of Milnor-Thurston [MT], who proved that the topological entropy of real quadratic polynomials depends continuously and monotonically on the parameter.

In order to generalize the entropy theory to complex polynomials and develop a "qualitative picture" of the parameter space of degree d polynomials, W. Thurston introduce a notion of entropy for complex polynomial around 2010. In his view, the entropy of a complex polynomial P should be defined as the topological entropy of the restriction of P on its "core", where a set as a candidate of a core of P should satisfy:

(1) it is compact and connected;
(2) it is a minimal P-invariant set; and
(3) it contains all critical points of P.

Such defined entropy is called, by Thurston, the "core entropy" of P.

At least in *postcritically-finte* case, i.e., the orbit of each critical point is finite, such cores for polynomials exist and known as *Hubbard tree*: the minimal invariant tree containing all critical points (see Section 2.2 below). This tree completely captures the dynamics of the corresponding polynomial (Poirier [Po2, Theorem 1.3]), and hence an appropriate candidate as a core. By Thurston, the *core entropy* of a polynomial P is defined as the topological entropy of the restriction of P on its Hubbard tree $\mathcal{H}_f$ if existing, i.e.,

$$h(P) := h_{top}(P|_{\mathcal{H}_P}), \tag{1.1}$$

We remark that the invariant segment for a real polynomial is just its Hubbard tree when considering it as a complex one. So the notion of core entropy is indeed a generalization of

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the topological entropy for real polynomials. Using core entropy, the classical entropy theory of real quadratic polynomial can be generalized to the complex case: the core entropy of quadratic polynomials depends continuously and monotonically along each *vein* of the Mandelbrot set ([Li], [Ti2],[Zeng]).

One can also consider the continuity of the entropy function on the whole space. Let $\mathcal{P}_d$ denote the set of monic, centered polynomials of degree $d \geq 2$, and $\mathcal{P}_d^{\mathrm{pf}}$ the subspace of $\mathcal{P}_d$ consisting of only postcritically-finite ones. Then we get an entropy function

$$h : \mathcal{P}_d^{\mathrm{pf}} \to \mathbb{R},$$

mapping f to its core entropy $h(f)$. Thurston asked whether this function is continuous. The answer is YES. The result are independently proved by Tiozzo [Ti2] and Dudko-Schleicher [DS] in quadratic case, and proved by Gao-Tiozzo [GT] in general case.

Following Thurston's spirit, in more general case, one can define the core entropy of a rational map if its core is found. Recently, Lodge, Mikulich and Schleicher construct the core for any postcritically-finite Newton map.

A rational map of degree $d \geq 3$ is called a *Newton map* if there exists a polynomial P so that

$$f = z - \frac{P(z)}{P'(z)} \text{ for all } z \in \mathbb{C}.$$

We denote by $\mathcal{N}_d$ the space of Newton maps of degree d, and by $\mathcal{N}_d^{\mathrm{pf}}$ the subspace of $\mathcal{N}_d$ consisting of postcritically-finite ones.

The cases $d < 3$ are excluded because they are trivial. Observed that f arises naturally when Newtons method is applied to find the roots of P. Each root of P is an attracting fixed point of f, and the point at infinity is a repelling fixed point of f. The degree d coincides with the number of distinct roots of P. Since the relation with the root-finding problem, the study of Newton maps became one of the major themes with general interest, both in discrete dynamical system (pure mathematics), and in root-finding algorithm (applied mathematics), see for example [AR, HSS, Ro, RS, RWY, Tan, WYZ].

In recent works, Lodge, Mikulich and Schleicher solve the long-standing classification problem for postcritically-finite Newton maps. The authors first specifically construct for any postcritically-finite Newton map f a finite connected f-invariant graph which satisfies a sequence of properties, called an *extended Newton graph* (see the proof of Theorem 6.2, Definition 7.3 and the proof of Theorem 1.2 in [LMS1]), and then prove that the postcritically-finite Newton maps (up to affine conjugation) can be classified by the equivalence classes of these extended Newton graphs ([LMS2, Theorems 1.4, 1.5]).

By their results, the extended Newton graphs completely capture the dynamics of postcritically-finite Newton maps, analogy to the status of Hubbard trees in postcritically-finite polynomial family. Therefore, it is reasonable to consider the extended Newton graph as a "core" of a postcritically-finite Newton map, and define the *core entropy* of a postcritically-finite Newton maps f as the topological entropy of the restriction of f on its extended Newton graph G_f, i.e.,

$$h(f) := h_{top}(f|_{G_f}). \tag{1.2}$$

The goal of this paper is to study the continuity of the core entropy of postcritically-finite Newton maps.

To present the main results of the paper, we give a brief overview of the structure of extended Newton graphs. All materials come from [LMS1], referring to Section 7 for details.

Let f be a postcritically-finite Newton map of degree $d \geq 3$. The *channel diagram* Δ of f is the union of the accesses from finite fixed points of f to ∞. The *Newton graph of level n* is constructed to be the connected component of $f^{-n}(\Delta)$ containing ∞ and is denoted by Δ_n. For a sufficiently high level n, the Newton graph captures the critical/postcritical points mapping onto fixed points. The remaining critical/postcritical points, if any, are disjoint from the Newton graph Δ_n for any level n. Factually, they are captured by another f-invariant combinatorial object called the *canonical Hubbard forest* of f. It is the disjoint union of finite trees which contain critical/postcritical point of f.

Thus far, all critical/postcritical points are contained in either the Newton graph or the Hubbard forest, but the Hubbard forest are disjoint from the Newton graph. To remedy this, preperiodic *Newton ray* are used to connect each component of the Hubbard forest to the Newton graph. Therefore, an *extended Newton graph* is a finite connected graph composed of:

- the Newton graph, which contains all critical/postcritical points mapping to fixed points;
- canonical Hubbard forest $\mathcal{H}_f$, which contains the remaining critical/postcritical points;
- preperiodic Newton rays connecting each subtree of $\mathcal{H}_f$ to the Newton graph.

We will say more about the canonical Hubbard forest. In fact, each non-trivial periodic component H of $\mathcal{H}_f$ corresponds a renormalization triple $\rho_H = (f^p, U, V)$ of f, where p is the renormalization period of ρ_H, such that H is an extended Hubbard tree of ρ_H (see Section 2.4 for the related concepts). By Straightening Theorem [DH2, Theorem 1], the polynomial-like map $f^p : U \to V$ is hybrid equivalent to a postcritically-finite polynomial $P_H \in \mathcal{P}_\delta$ with $\delta := \deg(f^p|_U)$, and the image of H under the conjugation is an extended Hubbard tree of P_H. It follows immediately that $h_{top}(f^p|_H) = h(P_H)$. The polynomial P_H is called a *renormalization polynomial* of f (associated to H) and p called the *renormalization period* of P_H. We denote

$$\mathfrak{R}(f) := \{P_H : \ H \text{ is a non-trivial periodic component of } \mathcal{H}_f.\} \tag{1.3}$$

The Newton map f is called *renormalizable* if $\mathfrak{R}(f) \neq \emptyset$ and *non-renormalizable* otherwise.

The first result in this paper is a formula to compute the core entropy of postcritically-finite Newton maps.

Proposition 1.1. *Let f be a postcritically-finite Newton map of degree $d \geq 3$. Then we have the core entropy formula*

$$h(f) = \begin{cases} \max_H \frac{1}{p_H} h_{top}(f^{p_H}|_H) = \max_{P \in \mathfrak{R}(f)} \frac{1}{p} h(P),, & \text{if } f \text{ is renormalizable;} \\ 0, & \text{otherwise,} \end{cases}$$

where H go though all non-trivial periodic components of $\mathcal{H}_f$ with period p_H, and p denotes the renormalization period of P. The renormalization polynomials achieving the maximum are called entropy-maximal.

Let f, $f_n, n \geq 1$ be postcritically-finite Newton maps of degree $d \geq 3$ such that $f_n \to f$ as $n \to \infty$. To study the continuity of the entropy function, we need to compare $h(f)$ and the limit behavior of $h(f_n)$ as $n \to \infty$. Let H be a non-trivial periodic component with period p of the canonical Hubbard forest $\mathcal{H}_f$, and $\rho_H = (f^p, U, V)$ a renormalization triple associated to H. As $f_n \to f$, we know that there exist polynomial-like maps $f_n^p : U_n \to V_n$ for all sufficiently large n such that $f_n^p : U_n \to V_n$ converge to $f^p : U \to V$ as $n \to \infty$. Note that the filled-in Julia set of (f_n^p, U_n, V_n) is not necessarily connected. Let H_n denote the intersection of the canonical Hubbard forest $\mathcal{H}_{f_n}$ of f_n and the filled-in Julia set of (f_n^p, U_n, V_n). We get the topological entropy $h_{top}(f_n^p|_{H_n})$. According to Proposition 1.1, the continuity of the entropy function at f is quite related to the limit behavior of $h_{top}(f_n^p|_{H_n})$ compared with $h_{top}(f^p|_H)$ for every non-trivial periodic component H of $\mathcal{H}_f$.

Using Straightening Theorem, the polynomial-like map $f^p : U \to V$ is hybrid equivalent to a polynomial $P_H \in \mathcal{P}_\delta$, and $f_n^p : U_n \to V_n$ is hybrid equivalent to a polynomial $P_{H,n} \in \mathcal{P}_\delta$. By suitable choice of the conjugations, we can assume $P_{H,n} \to P_H$ as $n \to \infty$. Note that P_H is postcritically-finite and $h(P_H) = h_{top}(f^p|_H)$, but $P_{H,n}$ is possibly not postcritically-finte: it may have disconnected Julia set although its critical points in the filled-in Julia set have finite orbits, and we haven't defined the core entropy for such polynomials. This motivate us to define the core entropy for some kinds of polynomials with disconnected filled-in Julia sets, including these $P_{H,n}$, so that $h(P_{H,n}) = h_{top}(f_n^p|_{H_n})$.

A polynomial is called *partial postcritically-finite* if its critical points in the filled-in Julia set have finite orbits. For any partial postcritically-finite polynomial P, we define its *Hubbard forest* $\mathcal{H}_P$ to be the minimal P-invariant forest containing all critical points in the filled-in Julia set, and define its core entropy $h(P)$ as the topological entropy of the restriction of P on $\mathcal{H}_P$, i.e.,

$$h(P) := h_{top}(P|_{\mathcal{H}_P}).$$

Note that $\mathcal{H}_P = \emptyset$ if and only if all critical points escape to ∞, and we define $h(P) = 0$ in this case. We denote by $\mathcal{P}_d^{\mathrm{ppf}}$ the subspace of $\mathcal{P}_d$ consisting of all partial postcritically-finite polynomials. Then $\mathcal{P}_d^{\mathrm{pf}} \subseteq \mathcal{P}_d^{\mathrm{ppf}}$ and the definition domain of the entropy function is enlarged from $\mathcal{P}_d^{\mathrm{pf}}$ to $\mathcal{P}_d^{\mathrm{ppf}}$.

It is easy to check that the polynomials $P_{H,n}, n \geq 1$ discussed above are partial postcritically-finite and $h(P_{H,n}) = h_{top}(f_n^p|_{H_n})$. It then follows from the discussion above that the continuity at f of the entropy function within the postcritically-finite Newton family has a close relationship to the limit behavior of $h(P_{H,n})$ compared with $h(P_H)$. In other words, to study the continuity of the core entropy of postcritically-finite Newton maps, we should first make clear of the continuity of the entropy function $h : \mathcal{P}_d^{\mathrm{ppf}} \to \mathbb{R}$ at postcritically-finite parameters. The following result completely solve this problem, and it plays an essential role in this paper.

Proposition 1.2. *Let P be a postcritically-finite polynomial in $\mathcal{P}_d$. Then the entropy function $h : \mathcal{P}_d^{\mathrm{ppf}} \to \mathbb{R}$ is upper semi-continuous at P, i.e.,*

$$\limsup_{\mathcal{P}_d^{\mathrm{ppf}} \ni Q \to P} h(Q) \leq h(P).$$

Furthermore, we can find a P-invariant forest $\mathcal{H}_P^*$ in $\mathcal{H}_P$ such that

$$\liminf_{\mathcal{P}_d^{\mathrm{ppf}} \ni Q \to P} h(Q) = h_{top}(P|_{\mathcal{H}_P^*}).$$

As a consequence, the function h is continuous at P if and only if $h(P) = h_{top}(P|_{\mathcal{H}_P^*})$.

We remark that by Gao-Tiozzo's result [GT], the entropy function is continuous within the postcritically-finite polynomial family. But the proposition above shows that in a larger family $\mathcal{P}_d^{\mathrm{ppf}}$, the entropy function is not continuous, for example at a Misiurewicz parameter P with positive core entropy, since $h_{top}(P|_{\mathcal{H}_P^*}) = 0$ in this case.

As discussed before, we can learn the continuity of the core entropy of postcritically-finite Newton maps by application of Proposition 1.2 and Straightening Theorem. In this process, a method of perturbing rational maps, called *capture surgery* (see Section 3), is also needed. We now state the main result in this paper. A postcritically-finite Newton map is called *generic* if the orbits of its critical points avoid ∞.

Theorem 1.3. *Let f be a generic postcritically-finite Newton map f of degree d. Then the entropy function $h : \mathcal{N}_d^{\mathrm{pf}} \to \mathbb{R}$ is upper semi-continuous at f, i.e.,*

$$\limsup_{\mathcal{N}_d^{\mathrm{ppf}} \ni g \to f} h(g) \leq h(f).$$

Furthermore, if f is non-renormalizable, then the function h is continuous at f; otherwise, the entropy function is continuous at f if and only if: there exists an entropy-maximal renormalization polynomial $P \in \mathfrak{R}(f)$ such that the entropy function $h : \mathcal{P}_\delta^{\mathrm{ppf}} \to \mathbb{R}$ is continuous at P with $\delta := \deg(P)$.

This theorem only tell us the continuity of the entropy function $h : \mathcal{N}_d^{\mathrm{pf}} \to \mathbb{R}$ at generic parameters, but we conjecture that it is correct for all parameters

Conjecture 1.4. *Theorem 1.3 holds for any postcritically-finite Newton map f.*

We are able to prove the conjecture in the cubic case, and it display a simple form.

Theorem 1.5. *The entropy function $h : \mathcal{N}_3^{\mathrm{pf}} \to \mathbb{R}$ is continuous at f if and only if the map f is either hyperbolic or non-renormalizable.*

For the completeness of the paper, we finally describe, as a byproduct, of the continuity of the entropy function $h : \mathcal{P}_d^{\mathrm{ppf}} \to \mathbb{R}$ at all parameters, not just at postcritically-finite ones as given in Proposition 1.2. The statement is parallel to that of Theorem 1.3 since the partial postcritically-finite polynomials are very similar to the postcritically-finite Newton maps in the view of entropy: both of them have the Hubbard forests on which the core entropy concentrate, and every non-trivial periodic component of the Hubbard forest induces a renormalization triple.

Let P be a partial postcritically-finite polynomial. We call P *renormalizable* if $\mathcal{K}_P$ has non-trivial components, and *non-renormalizable* otherwise. Suppose that P is renormalizable. Then for any non-trivial periodic component K of $\mathcal{K}_P$ with period p, there exists a monic,centered polynomial P_K of degree $\deg(P^p|_K)$ such that the restriction of P^p on K conjugates to the restriction of P_K on its filled-in Julia set. Using these notations, we state the following result.

Theorem 1.6. *Let P be any partial postcritically-finite polynomial in $\mathcal{P}_d^{\mathrm{ppf}}$. Then the entropy function $h : \mathcal{P}_d^{\mathrm{ppf}} \to \mathbb{R}$ is upper semi-continuous at P. Furthermore, if P has connected Julia set, the continuity of h at P is characterized by Proposition 1.2; in the case that $\mathcal{J}_P$ is not connected, if P is non-renormalizable, the function h is continuous at P, otherwise, the entropy function is continuous at P if and only if : there exists a non-trivial component K of $\mathcal{K}_P$ of period p such that $h(P) = h(P_K)/p$ and the entropy function $h : \mathcal{P}_\delta^{\mathrm{ppf}} \to \mathbb{R}$ is continuous at P_K, where δ denotes the degree of P_K.*

Organization of the paper. The paper is organized as follows:

In Section 2, we summarize some basic facts used in the paper.

In Section 3, we develop a method to perturb a sub-hyperbolic rational map to a hyperbolic one, called *capture surgery*, which is used in the proof of Proposition 1.2 and Theorem 1.3.

In Section 4, we study the convergence of internal/external rays when perturbing rational maps.

In Section 5, we will consider the dynamics of partial postcritically-finite polynomials. We focus on the critical markings of partial postcritically-finite polynomials, including its construction, properties and convergence.

In Section 6, we describe the continuity of the entropy function $h : \mathcal{P}_d^{\mathrm{ppf}} \to \mathbb{R}$ at the postcritically-finite parameters and prove Proposition 1.2. This is the key part of the paper. The proof of Proposition 1.2 is divided into two parts: we first show that the entropy function within the partial postcritically-finite polynomial family is upper semi-continuous at any postcritically-finite parameter P; and then construct a sequence $\{P_n, n \geq 1\} \subseteq \mathcal{P}_d^{\mathrm{ppf}}$ converging to P by capture surgery such that the limit of their core entropies obtain the limit inferior $\liminf_{Q \to P} h(Q)$.

In Section 7, we introduce the concept of core entropy for postcritically-finite Newton maps, and prove Proposition 1.1, which provides a formula for the computation of the core entropy.

In Section 8, we explore the continuity of the entropy function on postcritically-finite Newton family. We prove Theorem 1.3 by applying Proposition 1.2 and Straightening Theorem, and testify Theorem 1.5 with the help of Theorem 1.3 and some specific properties of cubic Newton maps given in [Ro].

Finally, in Section 9, we will give an outline of the proof of Theorem 1.6, which is parallel to the proof of Theorem 1.3.

2. Preliminary

2.1. **Topological entropy on graphs.** We will not use the general definition of the topological entropy in the paper (see [AKM]). Instead, we summarize some basic results about the topological entropy that will be applied below.

Let $h_{top}(f|_X)$ denote the topological entropy of f on X. Throughout the paper, for the uniform of the statement, we stipulate $h_{top}(f|_\emptyset) := 0$. The following propositions can be found in [Do1].

Proposition 2.1. *For any positive integer k we have $k \cdot h_{top}(f|_X) = h(f^k|_X)$.*

Proposition 2.2. *If $X = X_1 \cup X_2$, with X_1 and X_2 compact, $f(X_1) \subset X_1$ and $f(X_2) \subset X_2$, then $h_{top}(f|_X) = \sup\big(h_{top}(f|_{X_1}), h_{top}(f|_{X_2})\big)$.*

Proposition 2.3. *Let Z be a closed subset of X such that $f(Z) \subset Z$. Suppose that for any $x \in X$, the distance of $f^n(x)$ to Z tends to 0, uniformly on any compact set in $X - Z$. Then $h_{top}(f|_X) = h_{top}(f|_Z)$.*

Proposition 2.4. *Assume that π is a surjective semi-conjugacy*

$$
\begin{array}{ccc}
Y & \xrightarrow{\;g\;} & Y \\
\pi \downarrow & & \downarrow \pi \\
X & \xrightarrow{\;f\;} & X.
\end{array}
$$

Then $h_{top}(f|_X) \leq h_{top}(g|_Y)$. Furthermore, if $\sup_{x \in X} \#\pi^{-1}(x) < \infty$ then $h_{top}(f|_X) = h_{top}(g|_Y)$.

A (finite topological) *graph G* is a compact Hausdorff space which contains a finite non-empty set V_G, called *vertex set* of G, such that every connected component of $G \setminus V_G$ is homeomorphic to an open interval of the real line. The closure of each component of $G \setminus V_G$ is called an *edge* of G. There is a special kind of connected graphs, called *trees*, which are graphs without cycles. A finite disjoint union of trees is called a *forest*.

Let G be a finite graph with vertex set V_G. A continuous map $f : G \to G$ is called a *Markov map* if $f(V_G) \subseteq V_G$ and the restriction of f on each edge is injective.

Let $f : G \to G$ be a Markov graph map. By the definition, any edge of G is mapped to the union of several edges of G. Enumerate the edges of G by e_i , $i = 1, \cdots, k$. We then obtain an *incidence matrix* $D_{(G,f)} = (a_{ij})_{k \times k}$ of (G, f) such that $a_{ij} = 1$ if $f(e_i)$ covers e_j and 0 otherwise. Note that choosing different enumerations of the edges gives rise to conjugate incidence matrices, so in particular, the eigenvalues are independent of the choices.

Denote by λ the greatest non-negative eigenvalue of $D_{(G,f)}$. By the Perron-Frobenius theorem such an eigenvalue exists and equals the growth rate of $\|D_{(G,f)}^n\|$ for any matrix norm. The following result is classical (see [AM, MS]):

Proposition 2.5. *The topological entropy $h_{top}(f|_G)$ is equal to 0 if $D_{(G,f)}$ is nilpotent, i.e., all eigenvalues of $D_{(G,f)}$ are zero; and equal to $\log \lambda$ otherwise.*

2.2. **Dynamics of polynomials and rational maps.**

2.2.1. *General dynamical properties of rational maps.* Let f be a rational map of degree $d \geq 2$. We denote by $\mathcal{J}_f$ and $\mathcal{F}_f$ the Fatou set and Julia set of f respectively. Let Crit_f denote the set of critical points of f, and the set

$$\mathrm{Post}_f := \{f^n(c) : c \in \mathrm{Crit}_f, n \geq 1\}$$

is called the *postcritical set* of f. A point $z \in \overline{\mathbb{C}}$ is called a *Fatou point/Julia point* if $z \in \mathcal{F}_f/\mathcal{J}_f$, and called a *pre-critical point* if $f^n(z)$ is a critical point of f for some $n \geq 0$.

The map f is called *postcritically-finite* if $\#\mathrm{Post}_f < \infty$; called *hyperbolic* if all its critical points are attracted by the attracting cycles; and called *sub-hyperbolic* if all its Fatou critical points are attracted by the attracting cycles and all its Julia critical points have finite orbits.

Lemma 2.6. *Let f be a hyperbolic (resp. sub-hyperbolic) rational map. Then there exists a conformal metric (resp. orbifold metric) ω in a neighborhood of $\mathcal{J}_f$ such that f is* uniformly expanding *with respect to ω, i.e., $\exists \lambda > 1$, s.t $\|Df_z\|_\omega > \lambda$ for all z in this neighborhood (resp. whenever z and $f(z)$ are not ramified points). Moreover, each connected component of $\mathcal{J}_f$ is locally-connected.*

Proof. The existence of uniformly expanding (orbifold) metric can be found in [Mil, Section 19], and the local connectivity of Julia components is duo to the results of McMullen [Mc2] and Tan-Pilgrim [PT]. □

Let f be a rational map, and U a periodic Fatou component of f. Assume that U contains a unique postcritical point of f. Then U must be simply connected and there is a system of Riemann mappings

$$\left\{ \phi_W : \mathbb{D} \to W \,\middle|\, W \text{ Fatou component with } f^n(W) = f^m(U) \text{ for some } m, n \geq 0 \right\}$$

so that the following diagram commutes for all W:

$$
\begin{array}{ccc}
\mathbb{D} & \xrightarrow{\text{power map } z^{d_W}} & \mathbb{D} \\
{\scriptstyle \phi_W}\downarrow & & \downarrow{\scriptstyle \phi_{f(W)}} \\
W & \xrightarrow{\quad f \quad} & f(W),
\end{array}
$$

where d_W denotes the degree of f on W. The conformal map ϕ_W^{-1} is called a *Böttcher coordinate* of W, the image $\phi_W(0)$ is called the *center* of W and the image in W under ϕ_W of radial lines in $\mathbb{D}$ are called *internal rays* of W. Note that f maps internal rays of W to those of $f(W)$.

2.2.2. *Dynamics of Polynomials.* Let P be a monic polynomial of degree $d \geq 2$. Basic tools to understand the dynamics of P are the Green function g_P and the Böttcher map ϕ_P.

The *Green function g_P* of P is defined as

$$g_P(z) := \lim \log^+ |P^n(z)|/d^n, \text{ for all } z \in \mathbb{C}.$$

It is a well-defined continuous function which vanishes on the *filled-in Julia set*

$$\mathcal{K}_P := \{z \in \mathbb{C} : P^n(z) \nrightarrow \infty \text{ as } n \to \infty\},$$

and satisfies the functional relation $g_P(P(z)) = dg_P(z)$. In the basin of infinity $\Omega(P) := \mathbb{C} \setminus \mathcal{K}_P$, the derivative of g_P vanishes at z if and only if z is a pre-critical points of P. We say that z is a singularity of g_P.

The Böttcher map ϕ_P conjugates P with $z \mapsto z^d$ in a neighborhood of z. Since P is monic, we can normalize ϕ_P so that $\phi_P(z)/z \to 1$ as $z \to \infty$. We extend, along flow lines, to the basin of infinity under the gradient flow ∇g_P. Following Levin and Sodin [LS], we define the *reduced basin of infinity* $\Omega^*(P)$ to be the maximal basin in which the gradient flow ∇g_P is smooth. Now

$$\phi_f : \Omega^*(P) \to D_P \subseteq \mathbb{C} \setminus \overline{\mathbb{D}} \tag{2.1}$$

is a conformal isomorphism from $\Omega^*(P)$ onto a starlike (around ∞) domain D_P. We have the commutative diagram:

$$
\begin{array}{ccc}
\Omega^*(P) & \xrightarrow{\ P\ } & \Omega^*(P) \\
\phi_P \downarrow & & \downarrow \phi_P \\
D_P & \xrightarrow{z \mapsto z^d} & D_P.
\end{array}
$$

A flow line of ∇g_P in $\Omega^*(P)$ is called an *external radius*. External radii are parameterized $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. More precisely, for $t \in \mathbb{T}$, let $(r, \infty)e^{2\pi it}$ be the maximal portion $(1, \infty)e^{2\pi it}$ contained in D_P. The *external radius of argument t* is defined as

$$\mathcal{R}_P^*(t) := \phi_P^{-1}\big((r, \infty)e^{2\pi it}\big).$$

In the case of $r > 1$, the radius terminate at a singularity of g_P. While in the case of $r = 1$, the radius accumulate at the Julia set, and we call it a *smooth external ray*, writing also as $\mathcal{R}_P(t)$.

Let $\tau = \tau_d : \mathbb{T} \to \mathbb{T}$ denote the mapping which sends θ to $d \cdot \theta$ (mod$\mathbb{Z}$). Now let $\theta_1, \ldots, \theta_r$ be the arguments of the external radii that terminate at critical points of P. Since every pre-critical point of P is a singularity of g_P, the external radii with arguments in

$$\Sigma = \bigcup_{n \geq 0} \tau_d^{-n}(\{\theta_1, \ldots, \theta_r\})$$

also terminate at singularities. Since every singularity is a pre-critical point, we have smooth external rays defined for arguments in $\mathbb{T} \setminus \Sigma$. Following Goldberg and Milnor, for $t \in \mathbb{T}$, let

$$\mathcal{R}_P^{\pm}(t) := \lim_{\mathbb{T} \setminus \Sigma \ni s \to t^{\pm}} \mathcal{R}_P(s).$$

If $t \notin \Sigma$ then $\mathcal{R}_P^{\pm}(t) = \mathcal{R}_P(t)$ is a smooth external ray. If $t \in \Sigma$, then $\mathcal{R}_P^{\pm}(t)$ do not agree, and we say that they are *non-smooth external ray*. By *external rays* we mean either smooth or non-smooth external rays. An external ray is called *landing* at z if its accumulation set on $\mathcal{J}_P$ is $\{z\}$. We denote by $\arg_P(z)$ the set of angles such that $\theta \in \arg_P(z)$ if and only if an external ray $\mathcal{R}_P^{\sigma}(\theta)$ lands at z, with $\sigma \in \{+, -\}$.

Definition 2.7 (supporting ray/argument)**.** *Let U be a bounded Fatou component of a polynomial P, and $z \in \partial U$. The external rays landing at z (if existing) divide the plane into finite regions. We label the arguments of these rays by $\theta_1, \ldots, \theta_k$ in counterclockwise cyclic order, so that U belongs to the region delimited by $\mathcal{R}_P^{\sigma_1}(\theta_1)$ and $\mathcal{R}_P^{\sigma_2}(\theta_2)$ with $\sigma_1, \sigma_2 \in \{+, -\}$. The ray $\mathcal{R}_P^{\sigma_1}(\theta_1)$ (resp. $\mathcal{R}_P^{\sigma_2}(\theta_2)$) is called the* left-supporting *(resp.* right-supporting*) ray of U at z, and the argument θ_1 (resp. θ_2) is called the* left-supporting *(resp.* right-supporting*) argument of U at z.*
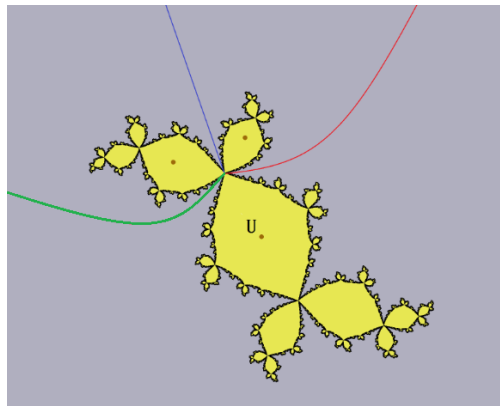

FIGURE 1. The red and green rays are right and left supporting rays of the Fatou component U, but the blue one does not support U.

According to Levin and Przytycki [LP], the landing theorem stated for connected Julia sets generalizes as follows.

Proposition 2.8. *Let P be a polynomial. Then every periodic external ray lands at a parabolic or repelling periodic point. On the other hand, let z be a parabolic or repelling periodic point. Then there exists at least one external ray landing at z. Moreover, either*

   (1) *all the external rays, smooth and non-smooth, landing at z are periodic of the same period, or*
   (2) *the arguments of the external rays, smooth and non-smooth, landing at z are irrational and form a Cantor set, and moreover, $\{z\}$ is a connected component of $\mathcal{J}_P$ and there are non-smooth rays landing at z; furthermore, there exists a non-smooth ray containing a critical point landing at a point in the orbit of z.*

Recall that $\Omega^*(P)$ is the reduced basin of infinity of P. According to Levin and Sodin [LS], the set $\mathcal{J}_P^* := \partial \Omega^*(P)$ is called the *extended Julia set* of P. It is known that $\mathcal{J}_P^*$ is a connected set such that $\mathcal{J}_P \subseteq \mathcal{J}_P^*$ and $P^{-1}(\mathcal{J}_P^*) \subseteq \mathcal{J}_P^*$. As all polynomials in the paper are sub-hyperbolic, we include here the following result (see [LS, Propositions 2.1,2.2]).

Proposition 2.9. *Let P be a sub-hyperbolic polynomial. Then every external ray, smooth and non-smooth, lands at the Julia set. Moreover, the extended Julia set $\mathcal{J}_P^*$ is locally connected such that every external radius terminates or lands at a point in $\mathcal{J}_P^*$.*

We remark that if P has connected Julia set, the Böttcher map ϕ_P extends to the whole $\Omega(P)$, so that $\Omega^*(P) = \Omega(P)$ and all external rays are smooth.

2.2.3. *Postcritically-finite polynomials and the Hubbard trees.* Let P be a postcritically-finite polynomial. Then it has connected and locally arc-connected filled-in Julia set. Since $\mathcal{K}_P$ is

arc-connected, given two points $z_1, z_2 \in \mathcal{K}_P$, there is an arc $\gamma : [0,1] \to \mathcal{K}_P$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. It is proved in [DH1] that the arc γ can be chosen in a unique way so that the intersection with the closure of a Fatou component consists of segment of internal rays. We call such an arc *regulated* and denote it by $[z_1, z_2]$. By [DH1, Proposition 2.7], the set

$$\mathcal{H}_P := \cup_{p,q \in \text{Post}_P \cup \text{Crit}_P}[p,q]$$

is a finite connected tree, called the *Hubbard tree* of P. The vertex set $V(\mathcal{H}_P)$ consists of the critical/postcritical points of P and the branched points of $\mathcal{H}_P$. It is well known that $\mathcal{H}_P$ is P-invariant and $P : \mathcal{H}_P \to \mathcal{H}_P$ is Markov (see [Po2, Section 1]). Furthermore, Poirier proved that the postcritically-finite polynomials can be dynamically classified by their Hubbard trees.

Following Thurston, the *core entropy* of P is defined as $h(P) := h_{top}(P|_{\mathcal{H}_P})$. A regulated tree within $\mathcal{K}_P$ is called an *extended Hubbard tree* if it is P-invariant and contains $\mathcal{H}_P$.

