



**HAL**  
open science

# DIRECT COMPUTATION OF ABSORBING BOUNDARY CONDITIONS AT THE DISCRETE LEVEL

Denis Duhamel

► **To cite this version:**

Denis Duhamel. DIRECT COMPUTATION OF ABSORBING BOUNDARY CONDITIONS AT THE DISCRETE LEVEL. 13th World Congress on Computational Mechanics (WCCM XIII), Jul 2018, New York, United States. hal-02142455

**HAL Id: hal-02142455**

**<https://hal.science/hal-02142455>**

Submitted on 28 May 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# DIRECT COMPUTATION OF ABSORBING BOUNDARY CONDITIONS AT THE DISCRETE LEVEL

D. DUHAMEL\*

\*Laboratoire Navier, Ecole des Ponts ParisTech,  
IFSTTAR, CNRS, UPE, UMR 8205, Champs-sur-Marne, France

**Key words:** Finite Element, absorbing boundary, wave, Helmholtz equation, discrete method

**Abstract.** The calculation of wave radiation in exterior domains by finite element methods can lead to large computations. A large part of the exterior domain is meshed and this computational domain is truncated at some distance where local or global boundary conditions are imposed at this artificial boundary. These conditions at finite distance must simulate as closely as possible the exact radiation condition at infinity and are generally obtained by discretizing an operator on the boundary.

Here, we propose a different approach, still based on the finite element method. Instead of finding an absorbing operator and then discretizing it, we will estimate the absorbing operator directly at the discrete level and build a sparse matrix approximating the absorbing condition. This discrete absorbing matrix is added to the dynamic stiffness matrix of the problem which is then solved in a classical way. The problem is considered for acoustics in the frequency domain and is described by the Helmholtz equation. The coefficients of the absorbing matrix are found from the solutions of small size linear systems for each node on the radiating boundary. This is done using a set of radiating functions for which a boundary condition is written. The precision of the method is estimated from the number of functions in the test set and from the number of coefficients allowed in the sparse matrix. Finally, some examples are computed to validate the method.

## 1 INTRODUCTION

Solving the Helmholtz equation in unbounded domains is important in many problems of mechanics and physics, for instance for the acoustic radiation or the diffraction around a body immersed in a fluid. Using the finite element method to solve the problem, one has to define a finite truncated domain on which the solution should be as close as possible to the solution on the unbounded domain. For this, it is necessary to define a boundary condition at the exterior of this truncated domain. These conditions at finite distance must simulate as closely as possible the exact radiation condition at infinity. This boundary condition could be global or local depending if all the degrees of freedom on the boundary are connected or if a given node is only coupled to a limited number of nodes around it. Among the global approaches we find the Dirichlet to Neumann (DtN)

proposed by [1, 2], or the boundary element method described in many classical textbooks like [3, 4, 5]. Both methods lead to full matrices and generate heavy computations.

In local methods, on the contrary, the condition at a boundary node involves only a limited number of neighbouring nodes. They can be classified into mainly three sets: those involving only the degrees of freedom of the domain, those with additional degrees of freedom at nodes on the boundary and those with an additional domain. Concerning the absorption conditions which do not involve additional variables or domains, a first possibility is using infinite elements as proposed by [6, 7, 8, 9]. These are elements extending at infinity and satisfying the Sommerfeld radiation condition. However, it needs the development of special elements based on functions with outwarding propagation wave-like behaviour in the radial direction. Other absorbing boundary conditions involving differential operators of different orders on the boundary were proposed by different authors [10, 11, 12]. These relations were improved by Bayliss and Turkel [13, 14] using sequences of local non-reflecting boundary conditions in spherical and cylindrical coordinates. However, all these conditions are difficult to implement above the second order because of the high order derivatives involved in their formulations. More efficient boundary conditions can be obtained by the addition of variables on the exterior surface such as in [15, 16]. They involve only second derivatives of the auxiliary variables and so can be efficiently implemented. Surrounding the computational domain by absorbing layers was also proposed by [17, 18] with the perfectly matched layer in which the wave equation is analytically continued into complex coordinates. This however can add a non negligible number of degrees of freedom to the problem and the optimal parameters in the absorbing layer are not so easy to find.

