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Abstract. Based on the finite element method, the wave finite element method (WFE) permits to analyze the dynamics of a periodic structure by using a wave decomposition of one period. This method reduces the number of DOF and it has advantages in calculation time. However, it cannot be applied easily to a railway track because this structure is subjected by moving loads which are not considered in a classical WFE. In this article, we present a technique to deal with moving loads applying in a railway track where the track components are modeled by 3D continuous media. By using the classical WFE for one track period in frequency domain, we can rewrite the vector of DOF and loads in a wave base. Then, we can calculate the wave amplitudes of the moving loads from their representation in this base. Thereafter, we apply the wave analyze of WFE to the hold structure. The result shows that the moving loads lead to a sum of wave amplitudes. Finally, we apply this method for a railway track subjected to constant moving loads with numerical application. The new technique permits to analyze the dynamic of railway tracks by considering only one track period.

1 INTRODUCTION

The dynamic of railway track has been studied with different methods. The analytical methods permit to calculate very fast track responses for different types of tracks [1-6]. These methods base on the model of beams supported by viscoelastic foundation or the model of periodically supported beams [6]. The numerical methods have been developed by using the finite element method (FEM). The advantage of this method is to give a complete analysis. However, it can not take easily the whole length of the railway track and the calculation time is long. Some reduced techniques have been developed for the FEM. Recently, the wave finite element method [7-10] has been applied to the railway track analysis. This method is based on the wave decomposition of one track period then using the wave analysis to compute the response of the whole track. However, the classical WFE can not take into account the moving loads of a train applied to the rail and this article deals with this problem.

By rewriting the dynamic equation of one period (a substructure) of the railway track subjected to a load in the frequency domain, we obtain a relation of the vector of DOF and nodal loads at the left and right boundary of the substructure and this relation is including the moving loads. Then, we use the WFE technique to rewrite the expression of the track response. In the other side, we present a bounded limit for an infinite periodic railway track which leads to a simple expression of the response. The applications have been developed then for a normal railway track and ones with a defect zone or a transition zone.

1.1 Dynamic equation

Consider a railway track with a periodic interval as shown in Figure 1. This interval is represented by one period (a substructure) with all components of the tracks (rails, pads, supports, fondations...). The track is subjected to dynamic loads of a train and we consider that all loads are given.

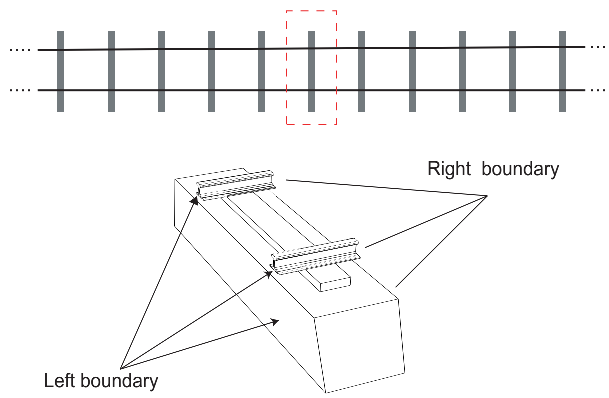

Figure 1: Periodic railway track represented by a substructure

By using the finite element method for a substructure, the dynamic equation can be written as follows

$$\mathbf{D}_{tot}\mathbf{q} = \mathbf{F} \tag{1}$$

where $\mathbf{D}_{tot} = \mathbf{K} + j\omega\mathbf{C} - \omega^2\mathbf{M}$ is the dynamic stiffness matrix (DSM) of the substructure. We note that the aforementioned equation holds for all type of modeling (beam, 2D or 3D). Then, we can rewrite the DSM as follows

$$\begin{bmatrix} \tilde{\mathbf{D}}_{II} & \tilde{\mathbf{D}}_{IL} & \tilde{\mathbf{D}}_{IR} \\ \tilde{\mathbf{D}}_{LI} & \tilde{\mathbf{D}}_{LL} & \tilde{\mathbf{D}}_{LR} \\ \tilde{\mathbf{D}}_{RI} & \tilde{\mathbf{D}}_{RL} & \tilde{\mathbf{D}}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{F}_I \\ \mathbf{F}_L \\ \mathbf{F}_R \end{bmatrix} \tag{2}$$

where L, R and I denote for the left, right boundaries and inner nodes of the substructure as shown in Figure 1. Then, we can reduce the inner nodes $\mathbf{q}_I$ of the cell from the first row of equation (2) and we obtain