Lemma 2.10. *Let T be any extend Hubbard tree of a postcritically-finite polynomial P. Then we have $h_{top}(P|_T) = h(P)$.*

Proof. Since every point, except the endpoints, of T are iterated to $\mathcal{H}_P$, the lemma follows directly from Lemma 2.3. $\square$

2.3. **Critical portraits and entropy algorithm.** To compute the core entropy of polynomials, W. Thurston developed a purely combinatorially algorithm (avoid knowing the topology of Hubbard trees) using the combinatorial data called *critical portraits.*

A finite collection $\Theta := \{\ell_1, \ldots, \ell_m\}$ of finite subsets of the unit circle is called a *critical portrait of degree d* if

(1) each $\tau_d(\ell_i)$ is a singleton, $1 \le i \le m$;
(2) the convex hulls $hull(\ell_i), hull(\ell_j)$ in the closed unit disk intersect at most at one point of $\mathbb{T}$ for any $i \ne j \in \{1, \ldots, m\}$;
(3) each $\#\ell_i \ge 2$, and $\sum_{i=1}^{m}(\#\ell_i - 1) = d - 1$.
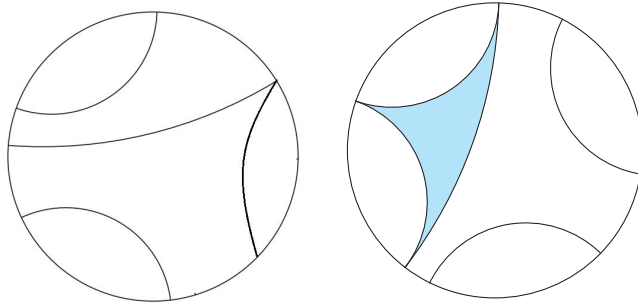
FIGURE 2. Two critical portraits of degree 5

Thurston's entropy algorithm endues every critical portrait Θ a non-negative real number $h(\Theta)$, called the *output* of the algorithm (see [G, Section 5.2] for detail). Gao-Tiozzo prove that $h(\Theta)$ varies continuously with respect to Θ.

Proposition 2.11 ([GT], Theorem 1.1)**.** *Let $\{\Theta_n, n \ge 1\}$ be a sequence of critical portraits Hausdorff converging to Θ. Then $\lim_{n\to\infty} h(\Theta_n) = h(\Theta)$.*

In fact, any postcritically-finite polynomial P of degree d induces a critical portrait Θ_P of degree d, which connects the core entropy of P and the output $h(\Theta)$ of Thurton's entropy algorithm. To show the construction of Θ_P, we first define $\Theta(U)$ as follows for each bounded critical Fatou component U. Denote $\delta_U = \deg(f|_U)$.

- **Case 1**: We first consider the case when U is a periodic, critical Fatou component. Let

$$U \mapsto f(U) \mapsto \cdots \mapsto f^n(U) = U$$

be a critical Fatou cycle of period n. We will construct the associated set $\Theta(U')$ for every critical Fatou component U' in this cycle simultaneously. Let $z \in \partial U$ be a periodic point with period less than or equal to n. Let θ denote the left-supporting argument of U at z. Clearly, θ is periodic with period n. We call θ a *preferred* angle for U. Note that this choice naturally determines a left-supporting argument of each Fatou component $f^k(U)$ for $k \in \{0, \ldots, n-1\}$, which is called a *preferred angle* of $f^k(U)$. Let U' be a critical Fatou component in the cycle and θ' its preferred angle. We now define $\Theta(U')$ as any set of $\delta_{U'}$ angles such that:
  (a) $\theta' \in \Theta(U')$;
  (b) the rays corresponding to the elements of $\Theta(U')$ land at $\delta_{U'}$ distinct points of $\partial U'$ and are inverse images of $f(R(\theta'))$.
- **Case 2**: U is a strictly preperiodic Fatou component. Let k be the minimal number such that $U' = f^k(U)$ is a critical Fatou component. We may assume that $\Theta(U')$ is already chosen, according to the previous case. Choose an angle $\theta' \in \Theta(U')$. We define $\Theta(U)$ to be the set of arguments of the δ_U rays landing at δ_U distinct points of ∂U that are k-th inverse images of $R(\theta')$.

Definition 2.12 (weak critical marking)**.** *A collection of finite subsets of $\mathbb{T}$*

$$\Theta_P := \{\Theta_1(c_1), \ldots, \Theta_m(c_m); \Theta(U_1), \ldots, \Theta(U_s)\} \tag{2.2}$$

is called a weak critical marking *of P if*

(1) *Θ_P is a critical portrait of degree d;*
(2) *each $\Theta(U_k)$ is defined as above for $k \in \{1, \ldots, n\}$ ($U_1, \ldots, U_s$ are pairwise distinct);*
(3) *each $c_j, 1 \le j \le m$, is a Julia critical point of P ($c_1, \ldots, c_m$ are not necessary pairwise distinct), and $\#\Theta_j(c_j) \ge 2$ for every $j = 1, \ldots, m$;*
(4) *for each critical point $c \in \mathcal{J}_f$,*

$$\deg(f|_c) - 1 = \sum_{c_j = c} \left(\#\Theta_j(c_j) - 1 \right).$$

We call $\mathscr{F}_P := \{\Theta(U_i) : 1 \le i \le s\}$ a weak Fatou critical marking *of P, and $\mathscr{J}_P := \{\Theta_j(c_j) : 1 \le j \le m\}$ a* weak Julia critical marking *of P (see Figure 3).*

The concept of critical markings of polynomials was introduced by Bielefeld-Fisher-Hubbard [BFH] for Misiurewicz postcritically-finite polynomials, and then generated by Poirier [Po1] to the general postcritically-finite case, to classify the dynamics of postcritically-finite polynomials. Our definition of weak critical marking, first given in [G] (also in [GT]), is much less restrictive than that of critical marking (see [Po1] or Section 5.2) because we just use it to compute core entropy. The following result was assured by Thurston and proved by Gao.

Proposition 2.13 ([G], Theorem 1.2)**.** *Let P be a postcritically-finite polynomial and Θ_P a weak critical marking of P. Then the core entropy $h(P)$ of P equals to the output $h(\Theta_P)$ in Thurston's entropy algorithm.*
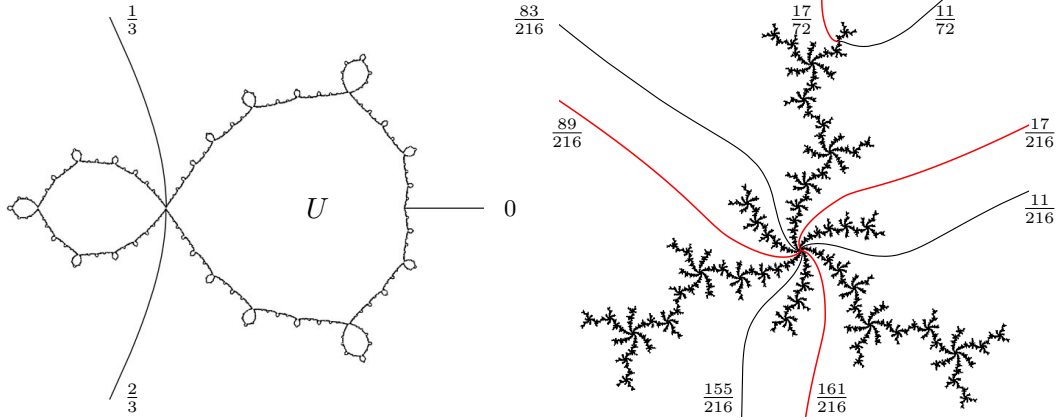
FIGURE 3. In the left picture, the polynomial $P(z) = z^3 - 3z^2/2$ has a fixed critical Fatou component U and another critical point $-1 \in \partial U$. A weak critical marking of P is $\Theta_P = \{ \Theta(U) = \{0, 1/3\}, \Theta(-1) = \{1/3, 2/3\} \}$. The right one shows the Julia set of $P(z) = z \mapsto z^3 + 0.22036 + 1.18612i$, and the external rays landing at its critical point and critical value. One can choose $\Theta_P = \{ \Theta_1(0) := \{11/216, 83/216\}, \Theta_2(0) := \{89/216, 161/216\} \}$ as a weak critical marking of P.

2.4. **Polynomial-like maps and renormalization.** Polynomial-like maps were introduced by Douady and Hubbard [DH2] and have played an important role in complex dynamics ever since.

A *polynomial-like* map of degree $d \geq 2$ is a triple (g, U, V) where U, V are topological disks in $\mathbb{C}$ with $\overline{U} \subseteq V$, and $f : U \to V$ is a holomorphic proper map of degree d. The *filled-Julia set* of g is the set of points in U that never leave V under iteration of g, i.e.,

$$K_g := \bigcap_{n \geq 0} g^{-n}(V),$$

and its *Julia set* is defined as $J_g := \partial K_g$. We can similarly define the *postcritical points* of g and the concept of *postcritical-finite*. If g is postcritically-finite, then its *Hubbard tree* and *extended Hubbard* tree is defined similar as the polynomial case.

Two polynomial-like maps f and g are *hybrid equivalent* if there is a quasiconformal conjugacy ψ between f and g that is defined on a neighborhood of their respective filled-in Julia sets so that $\partial \overline{\psi} = 0$ on K_f. The crucial relation between polynomial-like maps and polynomials is explained in the following theorem, due to Douady and Hubbard [DH2].

Theorem 2.14 (Straightening Theorem). *Let $f : U \to V$ be a polynomial-like map of degree $d \geq 2$. Then f is hybrid equivalent to a polynomial P of the same degree. Moreover, if K_f is connected, then P is unique up to affine conjugation.*

Definition 2.15 (renormalization). *Let f be a rational map. A triple $\rho = (f^p, U, V)$ is called a* renormalizaton triple *of f if (f^p, U, V) is a polynomial-like map with connected filled-in Julia set, and the number p is called the* renormalization period *of ρ.*

The filled Julia set of ρ is denoted by K_ρ, the Julia set by J_ρ, the critical/postcritical sets by $\mathrm{Crit}_\rho/\mathrm{Post}_\rho$, and the Hubbard tree by H_ρ, respectively.

2.5. **Basic results of Newton maps.** Let P be a complex polynomial, factored as

$$P(z) = (z - a_1)^{n_1} \cdots (z - a_d)^{n_d},$$

and $a_1, \ldots, a_d$ ($d \geq 3$) are distinct roots of P, with multiplicities $n_1, \ldots, n_d \geq 1$, respectively.

Its Newton map

$$f = f_P := z - \frac{P(z)}{P'(z)}$$

has degree d and fixes each root a_i with multiplier $f'(a_i) = (n_i - 1)/n_i$. Therefore, each root a_i of P corresponds to an attracting fixed point of f with multiplier $1 - 1/n_i$. One may verify that ∞ is a repelling fixed point of f with multiplier $d/(d-1)$. This discussion shows that a degree d Newton map has $d + 1$ distinct fixed points with specific multipliers. On the other hand, a well-known theorem of Head states that the fixed points together with the specific multipliers can determine a unique Newton map:

Proposition 2.16 (Head). *A rational map f of degree $d \geq 3$ is a Newton map if and only if f has $d + 1$ distinct fixed points $a_1, \ldots, a_d, \infty$, such that for each fixed point a_i, the multiplier takes the form $1 - 1/n_i$ with $n_i \in \mathbb{N}, 1 \leq i \leq d$.*

According to Shishikura [Sh], the Julia set of a Newton map is always connected, or equivalently, all Fatou components are simply connected.

3. Capture surgery for sub-hyperbolic rational maps

In the section, we develop a method, called *capture surgery*, to perturb sub-hyperbolic rational maps such that some of the Julia critical points are captured by attracting cycles after the perturbation. This surgery will be used to construct sequences of polynomials or Newton maps on which the entropy function achieving the inferior (in the proof of Proposition 1.2 and Theorem 1.3). A similar idea is used in [CT3, GZ] for a very special case.

Let f be a sub-hyperbolic rational map of degree d such that $\mathcal{F}_f \neq \emptyset$. Fix a vector $\mathbf{c} = (c_1, \ldots, c_r)$ of distinct Julia critical points of f. The perturbation of f by capture surgery near $\mathbf{c}$ includes three steps.

I. The topological surgery.

For each $c \in \mathbf{c}$, we choose a small open disk D_c containing c such that these $D_c's$ are pairwise disjoint and have iteration relation: if $f^m(c') = c$ for $c, c' \in \mathbf{c}$, then $D(c')$ is the component of $f^{-m}(D(c))$ containing c'. We call the vector $\mathbf{D} := (D_c : c \in \mathbf{c})$ a *perturbation domain*. In each D_c, we choose an open set (not necessarily connected) $Z_c \subseteq \mathcal{F}_f$ such that

(1) if $f^m(c') = c$ for $c, c' \in \mathbf{c}$, then $Z(c') = f^{-m}(Z(c)) \cap D(c')$;
(2) the forward orbit of any point in Z_c never return to D_c.

The vector $\mathbf{Z} := (Z_c : c \in \mathbf{c})$ is called an *invariant domain*. Finally, we define a *perturbation mapping* $\mathbf{X} = (\chi_c : c \in \mathbf{c})$ such that for each $c \in \mathbf{c}$,

(1) $\chi_c : D_c \to f(D_c)$ is a *quasi-regular map*, i.e, the composition of a rational map and a quasi-conformal map, of degree $\deg(f|_c)$;
(2) the map χ_c coincides with f on the invariant domain $Z_c \subseteq D_c$, the boundary ∂D_c and the point c;
(3) the critical values of χ_c belong to $f(Z_c) = \chi_c(Z_c)$.

The vector $\vec{\mathbf{s}} := (\mathbf{c}, \mathbf{D}, \mathbf{Z}, \mathbf{X})$ is called the *perturbation data* for the capture surgery.

We now define the *topological perturbation* $F = F_{\vec{s}}$ of f by capture surgery with a given perturbation date $\vec{s}$, as

$$F(z) = \begin{cases} \chi_c(z), & \text{if } z \in D(c) \text{ for some } c \in \mathbf{c}; \\ f(z), & \text{otherwise.} \end{cases} \tag{3.1}$$

By the definition, the perturbation map F differs with f only at $\cup_{c \in \mathbf{c}} D(c)$, and each Julia critical point c of f in $\mathbf{c}$ splits into several critical points of F captured by attracting cycles of F.

II. The rational realization.

To realize the topological perturbation F as a rational map, we need Thurston-Cui-Tan's result about the topological characterization of sub-hyperbolic rational maps [DH3, CT1].

Let G be a branched covering of degree $d \geq 2$. Its *postcritical set* is defined as

$$\mathrm{Post}_G = \overline{\{G^n(c) : n \geq 1, \ c \text{ is a critical point of } G\}}.$$

We say that G is a *sub-hyperbolic semi-rational map* if the *accumulation set* Post'_G of Post_G is finite (or empty); and in the case $\mathrm{Post}'_G \neq \emptyset$, the map G is holomorphic in a neighborhood of Post'_G and every periodic point Post'_G is (super-)attracting.

By a *marked sub-hyperbolic, semi-rational map* $(G, \mathcal{Q})$, we mean that $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a sub-hyperbolic, semi-rational map and the *marked set* $\mathcal{Q} \subseteq \overline{\mathbb{C}}$ is a closed set such that $\mathrm{Post}_G \subseteq \mathcal{Q}$, $G(\mathcal{Q}) \subseteq \mathcal{Q}$ and $\#(\mathcal{Q} \setminus \mathrm{Post}_G) < \infty$.

A Jordan curve $\gamma \subseteq \overline{\mathbb{C}} \setminus \mathcal{Q}$ is called *peripheral* in $\overline{\mathbb{C}} \setminus \mathcal{Q}$ if one of its complementary components contains at most one point of Post_G; and is otherwise called *non-peripheral* in $\overline{\mathbb{C}} \setminus \mathcal{Q}$. We say that $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ is a *multicurve* in $\overline{\mathbb{C}} \setminus \mathcal{Q}$, if each γ_i is a non-peripheral Jordan curve in $\overline{\mathbb{C}} \setminus \mathcal{Q}$, these curves are pairwise disjoint and pairwise non-homotopic in $\mathbb{C} \setminus \mathcal{Q}$. Its $(G, \mathcal{Q})$-*transition matrix* $D_\Gamma = (a_{ij})$ is defined by:

$$a_{ij} = \sum_\alpha \frac{1}{\deg(G : \alpha \to \gamma_j)},$$

where the summation is taken over the components α of $G^{-1}(\gamma_j)$ homotopic to γ_i in $\overline{\mathbb{C}} \setminus \mathcal{Q}$. The leading eigenvalue of D_Γ is also called the *leading eigenvalue of* Γ, denoted by λ_Γ.

We say that a multicurve $\Gamma \subseteq \overline{\mathbb{C}} \setminus \mathcal{Q}$ is G-*stable* if, for any curve $\gamma \in \Gamma$, every component of $G^{-1}(\gamma)$ is either peripheral or homotopic in $\overline{\mathbb{C}} \setminus \mathcal{Q}$ to a curve in Γ. A multicurve $\Gamma \subseteq \overline{\mathbb{C}} \setminus \mathcal{Q}$ is called a *Thurston obstruction* of $(G, \mathcal{Q})$ if it is G-stable and $\lambda_\Gamma \geq 1$.

Two marked sub-hyperbolic, semi-rational maps $(G_1, \mathcal{Q}_1)$ and $(G_2, \mathcal{Q}_2)$ are called *c-equivalent*, if there is a pair (ϕ, ψ) of homeomorphisms of $\overline{\mathbb{C}}$, and a neighborhood U_0 of $\mathcal{Q}'_1$ such that

- $\phi \circ G_1 = G_2 \circ \psi$;
- ϕ is holomorphic in U_0;
- the maps ϕ and ψ are equal on $\mathcal{Q}_1$, and thus on $\mathcal{Q}_1 \cup \overline{U_0}$ (by the isolated zero theorem);
- the two maps ϕ and ψ are isotopic to each other relatively to $\mathcal{Q}_1 \cup \overline{U_0}$.

If $\mathcal{Q}_1 = \mathrm{Post}_{G_1}$ (hence $\mathcal{Q}_2 = \mathrm{Post}_{G_2}$), we say that G_1 is *c-equivalent* to G_2.

Theorem 3.1 (Thurston-Cui-Tan[DH3, CT1])**.** *Let* $(G, \mathcal{Q})$ *be a sub-hyperbolic, semi-rational marked map, not Lattès type. Then* $(G, \mathcal{Q})$ *is c-equivalent to a rational marked map if and only if* $(G, \mathcal{Q})$ *has no Thurston obstructions. In this case the rational map is unique up to Möbius conjugation.*

According to the requirements on the perturbation data, any topological perturbation F of a sub-hyperbolic rational map f by capture surgery is sub-hyperbolic, semi-rational. We believe that F is c-equivalent to a rational map in general case. But in this paper, we just prove this point in the polynomial and Newton-map cases, since these are the only cases we encounter and the argument is relatively simple. We will deal with the polynomial case in this section, and leave the Newton-map case to Section 7.5.

Lemma 3.2. *Let f be a sub-hyperbolic polynomial, and the invariant domain $\mathbf{Z}$ belong to the basin of infinity. Then the topological perturbation F of f defined in Step I is c-equivalent to a polynomial.*

Proof. Let $\mathcal{Q} := \mathrm{Post}_F \cup \mathrm{Post}_f$. According to Property (2) of the perturbation mapping $\mathbf{X}$, the set $\mathcal{Q}$ is a marked set of F. By Theorem 3.1, we just need to check that the marked map $(F, \mathcal{Q})$ has no Thurston obstructions. In this case the map F is a topological polynomial, then Thurston obstructions are equivalent to *levy cycles*, i.e., a collection of non-peripheral Jordan curves $\Gamma = \{\gamma_1, \ldots, \gamma_k\} \subseteq \overline{\mathbb{C}} \setminus \mathcal{Q}$ such that, for each γ_i, there exists a unique component $\widetilde{\gamma}_i$ of $F^{-1}(\gamma_{i+1})$ so that $\widetilde{\gamma}_i$ is homotopic to γ_i in $\overline{\mathbb{C}} \setminus \mathcal{Q}$ and $F : \widetilde{\gamma}_i \to \gamma_{i+1}$ is a homeomorphism.

On the contrary, assume that $(F, \mathcal{Q})$ has a Levy cycle Γ. For the simplicity of the statment, we assume that $\Gamma = \{\gamma\}$. Throughout the proof, the curve $\widetilde{\gamma}$ denotes the component of $F^{-1}(\gamma)$ isotopic to γ in $\overline{\mathbb{C}} \setminus \mathcal{Q}$.

Claim 1. By suitably choosing γ in its isotopic class in $\overline{\mathbb{C}} \setminus \mathcal{Q}$, we have $\widetilde{\gamma}$ is disjoint with $\cup_{c \in \mathbf{c}} D_c$.

Proof of Claim 1. For the simplicity of the statement, we also assume that all critical points in $\mathbf{c}$ lie in one grand orbit and the periodic points in this orbit are fixed by F, which we denote by z_0. We specify the construction of $\mathbf{D}$ and $\mathbf{Z}$ as follows. Let $D_{z_0} \ni z_0$ be a open disk disjoint with all postcritical points of f except z_0. For each $c \in \mathbf{c}$ with $f^m(c) = z_0$, we define D_c the component of $f^{-m}(D_{z_0})$ containing c. In each component of $D_{z_0} \setminus \mathcal{K}_f$, we choose a disk in the basin of infinity such that its image by f leaves D_{z_0}, and converge to ∞ under the iteration of f. We denote by Z_{z_0} the union of all such disks. For each $c \in \mathbf{c}$ with $f^m(c) = z_0$, we define $Z_c := f^{-m}(Z_{z_0}) \cap D_c$. Let $\mathbf{D} := (D_c : c \in \mathbf{c})$ and $\mathbf{Z} := (Z_c : c \in \mathbf{c})$ be the perturbation domain and invariant domain of the perturbation map F.

Observe that if $\widetilde{\gamma}$ avoids all components of $F^{-1}(D_{z_0})$ disjoint with D_{z_0}, then the claim holds. To see this, we call D a k-th preimage of D_{z_0} ($k \geq 1$) if D is a component of $F^{-k}(D_{z_0})$, but $F^{k-1}(D) \cap D_{z_0} = \emptyset$. If $\widetilde{\gamma}$ avoids all 1-th preimages of D_{z_0}, then

$$F : \widetilde{\gamma} \cap \widetilde{D}_{z_0} \to \gamma \cap D_{z_0} \text{ is a homeomorphism,} \tag{3.2}$$

where $\widetilde{D}_{z_0}$ denotes the component of $F^{-1}(D_{z_0})$ contained in D_{z_0}. We modify γ outside D_{z_0} in its isotopic class so that γ is disjoint with all 1-th preimages of D_{z_0}. Then, on one hand, since the new $\widetilde{\gamma}$ differs from the original one outside $\widetilde{D}_{z_0}$, it still avoids the 1-th preimages of D_{z_0} by (3.2); on the other hand, the property that γ is disjoint with 1-th preimages of D_{z_0} implies that $\widetilde{\gamma}$ is disjoint from 2-th preimages of D_{z_0}. Therefore the new $\widetilde{\gamma}$ avoids 1-th, 2-th preimages of D_{z_0}. Then the claim holds by inductively use this argument. So we just need to show $\widetilde{\gamma} \cap (F^{-1}(D_{z_0}) \setminus D_{z_0}) = \emptyset$ by suitably choosing γ in its isotopic class.

We say that γ *essential intersects* a set S if all curves in the isotopic class of γ intersects S. If γ does not essentially intersect D_{z_0}, then the conclusion above obviously holds. So, in the following, we always assume γ essentially intersects D_{z_0}. Note that $\mathcal{Q} \cap D_{z_0}$ is contained in $\{c\} \cup Z_{z_0}$. We first show that γ does not essentially intersects Z_{z_0}. If not, the curve $\widetilde{\gamma}$ also essentially intersects Z_{z_0}. It implies that $\gamma = F(\widetilde{\gamma})$ essentially intersects $F(Z_{z_0})$. Inductively, we

have $\infty \in \gamma$ since $F^n(Z_{z_0}) \to \infty$, a contradiction. Thus, the γ must separate the components of Z_{z_0} which contain postcritical points of F.

Without loss of generality, we assume that the rotation number of z_0 is 0, i.e., all external rays of f landing at z_0 is fixed. Then we can construct an F-invariant ray for each component of Z_{z_0} intersecting $\mathcal{Q}$ such that the ray contains a point of $\mathcal{Q}$ in this component. Let B be a component of Z_{z_0} and $a \in B \cap \mathcal{Q}$. Since D_{z_0} is in the linearizable domain of z_0, then there exists an arc $\delta \subseteq D_{z_0} \cap \Omega(f)$ such that δ joins z_0 and a, $\delta \subseteq f(\delta)$ and the gradients of the points in δ are pairwise disjoint. It follows immediately that $f : \delta \to f(\delta)$ is a homeomorphism since the gradients of the points in $f(\delta)$ are also pairwise disjoint by $g_f(f(z)) = dg_f(z)$. Inductively, the arc $\mathcal{R}_B := \cup_{k \geq 0} f^k(\delta)$ is f-invariant. Note that such constructed $\mathcal{R}_B$ is disjoint with the perturbation domain $\mathbf{D}$, then $\mathcal{R}_B$ is also F-invariant. We denote by $\mathcal{R}_{z_0}$ the union of $\mathcal{R}_B$ with B going through all components of Z_{z_0} intersecting $\mathcal{Q}$.

By suitably choosing γ in its isotopic class, we can assume that the number of the components of $\gamma \cap D_{z_0}$ is minimal, and the number of $\gamma \cap \mathcal{R}_{z_0}$ is minimal. By the minimality of $\#(\gamma \cap \mathcal{R}_{z_0})$, we have $\#(\widetilde{\gamma} \cap \mathcal{R}_{z_0}) \geq \#(\gamma \cap \mathcal{R}_{z_0})$. As $F : \widetilde{\gamma} \to \gamma$ is a homeomorphism and $F(\mathcal{R}_{z_0}) = \mathcal{R}_{z_0}$, then $\#(\widetilde{\gamma} \cap \mathcal{R}_{z_0}) \leq \#(\gamma \cap \mathcal{R}_{z_0})$ and $F(\widetilde{\gamma} \cap \mathcal{R}_{z_0}) \subseteq \gamma \cap \mathcal{R}_{z_0}$. Combining these two aspects, we see that $F(\widetilde{\gamma} \cap \mathcal{R}_{z_0}) = \gamma \cap \mathcal{R}_{z_0}$. Hence, for each component ξ of $\gamma \cap D_{z_0}$ which intersects $\mathcal{R}_{z_0}$, there exists a component $\widetilde{\xi}$ of $\widetilde{\gamma} \cap D_{z_0}$ such that $\xi \subseteq F(\widetilde{\xi})$. Assume that ξ' is a component of $\widetilde{\gamma} \cap D$ with D a component of $F^{-1}(D_{z_0})$ disjoint with D_{z_0}, then $F(\xi')$ is a component of $\gamma \cap D_{z_0}$. To get a contradiction, we just need to show that $F(\xi')$ intersects $\mathcal{R}_{z_0}$ (this implies $F : \widetilde{\gamma} \to \gamma$ is not homeomorphic by the argument above). In fact, every component of $\gamma \cap D_{z_0}$ essentially intersects $\mathcal{R}_{z_0}$. This is because each component of $\gamma \cap D_{z_0}$ separates the components of Z_{z_0} which intersect $\mathcal{Q}$, since γ does not essentially intersect Z_{z_0} and $\#\mathrm{Comp}(\gamma \cap D_{z_0})$ is minimal. We then proved Claim 1.

We now assume that $\widetilde{\gamma}$ is disjoint with $\cup_{c \in \mathbf{c}} D_c$.

Claim 2. Let $U \subseteq \mathbb{C}$ be the disk bounded by γ. If $a \in \mathcal{Q}$ belongs to U, then $F(a) \in U$.

Proof of Claim 2. Since $\widetilde{\gamma}$ is disjoint with $\cup_{c \in \mathbf{c}} D_c$ and $f = F$ outside $\cup_{c \in \mathbf{c}} D_c$, the curve $\widetilde{\gamma}$ is also a component of $f^{-1}(\gamma)$. Since $\mathrm{Post}_f \subseteq \mathcal{Q}$, then $\widetilde{\gamma}$ and γ are homotopic in $\overline{\mathbb{C}} \setminus \mathrm{Post}_f$. It follows immediately that U contains at most one postcritical point of f: otherwise γ is a Levy cycle of the polynomial f, a contradiction. Hence, each component of $f^{-1}(U)$ is a disk and its boundary is a component of $f^{-1}(\gamma)$. We denote by $\widetilde{U}$ the component of $f^{-1}(U)$ bounded by $\widetilde{\gamma}$. As γ is isotopic to $\widetilde{\gamma}$ in $\overline{\mathbb{C}} \setminus \mathcal{Q}$, we have $a \in \widetilde{U}$. Note that $\widetilde{\gamma}$ avoids $\cup_{c \in \mathbf{c}} D_c$, then $F(\widetilde{U}) = f(\widetilde{U}) = U$, and hence $F(a) \in U$, which complete the proof of Claim 2. $\square$

By repeatedly using Claim 2, we see that U can not contain the escaping postcritical points of F. Since the remaining points of $\mathcal{Q}$ are postcritical points of f and γ is peripheral in $\overline{\mathbb{C}} \setminus \mathrm{Post}_f$, then γ is also peripheral in $\overline{\mathbb{C}} \setminus \mathcal{Q}$, a contradiction. $\square$

Fix three points $a_1, a_2, a_3 \in \mathrm{Post}_f \cap \mathcal{F}_f$. Then a_1, a_2, a_3 are also postcritical points of $F = F_{\vec{\mathbf{s}}}$. If F has no Thurston obstructions, by Theorem 3.1, it is c-equivalent to a rational map, denoted as $f_{\vec{\mathbf{s}}}$, by a pair of homeomorphisms (ϕ_0, ϕ_1). We normalize $f_{\vec{\mathbf{s}}}$ such that ϕ_0 fixes a_1, a_2, a_3, and call it a *normalized rational perturbation of f by capture surgery* with perturbation data $\vec{\mathbf{s}}$.

III. The convergence of perturbation maps by capture surgery.

We fix the vector of perturbation critical points $\mathbf{c}$, and choose a sequence of perturbation domains $\{\mathbf{D}_n = (D_{n,c} : c \in \mathbf{c}), n \geq 1\}$ such that

$$\mathrm{diam}(\mathbf{D}_n) := \max_{c \in \mathbf{c}} \mathrm{diam}(D_{n,c}) \to 0 \text{ as } n \to \infty, \tag{3.3}$$

a sequence of invariant domains $\{\mathbf{Z}_n = (Z_{n,c} : c \in \mathbf{c},), n \geq 1\}$, and a sequence of perturbation mappings $\{\mathbf{X}_n = (\chi_{n,c} : c \in \mathbf{c}), n \geq 1\}$. We then get a sequence of perturbation data $\{\vec{\mathbf{s}}_n := (\mathbf{c}, \mathbf{D}_n, \mathbf{Z}_n, \mathbf{X}_n), n \geq 1\}$.

For each $n \geq 1$, let $F_n = F_{\vec{\mathbf{s}}_n}$ denote the topological perturbation of f by capture surgery with data $\vec{\mathbf{s}}_n$, and $f_n = f_{\vec{\mathbf{s}}_n}$ the normalized rational perturbation of f by capture surgery with data $\vec{\mathbf{s}}_n$ if F_n has no Thurston obstructions.

Proposition 3.3. *If all F_n have no Thurston obstruction, then the normalized rational maps $f_n, n \geq 1$, uniformly converge to f as $n \to \infty$.*

The crucial tool to prove this proposition is the following distortion theorem, which is a special case of [CT2, Theorem 8.8]

Lemma 3.4. *Let X be a finite set in the Julia set of f, and V an open set compactly contained in the Fatou set of f. Fix three points $x_1, x_2, x_3 \in V$. Then, for any $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\sup_{z \in V}\{\mathrm{dist}(\varphi(z), z)\} \leq \epsilon,$$

for any univalent map

$$\varphi : \overline{\mathbb{C}} \setminus \bigcup_{z \in X} \bigcup_{1 \leq i \leq m} f^{-i}(B_\delta(z)) \to \overline{\mathbb{C}}$$

fixing x_1, x_2, x_3 and any $m \geq 0$, where $B_\delta(z)$ denotes the ball with center z and diameter δ with respect to the standard sphere metric.

Proof of Proposition 3.3. For any $n \geq 1$, we can obtain a sequence of rational maps $\{f_{n,k}, k \geq 0\}$ by applying Thurston algorithm on F_n as follows. Let a_1, a_2, a_3 be three chosen points in $\mathrm{Post}_f \cap \mathcal{F}_f$ used to normalize f_n.

Set $\eta_{n,0} := \mathrm{id}$. Then $F_n \circ \eta_{n,0}$ defines a complex structure on $\overline{\mathbb{C}}$ by pulling back the standard complex structure. By Uniformization Theorem, there exists a unique homeomorphism $\eta_{n,1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ normalized by fixing a_1, a_2, a_3 such that $f_{n,0} := \eta_{n,0} \circ F_n \circ \eta_{n,1}^{-1}$ is holomorphic. Note that $\eta_{n,1}$ is holomorphic except on $\cup_{c \in \mathbf{c}} D_{n,c}$.

Recursively, for each $k \geq 0$, there exists a normalized homeomorphism $\eta_{n,k+1}$ and a rational map $f_{n,k+1}$, such that $\eta_{n,k} \circ F_n = f_{n,k} \circ \eta_{n,k+1}$ (see diagram below), and $\eta_{n,k+1}$ is univalent on $\overline{\mathbb{C}} \setminus \bigcup_{0 \leq j \leq k} F_n^{-j}(\cup_{c \in \mathbf{c}} D_{n,c}) = \overline{\mathbb{C}} \setminus \bigcup_{0 \leq j \leq k} f^{-j}(\cup_{c \in \mathbf{c}} D_{n,c})$.

$$
\begin{array}{ccccccccccc}
\overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{F_n} & \overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{F_n} & \overline{\mathbb{C}} & \xrightarrow{F_n} & \overline{\mathbb{C}} \\
\downarrow{\cdots} & & \downarrow{\eta_{n,k+1}} & & \downarrow{\eta_{n,k}} & & \downarrow{\eta_{n,2}} & & \downarrow{\eta_{n,1}} & & \downarrow{\eta_{n,0}} \\
\overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{f_{n,k}} & \overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{f_{n,1}} & \overline{\mathbb{C}} & \xrightarrow{f_{n,0}} & \overline{\mathbb{C}}
\end{array}
\qquad (3.4)
$$

The sequence of rational maps $\{f_{n,k}, k \geq 0\}$ is called a *Thurston sequence* of F_n.