Most of the previous absorbing boundary conditions are written at the continuous level, but it can be interesting to write them at the discrete level. For instance, boundary conditions at the discrete level using the properties of periodic media were proposed by [19]. In [20] boundary conditions based on the PLM were written after discretisation of the equations and were found to be more efficient than their continuous versions. Such a discrete approach is used in this paper. The following section presents the problem formulation and the building of the discrete absorbing matrix. Then some examples are presented before the conclusion.

## 2 PROBLEM FORMULATION

### 2.1 Helmholtz equation

We consider the two-dimensional acoustic equation in the frequency domain in the exterior  $\Omega_e$  of a bounded domain  $\Omega_i$  of boundary  $\Gamma_i$ , see Fig.1. The Helmholtz equation with a Neumann boundary condition on  $\Gamma_i$  and a radiation condition at infinity is

$$\begin{aligned} \Delta p + k^2 p &= f \text{ on } \Omega_e \\ \frac{\partial p}{\partial n} &= g \text{ on } \Gamma_i \\ \frac{\partial p}{\partial r} - ikp &= o\left(\frac{1}{\sqrt{r}}\right) \text{ when } r \rightarrow \infty \end{aligned} \tag{1}$$

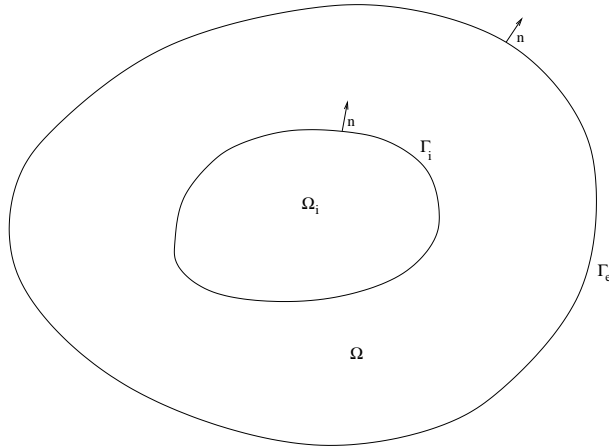


Figure 1: Exterior domain.

with  $f$  and  $g$  given functions representing the sources in the domain and at the boundary. The domain  $\Omega_e$  is truncated at some finite distance by the boundary  $\Gamma_e$  and the discrete problem is posed on the bounded domain  $\Omega$  located between the surfaces  $\Gamma_e$  and  $\Gamma_i$ . Its variational formulation is

$$\int_{\Omega} (\Delta p + k^2 p) q dx = \int_{\Omega} f q dx \quad (2)$$

$$- \int_{\Gamma_i} \frac{\partial p}{\partial n} q ds + \int_{\Gamma_e} \frac{\partial p}{\partial n} q ds + \int_{\Omega} (-\nabla p \cdot \nabla q + k^2 p q) dx = \int_{\Omega} f q dx \quad (3)$$

with  $q$  a test function and the exterior normals  $n$  on surfaces  $\Gamma_i$  and  $\Gamma_e$ . One assumes that the absorbing boundary condition can be written on the surface  $\Gamma_e$  as

$$\frac{\partial p}{\partial n} = Ap \quad (4)$$

where  $A$  is an operator acting on the pressure  $p$  inside  $\Omega$ . So the variational formulation is now

$$\int_{\Gamma_e} (Ap) q ds + \int_{\Omega} (-\nabla p \cdot \nabla q + k^2 p q) dx = \int_{\Omega} f q dx + \int_{\Gamma_i} \frac{\partial p}{\partial n} q ds \quad (5)$$

## 2.2 Discretisation of the absorbing operator

The absorbing operator is such that

$$\begin{aligned} \int_{\Gamma_e} \frac{\partial p}{\partial n}(s) q(s) ds &= \int_{\Gamma_e} (Ap)(s) q(s) ds \\ &= \int_{\Gamma_e} \int_{\Omega} A(s, x) p(x) q(s) dx ds \end{aligned} \quad (6)$$

The discrete form can be written as

$$\mathbf{M}^s \mathbf{q} = \mathbf{M}^s \mathbf{A} \mathbf{M}^v \mathbf{p} \quad (7)$$

with

$$\begin{aligned} M_{ij}^s &= \int_{\Gamma_e} N_i^s(s) N_j^s(s) ds \\ M_{im}^v &= \int_{\Omega} N_i^v(x) N_m^v(x) dx \end{aligned} \quad (8)$$

$\mathbf{q}$  is the vector of the normal derivatives of the pressure at the nodes of  $\Gamma_e$  and  $\mathbf{p}$  the vector of the pressures at nodes in  $\Omega$ .  $N^s$  and  $N^v$  are the usual interpolation functions on the boundary and in  $\Omega$  respectively. Finally the vectors  $\mathbf{p}$  and  $\mathbf{q}$  are linked by

$$\mathbf{q} = \mathbf{A}\mathbf{M}^v\mathbf{p} \quad (9)$$

and one has to identify the matrix  $\mathbf{A}\mathbf{M}^v$ .