$$\begin{bmatrix} \mathbf{D}_{LI}\mathbf{F}_I \\ \mathbf{D}_{RI}\mathbf{F}_I \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{F}_L \\ \mathbf{F}_R \end{bmatrix} \tag{3}$$

where

$$\begin{array}{ll} \mathbf{D}_{LL} = \tilde{\mathbf{D}}_{LL} - \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IL} & \mathbf{D}_{LR} = \tilde{\mathbf{D}}_{LR} - \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IR} \\ \mathbf{D}_{RL} = \tilde{\mathbf{D}}_{RL} - \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IL} & \mathbf{D}_{RR} = \tilde{\mathbf{D}}_{RR} - \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IR} \\ \mathbf{D}_{LI} = \tilde{\mathbf{D}}_{LI}\tilde{\mathbf{D}}_{II}^{-1} & \mathbf{D}_{RI} = \tilde{\mathbf{D}}_{RI}\tilde{\mathbf{D}}_{II}^{-1} \end{array} \tag{4}$$

We see that equation (3) presents a relation between the loads and displacements at the left and right boundaries of a cell. This equation contains a term of $\mathbf{F}_I$ which is zero when the substructure has non external loads.

1.2 Boundary conditions

We denote (n) for the substructure number n in the track interval. For the two consecutive substructures, the right boundary of (n) is the left boundary of $(n+1)$. Therefore, we have

$$\begin{array}{l} \mathbf{q}_R^{(n)} = \mathbf{q}_L^{(n+1)} \\ \mathbf{F}_R^{(n)} + \mathbf{F}_L^{(n+1)} = \mathbf{F}_{\partial R}^{(n)} \end{array} \tag{5}$$

where $\mathbf{F}_{\partial R}^{(n)}$ is the external load at the right boundary R of the substructure (n). By combining equations (3) and (5), we obtain

$$\begin{bmatrix} \mathbf{q}_L^{(n+1)} \\ -\mathbf{F}_L^{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{qI}\mathbf{F}_I^{(n)} \\ \mathbf{D}_{fI}\mathbf{F}_I^{(n)} - \mathbf{F}_{\partial R}^{(n)} \end{bmatrix} + \mathbf{S} \begin{bmatrix} \mathbf{q}_L^{(n)} \\ -\mathbf{F}_L^{(n)} \end{bmatrix} \tag{6}$$

where

$$\mathbf{S} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{LR}^{-1} \\ \mathbf{D}_{RL} - \mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{RR}\mathbf{D}_{LR}^{-1} \end{bmatrix}, \tag{7}$$

and

$$\begin{bmatrix} \mathbf{D}_{qI} \\ \mathbf{D}_{fI} \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LI} \\ \mathbf{D}_{RI} - \mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LI} \end{bmatrix} \tag{8}$$

We can rewrite equation (6) as follows

$$\mathbf{u}^{(n+1)} = \mathbf{S}\mathbf{u}^{(n)} + \mathbf{b}^{(n)} \tag{9}$$

where

$$\mathbf{u}^{(n)} = \begin{bmatrix} \mathbf{q}_L^{(n)} \\ -\mathbf{F}_L^{(n)} \end{bmatrix}, \quad \mathbf{b}^{(n)} = \begin{bmatrix} \mathbf{D}_{qI}\mathbf{F}_I^{(n)} \\ \mathbf{D}_{fI}\mathbf{F}_I^{(n)} - \mathbf{F}_{\partial R}^{(n)} \end{bmatrix} \tag{10}$$

Equation (9) presents a relation between DOF (displacements and loads) of a substructure (n) and its next substructures $(n+1)$. Here $\mathbf{b}^{(n)}$ presents the external loads on the substructures (n) (hence, when the substructure is free, $\mathbf{b}^{(n)} = 0$). For a series of substructures, this equation presents a geometric series which can be reduced to the following results

$$\mathbf{u}^{(n)} = \mathbf{S}^n \mathbf{u}^{(0)} + \sum_{k=1}^{n} \mathbf{S}^{n-k} \mathbf{b}^{(k)} \tag{11}$$

$$\mathbf{u}^{(0)} = \mathbf{S}^n \mathbf{u}^{(-n)} + \sum_{k=0}^{n-1} \mathbf{S}^{k} \mathbf{b}^{(-k)} \tag{12}$$

Equations (11) and (12) give the relations of the responses at the substructure (n) and $(-n)$ respectively, and the response at the reference origin. Note that the origin can be placed at any substructure because the railway track is periodic.

