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EQUIVARIANT TAMAGAWA NUMBER CONJECTURE FOR ABELIAN VARIETIES OVER GLOBAL FIELDS OF POSITIVE CHARACTERISTIC

FABIEN TRIHAN AND DAVID VAUCLAIR

ABSTRACT. We state and prove certain cases of the equivariant Tamagawa number conjecture of a semistable Abelian variety over an everywhere unramified finite Galois extension of a global field of characteristic $p > 0$ under a semisimplicity hypothesis.

1. INTRODUCTION

The Iwasawa theory and main conjecture for Abelian varieties over cyclotomic extensions of number fields first appeared in a celebrated paper of Mazur ([Ma]). In the recent years, Kato ([Ka]) and Skinner and Urban ([SU]) gave a full proof of the conjecture for elliptic curves over the rationals. Let us mention also the development of these conjectures to non commutative extensions. Meanwhile, it is possible to develop an analogous theory over function fields of characteristic $p > 0$. Up to now, and over such fields, the results obtained by both authors are as follows: the Iwasawa main conjecture is known in the following cases:

- (1) A/K is semistable and the extension is a profinite Galois extension without p -torsion, everywhere unramified and factoring through the unique \mathbb{Z}_p -extension everywhere unramified ([LLTT1], [TV2]).
- (2) A/K is a constant ordinary Abelian variety and the extension is a profinite Abelian Galois extension, ramified at a finite set of places ([LLTT2]).

Note that the first result is based on the use of crystalline methods while the second more or less can be reduced to the case of the trivial motives. Note also that in [BK], a Tamagawa number conjecture is moreover proved in case (2), using the previous results of the first author concerning this conjecture for the trivial motive, the Iwasawa main conjecture, as well as a descent theorem due to [BV2]. In this paper, we use again the descent theorem of [BV2] as well as our proof of the IMC ([TV2]) to establish, under some hypothesis on the μ -invariant of the Abelian variety as well as a weaker form of the hypothesis of the finiteness of the Tate-Shafarevich group, the equivariant Tamagawa number conjecture of a semistable Abelian variety over an everywhere unramified finite Galois extension of a global field of characteristic $p > 0$. In the first section we fix the notations and recall the tools of algebraic K -theory that will be used later on. In Section 2, we state the equivariant Tamagawa

number conjecture for a semistable Abelian variety and a finite Galois, unramified everywhere extension of our base field. In Section 3, we recall the results of [TV2] (using also [TV1]) and in particular the proof of the Iwasawa Main conjecture for a semistable Abelian variety and a profinite Galois, everywhere unramified extension. In the final section, we prove our main theorem (Theorem 5.2).

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2. PRELIMINARIES

In this section we shall, for the reader's convenience, recall some basic background material relating to relative algebraic K -theory, refined Euler characteristics, the Iwasawa algebras of certain kinds of non-commutative p -adic Lie groups.

2.1. Relative algebraic K -theory and Picard category. Throughout this article, modules are always to be understood, unless explicitly stated otherwise, as left modules.

For any associative, unital, left noetherian ring R we write $D(R)$ for the derived category of R -modules. We also write $D^p(R)$ for the full triangulated subcategories of $D(R)$ comprising complexes that are isomorphic to an object of the categories $C^p(R)$ of bounded complexes of finitely generated projective R -modules.

For any homomorphism $R \rightarrow R'$ of rings as above we write $K_0(R, R')$ for the relative algebraic K_0 -group that is defined in terms of explicit generators and relations by Swan in [Sw, p. 215]. We recall in particular that this group fits into a canonical exact sequence of Abelian groups of the form

$$(1) \quad K_1(R) \rightarrow K_1(R') \xrightarrow{\partial_{R,R'}} K_0(R, R') \rightarrow K_0(R) \rightarrow K_0(R').$$

Here, for any ring A , we write $K_1(A)$ for its Whitehead group and $K_0(A)$ for the Grothendieck group of the category of finitely generated projective A -modules, and the first and last arrows in (1) denote the homomorphisms that are naturally induced by the given ring homomorphism $R \rightarrow R'$. (For more details about this sequence, and a proof of its exactness, see [Sw, Chap. 15]).

We recall ([BV1], [BV2]) that one can define a Picard category \mathcal{C}_R of R which is endowed with a binary operation \cdot and a unit object $\mathbf{1}_R$ for which all objects are invertible. There exists also a functor called the determinant functor

$$d_R : D^p(R) \rightarrow \mathcal{C}_R$$

satisfying the following properties:

- (1) $\mathbf{1}_R = d_R(0)$.
- (2) For any exact sequence of perfect complexes

$$0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow C'[1] \rightarrow 0,$$

we have

$$d_R(C) = d_R(C').d_R(C'').$$

- (3) Any acyclic complex $C \in C^p(R)$ induces a canonical isomorphism $can : \mathbf{1}_R \rightarrow C$ in \mathcal{C}_R .
- (4) Any ring homomorphism $R \rightarrow R'$ induces a functor

$$R' \otimes_R . : \mathcal{C}_R \rightarrow \mathcal{C}_{R'}$$

sending the unit object to the unit object.

