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EA-Matrix integrals and cyclic cohomology

Serguei Barannikov

Abstract The localization of the EA-matrix integrals from [B4], see also [B1], is calculated via determinants and $\tau-$ functions of KP-type hierarchies in the case of associative algebras with even scalar product.

Introduction

In this paper I calculate the EA-matrix integrals from [B4] of associative algebras with scalar product via determinants and $\tau-$ functions of integrable hierarchies.

Notations

For a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $A = A_0 \oplus A_1$ denote via ΠA the parity inversed vector space, $(\Pi A)_0 = A_1$, $(\Pi A)_1 = A_0$. For an element a from $\mathbb{Z}/2\mathbb{Z}$-graded vector space A denote by $\pi a \in \Pi A$ the same element considerd with inversed parity.

1 Equivariantly closed matrix De Rham differential form.

Let $A = A_0 \oplus A_1$ denotes a $\mathbb{Z}/2\mathbb{Z}-$ graded associative algebra, $\dim_k A_0 = r < \infty$, $char(k) = 0$, with multiplication denoted by $m_2 : A^{\otimes 2} \to A$. Let A is endowed with *odd* invariant scalar product $\langle \cdot, \cdot \rangle : A_0 \otimes A_1 \to k$. The multiplication tensor can be viewed then as the $\mathbb{Z}/3\mathbb{Z}$ - cyclically invariant linear function on $(\Pi A)^{\otimes 3}$

2 The integral

I consider the integral of the closed differential form $\Psi(X)$ from [B4].

$$\mathcal{F}(Y) = \int_{\Gamma} \exp \frac{1}{i}(Tr\langle Y, X\rangle + \frac{1}{3!} m_{\tilde{A}\otimes gl_N}(X, \frac{\partial}{\partial X})) \vdash \prod_{\alpha,i,j} dX_i^{\alpha,j}$$

Consider also the normalized integral

$$\widehat{\mathcal{F}}(Y) = \int_{\Gamma} \exp \frac{1}{i}(Tr\langle Y, X\rangle + \frac{1}{3!} m_{\tilde{A}\otimes gl_N}(X, \frac{\partial}{\partial X})) \vdash \prod_{\alpha,i,j} dX_i^{\alpha,j} / \mathcal{F}_{[2]}(Y) \tag{2.1}$$

where $\mathcal{F}_{[2]}(Y)$ is the corresponding Gaussian integral of the quadratic part at critical point $(-2Y)^{\frac{1}{2}}$.

Let the algebra $\tilde{A}$ with odd scalar product is the tensor product $\tilde{A} = A \otimes Q(1)$ of the even associative algebra A with scalar product, denoted $\eta(y_1, y_2)$, and the algebra $Q(1) = \{1, \xi \mid \xi^2 = 1\}$ with the odd scalar product $\langle 1, \xi \rangle = 1$.

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2.1 The cycle of integration and the anti-involution

Let the associative algebra A has an *anti-involution* $a \to a^\dagger$

$$(ab)^\dagger = b^\dagger a^\dagger, (ca)^\dagger = \overline{c} a^\dagger, \operatorname{tr}\left(a^\dagger\right) = \overline{\operatorname{tr}(a)}, \left(a^\dagger\right)^\dagger = a \tag{2.2}$$

This anti-involution is extended naturally to $gl_N(\mathbb{C}) \otimes A$. Then the Lie subalgebra of anti-hermitian elements in

$$u_N(A) = \left\{Y^\dagger = -Y \mid Y \in A \otimes gl_N(\mathbb{C})\right\} \tag{2.3}$$

$gl_N(\mathbb{C}) \otimes A$ is a real form of $gl_N(\mathbb{C}) \otimes A$. And the space of hermitian elements in the dual space

$$\Gamma = \left\{X^\dagger = X \mid X \in A^\vee \otimes gl_N(\mathbb{C})\right\} \tag{2.4}$$

is invariant under the action of $u_N(A)$. Then the "real-slice" Γ is the natural choice of the cycle for the equivariant integration.