### 2.3 Determination of the absorbing matrix

The solution of the problem can be expanded as

$$p(r, \theta) = \sum_{-\infty}^{+\infty} a_n H_n(kr) e^{in\theta} \quad (10)$$

The completeness of the expansion on the boundary was proved by [21, 22, 23]. One now has to find an approximation of the matrix  $\tilde{\mathbf{A}} = \mathbf{A}\mathbf{M}^v$ . One looks for a discrete operator acting on the pressure at nodes inside  $\Omega$  such that the matrix  $\tilde{\mathbf{A}}$  is sparse and the relation (9) is satisfied for outgoing waves.

For a node  $i$  at point  $\mathbf{x}_i$  on the boundary, one considers nodes  $i_j$  at points  $\mathbf{x}_{i_j}$  in  $\Omega$  with  $j = 1 \dots n_i$  in the neighborhood of  $\mathbf{x}_i$  and such that  $\mathbf{x}_{i_1} = \mathbf{x}_i$ . So the line  $i$  of the matrix  $\tilde{\mathbf{A}}$  will have non zero coefficients only at nodes  $i_j$ . To find these coefficients, one writes equation (9) for Hankel functions of different orders  $n$ . Choosing a point  $\mathbf{o}$  interior to  $\Omega_i$ , one should have

$$\frac{\partial}{\partial n_i} (H_n(k|\mathbf{x}_i - \mathbf{o}|) e^{in\theta_i}) = \sum_{j=1 \dots n_i} a_j^i (H_n(k|\mathbf{x}_{i_j} - \mathbf{o}|) e^{in\theta_{i_j}}) \quad (11)$$

for  $-N \leq n \leq N$  and  $n_i$  the exterior normal at node  $i$ . Denoting the vectors

$$\mathbf{f}_i = \begin{bmatrix} \frac{\partial}{\partial n_i} (H_{-N}(k|\mathbf{x}_i - \mathbf{o}|) e^{-iN\theta_i}) \\ \dots \\ \frac{\partial}{\partial n_i} (H_0(k|\mathbf{x}_i - \mathbf{o}|)) \\ \dots \\ \frac{\partial}{\partial n_i} (H_N(k|\mathbf{x}_i - \mathbf{o}|) e^{iN\theta_i}) \end{bmatrix} \quad \text{and} \quad \mathbf{a}_i = \begin{bmatrix} a_{i_1}^i \\ \dots \\ a_{i_{n_i}}^i \end{bmatrix} \quad (12)$$

and the matrix

$$\mathbf{H}_i = \begin{bmatrix} H_{-N}(k|\mathbf{x}_{i_1} - \mathbf{o}|) e^{-iN\theta_{i_1}} & \dots & H_{-N}(k|\mathbf{x}_{i_{n_i}} - \mathbf{o}|) e^{-iN\theta_{i_{n_i}}} \\ \dots & \dots & \dots \\ H_0(k|\mathbf{x}_{i_1} - \mathbf{o}|) & \dots & H_0(k|\mathbf{x}_{i_{n_i}} - \mathbf{o}|) \\ \dots & \dots & \dots \\ H_N(k|\mathbf{x}_{i_1} - \mathbf{o}|) e^{iN\theta_{i_1}} & \dots & H_N(k|\mathbf{x}_{i_{n_i}} - \mathbf{o}|) e^{iN\theta_{i_{n_i}}} \end{bmatrix} \quad (13)$$

Relation (11) can be put under the form

$$\mathbf{f}_i = \mathbf{H}_i \mathbf{a}_i \quad (14)$$

Its solution is

$$\mathbf{a}_i = (\mathbf{H}_i^* \mathbf{H}_i)^{-1} \mathbf{H}_i^* \mathbf{f}_i \quad (15)$$

with  $*$  denoting the hermitian transpose of a matrix. The vector  $\mathbf{a}_i$  gives the  $i$ th line of the matrix  $\tilde{\mathbf{A}}$ . Considering these relations for all nodes at the boundary, one gets the sparse matrix  $\tilde{\mathbf{A}}$  describing an approximate absorbing boundary condition on  $\Gamma_e$ .