- (5) $K_0(R)$ is isomorphic to the isomorphism classes of objects of \mathcal{C}_R .
- (6) $K_1(R)$ is isomorphic to the group of automorphisms of the unit object of \mathcal{C}_R .
- (7) $K_0(R, R')$ is isomorphic to the isomorphism classes of objects of the fiber category $\mathcal{C}_R \times_{\mathcal{C}_{R'}} \mathcal{C}_0$. In particular this group is represented (via this isomorphism) by objects of the form $[L, t : R' \otimes_R L \rightarrow \mathbf{1}_{R'}]$, where $L \in \mathcal{C}_R$ and t is a map in $\mathcal{C}_{R'}$ called a trivialization.

Via the description of the K groups in terms of Picard categories, the connecting map $\partial_{R,R'}$ can be reinterpreted as the map sending an automorphism $\beta : \mathbf{1}_{R'} \rightarrow \mathbf{1}_{R'}$ to the object $[\mathbf{1}_R, \beta]$.

2.2. Euler characteristics. For a ring homomorphism $R \rightarrow R'$, we denote $D_{R'}^p(R)$ the full subcategory of $D^p(R)$ of object P such that $R' \otimes_R^L P$ is the zero object in $D^p(R')$.

We denote

$$\chi_0 : D_{R'}^p(R) \rightarrow K_0(R, R')$$

the functor sending an object P in $D_{R'}^p(R)$ to $[d_R(P), can^{-1} : d_{R'}(R' \otimes_R^L P) \rightarrow \mathbf{1}_{R'}]$.

We denote

$$\chi_1 : \{\text{automorphisms of } D^p(R')\} \rightarrow K_1(R')$$

sending an automorphism $\beta : P \rightarrow P$ to $d_{R'}(\beta)$ (via the identification of $Aut_{\mathcal{C}_{R'}}(\mathbf{1}_{R'})$ with $Aut_{\mathcal{C}_{R'}}(P)$).

Note that if Θ is an endomorphism of $D^p(R)$ with cone $D \in D_{R'}^p(R)$, then we have ([FK], 1.3.15)

$$(2) \quad \partial(\chi_1(\Theta \otimes_R R')) = -\chi_0(D).$$

2.3. Semisimplicity. Let A be a noetherian regular ring. Let $\Theta : P \rightarrow P$ be an endomorphism in $D^p(A)$ and denote $D := Cone(\Theta)$. Following [BV2], Appendix B, we say that Θ is semisimple if the chain complex (called the Bockstein complex of Θ) $H^i(D)$ with boundary operators the composed maps

$$\beta_{\Theta}^i : H^i(D) \rightarrow Ker H^{i+1}(\Theta) \rightarrow H^{i+1}(P) \rightarrow H^{i+1}(D)$$

is acyclic. Note that if Θ is semisimple, then $d_A(D)$ is endowed with a trivialization (see [BV2], (55))

$$t_{\Theta} : d_A(D) \rightarrow \mathbf{1}_A.$$

2.4. Iwasawa algebras. Let \mathcal{G} be a p -adic Lie group without p -torsion. For any finite extension \mathcal{O} of \mathbb{Z}_p we write $\Lambda_{\mathcal{O}}(\mathcal{G})$ for the \mathcal{O} -Iwasawa algebra $\varprojlim_U \mathcal{O}[G/U]$ of G , where U runs over the set of open normal subgroups of G (partially ordered by inclusion) and the limit is taken with respect to the obvious transition homomorphisms. We also write $\mathbb{Q}_{\mathcal{O}}(\mathcal{G})$ for the total quotient ring of $\Lambda_{\mathcal{O}}(\mathcal{G})$, and if $\mathcal{O} = \mathbb{Z}_p$, then we omit the subscripts ‘ \mathcal{O} ’ from both $\Lambda_{\mathcal{O}}(\mathcal{G})$ and $\mathbb{Q}_{\mathcal{O}}(\mathcal{G})$.

We assume that \mathcal{G} lies in a group extension of the form

$$(3) \quad \{1\} \rightarrow H \rightarrow \mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} \Gamma \rightarrow \{1\}$$

in which Γ is (topologically) isomorphic to the additive group of p -adic integers \mathbb{Z}_p . We also fix an algebraic closure \mathbb{Q}_p^c of \mathbb{Q}_p and write \mathcal{O} for the valuation ring of a finite extension of \mathbb{Q}_p in \mathbb{Q}_p^c .

We write S for the subset of $\Lambda_{\mathcal{O}}(\mathcal{G})$ comprising elements f for which the quotient $\Lambda_{\mathcal{O}}(\mathcal{G})/\Lambda_{\mathcal{O}}(\mathcal{G})f$ is finitely generated as a module over the ring $\Lambda_{\mathcal{O}}(H)$ and we also set $S^* := \bigcup_{i \geq 0} p^i S$.

We recall that in [CFKSV, §2] it is shown that S and S^* are both multiplicatively closed left and right Ore sets of non-zero divisors and so we can write $\Lambda_{\mathcal{O}}(\mathcal{G})_S$ and $\Lambda_{\mathcal{O}}(\mathcal{G})_{S^*} = \Lambda_{\mathcal{O}}(\mathcal{G})_S[\frac{1}{p}]$ for the corresponding localisations of $\Lambda_{\mathcal{O}}(\mathcal{G})$.