Let V be an open set compactly contained in $\mathcal{F}_f \setminus \overline{\cup_{c \in \mathbf{c}} D_{0,c}}$ and such that $a_1, a_2, a_3 \in V$. Because of the univalent property of $\eta_{n,k}$ described above, Lemma 3.4 gives a crucial distortion estimate

$$\sup_{k \geq 0, z \in V} \mathrm{dist}\left(\eta_{n,k}(z), z\right) \to 0 \text{ as } n \to \infty. \qquad (3.5)$$

Note that $F_n = f$ on $\overline{\mathbb{C}} \setminus \cup_{c \in \mathbf{c}} D_{n,c} \supseteq V$, then (3.4) and (3.5) imply

$$\sup_{k \geq 0, z \in V} \mathrm{dist}\left(f_{n,k}(z), f(z)\right) \to 0 \text{ as } n \to \infty. \qquad (3.6)$$

It was known that $f_n \to f$ on V implies $f_n \to f$ on $\overline{\mathbb{C}}$ (see e.g [CT2, Lemma 2.8]). Hence, by (3.6), this lemma will be proved provided we can show $f_{n,k} \to f_n$ on V as $k \to \infty$ for each $n \geq 1$.

To prove this point, let $(\phi_{n,0}, \phi_{n,1})$ be a pair of normalized homeomorphisms by which F_n and f_n are c-equivalent. We can further assume that $\phi_{n,0}$ is quasi-conformal on $\overline{\mathbb{C}}$ and holomorphic on a neighborhood U of all (supper-)attracting cycles of f_n. By Homotopy Lifting Lemma, we get a sequence of homeomorphisms $\{\phi_{n,k}\}_{k \geq 0}$ such that the following commutative graph holds:

$$
\begin{array}{ccccccccccc}
\overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{F_n} & \overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{F_n} & \overline{\mathbb{C}} & \xrightarrow{F_n} & \overline{\mathbb{C}} \\
\downarrow{\scriptstyle\cdots} & & \downarrow{\scriptstyle\phi_{n,k+1}} & & \downarrow{\scriptstyle\phi_{n,k}} & & \downarrow{\scriptstyle\phi_{n,2}} & & \downarrow{\scriptstyle\phi_{n,1}} & & \downarrow{\scriptstyle\phi_{n,0}} \\
\overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{f_n} & \overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{f_n} & \overline{\mathbb{C}} & \xrightarrow{f_n} & \overline{\mathbb{C}}
\end{array}
\qquad (3.7)
$$

For each $k \geq 0$, let $\psi_{n,k} := \eta_{n,k} \circ \phi_{n,k}^{-1}$. Combining diagrams (3.7) and (3.4), we get the following commutative graph

$$
\begin{array}{ccccccccccc}
\overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{f_n} & \overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{f_n} & \overline{\mathbb{C}} & \xrightarrow{f_n} & \overline{\mathbb{C}} \\
\downarrow{\scriptstyle\cdots} & & \downarrow{\scriptstyle\psi_{n,k+1}} & & \downarrow{\scriptstyle\psi_{n,k}} & & \downarrow{\scriptstyle\psi_{n,2}} & & \downarrow{\scriptstyle\psi_{n,1}} & & \downarrow{\scriptstyle\psi_{n,0}} \\
\overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{f_{n,k}} & \overline{\mathbb{C}} & \xrightarrow{\cdots} & \overline{\mathbb{C}} & \xrightarrow{f_{n,1}} & \overline{\mathbb{C}} & \xrightarrow{f_{n,0}} & \overline{\mathbb{C}}
\end{array}
$$

Note that $\psi_{n,0}$ is quasi-conformal on $\overline{\mathbb{C}}$ and holomorphic on U, then the diagram implies that each $\psi_{n,k}$ is quasi-conformal on $\overline{\mathbb{C}}$ with a uniformly bounded dilation K_n, and holomorphic on $\bigcup_{0 \leq i \leq k} f_n^{-i}(U)$. Thus $\{\psi_{n,k}, k \geq 0\}$ is a normal family since each $\psi_{n,k}$ fixes a_1, a_2, a_3. Let $\psi_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a limit of a subsequence of $\{\psi_{n,k}, k \geq 0\}$. By the argument above ψ_n is K_n-quasiconformal on $\overline{\mathbb{C}}$ and holomorphic on $\bigcup_{i \geq 0} f_n^{-i}(U) = \mathcal{F}_{f_n}$. It follows that ψ_n is holomorphic on $\overline{\mathbb{C}}$ since $\mathcal{J}_{f_n}$ is removable. Since ψ_n fixes a_1, a_2, a_3, we have $\psi_n = \mathrm{id}$. Applying this argument to any convergence subsequence of $\{\psi_{n,k}, k \geq 0\}$, one see that the entire sequence $\{\psi_{n,k}, k \geq 0\}$ uniformly converges to the identity on $\overline{\mathbb{C}}$. As a consequence, the Thurston sequence $\{f_{n,k}, k \geq 1\}$ uniformly converges to f_n. We then complete the proof of Proposition 3.3. $\qquad\square$

4. Perturbation of rational maps

Let f be a rational map and z be a repelling preperiodic point of f such that its orbit avoids all critical points of f. By implicit function theorem, there exists a neighborhood Λ of f and a holomorphic map $\zeta_z : \Lambda \to \mathbb{C}$ such that $\zeta_z(f) = z$ and $\zeta_z(g)$ is the unique repelling preperiodic points g near z with the same preperiod and period as those of z for all $g \in \Lambda$.

Definition 4.1 (continuation of repelling points). *Under the assumption and notations above, the point $\zeta_z(g)$ is called the* continuation of z at g.

Suppose that $\{f_n, n \geq 1\}$ is a sequence of rational maps converging to f, and U an (supper-)attracting periodic Fatou component of f with an attracting periodic point $a \in U$. The point a belongs to a unique attracting domain U_n of f_n for large n, which we call the *deformation of U at f_n*. Recall that $GO(U)$ denotes the collection of Fatou components of f contained in the grand orbit of U, and f is said to be *postcritically-finite in $GO(U)$* if each $U' \in GO(U)$ contains at most one postcritical points of f.

Lemma 4.2. *Follow the notations above and assume that f, f_n are postcritically-finite in $GO(U)$, $GO(U_n)$ respectively. Let U' be any element of $GO(U)$ with the center x. Then there exists a unique element $U'_n \in GO(U_n)$, such that the center x_n of U'_n converges to x as $n \to \infty$, and*

$\deg(f_n|_{U'_n}) = \deg(f|_{U'})$ *for all sufficiently large n. We call x_n the* continuation *of x at f_n. Furthermore, for any repelling preperiodic point $z \in \partial U'$, there is a unique point $z_n \in \partial U'_n$, having the same preperiod and period as z, such that $z_n \to z$ as $n \to \infty$. The point z_n is called the* continuation *of z at $\partial U'_n$.*

Proof. Let x, x_n be the centers of U', U'_n respectively. If x is periodic, the continuation y_n of x at f_n is an attracting periodic point contained in U'_n. Hence $y_n = x_n$. Let us now deal with the preperiodic case by induction. Let us assume that $f_n(x_n) \to f(x)$ as $n \to \infty$: we need to show that $x_n \to x$ and $\deg(f_n|_{U'_n}) = \deg(f|_{U'})$ as $n \to \infty$. Set $\delta := \deg(f|_{U'})$. By Rouché's theorem, any given small neighborhood of x contains exactly δ preimages by f_n of $f_n(x_n)$ (counting with multiplicity) for every sufficiently large n. Note that all these preimages belong to U_n, and are the centers of some Fatou components of f_n. So these preimages must coincide with x_n. It follows that $x_n \to x$ as $n \to \infty$ and $\deg(f_n|_{U_n}) = \delta$ for all large n.

For the remaining result of this lemma, we first assume that z is periodic. In this case, the conclusion holds by Goldberg and Milnor's proof in [GM, Appendix B]. Now, let $z \in \partial U'$ be a preperiodic point. Set $v := f(z) \in \partial f(U')$. Inductively, we assume that v_n is the unique preperiodic point of f_n in $\partial f_n(U'_n)$ such that v_n has the same preperiod and period as v, and $v_n \to v$ as $n \to \infty$. Since f_n uniformly converges to f, given any small disk neighborhood W_z of z, there is a disk neighborhood V_v of v such that the component of $f_n^{-1}(V_v)$ that contains z, denoted by $D_{n,z}$, belong to W_z, for all sufficiently large n. Given any sufficiently large n, choose a point $a_n \in D_{n,z} \cap U'_n$ and set $b_n := f_n(a_n)$. Then $b_n \in V_v \cap f_n(U'_n)$. By the inductive assumption, the point v_n belongs to $\partial f_n(U'_n) \cap V_v$. One can then choose an arc $\gamma_n \subset f_n(U'_n) \cap V_v$ joining b_n and v_n. Lifting γ_n by f_n with the starting point a_n, we get an arc $\widetilde{\gamma}_n \subset D_{n,z} \cap U'_n$. Its ending point, denoted by z_n, belongs to $\partial U'_n$ and satisfies that $f_n(z_n) = v_n$. By the argument above, we in fact proved that for any point $z' \in \partial U'$ with $f(z') = v$, and any small neighborhood $W_{z'}$ of z', there exists a point $z'_n \in \partial U'_n$ with the property that $z'_n \in W_{z'}$ and $f_n(z'_n) = v_n$ for all large n. Since $\deg(f_n|_{U'_n}) = \deg(f|_{U'})$, the points which have the same properties as z'_n are unique. This completes the proof of the lemma. $\square$

Let now $\{S_n\} \subset \mathbb{C}$ be a sequence of sets. We denote as $\limsup_{n\to\infty} S_n$ the set of points $z \in \mathbb{C}$ such that every neighborhood of z intersects infinitely many S_n. It follows immediately from the definition that $\limsup S_n$ is closed.

Lemma 4.3. *Under the assumption of Lemma 4.2, for each large n, let I_n be an preperiodic internal ray of f_n in U'_n with fixed preperiod $k \geq 0$ and period $p \geq 1$. If the landing point z_n of I_n converge to z, then $\limsup_{n\to\infty} I_n = I$, where I is the internal ray of f in U' landing at z.*

Proof. If I is period, the conclusion holds by Goldberg and Milnor's proof in [GM, Appendix B]. By induction, it suffices to prove $\limsup I_n = I$ provided that $\limsup f_n(I_n) = f(I)$. Since $f_n \to f$, we can choose Böttcher coordinates φ of U' and φ_n of U_n such that $\varphi_n^{-1} : D \to U_n$ converge uniformly on compact sets to $\varphi^{-1} : D \to U$. It follows that $I' := \limsup I_n \cap U$ is an internal ray of U. On the other hand, note that $\limsup I_n \cap \partial U$ is compact, connected and contains the point z. The map f sends $\limsup I_n \cap \partial U$ into the set $\limsup f_n(I_n) \cap \partial f(U)$, which is by induction a singleton. Then we get $\limsup I_n \cap \partial U = \{z\}$, and hence $I' = I$. $\square$

Lemma 4.4 (Goldberg-Milnor [GM])**.** *Consider a monic polynomial P and an external ray $\mathcal{R}_P(\theta)$ which lands at a repelling preperiodic point z such that the orbit of z avoids the critical points of P. Then $\mathcal{R}_Q(\theta)$ lands at the analytic continuation of z, for all monic polynomials Q in a sufficiently small neighborhood of P, and $\limsup_{Q\to P} \overline{\mathcal{R}_{\widetilde{P}}(\theta)} = \overline{\mathcal{R}_P(\theta)}$.*

5. Partial postcritically-finite polynomials

5.1. Hubbard forest and core entropy of partial postcritically-finite polynomials.

Let P be any polynomial of degree d. A critical/postcritical point of P is called *bounded* if it lies in $\mathcal{K}_P$, and *escaping* if it belongs to $\Omega(f)$. The polynomial P is called *partial postcritically-finite* if each bounded critical point has finite orbit.

Let P be any partial postcritically-finite polynomial. Then it is hyperbolic or sub-hyperbolic. By Lemma 2.6, every component of $\mathcal{K}(P)$ is a full, locally-connected compact set. As a consequence, given any two points z, w in a component of $\mathcal{K}_P$, there exists a unique arc $[z, w]$ within this component joining z, w such that its intersection with every bounded Fatou components consisting of (segments) of internal rays. Such an arc is called a *regular* arc. If z, w lie in different components of $\mathcal{K}_P$, denote $[z, w] := \emptyset$. We define a forest as

$$\mathcal{H}_P = \bigcup\{[z, w] \mid z, w \text{ are bounded critical/postcritical points of } P\},$$

and call it the *Hubbard forest* of P.

It is easily check that $\mathcal{H}_P$ is a P-invariant forest with the vertex set $V(\mathcal{H}_P)$ consisting of bounded critical/postcritical points of P and the branched points of $\mathcal{H}_P$, and the map $P : \mathcal{H}_P \to \mathcal{H}_P$ is Markov. Note that if P has connected Julia set, i.e., P is postcritically-finite, then $\mathcal{H}_P$ is just the Hubbard tree of P.

We define the *core entropy* of P as the topological entropy of P restricted on its Hubbard forest, i.e., $h(P) := h_{top}(P|_{\mathcal{H}_P})$. Then the defining domain of the entropy function h is enlarged from postcritically-finite polynomials to partial postcritically-finite polynomials.

5.2. Critical markings of partial postcritically-finite polynomials.
The objective here is to generalize the combinatorial data "critical markings" from the postcritically-finite case to the partial postcritically-finite case.

A partial postcritically-finite polynomial is called *visible* if every escaping critical point is terminated by several external radii (see Figture 4, compare with [Ki2, Definition 3.1]). The
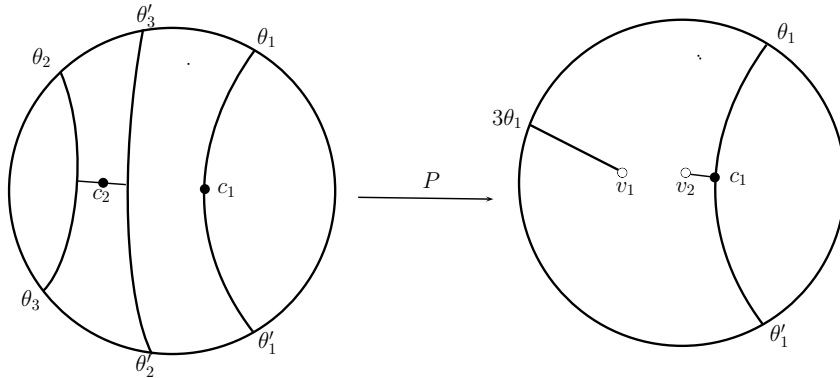

FIGURE 4. A non-visible example: on the left, the possible configurations of gradient flow lines connecting the critical points of a cubic polynomial, and on the right, their image under τ_3; here $f(c_i) = v_i$, $\theta_1 = 3\theta_2 = 3\theta'_2$, $\theta'_1 = 3\theta_3 = 3\theta'_3$.

number of external radii terminating at c is an integer multiple of the the local degree $\deg(P|_c)$, and the multiple is 1 if and only if the orbit of c avoids critical points of P.

Let P be a visible partial postcritically-finite polynomial, such that $U_1, \ldots, U_s$ are pairwise distinct bounded critical Fatou components, and $c_1, \ldots, c_m$ are pairwise distinct Julia or escaping critical points. A *critical marking* of P is a collection

$$\Theta_P := \{\Theta(U_1), \ldots, \Theta(U_s), \Theta(c_1), \ldots, \Theta(c_r)\}$$

of finite subsets of $\mathbb{R}/\mathbb{Z}$ constructed below:

The construction of $\Theta(U)$.

We first consider that U is periodic. Let

$$U \mapsto P(U) \mapsto \cdots \mapsto P^n(U) = c$$

be the critical cycle containing U. We will construct the associated set $\Theta(U')$ for every critical Fatou component U' in this cycle simultaneously. Let $z \in \partial U$ be a *root* of U, i.e., a periodic point with period less than or equal to n. Note that this choice naturally determines a root $P^k(z)$ for each Fatou component $P^k(U)$ for $0 \le k \le n-1$, which is called the *preferred root* of $P^k(U)$. Let z' be the preferred root of U'. We denote θ' the left-supporting angle of the component U' at z', and call it the *preferred angle* of U'. The angle θ' is periodic by Lemma 2.8, and has period n. We define $\Theta(U')$ the set of arguments of the rays in $P^{-1}\big(P(\mathcal{R}_P^\sigma(\theta))\big)$ which left-support U', where $\mathcal{R}_P^\sigma(\theta)$ denote the external ray left-supporting U' at z' with $\sigma \in \{+, -\}$.

In the case that U is a strictly preperiodic Fatou component, let k be the minimal number such that $P^k(U) =: U'$ is a critical Fatou component with $\Theta(U')$ and the preferred angle $\theta' \in \Theta(U')$ chosen. We define $\Theta(U)$ as the set of arguments of the rays in $P^{-k}(\mathcal{R}_P^\sigma(\theta'))$ which left-support U, where $\mathcal{R}_P^\sigma(\theta)$ denote the external ray left-supporting U'. We choose an angle $\theta \in \Theta(U)$ as the preferred angle of U.

The construction of $\Theta(c)$ for Julia critical points.

If the orbit of c avoids all critical points, we choose an angle $\theta \in \arg_P(c)$ as the preferred angle of c and define $\Theta(c)$ the set of arguments of rays in $P^{-1}(P(\mathcal{R}_P^\sigma(\theta)))$ which land at c, where $\mathcal{R}_P^\sigma(\theta)$ lands at c. Otherwise, let k be the minimal number such that $c' := f^k(c)$ is a critical point with $\Theta(c')$ and the preferred angle $\theta' \in \Theta(c')$ chosen. We define $\Theta(c)$ the set of arguments of the rays in $P^{-k}(\mathcal{R}_P^\sigma(\theta'))$ which land at c, where $\mathcal{R}_P^\sigma(\theta)$ denote the external ray landing at c'. We choose an angle $\theta \in \Theta(c)$ as the preferred angle of c.

The construction of $\Theta(c)$ for escaping critical points.

If the orbit of c avoids all critical points, by the visible assumption, there are $\deg(f|_c)$ external radii terminating at c. We define $\Theta(c)$ the set of arguments of all these radii and choose an angle $\theta \in \Theta(c)$ as a preferred angle. Otherwise, let k be the minimal number such that $c' := f^k(c)$ is a critical point with $\Theta(c')$ and the preferred angle $\theta' \in \Theta(c')$ chosen. Then there are $\deg(P|_c)$ external radii terminating at c such that they are mapped by P^k to the external radius $\mathcal{R}_P^*(\theta')$. We define $\Theta(c)$ the set of arguments of these radii, and choose an angle $\theta \in \Theta(c)$ as the preferred angle of c.

In the content below, we will use the following notations. We write a critical marking Θ_P as $\Theta_P := \{\mathscr{F}, \mathscr{L}\}$ such that

- $\mathscr{F} = \mathscr{F}_P := \{\Theta(U) : U \text{ is a bounded Fatou component}\}$;
- $\mathscr{J} = \mathscr{J}_P := \{\Theta(c) : c \text{ is a Julia critical point}\}$;
- $\mathscr{E} = \mathscr{E}_P := \{\Theta(c) : c \text{ is an escaping critical point}\}$;

and $\mathscr{L} = \mathscr{L}_P := \mathscr{J} \cup \mathscr{E}$. Note that the arguments participating in $\mathscr{F}$ are rational by Proposition 2.8, but those in $\mathscr{L}$ are not necessary.

5.3. **The properties of critical markings.** Let P be a visible partial postcritically-finite polynomial with a critical marking $\Theta = \{\mathscr{F}, \mathscr{L}\}$ such that

$$\mathscr{F} = \{\mathscr{F}_1, \ldots, \mathscr{F}_s\}$$
$$\mathscr{L} = \{\mathscr{L}_1, \ldots, \mathscr{L}_m\} \tag{5.1}$$

According to the construction, it is not difficult to check that Θ is a critical portrait of degree d. To make Θ satisfying some expected properties, we add some restrictions on P.

Definition 5.1. *A partial postcritically-finite polynomial P is called* admissible *if it is visible and the following properties hold:*

(1) *the escaping critical points have no iteration relation, and the external radii terminating at these points have strictly preperiodic angles;*
(2) *let c, c' be two escaping critical points such that $\theta \in \Theta(c), \theta' \in \Theta(c')$, then the landing points of $\mathcal{R}_P^\sigma(\theta)$ and $\mathcal{R}_P^{\sigma'}(\theta')$ have disjoint orbits for any $\sigma, \sigma' \in \{+, -\}$.*
(3) *let c be an escaping critical point with $\theta \in \Theta(c)$, and c' a Julia critical point, then the landing points of $\mathcal{R}_P^\sigma(\theta)$ and the point c' have disjoint orbits for any $\sigma \in \{+, -\}$.*

Lemma 5.2. *Let P be an admissible partial postcritically-finite polynomial with a critical marking Θ having the form as (5.1). Then Θ satisfies the following properties:*

(C1) *All arguments which participate Θ are rational.*
(C2) *$\mathscr{F}$ and $\mathscr{L}^-$ are* unlinked, *i.e., for $\epsilon > 0$ small enough, the collection of sets*

$$\mathscr{F}_1, \ldots, \mathscr{F}_s, \mathscr{L}_1 - \epsilon, \ldots, \mathscr{L}_m - \epsilon$$

have pairwise disjoint convex hulls in $\overline{\mathbb{D}}$.
(C3) *$\mathscr{F}$ (resp.$\mathscr{L}$) is* hierarchic, *i.e., for any two elements $\theta, \theta' \in \mathbb{R}/\mathbb{Z}$ that participate in $\mathscr{F}$ (resp. $\mathscr{L}$) such that $\tau^i(\theta)$ and $\tau^j(\theta')$ lie in $\mathscr{F}_k$ (resp. $\mathscr{L}_k$), for some $i, j > 0$; we have that $\tau^i(\theta) = \tau^j(\theta')$.*
(C4) *Given an argument that participates in $\mathscr{F}$, there exists a periodic argument $\tau^i(\theta)$ which also participates in $\mathscr{F}$.*
(C5) *None of the arguments that participate in $\mathscr{L}$ are periodic.*

Proof. According to Definition 5.1.(1), the arguments participating in $\mathscr{E}$ are rational. It then follows from Proposition 2.8 that the arguments participating $\mathscr{F}$ and $\mathscr{J}$ are also rational, hence property (C1) holds. Note that there is one preferred angle in every $\mathscr{F}_\ell, 1 \leq \ell \leq s$, and $\mathscr{L}_t, 1 \leq t \leq m$, and any argument participating in Θ is iterated to a preferred angle. Moreover, the preferred angle of $\mathscr{F}_\ell$ or $\mathscr{L}_t$ is periodic if and only if the critical point corresponding to $\mathscr{F}_\ell$ or $\mathscr{L}_t$ is periodic. It follows that properties (C3),(C4),(C5) hold. By the construction of Θ, we have that the elements within each of $\mathscr{F}, \mathscr{J}$ and $\mathscr{E}$ have pairwise disjoint convex hulls in $\overline{\mathbb{D}}$. Definition 5.1.(3) further assure that the elements in $\mathscr{L} = \mathscr{J} \cup \mathscr{E}$ have pairwise disjoint convex hulls in $\overline{\mathbb{D}}$. Then (C2) follows from the fact that all arguments participating in $\mathscr{F}$ are left-supporting. $\qquad\square$

In fact, this critical marking Θ also satisfies two other properties. To present these two properties, we need to introduce some notations. Let $\Theta = \{\ell_1, \ldots, \ell_m\}$ be a critical portrait of degree d. We say that θ and η are in the same Θ-*unlinked class* if and only if θ and η lie in the same connected component of $\mathbb{T} \setminus \ell_i$ for all $i = 1, \ldots, m$.

Lemma 5.3 ([Ki1],Lemma 6.6). *There are exactly d Θ-unlinked classes, each of which can be written as the union of finitely many open intervals:*

$$L = (\theta_0, \theta_1) \cup \ldots \cup (\theta_{2p-1}, \theta_{2p})$$

with subscripts mod $2p$ and respecting cyclic order. The total length of these intervals is $1/d$. Additionally, $\tau(\theta_{2i}) = \tau(\theta_{2i-1})$ for $i = 1, \ldots, p$. Furthermore, $\tau : L \to \tau(L)$ is a cyclic order preserving bijection and $\tau(L)$ consists of $\mathbb{T}$ with the finite set of points $\{\tau(\theta_0), \ldots, \tau(\theta_{2p-2})\}$ removed (refer to Figure 2).

Throughout we denote by $L_1, \ldots, L_d$ the Θ-unlinked classes.

Definition 5.4 (itinerary of angle)**.** *For any $t \in \mathbb{T}$, we call the symbol sequence $\ell_\Theta^+(t) = i_0 i_1 \ldots i_n \ldots$ right itinerary of t (associated to Θ) if for every $n \geq 0$, there exists $\epsilon > 0$ such that $(\tau^n(t), \tau^n(t) + \epsilon) \subseteq L_{i_n}$; and call the sequence $\ell_\Theta^-(t) = j_0 j_1 \ldots j_n \ldots$ left itinerary of t if for every $n \geq 0$, there exists $\epsilon > 0$ such that $(\tau^n(t) - \epsilon, \tau^n(t)) \subseteq L_{j_n}$.*

Note that $\ell_\Theta^+(\theta) = \ell_\Theta^-(\theta)$ if and only if the orbit of θ under τ avoids every ℓ_i.

Lemma 5.5. *Let P be a visible partial postcritically-finite polynomial with a critical marking Θ. Let $\theta_1, \theta_2 \in \mathbb{T}$ and $\sigma_1, \sigma_2 \in \{+, -\}$. If $\ell_\Theta^{\sigma_1}(\theta_1) = \ell_\Theta^{\sigma_2}(\theta_2)$, then $\mathcal{R}_P^{\sigma_1}(\theta_1), \mathcal{R}_P^{\sigma_2}(\theta_2)$ land at a common point.*

Proof. This lemma was proved in [Po1] in the postcritically-finite case by using the uniform expansionary of P with respect to the orbiford metric near the Julia set and the connectedness of the Julia set. In the partial postcritically-finite case, the polynomial P is still uniformly expanding with respect to the orbiford metric in some neighborhood of $\mathcal{J}_P$ (Lemma 2.6), and the extended Julia set $\mathcal{J}_P^*$ is connected, locally-connected and satisfies $P^{-1}(\mathcal{J}_P^*) \subseteq \mathcal{J}_P^*$ (Proposition 2.9). Then by substituting the Julia set with the extended Julia set, the argument in [Po1] also works in our case. $\qquad\square$

Lemma 5.6. *Under the assumption of Lemma 5.2, the critical marking Θ also satisfies the following two properties:*

(C6) *Consider a periodic argument t which participates in $\mathscr{F}$ and an argument $t' \in \mathbb{Q}/\mathbb{Z}$. If $\ell_\Theta^+(t) = \ell_\Theta^+(t')$, then $t = t'$.*

(C7) *Consider preperiodic angles $t \in \mathscr{L}_j$ and $t' \in \mathscr{L}_k$. If $\ell_\Theta^-(\tau^i(t)) = \ell_\Theta^-(t')$ for some i, then $\tau^i(t) = t'$.*

Proof. For (C6), by Lemma 5.5, we have that $\mathcal{R}_P^+(t)$ and $\mathcal{R}_P^+(t')$ lands at the same point. Since t is chosen a left-supporting angle, then $\ell_\Theta^+(t) = \ell_\Theta^+(t')$ also implies $t = t'$.

For (C7), still by Lemma 5.5, the rays $\mathcal{R}_P^-(\tau^i(t))$ and $\mathcal{R}_P^-(t')$ land at a common point z. The assumptions (2),(3) for admissible partial postcritically-finite polynomials imply that $\mathscr{L}_j, \mathscr{L}_k$ belong to $\mathscr{J}$. By the construction of $\mathscr{J}$, the angle $\tau^i(t)$ is the preferred angle in $\mathscr{L}_k$. If $\tau^i(t) \neq t'$, then their itineraries have distinct initial digits, a contradiction. $\qquad\square$

5.4. **Convergence of critical markings.** The main result in this subsection is the following convergence theorem of critical markings:

Proposition 5.7. *Let P be a monic postcritically-finite polynomial of degree d, and $P_n, n \geq 1$ be monic, visible partial postcritically-finite polynomials such that $P_n \to P$ as $n \to \infty$. Suppose that Θ_n is a critical marking of P_n for each n and $\Theta_n \to \Theta$ as $n \to \infty$ (in Hausdorff metric on $\mathbb{T}$). Then Θ is a weak critical marking of P (Definition 2.12).*

This result was proved in [GT, Proposition 9.16] when all P_n are postcritically-finite. The argument in the partial postcritically-finite case is similar, so we just give a sketch of the proof. The argument relies on Lemmas 5.8 and 5.9 below. These two lemmas corresponds to Lemmas

9.13 and 9.15 in [GT] (also for the postcritically-finite case) respectively, with completely the same argument. Hence we omit the proof.

Lemma 5.8. *Let P be a monic, postcritically-finite polynomial of degree d, and $P_n, n \geq 1$, be monic, partial postcritically-finite polynomials converging to P as $n \to \infty$. Assume that $\mathcal{R}_P(\theta)$ left/right-supports a Fatou component U at a periodic point z. Then for large n, the external ray $\mathcal{R}_{P_n}(\theta)$ left/right-supports the deformation U_n of U at the continuation z_n of z.*

Lemma 5.9. *Let P be a monic, postcritically-finite polynomial of degree d, and $P_n, n \geq 1$, be monic, partial postcritically-finite polynomials converging to P as $n \to \infty$. Assume that the arguments θ_n converge to θ, then $\limsup_{n \to \infty} \mathcal{R}_{P_n}^{\pm}(\theta_n) = \overline{\mathcal{R}_P(\theta)}$.*

Proof of Proposition 5.7. (Sketch) Let U be a critical Fatou component of P and denote by U_n the deformation of U at P_n. We write $\Theta(U_n)$ the critical marking of P_n associated to U_n.

In the periodic case, each $\Theta(U_n)$ contains a unique periodic angle θ_n with period equal to that of U_n and hence of U. By taking a subsequence if necessary, we can assume $\theta_n = \theta$ for large n. Note that any $\Theta(U_n)$ is a subset of $\tau^{-1}(\tau(\theta))$ and $\#\Theta(U_n) = \deg(P_n|_{U_n}) = \deg(P|_U)$ (by Lemma 4.2), then we can further assume by taking a subsequence that $\Theta(U_n)$ is constant for large n, which we write as $\Theta(U)$. According to Lemmas 4.3 and 5.9, the rays with arguments in $\Theta(U)$ land at the boundary of U. Furthermore, it follows from Lemma 5.8 that the periodic angle in $\Theta(U)$ left-supports U. The same situation happens in the strictly preperiodic case for U by a similar argument and the induction.

Let $U_1, \ldots, U_s$ be all the critical Fatou components of P. The discussion above shows that the collection of sets $\{\Theta(U_1), \ldots, \Theta(U_s)\}$ is

   (a) a weak Fatou critical marking of P (Definition 2.12.(1));
   (b) a part of the Fatou critical marking of P_n (contained in Θ_n) for large n.

Now, we write each Θ_n as $\Theta_n := \{\mathscr{F}, \mathscr{X}_n\}$ with $\mathscr{F} := \{\Theta(U_1), \ldots, \Theta(U_s)\}$ and $\mathscr{X}_n := \{\Theta_{n,1}, \ldots, \Theta_{n,m}\}$ for all large n, such that $\Theta_{n,i} \to \Theta_i$ as $n \to \infty$ and $1 \leq i \leq m$. It follows immediately that $\#\Theta_{n,i} = \#\Theta_i$ for any $1 \leq i \leq m$. Note that each $\Theta_{n,i}$ corresponds a critical point $c_{n,i}$ of P_n, and we can assume by taking subsequences that $c_{n,i}$ converge to a critical point c_i of P, which must belong to $\mathcal{J}_P$.

We have a fact that: if each $c_{n,i}$ is in the Fatou component $U(c_{n,i})$ of P_n, then the sequence of closed disks $\{\overline{U(c_{n,i})}, n \geq 1\}$ converge to c_i (see the corresponding discussion in [GT, Proposition 9.16]). Combining this fact and Lemma 5.9, all the external rays of P with arguments in Θ_i land at c_i. We then write Θ_i as $\Theta_i(c_i)$. In order to prove that

$$\Theta = \{\Theta(U_1), \ldots, \Theta(U_s), \Theta_1(c_1), \ldots, \Theta_m(c_m)\}$$

is a weak critical marking of P, we only need to check that $\Theta_1(c_1), \ldots, \Theta_m(c_m)$ satisfy Definition 2.12.(3), and this points is not difficult to check. $\square$

6. The continuity of entropy function: postcritically-finite parameters

The objective here is to prove Proposition 1.2. The proof is divided into two parts: we first show that the entropy function $h : \mathcal{P}_d^{\mathrm{ppf}} \to \mathbb{R}$ is upper semi-continuous at any postcritically-finite parameter P, and then determine the limit inferior $\liminf_{Q \to P} h(Q)$, as well as how to get it.