1.3 Load of a train

The load of a train applying on one period (n) of the track is presented by a dynamic force $f_n(x,t)$ with x is local position and t is the time. In the frequency domain, when using the finite element method, the nodal load on the rail at the period n in the moving reference is presented by $\mathbf{f}_n(\omega)$. In the fixed reference, the load on the period (n) is presented by $f_n\left(x, t+\frac{nl}{v}\right)$ with l is the length of one period and v is the train speed. Thus, the nodal load in the frequency domain is given by

$$\mathbf{F}_*^{(n)}(\omega) = \mathrm{e}^{\mathrm{i}\omega\frac{nl}{v}} \mathbf{f}_*^{(n)}(\omega) \tag{13}$$

where $\mathbf{F}_*$ can be $\mathbf{F}_I$ or $\mathbf{F}_{\partial R}$. We see that when the train load is stable, $\mathbf{f}_*^{(n)}(\omega)$ does not depend on n and the load on all period has the same spectrum but different phases given by the first term $\mathrm{e}^{\mathrm{i}\omega\frac{nl}{v}}$.

Bounded conditions: We suppose that the train move on an limited interval of the track and therefore, the track response at infinity are bounded

$$\lim_{n\to\pm\infty} \{\mathbf{q}^{(n)}, \mathbf{F}^{(n)}\} \text{ are bounded} \tag{14}$$

2 WAVE DECOMPOSITION

2.1 Calculation of wave base

We will now calculate the eigenvalues and eigenvectors $\{\mu_j, \phi_j\}_j$ of the matrix $\mathbf{S}$ given by equation (9). By definition, we have

$$\mathbf{S}\phi_j = \mu_j \phi_j \tag{15}$$

Due to the symplectic nature of the matrix $\mathbf{S}$, we consider the eigenproblem of the transformation $\mathbf{S} + \mathbf{S}^{-1}$ which yields eigenvalues of the form $\lambda_j = \mu_j + 1/\mu_j$ given by [9]

$$\left[\left(\mathbf{N}'\mathbf{J}\mathbf{L}'^T + \mathbf{L}'\mathbf{J}\mathbf{N}'^T\right) - \lambda_j \mathbf{L}'\mathbf{J}\mathbf{L}'^T\right] \mathbf{z}_j = 0 \tag{16}$$

where

$$\mathbf{L}' = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{D}_{LR} & \mathbf{0} \end{bmatrix}, \quad \mathbf{N}' = \begin{bmatrix} \mathbf{D}_{RL} & \mathbf{0} \\ -(\mathbf{D}_{LL} + \mathbf{D}_{RR}) & -\mathbf{I}_n \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{0} \end{bmatrix} \tag{17}$$

Thereafter, each pair of eigenvalues $(\mu_j, \mu_j^\star)$ can be computed analytically by the quadratic equation $(x^2 - \lambda_j x + 1 = 0)$. Also, the eigenvectors corresponding to these eigenvalues are computed by the closed-form expressions

$$\phi_j = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{D}_{RR} & \mathbf{I}_n \end{bmatrix} \mathbf{w'}_j, \quad \phi_j^\star = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{D}_{RR} & \mathbf{I}_n \end{bmatrix} \mathbf{w'}_j^\star \tag{18}$$

where $\mathbf{w'}_j = \mathbf{J}(\mathbf{L'}^T - \mu_j^\star \mathbf{N'}^T)\mathbf{z}_j$ and $\mathbf{w'}_j^\star = \mathbf{J}(\mathbf{L'}^T - \mu_j \mathbf{N'}^T)\mathbf{z}_j$.

If we note $\mathbf{\Phi} = [\phi_1 \cdots \phi_n]$ and $\mathbf{\Phi}^\star = [\phi_1^\star \cdots \phi_n^\star]$, we have a wave base $\{\mathbf{\Phi}, \mathbf{\Phi}^\star\}$ of the transformation $\mathbf{S}$. We can also separate the components of the wave base corresponding to $\mathbf{q}, \mathbf{F}$ as follows

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_q \\ \mathbf{\Phi}_F \end{bmatrix} \quad \mathbf{\Phi}^\star = \begin{bmatrix} \mathbf{\Phi}_q^\star \\ \mathbf{\Phi}_F^\star \end{bmatrix} \tag{19}$$