We often use the fact that the long exact sequence (1) is compatible with scalar extensions in the sense that there exists a commutative diagram

$$(4) \quad \begin{array}{ccccc} K_1(\Lambda_{\mathcal{O}}(\mathcal{G})) & \longrightarrow & K_1(\Lambda_{\mathcal{O}}(\mathcal{G})_{S^*}) & \xrightarrow{\partial} & K_0(\Lambda_{\mathcal{O}}(\mathcal{G}), \Lambda_{\mathcal{O}}(\mathcal{G})_{S^*}) \\ \parallel & & \uparrow & & \uparrow \\ K_1(\Lambda_{\mathcal{O}}(\mathcal{G})) & \xrightarrow{\alpha} & K_1(\Lambda_{\mathcal{O}}(\mathcal{G})_S) & \xrightarrow{\partial} & K_0(\Lambda_{\mathcal{O}}(\mathcal{G}), \Lambda_{\mathcal{O}}(\mathcal{G})_S) \end{array}$$

We recall that since \mathcal{G} has no element of order p , and Σ denotes either S or S^* , then the group $K_0(\Lambda_{\mathcal{O}}(\mathcal{G}), \Lambda_{\mathcal{O}}(\mathcal{G})_{\Sigma})$ is naturally isomorphic to the Grothendieck group of the category of finitely generated $\Lambda_{\mathcal{O}}(\mathcal{G})$ -modules M with the property that $\Lambda_{\mathcal{O}}(\mathcal{G})_{\Sigma} \otimes_{\Lambda_{\mathcal{O}}(\mathcal{G})} M$ vanishes (for an explicit description of this isomorphism see, for example, [BV2, §1.2]).

In particular, if \mathcal{G} is also Abelian then the determinant functor induces a natural isomorphism between $K_0(\Lambda_{\mathcal{O}}(\mathcal{G}), \Lambda_{\mathcal{O}}(\mathcal{G})_{\Sigma})$ and the multiplicative group of invertible $\Lambda_{\mathcal{O}}(\mathcal{G})$ -lattices in $\Lambda_{\mathcal{O}}(\mathcal{G})_{\Sigma}$. For any (finitely generated torsion) $\Lambda_{\mathcal{O}}(\mathcal{G})$ -module M as above, this isomorphism sends the element $\chi_0(M)$ defined above to the (classical) characteristic ideal of M .

2.4.1. Leading terms. We now fix a topological generator γ of the group Γ that occurs in the extension (3) and also an Ore set $\Sigma \in \{S, S^*\}$.

Then for the valuation ring \mathcal{O}' of any finite extension of \mathbb{Q}_p in \mathbb{Q}_p^c which contains \mathcal{O} , and any continuous homomorphism of the form

$$(5) \quad \rho : \mathcal{G} \rightarrow \mathrm{GL}_n(\mathcal{O}')$$

there is an induced ring homomorphism

$$\Lambda_{\mathcal{O}}(\mathcal{G})_{\Sigma} \rightarrow M_n(\mathcal{O}') \otimes_{\mathcal{O}'} \mathbb{Q}_{\mathcal{O}'}(\Gamma) \cong M_n(\mathbb{Q}_{\mathcal{O}'}(\Gamma))$$

that sends every element g of \mathcal{G} to $\rho(g) \otimes \pi_{\mathcal{G}}(g)$. This ring homomorphism in turn induces a homomorphism of Abelian groups

$$(6) \quad \Phi_{\mathcal{O}, \mathcal{G}, \Sigma, \rho} : K_1(\Lambda_{\mathcal{O}}(\mathcal{G})_{\Sigma}) \rightarrow K_1(M_n(\mathbb{Q}_{\mathcal{O}'}(\Gamma))) \cong K_1(\mathbb{Q}_{\mathcal{O}'}(\Gamma)) \\ \cong \mathbb{Q}_{\mathcal{O}'}(\Gamma)^{\times} \cong \mathbb{Q}(\mathcal{O}'[[u]])^{\times}$$

where we write $\mathcal{O}'[[u]]$ for the ring of power series over \mathcal{O}' in the formal variable u , the first isomorphism is induced by Morita equivalence, the second by taking determinants (over the ring $\mathbb{Q}_{\mathcal{O}'}(\Gamma)$) and the last by sending $\gamma - 1$ to u . In the sequel we shall abbreviate $\Phi_{\mathcal{O}, \mathcal{G}, \Sigma, \rho}$ to $\Phi_{\mathcal{G}, \rho}$ if we feel that \mathcal{O} and Σ are both clear from context.

For any element ξ of the Whitehead group $K_1(\Lambda_{\mathcal{O}}(\mathcal{G})_{\Sigma})$ and any representation ρ as in (5), one then defines the ‘value’ $\xi(\rho)$, resp. the ‘leading term’ $\xi^*(\rho)$, of ξ at ρ to be the value, resp. the leading term, at $u = 0$ of the series $\Phi_{\mathcal{O}, \mathcal{G}, \Sigma, \rho}(\xi)$. In particular, one has $\xi^*(\rho) \in \mathbb{Q}_p^{\times} \setminus \{0\}$ for all ρ and one regards the value $\xi(\rho)$ to be equal to ‘ ∞ ’ if the algebraic order of $\Phi_{\mathcal{O}, \mathcal{G}, \Sigma, \rho}(\xi)$ at $u = 0$ is strictly negative.