6.1. **Upper semi-continuity of the entropy function.**

Proposition 6.1. *Let $P \in \mathcal{P}_d$ be a postcritically-finite polynomial and $\{P_n, n \geq 1\}$ be a sequence of monic partial postcritically-finite polynomials converging to P. Then*

$$\limsup_{n \to \infty} h(P_n) \leq h(P).$$

The key to prove this proposition is the following lemma.

Lemma 6.2. *Let P be a partial postcritically-finite polynomial and $\Theta = \Theta_P$ a critical marking of P such that properties (C1)-(C7) hold for Θ. Then we have $h(P) \leq h(\Theta)$, where $h(\Theta)$ is the output of Thurston's entropy algorithm on Θ (see Section 2.3).*

Proof of Proposition 6.1 under Lemma 6.2. Firstly, using a standard quasi-conformal surgery, we can perturb each P_n to a nearby polynomial Q_n by twisting a sufficiently small angle for every escaping critical point of P_n, so that the external angles of the escaping critical points of Q_n satisfy the the properties below:

   (1) the external angles of each escaping critical point are strictly preperiodic;
   (2) the external angles of distinct escaping critical points have different periods;
   (3) the period of any external angle associate to escaping critical points is larger than the period of any external ray landing at a postcritical point.

It follows immediately that each Q_n is an admissible polynomial (see Definition 5.1). Note that Q_n conjugates to P_n on the filled-in Julia set, and can be chosen arbitrarily close to P_n. Then we have $h(P_n) = h(Q_n)$, and $Q_n, n \geq 1$, converge to P. So, without loss of generality, we can assume the initial $P_n's$ are admissible.

For each n, let Θ_n be a critical marking of P_n. By Lemmas 5.2 and 5.6, each Θ_n satisfies properties (C1)-(C7) in Section 5.3. If Θ_n converge to a critical portrait Θ, then Lemmas 6.2 and Propositions 2.11, 2.13, 5.7 together imply that

$$\limsup_{n \to \infty} h(P_n) \overset{\text{Lem.6.2}}{\leq} \limsup_{n \to \infty} h(\Theta_n) \overset{\text{Pro.2.11}}{=} h(\Theta) \overset{\text{Pro.5.7,2.13}}{=} h(P).$$

According to Proposition 5.7, the sequence $\{\Theta_n, n \geq 1\}$ can be divided into finitely many convergence subsequence. Then the proof is completed. $\square$

In the remaining part of this subsection, we focus on the proof of Lemma 6.2. The outline is as follows. Since Θ is assumed to satisfy properties (C1)-(C7), by Poirier's result (Theorem 6.3 below), there exists a monic, centered, postcritically-finite polynomial $\widetilde{P}$ such that Θ is a critical marking of $\widetilde{P}$. Based on the point that P and $\widetilde{P}$ have a common critical marking Θ, we can show $h(P) \leq h(\widetilde{P})$. It then follows from Proposition 2.13 that $h(P) \leq h(\widetilde{P}) = h(\Theta)$.

Theorem 6.3 (Poirier). *Let $\Theta = \{\mathscr{F}, \mathscr{L}\}$ be a degree d critical portrait satisfying properties (C1)-(C7). Then there is a unique monic centered postcritically-finite polynomial Q such that Θ is a critical marking of Q in the sense that $\mathscr{F}_Q = \mathscr{F}$ and $\mathscr{J}_Q = \mathscr{L}$.*

Let P be a partial postcritically-finite map with a critical marking $\Theta = \{\mathscr{F}, \mathscr{L}\}$ such that

$$\begin{aligned} \mathscr{F} = \mathscr{F}_P &= \{\mathscr{F}_1, \ldots, \mathscr{F}_s\} \\ \mathscr{L} = \mathscr{L}_P &= \mathscr{J}_P \cup \mathscr{E}_P = \{\mathscr{L}_1, \ldots, \mathscr{L}_m\}, \end{aligned}$$

where $\mathscr{F}_P, \mathscr{J}_P, \mathscr{E}_P$ denote the makings of bounded Fatou critical points, Julia critical points and escaping critical points, respectively (see Section 5.2). We further assume that Θ satisfies properties (C1)-(C7) in Section 5.3. According to Theorem 6.3, let $\widetilde{P}$ denote the unique postcritically-finite polynomial in $\mathcal{P}_d$ such that Θ is a critical marking of $\widetilde{P}$.

Factually, the dynamics of P and $\widetilde{P}$ are quite related. We indicate this point by the following three lemmas.

Lemma 6.4. *There exists a bijective ϕ_1 from the centers of Fatou components of P onto those of $\widetilde{P}$ such that $\widetilde{P} \circ \phi_1 = \phi_1 \circ P$.*

Proof. Since $\mathscr{F}_P = \mathscr{F}_{\widetilde{P}} = \mathscr{F}$, there is a natural bijection ϕ_1 which sends the critical point c_i of P associated to $\mathscr{F}_i$ to the critical point $\widetilde{c}_i$ of $\widetilde{P}$ associated to $\mathscr{F}_i$. We will extend ϕ_1 to all centers of bounded Fatou components of P.

Recall that the critical marking Θ induces a partition on $\mathbb{T}$ such that $\mathbb{T}$ is divided into d Θ-unlinked equivalence classes $L_1, \ldots, L_d$ (see Section 5.2). In fact, we have a corresponding partition in the dynamical plane of P. A ray is called an *extended ray* associated to a Fatou component U of P with argument θ, if it is the union of an external ray of angle θ and an internal ray in U which land at a common point on ∂U.

For each $\mathscr{F}_i \in \mathscr{F}_P$, we denote $\mathcal{R}_P(\mathscr{F}_i)$ the union of extended rays associated to U_i with arguments in $\mathscr{F}_i$, where U_i denote the critical Fatou component of P corresponding to $\mathscr{F}_i$; for each $\mathscr{J}_j \in \mathscr{J}_P$, we write $\mathcal{R}_P(\mathscr{J}_j)$ as the union of external rays with arguments in $\mathscr{J}_j$ which land at c_j together with c_j, where c_j denotes the critical point of P corresponding to $\mathscr{J}_j$; and for each $\mathscr{E}_k \in \mathscr{E}_P$, we let $\mathcal{R}_P(\mathscr{E}_k)$ be the union of external radii with arguments in $\mathscr{E}_k$ together with c_k, where c_k is the escaping critical point of P corresponding to $\mathscr{E}_k$. Thus, the union of $\mathcal{R}_P(\mathscr{F}_i), \mathcal{R}_P(\mathscr{J}_j)$ and $\mathcal{R}_P(\mathscr{E}_k)$ divide $\mathbb{C}$ into d parts, which we denote by $V_1, \ldots, V_d$ such that $\mathcal{R}_P^*(\theta) \subseteq V_i$ if and only if $\theta \in L_i$. We call $V_1, \ldots, V_d$ the Θ-*unlinked classes in the dynamical plane of P*. It is easy to see that the restriction of P on each V_i is injective and $\mathbb{C} \setminus P(V_i)$ consists of finitely many extended rays/external rays/segments of external rays, with arguments in $\tau_d(\Theta)$.

Let c be the center of a bounded Fatou component $U(c)$ of P, such that $c = c_0 \mapsto c_1 \mapsto \cdots \mapsto c_{k-1} \mapsto c_k$ are the first k terms in the orbit of c with $P'(c_i) \neq 0$ for $0 \leq i \leq k-1$, but $P'(c_k) = 0$. Then c_k corresponds to an element $\mathscr{F}_*$ of $\mathscr{F}$, with the preferred angle denoted as θ_k, and the ray $\mathcal{R}_P^\sigma(\theta_k)$ left-supports $U(c_k)$ with $\sigma \in \{+, -\}$. Note that each of $c_0, \ldots, c_{k-1}$ belongs to one of $V_1, \ldots, V_d$, so we write $c_i \in V_{\epsilon(i)}$ for each $0 \leq i \leq k-1$. By the argument in last paragraph, the ray $\mathcal{R}_P^\sigma(\theta_k)$, or the angle θ_k, can be lifted along to orbit of c, such that, for each $0 \leq i \leq k-1$, the ray $\mathcal{R}_P^\sigma(\theta_i)$ is the unique one in $V_{\epsilon(i)}$ with $P(\mathcal{R}_P^\sigma(\theta_i)) = \mathcal{R}_P^\sigma(\theta_{i+1})$. It is clear that the ray $\mathcal{R}_P^\sigma(\theta_i)$ left-supports $U(c_i)$ for each $0 \leq i \leq k-1$.

In the dynamical plane of $\widetilde{P}$, the ray $\mathcal{R}_{\widetilde{P}}(\theta_k)$ left-supports the Fatou component of $\widetilde{P}$ corresponding to $\mathscr{F}_* \in \mathscr{F}$. Then each $\mathcal{R}_{\widetilde{P}}(\theta_i)$ also left-supports a (unique) Fatou component of $\widetilde{P}$. We define $\phi_1(c)$ to be the center of the Fatou component of $\widetilde{P}$ left-supported by $\mathcal{R}_{\widetilde{P}}(\theta_0)$. We thus obtain a map ϕ_1 from the centers of Fatou components of P to those of $\widetilde{P}$. Note that all processes above can be reversed from the dynamical plane of $\widetilde{P}$ to that of P, so the map ϕ_1 is a bijection. The formula $\widetilde{P} \circ \phi_1 = \phi_1 \circ P$ follows directly from the construction of ϕ_1. $\square$

Lemma 6.5. *Let $\mathcal{R}_P^\sigma(\theta)$ and $\mathcal{R}_P^{\sigma'}(\theta')$ land at the same point in the dynamical plane of P with $\sigma, \sigma' \in \{+, -\}$. Then the rays $\mathcal{R}_{\widetilde{P}}(\theta)$ and $\mathcal{R}_{\widetilde{P}}(\theta')$ also land at the same point in the dynamical plane of $\widetilde{P}$. As a consequence, we get a surjection $\phi_2 : \mathcal{J}_P \to \mathcal{J}_{\widetilde{P}}$ with $\widetilde{P} \circ \phi_2 = \phi_2 \circ P$.*

Proof. We first claim that if θ, θ' belong to the closure of a Θ-unlinked class and $\mathcal{R}_{\widetilde{P}}(\tau(\theta)), \mathcal{R}_{\widetilde{P}}(\tau(\theta'))$ land at a common point, then $\mathcal{R}_{\widetilde{P}}(\theta), \mathcal{R}_{\widetilde{P}}(\theta')$ also land together.

We assume $\theta, \theta' \in \overline{L}$ with L a Θ-unlinked class write by

$$L = (\theta_0, \theta_1) \cup \ldots \cup (\theta_{2p-1}, \theta_{2p})$$

as in Lemma 5.3. In the case of $\tau(\theta) = \tau(\theta')$, by Lemma 5.3, there exists $1 \leq i \leq p$ such that $\{\theta, \theta'\} = \{\theta_{2i-1}, \theta_{2i}\}$. Combining the fact that $\mathcal{R}_P^\sigma(\theta), \mathcal{R}_P^{\sigma'}(\theta')$ land together, the angles θ, θ' must belong to an element of $\mathscr{L}$. It follows that $\mathcal{R}_{\widetilde{P}}(\theta), \mathcal{R}_{\widetilde{P}}(\theta')$ land at a Julia critical point of $\widetilde{P}$, proving the claim. In the case of $\tau(\theta) \neq \tau(\theta')$, let $\overline{\mathcal{R}_{\widetilde{P}}(\theta)} \cup \mathcal{R}_{\widetilde{P}}(\alpha)$ be a lift by $\widetilde{P}$ within $\overline{V}$ of the arc $\overline{\mathcal{R}_{\widetilde{P}}(\tau(\theta))} \cup \mathcal{R}_{\widetilde{P}}(\tau(\theta'))$, where V denote the Θ-unlinked class in the dynamical plane of $\widetilde{P}$ corresponding to L. We see that α is a preimage by τ of $\tau(\theta')$ within $\overline{L}$. If $\theta' \in L$, it is the unique preimage by τ of $\tau(\theta')$ within $\overline{L}$, and hence $\alpha = \theta'$, which complete the proof of the claim. Otherwise, by Lemma 5.3 again, we have $\alpha, \theta' \in \{\theta_{2i-1}, \theta_{2i}\}$ for some $1 \leq i \leq p$, and we just need to deal with the case of $\alpha \neq \theta'$. According to property (C2), there are three possibilities:

- θ_{2i-1} belongs to an element of $\mathscr{L}$ and θ_{2i} belong to an element of $\mathscr{F}$;
- $\theta_{2i-1}, \theta_{2i}$ belong to a common element of $\mathscr{F}$;
- $\theta_{2i-1}, \theta_{2i}$ belong to a common element of $\mathscr{L}$;

However, since $\mathcal{R}_P^\sigma(\theta), \mathcal{R}_P^{\sigma'}(\theta')$ land together, only the third case happens. It follows that $\mathcal{R}_{\widetilde{P}}(\alpha), \mathcal{R}_{\widetilde{P}}(\theta')$ land together, and the claim is then proved.

We start to prove the lemma. If θ is not preperiodic, the fact that $\mathcal{R}_P^\sigma(\theta), \mathcal{R}_P^{\sigma'}(\theta')$ land together implies θ and θ' have the same itineraries, and hence $\mathcal{R}_{\widetilde{P}}(\theta), \mathcal{R}_{\widetilde{P}}(\theta')$ land at the same point by Lemma 5.5. So we can assume that θ, θ' are preperiodic. The proof goes by induction on the preperiod of θ (hence θ').

Suppose θ is period. If the orbits of both θ and θ' avoids the arguments in Θ_P, then θ, θ' must have the same itinerary (because $\mathcal{R}_P^\sigma(\theta), \mathcal{R}_P^{\sigma'}(\theta')$ land together), and Lemma 5.5 shows that $\mathcal{R}_{\widetilde{P}}(\theta), \mathcal{R}_{\widetilde{P}}(\theta')$ land together. Otherwise, we can assume that $\mathcal{R}_P^\sigma(\theta)$ left-supports a periodic critical Fatou component U at $z \in \partial U$, and $\mathcal{R}_P^{\sigma'}(\theta')$ lands at z. In this case, we have $\ell_\Theta^-(\theta) = \ell_\Theta^-(\theta')$. Thus, using Lemma 5.5 again, the ray pair $\mathcal{R}_{\widetilde{P}}(\theta), \mathcal{R}_{\widetilde{P}}(\theta')$ land together.

Now, assume that $\mathcal{R}_{\widetilde{P}}(\theta), \mathcal{R}_{\widetilde{P}}(\theta')$ land together for all θ, θ' with preperiods strictly less than $n \geq 1$. Let θ, θ' have preperiod n. By the assumption of the induction, the rays $\mathcal{R}_{\widetilde{P}}(\tau(\theta)), \mathcal{R}_{\widetilde{P}}(\tau(\theta'))$ land together. Since $\mathcal{R}_P^\sigma(\theta)$ and $\mathcal{R}_P^{\sigma'}(\theta')$ land at a common point, then either the arguments θ, θ' belong to the closure of a Θ-unlinked class, or there exists an element $\mathscr{J}_*$ of $\mathscr{J}_P$ and two angles $\alpha, \alpha' \in \mathscr{J}_*$ such that θ, α (resp. θ', α') belong to the closure of a Θ-unlinked class. In the former case, we get that $\mathcal{R}_{\widetilde{P}}(\theta), \mathcal{R}_{\widetilde{P}}(\theta')$ land at a common point directly by the claim. In the latter case, following the claim above, we have that $\mathcal{R}_{\widetilde{P}}(\theta), \mathcal{R}_{\widetilde{P}}(\alpha)$ (resp. $\mathcal{R}_{\widetilde{P}}(\theta), \mathcal{R}_{\widetilde{P}}(\theta')$) land together. Since $\alpha, \alpha' \in \mathscr{J}_* \in \mathscr{J}_P \subseteq \mathscr{J}_{\widetilde{P}}$, then $\mathcal{R}_{\widetilde{P}}(\alpha), \mathcal{R}_{\widetilde{P}}(\alpha')$ land at a common point, and hence $\mathcal{R}_{\widetilde{P}}(\theta), \mathcal{R}_{\widetilde{P}}(\theta')$ also land together.

We now construct the map $\phi_2 : \mathcal{J}_P \to \mathcal{J}_{\widetilde{P}}$ as follows. Let $z \in \mathcal{J}_P$ and the ray $\mathcal{R}_P^\sigma(\theta)$ land at z. We define $\phi_2(z)$ the landing point of the external ray $\mathcal{R}_{\widetilde{P}}(\theta)$. This map is well-defined by the conclusion proved above, and is a surjection satisfying $\widetilde{P} \circ \phi_2 = \phi_2 \circ P$ by its definition. $\square$

Lemma 6.6. *Let U be a Fatou component of P and $\mathcal{R}_P(\theta)$ a preperiodic external ray lands on $z \in \partial U$ such that the orbit of z avoids the landing point of $\mathcal{R}_P^\sigma(t)$ for any t participating Θ_P*

and $\sigma \in \{+, -\}$. *Then the ray* $\mathcal{R}_{\widetilde{P}}(\theta)$ *lands on the boundary of the Fatou component* $\widetilde{U}$ *with the property that* ϕ_1 *sends the center of* U *to the center of* $\widetilde{U}$.

Proof. Recall that $L_1, \ldots, L_d$ denote the Θ-unlinked classes, and $V_1(P), \ldots, V_d(P)$, *resp.* $V_1(\widetilde{P})$, $\ldots, V_d(\widetilde{P})$, denote the corresponding Θ-unlinked classes in the dynamical plane of P (resp. $\widetilde{P}$). The assumption on z implies that any point in the orbit of z belongs to one of $V_1(P), \ldots, V_d(P)$. Then z has the same itinerary as that of θ in the sense that, if $\ell_\Theta(\theta) = \epsilon_0 \epsilon_1 \ldots$, then $P^i(z) \in V_{\epsilon_i}(P)$ for any $i \geq 0$.

Let $\widetilde{U}$ be the Fatou component of $\widetilde{P}$ such that ϕ_1 sends the center of U to the center of $\widetilde{U}$. Since P and $\widetilde{P}$ have the same critical marking Θ, it is not difficult to see that there is a unique prepriodic point $\widetilde{z} \in \partial\widetilde{U}$ such that $\widetilde{P}^i(\widetilde{z}) \in V_{\epsilon_i}(\widetilde{P})$ for any $i \geq 0$. Let $\mathcal{R}_{\widetilde{P}}(\alpha)$ be a external ray landing at $\widetilde{z}$. It then follows that α and θ have the same itineraries. By Lemma 5.5, the ray $\mathcal{R}_{\widetilde{P}}(\theta)$ lands at $\widetilde{z} \in \partial\widetilde{U}$. $\qquad\square$

Proof of Lemma 6.2. Let $\mathcal{H} = \mathcal{H}_P$ denote the Hubbard forest of P, with the vertex set $V(\mathcal{H})$ consisting of bounded critical/postcritical points of P and the branched points of $\mathcal{H}$. We establish a map $\phi : V(\mathcal{H}) \to \mathbb{C}$ such that

$$\phi(z) = \begin{cases} \phi_1(z), & \text{if } z \in \mathcal{F}_P; \\ \phi_2(z), & \text{if } z \in \mathcal{J}_P, \end{cases}$$

where ϕ_1, ϕ_2 are maps given in Lemmas 6.4, 6.5 respectively. It follows that $\widetilde{P} \circ \phi(z) = \phi \circ P(z)$.

For the discussion below, we represent each point in $V(\mathcal{H})$ and $\phi(V(\mathcal{H}))$ by an angle. Let $v \in V(\mathcal{H})$. If $v \in \mathcal{F}_P$, we define $\theta(v)$ the unique iterated preimage by τ of preferred angles participating $\mathscr{F}$ such that $\mathcal{R}_P^\sigma(\theta(v))$ left-supports $U(v)$ where $\sigma \in \{+, -\}$ and $U(v)$ the Fatou component of P containing v. According to the argument in Lemma 6.4, the ray $\mathcal{R}_{\widetilde{P}}(\theta(v))$ left-supports the Fatou component $\widetilde{U}(\phi(v))$ of $\widetilde{P}$ which contains $\phi(v)$, and we represent $\phi(v)$ also by $\theta(v)$. If $v \in \mathcal{J}_P$, let $\theta(v)$ be an arbitrary angle in $\arg_v(P)$. By Lemma 6.5, the angle $\theta(v)$ belong to $\arg_{\phi(v)}(\widetilde{P})$, and we represent $\phi(v)$ also by $\theta(v)$.

For every component H of $\mathcal{H}$, let $V(H) := V(\mathcal{H}) \cap H$ denote the vertex of H. We define by $\widetilde{H}$ the regulated hull of $\phi(V(H))$ within $\mathcal{K}_{\widetilde{P}}$. It is a tree in the dynamical plane of $\widetilde{P}$, and the vertex set $V(\widetilde{H})$ of $\widetilde{H}$ is defined as the union of $\phi(V(H))$ and the branched points of $\widetilde{H}$.

Claim 1. The restriction of ϕ on $V(H)$ is injective.

Proof. By Lemma 6.4, the restriction of ϕ on $V(H) \cap \mathcal{F}_P$ is injective. Let z, w be any distinct points in $V(H) \cap \mathcal{J}_P$. If the segment $[z, w]$ intersects a Fatou component U of P, we can find two prepriodic angles α, β such that $\mathcal{R}_P(\alpha), \mathcal{R}_P(\beta)$ land at the boundary of U, the periods of α, β are larger than the angles participating Θ and those in $\arg_z(P), \arg_w(P)$, and the arc

$$\Gamma := \mathcal{R}_P(\theta) \cup I_\alpha \cup I_\beta \cup \mathcal{R}_P(\beta)$$

separates z, w, with I_α, I_β the internal rays in U which have the same landing points as $\mathcal{R}_P(\alpha), \mathcal{R}_P(\beta)$ respectively. This arc also separates the rays $\mathcal{R}_P(\theta(z))$ and $\mathcal{R}_P(\theta(w))$ with $\theta(z), \theta(w)$ the angles representing z, w.

In the dynamical plane of $\widetilde{P}$, Lemma 6.6 implies that $\mathcal{R}_{\widetilde{P}}(\alpha), \mathcal{R}_{\widetilde{P}}(\beta)$ land on the same Fatou component $\widetilde{U}$ of $\widetilde{P}$ where ϕ sends the center of U to the center of $\widetilde{U}$. Moreover, the arc

$$\widetilde{\Gamma} := \mathcal{R}_{\widetilde{P}}(\theta) \cup \widetilde{I}_\alpha \cup \widetilde{I}_\beta \cup \mathcal{R}_{\widetilde{P}}(\beta)$$

separates $\mathcal{R}_{\widetilde{P}}(\theta(z))$ and $\mathcal{R}_{\widetilde{P}}(\theta(w))$, with $\widetilde{I}_{\alpha}, \widetilde{I}_{\beta}$ the internal rays in $\widetilde{U}$ which have the same landing points as $\mathcal{R}_{\widetilde{P}}(\alpha), \mathcal{R}_{\widetilde{P}}(\beta)$ respectively. Since the periods of α, β are chosen larger than $\theta(z), \theta(w)$, then the landing points of $\mathcal{R}_{\widetilde{P}}(\theta(z)), \mathcal{R}_{\widetilde{P}}(\theta(w))$, which are exactly $\phi(z), \phi(w)$, are disjoint with the $\widetilde{\Gamma}$. It means that $\widetilde{\Gamma}$ separates $\phi(z), \phi(w)$ and hence $\phi(z) \neq \phi(w)$.

If the segment $[z, w] \subseteq \mathcal{J}_P$, there exist preperiodic angles α, β such that $\mathcal{R}_P(\alpha), \mathcal{R}_P(\beta)$ land at the same point $x \in (z, w)$, the preperiods of α, β are larger than those of $\theta(z), \theta(w)$, and the arc

$$\Gamma := \mathcal{R}_P(\theta) \cup \{x\} \cup \mathcal{R}_P(\beta)$$

separates z, w. In the dynamical plane of $\widetilde{P}$, Lemma 6.5 implies that $\mathcal{R}_{\widetilde{P}}(\alpha), \mathcal{R}_{\widetilde{P}}(\beta)$ land on the same point $\widetilde{x}$, and the arc

$$\widetilde{\Gamma} := \mathcal{R}_{\widetilde{P}}(\theta) \cup \{\widetilde{x}\} \cup \mathcal{R}_{\widetilde{P}}(\beta)$$

separates $\mathcal{R}_{\widetilde{P}}(\theta(z))$ and $\mathcal{R}_{\widetilde{P}}(\theta(w))$. Since the preperiods of α, β are chosen larger than those of $\theta(z), \theta(w)$, then the landing points of $\mathcal{R}_{\widetilde{P}}(\theta(z)), \mathcal{R}_{\widetilde{P}}(\theta(w))$, which are exactly $\phi(z), \phi(w)$, are disjoint with $\{\widetilde{x}\}$. It means that $\phi(z) \neq \phi(w)$. We complete the proof of the injection of $\phi|_{V(H)}$. $\qquad\square$

The map $\phi|_{V(H)} : V(H) \to V(\widetilde{H})$ can be extended to a map, denoted by ϕ_H, from H to $\widetilde{H}$. Let $e = e(x, y)$ be an edge of H with endpoints x, y. We define its image $\phi_H(e)$ to be the regulated arc $[\phi(x), \phi(y)]$ within $\widetilde{H}$. Note that the endpoints of $\widetilde{H}$ belong to $\phi(V(H))$ (since $\widetilde{H}$ is defined as the regulated hull of $\phi(V(H))$ within $\mathcal{K}_{\widetilde{P}}$), then the map $\phi_H : H \to \widetilde{H}$ is surjective.

Claim 2. The map $\phi_H : H \to \widetilde{H}$ is a homeomorphism.

In the proof of Claim 1, we represent each vertex of $V(H)$ by an angle. To prove Claim 2, we will represent each edge of H by a subset of $\mathbb{T}$ consisting of two open arcs.

Let $e = e(z_1, z_2)$ be an arc of H with $z_1, z_2 \in V(H)$. In the case of $z_1 \in \mathcal{F}_P$, we choose two preperiodic angles α_1, α_1' with periods larger than those of the angles participating Θ, such that $\mathcal{R}_P(\alpha_1), \mathcal{R}_P(\alpha_1')$ land on the boundary of $U(z_1)$, and the region $W(\alpha_1, \alpha_1')$ contains all edges of H starting from z_1 except e, where $\widehat{\mathcal{R}}_P(\alpha_i)$ denote the extended ray of P associated to $U(z_1)$ with argument $\alpha_i, i = 1, 2$, and $W(\alpha_1, \alpha_1')$ is bounded by $\widehat{\mathcal{R}}_P(\alpha_1), \widehat{\mathcal{R}}_P(\alpha_2)$ such that $\mathcal{R}_P(\alpha) \in W(\alpha_1, \alpha_1')$ for all $\alpha \subseteq (\alpha_1, \alpha_1')$. In the case of $z_1 \in \mathcal{J}_P$, let $\alpha_1, \alpha_1' \in \arg_z(P)$ such that $\overline{W(\alpha_1, \alpha_1')}$ contains all external rays landing at z_1, where $W(\alpha_1, \alpha_1')$ is bounded by $\mathcal{R}_P^{\sigma_1}(\alpha_1), \mathcal{R}_P^{\sigma_1'}(\alpha_1')$ such that $\mathcal{R}_P^{\pm}(\alpha) \in W(\alpha_1, \alpha_1')$ for all $\alpha \subseteq (\alpha_1, \alpha_1')$ (see Figure 5). We similarly choose a pair of angles α_2, α_2' for the other endpoint z_2 of e.

Now, we define $A(e)$ the union of two disjoint arcs $(\alpha_1', \alpha_2), (\alpha_2', \alpha_1)$, which is called the *representation of e*. By the construction, it is easy to see that

Fact: *the representing angle $\theta(v)$ of a vertex $v \in V(H)$ belongs to $A(e)$ only if $v \in \mathcal{F}_P$ and v equals to either z_1 or z_2.*

Proof of Claim 2. Since $\phi_H : H \to \widetilde{H}$ is surjective and injective on $V(H)$, it is enough to show that ϕ_H maps an edge of H to an edge of $\widetilde{H}$. Let $e = e(z_1, z_2)$ be an edge of H with its representation $A(e) = (\alpha_1', \alpha_2) \cup (\alpha_2, \alpha_1)$. By Lemmas 6.5 and 6.6, the ray pair $\mathcal{R}_{\widetilde{P}}(\alpha_i), \mathcal{R}_{\widetilde{P}}(\alpha_i')$ either land on the boundary of the Fatou component $U(\phi_H(z_i))$ of $\widetilde{P}$ or on the common point $\phi_H(z_i)$, according to z_i belongs to $\mathcal{F}_P$ or not, for $i = 1, 2$. We then obtain a region W bounded by the four external or extended rays of $\widetilde{P}$ with arguments $\alpha_1, \alpha_1', \alpha_2, \alpha_2'$. Clearly, the open segment $(\phi_H(z_1), \phi_H(z_2))$ belongs to W.
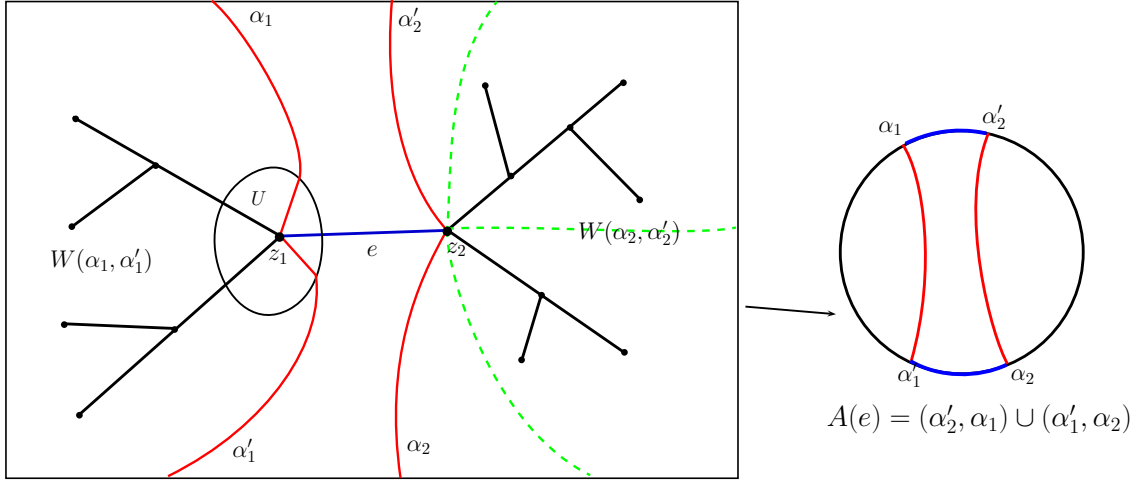
FIGURE 5. The construction of $W(\alpha_1, \alpha_1'), W(\alpha_2, \alpha_2')$ and $A(e)$. Here we assume that z_1 is the center of a Fatou component U and $z_2 \in \mathcal{J}_P$.

On the contrary, if $[\phi_H(z_1), \phi_H(z_2)]$ is not an edge of $\widetilde{H}$, there must be a point $z \in V(H)$ distinct with z_1, z_2 such that $\phi_H(z) \in W$. It implies the representation $\theta(z)$ of $\phi_H(z)$ belongs to $A(e)$, a contradiction to the Fact above. $\qquad\square$

Claim 3. Let H_1, H_2 be two distinct connected components of $\mathcal{H}$. Then their images $\widetilde{H_1}, \widetilde{H_2}$ by ϕ_{H_1}, ϕ_{H_2} intersect only possibly at their endpoints.

Proof. Since H_1, H_2 belongs to different components of $\mathcal{K}_P$, there exist a pair of angles η_1, η_2 such that the external radii of P with arguments η_1, η_2 terminates at a pre-critical point z, the arc $\mathcal{R}_P^*(\eta_1) \cup \{z\} \cup \mathcal{R}_P^*(\eta_2)$ separates H_1, H_2, and the rays $\mathcal{R}_P^+(\eta_1), \mathcal{R}_P^-(\eta_2)$ land together. As a consequence, the angles representing $V(H_1)$ and those representing $V(H_2)$ are contained in different components of $\mathbb{T} \setminus \{\eta_1, \eta_2\}$ except possibly the ones coincide with η_1, η_2.

In the dynamical plane of $\widetilde{P}$, the external rays of $\widetilde{P}$ with arguments η_1, η_2 land at a common point according to Lemma 6.5, and the union of these two rays separates the vertices of $\widetilde{H_1}$ and those of $\widetilde{H_2}$ except possibly the one coincide with the landing point. Then the claim is proved. $\qquad\square$

Now, let $\widetilde{\mathcal{H}}$ denote the union of all $\widetilde{H}$ with H going though the components of $\mathcal{H}$. By Claims 1, 2 and 3, the set $\widetilde{\mathcal{H}}$ is a forest contained in the Hubbard tree $\mathcal{H}_{\widetilde{P}}$, with its vertex set $V(\widetilde{\mathcal{H}})$ equal to $\phi(V(\mathcal{H}))$, and we have the formula

$$\widetilde{P} \circ \phi = \phi \circ P \text{ on } V(\mathcal{H}). \tag{6.1}$$

By this formula, we have $\widetilde{P}(V(\widetilde{\mathcal{H}})) \subseteq V(\widetilde{\mathcal{H}})$ and $\widetilde{P}(\widetilde{\mathcal{H}}) \subseteq \widetilde{\mathcal{H}}$. According to Claim 3, the interior of any edge of $\widetilde{\mathcal{H}}$ contains no critical points of $\widetilde{P}$, then $\widetilde{P} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ is a Markov map.

