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Abstract

We present a model for a class of non-local conservation laws arising in traffic flow modeling at road junctions. Instead of a single velocity function for the whole road, we consider two different road segments, which may differ for their speed law and number of lanes (hence their maximal vehicle density). We use an upwind type numerical scheme to construct a sequence of approximate solutions and we provide uniform L^∞ and total variation estimates. In particular, the solutions of the proposed model stay positive and below the maximum density of each road segment. Using a Lax-Wendroff type argument and the doubling of variables technique, we prove the well-posedness of the proposed model. Finally, some numerical simulations are provided and compared with the corresponding (discontinuous) local model.

Keywords: non-local scalar conservation laws, upwind scheme, macroscopic traffic flow models on networks.

AMS subject classifications: 35L65, 65M12, 90B20

1 Introduction

In recent years, conservation laws with non-local flux gained growing attention for a wide field of applications. Indeed, they turned out to be suitable to describe several phenomena: flux functions depending on space-integrals of the unknown appear for example in models for sedimentation [5], supply chains [19], conveyor belts [18], crowd motions [12] and traffic flows [8, 9, 17]. For this type of equations, general existence and uniqueness results have been established in [3, 8, 17] for specific classes of scalar equations in one space-dimension, and in [2, 9] for systems of equations coupled through the non-local term.

In this paper, we propose a one-dimensional scalar model, arising in traffic flow modeling. The main difference with respect to the above mentioned literature is that the flux function may involve different velocity functions on different parts of the road. The model focuses on a non-local mean downstream velocity and can therefore describe the behavior of drivers on two stretches of a road with different velocities and capacities, without violating the maximal density constraint on each road segment. Hence, we are modelling a 1-to-1 junction and this model can be seen as a first step towards a network formulation for traffic flow models with

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non-local flux. Note that the work [29] is dealing with a similar setting, where the author studies traveling waves profiles of conservation laws with non-local flux functions, describing traffic flow on a road with just different maximum velocities.
We approximate the solution using an adapted Godunov or rather upwind type scheme. Deriving several properties of the scheme and relying on a Kružkov type entropy condition, we are able to prove the well-posedness of the model.

Since it is still an open question whether the solution of the non-local model tends to the solution of the corresponding local equation when the support of the kernel function tends to zero, see for example [11] for an overview, we investigate this issue only from the numerical point of view.

The paper is organized as follows: In Section 2, we present our model and the main result of this work. In Section 3, we prove the Lipschitz continuous dependence of weak entropy solutions with respect to the initial data, which implies their uniqueness. In Section 4, we introduce an adapted upwind type scheme and derive important properties: the maximum principle, uniform total variation (BV) estimates and a discrete entropy inequality. Afterwards, we prove the convergence of the scheme and the main theorem in Section 5. In the last section, we present some numerical simulations of our non-local junction model and we numerically investigate the behaviour of the corresponding solutions as the support of the kernel function tends to zero.

2 Modeling

Based on the model presented in [17] we consider the following conservation law

$$\partial_t \rho(t,x) + \partial_x f(t,x,\rho) = 0, \quad x \in \mathbb{R},\, t > 0, \tag{2.1}$$

where

$$f(t,x,\rho) := \rho(t,x) V_1(t,x) + g(\rho(t,x)) V_2(t,x), \tag{2.2}$$

with

$$\begin{aligned} g(\rho) &:= \min\{\rho, \rho^2_{\max}\}, && (2.3)\\ V_1(t,x) &:= \int_{\min\{x,0\}}^{\min\{x+\eta,0\}} v_1(\rho(t,y))\omega_\eta(y-x)dy, && (2.4)\\ V_2(t,x) &:= \int_{\max\{x,0\}}^{\max\{x+\eta,0\}} v_2(\rho(t,y))\omega_\eta(y-x)dy, && (2.5)\end{aligned}$$

for any $\eta > 0$. We couple the equation (2.1) with the initial datum

$$\begin{aligned} &\rho(0,x) = \rho_0(x) \in \mathrm{BV}(\mathbb{R}),\\ &\text{s.t. } \rho_0(x) \in [0, \rho^1_{\max}] \text{ for } x<0 \text{ and } \rho_0(x) \in [0,\rho^2_{\max}] \text{ for } x>0. \end{aligned} \tag{2.6}$$

The model assumes that drivers adapt their speed based on a weighted mean of downstream velocities. In the considered setting, changes in road characteristics at $x = 0$ may translate in different velocity functions, v_1 and v_2, and in different road capacities, $\rho^1_{\max}$ and $\rho^2_{\max}$, for $x < 0$ and $x > 0$ respectively. An example of such a situation on a road is illustrated in Figure 1. Here, different maximum capacities and different velocity functions can be "seen" by the

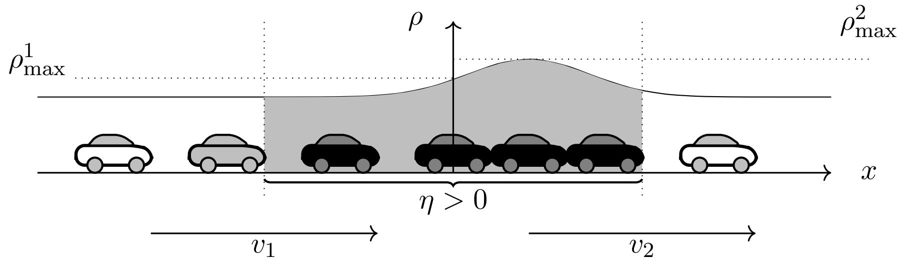

Figure 1: Illustration of cars on a road with different parameters on each segment. The grey area represents the non-local traffic downstream information of the car in light grey. In this model, this car slows down in advance with respect to the density of cars in front of it. The illustration describes a microscopic evolution but can also be used for our macroscopic model.

car in light grey. In (2.2), the flux also accounts for the maximum capacity of the second road segment.

The special structure of the flux function (2.2) does not fit into the framework proposed in e.g. [8, 17, 23]. Only for $v_1 \equiv v_2$ and therefore $\rho^1_{\max} = \rho^2_{\max}$ the model coincides with the one presented in [17]. Therefore, we have to investigate its well-posedness in the general case.

We impose the following reasonable hypotheses on $v_i,\ i \in \{1, 2\}$ and ω_η:

$$\begin{aligned} v_i \in C^2([0, \rho^i_{\max}]; \mathbb{R}^+)\colon v_i' \leq 0,\ v_i(\rho^i_{\max}) = 0, \\ \omega_\eta \in C^1([0, \eta]; \mathbb{R}^+)\colon \omega_\eta' \leq 0,\ \int_0^\eta \omega_\eta(x) dx = 1\ \forall \eta > 0, \end{aligned} \tag{2.7}$$

where η represents the look-ahead distance of the drivers.

Since the flux function (2.2) is continuous in x, entropy weak solutions of (2.1), (2.6) are intended in the following way:

Definition 1 (Entropy weak solution (see [25])). *A measurable function*

$$\rho \in C([0, +\infty); L^1(\mathbb{R})) : [0, +\infty) \times \mathbb{R} \to [0, \max\{\rho^1_{\max}, \rho^2_{\max}\}]$$

is an entropy weak solution of the initial value problem (2.1)–(2.6) *if for any test function* $\varphi \in C^1_c([0, +\infty) \times \mathbb{R}; \mathbb{R}^+)$ *and for any constant* $c \in \mathbb{R}$,

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} \big(|\rho - c|\partial_t \varphi + \mathrm{sgn}(\rho - c)(f(t, x, \rho) - f(t, x, c))\partial_x \varphi - \mathrm{sgn}(\rho - c)\partial_x f(t, x, c)\varphi\big)\, dxdt \\ + \int_{-\infty}^\infty |\rho_0(x) - c|\varphi(0, x)\, dx \geq 0. \end{aligned} \tag{2.8}$$

Remark 1. *We note that the entropy condition is essential to obtain the uniqueness of solutions in the framework of our proof's technique. Under suitable assumptions, an alternative could be to follow the approach considered in [23], where the uniqueness is obtained as a consequence of the Banach fixed point theorem and therefore no entropy condition is needed.*

The main result of this paper is the following theorem:

Theorem 1. *Let* $\rho_0 \in \mathrm{BV}(\mathbb{R}; [0, \max\{\rho^1_{\max}, \rho^2_{\max}\}])$ *such that* $\rho_0(x) \leq \rho^1_{\max}$ *for* $x < 0$ *and* $\rho_0(x) \leq \rho^2_{\max}$ *for* $x \geq 0$, *and hypotheses* (2.7) *hold. Then the Cauchy problem*

$$\begin{cases} \partial_t \rho(t,x) + \partial_x f(t,x,\rho) = 0, & x \in \mathbb{R},\, t > 0, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases}$$

admits a unique entropy weak solution in the sense of Definition 1 and

$$\begin{aligned} 0 \leq \rho(t,x) \leq \rho^1_{\max} \quad & \text{for a.e. } x < 0,\, t > 0, \\ 0 \leq \rho(t,x) \leq \rho^2_{\max} \quad & \text{for a.e. } x \geq 0,\, t > 0. \end{aligned} \tag{2.9}$$

Theorem 1 is proved at the end of Section 5.

3 Uniqueness

Let us start to prove the Lipschitz continuous dependence of weak entropy solutions with respect to the initial data, which ensures the uniqueness of entropy solutions of the model (2.1)–(2.6). We follow [6, 8, 17], using Kružkov's doubling of variables technique [25].