We recall finally that a continuous representation ρ as in (5) is said to be an ‘Artin representation’ if its image $\rho(\mathcal{G})$ is finite.

3. EQUIVARIANT TAMAGAWA NUMBER CONJECTURE FOR SEMISTABLE ABELIAN VARIETY AND FINITE UNRAMIFIED EXTENSIONS

3.1. Setting. In this section, we formulate the Equivariant Tamagawa Number conjecture for a semistable Abelian variety A/K over a global field of characteristic p with field of constant a finite field k and a finite Galois unramified extension K_G/K . Let C be the projective smooth geometrically connected curve with function field K . Then the extension K_G/K corresponds one to one to a finite étale covering $\pi : C_G \rightarrow C$ with Galois group G .

We denote \mathcal{A}/C the Néron model of A/K and Z the finite set of closed points of C where A/K has bad reduction. We denote C^{\sharp} the log scheme with underlying scheme C and log structure induced by Z . Similarly, we denote C_G^{\sharp} the log scheme with underlying scheme C_G and log structure induced by the closed points of C_G above those in Z . In [TV1] is associated to A/K a log Dieudonné crystal $D(A)$, that is a locally free finite rank crystal in the crystalline topos $(C^{\sharp}/\mathbb{Z}_p)_{crys}$ endowed with a Frobenius and Verschiebung operator F and V satisfying the relations $F \circ V = V \circ F = p$. It is moreover shown that one has a canonical epimorphism

$$D(A) \rightarrow Lie(\mathcal{A})$$

where $Lie(\mathcal{A})$ is naturally considered as an object of the crystalline topos. We denote $Fil^1 D(A)$ the kernel of this epimorphism.

3.2. In [TV2], the two following distinguished triangles of $D^p(\mathbb{Z}_p[G])$ have been constructed:

$$(7) \quad N_G \rightarrow I_G \xrightarrow{\mathbf{1}-\varphi} P_G \rightarrow N_G[1]$$

$$(8) \quad I_G \xrightarrow{\mathbf{1}} P_G \rightarrow L_G \rightarrow I_G[1]$$

where the objects P_G, I_G, L_G and N_G and operators $\mathbf{1}$ and φ are respectively defined as follows:

- (1) $P_G := \mathbb{R}\Gamma_{crys}(C_G^\sharp/\mathbb{Z}_p, \pi^* D(A)(-Z))$, that is the log-crystalline cohomology functor of the log Dieudonné crystal $D(A)$ associated to A/K and twisted by $-Z$ as defined in [TV1].
- (2) $I_G := \mathbb{R}\Gamma_{crys}(C_G^\sharp/\mathbb{Z}_p, \pi^* Fil^1 D(A)(-Z))$.
- (3) $L_G := \mathbb{R}\Gamma_{zar}(C_G, \pi^* Lie(\mathcal{A})(-Z))$.
- (4) $N_G = \mathbb{R}\Gamma^Z(C_G, \pi^* T_p \mathcal{A})$, where $\mathbb{R}\Gamma^Z(C_G, \cdot)$ is the derived functor of flat section vanishing at Z as defined in [TV1] and $T_p \mathcal{A} = \mathbb{R} \lim_{\leftarrow} \mathcal{A}[p^n]$.
- (5) $\mathbf{1}$ is induced by the canonical inclusion $Fil^1 D(A) \subset D(A)$ and is compatible with the identity map on $D(A)$ after inverting p .
- (6) φ is a linear operator corresponding after inverting p to the operator $p^{-1}F_{D(A)}$.

Note that the cohomology groups of L_G are finitely generated \mathbb{F}_p -vector spaces so that $\mathbb{Q}_p[G] \otimes_{\mathbb{Z}_p[G]} L_G$ is acyclic.

We make the following conjecture:

Conjecture 3.1. ETNC($A/K, K_G/K$)

- (1) The endomorphism $\Theta_G := \mathbb{Q}_p^c[G] \otimes_{\mathbb{Z}_p[G]} (\mathbf{1}-\varphi)$ on $\mathbb{Q}_p^c[G] \otimes_{\mathbb{Z}_p[G]} P_G$ is semisimple and induces therefore an object $[d_{\mathbb{Z}_p[G]} N_G, t_{\Theta_G}]$ in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p^c[G])$.
- (2) We denote $Irr(G)$ the \mathbb{Q}_p^c -valued irreducible finite dimensional characters of G . Then the Wedderburn decomposition induces (see [BV2], 1.5) an isomorphism

$$K_1(\mathbb{Q}_p^c[G]) \simeq \prod_{Irr(G)} \mathbb{Q}_p^{c,\times}.$$

Via this isomorphism, we define

$$L_{p,A,Z,G}^* := \left(\frac{L_Z^*(A, \rho, 1)}{(-\log(p))^{r_\rho}} \right)_{\rho \in Irr(G)} \in K_1(\mathbb{Q}_p^c[G]),$$

where $L_Z^*(A, \rho, s)$ is the leading coefficient at $s = 1$ of the Hasse-Weil L -function of A/K twisted by the character ρ and without Euler factors at Z

and r_ρ is the order of vanishing at $s = 1$ of $L_Z(A, \rho, s)$. Assuming that the first part of the conjecture holds, then

$$\partial_{\mathbb{Z}_p[G], \mathbb{Q}_p^c[G]}(L_{p,A,Z,G}^*) = [d_{\mathbb{Z}_p[G]}N_G, t_{\Theta_G}] + [d_{\mathbb{Z}_p[G]}L_G, \text{can}^{-1}]$$