3 The localization of the integral

The localization formula for equivariant cohomology reduces the integral of the equivariantly closed form Ω over Γ to the integral over the fixed locus F ,

$$\int_\Gamma \Omega = \int_F \frac{\Omega}{eu(N_F)} \tag{3.1}$$

where $eu(N_F)$ is the euler class of the normal bundle of F in Γ , see [BV],[AB].

Let the natural scalar product on the Lie algebra $u_1(A)$ of anti-hermitian elements in A is positive definite

$$-\eta(y,y) = \eta\left(y, y^\dagger\right) > 0, y \in u_1(A). \tag{3.2}$$

Proposition 1 *The Lie algebra $u_N(A)$ is reductive.*

Proof The pairing on the Lie algebra induced from the invariant nondegenerate scalar product on the associative algebra is invariant and nondegenerate. It follows that the adjoint representation of the Lie algebra is completely reducible.

Proposition 2 *The induced pairing on the subspace of hermitian elements is invariant and positive definite.*

3.1 Maximal abelian subalgebra of the reductive Lie algebra.

Let $H \subset A$ denotes maximal abelian Lie subalgebra of the Lie algebra $u_1(A)$.

Proposition 3 *The subspace H is a commutative subalgebra of A with respect to the associative product.*

Proof The product of any two elements $h_1 h_2$ commutes with all other elements in H . Since H is maximal abelian Lie subalgebra therefore $h_1 h_2 \in H$.

The Lie algebra H_N of anti-hermitian *diagonal* matrices with values in H

$$\mathbf{t} = \mathtt{diag}(Y_1^1, \ldots, Y_N^N),\ (Y_i^i)^\dagger = -Y_i^i, Y_i^i \in H, \tag{3.3}$$

acts naturally on Γ .

3.2 The fixed locus

The fixed locus $F \subset \Gamma$ of the $\mathbf{t}-$ action consists of hermitian diagonal matrices with values in iH , i.e. the subspace of Γ spanned by elements

$$\operatorname{diag}\left(X_1^{\alpha_H,1},\dots,X_N^{\alpha_H,N}\right)\otimes e_{\alpha_H} \tag{3.4}$$

Without loss of generality it can be assumed that the basis elements in A satisfy

$$(e^{\alpha})^{\dagger} = e^{\alpha} \tag{3.5}$$

Then the linear functions $X_i^{\alpha_H,i}$ are the $\mathbb{R}$ -valued coordinates on F and

$$X_i^{\alpha,j} = 0 \text{ on} F, \text{ for } i \neq j. \tag{3.6}$$

The euler class of the normal bundle $eu_{\mathbf{t}}(N_F)$ is invertible over some localization of the ring $H^*_{\mathbf{t}}(pt)$. Hence the pushforward ρ_* of the inclusion $\rho : F \hookrightarrow \Gamma$ on the equivariant cohomology is injective over the localized ring. Also the pushforward ρ_* is surjective, since $diag\,(i\lambda_1,\dots,i\lambda_N)\otimes 1_{A_0}$ and the maximal abelian subalgebra in which it is contained does not have fixed points outside of F . It follows, see [AB], that the integral of the $\mathbf{t}$ -equivariantly closed form equals the integral over F (3.1).

4 The localization formula in the commutative algebra case.

In this section for simplicity it is assumed that $H\otimes\mathbb{C} = A$ i.e. that the algebra is abelian.

4.1 The restriction of the differential form to F

The only nonzero terms in the restriction of

$$\exp\frac{1}{i}(Tr\langle Y,X\rangle + \frac{1}{3!}m_{A\otimes gl_N}(X,\frac{\partial}{\partial X}))\vdash \prod dX_i^{\alpha,j} \tag{4.1}$$

to F come from the terms which can contain only $X_i^{\alpha,j}$ with $i=j$ and in which all $dX_i^{\alpha,j}$ with $i\neq j$ are killed by the multiple insertions of the bivector

$$\frac{1}{2}\sum_{\alpha,\beta,\gamma}(m_A)_{\alpha}^{\beta\gamma}\sum_{i,j,l}(X_l^{\alpha,i}\frac{\partial}{\partial X_l^{\beta,j}}\wedge\frac{\partial}{\partial X_j^{\gamma,i}}). \tag{4.2}$$