Theorem 2. *Under hypotheses* (2.7), *let* ρ *and* $\tilde{\rho}$ *be two entropy solutions of* (2.1) *with initial datum* ρ_0 *and* $\tilde{\rho}_0$, *respectively. Then, for any* $T > 0$, *there holds*

$$\left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{L^1} \leq \exp(KT) \|\rho_0 - \tilde{\rho}_0\|_{L^1} \quad \forall t \in [0,T], \tag{3.1}$$

with K *given by* (3.7).

Proof. The functions ρ and $\tilde{\rho}$ are weak entropy solutions of

$$\begin{aligned} &\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) V_1(t,x) + g(\rho) V_2(t,x)\right) = 0, \quad \rho(0,x) = \rho_0(x), \\ &\partial_t \tilde{\rho}(t,x) + \partial_x \left(\tilde{\rho}(t,x) \tilde{V}_1(t,x) + g(\tilde{\rho}) \tilde{V}_2(t,x)\right) = 0, \quad \tilde{\rho}(0,x) = \tilde{\rho}_0(x), \end{aligned}$$

respectively. V_i, $\tilde{V}_i$ for $i = 1, 2$ are defined as in (2.4) and (2.5), where the convolution is computed over the velocity of ρ and $\tilde{\rho}$, respectively. They are bounded measurable functions and Lipschitz continuous w.r.t. x since ρ, $\tilde{\rho} \in (L^\infty \cap \mathrm{BV})(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$.
Using the classical doubling of variables technique, see [21, 25], we get the following inequality:

$$\begin{aligned} \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{L^1} \leq & \|\rho_0 - \tilde{\rho}_0\|_{L^1} + \int_0^T \int_{\mathbb{R}} \left|\partial_x \rho(t,x)\right| \left|V_1(t,x) - \tilde{V}_1(t,x)\right| dx dt \\ & + \int_0^T \int_{\mathbb{R}} \left|\partial_x \rho(t,x)\right| \left|V_2(t,x) - \tilde{V}_2(t,x)\right| dx dt \\ & + \int_0^T \int_{\mathbb{R}} |\rho| \left|\partial_x V_1 - \partial_x \tilde{V}_1\right| dx dt + \int_0^T \int_{\mathbb{R}} \left|g(\rho)\right| \left|\partial_x V_2 - \partial_x \tilde{V}_2\right| dx dt, \end{aligned} \tag{3.2}$$

where $\partial_x \rho$ must be understood in the sense of measures. Applying the mean value theorem and using the properties of the kernel function, we deduce

$$\left|V_i(t,x) - \tilde{V}_i(t,x)\right| \leq \omega_\eta(0) \left\|v_i'\right\|_\infty \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{L^1}, \quad \text{for } i = 1, 2. \tag{3.3}$$

Using the Leibniz integral rule and again the mean value theorem, we can also obtain for a.e. $x \in \mathbb{R}$

$$
\begin{aligned}
\left|\partial_x V_1(t,x) - \partial_x \tilde{V}_1(t,x)\right| &= \begin{cases} 0, & \text{if } x > 0, \\ \Big| \int_x^0 (v_1(\rho(t,y)) - v_1(\tilde{\rho}(t,y)))\omega_\eta'(y-x)dy & \\ + \left(v_1(\tilde{\rho}(t,x)) - v_1(\rho(t,x))\right)\omega_\eta(0)\Big|, & \text{if } -\eta < x < 0, \\ \Big| \int_x^{x+\eta} (v_1(\rho(t,y)) - v_1(\tilde{\rho}(t,y)))\omega_\eta'(y-x)dy & \\ - \left(v_1(\tilde{\rho}(t,x+\eta)) - v_1(\rho(t,x+\eta))\right)\omega_\eta(\eta) & \\ + \left(v_1(\tilde{\rho}(t,x)) - v_1(\rho(t,x))\right)\omega_\eta(0)\Big|, & \text{if } x < -\eta \end{cases} \\
&\leq \left\|\omega_\eta'\right\|_\infty \left\|v_1'\right\|_\infty \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{L^1} \qquad (3.4) \\
&\quad + \omega_\eta(0)\left\|v_1'\right\|_\infty \left(|\rho-\tilde{\rho}|(t,x+\eta) + |\rho-\tilde{\rho}|(t,x)\right).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\left|\partial_x V_2(t,x) - \partial_x \tilde{V}_2(t,x)\right| \leq{}& \left\|\omega_\eta'\right\|_\infty \left\|v_2'\right\|_\infty \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{L^1} \qquad (3.5) \\
&+ \omega_\eta(0)\left\|v_2'\right\|_\infty \left(|\rho-\tilde{\rho}|(t,x+\eta) + |\rho-\tilde{\rho}|(t,x)\right). \qquad (3.6)
\end{aligned}
$$

Plugging (3.3), (3.4), (3.5) into (3.2), we obtain

$$
\begin{aligned}
&\left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{L^1} \leq \left\|\rho_0 - \tilde{\rho}_0\right\|_{L^1} \\
&+ \max_{i=1,2}\left\{\left\|v_i'\right\|_\infty\right\} \int_0^T \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{L^1} dt \Bigg[2\omega_\eta(0) \sup_{t\in[0,T]} \left\|\rho(t,\cdot)\right\|_{\mathrm{BV}(\mathbb{R})} \\
&\qquad + \left\|\omega_\eta'\right\|_\infty \left(\sup_{t\in[0,T]} \left\|\rho(t,\cdot)\right\|_{L^1} + \sup_{t\in[0,T]} \left\|g(\rho(t,\cdot))\right\|_{L^1}\right)\Bigg] \\
&+ \max_{i=1,2}\left\{\left\|v_i'\right\|_\infty\right\} \omega_\eta(0) \left(\sup_{t\in[0,T]} \left\|\rho(t,\cdot)\right\|_\infty + \sup_{t\in[0,T]} \left\|g(\rho(t,\cdot))\right\|_\infty\right) \\
&\qquad \int_0^T \int_{\mathbb{R}} \left(|\rho-\tilde{\rho}|(t,x+\eta) + |\rho-\tilde{\rho}|(t,x)\right) dx dt \\
\leq{}& \left\|\rho_0 - \tilde{\rho}_0\right\|_{L^1} + K \int_0^T \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{L^1} dt,
\end{aligned}
$$

with

$$
\begin{aligned}
K := \max_{i=1,2}\left\{\left\|v_i'\right\|_\infty\right\} \Bigg[&2\omega_\eta(0) \sup_{t\in[0,T]} \left\|\rho(t,\cdot)\right\|_{\mathrm{BV}(\mathbb{R})} \\
&+ \left\|\omega_\eta'\right\|_\infty \left(\sup_{t\in[0,T]} \left\|\rho(t,\cdot)\right\|_{L^1} + \sup_{t\in[0,T]} \left\|g(\rho(t,\cdot))\right\|_{L^1}\right) \\
&+ 2\omega_\eta(0) \left(\sup_{t\in[0,T]} \left\|\rho(t,\cdot)\right\|_\infty + \sup_{t\in[0,T]} \left\|g(\rho(t,\cdot))\right\|_\infty\right)\Bigg]. \qquad (3.7)
\end{aligned}
$$

By Gronwall's lemma we get the thesis and for $\rho_0 = \tilde{\rho}_0$ the uniqueness of entropy solutions. □

Remark 2. *Note that we cannot directly apply previous results in the literature [10, 13, 21] to the present model, because it does not fit precisely the assumptions therein. Moreover, direct computations allow to recover sharper estimates on the coefficients.*

4 Numerical scheme

In order to prove the well-posedness of model (2.1)–(2.6), we prove the existence of solutions via a numerical scheme which is based on the scheme from [17]. Even though this scheme has been introduced in [17] as a Godunov type scheme, it reduces to an upwind type scheme.
For $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $x_{j-1/2} = j\Delta x$ be the cell interfaces, $x_j = (j+1/2)\Delta x$ the cells centers, corresponding to a space step Δx such that $\eta = N_\eta \Delta x$ for some $N_\eta \in \mathbb{N}$, and let $t^n = n\Delta t$ be the time mesh. In particular, $x = x_{-1/2} = 0$ is a cell interface. We aim at constructing a finite volume approximate solution $\rho^{\Delta x}$ such that $\rho^{\Delta x}(t,x) = \rho_j^n$ for $(t,x) \in [t^n, t^{n+1}[\,\times[x_{j-1/2}, x_{j+1/2}[$. To this end, we approximate the initial datum ρ_0 with the cell averages

$$\rho_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x)dx, \qquad j \in \mathbb{Z}.$$

Following [17], we consider the numerical flux function

$$F_{j+1/2}^n(\rho_j^n) := \rho_j^n V_j^{1,n} + g(\rho_j^n) V_j^{2,n} \tag{4.1}$$

with

$$V_j^{1,n} = \sum_{k=0}^{\min\{-j-2, N_\eta-1\}} \gamma_k v_1(\rho_{j+k+1}^n), \qquad V_j^{2,n} = \sum_{k=\max\{-j-1,0\}}^{N_\eta-1} \gamma_k v_2(\rho_{j+k+1}^n), \tag{4.2}$$

$$\gamma_k = \int_{k\Delta x}^{(k+1)\Delta x} \omega_\eta(x)dx, \qquad k = 0, \dots, N_\eta - 1, \tag{4.3}$$

where we set, with some abuse of notation $\sum_{k=a}^{b} = 0$ whenever $b < a$. In this way we can define the following finite volume numerical scheme

$$\rho_j^{n+1} = \rho_j^n - \lambda\left(F_{j+1/2}^n(\rho_j^n) - F_{j-1/2}^n(\rho_{j-1}^n)\right) \text{ with } \lambda := \frac{\Delta t}{\Delta x}. \tag{4.4}$$