- Remark 3.1.** (1) When G is the trivial group, the semisimplicity hypothesis in Θ is equivalent to say that the corank of the p -Selmer group is equal to the order of vanishing of the Hasse-Weil L -function of A/K at $s = 1$ (see [KT], section 3.5).
- (2) If we assume that the Tate-Shafarevich group of A/K is finite (which implies the semisimplicity hypothesis), then our conjecture is related to the equivariant form of the Birch and Swinnerton-Dyer conjecture. We refer the reader to [LLTT1], Theorem 1.4 where the precise relation between the Iwasawa Main Conjecture of A/K over the arithmetic extension (that is, the family of Equivariant Tamagawa Number Conjectures for A over the extensions $K/K.\mathbb{F}_q^{p^n}$, for $n \geq 0$) and the Birch and Swinnerton-Dyer conjecture has been worked out in detail.
- (3) We hope that this formulation in terms of Selmer complex is more motivic in the sense that it adapts easily to more general motives than the sole case of Abelian varieties (even when arithmetic invariants like Mordell-Weil or Tate-Shafarevich groups are not well-defined). For example, the reader can imagine another equivariant Tamagawa number conjecture by replacing A/K by a p -divisible group over C or more generally by any log-Dieudonné crystal over C . The most general form of the coefficients, would be that of non-degenerated crystals of level N (that is crystals endowed with Frobenius and Verschiebung operators F and V satisfying $FV = VF = p^N$). It seems however a difficult problem to construct Selmer (or syntomic) complex for such coefficients.

4. THE IWASAWA MAIN CONJECTURE FOR AN ABELIAN VARIETY OVER AN UNRAMIFIED PROFINITE EXTENSION OF GLOBAL FIELDS OF CHARACTERISTIC $p > 0$

In this section, we recall the main result of [TV2]. Let A/K be a semistable Abelian variety and consider the intermediate extension $K_{\mathcal{G}}/K_{ar}/K$ such that K_{ar}/K is the unique unramified $\Gamma = \mathbb{Z}_p$ -extension of K and $K_{\mathcal{G}}/K$ is a Galois extension with group \mathcal{G} a profinite group satisfying the hypothesis of the section 2.4.

By passing to the derived inverse limit functor over the open subgroups of \mathcal{G} , we can define two distinguished triangles in $D^p(\Lambda(\mathcal{G}))$

$$(9) \quad N_{\mathcal{G}} \rightarrow I_{\mathcal{G}} \xrightarrow{1-\varphi} P_{\mathcal{G}} \rightarrow N_{\mathcal{G}}[1]$$

$$(10) \quad I_{\mathcal{G}} \xrightarrow{1} P_{\mathcal{G}} \rightarrow L_{\mathcal{G}} \rightarrow I_{\mathcal{G}}[1]$$

as well as the analogue distinguished triangles in $D^p(\Lambda(\Gamma))$.

We recall that the p -Selmer group of A over any algebraic extension F/K is defined to be

$$\text{Sel}_p(A/F) := \text{Ker}(H_{fl}^1(F, A[p^\infty]) \rightarrow \prod_v H_{fl}^1(F_v, A)),$$

where H_{fl} denotes the cohomology in the big flat topos and in the product, v runs over all places of F . We denote $X_p(A/F)$ the Pontrjagin dual of $\text{Sel}_p(A/F)$. It has been proved in [OT] that $X_p(A/K_{ar})$ is a finitely generated torsion $\Lambda(\Gamma)$ -module.

We will need the following hypothesis:

Assumption 4.1. " $\mu \simeq 0$ "

The Abelian variety A/K is isogenous to an Abelian variety A'/K such that $X_p(A'/K)$ has trivial μ -invariant.

Theorem 4.1. ([TV2], 5.9)

- (1) $\Lambda(\mathcal{G})_{\mathbb{Q}_p} \otimes_{\Lambda(\mathcal{G})} L_{\mathcal{G}}$ is acyclic.
- (2) Assume that the assumption " $\mu \simeq 0$ " holds. Then $\Lambda(\mathcal{G})_{S^*} \otimes_{\Lambda(\mathcal{G})} N_{\mathcal{G}}$ is acyclic.

As a consequence, $[d_{\Lambda(\mathcal{G})} N_{\mathcal{G}}, \text{can}^{-1}]$ and $[d_{\Lambda(\mathcal{G})} L_{\mathcal{G}}, \text{can}^{-1}]$ are two well-defined elements of $K_0(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})_{S^*})$ and $\mathcal{L} := \chi_1((\mathbf{1} - \varphi)_{\Lambda(\mathcal{G})_{S^*}} \circ (\mathbf{1}_{\Lambda(\mathcal{G})_{S^*}})^{-1})$ a well-defined element of $K_1(\Lambda(\mathcal{G})_{S^*})$.