The differential $dX_j^{\beta,i}$ is killed by the insertion of $\frac{\partial}{\partial X_j^{\beta,i}}$ taken from the term

$$\frac{1}{2}\sum_{\beta_1,\beta_2,\alpha}(m_A)_{\alpha}^{\beta_1\beta_2}\sum_{i\neq j}(X_i^{\alpha,i}\frac{\partial}{\partial X_i^{\beta_1,j}}\wedge\frac{\partial}{\partial X_j^{\beta_2,i}}) =$$
$$= \frac{1}{2}\sum_{(\beta_1,\beta_2),\alpha}\sum_{i<j}((m_A)_{\alpha}^{\beta_1\beta_2}X_i^{\alpha,i} - (m_A)_{\alpha}^{\beta_2\beta_1}X_j^{\alpha,j})\frac{\partial}{\partial X_i^{\beta_1,j}}\wedge\frac{\partial}{\partial X_j^{\beta_2,i}} \tag{4.3}$$

The resulting coefficient in front of the top ($\dim F$) degree nonzero product is the product over pairs of $i<j$ of pfaffians of the matrices

$$\begin{pmatrix} 0 & (\sum_\alpha X_i^{\alpha,i}e_\alpha)(e^{\beta_1}e^{\beta_2}) - (\sum_\alpha X_j^{\alpha,j}e_\alpha)(e^{\beta_2}e^{\beta_1}) \\ (\sum_\alpha X_j^{\alpha,j}e_\alpha)(e^{\beta_1}e^{\beta_2}) - (\sum_\alpha X_i^{\alpha,i}e_\alpha)(e^{\beta_2}e^{\beta_1}) & 0 \end{pmatrix}$$

or

$$(-1)^{\frac{n(n-1)}{2}}\det_{\beta_1\beta_2}\left(X_i^i(e^{\beta_1}e^{\beta_2}) - X_j^j(e^{\beta_2}e^{\beta_1})\right)$$

which coincides with

$$(-1)^{\frac{n(n-1)}{2}} \det_{\beta_1\beta_2} \left((X_i^i - X_j^j)(e^{\beta_1} e^{\beta_2}) \right).$$

After identification $A^\vee = A$ induced by the even scalar product η, $\eta(e^\alpha, e^\beta)$:

$$\sum_\alpha X^\alpha e_\alpha(e^{\beta_1} e^{\beta_2}) = \sum_\alpha X^\alpha \eta(e^\alpha, e^{\beta_1} e^{\beta_2}) = X^\alpha \eta(e^\alpha e^{\beta_1}, e^{\beta_2}) = (l_X)^{\beta_1\beta_2}$$

where l_X is the operator of left multiplication by $X \in A$ acting on A. Therefore the restriction of the differential form 4.1 to the fixed locus F is

$$\exp \frac{1}{i}(\sum_k \langle Y_k, X_k \rangle + \frac{1}{3!} m^{\alpha\beta\gamma}(X_{k\alpha} X_{k\beta} X_{k\gamma})) \prod_{k<j} (-1)^{\frac{n(n-1)}{2}} \det_A ((X_k - X_j)*) \prod dX_{k\alpha} \tag{4.4}$$

where $Y_k \in U$, $X_k \in S$, $X_k = X_{k\alpha} e^\alpha$ are the entries at the $k-th$ places of the diagonal matrices $Y = \operatorname{diag}(Y_1, \ldots, Y_N)$, $Y \in \mathbf{t}$, $X = \operatorname{diag}(X_1, \ldots, X_N)$, $X \in F$.