Note that, due to the accurate calculation of the integral in (4.3) and the definition of the convoluted velocities in (4.2), there holds

$$0 \le V_j^{1,n} \le v_{\max}^1, \quad 0 \le V_j^{2,n} \le v_{\max}^2, \quad 0 \le V_j^{1,n} + V_j^{2,n} \le \max\{v_{\max}^1, v_{\max}^2\}, \ \forall\, j \in \mathbb{Z},\ n \in \mathbb{N}.$$

We set

$$\|v\| := \max\{\|v_1\|_\infty, \|v_2\|_\infty\}, \quad \|v'\| := \max\{\|v_1'\|_\infty, \|v_2'\|_\infty\}, \quad \|\rho\| := \max\{\rho_{\max}^1, \rho_{\max}^2\}$$

and consider the following CFL condition:

$$\lambda \le \frac{1}{\gamma_0\|v'\|\|\rho\| + \|v\|}. \tag{4.5}$$

We will show that, under this CFL condition, the numerical scheme (4.1)–(4.4) satisfies a maximum principle, uniform BV estimates and a discrete entropy inequality. Equipped with these properties, we will show in Section 5 that the sequence of approximate solutions $\left\{\rho^{\Delta x}\right\}$ converges towards the entropy solution of (2.1)–(2.6). Note that, for $v_1 \equiv v_2$, the scheme (4.1)–(4.4) coincides with the scheme in [17].

In the following proofs, we will omit the dependence on n of the flux function and the velocity whenever possible, in order to simplify the notation.

4.1 Maximum principle

The solutions generated by the numerical scheme (4.4) stay always positive and they are bounded by the maximum road capacity of each road segment as stated by the following lemma.

Lemma 1. *Under hypothesis* (2.6) *and the CFL condition* (4.5)*, the sequence generated by the numerical scheme* (4.1)–(4.4) *satisfies the following maximum principle:*

$$0 \le \rho_j^n \le \rho^1_{\max} \quad \textit{for} \quad j \le -1 \qquad \textit{and} \qquad 0 \le \rho_j^n \le \rho^2_{\max} \quad \textit{for} \quad j \ge 0, \qquad \forall n \in \mathbb{N}.$$

Proof. We start by showing the positivity. We directly obtain

$$\rho_j^{n+1} = \rho_j^n - \lambda \left(F^n_{j+\frac{1}{2}}(\rho_j^n) - F^n_{j-\frac{1}{2}}(\rho^n_{j-1}) \right) \ge \rho_j^n - \lambda F^n_{j+\frac{1}{2}}(\rho_j^n) \ge \rho_j^n - \lambda \|v\| \rho_j^n \ge 0.$$

Here we used the CFL condition (4.5) and $g(\rho_j^n) \le \rho_j^n$.
The rest of the proof follows closely the proof of [17, Theorem 3.1]. Therefore, we compute the differences of the velocities and obtain

$$V^{1,n}_{j-1} - V^{1,n}_j = \begin{cases} \sum_{k=1}^{N_\eta - 1} (\gamma_k - \gamma_{k-1}) v_1(\rho^n_{j+k}) - \gamma_{N_\eta - 1} v_1(\rho^n_{j+N_\eta}) + \gamma_0 v_1(\rho_j^n), & j \le -N_\eta - 1, \\ \sum_{k=1}^{-j-1} (\gamma_k - \gamma_{k-1}) v_1(\rho^n_{j+k}) + \gamma_0 v_1(\rho_j^n), & -N_\eta \le j \le -2, \\ \gamma_0 v_1(\rho^n_{-1}), & j = -1, \\ 0, & j \ge 0, \end{cases} \tag{4.6}$$

and

$$V^{2,n}_{j-1} - V^{2,n}_j = \begin{cases} 0, & j \le -N_\eta - 1, \\ -\gamma_{N_\eta - 1} v_2(\rho^n_0), & j = -N_\eta, \\ \sum_{k=-j}^{N_\eta - 1} (\gamma_k - \gamma_{k-1}) v_2(\rho^n_{j+k}) - \gamma_{N_\eta - 1} v_2(\rho^n_{j+N_\eta}), & -N_\eta + 1 \le j \le -1, \\ \sum_{k=1}^{N_\eta - 1} (\gamma_k - \gamma_{k-1}) v_2(\rho^n_{j+k}) - \gamma_{N_\eta - 1} v_2(\rho^n_{j+N_\eta}) + \gamma_0 v_2(\rho_j^n), & j \ge 0. \end{cases} \tag{4.7}$$

It is easy to see that the following estimates hold:

$$V^{1,n}_{j-1} - V^{1,n}_j \le \begin{cases} \gamma_0 v_1(\rho_j^n) & j \le -1, \\ 0, & j \ge 0, \end{cases} \quad \text{and} \quad V^{2,n}_{j-1} - V^{2,n}_j \le \begin{cases} 0 & j \le -1, \\ \gamma_0 v_2(\rho_j^n), & j \ge 0. \end{cases}$$

Using $v_1(\rho^1_{\max}) = v_2(\rho^2_{\max}) = 0$ and the mean value theorem we get

$$V^{1,n}_{j-1} - V^{1,n}_j \le \begin{cases} \gamma_0 \|v'\| (\rho^1_{\max} - \rho_j^n) & j \le -1, \\ 0, & j \ge 0, \end{cases} \quad \text{and} \quad V^{2,n}_{j-1} - V^{2,n}_j \le \begin{cases} 0 & j \le -1, \\ \gamma_0 \|v'\| (\rho^2_{\max} - \rho_j^n), & j \ge 0. \end{cases}$$

Now we consider the case $j \leq -1$ and multiply the first inequality by $\rho^1_{\max}$, subtract $V_j^{1,n}\rho_j^n$ and we get

$$V_{j-1}^{1,n}\rho^1_{\max} - V_j^{1,n}\rho_j^n \leq \left(\gamma_0\|v'\|\|\rho\| + V_j^{1,n}\right)(\rho^1_{\max} - \rho_j^n).$$

Similarly, we get

$$V_{j-1}^{2,n}g(\rho^1_{\max}) - V_j^{2,n}g(\rho_j^n) \leq V_j^{2,n}\left(g(\rho^1_{\max}) - g(\rho_j^n)\right) \leq V_j^{2,n}(\rho^1_{\max} - \rho_j^n).$$

Adding the last two inequalities we obtain,

$$V_{j-1}^{1,n}\rho^1_{\max} - V_j^{1,n}\rho_j^n + V_{j-1}^{2,n}g(\rho^1_{\max}) - V_j^{2,n}g(\rho_j^n) \leq (\gamma_0\|v'\|\|\rho\| + \|v\|)(\rho^1_{\max} - \rho_j^n).$$

Due to the CFL condition (4.5), we have for $j \leq -1$

$$\rho_j^{n+1} \leq \rho_j^n + \lambda\left(V_{j-1}^{1,n}\rho^1_{\max} - V_j^{1,n}\rho_j^n + V_{j-1}^{2,n}g(\rho^1_{\max}) - V_j^{2,n}g(\rho_j^n)\right) \leq \rho^1_{\max}.$$

For $j \geq 0$ the bound

$$V_{j-1}^{2,n}\rho^2_{\max} - V_j^{2,n}\rho_j^n \leq (\gamma_0\|v'\|\|\rho\| + \|v\|)(\rho^2_{\max} - \rho_j^n)$$

follows analogously to above. Note that $V_j^{1,n} = 0$ for $j \geq -1$. Since $g(\rho_{j-1}^n) \leq \rho^2_{\max}$ holds even for $j = 0$ and $g(\rho_j^n) = \rho_j^n$ for $j \geq 0$, we obtain

$$\rho_j^{n+1} \leq \rho_j^n + \lambda\left(V_{j-1}^{2,n}\rho^2_{\max} - V_j^{2,n}\rho_j^n\right) \leq \rho^2_{\max}.$$

This concludes the proof. □

Remark 3. *The role of the limiter g given by* (2.3) *in the flux function* (2.2) *is essential for the discrete maximum principle above. Indeed, let us consider the model without this limiter. In order to deal with meaningful velocities, we set $v_2(\rho) = 0$ if $\rho > \rho^2_{\max}$, such that we have*

$$V_2(t,x) = \int_{\max\{x,0\}}^{\max\{x+\eta,0\}} \max\{0, v_2(\rho(t,y))\}\omega_\eta(y-x)dy.$$

For this model, it is possible to prove a maximum principle on $[0, \max\{\rho^1_{\max}, \rho^2_{\max}\}]$ as above and similar BV estimates as below, so that the convergence to a solution is ensured. Nevertheless, this solution has an interesting behavior for $\eta \to \infty$ and an initial datum of compact support; in this case, $V_1(t,x) + V_2(t,x)$ converges pointwise to $v_2(0)$ and it is possible to prove that the solution will converge to the solution of a linear transport with velocity $v_2(0)$ (see [8, Corollary 1.3] for a similar result). Therefore, it is obvious that, if $\rho^1_{\max} > \rho^2_{\max}$, for any initial datum ρ_0 of compact support, such that $\rho^2_{\max} < \rho_0(x) \leq \rho^1_{\max}$ for $x < 0$, the corresponding solution does not satisfy (2.9)*, i.e. $\rho(x,t) > \rho^2_{\max}$ for $x \in]0, a[$, $a > 0$, and t and η large enough.*

4.2 BV estimate

In addition to the L^∞ bound, we also need a uniform estimate on the total variation of the sequence of approximate solutions. The crucial part here lies in the presence of the limiter g at $x = 0$.