The main theorem of [TV2], Theorem 5.14 is under the assumption " $\mu \simeq 0$ " the proof of the Iwasawa Main conjecture for A/K and the extension $K_{\mathcal{G}}/K$:

Theorem 4.2. (IMC($A/K, K_{\mathcal{G}}/K$))

- (1) For any Artin representation ρ of \mathcal{G} with field of definition a totally ramified extension of \mathbb{Q}_p , we have $\mathcal{L}(\rho) = L_Z(A, \rho, 1)$.
- (2) $\partial_{\Lambda(\mathcal{G}), \Lambda(\mathcal{G})_{S^*}}(\mathcal{L}) = [d_{\Lambda(\mathcal{G})} N_{\mathcal{G}}, \text{can}^{-1}] + [d_{\Lambda(\mathcal{G})} L_{\mathcal{G}}, \text{can}^{-1}]$.

5. THE MAIN THEOREM

We take the same setting as in the section 3. We denote **(HYP)** the following hypotheses and data:

- (1) A/K is a semistable Abelian variety.
- (2) $K_{\mathcal{G}}/K$ a finite Galois everywhere unramified extension with Galois group a finite p -group G .
- (3) We also assume given a p -adic Lie group \mathcal{G} satisfying the hypothesis of section 2.4 and admitting G as quotient.

The existence of such p -adic Lie group results of the following

Lemma 5.1. Consider a finite etale Galois covering $C_{\mathcal{G}}/C$ with group G . There exists a pro etale Galois covering $C_{\mathcal{G}}/C_{\mathcal{G}}$ such that $C_{\mathcal{G}}/C$ is p -adic Lie Galois without p -torsion.

Proof. For each non trivial $\sigma \in G$, says of order n we chose C_σ between C_G and C such that $\text{Gal}(C_G/C_\sigma) = \langle \sigma \rangle$ so that C_G/C_σ corresponds to a cyclic subgroup, say $\langle a \rangle$, of $H^1(C_\sigma, \mathbb{Z}/p^n)$. Since $H^2(C_\sigma, \mathbb{Z}_p) = 0$, we can lift a to some $\tilde{a} \in H^1(C_\sigma, \mathbb{Z}_p)$. Let us denote $b \in H^1(C_G, \mathbb{Z}_p)$ the restriction of \tilde{a} to C_G . The sub $\mathbb{Z}_p[G]$ -module of $H^1(C_G, \mathbb{Z}_p)$ generated by b corresponds to a multiple \mathbb{Z}_p -extension say C'_σ/C_G which is such that C'_σ/C is Galois, p -adic Lie and $\text{Gal}(C'_\sigma/C_\sigma)$ has no torsion (indeed C'_σ is a multiple \mathbb{Z}_p -extension of $C_{\tilde{a}}$ where $C_{\tilde{a}}$ is the \mathbb{Z}_p -extension of C_σ defined by \tilde{a}). Composing all C'_σ/C_G 's, we find a multiple \mathbb{Z}_p -extension C_G/C_G which is such that $\text{Gal}(C_G/C)$ is p -adic Lie without torsion. Indeed a non trivial torsion element of $\mathcal{G} := \text{Gal}(C_G/C)$ would have some non trivial σ as image in G but then its image in $\text{Gal}(C'_\sigma/C)$ would lie in $\text{Gal}(C'_\sigma/C_\sigma)$, which is torsion free. □

In this section, we will prove the following theorem:

Theorem 5.2. *Under the assumptions (HYP), " $\mu \simeq 0$ " and assuming that Θ_G is semisimple, the conjecture ETNC($A/K, K_G/K$) holds.*

We will need the following lemmas:

Lemma 5.3. *Let $\Sigma := \{n_1, \dots, n_k\}$ be a finite set of natural integers and set $\Gamma = \varprojlim_n \Gamma_n$. Then, the natural ring homomorphism*

$$i : \Lambda(\Gamma) \hookrightarrow \prod_{n \notin \Sigma} \mathbb{Z}_p[\Gamma_n] \rightarrow \prod_{n \notin \Sigma, \chi \in \hat{\Gamma}_n} O_\chi$$

factors through the localization of $\Lambda(\Gamma)$ by the multiplicative system

$$\mathcal{S} := \{h \in \Lambda(\Gamma), h(\chi) \neq 0, \text{ except for those } \chi \in \hat{\Gamma}_n, \text{ with } n \in \Sigma\}$$

and induce a ring monomorphism

$$i_{\mathcal{S}} : \Lambda(\Gamma)_{\mathcal{S}} \rightarrow \prod_{n \notin \Sigma, \chi \in \hat{\Gamma}_n} L_\chi,$$

with $L_\chi := \text{Frac}(O_\chi)$.