4.2 The equivariant Euler class of the normal bundle

The normal bundle to F in Γ is the direct sum of $\frac{N(N-1)}{2}$ real $\mathbf{t}$-invariant subspaces of dimension $2 \dim A$

$$V_{kj} = (E_k^j + E_j^k) \otimes S \oplus (iE_k^j - iE_j^k) \otimes S,\ j < k \tag{4.5}$$

where $S = iU$ denotes the subspace of hermitian elements in A and E_k^j are the elementary matrices. These subbundles are topologically trivial. The euler class is determined by the action of $\mathbf{t}$ on a generic fiber. The matrix of the action of

$$Y = \operatorname{diag}(iy_1, \ldots, iy_N),\ y_i \in S$$

on V_{kj} is

$$\begin{pmatrix} 0 & -(l_{y_k} - l_{y_j}) \\ (l_{y_k} - l_{y_j}) & 0 \end{pmatrix}$$

The euler class of V_{kj} is given by the pfaffian :

$$\mathrm{eu}_{\mathbf{t}}(V_{kj}) = (-1)^{\frac{n(n-1)}{2}} \det_S ((y_k - y_j)*) \tag{4.6}$$

4.3 The integral over the fixed point set F.

From (4.4)(4.6) I get the following formula for the integral:

$$\int_F \exp \frac{1}{i}(\sum_i \langle iy_i, X_i \rangle + \frac{1}{3!} m^{\alpha\beta\gamma}(X_{i\alpha} X_{i\beta} X_{i\gamma})) \frac{\prod_{i<j} \det_S ((X_i - X_j)*)}{\prod_{i<j} \det_S ((y_k - y_j)*)} \prod dX_{i\alpha}$$

The matrices of linear operators X_j* , $X_j \in S$ commute. Therefore the product $\prod_{i<j} \det_A ((X_i - X_j)*)$ coincides with the generalized Vandermond determinant consisting of blocks of matrices of linear operators $(X_j*)^l$

$$\prod_{i<j} \det_A ((X_i - X_j)*) = \det_{A \otimes \mathbb{C}^N} \begin{pmatrix} (X_1*)^0 & \ldots & (X_N*)^0 \\ \vdots & \ddots & \vdots \\ (X_1*)^{N-1} & \ldots & (X_N*)^{N-1} \end{pmatrix}$$

Then

$$\int_F \exp(\sum_k \langle y_k, X_k\rangle - \frac{i}{3!} m^{\alpha\beta\gamma}(X_{k\alpha}X_{k\beta}X_{k\gamma}))\prod_{k<j}\det_S((X_k - X_j)*)\prod_{k,\alpha} dX_{k\alpha} = \det{}_{A\otimes\mathbb{C}^N}\begin{pmatrix}\Phi_0(y_1) & \dots & \Phi_0(y_N)\\ \vdots & \ddots & \vdots\\ \Phi_{N-1}(y_1) & \dots & \Phi_{N-1}(y_N)\end{pmatrix}$$

where $\Phi_k(y)$, $iy \in U$ is the $\dim A \times \dim A$ matrix of the linear operator of multiplication by the element

$$\int_S \exp(\langle y, x\rangle - \frac{i}{3!} m^{\alpha\beta\gamma}(x_\alpha x_\beta x_\gamma))x^{*k}\prod_\alpha dx_\alpha$$

so that the localized integral is the ratio of determinants:

$$\det{}_{A\otimes\mathbb{C}^N}\begin{pmatrix}\Phi_0(y_1) & \dots & \Phi_0(y_N)\\ \vdots & \ddots & \vdots\\ \Phi_{N-1}(y_1) & \dots & \Phi_{N-1}(y_N)\end{pmatrix} / \prod_{i<j}\det_A((y_k - y_j)*) \tag{4.7}$$