The discretisation of the other parts of the variation formulation (5) leads to the final discrete equation.

$$(\mathbf{K} - \tilde{\mathbf{A}} - k^2 \mathbf{M}) \mathbf{p} = \mathbf{f} \quad (16)$$

which can be solved by classical solvers.

### 3 NUMERICAL EXAMPLES

#### 3.1 Test problem

As example we consider an annular domain limited by an interior circle of radius  $0.15m$  and an exterior circle of radius  $0.3m$  (see Fig.2). The sound velocity is  $c = 340m/s$ . A boundary condition is defined at the interior circle as the normal derivative of the sound pressure generated by a point source located at point  $\mathbf{x}_s = (0.1, 0)$  and is given by

$$q(\mathbf{x}) = -\frac{ik \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_s)}{4 |\mathbf{x} - \mathbf{x}_s|} H_1(k|\mathbf{x} - \mathbf{x}_s|) \quad (17)$$

with  $\mathbf{x}$  the position of a node on the interior boundary and  $\mathbf{x}_s$  the position of the point source. The analytical solution is given by

$$p(\mathbf{x}) = \frac{i}{4} H_0(k|\mathbf{x} - \mathbf{x}_s|) \quad (18)$$

and will be compared to various numerical solutions.

We define the errors  $e_g$  on the whole domain  $\Omega$  and  $e_b$  on the exterior boundary  $\Gamma_e$  by

$$\begin{aligned} e_g^2 &= \frac{\sum_{i \text{ node on } \Omega} |p_i^{num} - p_i^{ana}|^2}{\sum_{i \text{ node on } \Omega} |p_i^{ana}|^2} \\ e_b^2 &= \frac{\sum_{i \text{ node on } \Gamma_e} |p_i^{num} - p_i^{ana}|^2}{\sum_{i \text{ node on } \Gamma_e} |p_i^{ana}|^2} \end{aligned} \quad (19)$$

with the superscripts *ana* and *num* denoting respectively the analytical and numerical solutions.

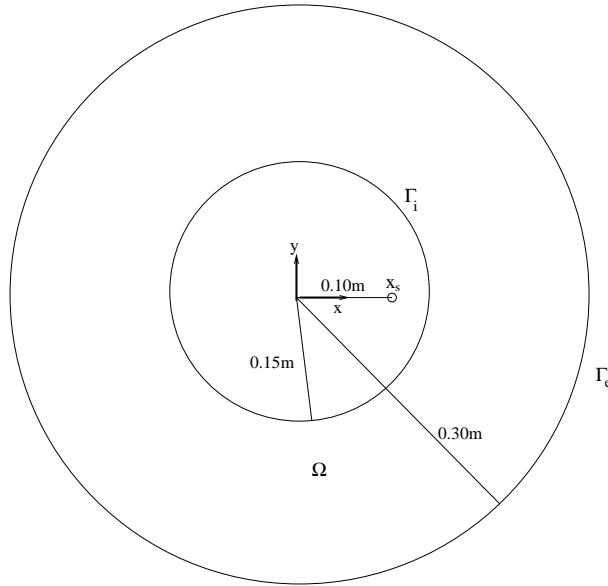


Figure 2: Annular domain.

### 3.2 Influence of different parameters

We begin by estimating the influence of the truncation order  $N$  on the error. In Fig.3 four solutions are plotted. The first one is obtained with the crude boundary condition  $\frac{\partial p}{\partial n} = ikp$  (denoted as the  $ik$  solution), two solutions are obtained by the present method with respectively  $N = 0$  and  $N = 1$  and the last one is the analytical solution for the frequency  $100Hz$ . These solutions are obtained by taking  $n_i = 20$  coefficients for each boundary node in the building of the matrix  $\tilde{A}$ . As can be seen the  $ik$  solution leads to large errors while the solutions with the present method lead to rather good solutions even for the simplest one with  $N = 0$ . The numerical errors are given in table 1 with similar conclusions.

condition	global error	boundary error
ik	0.837	0.956
N=0	0.059	0.078
N=1	0.003	0.005

Table 1: Errors for the different boundary condition at 100Hz

In Fig.4 the solution is plotted versus the number of points  $n_i$  used to build  $\tilde{A}$  with  $N = 1$ . Using  $n_i = 2$  is clearly not enough. However, one can see that  $n_i = 5$  gives a rather good solution which is still improved by using more points. Numerical values are given in table 2.