Lemma 2. *Let $\rho^{\Delta x}$ be constructed by (4.1)–(4.4) and let the CFL condition (4.5) hold, then for every $T > 0$ the following discrete space BV estimate is satisfied:*

$$TV(\rho^{\Delta x}(T, \cdot)) \leq \exp\left(Tw_\eta(0)\left(2\|v\| + \|v'\|\|\rho\|\right)\right)\left(TV(\rho_0) + T2\omega_\eta(0)\|v\|\|\rho\|\right) =: K(T). \tag{4.8}$$

Proof. For all $j \in \mathbb{Z}$, we set

$$\Delta_j^n := \rho_{j+1}^n - \rho_j^n.$$

In the following we consider a regularization of the function g defined in (2.3), namely

$$g_\varepsilon(\rho) = \frac{1}{2}\left(\rho + \rho_{\max}^2 - \sqrt{(\rho - \rho_{\max}^2)^2 + \varepsilon}\right), \qquad \varepsilon > 0. \tag{4.9}$$

The function g_ε is differentiable for every $\varepsilon > 0$ with $\|g_\varepsilon'\| \leq 1$ for all $\varepsilon > 0$. This will allow us to use the mean value theorem in the following computations. In particular, we will denote by ξ_j^n a value between ρ_j^n and ρ_{j+1}^n such that $g_\varepsilon'(\xi_j^n)\Delta_j^n = g_\varepsilon(\rho_{j+1}^n) - g_\varepsilon(\rho_j^n)$ holds. We obtain:

$$\begin{aligned}
\Delta_j^{n+1} =& \Delta_j^n - \lambda\left(F_{j+\frac{3}{2}}^n(\rho_{j+1}^n) - 2F_{j+\frac{1}{2}}^n(\rho_j^n) + F_{j-\frac{1}{2}}^n(\rho_{j-1}^n)\right)\\
=& \Delta_j^n - \lambda\left(\left(V_{j+1}^{1,n} + g_\varepsilon'(\xi_j^n)V_{j+1}^{2,n}\right)\Delta_j^n - \left(V_{j-1}^{1,n} + g_\varepsilon'(\xi_{j-1}^n)V_{j-1}^{2,n}\right)\Delta_{j-1}^n\right.\\
&\left. + \rho_j^n\left(V_{j+1}^{1,n} - 2V_j^{1,n} + V_{j-1}^{1,n}\right) + g_\varepsilon(\rho_j^n)\left(V_{j+1}^{2,n} - 2V_j^{2,n} + V_{j-1}^{2,n}\right)\right).
\end{aligned}$$

Let us now consider the differences of the velocities. With the differences already computed in (4.6) and (4.7) and the help of the mean value theorem, where ζ_j^n is a value between ρ_j^n and ρ_{j+1}^n for which $v_i'(\zeta_j^n)\Delta_j^n = v_i(\rho_{j+1}^n) - v_i(\rho_j^n)$ for $i \in \{1, 2\}$ holds, we derive

$$V_{j+1}^{1,n} - 2V_j^{1,n} + V_{j-1}^{1,n} =$$
$$\begin{cases}
\sum_{k=1}^{N_\eta-1}(\gamma_{k-1} - \gamma_k)v_1'(\zeta_{j+k}^n)\Delta_{j+k}^n + \gamma_{N_\eta-1}v_1'(\zeta_{j+N_\eta}^n)\Delta_{j+N_\eta}^n - \gamma_0 v_1'(\zeta_j^n)\Delta_j^n, & j \leq -N_\eta - 2,\\
\sum_{k=1}^{N_\eta-1}(\gamma_{k-1} - \gamma_k)v_1'(\zeta_{j+k}^n)\Delta_{j+k}^n - \gamma_{N_\eta-1}v_1(\rho_{-1}^n) - \gamma_0 v_1'(\zeta_j^n)\Delta_j^n, & j = -N_\eta - 1,\\
\sum_{k=1}^{-j-2}(\gamma_{k-1} - \gamma_k)v_1'(\zeta_{j+k}^n)\Delta_{j+k}^n + (\gamma_{-j-1} - \gamma_{-j-2})v_1(\rho_{-1}^n) - \gamma_0 v_1'(\zeta_j^n)\Delta_j^n, & -N_\eta \leq j \leq -3,\\
(\gamma_1 - \gamma_0)v_1(\rho_{-1}^n) - \gamma_0 v_1'(\zeta_j^n)\Delta_j^n, & j = -2,\\
\gamma_0 v_1(\rho_{-1}^n), & j = -1,\\
0, & j \geq 0,
\end{cases}$$

and

$$V_{j+1}^{2,n} - 2V_j^{2,n} + V_{j-1}^{2,n} =$$

$$
\begin{cases}
0, & j \leq -N_\eta - 2, \\
\gamma_{N_\eta - 1} v_2(\rho_0^n), & j = -N_\eta - 1, \\
\gamma_{N_\eta - 1} v_2'(\zeta_{j+N_\eta}^n) \Delta_{j+N_\eta}^n + (\gamma_{N_\eta-1} - \gamma_{N_\eta}) v_2(\rho_0^n), & j = -N_\eta, \\
\sum_{k=-j}^{N_\eta-1} (\gamma_{k-1} - \gamma_k) v_2'(\zeta_{j+k}^n) \Delta_{j+k}^n + \gamma_{N_\eta-1} v_2'(\zeta_{j+N_\eta}^n) \Delta_{j+N_\eta}^n + (\gamma_{-j-2} - \gamma_{-j-1}) v_2(\rho_0^n), & -N_\eta + 1 \leq j \leq -2, \\
\sum_{k=1}^{N_\eta-1} (\gamma_{k-1} - \gamma_k) v_2'(\zeta_{j+k}^n) \Delta_{j+k}^n + \gamma_{N_\eta-1} v_2'(\zeta_{j+N_\eta}^n) \Delta_{j+N_\eta}^n - \gamma_0 v_2(\rho_0^n), & j = -1, \\
\sum_{k=1}^{N_\eta-1} (\gamma_{k-1} - \gamma_k) v_2'(\zeta_{j+k}^n) \Delta_{j+k}^n + \gamma_{N_\eta-1} v_2'(\zeta_{j+N_\eta}^n) \Delta_{j+N_\eta}^n - \gamma_0 v_2'(\zeta_j^n) \Delta_j^n, & j \geq 0.
\end{cases}
$$

Putting everything together we have

$$
\begin{aligned}
\Delta_j^{n+1} = {} & \left(1 - \lambda \left(V_{j+1}^{1,n} + g_\varepsilon'(\xi_j^n) V_{j+1}^{2,n} - \gamma_0 a_j^n\right)\right) \Delta_j^n + \lambda \left(V_{j-1}^{1,n} + g_\varepsilon'(\xi_{j-1}^n) V_{j-1}^{2,n}\right) \Delta_{j-1}^n \\
& + \lambda \sum_{k=1}^{N_\eta-1} (\gamma_{k-1} - \gamma_k) b_{j+k}^n \Delta_{j+k}^n + \lambda \gamma_{N_\eta-1} c_{j+N_\eta}^n \Delta_{j+N_\eta}^n \\
& + \lambda d_j^n \left(\rho_j v_1(\rho_{-1}^n) - g_\varepsilon(\rho_j^n) v_2(\rho_0^n)\right),
\end{aligned} \tag{4.10}
$$

where

$$
a_j^n = \begin{cases}
v_1'(\zeta_j^n)\rho_j^n, & j \leq -2, \\
0, & j = -1, \\
v_2'(\zeta_j^n)\rho_j^n, & j \geq 0,
\end{cases}
\qquad
b_{j+k}^n = \begin{cases}
-v_1'(\zeta_{j+k}^n)\rho_j^n, & j + k \leq -2, \\
0, & j + k = -1, \\
-v_2'(\zeta_{j+k}^n) g_\varepsilon(\rho_j^n), & j + k \geq 0,
\end{cases}
$$

$$
c_{j+N_\eta}^n = \begin{cases}
-v_1'(\zeta_{j+N_\eta}^n)\rho_j^n, & j \leq -N_\eta - 2, \\
0, & j = -N_\eta - 1, \\
-v_2'(\zeta_{j+N_\eta}^n) g_\varepsilon(\rho_j^n), & j \geq -N_\eta,
\end{cases}
\qquad
d_j^n = \begin{cases}
0, & j \leq -N_\eta - 2, \\
\gamma_{N_\eta-1}, & j = -N_\eta - 1, \\
\gamma_{-j-2} - \gamma_{-j-1}, & -N_\eta \leq j \leq -2, \\
-\gamma_0, & j = -1, \\
0, & j \geq 0.
\end{cases}
$$