4.4 Tau function.

Let for $h_i \in A$, $i = 1, \dots, N$, define

$$t_{k,\alpha} = \frac{1}{k}\sum_{i=1}^{i=N}(h_i^{*k})_\alpha$$

and consider the operator

$$\rho(t_k) = \exp\left(\lambda^{-k} t_k *\right) = \prod_{i=1}^N \left(1 - \frac{(h_i *)}{\lambda}\right)^{-1}$$

acting on $A((\lambda))$, the space of series in variable λ with values in A , and on its semi-infinite wedge space

$$\Lambda^{\frac{\infty}{2}}(A((\lambda)))$$

The tau-function associated with a subspace $W \subset A((\lambda))$ with base given by series

$$w_j(\lambda) = \sum_{n=-j} w_{jn}\lambda^n,\ w_{jn} \in gl(A) \tag{4.8}$$

is defined in the standard way as the determinant:

$$\tau(t_{k,\alpha}) = \det\left[\rho(t_k)\begin{pmatrix}\dots & w_{j(-j)} & \dots & w_{0n}\\ & \vdots & & \vdots\\ \dots & w_{j0} & \dots & w_{00}\\ & \vdots & & \vdots\\ \dots & w_{jn} & \dots & w_{0n}\\ \dots & \vdots & \dots & \vdots\end{pmatrix}\right]_+$$

of the part of the matrix corresponding to the projection on the subspace $A\left[\left[\lambda^{-1}\right]\right]$.

Theorem 1 *The τ -function $\tau(t_{k,\alpha})$, $t_{k,\alpha} = \frac{1}{k}\sum_{i=1}^{i=N}(h_i^{*k})_\alpha$ associated with the subspace with base* (4.8) *with $w_{jn}(e^\gamma) = w^\gamma_{jn,\beta}(e^\beta *)$ coincides with the ratio of determinants*

$$\tau(t_{k,\alpha}) = \det{}_{A\otimes\mathbb{C}^N}(w_j(h_k)) / \prod_{i<j}\det_A((h_k - h_j)*)\,. \tag{4.9}$$

Define

$$\left(\boldsymbol{\partial}^2\right)_k = \sum_\alpha e^\alpha m_{\alpha\beta\gamma} \frac{\partial^2}{\partial t_{k,\beta}\partial t_{k,\gamma}}$$

and similarly $\left(\boldsymbol{\partial}^3\right)_k$, $\left(\boldsymbol{\partial}^4\right)_k$, etc. And similarly the operators D_k acting on the tensor products $\tau(t)\otimes\tau(s)$

Theorem 2 *The τ-function $\tau(t_{k,\alpha})$ satisfies the Hirota bilinear equation:*

$$(D_1^4 + 3D_2^2 - 4D_1D_3)\tau\cdot\tau = 0$$

4.5 The asymptotic series developpment of $\Phi_k(y)$ and the semi-infinite subspace.

Proposition 4 *Let $y = -\frac{1}{2}z^{*2}$ then $\Phi_k(y(z))/\Phi_{0,[2]}(z) = \tilde{\Phi}_k(z)$ is a function of z with values in A whose $(-n)-$ th coefficient in the asymptotic development is linear in z^{*n}*

Let $\tau_{W^\Phi}(t_{k,\alpha})$, $t_{k,\alpha} = \frac{1}{k}\sum_{i=1}^{i=N}(z_i^{*(-k)})_\alpha$ be the τ-function associated with the subspace with base given by the series $\tilde{\Phi}_k(\lambda) = \sum_n \lambda^n \tilde{\Phi}_{kn}$

Theorem 3 *The normalized integral*

$$\left(\mathcal{F}_{[2]}(Z)\right)^{-1}\int_\Gamma \exp\frac{1}{i}(-\frac{1}{2}Tr\langle Z^{*2}, X\rangle + \frac{1}{3!}m_{A\otimes gl_N}(X, \frac{\partial}{\partial X})) \vdash \prod dX_i^{\alpha,j}$$

coincides with the τ-function $\tau_{W^\Phi}(t_{k,\alpha})$.

4.6 The nonabelian case and the Zacharov-Shabat method.

In the case of A which is not abelian, the fixed locus of $\mathbf{t}-$ action to which the integral localizes is the subspace of hermitian diagonal matrices with values in the maximal abelian subalgebra of A. The localization formula gives the ratio of determinants similar to (4.7). The integrable hierarchy can be constructed in the same way, using the Zacharov-Shabat method, see[DS], for the abelian Lie subalgebra $H \subset A$, and the localized integral coincides with the τ-function of this hierarchy, and satisfies the Hirota quadratic equations.

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