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Necessary and sufficient conditions for the finiteness of the second moment of the measure of level sets

J-M. Azaïs[*] *J. R. León*[†]

Abstract

For a smooth vectorial stationary Gaussian random field $X : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, we give necessary and sufficient conditions to have a finite second moment for the number of roots of $X(t) - u$. The results are obtained by using a method of proof inspired on the one obtained by D. Geman for stationary Gaussian processes long time ago. Afterwards the same method is applied to the number of critical points of a scalar random field and also to the level set of a vectorial process $X : \Omega \times \mathbb{R}^D \to \mathbb{R}^d$ with $D > d$.

Keywords: Level Sets, Kac-Rice formula, Moments, Random fields
2000 *Mathematics Subject Classification*: 60G15, 60G60

1 Introduction

In the 1940s three articles with apparently different orientations appeared in mathematical literature. Firstly was Mark Kac's paper [11] "On the average number of real roots of a random algebraic equation" and secondly two papers written by S.O. Rice [17], [18]"Mathematical analysis of random noise I and II". In the work of Kac and in the second of Rice the zeros of Gaussian random functions were

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studied. In particular they established with precision a formula, known today as the Kac-Rice formula, which allows to compute the expectation of the number of zeros (or crossings by any level) of a Gaussian random function. In spite of the apparently separated that seem the works, M. Kac in the review of the article affirms that "All these results (of Rice) can also be derived using the methods introduced by the reviewer (M. Kac)".

After these two works an intense research activity has been developed. In particular, the interest in these subjects had a great impulse after the appearance of the book written by H. Cramer and M. R. Leadbetter [7]. In this work, there is not only a general demonstration of the Kac-Rice formula for the number of crossings of a Gaussian processes, but also formulas for the factorial moments of this last random variable. An important fact to notice is that in the book a sufficient condition for the second moment of the number of crossings of zero to be finite is established. Then a little time later D. Geman in [10] showed that this condition was also necessary. This condition is now known as " the Geman condition". This result has been extended to any level at [12].

The theme gained a new impulse when appear in the eighties two books, the first one written by R. Adler [1] "The geometry of random fields" and the second one a Lecture Notes [19] written by M. Wschebor "Surfaces aléatoires. Mesure géométrique des ensembles de niveau". Both books focus their study on crossings or geometric invariants of the level sets, for random fields having a multidimensional domain and taking scalar or vector values. The problems studied by Cramer & Leadbetter were extended to this new context. In particular we must point out the Adler & Hasofer's article [2] in which conditions are established so that the number of stationary points for a Gaussian field of $X : \mathbb{R}^2 \to \mathbb{R}$ have a second moment. It is important to observe that studying the stationary points of a scalar field leads to study the zeros of its gradient, which is a vector field.

The twenty-first century saw two books appear [3] and [5] that gave a new impetus to the subject. New fields of application of the formulas appeared in the literature and the area has become a large domain of research. We can point out for instance the applications to the number of roots of random polynomial systems (algebraic or trigonometric) and also to the volume of nodal sets when the systems are rectangular [16]. Also Kac-Rice formulas are a basic tool to study the sets of zeros of random waves and it has been much effort to prove or disprove Berry's conjectures [6], see [16] and the references therein. A field of applications where the formulas have been very useful is in random sea modeling, the Lund's School of probability has been very active in these matters, see for instance the paper [15] and the references therein. In addition, the processes to which the crossings are studied can have their domain in a manifold of finite dimension see [14]. A very interesting case of this last situation is the article [4] where the domain of the

random filed is the sphere in large dimension.

In the present paper we obtain necessary and sufficient conditions to have a finite second moment for the number of roots of $X(t) - u$, for a stationary, mean zero Gaussian field $X : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$. The proofs of the main results are rather simple using the case $d = 1$ as inspiration. Our results can be extended to the number of critical points of a stationary mean zero scalar Gaussian field. We must note that recently in [8] a sufficient condition for the critical points of a scalar field has finite second moment was given, however our method is rather different. Finally let us point us that as a bonus our method of proof allows obtaining a very simple result for the volume of level sets for Gaussian fields $X : \Omega \times \mathbb{R}^D \to \mathbb{R}^d$ with $D > d$. Under condition of stationarity and diferentiability, the second moment is always finite.