Orthogonality and normalisation: the wave base is symplectic orthogonal in the meaning of $\phi_j^T \mathbf{J} \phi_i = \phi_j^{\star T} \mathbf{J} \phi_i^\star = 0$ $(\forall i, j)$ and $\phi_j^{\star T} \mathbf{J} \phi_i = \phi_j^T \mathbf{J} \phi_i^\star = 0$ $(\forall i \neq j)$(see [9]). However, the base is not normalized automatically after computation of the eigenproblem. We can calculate the weighting matrix as follows

$$\begin{aligned} \mathbf{\Psi} &= \mathbf{\Phi}^{\star T} \mathbf{J} \mathbf{\Phi} = \mathbf{\Phi}_q^{\star T} \mathbf{\Phi}_F - \mathbf{\Phi}_F^{\star T} \mathbf{\Phi}_q \\ \mathbf{\Psi}^\star &= \mathbf{\Phi}^T \mathbf{J} \mathbf{\Phi}^\star = \mathbf{\Phi}_q^T \mathbf{\Phi}_F^\star - \mathbf{\Phi}_F^T \mathbf{\Phi}_q^\star \end{aligned} \tag{20}$$

The matrices $\mathbf{\Psi}, \mathbf{\Psi}^\star$ are diagonal and satisfying $\mathbf{\Psi}^\star = -\mathbf{\Psi}^T$. Thus, we can normalize the wave base by calculating a matrix $\mathbf{T} = \mathbf{\Psi}^{1/2}$ where $\mathbf{T}$ is a diagonal matrix with the diagonal equals to the square root of the diagonal of $\mathbf{\Psi}$. The normalized wave base is presented by $\mathbf{\Phi T}$ and $\mathbf{\Phi}^\star \mathbf{T}$.

2.2 Wave decomposition

We can decompose each vector of equation (9) in this wave base as follows

$$\begin{aligned} \mathbf{u}^{(n)} &= \mathbf{\Phi} \mathbf{Q}^{(n)} - \mathbf{\Phi}^\star \mathbf{Q}^{\star(n)} \\ \mathbf{b}^{(n)} &= \mathbf{\Phi} \mathbf{Q}_E^{(n)} - \mathbf{\Phi}^\star \mathbf{Q}_E^{\star(n)} \end{aligned} \tag{21}$$

where $\mathbf{Q}^{(n)}, \mathbf{Q}^{\star(n)}$ are the wave amplitudes of $\mathbf{u}^{(n)}$ and $\mathbf{Q}_E^{(k)}, \mathbf{Q}_E^{\star(k)}$ are the wave amplitudes of the external loads on the intermediate substructures $\mathbf{b}^{(n)}$.

Remark: the wave decomposition in equation (21) is different to usual expression for WFE by the minus sign on the right to left waves. The avantage of this expression is that we can calculate directly the wave amplitudes by using the symplectic orthogonality of the wave base as the following

$$\begin{aligned} \mathbf{Q}^{(n)} &= \mathbf{\Phi}^{\star T} \mathbf{J} \mathbf{u}^{(n)}, \quad \mathbf{Q}^{\star(n)} = \mathbf{\Phi}^T \mathbf{J} \mathbf{u}^{(n)} \\ \mathbf{Q}_E^{(n)} &= \mathbf{\Phi}^{\star T} \mathbf{J} \mathbf{b}^{(n)}, \quad \mathbf{Q}_E^{\star(n)} = \mathbf{\Phi}^T \mathbf{J} \mathbf{b}^{(n)} \end{aligned} \tag{22}$$

By substituting equation (10) into equation (22), we obtain

$$
\begin{aligned}
\mathbf{\Phi}^{\star T}\mathbf{J}\mathbf{b}^{(n)} &= \left(\mathbf{\Phi}_q^{\star T}\mathbf{D}_{fI} - \mathbf{\Phi}_F^{\star T}\mathbf{D}_{qI}\right)\mathbf{F}_I^{(k)} - \mathbf{\Phi}_q^{\star T}\mathbf{F}_{\partial R}^{(k)} \\
\mathbf{\Phi}^{T}\mathbf{J}\mathbf{b}^{(n)} &= \left(\mathbf{\Phi}_q^{T}\mathbf{D}_{fI} - \mathbf{\Phi}_F^{T}\mathbf{D}_{qI}\right)\mathbf{F}_I^{(k)} - \mathbf{\Phi}_q^{T}\mathbf{F}_{\partial R}^{(k)}
\end{aligned}
\tag{23}
$$

In addition, we have the relation between the $\mathbf{\Phi}_q$ and $\mathbf{\Phi}_F$ as follows (see [?])