It is known from Proposition 2.13 that $h(\widetilde{P}) = h(\Theta)$, and from Lemma 2.2 that $h_{top}(\widetilde{P}|_{\widetilde{\mathcal{H}}}) \leq h(\widetilde{P})$. Thus, to complete the proof of Lemma 6.2, we only need to show $h(P) := h_{top}(P|_{\mathcal{H}}) = h_{top}(\widetilde{P}|_{\widetilde{\mathcal{H}}})$.

By enumerating the edges of $\mathcal{H}$, we obtain an incidence matrix $D = D_{(\mathcal{H}, P)}$ of $(\mathcal{H}, P)$, and the topological entropy $h_{top}(P|_{\mathcal{H}})$ equals to $\log \rho(D)$ with $\rho(D)$ the leading eigenvalue of D (see

Section 2.1). By Claims 2 ,3, we get a bijection $\phi_* : E(\mathcal{H}) \to E(\widetilde{\mathcal{H}})$ such that $\phi_*(e) = \phi_H(e)$ if e is an edge of $\mathcal{H}$ contained in a component H of $\mathcal{H}$. Thus, the enumeration on $E(\mathcal{H})$ induces an enumeration on $E(\widetilde{\mathcal{H}})$ such that e and $\phi_*(e)$ have the same label.

Let $e = e(x, y)$ be any edge of $\mathcal{H}$ with endpoints $x, y \in V(\mathcal{H})$. Since $P : \mathcal{H} \to \mathcal{H}$ is a Markov map, then $P(e) = [P(x), P(y)]$ consists of several edges of $\mathcal{H}$, denoted as $e_1, \ldots, e_r$. The edge $\phi_*(e)$ of $E(\widetilde{H})$ has the endpoints $\phi(x), \phi(y)$. Since the map $\widetilde{P} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ is also Markov, the image $\widetilde{P}(\phi(e))$ of $\phi(e)$ equals to $[\widetilde{P} \circ \phi(x), \widetilde{P} \circ \phi(y)]$. Using Claim 2 successively on $e_1, \ldots, e_r$, we get that the segment $\widetilde{P}(\phi(e))$ consists of exactly the edges $\phi_*(e_1), \ldots, \phi_*(e_r)$ of $E(\widetilde{\mathcal{H}})$. It means that the two Markov maps $(\mathcal{H}, P)$ and $(\widetilde{\mathcal{H}}, \widetilde{P})$ have the same incidence matrix, and hence the same topological entropy. We then complete the proof of Lemma 6.2. $\square$

6.2. **The limit inferior of the entropy function.** According to Proposition 6.1, the continuity of the entropy function at a postcritically-finite parameter P depends only on the relation of $h(P)$ and the limit inferior $\liminf_{Q \to P} h(Q)$: the entropy function is continuous at P if and only if $h(P) \leq \liminf_{Q \to P} h(Q)$. In this subsection, we will find this limit inferior by the dynamics of P and prove Proposition 1.2.

Let P be a monic postcritically-finite polynomial, and $\mathscr{J} := \{\Theta_1(c_1), \ldots, \Theta_r(c_r)\}$ a weak Julia critical marking of P (see Definition 2.12). We are able to construct a puzzle induced by $\mathscr{J}$. Let $W := \{z \in \mathbb{C} : G_P(z) < 2\}$, and $\mathcal{R}_i$ denote the union of external rays of P with arguments in $\Theta_i(c_i)$ and the critical points c_i for every $i = 1, \ldots, m$. We say that a subset $B \subseteq \overline{W}$ is a $\mathscr{J}$-*puzzle piece of level* 0 if B is maximal such that any two points in B are not separated by $\mathcal{R}_i$ for all $i = 1, \ldots, m$. Then each $\mathscr{J}$-puzzle piece B of level 0 is a full continuum (i.e., a non-trivial, connected, compact set in $\overline{\mathbb{C}}$ such that its complementary is also connected), with boundary consisting of segments of external rays and equipotential line of potential 2, and $P(B) = \{z \in \mathbb{C} : g_P(z) \leq 2^d\}$ (see the left one in Figure 6).
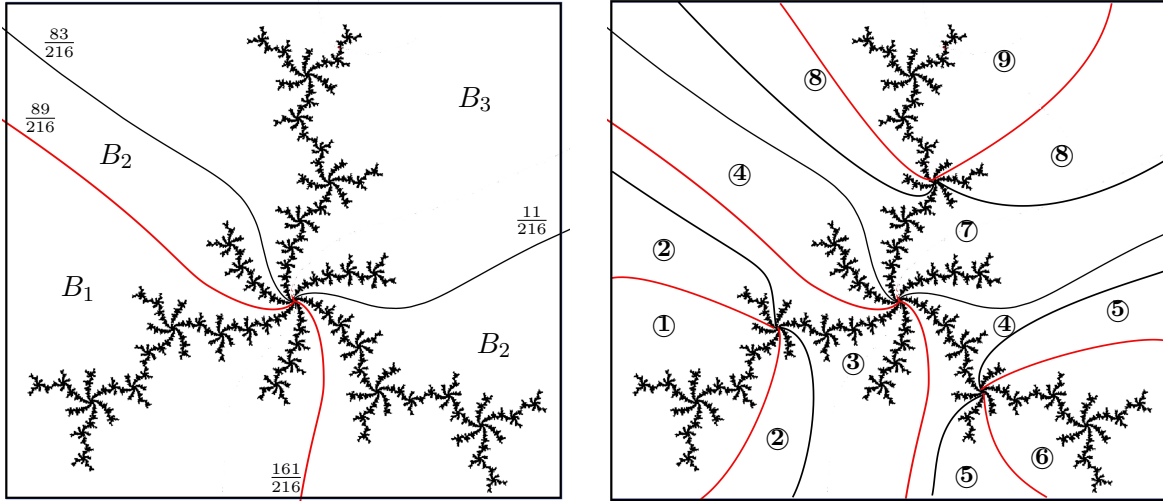

FIGURE 6. Consider the polynomial $P(z) = z \mapsto z^3 + 0.22036 + 1.18612i$ in Figure 3. Then $\mathscr{J} = \{ \{11/216, 83/216\}, \{89/216, 161/216\} \}$ is a Julia weak critical marking of P. Its $\mathscr{J}$-puzzle pieces of level 0 are label in the left figure by B_1, B_2, B_3 (including the boundary rays). In the right figure, the closure of each region labeled by ①, $\ldots$, ⑨ corresponds to a $\mathscr{J}$-puzzle piece of level 1.

The $\mathscr{J}$-puzzle pieces of level 1 is defined as follows. Let B, B' be any pair of $\mathscr{J}$-puzzle piece of level 0. Since $P(B)$ covers B', we call each component of $P|_B^{-1}(B')$ a $\mathscr{J}$-puzzle piece of level 1 (see the right one in Figure 6). By this definition, the following results hold:

(1) each $\mathscr{J}$-puzzle piece of level 1 is a full continuum, with boundary consisting of segments of external rays and equipotential line of potential $2/d$, and different puzzle pieces of level 1 have pairwise disjoint interiors;

(2) each puzzle piece of level 1 is contained in a puzzle piece of level 0;

(3) the map P sends a puzzle piece of level 1 onto a puzzle piece of level 0.

Inductively, we can define the $\mathscr{J}$-puzzle pieces of level $k+1$ ($k \geq 1$) as the components of $(P|_{B_k})^{-1}(B'_k)$ for all pairs B_k, B'_k of puzzle pieces of level k with $B'_k \subseteq P(B_k)$, and the following properties can be inductively checked:

(1) each $\mathscr{J}$-puzzle piece of level $k+1$ is a full continuum, with boundary consisting of segments of external rays and equipotential line of potential $2/d^k$, and different puzzle pieces of level $k+1$ have pairwise disjoint interiors;

(2) each puzzle piece of level $k+1$ is contained in a puzzle piece of level k;

(3) the map P sends a puzzle piece of level $k+1$ onto a puzzle piece of level k;

Let $\{B_n, n \geq 0\}$ be a sequence of nested $\mathscr{J}$-puzzle pieces such that B_n has level n. The intersection $E := \cap_{n \geq 0} B_n$ is called a $\mathscr{J}$-end. The $\mathscr{J}$-ends can be characterized by $\mathscr{J}$-itinerary. We say that $z, w \in \mathbb{C}$ have the same $\mathscr{J}$-itinerary if the pair $\{P^k(z), P^k(w)\}$ are not separated by $\mathcal{R}_i$ for all $k \geq 0$ and $1 \leq i \leq m$.

Lemma 6.7 (properties of $\mathscr{J}$-ends). *Let E be a $\mathscr{J}$-end. Then we have*

(1) *the end E is a full continumm in $\mathcal{K}_P$ with its boundary in $\mathcal{J}_P$;*

(2) *the image $P(E)$ is an $\mathscr{J}$-end;*

(3) *E is a maximal connected set in $\mathcal{K}_P$ with the same $\mathscr{L}$-itinerary, i.e., if $E' \subseteq \mathcal{K}_P$ is a connected set containing E such that any two points in E' have the same $\mathscr{L}$-itinerary, then $E = E'$.*

(4) *the end E is non-trivial if and only if it contains Fatou components of P.*

Proof. Let $\{B_n, n \geq 1\}$ be a sequence of nested $\mathscr{J}$-puzzle pieces such that $E = \cap_{n \geq 1} B_n$. Point (1) follows directly from Property (1) of the $\mathscr{J}$-puzzle pieces. By Property (3) of the $\mathscr{J}$-puzzle pieces, $\{P(B_n), n \geq 1\}$ is also a sequence of nested $\mathscr{J}$-puzzle pieces. Then $P(E) \subseteq E' := \cap_{n \geq 1} P(B_n)$. To get the equality, let y be a point in E' and $x_n \in B_n$ a preimage of y by P. Since x_n has finitely many choices, and $\{B_n, n \geq 1\}$ a nested sequence, there exists a preimage x by P of y which belongs to B_n for all large n. Then point (2) is proved.

To see point (3), we first show that any two points in E have a common $\mathscr{L}$-itinerary. Let x, y be any two distinct points in E. Then $x, y \in B_n$ for any $n \geq 1$. According to Properties (2),(3) of the $\mathscr{J}$-puzzle pieces, the pair $\{P^i(x), P^i(y)\}$ is not separated by any $\mathcal{R}_i$ for $i = 0, \dots, n$. As n is arbitrary, the points x, y have the same $\mathscr{J}$-itinerary.

Let $E' \subseteq \mathcal{K}_P$ be a connected set such that $E \subseteq E'$ and $E' \setminus E \neq \emptyset$. Then there exists a minimal level k such that E belongs to a $\mathscr{J}$-puzzle pieces B_k of level k and $E' \setminus B_k \neq \emptyset$. Then there exists an arc Γ consisting of two external rays and their common landing point such that Γ separates E', i.e., both components of $\mathbb{C} \setminus \Gamma$ intersects E', and $P^k : \Gamma \to \mathcal{R}_i$ is injective for some $1 \leq i \leq m$. It follows that $P^k(\Gamma) \subseteq \mathcal{R}_i$ separates $P^k(E')$, which complete the proof of point (3).

For (4), we just need to prove that if $E \subseteq \mathcal{J}_P$, then E is a singleton. On the contrary, let $z, w \subseteq E$ be distinct points such that $[z, w] \subseteq E$. For each $j \geq 0$, the set $P^j(E)$ belongs to $\mathcal{J}_P$ and is not separated by $\mathcal{R}_1, \ldots, \mathcal{R}_m$. Then, for each $j \geq 1$, the map $P^j : [z, w] \to P^j([z, w])$ is a homeomorphism although some image $P^j([z, w])$ possibly intersects critical points of P. By the uniform expansionary of P near $\mathcal{J}_P$ (Lemma 2.6), the length of $P^j([z, w]) = [P^j(z), P^j(w)]$ converge to ∞ as $j \to \infty$, a contradiction. $\qquad \square$

Let $\mathcal{E}_{\mathcal{J}}$ denotes the collection of periodic $\mathcal{J}$-ends which contain Fatou critical/postcritical points of P. Then $\mathcal{E}_{\mathcal{J}}$ is P-invariant. By Lemma 6.7.(4), the set $\mathcal{E}_{\mathcal{J}}$ is empty if and only if $\mathcal{K}_P = \mathcal{J}_P$, i.e., P is Misiurewicz . For each element E of $\mathcal{E}_{\mathcal{J}}$, we define the tree H_E as the regulated convex hull within $\mathcal{K}_P$ of the Fatou critical/postcritical points contained in E. We claim that $H_E \subseteq E$: because for each $\mathcal{J}$-puzzle piece B containing E, the tree H_E lie in B by Property (1) of $\mathcal{J}$-puzzle pieces, and then $H_E \subseteq E$. The tree H_E is a subtree of the Hubbard tree $\mathcal{H}_P$, and we define its vertex set $V(H_E)$ as the union of critical/postcritical points of P contained in H_E and the branched points of H_E.

By the definition, if E, E' are two elements of $\mathcal{E}_{\mathcal{J}}$ with $P(E) = E'$, then we have $P(V(H_E)) \subseteq V(H_{E'})$ and $P(H_E) \subseteq H_{E'}$. So each H_E is periodic under P. We claim that $E_1 \cap E_2 = \emptyset$ for any distinct $E_1, E_2 \in \mathcal{E}_{\mathcal{J}}$. Otherwise, the set $E_1 \cup E_2$ is connected and contains E_1, E_2. Thus, by Lemma 6.7.(3), there exists a minimal k such that $P^k(E_1)$ and $P^k(E_2)$ are separated by $\mathcal{R}_i$ for some $1 \leq i \leq m$. Without loss of generality, we assume that $k = 0$. Then the intersection $E_1 \cap E_2$ coincides with c_i. On the other hand, since E_1, E_2 are periodic, there exists p such that $P^p(E_j) \subseteq E_j$ for $j = 1, 2$. It follows that $P^p(c_i)$ belongs to $E_1 \cap E_2$, and hence equal to c_i. This is impossible because c_i is strictly preperiodic.

Let $\mathcal{H}_{\mathcal{J}}$ denote the union of H_E for all elements E of $\mathcal{E}_{\mathcal{J}}$. Here we allow $\mathcal{H}_{\mathcal{J}} = \emptyset$, and this happens if and only if $\mathcal{E}_{\mathcal{J}} = \emptyset$, and if and only if $\mathcal{J}_P = \mathcal{K}_P$ by Lemma 6.7.(4). By the claim above, each H_E with $E \in \mathcal{E}_{\mathcal{J}}$ is a component of $\mathcal{H}_{\mathcal{J}}$. So we define the vertex set $V(\mathcal{H}_{\mathcal{J}})$ of $\mathcal{H}_{\mathcal{J}}$ as the union of $V(H_E)$ for all $E \in \mathcal{E}_{\mathcal{J}}$. Moreover, the forest $\mathcal{H}_{\mathcal{J}}$ is a P-invariant, contained in $\mathcal{H}_P$, and the map $P : \mathcal{H}_{\mathcal{J}} \to \mathcal{H}_{\mathcal{J}}$ is Markov.

Let $\{P_n, n \geq 1\}$ be a sequence of partial postcritically-finite polynomials converging to P. We say the sequence $\{P_n, n \geq 1\}$ has *type* $\mathcal{J}$ if there exists a subset $\mathscr{X}_n$ of a critical marking Θ_n of P_n such that $\mathscr{X}_n \to \mathcal{J}$ as $n \to \infty$. We call $\{P_n, n \geq 1\}$ a *maximal-hyperbolic* sequence if each P_n is hyperbolic and the bounded critical points of P_n converge to Fatou critical points of P.

Proposition 6.8. *Let $\{P_n, n \geq 1\}$ be a sequence of monic partial postcritically-finite polynomials converging to P with type $\mathcal{J}$. Then*

$$h_{top}(P|_{\mathcal{H}_{\mathcal{J}}}) \leq h(P_n), \tag{6.2}$$

for large n, and the equality holds if $\{P_n, n \geq 1\}$ is maximal-hyperbolic. As a consequence, let μ_P denote the minimality of $h_{top}(P|_{\mathcal{H}_{\mathcal{J}}})$ with $\mathcal{J}$ going through all weak Julia critical markings of P, then we have $\mu_P \leq \liminf\limits_{Q \to P} h(Q)$ with Q chosen in monic partial postcritically-finite polynomials.

Proof. The idea of the proof is similar to that of Proposition 6.1: we will embed $\mathcal{H}_{\mathcal{J}}$ into the Hubbard forest $\mathcal{H}_{P_n}$ by an injective $\phi_n : \mathcal{H}_{\mathcal{J}} \to \mathcal{H}_{P_n}$ for all large n, and prove that $h_{top}(P|_{\mathcal{H}_{\mathcal{J}}}) = h_{top}(P_n|_{\phi_n(\mathcal{H}_{\mathcal{J}})})$. As explained in the proof of Proposition 6.1, we can assume these $P'_n s$ are visible so that their Julia critical markings can be defined.

We first show the inequality (6.2). If $\mathcal{H}_{\mathcal{J}} = \emptyset$, the conclusion is obvious (because $h_{top}(P|_{\mathcal{H}_{\mathcal{J}}}) = 0$). So we assume in the following that $\mathcal{H}_{\mathcal{J}} \neq \emptyset$.

Since the sequence $\{P_n, n \geq 1\}$ has type $\mathscr{J} = \{\Theta_i(c_i) : 1 \leq i \leq m\}$, there exist a critical marking $\Theta_n := \{\mathscr{F}_n, \mathscr{X}_n\}$ of P_n with

$$\mathscr{F}_n := \{\Theta(U_{n,1}), \ldots, \Theta(U_{n,s})\}; \ \mathscr{X}_n := \{\Theta(c_{n,1}), \ldots, \Theta(c_{n,m})\}, \tag{6.3}$$

such that $\Theta(c_{n,i}) \to \Theta_i(c_i)$ for each $1 \leq i \leq m$; where $U_1, \ldots, U_s$ denote all critical Fatou components of P, and $U_{n,j}$ is the continuation of U_j at P_n for $j = 1, \ldots, s$. In the dynamical plane of P_n, we define $\mathcal{R}_{n,i}$ as the closures of the union of external rays/extended rays/external radii of P_n with arguments in $\Theta(c_{n,i})$ which land or terminate at $c_{n,i}$, for $i = 1, \ldots, m$. Two points $z, w \in \mathbb{C}$ is said to have the same $\mathscr{X}_n$-itineraries if $P_n^k(z), P_n^k(w)$ are not separated by $\mathcal{R}_{n,i}$ for all $k \geq 0$ and $1 \leq i \leq m$.

Fix an element E of $\mathcal{E}_{\mathscr{J}}$.

Claim 1. Let x, y be the centers of distinct Fatou components in E, and x_n, y_n denote their continuations at P_n (Lemma 4.2). Then x_n, y_n have a common $\mathscr{X}_n$-itinerary for large n.

Proof of Claim 1. On the contrary, assume that there exist infinite n such that x_n, y_n have different $\mathscr{X}_n$-itinerary. For each such n, there exists a minimal number k_n such that $P_n^{k_n}(x_n), P_{n,E}^{k_n}(y_n)$ are separated by some $\mathcal{R}_{n,i_n}$. Since all x_n (resp. y_n) have the same preperiod and period, the number k_n is uniformly bounded above. So, without loss of generality and by taking subsequences, we can assume that x_n, y_n are separated by $\mathcal{R}_{n,i}$ for some $1 \leq i \leq m$ and all large n. Note that $x_n \to x, y_n \to y$ and $\mathcal{R}_{n,i} \to \mathcal{R}_i$ (by Proposition 5.7), then $\mathcal{R}_i$ separates x, y, contradicting that x, y belong to a common $\mathscr{J}$-end. $\qquad\square$

We denote by M_E the Fatou-type vertices of H_E. For each large n, we define a map $\phi_{n,E} : M_E \to \mathbb{C}$ as $\phi_{n,E}(z) =: z_n$ with z_n the continuation of z at P_n. Claim 1 implies the points in $\phi_{n,E}(M_E)$ have a common $\mathscr{X}_n$-itinerary.

Claim 2. Let z be a Juila preperiodic point in E. Then there exists a unique point z_n such that z_n has the same preperiod and period by P_n as those of z by P, $z_n \to z$ as $n \to \infty$ and z_n has the same $\mathscr{X}_n$-itinerary as the points in $\phi_{n,E}(M_E)$.

Proof of Claim 2. We first assume that z is periodic. Since z is repelling, then there exists a unique continuation z_n of z by Implicity Function Theorem. We just need to show that z_n has the same $\mathscr{X}_n$-itinerary as the points in $\phi_{n,E}(M_E)$. On the contrary, assume that there exists $x \in M_E$ such that $\phi_{n,E}(x)$ and z_n have different $\mathscr{X}_n$-itineraries for all large n. As explained in the proof of Claim 1, one can assume that $\phi_{n,E}(x)$ and z_n are separated by $\mathcal{R}_{n,i}$ for some $1 \leq i \leq m$. Since $\phi_{n,E}(x) \to x, z_n \to z$ and $\mathcal{R}_{n,i} \to \mathcal{R}_i$, then either $z = c_i$, or $\mathcal{R}_i$ separates x, z. Both of the cases lead to a contradiction: in the former case, this is because z is periodic but $c_i \in \mathcal{J}_P$ is strictly preperiodic; and in the latter case, it contradict that x, z belong to a common $\mathscr{J}$-end.

Without loss of generality, we assume that E is P-invariant. Let z' be a Julia preperiodic point in E. Inductively, we will prove that Claim 2 holds for z' under the assumption that Claim 2 holds for $z := P(z')$.

Let z_n be the unique continuation of z satisfying the requirements in Claim 2. If z' is not a critical point of P, then M_E and z' are disjoint with $\cup_{i=1}^m \mathcal{R}_i$ and not separated by it. In this case, there is a unique preimage z'_n of z_n by P_n such that $z'_n \to z'$. As $\cup_{i=1}^n \mathcal{R}_{n,i} \to \cup_{i=1}^n \mathcal{R}_i$, then z'_n and $\phi_{n,E}(M_E)$ are not separated by $\cup_{i=1}^m \mathcal{R}_{n,i}$. Combining the assumption on z_n, then the point z'_n have the same $\mathscr{X}_n$-itinerary as $\phi_{n,E}(M_E)$.

Now let $c = z'$ be a Julia critical point of P, and denote $I(c) = \{i : 1 \leq i \leq m, c_i = c\}$. The plane $\mathbb{C}$ is divided into $\deg(P|_c)$ closed sets, called $\mathscr{J}(c)$-puzzle pieces, such that the points

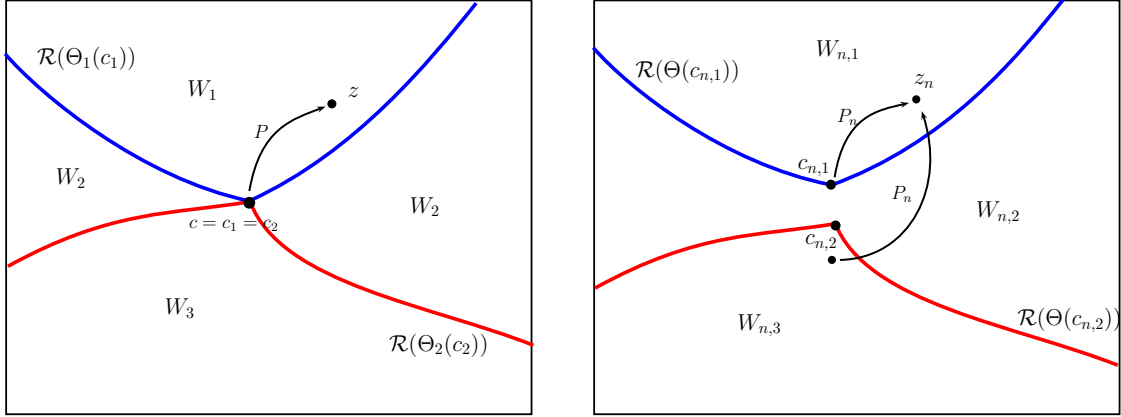in each of the puzzle pieces are not separated by $\cup_{i \in I(c)} \mathcal{R}_i$. Note that $z = P(c)$ has exactly

FIGURE 7. $\mathscr{J}(c)$-puzzle pieces and $\mathscr{X}_n(c)$-puzzle pieces. Here $I(c) = \{1, 2\}$, $\deg(P|_c) = 3$, $\Theta(c_{n,i}) \to \Theta_i(c_i)$ for $i = 1, 2$. The $\mathscr{J}(c)$-puzzle pieces of P are $\overline{W_1}, \overline{W_2}, \overline{W_3}$; and the $\mathscr{X}_n(c)$-puzzle pieces of P_n are $\overline{W_{n,1}}, \overline{W_{n,2}}, \overline{W_{n,3}}$.

one preimage by P near c in each $\mathscr{J}(c)$-puzzle piece, and all these preimages coincide with c. In the dynamical plane of P_n, the plane $\mathbb{C}$ can be also divided into $\deg(P|_c)$ closed sets, called $\mathscr{X}_n(c)$-puzzle pieces, such that the points in each of the puzzle pieces are not separated by $\cup_{i \in I(c)} \mathcal{R}_{n,i}$. Since $\phi_{n,E}(M_k)$ is not separated by $\cup_{i=1}^m \mathcal{R}_{n,i}$, there exists a unique $\mathscr{X}_n(c)$-puzzle pieces A_n with $\phi_{n,E}(M_k) \subseteq A_n$. Similarly, the point z_n has exactly one preimage by P_n near c in each $\mathscr{X}_n(c)$-puzzle piece, and we denote the one in A_n by z'_n. Then z'_n and $\phi_{n,E}(M_k)$ are not separated by $\cup_{i=1}^m \mathcal{R}_{n,i}$. Combining the assumption on z_n, the point z'_n has the same $\mathscr{X}_n$-itinerary as $\phi_{n,E}(M_E)$. $\square$

By Claim 2, we can extend $\phi_{n,E}$ to $V(H_E)$ such that $\phi_{n,E}(v) := v_n$ satisfy the properties in Claim 2 for all $v \in V(H_E)$. Then all points in $\phi_{n,E}(V(H_E))$ have a common $\mathscr{X}_n$-itinerary.

Claim 3. The set $\phi_{n,E}(V(H_E))$ is contained in a component of $\mathcal{K}_{P_n}$ for large n.

Proof of Claim 3. We first observe that, for any distinct Fatou centers $x, y \in E$, their continuations x_n, y_n are contained in the same component of $\mathcal{K}_{P_n}$. On the contrary, by taking subsequences if necessary, one can find an arc Γ_n consisting of two external radii and their common terminating point, such that Γ_n separates x_n, y_n, and $P_n^k : \Gamma_n \to \mathcal{R}_{n,i}$ is injective for some $k \geq 0$ and some $i \in \{1, \dots, m\}$. By Lemma 5.9, the arc Γ_n, by taking subsequences if necessary, converges to an arc Γ which consisting of two external rays of P and their common landing point denoted by z, and the map $P^k : \Gamma \to \mathcal{R}_i$ is injective. Since $z \notin \{x, y\}$, the arc Γ separates E, and thus $\mathcal{R}_i$ separates $P^k(E)$, a contradiction.

This observation shows that $\phi_{n,E}(M_E)$ belong to a common component of $\mathcal{K}_{P_n}$, denoted as $K_{n,E}$. Let z be a Julia vertex of H_E. If z is on the boundary of a Fatou component in E, then $\phi_{n,E}(z) \in K_{n,E}(z)$ according to the observation above. We now assume that z is not on the boundary of Fatou components in E and $\phi_{n,E}(z) \notin K_{n,E}$ for large n. Then by taking subsequences if necessary, one can find an arc Γ_n consisting of two external radii and their common terminating point, such that Γ_n separates $K_{n,E}, \phi_{n,E}(z)$, and $P_n^k : \Gamma_n \to \mathcal{R}_{n,i}$ is injective for some $k \geq 0$ and some $i \in \{1, \dots, m\}$. For simplicity, we assume $P(E) \subseteq E$.

Let $\Theta = \{\Theta_1, \dots, \Theta_r\}$ be any weak critical marking of P. We denote by $\mathcal{R}(\Theta_i)$ the union of extended/external rays of P with arguments in Θ_i, and $\mathcal{R}(\Theta) := \cup_{i=1}^r \mathcal{R}(\Theta_i)$. By the assumption on z, one can choose a Fatou center $a \in E$ sufficiently near z, such that the pair $\{P^j(a), P^j(z)\}$

is not separated by $\mathcal{R}(\Theta)$ for any $j \in \{0, \ldots, k\}$ and any weak critical marking Θ of P. By the observation, we get $\phi_{n,E}(a) \in K_{n,E}$.

Recall (6.3), for every $i \in \{1, \ldots, s\}$, we denote $\mathcal{R}_n(U_{n,i})$ the union of extended rays of P_n with arguments in $\Theta(U_{n,i})$. We claim that for each $0 \leq j \leq k$, the points $\phi_{n,E}(P^j(a)), \phi_{n,E}(P^j(z))$ are not separated by $\cup_{i=1}^s \mathcal{R}_n(U_{n,i})$ for large n. If not, by taking subsequences, we have $\Theta(U_{n,i}) \to \Theta_i$ for $1 \leq i \leq s$. By Proposition 5.7, the collection $\{\Theta_1, \ldots, \Theta_s\}$ is a Fatou weak critical marking of P, and $\mathcal{R}_n(U_{n,i}) \to \mathcal{R}(\Theta_i)$ for $i = 1, \ldots, s$. Since $\phi_{n,E}(P^j(a)), \phi_{n,E}(P^j(z))$ is supposed to be separated by $\cup_{i=1}^s \mathcal{R}_n(U_{n,i})$ for infinite n, and z is not on the boundary of Fatou components in E, it follows that $P^j(a), P^j(z)$ are separated by $\cup_{i=1}^s \mathcal{R}(\Theta_i)$, a contradiction to the choice of a.

By the claim above and the fact that $\phi_{n,E}(a), \phi_{n,E}(z)$ have the same $\mathscr{X}_n$-itinerary (Claims 1, 2), the pairs of points $\{\phi_{n,E}(P^j(a)), \phi_{n,E}(P^j(z))\}$, $j = 0, \ldots, k$, factually belong to the closure of a Θ_n-unlinked class in the dynamical plane of P_n. As Γ_n can not cross any $\mathcal{R}_n(U_{n,i})$ and $\mathcal{R}_{n,j}$ for $1 \leq i \leq s, 1 \leq j \leq m$, then Γ_n belongs to the same equivalence class as $\phi_{n,E}(a), \phi_{n,E}(z)$ (because Γ_n separates them). Note that P_n is injective on each Θ_n-unlinked class, then $P_n(\Gamma_n)$ separates $P_n \circ \phi_{n,E}(z) = \phi_{n,E}(P(z))$ and $P_n \circ \phi_{n,E}(a) = \phi_{n,E}(P(a))$. Inductively use the argument above, we get that $P_n^k(\Gamma_n) = \mathcal{R}_{n,i}$ separates $\phi_{n,E}(P^k(z))$ and $\phi_{n,E}(P^k(a))$. It means that a and z have distinct $\mathscr{X}_n$-itinerary, a contradiction. $\qquad\square$

By Claim 3, we define the tree H_E^n in the dynamical plane of P_n as the convex hull of $\phi_{n,E}(V(H_E))$ within the component of $\mathcal{K}_{P_n}$ containing $\phi_{n,E}(V(H_E))$. The vertex set of H_E^n is defined as the union of $\phi(V(H_E))$ and the branched points of H_E^n. The map $\phi_{n,E} : V(H_E) \to V(H_E^n)$ can be extended to a map, also denoted by $\phi_{n,E}$, from H_E to H_E^n, such that it maps an edge $e(x, y)$ of H_E to an segment $[\phi_{n,E}(x), \phi_{n,E}(y)]$ in H_E^n. Since the endpoints of H_E^n are contained in $\phi_{n,E}(V(H_E))$, the map $\phi_{n,E} : H_E \to H_E^n$ is thus surjective.

Claim 4. Then map $\phi_{n,E} : H_E \to H_E^n$ is a homeomorphism.

Proof of Claim 4. Since $\phi_{n,E}$ is surjective and injective on $V(H_E)$, it is enough to show that $\phi_{n,E}$ maps an edge of H_E to an edge of H_E^n. Let $e = e(x, y)$ be an edge of H_E. By definition, its image $\phi_{n,E}(e)$ is the segment $[\phi_{n,E}(x), \phi_{n,E}(y)]$. By taking subsequences if necessary, we only need to exclude the following two cases.

Case 1. There exists $z \in V(H_E)$ such that $\phi_{n,E}(z)$ belongs to the open arc $(\phi_{n,E}(x), \phi_{n,E}(y))$ for large n.