Suppose that we have a way to check easily that the measure of the level set of a Gaussian field has finite second moment. Then it is ready to obtain an Itô-Wiener expansion for this functional. Two consequences of this representation are important to remark: firstly the asymptotic variance of the level functional can be computed and also the speed of the divergence of this quantity can be estimated, secondly the fourth moment theorem can be used to obtain diverse CLT. This has been done some time ago in [13] and more recently in a lot of papers. We can cite by instance the article [16] where one can also consult some recent references.

The organisation of the paper is the following: in Section 2 we revisit the results of [12] in dimension 1. Section 3 studies the number of points of levels sets for a random field $X : \mathbb{R}^d \to \mathbb{R}^d$, $d > 1$. The subsection 3.4 is devoted to the study of the number of critical points of a random field $X : \mathbb{R}^d \to \mathbb{R}$. Section 4 studies the measure of levels sets for a random field $X : \mathbb{R}^D \to \mathbb{R}^d$, $D > d$. The proof of the different lemmas are given in the appendix.

2 Real valued process on the line, Geman's condition

The results of this section are contained in the paper [12]. However, we present a new proof as an introduction to the next section.
Consider a process $X : \mathbb{R} \to \mathbb{R}$ and assume

- It is Gaussian stationary, normalized this is:

$$\mathbb{E}(X(0)) = 0; \quad \mathrm{Var}(X(t)) = 1.$$

  This last point is without loss of generality.

- The second spectral moment λ_2 is positive and finite.

Let $N_u([0,T]) := \#\{t \in [0,T] : X(t) = u\}$ for a given level $u \in \mathbb{R}$. Moreover we define the covariance

$$r(\tau) = \mathbb{E}[X(0)X(\tau)].$$

And let set

$$\sigma^2(\tau) := \mathrm{Var}(X'(0)|\, X(0) = X(\tau) = 0) = \lambda_2 - \frac{(r'(\tau))^2}{1 - r^2(\tau)}.$$

In what follows $(Const)$ will denote a generic positive constant, its value can change from one occurence to another.
The relation $x \leq (Const)y, y \leq (Const)x$ is denoted $x \asymp y$.

The object of this section is to prove the next theorem:

Theorem 1. *The following statements are equivalent*

(a) $\mathbb{E}(N_u([0,T])^2)$ *is finite for some* u *and* T.

(b) $\mathbb{E}(N_u([0,T])^2)$ *is finite for all* u *and all finite* T.

(c) The intergral $\int \frac{\sigma^2(\tau)}{\tau} d\tau$ *converges at zero.*

Remark: Integrating by parts in (c) we get the classical Geman's condition by using the following lemma, whose proof (as well as the proofs of all lemmas) is referred to the appendix.

Lemma 2. *There is equivalence between the convergence at zero of the two following integrals*

$$\int \frac{\lambda_2 + r''(\tau)}{\tau} d\tau \quad \text{and} \quad \int \frac{\sigma^2(\tau)}{\tau} d\tau.$$

Before the proof of the theorem we need some notation and two lemmas.

Lemma 3. *For* τ *sufficiently small we set the following definitions and we have the following relations.*

(a) $\mu_{1,\tau,u} := \mathbb{E}(X(\tau)|\, X(0) = X(\tau) = u) = \frac{r'(\tau)u}{1+r(\tau)}$.

(b) $\mu_{2,\tau,u} := \mathbb{E}(X(0)|\, X(0) = X(\tau) = u) = -\mu_{1,\tau,u}$.

(c) Recall that $\sigma^2(\tau) = \mathrm{Var}(X(0)|\, X(0),\, X(\tau)) = \lambda_2 - \frac{(r'(\tau))^2}{1-r^2(\tau)}$.

(d) $\det(\mathrm{Var}(X(0); X(\tau)) = 1 - r^2(\tau) \asymp \tau^2$

(e) *if the fourth spectral moment* λ_4 *satisfies* $\lambda_2^2 < \lambda_4 \leq +\infty$, *then* $\frac{|\mu_{1,\tau,u}|}{\sigma(\tau)} \leq (Const)u$.

Lemma 4. *Assume that* $|m_1|, |m_2| \leq K$ *for some constant* K *and that* $(Y_1; Y_2) \stackrel{\mathcal{L}}{=} N\big((m_1; m_2), \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\big)$. *Then* $\mathbb{E}|Y_1 Y_2| \asymp 1$. *Where the two constants implied in the symbol* $\asymp$ *depend on* K.