$$
\begin{aligned}
\mathbf{\Phi}_F &= \mathbf{D}_{RR}\mathbf{\Phi}_q + \mathbf{D}_{RL}\mathbf{\Phi}_q\boldsymbol{\mu}^\star = -\left(\mathbf{D}_{LL}\mathbf{\Phi}_q + \mathbf{D}_{LR}\mathbf{\Phi}_q\boldsymbol{\mu}\right) \\
\mathbf{\Phi}_F^\star &= \mathbf{D}_{RR}\mathbf{\Phi}_q^\star + \mathbf{D}_{RL}\mathbf{\Phi}_q^\star\boldsymbol{\mu} = -\left(\mathbf{D}_{LL}\mathbf{\Phi}_q^\star + \mathbf{D}_{LR}\mathbf{\Phi}_q^\star\boldsymbol{\mu}^\star\right)
\end{aligned}
\tag{24}
$$

By combining equations (22) and (23) into equation (24), we obtain

$$
\begin{aligned}
\mathbf{Q}_E^{(k)} &= \left[\left(\boldsymbol{\mu}\mathbf{\Phi}_q^{\star T}\mathbf{D}_{LI} + \mathbf{\Phi}_q^{\star T}\mathbf{D}_{RI}\right)\mathbf{F}_I^{(k)} - \mathbf{\Phi}_q^{\star T}\mathbf{F}_{\partial R}^{(k)}\right] \\
\mathbf{Q}_E^{\star(k)} &= \left[\left(\boldsymbol{\mu}^\star\mathbf{\Phi}_q^{T}\mathbf{D}_{LI} + \mathbf{\Phi}_q^{T}\mathbf{D}_{RI}\right)\mathbf{F}_I^{(k)} - \mathbf{\Phi}_q^{T}\mathbf{F}_{\partial R}^{(k)}\right]
\end{aligned}
\tag{25}
$$

Equation (25) show that the wave amplitude of external loads on one substructure can be calculated directly from its loads and it does not depend on the other substructures.

Now we will calculate the amplitude from the wave amplitudes $\{\mathbf{Q}^{(n)}, \mathbf{Q}^{\star(n)}\}$. By replacing equation (21) with $n = 0$ into equation (11), we obtain

$$
\mathbf{u}^{(n)} = \mathbf{\Phi}\mathbf{Q}^{(n)} - \mathbf{\Phi}^\star\mathbf{Q}^{\star(n)} = \mathbf{\Phi}\boldsymbol{\mu}^n\mathbf{Q} - \mathbf{\Phi}^\star\boldsymbol{\mu}^{\star n}\mathbf{Q}^\star + \sum_{k=1}^{n}\mathbf{\Phi}\boldsymbol{\mu}^{n-k}\mathbf{Q}_E^{(k)} - \mathbf{\Phi}^\star\boldsymbol{\mu}^{\star n-k}\mathbf{Q}_E^{\star(k)}
\tag{26}
$$

Then, by substituting equation (22) into the aforementioned equation, we obtain

$$
\begin{aligned}
\mathbf{Q}^{(n)} &= \boldsymbol{\mu}^n\left(\mathbf{Q} + \sum_{k=1}^{n}\boldsymbol{\mu}^{\star k}\mathbf{Q}_E^{(k)}\right) \\
\mathbf{Q}^{\star(n)} &= \boldsymbol{\mu}^{\star n}\left(\mathbf{Q}^\star + \sum_{k=1}^{n}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{\star(k)}\right)
\end{aligned}
\tag{27}
$$

In a similar way, by combining equations (21) into equation (12), we obtain the following result

$$
\begin{aligned}
\mathbf{Q}^{(-n)} &= \boldsymbol{\mu}^{\star n}\left(\mathbf{Q} - \sum_{k=0}^{n-1}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{(-k)}\right) \\
\mathbf{Q}^{\star(-n)} &= \boldsymbol{\mu}^{n}\left(\mathbf{Q}^\star - \sum_{k=0}^{n-1}\boldsymbol{\mu}^{\star k}\mathbf{Q}_E^{\star(-k)}\right)
\end{aligned}
\tag{28}
$$

Equations (27) and (28) present the relations between the wave amplitudes of the external loads and the amplitudes $\{\mathbf{Q}, \mathbf{Q}^\star\}$ of $\mathbf{u}^{(0)}$. In the next section, we will develop the wave analysis by using this result and the boundary condition for an infinite periodic track.