Finally Fig.5 compares the analytical solutions and the numerical ones for different frequencies. Only  $n_i = 2$  points are used which leads to crude estimates. While the

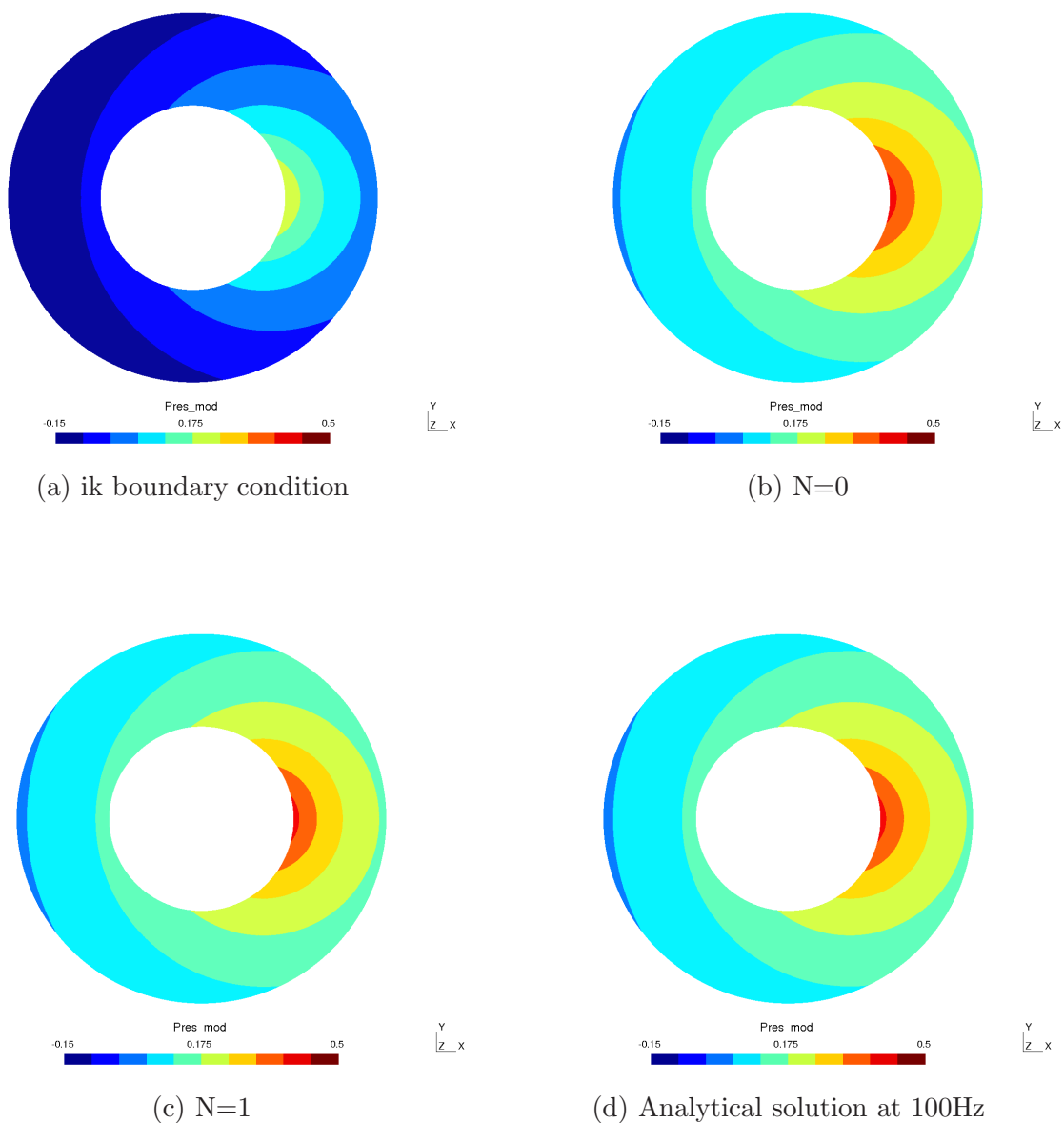


Figure 3: Comparison of solutions at 100Hz

solution at  $100Hz$  shows important errors, the results at  $300Hz$  and  $1000Hz$  are much better. This shows that the condition is more efficient as the frequency increases as for other absorbing boundary conditions.