We first suppose $z \in \mathcal{F}_P$, and let the preperiod of $\phi_{n,E}(z)$ is k. Let p be sufficiently large integer. We can choose a preperiodic point of preperiod k and period p in each of the two components $\partial U(\phi_{n,E}(z)) \setminus (\phi_{n,E}(x), \phi_{n,E}(y))$, denoted as u_n, v_n. By taking subsequences if necessary, we assume $u_n \to u, v_n \to v$. By Lemma 4.3, the points u, v are preperiod points of P contained in $\partial U(z)$ with preperiod k and period p. Since p is large enough, the orbits of u, v avoid the critical points of P. Hence u_n, v_n are the continuation of u, v respectively.

Let $\alpha \in \arg_P(u)$ and $\beta \in \arg_P(v)$. By Lemma 4.4, the external rays $\mathcal{R}_{P_n}(\alpha)$ and $\mathcal{R}_{P_n}(\beta)$ land at u_n and v_n respectively for all large n. Let $\widehat{\mathcal{R}}_{P_n}(\alpha), \widehat{\mathcal{R}}_{P_n}(\beta)$ denote the corresponding extended rays associated to $U(\phi_{n,E}(z))$. According to Lemmas 4.4 and 4.3, the arcs $\Gamma_n := \widehat{\mathcal{R}}_{P_n}(\alpha) \cup \widehat{\mathcal{R}}_{P_n}(\beta)$ converge to the arc $\Gamma := \widehat{\mathcal{R}}_P(\alpha) \cup \widehat{\mathcal{R}}_P(\beta)$. Note that every Γ_n separates x_n and y_n, then Γ separates x and y. It implies z belongs to the open arc (x, y), a contradiction.

The situation of $z \in \mathcal{J}_P$ is similar. By taking subsequences, there exists an arc $\Gamma_n := \mathcal{R}_{P_n}(\alpha) \cup \{\phi_{E,n}(z)\} \cup \mathcal{R}_{P_n}(\beta)$ separating $\phi_{E,n}(x)$ and $\phi_{E,n}(y)$ for all large n. By Lemma 5.9, the arc Γ_n converge to the arc $\Gamma := \mathcal{R}_P(\alpha) \cup \{z\} \cup \mathcal{R}_P(\beta)$, which separates x, y, also a contradiction.

Case 2. There exists a point $z_n \in V(H_E^n) \backslash \phi_{E,n}(V(H_E))$ contained in the open arc $(\phi_{n,E}(x), \phi_{n,E}(y))$ for every large n.

By the definition of $V(H_E^n)$, the point z_n is a branched point. Suppose on the contrary that this case happens. Then, for each n, we can find three external rays/extended rays of P_n with arguments $\alpha_n, \beta_n, \eta_n$ landing together at z_n, and a endpoint v_n of H_E^n disjoint with $\phi_{n,E}(x), \phi_{n,E}(y)$, such that the three components of $\mathbb{C} \backslash \mathcal{R}_{P_n}(\alpha_n, \beta_n, \eta_n)$ contains $\phi_{n,E}(x), \phi_{n,E}(y)$ and v_n respectively, where $\mathcal{R}_{P_n}(\alpha_n, \beta_n, \eta_n)$ denotes the union of the three rays with arguments $\alpha_n, \beta_n, \eta_n$ together with their common landing point z_n. By taking subsequences, we can assume $\alpha_n \to \alpha, \beta_n \to \beta, \eta_n \to \eta, z_n \to z$ and $v_n = \phi_{n,E}(v)$ for a point $v \in V(H_E)$ since the endpoints of H_E^n belong to $\phi_{n,E}(V(H_E))$.

As explained in Case 1, one can suitably choose the rays in $\mathcal{R}_{P_n}(\alpha_n, \beta_n, \eta_n)$ such that
$$\limsup_{n\to\infty} \mathcal{R}_{P_n}(\alpha_n, \beta_n, \eta_n) = \mathcal{R}_P(\alpha, \beta, \eta),$$
where $\mathcal{R}_P(\alpha, \beta, \eta)$ consists of external/extended rays of P with arguments α, β, η, together with their common landing point z. If $z \notin \{x, y, v\}$, then $\mathcal{R}_P(\alpha, \beta, \eta)$ separates x, y, v, and hence z is a branched point of H_E contained in (x, y), a contradiction. Otherwise, one of $\{x, y, v\}$, say x, coincides with z. In this case, the distance of $\phi_{n,E}(x)$ and z_n converge to 0, so that the length of the orbit of z_n converge to ∞. Since z_n is a branched point of H_E^n, it follows that the number of the branched points of H_E^n goes to infinity, which contradicts that the number of the endpoints of H_E^n is bounded by $\#V(H_E)$ for all n. $\qquad\square$

By this claim, we see that $V(H_E^n) = \phi_{n,E}(V(H_E))$, and the endpoints of H_E^n are contained in $\phi_{n,E}(M_E)$. Hence H_E^n is a subtree of $\mathcal{H}_{P_n}$.

Claim 5. Let E, E' be two elements of $\mathcal{E}_{\mathscr{J}}$ such that $E' = P(E)$. Then $P_n(H_E^n) \subseteq H_{E'}^n$ and $P_n : H_E^n \to H_{E'}^n$ is Markov.

Proof. By the definition of $\phi_{n,E}$, we have $P_n \circ \phi_{n,E} = \phi_{n,E'} \circ P$ on $V(H_E)$. Since $P(V(H_E)) \subseteq V(H_{E'})$, it follows that
$$P_n(V(H_E^n)) = P_n \circ \phi_{n,E}(V(H_E)) = \phi_{n,E'} \circ P(V(H_E)) \subseteq \phi_{E',n}(V(H_{E'})) = V(H_{E'}^n).$$
Note that the image $P_n(H_E^n)$ equals to the regulated hull of $P_n(V(H_E^n))$ in $\mathcal{K}_{P_n}$, then one get $P_n(H_E^n) \subseteq H_{E'}^n$. To prove that $P_n : H_E^n \to H_{E'}^n$ is Markov, we just need to check that the restriction of P_n on each edge of H_E^n is injective.

Let $e = e(\phi_{n,E}(x), \phi_{n,E}(y))$ be an edge of H_E^n with endpoints $\phi_{n,E}(x), \phi_{n,E}(y)$. Observe that the restriction of P_n on e is not injective only if e is separated by $\mathcal{R}_{n,i}$ for some $1 \leq i \leq m$. In this case $\phi_{n,E}(x)$ and $\phi_{n,E}(y)$ are separated by $\mathcal{R}_{n,i}$. However, it is impossible because $\phi_{n,E}(x)$ and $\phi_{n,E}(y)$ have the same $\mathscr{X}_n$-itinerary. $\qquad\square$

Claim 6. For any different elements E_1, E_2 of $\mathcal{E}_{\mathscr{J}}$, we have $H_{E_1}^n \cap H_{E_2}^n = \emptyset$ for all large n.

Proof. By Lemma 6.7.(3), there exists a minimal k such that $P^k(E_1)$ and $P^k(E_2)$ are separated by $\mathcal{R}_i$ for some $1 \leq i \leq m$. Without loss of generality, we assume that $k = 0$. As $V(H_{E_j}^n) \to V(H_{E_j})$ for $j = 1, 2$ and $\cup_{i=1}^m \mathcal{R}_{n,i} \to \cup_{i=1}^m \mathcal{R}_i$, then trees $H_{E_1}^n$ and $H_{E_2}^n$ are separated by $\mathcal{R}_{n,i}$ for all large n. Consequently, if $H_{E_1}^n \cap H_{E_2}^n \neq \emptyset$, the intersection equals to $c_{n,i}$. On the other hand, since E_1, E_2 are periodic, by Claim 5, there exists p independent on n such that $P_n^p(H_{E_j}^n) \subseteq H_{E_j}^n$ for $j = 1, 2$. It follows that $P_n^p(c_{n,i})$ belongs to $H_{E_1}^n \cap H_{E_2}^n$, and hence equal to $c_{n,i}$. This is impossible because $c_{n,i}$ is either non-periodic, or periodic with period converges to ∞. $\qquad\square$

Now we define $\mathcal{H}_{\mathscr{J}}^n$ as the union of H_E^n with $E \in \mathcal{E}_{\mathscr{J}}$. Since each H_E^n is a connected component of $\mathcal{H}_{\mathscr{J}}^n$ (by Claim 6), the vertex set $V(\mathcal{H}_{\mathscr{J}}^n)$ is defined as the union of $V(H_E^n)$ for all $E \in \mathcal{E}_{\mathscr{J}}$.

By Claim 5, we have $P_n(V(\mathcal{H}^n_{\mathscr{J}})) \subseteq V(\mathcal{H}^n_{\mathscr{J}})$ the map $P_n : \mathcal{H}^n_{\mathscr{J}} \to \mathcal{H}^n_{\mathscr{J}}$ is Markov. Note that the trees $H_E, E \in \mathcal{E}_{\mathscr{J}}$ are pairwise disjoint, we then obtain a homeomorphism $\phi_n : \mathcal{H}_{\mathscr{J}} \to \mathcal{H}^n_{\mathscr{J}}$ for large n such that $\phi_n(z) := \phi_{n,E}(z)$ if $z \in H_E$ and $E \in \mathcal{E}_{\mathscr{J}}$.

To prove $h_{top}(P|_{\mathcal{H}_{\mathscr{J}}}) \leq h(P_n)$, it is enough to show that $h_{top}(P|_{\mathcal{H}_{\mathscr{J}}}) = h_{top}(P_n|_{\mathcal{H}^n_{\mathscr{J}}})$, since $\mathcal{H}^n_{\mathscr{J}} \subseteq \mathcal{H}_{P_n}$. Let D be the incidence matrix of $(\mathcal{H}_{\mathscr{J}}, P)$. By the homeomorphism $\phi_n : \mathcal{H}_{\mathscr{J}} \to \mathcal{H}^n_{\mathscr{J}}$, the enumeration of $E(\mathcal{H}_{\mathscr{J}})$ induces an enumeration on $E(\mathcal{H}^n_{\mathscr{J}})$ such that e and $\phi_n(e)$ have the same label. Let $e = e(x,y)$ be any edge of $\mathcal{H}_{\mathscr{J}}$ with endpoints $x, y \in V(\mathcal{H}_{\mathscr{J}})$. Then $P(e) = [P(x), P(y)]$ consists of several edges $\mathcal{H}_{\mathscr{J}}$, denoted as $e_1, \ldots, e_r$. The edge $\phi_n(e)$ of $\mathcal{H}^n_{\mathscr{J}}$ has the endpoints $\phi_n(x), \phi_n(y)$, and its image $P_n(\phi_n(e))$ under P_n equals to $[P_n \circ \phi_n(x), P_n \circ \phi_n(y)]$. Since ϕ_n is a homeomorphism, the segment $P_n(\phi_n(e))$ consists of exactly the edges $\phi_n(e_1), \ldots, \phi_n(e_r)$ of $\mathcal{H}^n_{\mathscr{J}}$. It means that $(\mathcal{H}_{\mathscr{J}}, P)$ and $(\mathcal{H}^n_{\mathscr{J}}, P_n)$ have the same incidence matrix, and hence the same topological entropy. We then complete the proof of (6.2).

To prove the equality, we just need to show that, if $\{P_n, n \geq 1\}$ is maximal-hyperbolic (of type $\mathscr{J}$), then any non-trivial periodic component H_n of $\mathcal{H}_{P_n}$ is contained in $\mathcal{H}^n_{\mathscr{J}}$ for large n. Let M_n denote the Fatou critical/postcritical points of P_n contained in H_n. Then H_n is the regulated convex hull of M_n within $\mathcal{K}_{P_n}$.

In the dynamical plane of P, we can find a set M such that each point in M_n is a continuation of a point in M. To complete the proof the proposition, it is enough to show that M belong to a $\mathscr{J}$-end. On the contrary, we postulate that $x, y \in M$ belong to different $\mathscr{J}$-ends. Then one can find a minimal k and a $\mathscr{J}$-puzzle piece B_k such that $x \in B_k$ and $y \notin B_k$. It follows that there exist an arc Γ consisting of two external rays and their common landing point, and a index $1 \leq i \leq m$ such that Γ separates x, y and $P^k(\Gamma) \subseteq \mathcal{R}_i$. Correspondingly, in the dynamical plane of P_n, we can find an arc Γ_n consisting of two external radii and their common terminating point such that $P_n^k(\Gamma_n) \subseteq \mathcal{R}_{n,i}$ and $\Gamma_n \to \Gamma$ as $n \to \infty$. Since $x_n \to x, y_n \to y$, we then get that Γ_n separates x_n, y_n, contradicting that x_n, y_n belongs to the same component of $\mathcal{K}_{P_n}$.

The remaining part of the proposition follows directly from Proposition 5.7 because the space of critical portraits of degree d is compact (under the Hausdorff metric). $\qquad\square$

Proposition 6.9. *Let P be a monic, centered postcritically-finite polynomial. Then we can find a sequence $\{P_n, n \geq 1\}$ of partial postcritically-finite polynomials converging to P by perturbing P with capture surgery such that $\mu_P = h(P_n)$ for large n, where μ_P is defined in Proposition 6.8, and the number μ_P equals to the limit inferior $\liminf_{Q \to P} h(Q)$ with Q chosen in $\mathcal{P}^{\mathrm{ppf}}_d$.*

Proof. Let $\mathscr{J}$ be a weak Julia critical marking of P such that $h(P|_{\mathcal{H}_{\mathscr{J}}}) = \mu_P$. By Proposition 6.8, we just need to construct a maximal-hyperbolic polynomial sequence converging to P of type $\mathscr{J}$ by perturbing P with capture surgery.

Denote $\mathscr{J} := \{\Theta_1(c_1), \ldots, \Theta_m(c_m)\}$ as in Section 5.2. Let $\mathbf{c}$ be the vector consists all Julia critical points of P, and $\vec{\mathbf{s}}_n = (\mathbf{c}, \mathbf{D}_n, \mathbf{Z}_n, \mathbf{X}_n), n \geq 1$ be a sequence of perturbation data for capture surgery defined in Section 3 such that

- $\mathrm{diam}(\mathbf{D}_n) := \max_{c \in \mathbf{c}} \mathrm{diam}(D_n(c)) \to 0$ as $n \to \infty$;
- for each n and $c \in \mathbf{c}$, the invariant domain $Z_n(c) \in \mathbf{Z}_n$ has one component in each component of $D_n(c) \setminus \mathcal{K}_P$.

To get an expected polynomial sequence by capture surgery, we need to specifically define the surgery mappings $\mathbf{X}_n, n \geq 1$.

Let c be a Julia critical point of P, and $I_c := \{i : 1 \leq i \leq m, c_i = c\}$ where the notations c_i come from $\mathscr{J} = \{\Theta_1(c_1), \ldots, \Theta_m(c_m)\}$. For any index $i \in I_c$, we denote by $V_{n,i}$ the component of $P(D_{n,c}) \setminus \mathcal{K}_P$ such that the external ray of P with argument $\theta_i := \tau(\Theta_i(c_i))$ lands at $P(c)$ through $V_{n,i}$, and denote $Y_{n,i} := P(Z_n(c)) \cap V_{n,i}$. We also denote by $v_{n,i}$ the intersection of $\mathcal{R}_P(\theta_i)$ and $\partial P(D_{n,c})$, and by $w_{n,i,j}, j = 1, \ldots, k_i$ the intersections of $\mathcal{R}_P(\alpha_{i,j})$ and $\partial D_{n,c}$ where $\Theta_i(c_i) := \{\alpha_{i,1}, \ldots, \alpha_{i,k_i}\}$ (see Figure 8). Then all $w_{n,i,j}, j = 1, \ldots, k_i$ are preimages of $v_{n,i}$ by
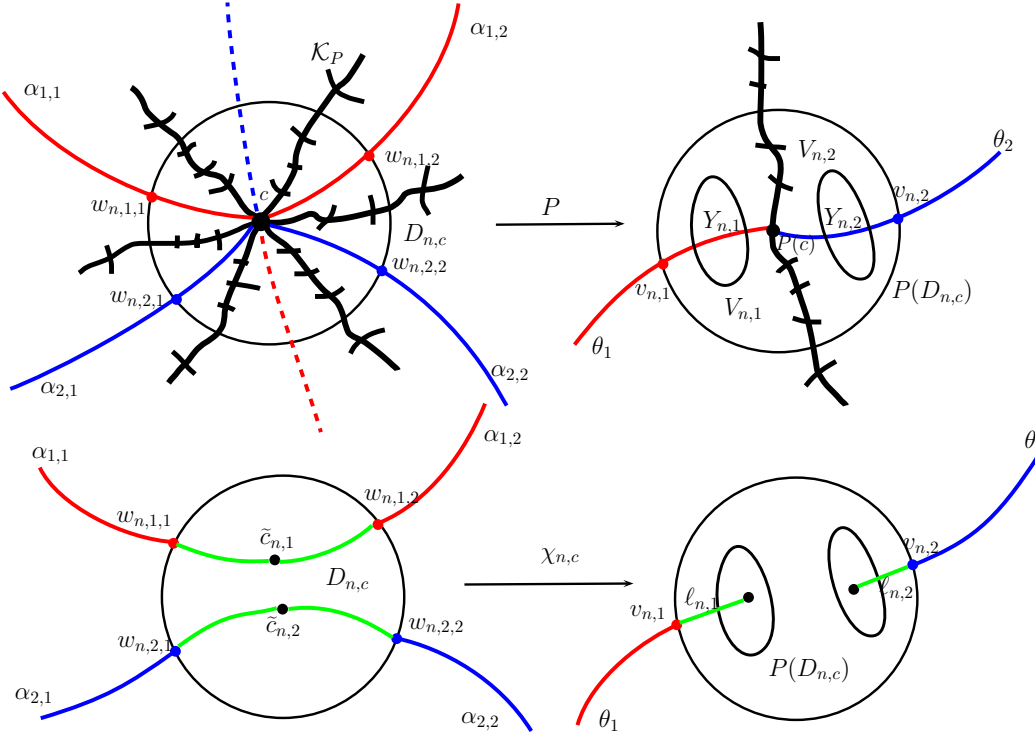

FIGURE 8. The construction of perturbation mappings. The picture above shows the local behavior of P near c and the one below illustrates the perturbation of P near c. Here $c = c_1 = c_2$, $\Theta_1(c_1) = \{\alpha_{1,1}, \alpha_{1,2}\}$ and $\Theta_2(c_2) = \{\alpha_{2,1}, \alpha_{2,2}\}$.

$\chi_{n,i}$ since $P = \chi_{n,c}$ on $\partial D_{n,c}$. For the definition of $\chi_{n,c}$, we require that the critical point c of P splits into $\#I(c)$ critical points $\widetilde{c}_{n,i}, i \in I(c)$, of the surgery map $\chi_{n,c}$, such that, for each i,

- the critical point $\widetilde{c}_{n,i}$ (of $\chi_{n,c}$) has the multiplicity $\#\Theta_i(c_i) - 1$ and $\chi_{n,c}(\widetilde{c}_{n,i}) \in Y_{n,i}$;
- let $\ell_{n,i}$ be an arc in $V_{n,i}$ joining $\chi_{n,i}(\widetilde{c}_{n,i})$ and $v_{n,i}$, then the lift of $\ell_{n,i}$ by $\chi_{n,c}$ based at each $w_{n,i,j}$, with $j = 1, \ldots, k_i$, has the terminal point $\widetilde{c}_{n,i}$.

Let $\widetilde{P}_n$ be the topological perturbation of P by capture surgery with data $\vec{s}_n$ specified above. By Lemma 3.2, there exists a unique polynomial P_n c-equivalent to $\widetilde{P}_n$ by a pair of normalized homeomorphisms $(\phi_{n,0}, \phi_{n,1})$ such that they fix two points near ∞ independent on n. This normalized property implies that $\phi_{n,0}(z)/z \to 1$ as $z \to \infty$, and hence P_n is monic. For each Julia critical point c of P and any $i \in I(c)$, we denote $c_{n,i} := \phi_{n,0}(\widetilde{c}_{n,i})$. Then the Julia critical point c of P splits into $\#I(c)$ escaping critical points $c_{n,i}, i \in I(c)$ of P_n. By Proposition 3.3, it follows that the sequence $\{P_n, n \geq 1\}$ is maximal-hyperbolic and converge to P.

To prove $h(P_n) = \mu_P$ for large n, we remains to show that the sequence $\{P_n, n \geq 1\}$ has type $\mathscr{J}$. Let $c_{n,i}$ be a escaping critical point of P_n with $i \in I(c)$, where c is a Julia critical point of P. We define $\Theta(c_{n,i})$ as the set of the arguments of the external radii of P_n which

terminate at $c_{n,i}$. By taking subsequence if necessary, we assume that $\Theta(c_{n,i}) \to \Theta_i$ as $n \to \infty$. According to Lemma 5.9, the external rays of P with arguments in Θ_i land at c. Our specific construction of $\mathbf{X}_n$ ensure that the set Θ_i is exactly $\Theta_i(c_i)$. Hence the Julia critical marking $\{\Theta(c_{n,i}) : 1 \leq i \leq m\}$ of P_n converge to $\mathscr{J}$ as $n \to \infty$.

Notice that the polynomials $P_n, n \geq 1$, constructed above by capture surgery are monic, but not necessary centered. To testify μ_P the limit inferior $\liminf_{Q \to P} h(Q)$ with $Q \in \mathcal{P}_d^{\mathrm{ppf}}$, we will find a sequence $\{Q_n, \geq 1\} \subseteq \mathcal{P}_d^{\mathrm{ppf}}$ such that $h(Q_n) = h(P_n)$ and $Q_n \to P$. Let T_n be a the translation such that $Q_n := T_n \circ P_n \circ T_n^{-1}$ is a monic, centered polynoimal. Then $h(P_n) = h(Q_n)$. By taking a subsequence, we can assume $T_n \to T$. Hence $Q_n, n \geq 1$, converge to a monic, centered polynomial $Q = T \circ P \circ T^{-1}$. Since P itself is monic, centered, and T is a translation, if follows that $T = id$, and then $P = Q$. $\square$

Proof of Proposition 1.2. The upper semi-continuous of h at P is duo to Proposition 6.1. Let $\mathscr{J}$ be a Julia weak critical marking of P such that $\mu_P = h_{top}(P|_{\mathcal{H}_{\mathscr{J}}})$. We define $\mathcal{H}_P^* := \mathcal{H}_{\mathscr{J}}$. Then the remaining part of Proposition 1.2 follows directly from Propositions 6.8 and 6.9. $\square$

7. The core entropy for postcritically-finite Newton maps

In this section, we summarize the construction of *extended Newton graphs* given in [LMS1], define the core entropy for postcritically-finite Newton maps, and give a formula to compute the core entropy. Throughout this subsection, the map f denote a postcritically-finite Newton map of degree $d \geq 3$.

7.1. **Newton graphs and the induced puzzles.** Let $B_1, \ldots, B_d$ are the fixed Fatou components of f, with centers $a_1, \ldots, a_d$, respectively. Let $\{(B_i, \Phi_i)_{1 \leq i \leq d}\}$ be the Böttcher coordinates of $B_1, \ldots, B_d$. For each B_k, the $d_k - 1$ fixed internal rays

$$I_{B_k}\left(\frac{i}{d_k - 1}\right), 1 \leq i \leq d_k - 1,$$

must land at fixed points in ∂B_k, which can only possible be ∞. Thus these rays possesses a common landing point. The union of all these $\sum_k (d_k - 1)$ fixed internal rays together with their landing point ∞, usually denoted by Δ_0:

$$\Delta_0 := \bigcup_{k=1}^{d} \bigcup_{i=1}^{d_k - 1} \overline{I_{B_k}\left(i/(d_k - 1)\right)}.$$

The set Δ_0 is called the *channel diagram* of f, following the notation in [HSS]. Clear $f(\Delta_0) = \Delta_0$. For any $n \geq 0$, denote by Δ_n the connected component of $f^{-n}(\Delta_0)$ that contains Δ_0. Following [LMS1], we call Δ_n the *Newton graph* of f at level n. The vertex set $V(\Delta_n)$ of Δ_n consists of the points in Δ_n which are iterated by f to the fixed points.

Lemma 7.1 ([MRS],Theorem 3.4). *There exists $N \geq 0$ such that the Newton graph Δ_N contains all poles of f. Then $\Delta_{n+1} = f^{-1}(\Delta_n)$ and $\Delta_n \subseteq \Delta_{n+1}$ for any $n \geq N$.*

By this lemma, the Newton graphs naturally induce a puzzle as follows. Let N be the minimal number such that Δ_N contains all poles of f. For each $k \geq 0$, let $\mathcal{W}_k$ denote the collection of all connected components of $\overline{\mathbb{C}} \setminus \Delta_{N+k}$. We call $\mathcal{W} = \cup_{k \geq 0} \mathcal{W}_k$ the *Δ-puzzle of f*, or *$\Delta(f)$-puzzle*, and call each element of $\mathcal{W}_k$ a *Δ-puzzle piece (of f) at level k*. By Lemma 7.1, we know that

   (1) each Δ-puzzle piece is a simply-connected domain;

(2) two distinct Δ-puzzle pieces are either disjoint or nested;

(3) each Δ-puzzle piece of level $k + 1$ ($k \geq 0$) is contained in a Δ-puzzle piece of level k;

(4) each Δ-puzzle piece of level $k + 1$ ($k \geq 0$) is a connected component of the preimage by f of a Δ-puzzle piece of level k.

7.2. Puzzle-renormalization of Newton maps.

We call $\rho = (f^p, W, W')$ a *puzzle-renormalization triple* (of period p) if the following properties hold:

(1) $W \subseteq W'$ are distinct Δ-puzzle pieces;

(2) $f^i(W), 0 \leq i \leq p-1$, are pairwise disjoint Δ-puzzle pieces, $f^p(W) = W'$, and $\deg(f^p|_W) \geq 2$;

(3) the *filled-in Julia set* of ρ, defined as
$$K_\rho := \{z \in W \mid f^n(z) \in W \text{ for all } n \geq 0\}$$
is connected.

Note that the definition of puzzle-renormalization is different from that of renormalization given in Definition 2.15, because it is possible that either $\overline{W} \not\subseteq W'$ (although $W \subseteq W'$) or W, W' are not disks (although they are simply-connected). However, the following result shows that puzzle-renormalization implies renormalization.

Proposition 7.2 ([LMS1], Lemma 4.19). *Let (f^p, W, W') be a puzzle-renormalization triple. Then there exists a pair of Jordan domain $U \subseteq V$, such that (f^p, U, V) is a renormalization triple of f in the sense of Definition 2.15, and the filled-in Julia set of (f^p, U, V) equals to that of (f^p, W, W').*

Notice that, by definition, the filled-in Julia set of any puzzle-renormalization triple is disjoint from the Newton graphs Δ_n for all $n \geq 0$. So the filled-in Julia sets of different puzzle-renormalization triples either coincides or are disjoint. On the other hand, let $\rho_0 = (f^p, W, W')$ be a puzzle-renormalization triple of period p, such that the level of W' is larger than p. Then, for each $0 \leq i \leq p$, the triple $\rho_i := (f^p, f^i(W), f^i(W'))$ is also a puzzle-renormalization triple with its filled-in Julia set K_{ρ_i} equal to $f^i(K_{\rho_0})$. Therefore, the distinct filled-in Julia sets of all puzzle-renormalization triples are pairwise disjoint, and can be divided into finitely many orbits under the iteration of f.

Let $\rho = (f^p, W, W')$ be any puzzle-renormalization triple. By Proposition 7.2, we can define an *canonical extend Hubbard tree* H_ρ of ρ as the regular convex hull within K_ρ of all critical/postcritical points and fixed points of $f^p|_{K_\rho}$ (following [LMS1]). It is clear that $f(H_{\rho_i}) \subseteq H_{\rho_{i+1}}$ with $\rho_i := f^i(\rho)$ for $i = 0, \ldots, p-1$.

A critical/postcritical point of f is called *free* if it is not iterated by f to the fixed points. So the free critical/postcritical points are disjoint with Newton graphs of any level. Let c be a free critical/postcritical point of f. If there exists a puzzle-renormalization triple ρ such that $f^k(c) \in H_\rho$ for some minimal $k \geq 0$, let H_c be the component of $f^{-k}(H_\rho)$ containing c; otherwise we define $H_c = \{c\}$. It is obvious that, for any two free critical/postcritical points c, c', the trees H_c and $H_{c'}$ either coincide or are disjoint.

Definition 7.3 (canonical Hubbard forest). *We define $\mathcal{H}_f$ as the union of H_c (constructed above) for all free critical/postcritical points of f, and call it the* canonical (puzzle-renormalization) Hubbard forest *of f.*

The vertex set $V(\mathcal{H}_f)$ of $\mathcal{H}_f$ consists of the endpoints and branched points of $\mathcal{H}_f$, together with the free critical/postcritical points of f. Then the Hubbard forest $\mathcal{H}_f$ is f-invariant, and

$f : \mathcal{H}_f \to \mathcal{H}_f$ is Markov. Note also that every non-trivial periodic component of $\mathcal{H}_f$ is the canonical extended Hubbard tree of a puzzle renormalization triple.

Lemma 7.4. *Let Δ_n be the Newton graph of level $n \geq 0$, and H a non-trivial, periodic component of $\mathcal{H}_f$ with period p. Then there exists a renormalization triple (f^p, U_H, V_H) (in the sense of Definition 2.15) such that $V_H \cap \Delta_n = \emptyset$ and H is an extended Hubbard tree of the renormalization triple.*

Proof. It follows directly from Proposition 7.2 and the fact that the filled-in Julia set of any puzzle-renormalization triple of f is disjoint with the Newton graph of any level. $\qquad\square$

7.3. **The construction of extended Newton graphs.** By now we have two disjoint f-invariant objectives: one is a Newton graph Δ_n with Δ_n containing all poles of f; and the other one is the canonical Hubbard forest $\mathcal{H}_f$ of f. To obtain a connected f-invariant graph, one need to find some invariant arcs joining the two objectives together. These invariant arcs are constructed in [LMS1] as *Newton rays*.

A continuous injective map $\gamma : [0, 1) \to \overline{\mathbb{C}}$ is called a *ray*. Let Δ_n ($n \geq 0$) be any Newton graph of f. A ray $\gamma : [0, 1) \to \overline{\mathbb{C}}$ is called a *Newton ray* (with respect to Δ_n) if $\gamma(0) \in \Delta_n$, $\gamma(0, 1) \cap \Delta_n = \emptyset$ and $\gamma(t)$ is iterated to Δ_0 by f for any $t \in [0, 1)$. In other words, a Newton ray is a ray with image in $(\cup_{m \geq 0} \Delta_m) \setminus \Delta_n$ except its starting point. The sub-hyperbolicity of f implies that any Newton ray converges to a unique point as t tends to 1. We say that γ *lands* on this point.

A Newton ray γ with respect to Δ_n is said to be *periodic* if there exists an integer $m \geq 1$ such that $f^m(\gamma) = \gamma \cup \mathcal{E}$ with $\mathcal{E}$ a subset of Δ_n. The smallest such m is called the *period* of γ.

Lemma 7.5 ([LMS1], Lemma 4.14). *For any sufficiently large n, the Newton graph Δ_n satisfies the following two properties:*

(1) *all poles and the non-free critical/postcritical points of f are contained in Δ_n;*
(2) *any two components of $\mathcal{H}_f$ lie in different complementary components of Δ_n.*

Let $N = N_f$ be the minimal integer such that the Newton graph Δ_N satisfies the two properties in Lemma 7.5. Then Δ_N is called the *canonical Newton graph* of f. Note that the canonical Newton graph and the canonical Hubbard forest are uniquely determined by f.

Roughly speaking, an extended Newton graph of f consists of three parts: the canonical Newton graph Δ_f, the canonical Hubbard forest $\mathcal{H}_f$, and finitely many preperiodic Newton rays which join each compnent of $\mathcal{H}_f$ to Δ_f. We do not intend to write the specific construction of an extended Newton graph here (one can refer to the proof of [LMS1, Theorems 6.2]). Instead, we list some properties of extended Newton graphs that will be used below.

Proposition 7.6 ([LMS1], Theorem 6.2). *Let $G = G_f$ be an extended Newton graph of a given Newton map f. Then the pair (G, f) satisfies the following properties.*

(1) *The graph G is a finite connected graph consisting of the canonical Newton graph, the canonical Hubbard forest of f and several preperiodic Newton rays. As a consequence, all critical/postcritical points of f belong to G.*
(2) *The vertex set of G consists of the vertices of the canonical Newton graph and the vertices of the canonical Hubbard forest. In other words, each Newton ray in G is an edge of G with one endpoint in $V(\Delta_f)$ and the other one in $V(\mathcal{H}_f)$.*
(3) *The edge set of G consists of: the edges of the canonical Newton graph, the edges of the canonical Hubbard forest, and the Newton rays in G.*

(4) *The graph G is f-invariant and $f|_G : G \to G$ is a Markov map.*

According to the proposition, the only vague and possibly non-unique part of an extended Newton graph comes from the Newton rays. In fact, the Newton rays satisfying the properties of Proposition 7.6 are not unique.

7.4. **The entropy formula for postcritically-finite Newton maps.** Let f be a given postcritically-finite Newton map of degree $d \geq 3$, and G be an extended Newton graph of f. Let $\Delta = \Delta_f$ be the canonical Newton graph of f, $\mathcal{H} = \mathcal{H}_f$ be the canonical Hubbard forest, and $\mathcal{R} = \mathcal{R}(G)$ the union of Newton rays contained in G. We know from Proposition 7.6 that G is the disjoint union of $\Delta, \mathcal{H}, \mathcal{R}$ except the vertices, $f(\Delta) \subseteq \Delta$, $f(\mathcal{R}) \subseteq \Delta \cup \mathcal{R}$ and $f(\mathcal{H}) \subseteq \mathcal{H}$. Recall we stipulate in the paper that $h_{top}(f|_\emptyset) := 0$.