3 ANALYSIS OF A COMPLETE RAILWAY TRACK

3.1 Response of an infinite periodic railway track

We consider a railway without defects as show in Figure 1. The track is an infinite periodic structure subjected to moving dynamic loads which can be different from one period to another. We will find the response by using the bounded condition.

By substituting equations (27) and (28) into equation (21), we obtain

$$\begin{aligned} \mathbf{u}^{(n)} &= \mathbf{\Phi}\boldsymbol{\mu}^{n}\left(\mathbf{Q}+\sum_{k=1}^{n}\boldsymbol{\mu}^{\star k}\mathbf{Q}_E^{(k)}\right)-\mathbf{\Phi}^{\star}\boldsymbol{\mu}^{\star n}\left(\mathbf{Q}^{\star}+\sum_{k=1}^{n}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{\star(k)}\right) \\ \mathbf{u}^{(-n)} &= \mathbf{\Phi}\boldsymbol{\mu}^{\star n}\left(\mathbf{Q}-\sum_{k=0}^{n-1}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{(-k)}\right)-\mathbf{\Phi}^{\star}\boldsymbol{\mu}^{n}\left(\mathbf{Q}^{\star}-\sum_{k=0}^{n-1}\boldsymbol{\mu}^{\star k}\mathbf{Q}_E^{\star(-k)}\right) \end{aligned} \tag{29}$$

For a damped structure, $\|\boldsymbol{\mu}\| < 1$ and $\|\boldsymbol{\mu}^{\star}\| > 1$ (see [8]). Therefore, we have $\boldsymbol{\mu}^{n} \to 0$ and $\boldsymbol{\mu}^{\star n} \to \infty$ when n tends to infinity. Therefore, in order to get $\mathbf{u}^{(n)}$ and $\mathbf{u}^{(-n)}$ in equation (29) satisfying the bounded condition (14), the coefficients corresponding to $\boldsymbol{\mu}^{\star n}$ must tend to zeros when $n \to \infty$. That means

$$\mathbf{Q} = \sum_{k=0}^{\infty}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{(-k)}, \quad \mathbf{Q}^{\star} = -\sum_{k=1}^{\infty}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{\star(k)} \tag{30}$$

By substituting the aforementioned equation into equation (21) with $n = 0$, we obtain

$$\begin{aligned} \mathbf{q}^{(0)} &= \mathbf{\Phi}_q\sum_{k=0}^{\infty}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{(-k)}+\mathbf{\Phi}_q^{\star}\sum_{k=1}^{\infty}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{\star(k)} \\ \mathbf{F}^{(0)} &= \mathbf{\Phi}_F\sum_{k=0}^{\infty}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{(-k)}+\mathbf{\Phi}_F^{\star}\sum_{k=1}^{\infty}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{\star(k)} \end{aligned} \tag{31}$$

Equation (31) is the expression of the response of the substructure at $n = 0$. For the others substructures $n \neq 0$, we can calculate the response by using equation (29). There is another way to calculate the responses $\mathbf{u}^{(n)}$ by translating the reference origin and using again the formulation (31).

3.2 Remarks

- We see that the response $\mathbf{u}^{(0)}$ is the combination of two terms corresponding to the sum of left and right waves generated by the external forces from the two sides. If the structure is subjected by an external load at only one side of a period, the expressions in equation (31) have only one term and the eigenvalue $\boldsymbol{\mu}$ is exactly the rate of the wave amplitudes between the left and right boundary of one period. Therefore, the eigenvalues $\boldsymbol{\mu}$ plays a role as a **structural damping factor** of a periodic structure.

Table 1: Parameters of a periodically supported beam

Rail section mass (ρS)	kg/m	60
Rail stiffness (EI)	MNm^2	6.3
Distance of sleepers (l)	m	0.6
Block mass (M)	kg	100
Damping factor of rail pad (η_1)	MNsm^{-1}	1.97
Stiffness of rail pad (k_1)	MNm^{-1}	192
Damping coeff. under support (η_2)	MNsm^{-1}	0.17
Stiffness under support (k_2)	MNm^{-1}	26.4

- This method does not need to inverse any matrix, the responses can be calculated directly from the wave decompositions. Therefore, we have reduced the DOF of all structure to only one substructure.