Since the coefficients in (4.10) are positive due to the CFL condition (4.5), we take absolute values, sum over j and rearrange the indices, which gives us

$$
\begin{aligned}
\sum_j |\Delta_j^{n+1}| \leq {} & \sum_j \Bigg[\left(1 - \lambda \left(V_{j+1}^{1,n} + g_\varepsilon'(\xi_j^n) V_{j+1}^{2,n} - \gamma_0 a_j^n\right)\right) |\Delta_j^n| + \lambda \left(V_{j-1}^{1,n} + g_\varepsilon'(\xi_{j-1}^n) V_{j-1}^{2,n}\right) |\Delta_{j-1}^n| \\
& + \lambda \sum_{k=1}^{N_\eta-1} (\gamma_{k-1} - \gamma_k) b_{j+k}^n |\Delta_{j+k}^n| + \lambda \gamma_{N_\eta-1} c_{j+N_\eta}^n |\Delta_{j+N_\eta}^n| \\
& + \lambda |d_j^n| \left|\rho_j v_1(\rho_{-1}^n) - g_\varepsilon(\rho_j^n) v_2(\rho_0^n)\right| \Bigg] \\
= {} & \sum_j \Bigg[1 - \lambda \left(V_{j+1}^{1,n} + g_\varepsilon'(\xi_j^n) V_{j+1}^{2,n} - V_j^{1,n} - g_\varepsilon'(\xi_j^n) V_j^{2,n}\right) \\
& + \lambda \Bigg(\gamma_0 a_j^n + \sum_{k=1}^{N_\eta-1} (\gamma_{k-1} - \gamma_k) b_j^n + \gamma_{N_\eta-1} c_j^n \Bigg) \Bigg] |\Delta_j^n| \\
& + \sum_j \lambda |d_j^n| \left|\rho_j v_1(\rho_{-1}^n) - g_\varepsilon(\rho_j^n) v_2(\rho_0^n)\right|.
\end{aligned}
$$

Now we use that $V_j^{i,n} - V_{j+1}^{i,n} \leq \gamma_0 \|v\|$ and $\|g_\varepsilon'\| \leq 1$ for the first term and for the second term we have $a_j^n \leq 0$ and $b_j^n, c_j^n \leq \|v'\|\|\rho\|$, which gives us

$$\begin{aligned}\sum_j |\Delta_j^{n+1}| \leq &\Big(1 + \lambda\gamma_0 \left(2\|v\| + \|v'\|\|\rho\|\right)\Big) \sum_j |\Delta_j^n| \\ &+ \sum_j \lambda |d_j^n| \left| \rho_j v_1(\rho_{-1}^n) - g_\varepsilon(\rho_j^n) v_2(\rho_0^n) \right|.\end{aligned}$$

Since $\sum_j |d_j^n| = 2\gamma_0$ holds, using also $\lambda\gamma_0 \leq \Delta t \omega_\eta(0)$ we finally obtain

$$\sum_j |\Delta_j^{n+1}| \leq \Big(1 + \Delta t \omega_\eta(0) \left(2\|v\| + \|v'\|\|\rho\|\right)\Big) \sum_j |\Delta_j^n| + \Delta t 2\omega_\eta(0) \|v\| (\|\rho\| + \frac{\sqrt{\varepsilon}}{2}).$$

This estimate holds for any $\varepsilon > 0$ and for $\varepsilon \to 0$ we obtain the following estimate for the total variation

$$\begin{aligned}TV(\rho(T,\cdot)) &\leq \Big(1 + \Delta t \omega_\eta(0) \left(2\|v\| + \|v'\|\|\rho\|\right)\Big)^{T/\Delta t} \left(TV(\rho_0) + T 2\omega_\eta(0)\|v\|\|\rho\|\right) \\ &\leq \exp\Big(\omega_\eta(0) \left(2\|v\| + \|v'\|\|\rho\|\right) T\Big) \left(TV(\rho_0) + T2\omega_\eta(0)\|v\|\|\rho\|\right).\end{aligned}$$

□

To finally apply Helly's Theorem we also need an estimate for the discrete total variation in space and time, which we are now able to provide.

Lemma 3. *Let $\rho^{\Delta x}$ be constructed by (4.1)–(4.4) and let the CFL condition (4.5) hold, then for every $T > 0$ the following discrete space and time total variation estimate is satisfied:*

$$TV(\rho^{\Delta x}; \mathbb{R} \times [0,T]) \leq T K(T) (1 + \|v'\|\|\rho\| + \|v\|)$$

with $K(T)$ defined as in (4.8).

Using the regularization of g given by (4.9), the proof is entirely analogous to the one of [17, Theorem 3.3].

4.3 Discrete Entropy Inequality

In the following, we use the notation $a \wedge b = \max\{a,b\}$, $a \vee b = \min\{a,b\}$ and follow [3, 8, 17].

Lemma 4. *Let $\rho^{\Delta x}$ be constructed by (4.1)–(4.4). If the CFL condition (4.5) holds, then for $c \in \mathbb{R}$ we have the following discrete entropy inequality*

$$\begin{aligned}\left|\rho_j^{n+1} - c\right| \leq &\left|\rho_j^n - c\right| - \lambda \left(H_{j+1/2}^n(\rho_j^n) - H_{j-1/2}^n(\rho_{j-1}^n)\right) \\ &- \lambda \operatorname{sgn}(\rho_j^{n+1} - c) \left(F_{j+1/2}^n(c) - F_{j-1/2}^n(c)\right),\end{aligned} \tag{4.11}$$

where

$$H_{j+1/2}^n(u) = F_{j+1/2}^n(u \wedge c) - F_{j+1/2}^n(u \vee c).$$

Proof. Let

$$G_j^n(u,w) = w - \lambda(F_{j+1/2}^n(w) - F_{j-1/2}^n(u)).$$

Under the CFL condition (4.5) and using the regularization (4.9) of g, G_j is monotone in both its arguments, since we obtain

$$\frac{\partial G_j^n}{\partial w} = 1 - \lambda(V_{j+\frac{1}{2}}^{1,n} + g_\varepsilon'(w)V_{j+\frac{1}{2}}^{2,n}) \geq 0, \quad \frac{\partial G_j^n}{\partial u} = \lambda(V_{j-\frac{1}{2}}^{1,n} + g_\varepsilon'(u)V_{j-\frac{1}{2}}^{2,n}) \geq 0.$$

The monotonicity implies that

$$G_j^n(\rho_{j-1}^n \wedge c, \rho_j^n \wedge c) \geq G_j^n(\rho_{j-1}^n, \rho_j^n) \wedge G_j^n(c,c) \tag{4.12}$$

$$G_j^n(\rho_{j-1}^n \vee c, \rho_j^n \vee c) \leq G_j^n(\rho_{j-1}^n, \rho_j^n) \vee G_j^n(c,c). \tag{4.13}$$

Subtracting (4.13) from (4.12), we obtain

$$\left|G_j^n(\rho_{j-1}^n, \rho_j^n) - G_j^n(c,c)\right| \leq \left|\rho_j^n - c\right| - \lambda\left(H_{j+1/2}^n(\rho_j^n) - H_{j-1/2}^n(\rho_{j-1}^n)\right). \tag{4.14}$$

The left side of (4.14) is $\left|\rho_j^{n+1} - c + \lambda(F_{j+1/2}^n(c) - F_{j-1/2}(c))\right|$, and we get

$$\begin{aligned}
&\left|\rho_j^{n+1} - c + \lambda(F_{j+1/2}^n(c) - F_{j-1/2}^n(c))\right| \\
&\qquad \geq \operatorname{sgn}(\rho_j^{n+1} - c)\left(\rho_j^{n+1} - c + \lambda(F_{j+1/2}^n(c) - F_{j-1/2}^n(c))\right) \\
&\qquad = \left|\rho_j^{n+1} - c\right| + \lambda \operatorname{sgn}(\rho_j^{n+1} - c)\left(F_{j+1/2}^n(c) - F_{j-1/2}^n(c)\right).
\end{aligned} \tag{4.15}$$

The proof is completed by combining (4.14) and (4.15). □

5 Convergence

Lemma 5. *Let $\rho = \rho(t,x) \in L^\infty \cap \mathrm{BV}([0,+\infty) \times \mathbb{R}; [0, \max\{\rho_{\max}^1, \rho_{\max}^2\})$ be the L_{loc}^1-limit of approximations $\rho^{\Delta x}$ generated by the upwind scheme (4.4) and let $c \in \mathbb{R}$, $\varphi \in C_c^1([0,+\infty)\times\mathbb{R})$. Then ρ satisfies the entropy inequality given by (2.8).*

Proof. Let $\varphi \in C_c^1([0,+\infty) \times \mathbb{R})$ and set $\varphi_j^n = \varphi(t^n, x_j)$. We multiply the discrete entropy inequality (4.11) by $\varphi_j^n \Delta x$, and then apply summation by parts to get

$$\Delta x \Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \left|\rho_j^{n+1} - c\right|(\varphi_j^{n+1} - \varphi_j^n)/\Delta t + \Delta x \sum_j \left|\rho_j^0 - c\right|\varphi_j^0 \tag{5.1}$$

$$+ \Delta x \Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} H_{j-1/2}^n(\varphi_j^n - \varphi_{j-1}^n)/\Delta x \tag{5.2}$$

$$- \Delta x \Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \operatorname{sgn}(\rho_j^{n+1} - c)\left(F_{j+1/2}^n(c) - F_{j-1/2}^n(c)\right)\varphi_j^n/\Delta x \geq 0. \tag{5.3}$$