Proof of the Theorem. First we have to consider the particular case $\lambda_4 = \lambda_2^2$. This corresponds to the Sine-Cosine process: $X(t) = \xi_1 \sin(wt) + \xi_2 \cos(wt)$ where ξ_1, ξ_2 are independent standard normals. In this case a direct calculation shows that (a)-(c) hold true.

We consider now the other cases assuming that $\lambda_2^2 < \lambda_4$. We start from (c): we assume that

$$\int_0^T \frac{\sigma^2(\tau)}{\tau} d\tau < +\infty \qquad \text{with } T \text{ sufficiently small.}$$

The expectation of the number of crossings is finite because the second spectral moment is, see [7]. Thus it is enough to work with the second factorial moment. The Kac-Rice formula for this quantity [7] writes

$$\begin{aligned} &\mathbb{E}(N_u([0,T])(N_u([0,T]) - 1)) = \\ &\frac{1}{\pi} \int_0^T (T - \tau) \mathbb{E}[|X'(0)||X'(\tau)| \, |X(0) = X(\tau) = u) \frac{e^{-\frac{u^2}{1+r}}}{\sqrt{1 - r^2}} d\tau. \\ &\leq (Const) \int_0^T \mathbb{E}[|\frac{X'(0)}{\sigma(\tau)}||\frac{X'(\tau)}{\sigma(\tau)}| \, \Big| X(0) = X(\tau) = u) \frac{\sigma^2(\tau)}{\tau^2} d\tau, \end{aligned} \tag{1}$$

using Lemma 3 (d). By Lemma 3 (e), $\frac{X'(0)}{\sigma(\tau}$ and $\frac{X'(\tau)}{\sigma(\tau)}$ have a bounded conditional mean, then applying now Lemma 4:

$$\mathbb{E}(N_u([0,T])(N_u([0,T]) - 1) \leq (Const) \int_0^T \frac{\sigma^2(\tau)}{\tau^2} d\tau. \tag{2}$$

This give the finiteness of the second moment form T sufficiently small. By the Minkowsky inequality it is also the case for every T giving (b).

In the other direction we start from (a) with $u = 0$ and T sufficiently small (which is weaker than (b)) and we prove (c) .

Again we can consider the second factorial moment and apply the Kac-Rice formula to get that

$$\mathbb{E}(N_u([0,T])(N_u([0,T]) - 1)$$
$$\geq (Const) \int_0^{T/2} \mathbb{E}\Big(|\frac{X'(0)}{\sigma(\tau)}| |\frac{X'(\tau)}{\sigma(\tau)}| \Big| X(0) = X(\tau) = u \Big) \frac{\sigma^2(\tau)}{\tau^2} d\tau.$$

It suffices to apply Lemma 4 in the other direction. □

Remark 5. *We can also obtain* (2) *with an explicit constant by use of the Cauchy-Schwarz inequality.*

3 Random fields $\mathbb{R}^d \to \mathbb{R}^d$, $d > 1$

3.1 Position of the problem

Let us consider a random field $X : \mathbb{R}^d \to \mathbb{R}^d$. We assume (H_1):

- The field is Gaussian and stationary and has a continuous derivative.
- The distribution of $X(0)$ (respectively $X'(0)$) is non degenerate (N.D.).

By a rescaling in space we can assume without loss of generality that

$$\mathbb{E}[X(t)] = 0 \quad \text{and} \quad \mathrm{Var}(X(t)) = I_d,$$

where Var denotes for us the variance-covariance matrix. We keep the notation Cov for the matrix

$$\mathrm{Cov}(X,Y) := \mathbb{E}\Big(\big(X - \mathbb{E}(X)\big) \big(Y - \mathbb{E}(Y)\big)^\top \Big).$$

We also define the following additional hypothesis

$$\text{The coordinates } X_i \text{ of } X \text{ are independent and isotrope} \qquad (H_2)$$

We define

$$\sigma^2_{i,\lambda}(r) := \mathrm{Var}\big(X'_{i\lambda} \big| X(0), X(\lambda r)\big),$$
$$\sigma^2_{max}(r) := \max_{i=1,\dots,d} \max_{\lambda \in \mathbb{S}^{d-1}} \sigma^2_{i,\lambda}(r),$$

where $X'_{i\lambda}$ denotes the derivative of X_i in the direction $\lambda \in \mathbb{S}^{d-1}$.