#### 4 CONCLUSION

A new numerical method has been presented for computing absorbing boundary conditions for the Helmholtz equation. It builds a discrete absorbing matrix directly from



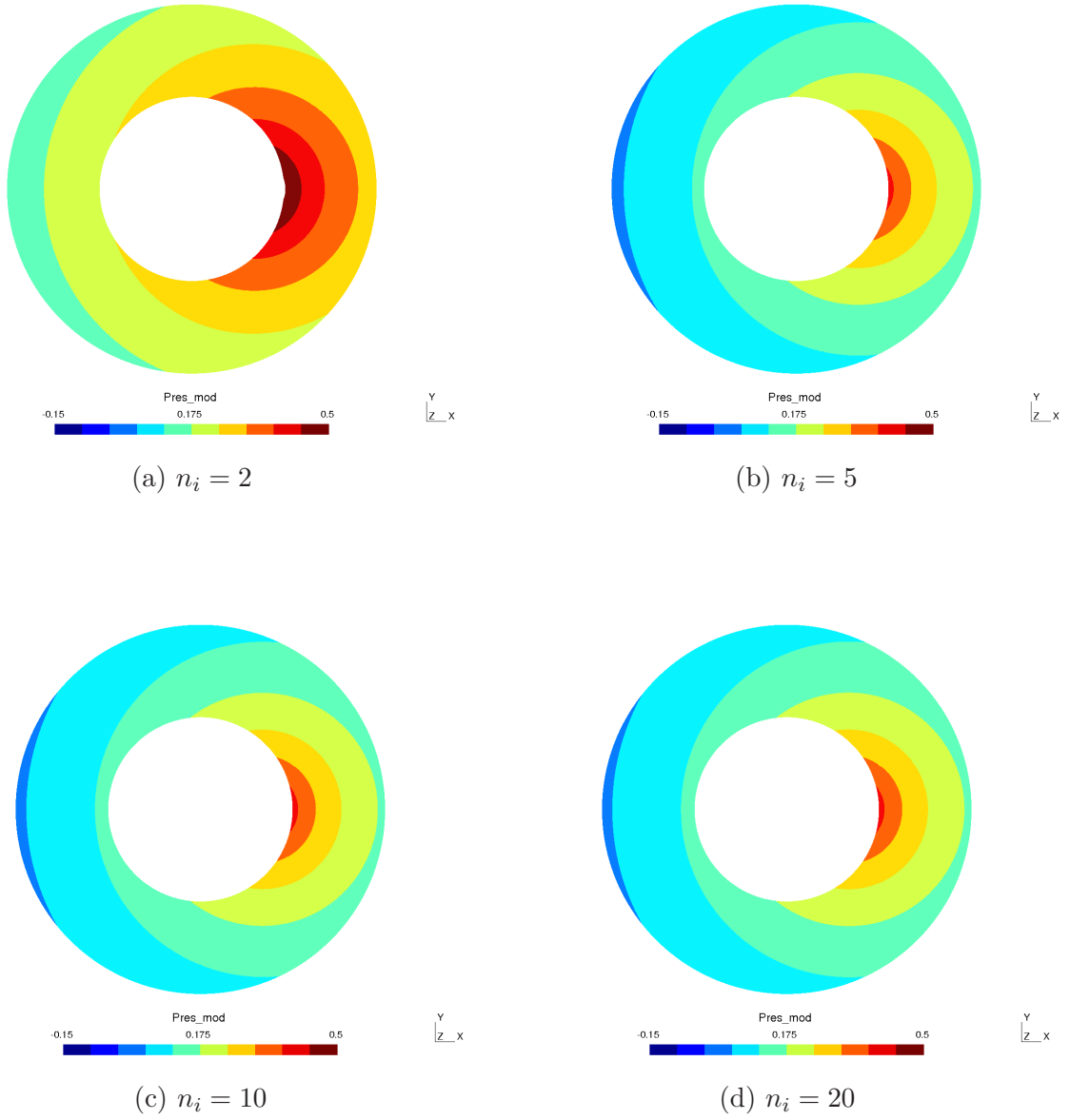


Figure 4: Solutions for different numbers of nodes used to define the matrix  $\tilde{A}$

the finite element discretisation of the problem. This can be applied to any shape and does not require additional variables or additional domains. So the number of degrees of freedom is the same as for the problem without absorbing boundary conditions. Examples show the accuracy of the method. Similar approaches could be used for other wave propagation problems such as for the propagation of elastic waves. Future works will include comparisons with other classical absorbing boundary conditions and the consideration of more complex domains such as domains with corners.