By Lebesgue's dominated convergence theorem, as $\Delta x \to 0$, we have

$$(5.1) \to \int_0^\infty \int_{\mathbb{R}} |\rho - c|\varphi_t dxdt + \int_{-\infty}^\infty |\rho_0(x) - c|\varphi(0,x)dx.$$

As $\Delta x \to 0$, the sums in (5.2) converge by standard arguments, see [5], [6, Sec. 4 Proof of Theorem 1], [20], to

$$\int_0^\infty \int_{\mathbb{R}} \operatorname{sgn}(\rho - c)(f(t,x,\rho) - f(t,x,c))\varphi_x \, dxdt.$$

Now let us study the sum (5.3) and we have

$$\begin{aligned}
(5.3) = &-\Delta x \Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \operatorname{sgn}(\rho_j^{n+1} - c)\left(cV_j^1 + g(c)V_j^2 - cV_{j-1}^1 - g(c)V_{j-1}^2\right)\varphi_j^n/\Delta x \\
= &-\Delta x \Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \operatorname{sgn}(\rho_j^{n+1} - c)\left(c\frac{V_j^1 - V_{j-1}^1}{\Delta x} + g(c)\frac{V_j^2 - V_{j-1}^2}{\Delta x}\right)\varphi_j^n \\
= &-\Delta x \Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} (\operatorname{sgn}(\rho_j^{n+1} - c) - \operatorname{sgn}(\rho_j^n - c))\left(c\frac{V_j^1 - V_{j-1}^1}{\Delta x} + g(c)\frac{V_j^2 - V_{j-1}^2}{\Delta x}\right)\varphi_j^n \\
&-\Delta x \Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \operatorname{sgn}(\rho_j^n - c)\left(c\frac{V_j^1 - V_{j-1}^1}{\Delta x} + g(c)\frac{V_j^2 - V_{j-1}^2}{\Delta x}\right)\varphi_j^n.
\end{aligned}$$

The second term in the last equality clearly converges to

$$-\int_0^\infty \int_{-\infty}^\infty \operatorname{sgn}(\rho - c)\left(c(V_1)_x + g(c)(V_2)_x\right)\varphi)dxdt.$$

We will show now that the first term vanishes as $\Delta x \to 0$. We follow here [5, 6] and we perform a summation by parts, which gives us:

$$\begin{aligned}
&\Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \operatorname{sgn}(\rho_j^{n+1} - c)\varphi_j^n \left[c\left[\left(V_j^{1,n+1} - V_{j-1}^{1,n+1}\right) - \left(V_j^{1,n} - V_{j-1}^{1,n}\right)\right]\right. \\
&\left. + g(c)\left[\left(V_j^{2,n+1} - V_{j-1}^{2,n+1}\right) - \left(V_j^{2,n} - V_{j-1}^{2,n}\right)\right]\right] \\
&+ \Delta t \Delta t \Delta x \sum_{n\geq 0}\sum_{j<0} \operatorname{sgn}(\rho_j^{n+1} - c)\left[c\frac{\left(V_j^{1,n+1} - V_{j-1}^{1,n+1}\right)}{\Delta x} + g(c)\frac{\left(V_j^{2,n+1} - V_{j-1}^{2,n+1}\right)}{\Delta x}\right]\frac{\left(\varphi_j^{n+1} - \varphi_j^n\right)}{\Delta t}.
\end{aligned}$$

As can be seen in (4.6) and (4.7) $V_j^{i,n+1} - V_{j-1}^{i,n+1} \leq \Delta x \omega_\eta(0)\|v\|$ holds and due to the compactness of the support function the second term vanishes as $\Delta x, \Delta t \to 0$. For the first term we first obtain that

$$\begin{aligned}
&\left(V_j^{1,n+1} - V_{j-1}^{1,n+1}\right) - \left(V_j^{1,n} - V_{j-1}^{1,n}\right) \\
&= \begin{cases}
\displaystyle\sum_{k=1}^{N_\eta - 1} (\gamma_{k-1} - \gamma_k)(v_1(\rho_{j+k}^{n+1} - v_1(\rho_{j+k}^n)) & \\
\quad + \gamma_{N_\eta - 1}(v_1(\rho_{j+N_\eta}^{n+1}) - v_1(\rho_{j+N_\eta}^n)) - \gamma_0(v_1(\rho_j^{n+1}) - v_1(\rho_j^n)), & j \leq -N_\eta - 1, \\
\displaystyle\sum_{k=1}^{-j-1} (\gamma_{k-1} - \gamma_k)(v_1(\rho_{j+k}^{n+1}) - v_1(\rho_{j+k}^n)) - \gamma_0(v_1(\rho_j^{n+1}) - v_1(\rho_j^n)), & -N_\eta \leq j \leq -2, \\
\gamma_0(v_1(\rho_{-1}^{n+1}) - v_1(\rho_{-1}^n)), & j = -1,
\end{cases}
\end{aligned}$$

and

$$
\begin{aligned}
&\left(V_j^{2,n+1}-V_{j-1}^{2,n+1}\right)-\left(V_j^{2,n}-V_{j-1}^{2,n}\right)\\
&=\begin{cases}
0, & j\le -N_\eta-1,\\
\gamma_{N_\eta-1}(v_2(\rho_{j+N_\eta}^{n+1})-v_2(\rho_{j+N_\eta}^{n})), & j=-N_\eta,\\
\displaystyle\sum_{k=-j}^{N_\eta-1}(\gamma_{k-1}-\gamma_k)(v_2(\rho_{j+k}^{n+1})-v_2(\rho_{j+k}^{n}))+\gamma_{N_\eta-1}(v_2(\rho_{j+N_\eta}^{n+1})-v_2(\rho_{j+N_\eta}^{n})), & -N_\eta+1\le j\le -1.
\end{cases}
\end{aligned}
$$

Now we use the compact support of the test function. There exist $T>0$ and $R>0$ such that $\varphi(t,x)=0$ for $t>T$ and $|x|>R$. Let $N_T\in\mathbb{N}$ and $j_0,j_1\in\mathbb{Z}$ be such that $T\in\,]n_T\Delta t,(n_T+1)\Delta t]$, $-R\in]x_{j_0-\frac{1}{2}},x_{j_0+\frac{1}{2}}]$, $R\in]x_{j_1-\frac{1}{2}},x_{j_1+\frac{1}{2}}]$. We only consider $j_0<0$, since otherwise the term is already 0. In addition, similar to [17, Theorem 3.3], the following estimate is derived during the proof of Lemma 3:

$$
\sum_{n=0}^{N_T}\sum_j \Delta x|\rho_j^{n+1}-\rho_j^n|\le\tilde{K},
$$

By plugging in the equality obtained before, using the mean value theorem, the above mentioned estimate and $g(c)\le c$ we obtain

$$
\begin{aligned}
&\Delta t\sum_{n\ge0}\sum_{j<0}\operatorname{sgn}(\rho_j^{n+1}-c)\varphi_j^n\Bigg[c\left(\left(V_j^{1,n+1}-V_{j-1}^{1,n+1}\right)-\left(V_j^{1,n}-V_{j-1}^{1,n}\right)\right)\\
&+g(c)\left(\left(V_j^{2,n+1}-V_{j-1}^{2,n+1}\right)-\left(V_j^{2,n}-V_{j-1}^{2,n}\right)\right)\Bigg]\\
&\le\frac{\Delta t}{\Delta x}\|\varphi\|\|v'\|c\Bigg[\gamma_{N_\eta-1}\sum_{n=0}^{N_T}\sum_{j=j_0}^{\min\{-1,j_1\}}\Delta x|\rho_{j+N_\eta}^{n+1}-\rho_{j+N_\eta}^{n}|+\\
&\sum_{k=1}^{N_\eta-1}(\gamma_{k-1}-\gamma_k)\sum_{n=0}^{N_T}\sum_{j=j_0}^{\min\{-1,j_1\}}\Delta x|\rho_{j+k}^{n+1}-\rho_{j+k}^{n}|+\gamma_0\sum_{n=0}^{N_T}\sum_{j=j_0}^{\min\{-1,j_1\}}\Delta x|\rho_j^{n+1}-\rho_j^n|\Bigg]\\
&\le\Delta t\|\varphi\|\|v'\|c\tilde{K}2\omega_\eta(0),
\end{aligned}
$$

which goes to zero as $\Delta x\to0$ (and then $\Delta t\to0$). This concludes the proof. □

Proof of Theorem 1.
Similar to [8, Theorem 1], [17, Theorem 2.3] or [6, Theorem 1], the convergence of the approximate solutions constructed by the upwind scheme (4.4) to the unique weak entropy solution can be proven by applying Helly's theorem, see [16, Lemma 5.6]. Due to Lemma 1 and Lemma 3, there exists a sub-sequence of approximate solutions that converges to some $\rho\in(L^\infty\cap\mathrm{BV})([0,+\infty)\times\mathbb{R};[0,\max\{\rho^1_{\max},\rho^2_{\max}\}])$. Lemma 5 shows that the limit function ρ is a weak entropy solution of (2.1)–(2.6) in the sense of Definition 1. Adding the uniqueness result in Theorem 2, we conclude the proof of Theorem 1. □

6 Numerical simulations

In this section, we show some simulation results to illustrate the numerical solutions of the non-local model (2.1). The behaviour of solutions is also studied as the look-ahead distance η tends to zero. To this end, we will consider Riemann initial data of the type

$$\rho_0(x) = \begin{cases} \rho_L, & \text{if } x < 0, \\ \rho_R, & \text{if } x > 0. \end{cases} \tag{6.1}$$