3.2 Zero level

We set $N(0,S)$ the number of roots of the field $X(\cdot)$ on some compact set S. The following result is new as well as all that follows.

Theorem 6. *Under* $(H1)$*, if*

$$\int \frac{\sigma^2_{max}(r)}{r} dr \text{ converges at } 0,$$

then for all compact $S \subset \mathbb{R}^d$ *:* $\mathbb{E}\big((N(0,S))^2\big)$ *is finite.*

The proof of the Theorem uses the following lemma.

Lemma 7. *Let* $T, (Z_n)_n$ *be in the same Gaussian space. Assume that* $Z_n \to Z$ *a.s. or in probability or in* $\mathbb{L}^2(\Omega)$ *and the random variable* Z *is (N.D.). Then*

$$\forall z, \mathbb{E}(T|\, Z_n = z) \to \mathbb{E}(T|\, Z = z),$$
$$\mathrm{Var}(T|Z_n) \to \mathrm{Var}(T|Z).$$

Proof of Theorem 6. Set

$$\mathcal{C} = \{X(0) = X(t) = 0\},$$

and let $E_{\mathcal{C}}$ denotes the expectation conditional to $\mathcal{C}$.

We consider the following quantity

$$\mathcal{A}(t,0) = \mathbb{E}_{\mathcal{C}}\big(|\det X'(0) \det X'(t)|\big). \tag{3}$$

By applying the Cauchy-Schwarz inequality and by symmetry of the roles of 0 and t:

$$\mathcal{A}(t,0) \le \mathbb{E}_{\mathcal{C}}\big(\det((X'(0))^\top X'(0))\big),$$

We define the Jacobian the matrix $X'(0)$ by $X'_{ij}(0) = \frac{\partial X_i}{\partial t_j}$.

We perform a change of basis so that $t = re_1 = |t|e_1$ where e_1 is the first vector of the new basis. We denote by $\bar{X}$ the expression of X in this basis. Let $\bar{X}'_{:j}$ denote the jth column of $\bar{X}'$. Using Gram representation of the semidefinite positive matrix $M = (M_{ij})$, we know that

$$\det(M) \le M_{1,1} \dots M_{d,d}. \tag{4}$$

This gives

$$\mathcal{A}(t,0) \le \mathbb{E}_{\mathcal{C}}\big(\|\bar{X}'_{:1}\|^2 \dots \|\bar{X}'_{:d}\|^2\big) = \sum_{1 \le i_1, \dots, i_d \le d} \mathbb{E}_{\mathcal{C}}((X'_{i_1,1})^2 \dots (X'_{i_d,d})^2). \tag{5}$$

Because the conditional expectation is contractive, for $j > 1$,

$$\mathbb{E}_{\mathcal{C}}\big((\bar{X}'_{i_j,j})^2\big) \leq \mathbb{E}\big((\bar{X}'_{i_j,j})^2\big) \leq (Const). \tag{6}$$

In addition

$$\mathbb{E}_{\mathcal{C}}\big((\bar{X}'_{i_1,1})^2\big) \leq \sigma^2_{max}(r). \tag{7}$$

If we consider a term of (5), we can apply Cauchy-Schwarz inequality to get that it is bounded by

$$\big(E_{\mathcal{C}}((X'_{i_1,1})^4\big)^{1/2}\big(E_{\mathcal{C}}((X'_{i_2,2})^4 \dots (X'_{i_d,d})^4)\big)^{1/2}.$$

Using (6) and (7), we get that this term is bounded by

$$(Const)\sigma^2_{max}(r).$$

As a consequence we get the same bound for the whole sum.

We now study the joint density

$$p_{X(0),X(t)}(0,0) = (Const)\big(\det \mathrm{Var}(X(0), X(t))\big)^{-\frac{1}{2}}.$$

Using the fact that a determinant is invariant by adding to some row (or column) a linear combination of the others rows (or columns) we get

$$\det(\mathrm{Var}(X(0), X(t))) = \det(\mathrm{Var}(X(0), X(t) - X(0))).$$

Using Lemma 7.

$$p_{X(0),X(t)}(0,0) \simeq (Const)r^{-d}(\det \mathrm{Var}(X(0), X'_\lambda(0)))^{-\frac{1}{2}} \asymp r^{-d}, \tag{8}$$

where $\lambda := t/\|t\|$.