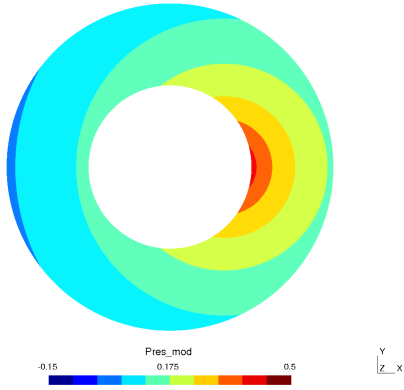
Number of nodes	global error	boundary error
$n_i = 2$	0.459	0.522
$n_i = 5$	0.010	0.013
$n_i = 10$	0.005	0.006
$n_i = 20$	0.003	0.005

Table 2: Error for different numbers of nodes used to build the matrix  $\tilde{A}$ 

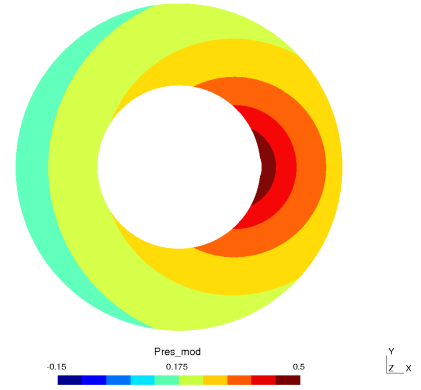
## REFERENCES

- [1] J.B. Keller and D. Givoli. Exact non-reflecting boundary conditions. *Journal of computational physics*, 82(1):172–192, 1989.
- [2] D. Givoli and J.B. Keller. A finite element method for large domains. *Computer Methods in Applied Mechanics and Engineering*, 76(1):41–66, 1989.
- [3] C.A. Brebbia and S. Walker. *Boundary Element Techniques in Engineering*. Newnes-Butterworths, 1980.
- [4] R.D. Ciskowski and C.A. Brebbia. In *Boundary element methods in acoustics*. Computational mechanics publications, Elsevier Applied Sciences, 1991.
- [5] M. Bonnet. *Boundary Integral Equation Methods for Solids and Fluids*. John Wiley and Sons, 1995.
- [6] P. Bettess. Infinite elements. *International Journal for Numerical Methods in Engineering*, 11(1):53–64, 1977.
- [7] P. Bettess. *Infinite Elements*. Penshaw Press, 1992.
- [8] D.S. Burnett. A three-dimensional acoustic infinite element based on a prolate spheroidal multipole expansion. *Journal of the Acoustical Society of America*, 96:2798–2816, 1994.
- [9] R.J. Astley. Infinite elements for wave problems: a review of current formulations and an assessment of accuracy. *International Journal for Numerical Methods in Engineering*, 49(7):951–976, 2000.
- [10] B. Engquist and A. Majda. Absorbing boundary conditions for numerical simulation of waves. *Proceedings of the National Academy of Sciences*, 74(5):1765–1766, 1977.
- [11] R. Clayton and B. Engquist. Absorbing boundary conditions for acoustic and elastic wave equations. *Bulletin of the Seismological Society of America*, 67(6):1529–1540, 1977.
- [12] A. C. Reynolds. Boundary conditions for the numerical solution of wave propagation problems. *Geophysics*, 43(6):1099–1110, 1978.