We take a spatial step size of $\Delta x = 10^{-3}$. Due to the CFL condition (4.5) the time step size Δt is given by $\Delta t \approx 0.9 \Delta x / (\gamma_0 \|v'\| \|\rho\| + \|v\|)$.
We divide this section into three parts. In the first part we analyze how our model behaves for a fixed look ahead distance $\eta > 0$. For non-local conservation laws, it is still an open question whether the model tends to the corresponding local equation for η tending to zero (see for example [11] for a recent overview). For this reason, we will investigate the limit question as $\eta \to 0$ from the numerical point of view in Section 6.2. Overall, we will consider the following settings:

Test 1: $v_i(\rho) = v^i_{\max}\left(1 - \left(\frac{\rho}{\rho^i_{\max}}\right)^2\right)$ for $i \in \{1,2\}$, with $v^1_{\max} = 1,\ v^2_{\max} = 2,\ \rho^1_{\max} = \rho^2_{\max} = 1,\ \rho_L = 0.75,\ \rho_R = 0.5$;

Test 2: as in Test 1, but with $v^1_{\max} = 2,\ v^2_{\max} = 1$;

Test 3: $v_i(\rho) = v^i_{\max}\left(1 - \frac{\rho}{\rho^i_{\max}}\right)$ for $i \in \{1,2\}$, with $v^1_{\max} = 2,\ v^2_{\max} = 1,\ \rho^1_{\max} = 0.5,\ \rho^2_{\max} = 1,\ \rho_L = 0.25,\ \rho_R = 0.5$;

Test 4: v_i as in Test 3, but with $v^1_{\max} = 1,\ v^2_{\max} = 2,\ \rho^1_{\max} = 1,\ \rho^2_{\max} = 0.5,\ \rho_L = 0.5,\ \rho_R = 0.25$.

The first two settings are used to show that the obtained solutions are reasonable also for non-linear velocity functions, while the last two settings turn out to be interesting in Section 6.2. For all the tests, the kernel function is given by $\omega_\eta(x) = 2(\eta - x)/\eta^2$ and the final simulation time is $T = 1$.
Finally, in Section 6.3, we will show that our model can be easily extended to more than two stretches and therefore to a sequence of 1-to-1 junctions to simulate traffic.

6.1 Fixed look-ahead distance η

We set $\eta = 0.1$. Let us consider the first test. Here we start with a congested situation on the first road segment. In addition, the maximum velocity on the first road is lower than the one on the second road segment. Therefore, the traffic jam resolves over time as can be seen in Figure 2, left. In contrast to Test 1, Test 2 presents the opposite situation: the velocity on the first road segment is now higher than the second one. Hence, the traffic jam can not resolve and we get a backward traveling increase of the density (see Figure 2, right).

In the last two settings we can see that the presence of the look ahead distance results in a smoothing of the density close to the end of the first and the beginning of the second road segment, see Figure 3.

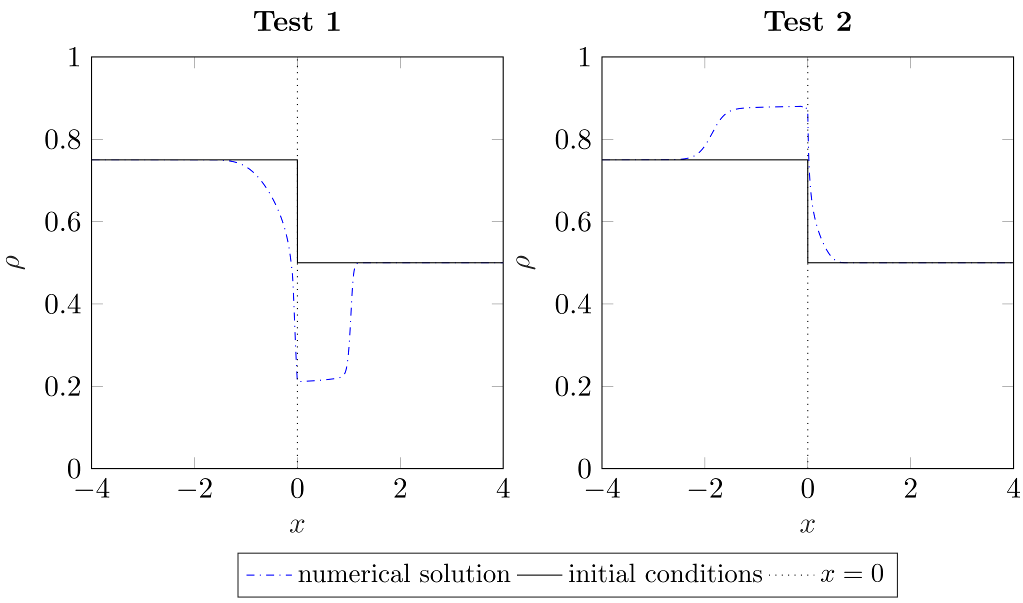

Figure 2: Numerical solutions at $T = 1$ corresponding to Test 1 (left) and Test 2 (right).

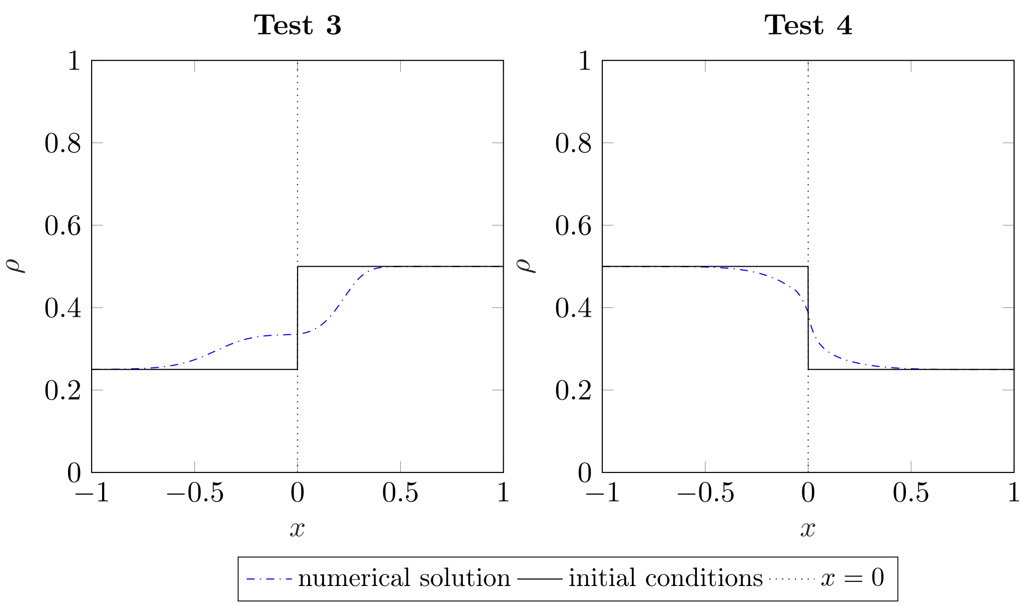

Figure 3: Numerical solutions at $T = 1$ for the Test 3 (left) and Test 4 (right).

6.2 Look-ahead distance η tending to zero

As mentioned above, the behaviour of solutions for η tending to zero is of special interest for non-local conservation laws. Concerning non-local LWR traffic flow models as in [8, 17], or model (2.1) with $v_1 \equiv v_2$, so far the convergence to the classical LWR traffic flow model [27, 28] can only be proven for monotone initial data (see [11, 24]), since the solution is monotonicity preserving and therefore has a strict maximum principle and a bounded total variation, uniformly in η. Unfortunately, similar results do not hold for model (2.1) with $v_1 \neq v_2$, since the model is, in general, not monotonicity preserving even for constant initial data. Therefore, we just investigate the limit numerically.

The local (discontinuous) conservation law corresponding to model (2.1) is given by:

$$\rho_t + f(x,\rho)_x = 0, \quad \text{with } f(x,\rho) := H(-x)\rho v_1(\rho) + H(x)\rho v_2(\rho), \tag{6.2}$$

where $H(x)$ is the Heaviside function. As pointed out in [1, 7], (6.2) admits many L^1 contraction semigroups, one for each so-called (A,B)-*connection*. The two most common connections are the one corresponding to the supply-demand approach [26], and the vanishing viscosity solution (see [22, Definition 3.1]), which is a weak solution satisfying, besides the Kruzkov entropy inequalities for $x < 0$ and $x > 0$, the Γ-condition of [14, 15], see also [22, Definition 3.1] and [4].

For instance, the vanishing viscosity solution can be obtained by a Godunov scheme considering a grid where $x = 0$ is a cell midpoint, see [22].

In the following, we will consider $\eta \in \{0.05, 0.01, 0.005, 0.001\}$ and $\Delta x = 10^{-4}$ to keep the nonlocal impact. We compare it to the solution of (6.2)–(6.1), which will be computed by the Godunov scheme as presented in [22], since we are interested in the vanishing viscosity solution. Note that, due to the different grids, we do not compute L^1-errors between the different solutions.

We will now investigate the previous four test cases. In the first two settings, as $\eta \to 0$ the solution of (2.1) with initial conditions (6.1) is very similar to the vanishing viscosity solution of the corresponding local problem, see Figure 4. We also remark that, in the parameters settings Test 1 and Test 2, the solution obtained by the supply-demand approach is equal to the vanishing viscosity solution.