We are now able to apply the Kac-Rice formula see, for example, [5], Theorem 6.3. As in the case $d = 1$ we can limit our attention to the second factorial moment. We have

$$\begin{aligned}
&\mathbb{E}(N(0,S)(N(0,S)-1) \\
&= \int_{S^2} \mathcal{A}(t-s,0)P_{X(s),X(t)}(0,0)dsdt \leq (Const)|S| \int_S \sigma^2_{max}(t)\|t\|^{-d}dt,
\end{aligned}$$

where $|S|$ is the Lebesgue measure of S. Passing in polar coordinates, including S in a centered ball with radius a, we get that the term above is bounded by

$$(Const)\int_0^a r^{d-1}r^{-d}\sigma^2_{max}(r)dr = (Const)\int_0^a \frac{\sigma^2_{max}(r)}{r}dr.$$

□

3.3 General level

In this section we assume (H_1) and (H_2). Note that $\sigma^2_{i,\lambda}(r)$ depends no more on λ. We denote by $\sigma^2_i(r)$ its value. We have $\sigma^2_{max}(r) = \max_{i=1,\dots,d} \sigma^2_i(r)$.

Our result is the following

Theorem 8. *Under the hypotheses above*
• *If*

$$\int \frac{\sigma^2_{max}(r)}{r} dr \text{ converges at } 0,$$

then for all compact $S \subset \mathbb{R}^d$ an all $u \in \mathbb{R}^d$: $\mathbb{E}\big((N(u,S))^2\big)$ is finite.

• *If $\mathbb{E}\big((N(u,S))^2\big)$ is finite for some u and some compact S with non-empty interior, then*

$$\int \frac{\sigma^2_{max}(r)}{r} dr \text{ converges at } 0.$$

Because of stationarity and isotropy we have

$$\mathrm{Cov}(X_i(s), X_i(t)) = \rho_i(\|s-t\|^2),$$

where ρ_i is some function of class $C^2(\mathbb{R})$.

Before the proof of the Theorem we state the following lemmas.

Lemma 9. *Let $\mathcal{F}$ be a family of Gaussian distributions for X, Y two $d \times d$ Gaussian matrices. Let Z be the $2d^2$ vector obtained by the elements of X, Y in any order.*

(a) Suppose that for all distribution in $\mathcal{F}$, $\mathbb{E}(Z) \in K_1$ and $\mathrm{Var}(Z) \in K_2$ where K_1, K_2 are two compacts sets .

Then there exists C such that :

$$\sup_{f \in \mathcal{F}} \mathbb{E}_f(|\det(X)\det(Y)|) \le C.$$

The constant C depends only on K_1, K_2 and d.

(b) Suppose in addition that for every $f \in \mathcal{F}$,

$$\mathbb{P}\{\det(X) = 0\} = 0, \mathbb{P}\{\det(Y) = 0\} = 0$$

then there exists c such that :

$$\mathbb{E}(|\det(X)\det(Y)|) \ge c.$$

The positive constant c depends only on K_1, K_2 and d.

For establishing the next lemma let introduce the following definitions

$$\sigma_i^2(r) = -2\rho_i'(0) - \frac{4r^2(\rho_i'(r^2))^2}{1-\rho_i^2(r^2)},$$

$$b_i(r)\sigma_i(r) = \big(-2\rho_i'(r^2) - 4r^2\rho_i''(r^2) - \frac{4r^2\rho_i(r^2)(\rho_i'(r^2))^2}{1-\rho_i^2(r^2)}\big).$$

Then we have the following, denoting $\mathrm{Var}_{\mathcal{C}}$ the variance-covariance matrix conditional to $\mathcal{C}$.

Lemma 10. *[see [5] p. 336]*

$$\mathrm{Var}_{\mathcal{C}}\big(X_i'(0); X_i'(re_1)\big) = \begin{bmatrix} \sigma_i(r) & 0 & \dots & 0 & b_i(r)\sigma_i(r) & 0 & \dots & 0 \\ 0 & -2\rho_i'(0) & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -2\rho_i'(0) & 0 & 0 & \dots & 0 \\ b_i(r)\sigma_i(r) & 0 & \dots & 0 & \sigma_i(r) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -2\rho_i'(0) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & -2\rho_i'(0) \end{bmatrix}.$$