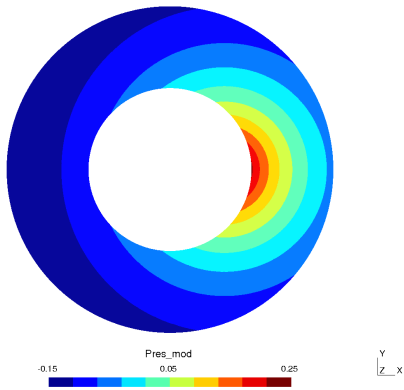
- [13] A. Bayliss and E. Turkel. Radiation boundary conditions for wave-like equations. *Communications on Pure and Applied Mathematics*, 33(6):707–725, 1980.
- [14] A. Bayliss, M. Gunzburger, and E. Turkel. Boundary conditions for the numerical solution of elliptic equations in exterior regions. *SIAM Journal on Applied Mathematics*, 42(2):430–451, 1982.
- [15] F. Collino. High-order absorbing boundary conditions for wave propagation models. straight line boundary and corner cases. In *Proc. 2nd Int. Conf. on Mathematical & Numerical Aspects of Wave Propagation*, R. Kleinman et al. SIAM, pages 161–171, Delaware, USA, 1993.
- [16] T. Hagstrom and S.I. Hariharan. A formulation of asymptotic and exact boundary conditions using local operators. *Applied Numerical Mathematics*, 27(4):403 – 416, 1998.
- [17] J.P. Berenger. A perfectly matched layer for the absorption of electromagnetic waves. *Journal of computational physics*, 114(2):185–200, 1994.
- [18] J.P. Berenger. Three-dimensional perfectly matched layer for the absorption of electromagnetic waves. *Journal of computational physics*, 127(2):363–379, 1996.
- [19] D. Duhamel and T.M. Nguyen. Finite element computation of absorbing boundary conditions for time-harmonic wave problems. *Computer Methods in Applied Mechanics and Engineering*, 198(37):3006–3019, 2009.
- [20] S. Thirunavukkarasu and M.N. Guddati. Absorbing boundary conditions for time harmonic wave propagation in discretized domains. *Computer Methods in Applied Mechanics and Engineering*, 200(33):2483 – 2497, 2011.
- [21] D. Colton. Runge's theorem and far field patterns for the impedance boundary value problem in acoustic wave propagation. *SIAM Journal on Mathematical Analysis*, 13(6):970–977, 1982.
- [22] R.F. Millar. On the completeness of sets of solutions to the helmholtz equation. *IMA Journal of Applied Mathematics*, 30(1):27–37, 1983.
- [23] Y. Liu and J.S. Bolton. On the completeness and the linear dependence of the cartesian multipole series in representing the solution to the helmholtz equation. *The Journal of the Acoustical Society of America*, 140(2):EL149–EL153, 2016.



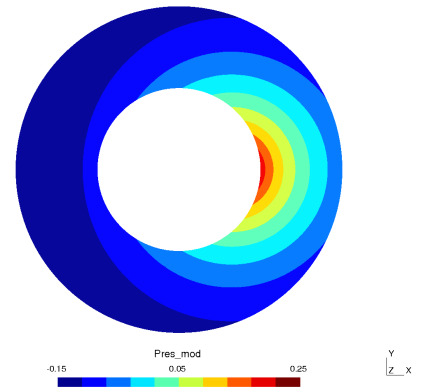
(a) Analytical solution 100Hz



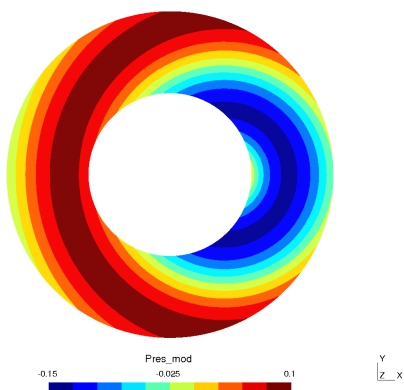
(b) Numerical solution 100 Hz



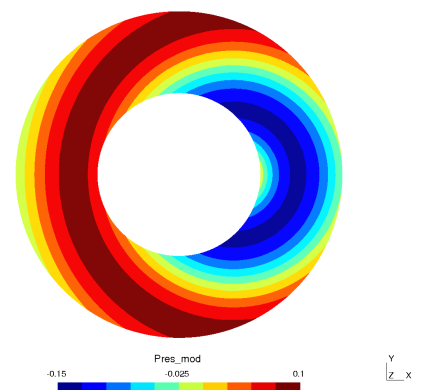
(c) Analytical solution 300Hz



(d) Numerical solution 300 Hz



(e) Analytical solution 1000Hz



(f) Numerical solution 1000 Hz

Figure 5: Solutions for different frequencies