Proof. Observe first that the map $i_{\mathcal{S}}$ is well-defined since $i(\mathcal{S}) \subset \prod_{n \notin \Sigma, \chi \in \hat{\Gamma}_n} L_\chi^\times$. Moreover the map $i_{\mathcal{S}}$ is injective. Indeed, if $\frac{a}{b} \in \Lambda_{\mathcal{S}}$ is sent to 0 by the map $i_{\mathcal{S}}$, then for all but finitely many χ 's, we must have $a^*(\chi) = 0$. By the Weierstrass preparation, we can write a as the product of a unit and a polynomial in one variable. Then this polynomial must be zero since it will admit infinitely many roots. □

Lemma 5.4. *For any Artin representation ρ of \mathcal{G} ,*

$$\mathcal{L}^*(\rho) = \frac{L_{\mathbb{Z}}^*(A, \rho, 1)}{(-\log(p))^{r_\rho}}.$$

Proof. We denote $F_\rho(u) = \Phi_\rho(\mathcal{L})$ and $G_\rho(u) := L_Z(A, \rho, p^{-1}u)$ both seen as elements of $Q_{O_\rho}(\Gamma)^\times \simeq Q_{O_\rho}(\mathbb{Z}_p[[u]])^\times$. To prove the lemma it is enough to prove that $F_\rho(u) = G_\rho(u)$. Indeed, on the one hand the leading term of $F_\rho(u)$ is $\mathcal{L}^*(\rho)$ by definition while on the other hand the leading term of $G_\rho(u)$ is

$$\begin{aligned} \lim_{u \rightarrow 0} u^{-r_\rho} G_\rho(u) &= (-\log(p))^{-r_\rho} \lim_{s \rightarrow 0} s^{-r_\rho} G_\rho(p^{-s} - 1) \\ &= (-\log(p))^{-r_\rho} L_Z^*(A, \rho, 1) \end{aligned} .$$

For all but finitely many characters $\chi \in \hat{\Gamma}$, $F_{\rho, \chi}(0) \neq 0$: to see that, write $F_\rho(u) = \frac{a_\rho(u)}{b_\rho(u)}$ so that $F_{\rho, \chi}(u) = \frac{a_{\rho, \chi}(u)}{b_{\rho, \chi}(u)}$; if $a_{\rho, \chi}(0) = 0$ for infinitely many χ , then $a_\rho(u) = 0$, ie. $F_\rho(u) = 0$. Choose Σ to be the finite set of integers n such that there exists some $\chi \in \hat{\Gamma}_n$ with $F_{\rho, \chi}(0) = 0$. Let us denote \mathcal{S} , the multiplicative set corresponding to Σ as constructed in Lemma 5.3. Then, by injectivity of the map $i_{\mathcal{S}}$, it is enough to prove that the respective image of $F_\rho(u)$ and $G_\rho(u) \in \Lambda(\Gamma)_{\mathcal{S}}$ coincides. But we have $i_{\mathcal{S}}(F_\rho(u)) = (L_Z(A, \rho, \chi, 1))_{n \notin \Sigma, \chi \in \hat{\Gamma}_n}$ by the first assertion of Theorem 4.2, which is also the image of $G_\rho(u)$ by the map $i_{\mathcal{S}}$ so we are done. □

6. PROOF OF THE MAIN THEOREM

Recall that we need to prove that

$$\partial_{\mathbb{Z}_p[G], \mathbb{Q}_p^c[G]}(L_{p, A, Z, G}^*) = [d_{\mathbb{Z}_p[G]} N_G, t_\Theta] + [d_{\mathbb{Z}_p[G]} L_G, \text{can}^{-1}].$$

Now by Lemma 5.4, we have

$$\partial_{\mathbb{Z}_p[G], \mathbb{Q}_p^c[G]}(L_{p, A, Z, G}^*) = \partial_{\mathbb{Z}_p[G], \mathbb{Q}_p^c[G]}(\{\chi_1((\mathbf{1} - \varphi)_{\Lambda(\mathcal{G})_{S^*}} \circ \mathbf{1}_{\Lambda(\mathcal{G})_{S^*}}^{-1})^*(\rho)\}_{\rho \in \text{Irr}(G)}).$$

By [TV2], Lemma 3.2, we know that there exists an endomorphism $\Phi_{\mathcal{G}} : P_{\mathcal{G}} \rightarrow P_{\mathcal{G}}$ such that

$$\Phi_{\mathcal{G}} \circ \mathbf{1} = d_{\mathcal{G}} \circ (\mathbf{1} - \varphi),$$

where $d_{\mathcal{G}} : P_{\mathcal{G}} \rightarrow P_{\mathcal{G}}$ is the multiplication by some power of p on $P_{\mathcal{G}}$. After extending the scalar to $\Lambda(\mathcal{G})_{S^*}$, we deduce that

$$(\mathbf{1} - \varphi)_{\Lambda(\mathcal{G})_{S^*}} \circ \mathbf{1}_{\Lambda(\mathcal{G})_{S^*}}^{-1} = d_{\mathcal{G}, \Lambda(\mathcal{G})_{S^*}}^{-1} \circ \Phi_{\mathcal{G}, \Lambda(\mathcal{G})_{S^*}}.$$