Let us now consider Tests 3 and 4. The initial datum in both of them is exactly the density corresponding to the maximum fluxes attainable on each road segment. Therefore, the solution of the supply and demand approach is given by a stationary discontinuity coinciding with the initial datum. As can be seen in Figure 5, in both tests the limit of model (2.1) behaves as the vanishing viscosity solution. In Test 4, the numerical results also coincide with supply-demand solution. The most interesting case is Test 3. For these parameters, the vanishing viscosity solution differs from the supply-demand solution and, as can be seen in Figure 5 (left picture) the solution of the model (2.1) seems to converge to the vanishing viscosity solution for η tending to zero.

6.3 Linear network scenario

Finally, we show that the model can be extended to more than two stretches of a road. We consider the case of road works on a highway, modeled by the segment $[0, L]$, with $L = 2$, where the road capacity and the maximal speed are smaller. Therefore, we have three different road segments, $]-\infty, 0[$, $[0, L[$ and $[L, \infty[$, and we consider the linear velocity function as in

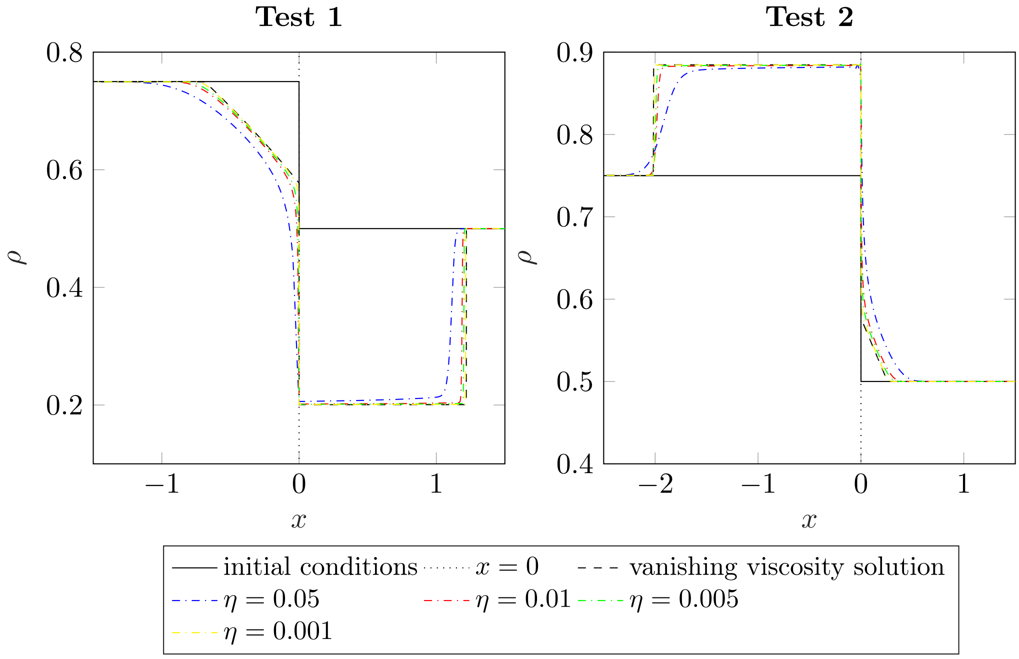

Figure 4: Numerical solutions at $T = 1$ corresponding to Test 1 (left) and Test 2 (right) and different values of η.

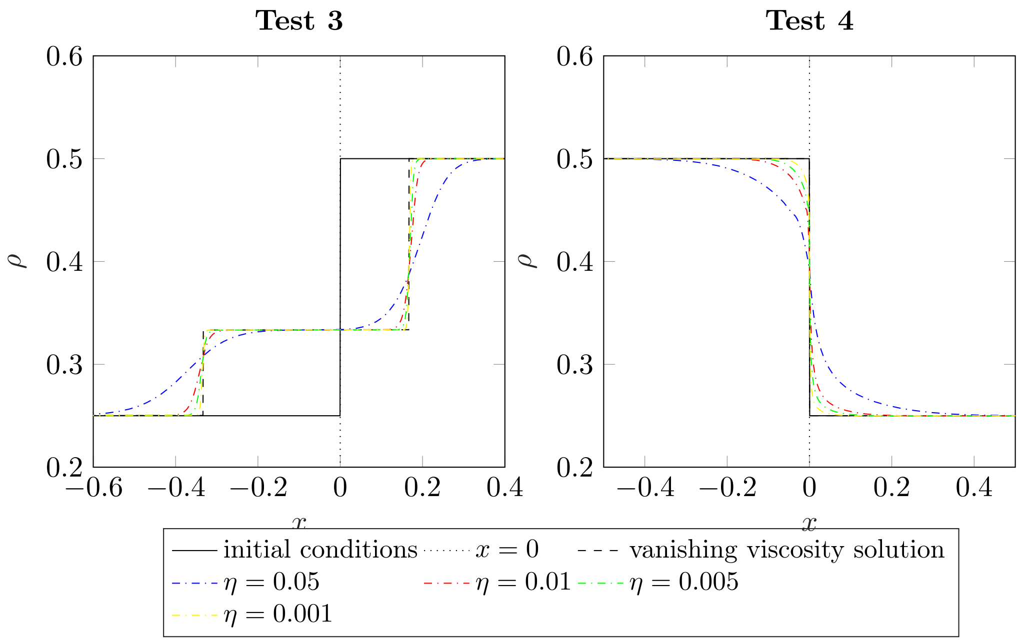

Figure 5: Numerical solutions at $T = 1$ corresponding to Test 3 (left) and Test 4 (right) and different values of η

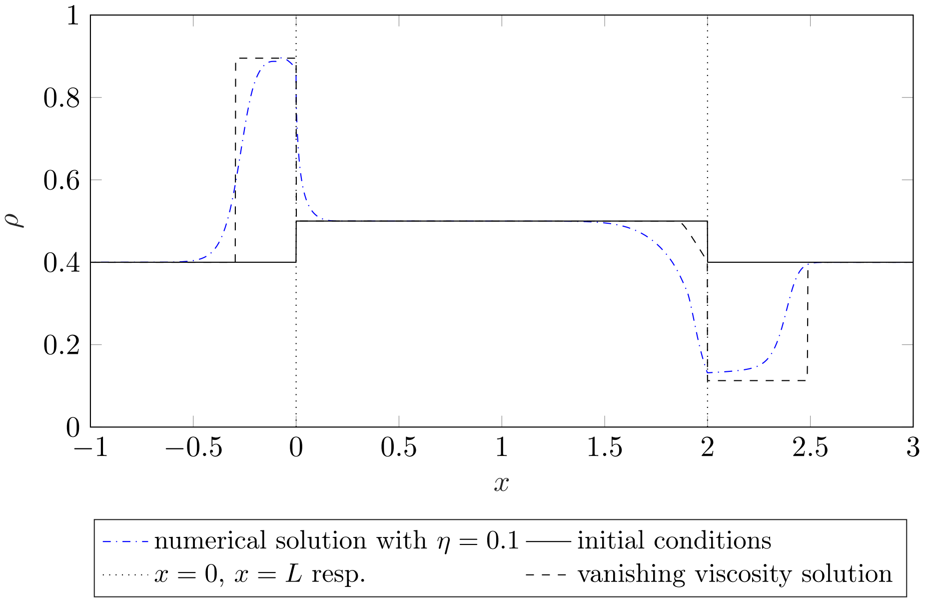

Figure 6: Numerical solution for three road segments at $T = 1$

Test 3, with $v^1_{\max} = v^3_{\max} = \rho^1_{\max} = \rho^3_{\max} = 1$ before and after the road works, and $v^2_{\max} = 0.5$ and $\rho^2_{\max} = 0.8$ for $x \in [0, L]$. We start with a higher density on the segment with the road works, i.e.

$$\rho_0(x) = \begin{cases} 0.4, & \text{if } x < 0, \\ 0.5, & \text{if } 0 < x < L, \\ 0.4, & \text{if } L < x. \end{cases} \tag{6.3}$$

As in Section 6.1, the look ahead distance is $\eta = 0.1$, and as in Section 6.2 we also present the vanishing viscosity solution obtained by the Godunov scheme of [22] to get an impression of the corresponding local problem. As can be seen in Figure 6, the presence of the road works results in a traffic jam upstream and a decrease of the density downstream. As noticed in Section 6.2, the numerical solution of the non-local problem tends for small η towards the vanishing viscosity solution.

7 Conclusion

In this work we have presented a non-local flux model, which can handle different maximum velocities and capacities, i.e. different velocity functions, on the road and therefore models a 1-to-1 junction. The model considers a non-local mean downstream velocity on both road segments and satisfies a maximum principle on each road segment. We have proven its well-posedness, i.e. existence, uniqueness and continuous dependence of solutions with respect to the initial data, via an upwind numerical scheme. Numerical examples suggest that the solution tends to the vanishing viscosity solution of the corresponding local conservation law as the look-ahead distance goes to 0. We intend to further investigate this question in future

work.
In addition, the model can be extended to more than two stretches to model traffic behavior on a more complex road segment, as shown in Section 6.3. Hence, this model can be seen as a first step towards non-local traffic flow models on networks. In the future, we aim to extend this model from the current simple network structure to a more general network formulation.

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