- When the moving load is the same for all substructure, by combining equations (13) and (25) we obtain

$$\mathbf{Q}_E^{\star(n)} = \mathrm{e}^{\mathrm{i}\omega\frac{nl}{v}}\mathbf{Q}_E^{\star(0)}, \quad \mathbf{Q}_E^{(k)} = \mathrm{e}^{\mathrm{i}\omega\frac{nl}{v}}\mathbf{Q}_E^{(0)} \quad \forall n \tag{32}$$

In addition, $\boldsymbol{\mu}$ is a diagonal matrix, we can use a formula of the geometric series $\sum_{k=0}^{\infty}(a\boldsymbol{\mu})^k = \frac{1}{1-a\boldsymbol{\mu}}$ provided that $\|a\boldsymbol{\mu}\| < 1$ (this is a diagonal matrix with the values calculated by the formulas in the expression). Therefore, we can rewritten equation (31) as follows

$$\begin{aligned}\mathbf{q}^{(0)} &= \boldsymbol{\Phi}_q\frac{1}{1-\boldsymbol{\mu}\mathrm{e}^{-\mathrm{i}\omega\frac{l}{v}}}\mathbf{Q}_E^{(0)} + \boldsymbol{\Phi}_q^{\star}\frac{\boldsymbol{\mu}\mathrm{e}^{\mathrm{i}\omega\frac{l}{v}}}{1-\boldsymbol{\mu}\mathrm{e}^{\mathrm{i}\omega\frac{l}{v}}}\mathbf{Q}_E^{\star(0)}\\ \mathbf{F}^{(0)} &= \boldsymbol{\Phi}_F\frac{1}{1-\boldsymbol{\mu}\mathrm{e}^{-\mathrm{i}\omega\frac{l}{v}}}\mathbf{Q}_E^{(0)} + \boldsymbol{\Phi}_F^{\star}\frac{\boldsymbol{\mu}\mathrm{e}^{\mathrm{i}\omega\frac{l}{v}}}{1-\boldsymbol{\mu}\mathrm{e}^{\mathrm{i}\omega\frac{l}{v}}}\mathbf{Q}_E^{\star(0)}\end{aligned} \tag{33}$$

- We can decompose the real loads of a train into dynamic and statique loads. Because the statique loads are constant, we can apply the formula (33) for this term and then apply the formula (31) for the dynamic loads.

4 NUMERICAL APPLICATIONS

Consider a periodically supported beam subjected to a constant moving load as shown in Figure 2, where the rail is modeled by Euler-Bernoulli beam and the support systems are the mass-springs. The beam is subjected to a constant moving load $Q = 100kN$.The parameters of the railway track are given in Table 1. Now we will compare the analytic solution [6] and the numerical method. From the finite element method with element type B21 in Abaqus, we obtain the dynamic stiffness matrix of the beam for one period of length l. In order to take into account the supports, we add the support dynamic stiffness into the beam stiffness matrix at the term in the diagonal corresponding to the DOF of the contact point between the beam and the spring-mass. Figure 3 shows a comparison

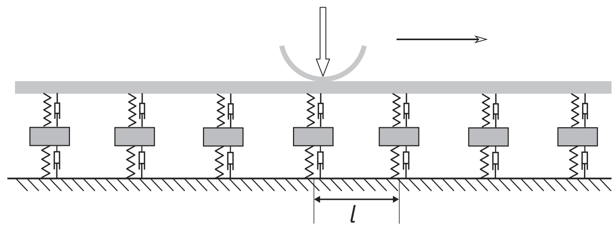

Figure 2: Periodically supported beam subjected to a moving load

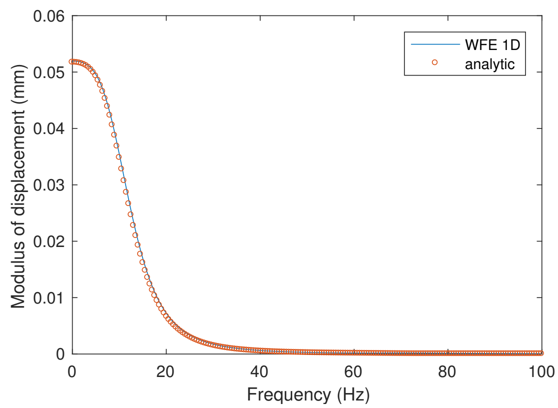

Figure 3: Response of the periodically supported beam by the analytical and numerical methods

of the analytic and numerical results with element B21 of size 1cm in Abaqus. In this example, the calculation times of the numerical method is almost the same time of the analytical method. We note that this result is for an infinite beam for the both two methods and it does not exist for the classical FEM. In addition, Figure 4 presents the structural damping factor $\boldsymbol{\mu}$.