Lemma 7.7. *The topological entropy $h_{top}(f|_G)$ equals to $h_{top}(f|_\mathcal{H})$.*

Proof. Note that $f(\mathcal{H}) \subseteq \mathcal{H}$ and $f(\Delta \cup \mathcal{R}) \subseteq \Delta \cup \mathcal{R}$, by Propositions 2.2, 2.3, we have
$$h_{top}(f|_G) = \max\{h_{top}(f|_{\Delta \cup \mathcal{R}}), h_{top}(f|_\mathcal{H})\}.$$
Let $D_{\Delta \cup \mathcal{R}}$ denote the incidence matrix of $(\Delta \cup \mathcal{R}, f)$. Since Δ is f-invariant, then the matrix $D_{\Delta \cup \mathcal{R}}$ has the form
$$\begin{pmatrix} D_\mathcal{R} & O \\ * & D_\Delta \end{pmatrix},$$
where D_Δ denotes the incidence matrix of (Δ, f) and O denotes a zero matrix. As each Newton ray in $\mathcal{R}$ is preperiodic with respect to Δ, and any two rays in $\mathcal{R}$ are disjoint except at their endpoints, then the leading eigenvalue of $D_\mathcal{R}$ is 1. It follows that $h_{top}(f|_{\Delta \cup \mathcal{R}}) = h_{top}(f|_\Delta)$. Note also that Δ is finally iterated to the channel diagram Δ_0 and $f : \Delta_0 \to \Delta_0$ is a homeomorphism, then $h_{top}(f|_{\Delta \cup \mathcal{R}}) = h_{top}(f|_\Delta) = h_{top}(f|_{\Delta_0}) = 0$. $\square$

By this lemma, although the construction of extended Newton graphs G is not unique, the topological entropy $h_{top}(f|_G)$ is independent on the choices of G since $\mathcal{H}$ is uniquely determined by f. Therefore, the following definition is well-defined.

Definition 7.8 (core entropy of Newton map). *Let f be a postcritically-finite Newton graph, and G an extended Newton graph of f. The core entropy $h(f)$ of f is defined as the topological entropy of the restriction of f on G, i.e., $h(f) = h_{top}(f|_G)$.*

In the following, we will develop some precise formulas to compute the core entropy of Newton maps. A collection $\mathcal{O} = \{H_i, 0 \leq i \leq p-1\}$ of components of $\mathcal{H}_f$ is called a *cycle in* $\mathrm{Comp}(\mathcal{H}_P)$ *of period* p if p is the minimal number such that $f(H_i) \subseteq H_{i+1}$ for each $0 \leq i \leq p-1$ and $H_p := H_0$. Let $\mathcal{H}_\mathcal{O}$ denote the union of $H_0, \ldots, H_{p-1}$. Then, by Propositions 2.2 and 2.3, we have the entropy formula
$$h(f) = \max_{\mathcal{O} \subseteq \mathrm{Comp}(\mathcal{H}_f)} h_{top}(f|_{\mathcal{H}_\mathcal{O}}). \tag{7.1}$$

Lemma 7.9. *Let $\mathcal{O} = (H_0, \ldots, H_{p-1})$ be a cycle in $\mathrm{Comp}(\mathcal{H}_f)$ of period p. Then for any element H of $\mathcal{O}$, we have*
$$h_{top}(f|_\mathcal{O}) = h_{top}(f^p|_H)/p.$$

Proof. By Propositions 2.1 and 2.2, we have
$$h_{top}(f|_{\mathcal{H}_O}) = \frac{1}{p} h_{top}(f^p|_{\mathcal{H}_\mathcal{O}}) = \frac{1}{p} \max_{0 \leq i \leq p-1} h_{top}(f^p|_{H_i}).$$

So it is enough to prove that $h_{top}(f^p|_{H_i}) = h_{top}(f^p|_{H_{i+1}})$ for every $0 \leq i \leq p-1$. Let $0 \leq i \leq p-1$. We have the commutative diagram

$$
\begin{array}{ccc}
H_i & \xrightarrow{f^p} & H_i \\
f \downarrow & & \downarrow f \\
H_{i+1} & \xrightarrow{f^p} & H_{i+1}.
\end{array}
$$

Let $\widetilde{H}_i$ be the component of $f^{-1}(H_{i+1})$ containing H_i. Then the commutative graph

$$
\begin{array}{ccc}
\widetilde{H}_i & \xrightarrow{f^p} & \widetilde{H}_i \\
f \downarrow & & \downarrow f \\
H_{i+1} & \xrightarrow{f^p} & H_{i+1}
\end{array}
$$

holds. In this case $f : \widetilde{H}_i \to H_{i+1}$ is surjective and $\#f^{-1}(y)$ is uniformly bounded for all $y \in H_{i+1}$. According to Proposition 2.4, we have $h_{top}(f^p|_{\widetilde{H}_i}) = h_{top}(f^p|_{H_{i+1}})$.

By Lemma 7.4, there exists a renormalization triple (f^p, U_{H_i}, V_{H_i}) such that H_i, hence $\widetilde{H}_i$, are extended Hubbard trees of this triple. Then, following Straightening Theorem and Lemma 2.10, the topological entropies $h_{top}(f^p|_{H_i})$ and $h_{top}(f|_{\widetilde{H}_i})$ are equal. □

For each periodic component H of $\mathcal{H}_f$, we denote p_H the period of H. In each cycle $\mathcal{O}$ in $\mathrm{Comp}(\mathcal{H}_f)$, we assign any element $H_{\mathcal{O}}$. By Lemma 7.9, we obtain two finer formulas for $h(f)$ as

$$
h(f) \quad = \quad \max_{\substack{H \in \mathrm{Comp}(\mathcal{H}_f) \\ \mathrm{non-trivial} \\ \mathrm{periodic}}} \frac{1}{p_H} h_{top}(f^{p_H}|_H) \tag{7.2}
$$

$$
= \quad \max_{\substack{\mathcal{O} \subseteq \mathrm{Comp}(\mathcal{H}_f) \\ \mathrm{non-trivial\ cycle}}} \frac{1}{p_{H_{\mathcal{O}}}} h_{top}(f^{p_{H_{\mathcal{O}}}}|_{H_{\mathcal{O}}}). \tag{7.3}
$$

Let H be a non-trivial periodic component of $\mathcal{H}_f$ of period p. By Lemma 7.4 and Straightening Theorem, there exist a postcritically-finite monic, centered polynomial P_H of degree $\delta := \deg(f^p|_{K_\rho})$ such that $f^p : U_H \to V_H$ is hybrid conjugate to P_H. Such polynomial P_H is unique up to conjugation by $\delta - 1$ roots of unit, and called a *renormalization polynomial* of f (associated to H). We denote by

$$\mathfrak{R}(f) := \{P_H : H \text{ is a non-trivial periodic component of } \mathcal{H}_f\}.$$

The map f is called *(puzzle-)renormalizalbe* if $\mathfrak{R}(f) \neq \emptyset$, and *non-renormalizable* otherwise.

Proof of Proposition 1.1. It follows directly from Lemma 7.7, equation (7.2) and the fact of $h(P_H) = h_{top}(f^p|_H)$ for any non-trivial periodic component of $\mathcal{H}_f$. □

7.5. Capture surgery for postcritically-finite Newton maps.

Let f be a postcritically-finite Newton map, and $\vec{s} = (\mathbf{c}, \mathbf{D}, \mathbf{Z}, \mathbf{X})$ a perturbation data for capture surgery on f defined in Section 3. To obtain an expected perturbation, we restrict two further properties on $\vec{s}$:

- the critical points in $\mathbf{c}$ are not iterated to ∞, i.e., all critical points in $\mathbf{c}$ belong to $\mathcal{H}_f$;
- the critical values of each χ_c in $\mathbf{X}$ are the iterated preimages of the fixed critical points of f.

The first property means that we only perturb some free critical points of f, and the second one ensures that the perturbation map $F_{\vec{s}}$ is a "topological postcritically-finite Newton map".

Lemma 7.10. *Let $F = F_{\vec{\mathbf{s}}}$ be the topological perturbation of f by capture surgery with data $\vec{\mathbf{s}}$. Then F is c-equivalent to a postcritically-finite Newton map.*

Proof. By the second property above and Proposition 2.16, any rational map c-equivalent to F must be a postcritically-finite Newton maps up to conformal conjugation. So, by Theorem 3.1, we just need to check that the marked map $(F, \mathcal{Q})$ has no Thurston obstructions for some marked set $\mathcal{Q}$ of F.

Let Δ_N be the Newton graph of level N such that all critical/postcritical points of f not in $\mathcal{H}_f$ belong to Δ_N, and different components of $\mathcal{H}_f$ are contained in different components of $\mathbb{C} \setminus \Delta_N$. Then $F = f$ outside the components of $\mathbb{C} \setminus \Delta_N$ at which the surgery happens. We define the marked set $\mathcal{Q}$ as the union of the vertex set of Δ_N and Post_F.

On the contrary, let $\widetilde{\Gamma}$ be a Thursotn obstruction of $(F, \mathcal{Q})$, then there exists a sub-multicurve $\Gamma \subseteq \widetilde{\Gamma}$, such that

- the leading eigenvalue $\lambda(\Gamma)$ of the transition matrix of Γ is at least 1;
- for each curve $\gamma \in \Gamma$, there exists a component in $F^{-1}(\Gamma)$ homotopic to γ in $\overline{\mathbb{C}} \setminus \mathcal{Q}$;
- for each curve $\gamma \in \Gamma$, there exists a component in $F^{-1}(\gamma)$ which is homotopic to an element of Γ in $\overline{\mathbb{C}} \setminus \mathcal{Q}$.

We assume that for each $\gamma \in \Gamma$, $\#(\gamma \cap \Delta_N)$ is minimal in its homotopic class in $\overline{\mathbb{C}} \setminus \mathcal{Q}$. For each $0 \le i \le N$, let Γ_i consists of all curves in Γ which intersect Δ_i, but not intersect $\Delta_{i-1}, \dots, \Delta_0$, and let Γ_∞ consists the curves in Γ disjoint with Δ_N. Then the multicurve Γ equals to the disjoint union of $\Gamma_0, \dots, \Gamma_N, \Gamma_\infty$.

Note that for each $1 \le i \le N$, any component in $F^{-1}(\Gamma_i)$ can not be homotopic in $\overline{\mathbb{C}} \setminus \mathcal{Q}$ to a curve in $\Gamma_0, \dots, \Gamma_i$, and any component in $F^{-1}(\Gamma_\infty)$ can not be homotopic in $\overline{\mathbb{C}} \setminus \mathcal{Q}$ to a curve in $\cup_{i=0}^N \Gamma_i$ (by the minimality of $\#(\Gamma \cap \Delta_N)$). Then the incidence matrix of Γ has the lower-triangle form

$$
\begin{pmatrix}
D_0 & & & & \\
* & O & & & \\
\vdots & \ddots & \ddots & & \\
* & \cdots & * & O & \\
* & \cdots & \cdots & * & D_\infty
\end{pmatrix},
$$

where D_0, D_∞ denote the incidence matrix of Γ_0, Γ_∞ respectively, and O denotes the zero metric. To deduce a contradiction, we will show that $\lambda_F(D_0), \lambda_F(D_\infty) < 1$.

We first suppose $\lambda_F(D_\infty) \ge 1$. Since the curves in Γ_∞ are disjoint with Δ_N, for each non-trivial component H of $\mathcal{H}_f$, we denote by Γ_H the curves in Γ_∞ belonging to the components of $\mathbb{C} \setminus \Delta_N$ that contains H. Since $\lambda_F(D_\infty) \ge 1$, then there exists a non-trivial periodic component H of $\mathcal{H}_f$ with period p, such that $\lambda_{F^p}(\Gamma_H) \ge 1$. If the surgery does not happens in the orbit of H, then F^p induces a polynomial-like map in a neighborhood of H, and hence $\lambda_{F^p}(\Gamma_H) < 1$, a contradiction. Otherwise, the map F^p induces a topological polynomial-like map in a neighborhood of H, which is exactly a topological perturbation by capture surgery of a polynomial-like map induced by f^p near H. By Lemma 3.2, we have $\lambda_{F^p}(\Gamma_H) < 1$, also a contradiction.

We then assume that $\lambda_F(D_0) \ge 1$. Keep in mind that $\#(\gamma \cap \Delta_0)$ is minimal for each $\gamma \in \Gamma_0$. Since $F : \Delta_0 \to \Delta_0$ is bijective, the number $0 < \#(\gamma \cap \Delta_0) < \infty$ is a constant for $\gamma \in \Gamma_0$, and then there exists at most one component of $F^{-1}(\gamma)$ homotopic in $\overline{\mathbb{C}} \setminus \mathcal{Q}$ to a curve in Γ_0 for each $\gamma \in \Gamma_0$. Thus for each $\gamma \in \Gamma_0$, there is one curve $\beta \in \Gamma$ such that $F^{-1}(\beta)$ has one component homotopic in $\overline{\mathbb{C}} \setminus \mathcal{Q}$ to γ (combining properties (2),(3) of Thurston obstruction). Therefore each

entry of the transition matrix D_0 is less than or equal to 1. Because $\lambda(\Gamma_0) \geq 1$, there is a Levy cycle in Γ_0. For simplicity of the statement, we assume that $\Gamma_0 = \{\gamma\}$ is the Levy cycle, and $\widetilde{\gamma}$ denote the component of $F^{-1}(\gamma)$ homotopic to γ in $\overline{\mathbb{C}} \setminus \mathcal{Q}$.

By the minimality of $\#(\gamma \cap \Delta_0)$, we have $\#(\widetilde{\gamma} \cap \Delta_0) \geq \#(\gamma \cap \Delta_0)$. On the other hand, since $F : \widetilde{\gamma} \to \gamma$ is a homeomorphism and $F(\Delta_0) = \Delta_0$, then $\#\widetilde{\gamma} \cap \Delta_0 \leq \#\gamma \cap \Delta_0$ and $F(\widetilde{\gamma} \cap \Delta_0) \subseteq \gamma \cap \Delta_0$. Combining these two aspects, we get that $F : \widetilde{\gamma} \cap \Delta_0 \to \gamma \cap \Delta_0$ is a homeomorphism. To deduce a contradiction, we only need to show that $\widetilde{\gamma}$ intersects $\Delta_1 \setminus \Delta_0$ (contradicting that $F : \widetilde{\gamma} \to \gamma$ is a homeomorphism).

Observe that γ must intersects the boundaries of the components of $\overline{\mathbb{C}} \setminus \Delta_N$ intersecting $\mathcal{H}_f$: otherwise γ is a Levy cycle of the marked map $(f, V(\Delta_N) \cup \mathrm{Post}_f)$ since $(F, \mathcal{Q}) = (f, V(\Delta_N) \cup \mathrm{Post}_f)$ outside these components, a contradiction. It implies that γ intersects $\Delta_j \setminus \Delta_{j-1}$ for some $1 \leq j \leq N$. By the minimality of $\#(\gamma \cap \Delta_N)$, the curve $\widetilde{\gamma}$ also intersects $\Delta_j \setminus \Delta_{j-1}$, and hence $\gamma = F(\widetilde{\gamma})$ intersects $\Delta_{j-1} \setminus \Delta_{j-2}$. Inductively, we have $\widetilde{\gamma}$ intersects $\Delta_1 \setminus \Delta_0$, a contradiction. $\square$

8. Continuity of the entropy function:postcritically-finite Newton family

A postcritically-finite Newton map is called *generic* if $\infty \notin \mathrm{Post}_f$. We will study the continuity of the entropy function within the postcritically-finite Newton family at generic parameters. Throughout this section, we always assume that f is a generic postcritically-finite Newton map, and $\{f_n, n \geq 1\}$ a sequence of postcritically-finite Newton map converging to f.

8.1. **Perturbation of Newton graphs.** The reason we require f generic is the following perturbation lemma.

Lemma 8.1 (perturbation of Newton graphs). *Fix any $m \geq 0$. Let Δ_m and Δ_m^n denote the Newton graphs of level m associated to f and f_n respectively. Then Δ_m is homeomorphic to Δ_m^n for all sufficiently large n and $\limsup_{n \to \infty} \Delta_m^n = \Delta_m$.*

Proof. Fix any $m \geq 0$. We first establish an injection $\xi_n : \Delta_m \to \Delta_m^n$ for all large n such that $\xi_n(\Delta_m) \to \Delta_m$ as $n \to \infty$.

Note that the vertices of Δ_m are iterated preimages of fixed points of f. Let $v \in V(\Delta_m)$. If $v \in \mathcal{F}_f$, it is a center of a Fatou component U of f. By Lemma 4.2 v has a unique continuation v_n at f_n such that v_n is the center of U_n, where U_n is the deformation of U at f_n. In this case, we define $\xi_n(v) =: v_n$. If $v \in \mathcal{J}_f$, since its orbit is supposed to avoid Crit_f, then v has a unique continuation v_n at f_n, and we define $\xi_n(v) =: v_n$. Thus we obtain an injection $\xi_n : V(\Delta_m) \to \mathbb{C}$ such that $\lim_{n \to \infty} \xi_n(V(\Delta_m)) = V(\Delta_m)$. Since $\xi_n(v)$ is the unique continuation of v for each vertex v of Δ_m, we have $f_n \circ \xi_n = \xi_n \circ f$ on $V(\Delta_m)$. It follows that $\xi_n(V(\Delta_m)) \subseteq V(\Delta_m^n)$ for all large n.

We begin to extend the map ξ_n to Δ_m. Let e be an edge of Δ_m with endpoints x, y. Then one point of x and y, say x, is the center of a Fatou component U of f, $y \in \partial U$ and e is the internal ray in U landing at y. We extend ξ_n homeomorphically on e such that its image is the internal ray in U_n landing at $\xi_n(y)$ where U_n denotes the deformation of U at f_n. Clearly, the map $\xi_n : \Delta_m \to \mathbb{C}$ is injective, and by Lemma 4.3, we have $\limsup_{n \to \infty} \xi_n(\Delta_m) = \Delta_m$. Notice that the edges of $\xi_n(\Delta_m)$ are all internal rays of f_n. Then, according to the formula $f_n \circ \xi_n = \xi_n \circ f$ on $V(\Delta_m)$, each edge of $\xi_n(\Delta_m)$ is mapped by f_n^m to an fixed internal ray in some fixed Fatou component of f_n. It means that $\xi_n(\Delta_m) \subseteq \Delta_m^n$.

To complete the proof of the lemma, we remain to show $\xi_n(\Delta_m) = \Delta_m^n$. On the contrary, by taking subsequences, we assume that there exists a vertex v of Δ_m and an vertex $u_n \in$

$V(\Delta_m^n) \setminus \xi_n(\Delta_m)$ such that $\xi_n(v)$ and u_n are the endpoints of an edge e_n of Δ_m^n. By Lemma 4.3, the edge e_n converge to an internal ray e of f which joins v and $u = \lim_{n\to\infty} u_n$. Since $u_n \in V(\Delta_m^n)$, then $f_n^m(u_n)$ is a fixed point of f_n. It follows that $f^m(u)$ is a fixed point of f. Combining the fact that $\Delta_m \cup e$ is connected, we get $u \in V(\Delta_m)$, and hence $u_n \in \xi_n(V(\Delta_m))$, a contradiction to the choice of u_n. $\qquad\square$

As a consequence, we see that for any $\Delta(f)$-puzzle piece W of level $k \geq 0$, there exists a unique $\Delta(f_n)$-puzzle piece W_n of level k such that $\overline{W_n} \to \overline{W}$ as $n \to \infty$.

Definition 8.2 (deformation of puzzle pieces). *Under the notations above, the $\Delta(f_n)$-puzzle piece W_n is called the* deformation of W at f_n.

8.2. **Continuity of entropy function in non-renormalizable case.** In this subsection, we prove that the entropy function is continuous at generic, non-renormalizable parameters.

Lemma 8.3. *Let T_n be a non-trivial periodic component of the Hubbard forest $\mathcal{H}_{f_n}$ for all large n. Assume that a critical c_n of f_n belong to T_n and $\lim_{n\to\infty} c_n = c$. Then, if c does not belong to a periodic component of $\mathcal{H}_f$, the period p_n of T_n goes to infinity.*

Proof. On the contrary, assume that p_n is uniformly bounded above. Then there exists an integer M such that $f_n^M(T_n) \subseteq T_n$ for all $n \geq 0$. Set $b_n := f_n^M(c_n) \in T_n$ and $b := \lim_{n\to\infty} b_n$. Then $f^M(c) = b$. By increasing M if necessary, we assume that b is periodic. Hence $b \neq c$ since c is not period.

Let Δ_N, Δ_N^n be the Newton graphs of f and f_n respectively, with level N, such that all critical/postcritical points of f not in $\mathcal{H}_f$ belong to Δ_N, and different components of $\mathcal{H}_f$ lie in different components of $\mathbb{C} \setminus \Delta_N$. If $c \in \Delta_N$ the point b must be a fixed point of f. Since f is generic, then $b \neq \infty$ and hence a fixed critical point of f. In this case, the point b_n is also a fixed critical point of f_n by Lemma 4.2. It contradicts that $b_n \in T_n \subseteq \mathcal{H}_{f_n}$. In the case of $c \in \mathcal{H}_f$, the points b, c belong to different components of $\mathcal{H}_f$, since c is not in a periodic component of $\mathcal{H}_f$ (by the assumption of the lemma) but b is (because b is period). Then the graph Δ_N separates b, c. It follows from Lemma 8.1 that the graphs Δ_N^n also separate b, c for all large n. On the other hand, note that T_n contains b_n and c_n which converges to b and c respectively, then T_n intersects Δ_N^n for all large n. It also contradicts to $T_n \subseteq \mathcal{H}_{f_n}$ (since $\mathcal{H}_{f_n} \cap \Delta_N^n = \emptyset$). $\qquad\square$

Proposition 8.4. *Let f be a generic, non-renormalizable postcritically-finite Newton map. Then the entropy function $h : \mathcal{N}_d^{\mathrm{pf}} \to \mathbb{R}$ is continuous at f.*

Proof. Let $\{f_n, n \geq 1\}$ be an arbitrary sequence of postcritically-finite Newton map converging to f. Since $h(f) = 0$ (by Proposition 1.1), we just need to show $\lim_{n\to\infty} h(f_n) = 0$. Without loss of generality, we can assume that all f_n are renormalizable.

We first claim that, for any non-trivial periodic component T_n of $\mathcal{H}_{f_n}$ with period p_n, the topological entropy $h_{top}(f_n^{p_n}|_{T_n})$ is bounded by $(2d-2)\log d$. To see this point, let $\rho_n = (f_n^{p_n}, U_n, V_n)$ be a renormalization triple such that T_n is an extended Hubbard tree of ρ_n (Lemma 7.4). Note that the orbit of the filled-in Julia set K_{ρ_n} covers the critical points of f_n at most $2d-2$ times, and the degree of $f_n|_{f_n^k(K_{\rho_n})}$ is at most d ($k \geq 0$), then the degree of $f_n^{p_n}|_{K_{\rho_n}}$ is bounded by d^{2d-2}. As a consequence, the topological entropy $h_{top}(f_n^{p_n}|_{T_n})$ is bounded by $(2d-2)\log d$.

Let T_n be any non-trivial periodic component of the $\mathcal{H}_{f_n}$, such that a critical point c_n of f_n belong to T_n and $c_n \to c$ as $n \to \infty$. Since f is postulated non-renormalizable, then c is not in a periodic component of $\mathcal{H}_f$ (in this case, any periodic component of $\mathcal{H}_f$ is a repelling periodic point). By Lemma 8.3, the period p_n of T_n converge to ∞. Combining the claim above, we have

$h_{top}(f_n^{p_n}|_{T_n})/p_n \to 0$ as $n \to \infty$. On the other hand, according to formula (7.3), the core entropy $h(f_n)$ equals to the maximum of $h_{top}(f_n^{p_n}|_{T_n})/p_n$ with T_n going though all periodic components of $\mathcal{H}_{f_n}$ containing critical points. Hence $h(f_n) \to 0$ as $n \to \infty$.   □

8.3. Continuity of the entropy function in renormalizable case.
In this part, we always assume that f is renormalizable, i.e., $\mathcal{H}_f$ contains non-trivial components.

By Lemma 7.5, we can choose, for each non-trivial periodic component H of $\mathcal{H}_f$ (with period p), a pair of $\Delta(f)$-puzzle pieces (W_H, W'_H) containing H such that (f^p, W_H, W'_H) is a puzzle-renormalization triple, and the $\Delta(f)$-puzzle pieces $\{W'_H : H \in \mathrm{Comp}(\mathcal{H}_f), \text{periodic and non-trivial}\}$ are pairwise disjoint.

In the dynamical plane of f_n, for each non-trivial periodic component H of $\mathcal{H}_f$, we denote by $W_{H,n}, W'_{H,n}$ the $\Delta(f_n)$-puzzle piece deformed from W_H, W'_H respectively (see Definition 8.2). Then the pairs of $\Delta(f_n)$-puzzle pieces $(W_{H,n}, W'_{H,n})$ satisfy the following properties:

(1) the $\Delta(f_n)$-puzzle pieces $\{W'_{H,n} : H \in \mathrm{Comp}(\mathcal{H}_f), \text{periodic and non-trivial}\}$ are pairwise disjoint;

(2) for each non-trivial, periodic component H of $\mathcal{H}_f$ with period p, we have $W_{H,n} \subseteq W'_{H,n}$ and the map $f_n^p : W_{H,n} \to W'_{H,n}$ is a branched covering of degree equal to $\deg(f^p|_{W_H})$.

To compare the core entropy $h(f)$ and the limits of $h(f_n)$ as $n \to \infty$, we divide the canonical Hubbard forest $\mathcal{H}_{f_n}$ into two parts: let $\mathcal{H}_{\mathrm{bd}}^n$ denote the union of the components of $\mathcal{H}_{f_n}$ which stay in $\cup_H W_{H,n}$ under the iteration of f_n; and $\mathcal{H}_{\mathrm{esc}}^n$ denote the union of the periodic components of $\mathcal{H}_{f_n}$ not contained in $\mathcal{H}_{\mathrm{bd}}^n$, where the subscripts "bd" and "esc" mean "bounded" and "escape" respectively. It is clear that both $\mathcal{H}_{\mathrm{bd}}^n$ and $\mathcal{H}_{\mathrm{esc}}^n$ are f_n-invariant, and by Propositions 2.2 and 2.3, we have

$$h(f_n) = \max\{\ h_{top}(f_n|_{\mathcal{H}_{\mathrm{bd}}^n}), h_{top}(f_n|_{\mathcal{H}_{\mathrm{esc}}^n})\ \}. \tag{8.1}$$

Lemma 8.5. *Let p_n denote the minimal period among the non-trivial components of $\mathcal{H}_{\mathrm{esc}}^n$, then $p_n \to \infty$ as $n \to \infty$. As a consequence, we have $h_{top}(f_n|_{\mathcal{H}_{\mathrm{esc}}^n}) \to 0$ as $n \to \infty$.*

Proof. On the contrary, by taking subsequences if necessary, we assume that all p_n have a uniform upper bound. Without loss of generality, we can assume $p_n = p$ for all n. Let T_n be a component of $\mathcal{H}_{\mathrm{esc}}^n$ with $f^p(T_n) \subseteq T_n$. Note that at least one of $T_n, f(T_n), \ldots, f^{p-1}(T_n)$ contains a critical point of f_n. So, without loss of generality and by taking subsequences if necessary, we can assume that T_n contains a critical point c_n of f_n and c_n converge to a critical point c of f.

By Lemma 8.3, the point c must belong to a non-trivial periodic component H of $\mathcal{H}_f$. Then $c \in W_H$. Let $\widetilde{W}_H \subseteq W_H$ be a $\Delta(f)$-puzzle piece containing H such that level difference between $\widetilde{W}_H$ and W_H is sufficiently large. It follows that, for each $0 \le i \le p$, the $\Delta(f)$-puzzle piece $f^i(\widetilde{W}_H)$ belongs to W_{H_i}, where H_i is the component of $\mathcal{H}_f$ which contains $f^i(H)$. Let $\widetilde{W}_{H,n}$ denote the deformation of $\widetilde{W}_H$ for large n. Since $c_n \to c$ and $T_n \cap \partial \widetilde{W}_{H,n} = \emptyset$, the tree T_n is contained in $\widetilde{W}_{H,n}$ for all large n. Moreover, the $\Delta(f_n)$-puzzle piece $f_n^i(\widetilde{W}_{H,n})$ belongs to $W_{H_i,n}$ for each $0 \le i \le p$ and all large n. It implies that the entire orbit of T_n stay in $\cup_H W_{H,n}$, and hence $T_n \subseteq \mathcal{H}_{\mathrm{bd}}^n$, a contradiction. So we prove that $\lim_{n\to\infty} p_n = \infty$.

By the same argument as that in the proof of Lemma 7.9, we have

$$h_{top}(f_n|_{\mathcal{H}_{\mathrm{esc}}^n}) = \max_{T_n \in \mathrm{Comp}(\mathcal{H}_{\mathrm{esc}}^n)} \frac{1}{p_{T_n}} h_{top}(f_n^{p_{T_n}}|_{T_n}) \le \frac{1}{p_n} \max_{T_n \in \mathrm{Comp}(\mathcal{H}_{\mathrm{esc}}^n)} h_{top}(f_n^{p_{T_n}}|_{T_n}),$$

where p_{T_n} denote the period of T_n. Since $h_{top}(f_n^{p_{T_n}}|_{T_n}) \leq (2d-2)\log d$ (see the claim in the proof Proposition 8.4) and $p_n \to \infty$, then $\lim_{n\to\infty} h_{top}(f_n|_{\mathcal{H}_{esc}^n}) \to 0$. $\square$

For any non-trivial, periodic component H of $\mathcal{H}_f$, we define a forest

$$H_n := \mathcal{H}_{bd}^n \cap W_{H,n}. \tag{8.2}$$

It is easy to see that $f_n(H_n) \subseteq H'_n$ if and only if $f(H) \subseteq H'$ where H' is a non-trivial periodic component of $\mathcal{H}_f$. By a similar argument as previous, we have an entropy formula

$$h_{top}(f_n|_{\mathcal{H}_{bd}^n}) = \max_{\substack{H \in \mathrm{Comp}(\mathcal{H}_f) \\ \mathrm{non-trivial} \\ \mathrm{period}\ p_H}} \frac{1}{p_H} h_{top}(f_n^{p_H}|_{H_n}). \tag{8.3}$$

Lemma 8.6. *The limit/limit superior/limit inferior of $h(f_n)$ is equal to that of $\max_H \frac{1}{p_H} h_{top}(f_n^{p_H}|_{H_n})$ with H going through all non-trivial periodic components of $\mathcal{H}_f$, as $n \to \infty$.*

Proof. It follows directly from formulas (8.1), (8.3), and Lemma 8.5. $\square$

Let H be a non-trivial periodic component of $\mathcal{H}_f$ with period p. By Lemma 7.4, there exists a renormalization triple (f^p, U_H, V_H) such that $\overline{V_H} \subseteq W_H$, and $f^p : U_H \to V_H$ has the same filled-in Julia set as $f^p : W_H \to W'_H$. Therefore, by shrinking V_H, we can assume that $\overline{f^i(U_H)} \subseteq W_{H_i}$ for $0 \leq i \leq p$, where H_i denotes the component of $\mathcal{H}_f$ such that $f^i(H) \subseteq H_i$. As $f_n \to f$, then in the dynamical plane of f_n for large n, there exists a polynomial-like map $f_n^p : U_{H,n} \to V_{H,n}$ converging to $f^p : U_H \to V_H$ in the sense that $\overline{U_{H,n}} \to \overline{U_H}$ and $\overline{V_{H,n}} \to \overline{V_H}$. Notice also that $W_{H,n} \to W_H$ for each H, we then get a property for all large n:

$$\overline{f_n^i(U_{H,n})} \subseteq W_{H_i,n} \text{ for each } 0 \leq i \leq p. \tag{8.4}$$

Lemma 8.7. *Let $K_{H,n}$ denote the filled-in Julia set of the polynomial map $f_n^p : U_{H,n} \to V_{H,n}$. Then a point belongs to $K_{H,n}$ if and only if this point belongs to $W_{H,n}$ and its orbit stay in $\cup_H W_{H,n}$. As a consequence, we have that $H_n := \mathcal{H}_{bd} \cap W_{H,n}$ is an extended Hubbard forest of $(f_n^p, U_{H,n}, V_{H,n})$ (for large n).*

Proof. If $x \in K_{H,n}$, by property (8.4), we have $x \in W_{H,n}$ and its orbit belongs to $\cup_{i=0}^{p-1} W_{H_i,n}$. To prove the other side, since $f^p : W_H \to W'_H$ and $f^p : U_H \to V_H$ have the same filled-in Julia set, denoted as K_H, we can choose a $\Delta(f)$-puzzle piece $\widetilde{W}_H$ containing K_H, such that for each $0 \leq i \leq p$, the closure of $f^i(\widetilde{W}_H)$ is contained in U_{H_i}. Let $\widetilde{W}_{H,n}$ denote the $\Delta(f_n)$-puzzle piece deformed from $\widetilde{W}_H$. As $\widetilde{W}_{H,n} \to \widetilde{W}_H$ and $U_{H_i,n} \to U_{H_i}(0 \leq i \leq p-1)$, it follows that

$$f_n^i(\widetilde{W}_{H,n}) \subseteq U_{H_i,n} \text{ for each } 0 \leq i \leq p-1. \qquad (*)$$

Let $x \in W_{H,n}$ such that its orbit stay in $\cup_{i=0}^{p-1}(W_{H_i,n})$. Since the points in $W_{H,n} \setminus \widetilde{W}_{H,n}$ will eventually leave $W_{H,n}$ under the iteration of f_n^p, then $x \in \widetilde{W}_{H,n}$. It follows from property $(*)$ that the point x belongs to $K_{H,n}$.