We want to apply the descent theorem [BV2] 2.2 to $\xi = -\chi_1(d_{\mathcal{G}} \otimes \Lambda(\mathcal{G})_{S^*})$ and to $\xi = \chi_1(\Phi_{\mathcal{G}} \otimes \Lambda(\mathcal{G})_{S^*})$. Using the equality $\Phi_{\mathcal{G}} \circ \mathbf{1} = d_{\mathcal{G}} \circ (\mathbf{1} - \varphi)$, an easy chase in the following diagram of distinguished triangles

$$\begin{array}{ccccc}
 I_{\mathcal{G}} & \xrightarrow{1-\phi} & P_{\mathcal{G}} & \longrightarrow & N_{\mathcal{G}}[1] \xrightarrow{+1} \longrightarrow \\
 \downarrow \mathbf{1} & & \downarrow d_{\mathcal{G}} & & \downarrow \\
 P_{\mathcal{G}} & \xrightarrow{\Phi_{\mathcal{G}}} & P_{\mathcal{G}} & \longrightarrow & C(\Phi_{\mathcal{G}}) \xrightarrow{+1} \longrightarrow \\
 \downarrow & & \downarrow & & \downarrow \\
 L_{\mathcal{G}} & \longrightarrow & C(d_{\mathcal{G}}) & \longrightarrow & (\star) \xrightarrow{+1} \longrightarrow \\
 \downarrow +1 & & \downarrow +1 & & \downarrow +1
 \end{array}$$

shows that the cones of both maps are in $D_{\Lambda(\mathcal{G})_{S^*}}^p(\Lambda(\mathcal{G}))$. Moreover, we deduce from the same diagram the following relation in $K_0(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})_{S^*})$

$$(11) \quad [C(d_{\mathcal{G}}), can^{-1}] - [L_{\mathcal{G}}, can^{-1}] = [C(\Phi_{\mathcal{G}}), triv] + [N_{\mathcal{G}}, can^{-1}]$$

We are thus reduced to check that those two cones are *semisimple at ρ* , for any $\rho \in Irr(G)$ in the sense of [BV1], 3.11. The latter condition is described in [BV2], Lemma 5.5, (iii): the multiplication by $\gamma - 1$ on $\Lambda(G \times \Gamma)$ induces for any $M \in D^p(\Lambda(\mathcal{G}))$ a distinguished triangle in $D^p(\Lambda(G \times \Gamma))$

$$tw_G(M)_H \xrightarrow{\gamma-id} tw_G(M)_H \rightarrow \mathbb{Z}_p[G] \otimes_{\Lambda(\mathcal{G})} M \rightarrow tw_G(M)_H[1]$$

with $tw_G(M)_H := \Lambda(G \times \Gamma) \otimes_{\Lambda(G \times \mathcal{G})}^{\mathbf{L}} tw_G(M)$ and $tw_G(M) = \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} M$. Then M is called semisimple at ρ for any $\rho \in Irr(G)$ if and only if $(H^i(\mathbb{Z}_p[G] \otimes_{\Lambda(\mathcal{G})} M) \otimes \mathbb{Q}_p, \beta_{\gamma-id})$ is acyclic. This is obvious for the cone of $d_{\mathcal{G}}$ and for $L_{\mathcal{G}}$ since both complexes become acyclic after inverting p . We deduce from the relation (11) and from the multiplicativity of Bockstein complexes in distinguished triangles that the cone of $\Phi_{\mathcal{G}}$ is semisimple at ρ , for any $\rho \in Irr(G)$ if and only if the same assertion holds for $N_{\mathcal{G}}[1]$. We prove the latter assertion in the next lemma:

Lemma 6.1. *$N_{\mathcal{G}}[1]$ is semisimple at ρ , for any $\rho \in Irr(G)$.*

Proof. Thanks to our semi-simplicity hypothesis, we know that the endomorphism $1 - \varphi$ is semisimple on $P_{\mathcal{G}} \otimes \mathbb{Q}_p^c[G]$, which means that the complex $(H^i(N_{\mathcal{G}}[1]) \otimes \mathbb{Q}_p, \beta_{1-\varphi})$ is acyclic (note that the ring $\mathbb{Q}_p^c[G]$ is regular noetherian). By [KT], Lemma 6.10 applied to the Abelian variety A/K_G we deduce that $(H^i(N_{\mathcal{G}}[1]) \otimes \mathbb{Q}_p, \beta_{\gamma-id})$ is also acyclic. By Theorem 2.23 (ii) of [TV2], $\mathbb{Z}_p[G] \otimes_{\Lambda(\mathcal{G})} N_{\mathcal{G}}[1] = N_{\mathcal{G}}[1]$ and so we conclude that $N_{\mathcal{G}}[1]$ is semisimple at ρ , for any $\rho \in Irr(G)$. □

Applying the descent theorem 2.2 of [BV2] to both cones, we deduce (for simplicity, we omit the trivializations):

$$\begin{aligned} \partial_{\mathbb{Z}_p[G], \mathbb{Q}_p^c[G]}(L_{p,A,,Z,G}^*) &= [d_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G] \otimes_{\Lambda(G)}^{\mathbf{L}} C(d_G)] - [d_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G] \otimes_{\Lambda(G)}^{\mathbf{L}} C(\Theta_G)] \\ &= [d_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G] \otimes_{\Lambda(G)}^{\mathbf{L}} L_G] + [d_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G] \otimes_{\Lambda(G)}^{\mathbf{L}} N_G] \end{aligned}$$

and we conclude by applying again the descent theorem 2.23 of [TV2] to L_G and N_G .

□

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