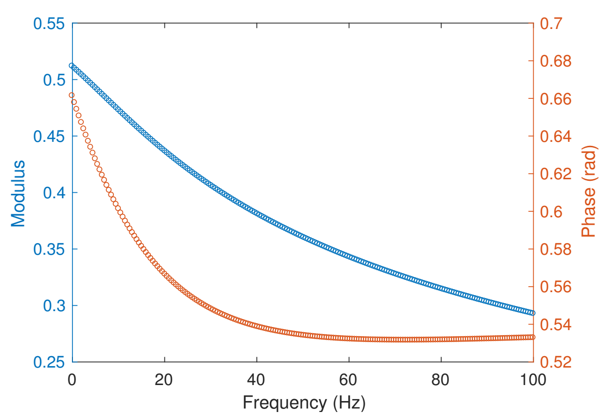

Figure 4: Structural damping factor of a railway track

Moreover, the numerical method permits to calculate the response of the track subjected to dynamic forces as shown in Figure 5. Here we consider a random dynamic force occurs at one point on the track with an amplitude $Q_{dyna} = 100\text{kN}$ in an frequency bandwidth $[25 - 50]\text{Hz}$. The track response is the sum of the response to the static load calculated by equation (33) and the dynamic load calculated by equation (31).

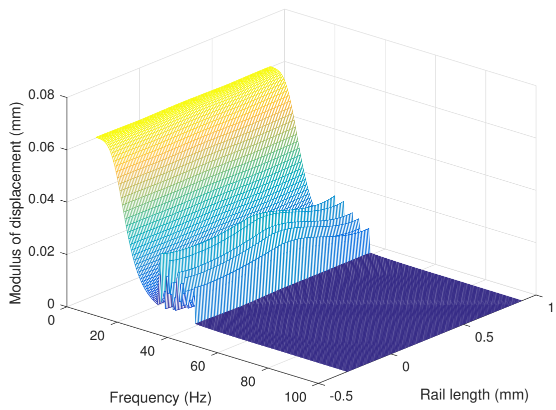

Figure 5: Response of the track to a dynamic load

Now we consider an example of a non-ballasted railway track where the supports contains a rail pad, a concrete block and an elastic pad under the block. All the component of the support and the concrete slab are simulated by the element finit method with the mesh shown in Figure 6. In this model, we have 16.075 elements C3D8R. The base of the slab is fixed and the track is subjected by moving force $Q = 100\text{kN}$ at a row of nodes on the rail.

Figure 7 presents the response of the loaded point by WFE for beam and 3D models and the analytic result. The calculation time for 3D model is 21.6h and for other model is about 0.2s. We see that the responses by different methods are coherent. The 3D model response has noise because it cause by the error of the eigenvalue calculation.

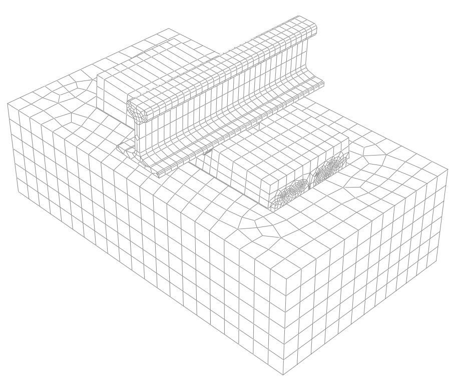

Figure 6: Example of a non-ballasted railway track

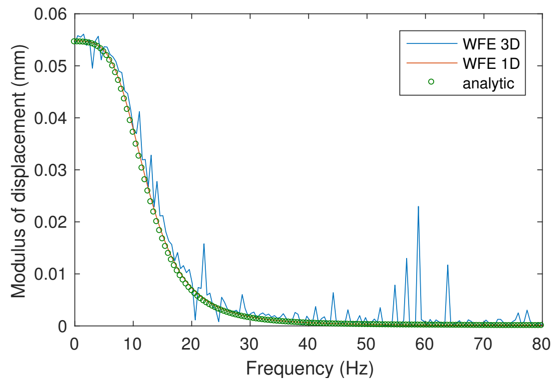

Figure 7: Response of the track subjected to a dynamic load

5 CONCLUSION

By using the wave finite element method, we demonstrate that the wave amplitude at the boundary of one period of the track can be represented by a sum of the wave amplitude which corresponds to the moving loads. This characteristics comes from the bounded condition of the infinite periodic structure. This method permits to reduce the DOF of the railway track to only one track period while remaining all the dynamic load.

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