By the result we just proved and the definition of H_n, we have that H_n is an f_n^p-invariant forest in $K_{H,n}$ and contains all bounded critical points of $f_n^p|_{U_{H,n}}$, hence an extended Hubbard forest of $(f_n^p, U_{H,n}, V_{H,n})$. $\square$

Now, let H be a non-trivial periodic component of $\mathcal{H}_f$, and (f^p, U_H, V_H) the renormalization triple near H given before Lemma 8.7. Recall that a monic, centered postcritically-finite polynomial P_H hybrid equivalent to (f^p, U_H, V_H) is called a renormalization polynomial of f

associated to H. By the discussion above, we also get a polynomial-like map $(f_n^p, U_{H,n}, V_{H,n})$ for every large n such that $(f_n^p, U_{H,n}, V_{H,n})$ converges to (f^p, U_H, V_H).

Definition 8.8 (perturbation of P_H). *A monic, centered polynomial $P_{H,n}$ hybrid equivalent to $f_n^p : U_{H,n} \to V_{H,n}$ is called a* straightening-perturbation polynomial of P_H at f_n.

Observe that the orbit of every bounded critical point of $(f_n^p, U_{H,n}, V_{H,n})$ is finite (since f_n is postcritically-finite) and H_n is an extended Hubbard forest of $(f_n^p, U_{H,n}, V_{H,n})$ (Lemma 8.7), then the polynomial $P_{H,n}$ is partial postcritically-finite, and $\varphi_{H,n}(H_n)$ is an extended Hubbard forest of $P_{H,n}$, where $\varphi_n : V_{H,n} \to \mathbb{C}$ is a quasi-conformal map realizing the hybrid equivalent between $(f_n^p, U_{H,n}, V_{H,n})$ and $P_{H,n}$. By a generalized version of Lemma 2.10, we have $h_{top}(f_n^p|_{H_n}) = h(P_{H,n})$. The following fact will be repeatedly used.

Lemma 8.9. *Let H be a non-trivial periodic component of $\mathcal{H}_f$ with period p, and H_n the forest defined in (8.2). Then we have $h_{top}(f^p|_H) = h_{top}(P_H)$ and $h_{top}(f_n^p|_{H_n}) = h(P_{H,n})$.*

Lemma 8.10. *Let H be a non-trivial periodic component of $\mathcal{H}$, with P_H a renormalization polynomial of f associated to H (Definition 2.15), and $P_{H,n}$ a straightening-perturbation of P_H at f_n (Definition 8.8). By suitably choice of $P_{H,n}$, we have $P_{H,n} \to P_H$ as $n \to \infty$.*

Proof. Let $\varphi : V_H \to \mathbb{C}, \varphi_n : V_{H,n} \to \mathbb{C}$ be two quasi-conformal maps, by which the polynomial-like maps $(f^p : U_H, V_H), (f_n^p : U_{H,n}, V_{H,n})$ are hybrid equivalent to $P_H, P_{H,n}$ respectively. We are able to extend the polynomial-like maps $f_n^p : U_{H,n} \to V_{H,n}$ and $f^p : U_H \to V_H$ to topological polynomials $F_n : \mathbb{C} \to \mathbb{C}$ and $F : \mathbb{C} \to \mathbb{C}$ respectively, and extend the quasi-conformal maps φ_n and φ to quasi-conformal maps on $\mathbb{C}$, also denoted by φ_n and φ, such that

- F_n and F are holomorphic and coincide in a neighborhood of ∞;
- φ_n and φ are conformal in a neighborhood of ∞ and normalized by fixing two given points in U_H (and also ∞);
- $\varphi_n \circ F_n = P_{H,n} \circ \varphi_n$ and $\varphi \circ F = P_H \circ \varphi$ on $\mathbb{C}$.

The convergence of $(f_n^p, U_{H,n}, V_{H,n})$ to (f^p, U_H, V_H) ensures that one can construct F_n, F, φ_n such that F_n uniform converge to F and all φ_n are K-quasi-conformal for some constant $K > 1$. Hence the collection of maps $\{\varphi_n, n \geq 1\}$ is a normal family.

Let $\varphi_{n_k}, k \geq 1$, be an arbitrary convergent subsequence in this family with the limit map ψ. Then the map ψ is K-quasiconformal on $\mathbb{C}$, conformal near ∞, and the polynomials $P_{H,n_k} = \varphi_{n_k} \circ F_n \circ \varphi_{n_k}^{-1}$ local-uniformly converge to a monic, centered polynomial $P := \psi \circ F \circ \psi^{-1}$ on $\mathbb{C}$. Note that $P_H = \varphi \circ F \circ \varphi^{-1}$. So P and P_H are quasi-conformal conjugate on $\mathbb{C}$, and conformal conjugate near ∞. According to Theorem 3.1, the monic, centered polynomials P and P_H are conjugate under a $(\delta - 1)$-th root of unit u with $\delta := \deg(P_H)$. Therefore, the monic, centered polynomials $\widetilde{P}_{H,n_k} := u \circ P_{H,n_k} \circ u^{-1}, n \geq 1$, converge to P_H. Since $\{\varphi_{n_k}, k \geq 1\}$ is arbitrary, then the conclusion of the lemma holds $\qquad\square$

Definition 8.11 (entropy-minimal sequence). *Let g be a partial postcritically-finite polynomial (resp. postcritically-finite Newton map), and $\{g_n, n \geq 1\}$ a sequence of partial postcritically-finite polynomials (resp. postcritically-finite Newton maps) converging to g. We say that the sequence is* entropy-minimal *if $\lim_{n \to \infty} h(g_n) = \liminf_{\widetilde{g} \to g} h(g)$, where $\widetilde{g}$ are chosen in the partial postcritically-finite polynomial family (resp. postcritically-finite Newton-map family).*

Proposition 8.12. *Let f be a postcritically-finite Newton map, and $H_1, \ldots, H_k$ a collection of non-trivial periodic components of $\mathcal{H}_f$ in pairwise distinct cycles. Let $P_{H_i}, i = 1, \ldots, k$, be a renormalization polynomial of f associated to H_i (Definition 2.15). Then we can construct a sequence of postcritically-finite Newton map $\{f_n, n \geq 1\}$ by perturbation of f with capture*

surgery, such that $f_n \to f$ and the polynomials $\{P_{H_i,n}, n \geq 1\}$ (defined in Definition 8.8) is a entropy-minimal sequence converging to P_{H_i} for each $1 \leq i \leq k$.

Proof. Note that different cycles in $\mathrm{Comp}(\mathcal{H}_f)$ are pairwise disjoint and the capture surgery only happens in sufficiently small neighborhood of each cycle, then, without loss of generality, we can assume that the surgery is carried out at one non-trivial cycle in $\mathrm{Comp}(\mathcal{H}_f)$. Let H be any element in this cycle of period p. We just need to construct a sequence of postcritically-finite Newton maps $\{f_n, n \geq 1\}$ converging to f such that the strengthening-perturbation polynomials $\{P_{H,n}, n \geq 1\}$ of P_H is an entropy-minimal sequence converging to P_H.

As previous, let $\rho = (f^p, W_H, W'_H)$ be a puzzle renormalization triple of f with H an extended Hubbard tree of ρ, and (f, U_H, V_H) a renormalization triple such that $\overline{V} \subseteq W_H$ and (f^p, U_H, V_H) has the same filled in Julia set as ρ (by Lemma 7.4). Denote by P_H a renormalization polynomial of f associated to H. Let $\varphi : V_H \to \mathbb{C}$ be a quasi-conformal map by which $f : U_H \to V_H$ is hybrid equivalent to P_H. We denote $U := \varphi(U_H)$ and $V := \varphi(V_H)$. Then $P_H : U \to V$ is a polynomial-like map with its filled-in Julia set equal to $\mathcal{K}_{P_H}$.

For each n, we construct a rational perturbation f_n of f by capture surgery as follows. Let $\{\widetilde{Q}_{H,n}, n \geq 1\}$ be a sequence of topological perturbation of P_H by capture surgery, such that

(1) the surgery domain is compactly contained in U;
(2) each escaping critical value of $\widetilde{Q}_{H,n}$ is the image by φ of an iterated preimage of a fixed critical point of f;
(3) the rational realization $\{Q_{H,n}, n \geq 1\}$ of $\{\widetilde{Q}_{H,n}, n \geq 1\}$ is an entropy-minimal sequence converging to P_H (by Proposition 6.9).

Each capture surgery above on P_H induces a topological perturbation F_n of f by capture surgery near the orbit $H, \ldots, f^{p-1}(H)$, such that $F_n^p(z) = \varphi^{-1} \circ \widetilde{Q}_{H,n} \circ \varphi(z)$ for all $z \in U_H$. By Lemma 7.10 and Proposition 3.3, each F_n is c-equivalent to a postcritically-finite Newton map f_n such that the sequence $\{f_n, n \geq 1\}$ converges to f.

For each n, let $(\phi_{n,o}, \phi_{n,1})$ be a pair of normalized homeomorphisms by which F_n is c-equivalent to f_n. According to the homotopy lifting lemma, we get a sequence of homeomorphism $\{\phi_{n,k}, k \geq 1\}$ such that $\phi_{n,k} \circ F_n = f_n \circ \phi_{n,k+1}$ on $\overline{\mathbb{C}}$ for each $k \geq 0$. Assume W_H has level k as a $\Delta(f)$ puzzle piece. We can choose $\phi_{n,0}$ in its isotopic class so that $\phi_{n,0} : \Delta_0(f) \to \Delta_0(f_n)$ is a homeomorphism and $\phi_{n,0} \circ f(= \phi_{n,0} \circ F_n) = f_n \circ \phi_{n,0}$ on $\Delta_0(f)$. Since f coincides with F_n outside $U_H, \ldots, f^{p-1}(U_H)$, it follows that $\phi_{n,k} : \Delta_k(f) \to \Delta_k(f_n)$ is a homeomorphism and $\phi_{n,k} \circ f = f_n \circ \phi_{n,k}$ on $\Delta_k(f)$. So we can assume the initial $\phi_{n,0}$ has this property.

Let $W_{H,n}$ and $W'_{H,n}$ be the image of W_H, W'_H by $\phi_{n,0}$. Then $(f_n^p, W_{H,n}, W'_{H,n})$ is the deformation of (f^p, W_H, W'_H) at f_n (in the sense of Definition 8.2). As $\overline{V_H} \subseteq W_H$, there exists polynomial-like maps $f_n^p : U_{H,n} \to V_{H,n}$ which converge to $f^p : U_H \to V_H$ and satisfy $\overline{V_{H,n}} \subseteq W_{H,n}$. Each excepted polynomial $P_{H,n}$ is hybrid equivalent to $f_n^p : U_{H,n} \to V_{H,n}$ by φ_n (see the following commutative graph).

$$
\begin{array}{ccccccccccc}
\mathbb{C} & \xleftarrow{\varphi_n} & U_{H,n} & \xhookrightarrow{} & W_{H,n} & \xleftarrow{\phi_{n,p}} & W_H & \xleftarrow{} & U_H & \xrightarrow{\varphi} & \mathbb{C} & \xrightarrow{\psi_{n,1}} & \mathbb{C} \\
\downarrow{\scriptstyle P_{H,n}} & & \downarrow{\scriptstyle f_n^p} & & \downarrow{\scriptstyle f_n^p} & & \downarrow{\scriptstyle F_n^p} & & \downarrow{\scriptstyle F_n^p} & & \downarrow{\scriptstyle \widetilde{Q}_{H,n}} & & \downarrow{\scriptstyle Q_{H,n}} \\
\mathbb{C} & \xleftarrow{\varphi_n} & V_{H,n} & \xhookrightarrow{} & W'_{H,n} & \xleftarrow{\phi_{n,0}} & W'_H & \xleftarrow{} & V_H & \xrightarrow{\varphi} & \mathbb{C} & \xrightarrow{\psi_{n,0}} & \mathbb{C}
\end{array}
\tag{8.5}
$$

where $(\psi_{n,0}, \psi_{n,1})$ is a pair of homeomorphisms by which $\widetilde{Q}_{H,n}$ is c-equivalent to $Q_{H,n}$.

Let T_n be the Hubbard forest of $Q_{H,n}$ and set $H_n := \varphi_n \circ \phi_{n,0} \circ \varphi^{-1} \circ \psi_{n,0}^{-1}(T_n)$. Using Lemma 8.7, we can see the points in H_n do not escape under the iteration of $P_{H,n}$, hence $H_n \subseteq \mathcal{K}_{P_{H,n}}$. By revising $\phi_{n,0}$ its isotopic class, we can assume H_n is regulated. Clearly, H_n contains all bounded critical/postcritical points of $P_{H,n}$, then H_n is the Hubbard forest of $P_{H,n}$. The combinatorial equivalence between $P_{H,n} : H_n \to H_n$ and $Q_{H,n} : T_n \to T_n$ implies $h(P_{H,n}) = h(Q_{H,n})$. Since the sequence $\{Q_{H,n}, n \geq 1\}$ is entropy-minimal, so is $\{P_{H,n}, n \geq 1\}$. $\qquad\square$

Proof of Theorem 1.3. Let $\{f_n, n \geq 1\}$ be an arbitrary sequence in $\mathcal{N}_d^{\mathrm{pf}}$ converging to f. We first claim that

$$\limsup_{n\to\infty} h_{top}(f_n^{p_H}|_{H_n}) \leq h_{top}(f^{p_H}|_H)$$

for any non-trivial periodic component H of $\mathcal{H}_f$, and H_n is defined in (8.2). By Lemma 8.9, the inequality above is equivalent to

$$\limsup_{n\to\infty} h(P_{H,n}) \leq h(P_H),$$

where P_H is a renormalization polynomial of f associated to H (Definition 2.15), and $P_{H,n}$ a straightening-perturbation of P_H at f_n (Definition 8.8). Now, this inequality follows directly from Lemma 8.10 and Proposition 1.2. Then the upper semi-continuous of h at f is duo to the claim above, formulas (7.2), and Lemma 8.6.

The entropy function is continuous at non-renormalizable parameters duo to Proposition 8.4. So we only need to deal with the renormalizable case.

Suppose first that h is continuous at f. According to Proposition 8.12, we can find a sequence of postcritically-finite Newton maps $\{f_n, n \geq 1\}$ by perturbing f with capture surgery such that $f_n \to f$, and the associated polynomials $\{P_{H,n}, n \geq 1\}$ (Definition 8.8) is an entropy-minimal sequence converging to P_H for every non-trivial periodic component H of $\mathcal{H}_f$ with $h(f) = h_{top}(f^{p_H}|_H)/p_H$. By the claim in first paragraph and Lemma 8.6, by taking subsequences if necessary, the continuity of h at f implies that there exists such a component H so that $\lim_{n\to\infty} h_{top}(f_n^{p_H}|_{H_n}) = h_{top}(f^{p_H}|_H)$. Combining Lemma 8.9, we have $\lim_{n\to\infty} h(P_{H,n}) = h(P_H)$. Note that $\{P_{H,n}, n \geq 1\}$ is an entropy-minimal sequence converging to P_H, then the entropy function $h : \mathcal{P}_\delta^{\mathrm{pf}} \to \mathbb{R}$ is continuous at P_H duo to Proposition 1.2, where $\delta := \deg(P_H)$.

On the other hand, suppose that H is a non-trivial periodic component of $\mathcal{H}_f$ with period p, such that $h(f) = h_{top}(f^p|_H)/p$ and the entropy function on $\mathcal{P}_\delta^{\mathrm{pf}}$ is continuous at the renormalization polynomial P_H associated to H, where $\delta := \deg(P_H)$. Let $\{f_n, n \geq 1\}$ be an entropy-minimal sequence in $\mathcal{N}_d^{\mathrm{pf}}$ converging to f. By Lemma 8.10, there exists a sequence $\{P_{H,n}, n \geq 1\}$ of straightening-perturbation polynomials of P_H at f_n converging to P_H. It follows that

$$h(f) = h_{top}(f^p|_H)/p = h(P_H)/p = \lim_{n\to\infty} h(P_{H,n})/p = \lim_{n\to\infty} h_{top}(f_n^p|_{H_n})/p \leq \liminf_{n\to\infty} h(f_n),$$

where the third equality is duo to the continuity of h at P_H, the second and fourth equality follows form Lemma 8.9, and the last inequality is by formula (7.2). Remember that the sequence $\{f_n, n \geq 1\}$ is entropy-minimal, then h is lower semi-continuous at f, and hence continuous since h is proved upper semi-continuous. $\qquad\square$

Proof of Theorem 1.5. Let f be a postcritically-finite cubic Newton map. We first show the "if" part. The conclusion is obvious if f is hyperbolic; and follows directly from Theorem 1.3 when f is generic and non-renormalizable. So it remains to show that h is continuous at f in the non-generic, non-renormalizable case, for which we need to use some particular results for cubic Newton maps given in [Ro].

Let $\{f_n, \geq 1\}$ be any sequence of postcritically-finite cubic Newton maps converging to f. Since $h(f) = 0$ (by the non-renormalizable property), one can assume that all f_n are renormalizable. Denote by p_n the period of the unique periodic component T_n of $\mathcal{H}_{f_n}$ which contains a critical point of f_n. Since $h(f_n) = h_{top}(f_n^{p_n}|_{T_n})/p_n$ and $h_{top}(f_n^{p_n}|_{T_n}) \leq \log 2$, we only need to prove $\lim_{n \to \infty} p_n = \infty$.

On the contrary, suppose all p_n have a uniform upper bound. Let c_n be the unique (free) critical point of f_n contained in T_n, with $c_n \to c$ as $n \to \infty$. By the assumption, there exist a integer p such that $f_n^p(T_n) \subseteq T_n$ for all large n. Let $b_n := f_n^p(c_n)$. Then $f^p(c) = b$. By increasing p if necessary, we can assume that b is period. Since f is postulated non-generic, then $b = \infty$.

By Lemmas 3.14, 3.15 and Corollary 3.16 in [Ro], there exists a Jordan curve γ consisting of the closure of finitely many preperiodic internal rays of f which are iterated to the fixed Fatou components of f, such that γ separates c and $b = \infty$, and the periods of the rays in γ are sufficiently large. Note that the landing point of the rays in γ are not pre-critical. Then, by a similar argument as that in Lemma 8.1, we can find a Jordan curve γ_n consisting of the closure of internal rays of f_n perturbed from the internal rays of f contained in γ. As $\gamma_n \to \gamma$, the curve γ_n also separates $b = \infty, c$, and hence separates b_n, c_n for all large n. It follows that the tree T_n intersects γ_n for every large n. Since T_n is disjoint with the iterated preimages of fixed Fatou components of f_n, the intersection of T_n and γ_n are landing points of the internal rays in γ_n. Notice that all internal rays in γ_n are iterated to fixed Fatou components, then T_n intersects the boundary of a fixed Fatou component U_n of f_n at a preperiodic point x_n. It follows that $f_n^p(x_n)$ also belongs to $T_n \cap \partial U_n$ and distinct with x_n by the large period of x_n. It contradicts [Ro, Lemma 6.3]: which says that the filled-in Julia set of any renormalization triple of a cubic Newton map intersects at most one point with the closure of a fixed Fatou component.

For the "only if" part, we need to show that if f is non-hyperbolic and renormalizable, then h is not continuous at f. Note that in this case, the map f must be generic, since its unique free critical point belongs to the filled-in Julia set of a renormalization triple. As f is not hyperbolic, the unique (quadratic) renormalization polynomial in $\mathfrak{R}(f)$ is Misiurewicz. However, the entropy function $h : \mathcal{P}_2^{\mathrm{ppf}} \to \mathbb{R}$ is not continuous at Misiurewicz parameters, because any Misiurewicz quadratic polynomial has positive core entropy (see [Do1]), but can be estimated by hyperbolic polynomials with Cantor Julia set (having 0 core entropy). Following Theorem 1.3, the function h is not continuous at f. $\square$

9. Continuity of entropy fucntion: partial postcritically-finite parameters

The objective here is to prove Theorem 1.6. Its proof is completely the same as that of Theorem 1.3: combining Proposition 1.2 and Straightening Theorem. In fact, every Proposition or Lemma in Sections 7 and 8 for Newton maps has a parallel version in partial postcritically-finite case with a similar argument. So we just state these parallel results without giving proofs, from which we deduce Theorem 1.6.

9.1. **The computation of the core entropy.**
Let P be a partial postcritically-finite polynomials with the Hubbard forest $\mathcal{H}_P$. Then $\mathcal{H}_P = \emptyset$ if and only if P is hyperbolic with Cantor Julia set. In this case the core entropy $h(P) = 0$.

In the case of $\mathcal{H}_P \neq \emptyset$, a collection $\mathcal{O} = \{H_i, 0 \leq i \leq p - 1\}$ of components of $\mathcal{H}_P$ is called a *cycle in* $\mathrm{Comp}(\mathcal{H}_P)$ *of period* p if p is the minimal number such that $P(H_i) \subseteq H_{i+1}$ for each $0 \leq i \leq p - 1$ and $H_p := H_0$. Let $\mathcal{H}_\mathcal{O}$ denote the union of $H_0, \ldots, H_{p-1}$. Then, by Lemma 2.2, the core entropy $h(P)$ equals to the maximum of $h(P|_{\mathcal{H}_\mathcal{O}})$ with $\mathcal{O}$ going through all cycles in $\mathrm{Comp}(\mathcal{H}_P)$. Furthermore, we have the following results (remember $h_{top}(P|_\emptyset) := 0$).

Lemma 9.1 (analogy to Lemma 7.9)**.** *Let $\mathcal{O}$ be a cycle in $\mathrm{Comp}(\mathcal{H}_P)$ of period p. Then $h_{top}(P|_{\mathcal{H}_{\mathcal{O}}}) = h_{top}(P^p|_H)/p$ for any element H of $\mathcal{O}$. As a consequence, we have entropy foumulas (p_H denotes the period of H, and $H_{\mathcal{O}}$ is an arbitrary element of $\mathcal{O}$):*

$$h(P) \quad = \quad \max_{\substack{H \in \mathrm{Comp}(\mathcal{H}_P) \\ non-trivial \\ periodic}} \frac{1}{p_H} h_{top}(P^{p_H}|_H) \tag{9.1}$$

$$= \quad \max_{\substack{\mathcal{O} \subseteq \mathrm{Comp}(\mathcal{H}_P) \\ non-trivial\ cycle}} \frac{1}{p_{H_{\mathcal{O}}}} h_{top}(P^{p_{H_{\mathcal{O}}}}|_{H_{\mathcal{O}}}). \tag{9.2}$$

9.2. **Branner-Hubbard puzzles and renormalization.**

According to Branner and Hubbard [BH], one can construct *BH-puzzle pieces* for any polynomial with disconnected Julia set.

Recall that g_P is the Green function of P. We denote by $a_P > 0$ the maximum of $g_P(c)$ with $c \in \mathrm{Crit}_P$. Choose r_0 such that $a_P < r_0 < da_P$, and $r_0 \neq d^k g_P(c)$ for any $k \geq 0$ and $c \in \mathrm{Crit}_P$. We define $V_0(P) := \{z \in \mathbb{C} : g_P(z) < r_0\}$, and $\mathcal{V}_n(P)$ the collection of bounded components of $P^{-n}(V_0(P))$ for each $n \geq 0$. We call $\mathcal{V}(P) := \cup_{n \geq 0} \mathcal{V}_n(P)$ the *BH-puzzle* of P, and any element of $\mathcal{V}_n(P)$ a *BH-puzzle piece of level n*. The BH-puzzle pieces satisfy the following properties.

(1) Each puzzle piece is a Jordan disk with its boundary contained in $\Omega(P)$.
(2) Each puzzle piece of level $n \geq 1$ is compactly contained in a puzzle piece of level $n - 1$.
(3) The map P sends any puzzle piece of level $n \geq 1$ to a puzzle piece of level $n - 1$ as a branched covering.
(4) The intersection of a sequence of nested puzzle pieces is a component of $\mathcal{K}_P$.

Let K be a non-trivial periodic component of $\mathcal{K}_P$ with period m. Since P is uniform expanding near $\mathcal{J}_P$, we have $\deg(P^m|_K) \geq 2$. By properties above, there exists a renormalization triple $\rho = (P^m, U, V)$ such that U, V are BH-puzzle pieces of P and $K_\rho = K$. By Straightening Theorem, the map $P^m : U \to V$ is hybrid equivalent to a unique (up to a conjugation by $\deg(P^m|_K) - 1$-th roots of unit) monic, centered postcritically-finite polynomial, denoted as Q_K, called the *renormalization polynomial associated to K*. The polynomial P is called *renormalizable* if $\mathcal{K}_P$ has non-trivial periodic components.

For each $n \geq 0$, we denote by $\Gamma_n(P)$ the union of the boundaries of all BH-puzzle pieces of level $0 \leq i \leq n$. Then $\Gamma_n(P)$ is a disjoint union of circles in $\Omega(P)$. Given any integer $N > 0$, the number r_0 is not included in $\{g_Q(z) : z \in \cup_{c \in \mathrm{Crit}_Q} \cup_{i=0}^{N} Q^i(c)\}$ for all Q closed enough to P. Hence the set $\Gamma_N(Q)$ is also the disjoint union of circles for all Q close enough to P.

Lemma 9.2 (anology to Lemma 8.1)**.** *The graph $\Gamma_N(Q)$ is homeomorphic to $\Gamma_N(P)$ for any Q close enough to P and $\Gamma_N(Q) \to \Gamma_N(P)$ as $Q \to P$.*

Proof. The proof goes by induction. Note that the definition domains of ϕ_P, ϕ_Q contains $\Gamma_0(P), \Gamma_0(Q)$ respectively for Q close to P, where ϕ_P, ϕ_Q denotes the Böttcher coordinates of P, Q. Then $\Gamma_0(Q) \to \Gamma_0(P)$ as $Q \to P$ since $\Gamma_0(P) = \phi_P^{-1}\{z : |z| = r_0\}$, $\Gamma_0(Q) = \phi_Q^{-1}\{z : |z| = r_0\}$ and ϕ_Q^{-1} uniformly converge to ϕ_P^{-1} in a compact set containing $\{z : |z| = r_0\}$.

Assume that $\Gamma_k(Q)$ is homeomorphic to $\Gamma_k(P)$ and $\Gamma_k(Q) \to \Gamma_k(P)$ for some $0 \leq k < N$. Let γ_P be any component of $\Gamma_{k+1}(P)$. Then its image $P(\gamma_P) =: \gamma'_P$ is a component of $\Gamma_k(P)$. By the assumption of induction, we can find a unique component γ'_Q of $\Gamma_k(Q)$ such that γ'_Q is in a sufficiently small neighborhood of γ'_P for Q close enough to P. Hence, by Rouché Theorem, there is a unique preiamge γ_Q by Q of γ'_Q which lies in arbitrarily small neighborhood of γ_P provided Q is close enough to P. Since γ_P, γ_Q are circles, we then have $\gamma_Q \to \gamma_P$. $\qquad\square$

As a consequence, we see that for any BH-puzzle piece V_P of P with level $k \geq 0$, there exists a unique BH-puzzle piece V_Q of Q with level k such that $\overline{V_Q} \to \overline{V_P}$ as $Q \to P$. Such V_Q is called the *deformation* of V_P at Q.

9.3. Continuity of the entropy function in partial postcritically-finite polynomial family.

Throughout this subsection, we always assume that P is a partial postcritically-finite polynomial with disconnected Julia set, and $\{P_n, n \geq 1\}$ a sequence of partial postcritically-finite polynomials converging to P.

Proposition 9.3 (analogy to Proposition 8.4). *Let P be a non-renormalizable partial postcritically-finite polynomial. Then the entropy function $h : \mathcal{P}_d^{\mathrm{ppf}} \to \mathbb{R}$ is continuous at P.*

In the following, we only deal with the renormalizable case. One can choose a BH-puzzle piece V_H of P for each non-trivial periodic component H of $\mathcal{H}_P$ such that $H \subseteq V_H$ and $V_H \cap V_{H'} = \emptyset$ for any two different components H, H' of $\mathcal{H}_P$. For each non-trivial periodic component H of $\mathcal{H}_f$ and each large n, we denote by $V_{H,n}$ the BH-puzzle piece of P_n deformed from V_H. These puzzle pieces are pairwise disjoint and $\deg(P|_{V_H}) = \deg(P_n|_{V_{H,n}})$.

As in the case of postcritically-finite Newton maps, the Hubbard forest $\mathcal{H}_{P_n}$ can be divided into two parts: let $\mathcal{H}_{\mathrm{bd}}^n$ denote the union of the components of $\mathcal{H}_{P_n}$ which stay in $\cup_H V_{H,n}$ under the iteration of P_n, and $\mathcal{H}_{\mathrm{esc}}^n$ the union of periodic components of $\mathcal{H}_{P_n}$ not contained in $\mathcal{H}_{\mathrm{bd}}^n$, where the subscripts "bd" and "esc" mean "bounded" and "escape" respectively. Then both $\mathcal{H}_{\mathrm{bd}}^n$ and $\mathcal{H}_{\mathrm{esc}}^n$ are P_n-invariant, and by Propositions 2.2, 2.3, we have

$$h(P_n) = \max\{ \ h_{top}(P_n|_{\mathcal{H}_{\mathrm{bd}}^n}), h_{top}(P_n|_{\mathcal{H}_{\mathrm{esc}}^n}) \ \}. \tag{9.3}$$

Lemma 9.4 (analogy to Lemma 8.5). *The topological entropy $h_{top}(P_n|_{\mathcal{H}_{\mathrm{esc}}^n})$ tends to 0 as $n \to \infty$.*

For any non-trivial periodic component H of $\mathcal{H}_P$, we define a forest

$$H_n := \mathcal{H}_{\mathrm{bd}}^n \cap V_{H,n}. \tag{9.4}$$

It is easy to see that $f_n(H_n) \subseteq H_n'$ if and only if $f(H) \subseteq H'$, and we have the entropy formula (apology to formula (8.3)):

$$h_{top}(P_n|_{\mathcal{H}_{\mathrm{bd}}^n}) = \max_{\substack{H \in \mathrm{Comp}(\mathcal{H}_P) \\ \mathrm{non-trivial} \\ \mathrm{period} \ p_H}} \frac{1}{p_H} h_{top}(P_n^{p_H}|_{H_n}). \tag{9.5}$$

Combining formulas (9.3),(9.5) and Lemma 9.4, we get a result analogous to Lemma 8.6

Lemma 9.5 (analogy to Lemma 8.6). *The limit/limit superior/limit inferior of $h(P_n)$ is equal to that of $\max_H \frac{1}{p_H} h_{top}(P_n^{p_H}|_{H_n})$ with H going through all non-trivial periodic components of $\mathcal{H}_P$, as $n \to \infty$.*

Let H be a non-trivial periodic component of $\mathcal{H}_f$ with period p. Then $\rho_H := (P^p, V_H, P^p(V_H))$ is a renormalization triple. By Straightening Theorem, there exist a monic, centered polynomial P_H of degree $\delta = \deg(P|_{V_H})$ hybrid equivalent to $P^p : V_H \to P^p(V_H)$, called a *renormalization polynomial of P associated to H*. It is clear that $h_{top}(P^p|_H) = h(P_H)$.

Similarly, for each large n, the map $P_n^p : V_{H,n} \to f(V_{H,n})$ is a polynomial-like map of degree δ; By Straightening Theorem again, we get for each large n a monic, centered polynomial $P_{H,n}$ of degree δ hybrid equivalent to $P_n^p : V_{H,n} \to f(V_{H,n})$, which we call a *straightening-perturbation of P_H at P_n*.

Lemma 9.6 (analogy to Lemma 8.10). *Every polynomial $P_{H,n}$ is partial postcritically-finite and $h_{top}(P_n^p|_{H_n}) = h(P_{H,n})$ (H_n is defined in (9.4)). Furthermore, by suitably choice of $P_{H,n}$, we have $P_{H,n} \to P_H$ as $n \to \infty$.*

Proposition 9.7 (analogy to Proposition 8.12). *Let P be a partial postcritically-finite polynomial, and $H_1, \ldots, H_k$ a collection of non-trivial periodic components of $\mathcal{H}_P$ in pairwise distinct cycles. Let $P_{H_i}, i = 1, \ldots, k$, be a renormalization polynomial of P associated to H_i. Then there exists a sequence of polynomials $\{P_n, n \geq 1\} \subseteq \mathcal{P}_d^{\mathrm{ppf}}$ such that $P_n \to P$ and the polynomials $\{P_{H_i,n}, n \geq 1\}$ is a entropy-minimal sequence converging to P_{H_i} for each $1 \leq i \leq k$.*

Proof of Theorem 1.6. This proof is completely the same as that of Theorem 1.3. One just need to replace the formula (7.2); Lemmas 8.10, 8.6 and Propositions 8.4, 8.12 used in the proof of Theorem 1.3, by the formula (9.1); Lemmas 9.6, 9.5 and Propositions 9.3, 9.7 respectively. The detail is omitted. $\square$

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