Proof of the Theorem 8. We begin by considering instead of (9) the quantity

$$\mathcal{A}(t,u) = \mathbb{E}_{\mathcal{C}}\big(|\det X'(0) \det X'(t)|\big), \tag{9}$$

where now

$$\mathcal{C} = \{X(0) = X(t) = u\}.$$

Because of isotropy we can assume, without loss of generality, that $t = re_1$. Because of the independence of each coordinates assumed in (H_2)

$$\mathbb{E}_{\mathcal{C}}(X_{i,1}'(0)) = \mathbb{E}(X_{i,1}'(0)\big|\, X_i(0) = X_i(re_1) = u_i).$$

So we have to consider a one dimensional problem as in Section 2. In addition the spectral measure of each X_i is invariant by isometry so its projection on the first axis cannot be reduced to one point (or two taking into account symmetry). As a consequence Lemma 3 (e) holds true implying that

$$|\mathbb{E}_{\mathcal{C}}(X_{i,1}'(0)| \le (Const) u_i \sigma_i(r).$$

Let us consider now $E_{\mathcal{C}}(X_{i,j}') = \mathbb{E}\big(X_{i,j}'\big|\, X_i(0) = u_i, \frac{X_i(t)-X_i(0)}{r} = 0\big)$ for $j \neq 1$. From Lemma 7

$$E_{\mathcal{C}}(X_{i,j}') \simeq \mathbb{E}\big(X_{i,j}'\big|\, X_i(0) = u_i, X_{i,1}'(0) = 0\big).$$

By independence

$$\mathbb{E}_{\mathcal{C}}(X_{i,j}') \simeq \mathbb{E}\big(X_{i,j}'\big|\, X_{i,1}'(0) = 0\big) = 0.$$

Of course we have the same kind of result for $X'(re_1)$.

So, if we divide the first column of $X'(0)$ and $X'(re_1)$ by $\sigma_{max}(r)$ to obtain $\tilde{X}'(0)$ and $\tilde{X}'(re_1)$, Lemma 10 implies that all the terms of the variance-covariance matrix are bounded, the expectation is bounded. Using lemma 9 we get that

$$\mathcal{A}(t,u) \leq (Const)\sigma^2_{max}(r). \tag{10}$$

The end of the proof of the first assertion is similar to the one of Theorem 6.

We turn now to the proof of the second assertion. To get the inequality on the other direction we must carefully apply (b) of Lemma 9. For this purpose we need to describe the compact sets K_1 and K_2.

We know, from the proof of the first assertion, that all the expectations are bounded so K_1 is just $[-a,a]^{2d^2\times 2d^2}$ for some a. For the domain K_2 of the variance-covariance matrix of $\tilde{X}'(0), \tilde{X}'(t)$:

- First we have an independence between the coordinates X_i. Denoting by $X'_{i,:}(t)$ the i row of X' i.e. the gradient of X_i at t, then the variables $\big(X'_{i,:}(0), X'_{i,:}(t)\big), i = 1,\dots,d$ are independent.

- If we consider $\big(X'_{i,:}(0), X'_{i,:}(t)\big)$ for some fixed i, we see from lemma 10 that (i) only one variance varies : $\sigma_i(r)$, (ii) the only non-zero covariance is between $X'_{i,1}(0)$ and $X'_{i,1}(t)$.

- After dividing $X'_{i,1}(0)$ and $X'_{i,1}(t)$ by $\sigma_{max}(r)$ to obtain $\tilde{X}'_{i,1}(0), \tilde{X}'_{i,1}(t)$ the variance becomes
$$\tilde{\sigma}_i(r) := \frac{\sigma_i(r)}{\sigma_{\max}(r)}.$$
The domain for these variance, when i varies is
$$K'_2 := \{\tilde{\sigma}(r) \in \mathbb{R}^d \,:\, 0 \leq \tilde{\sigma}_i(r) \leq 1, \text{ at least one } \tilde{\sigma}_i(r) = 1\}.$$
That is a compact set.

- The domain for the covariance between $\tilde{X}'_{i,1}(0)$ and $\tilde{X}'_{i,1}(t)$ is given by Cauchy-Schwarz inequality :
$$\mathrm{Cov}\big(\tilde{X}'_{i,1}(0), \tilde{X}'_{i,1}(t)\big) \leq \tilde{\sigma}_i(r),$$
that defines another compact set.

- The other variables are independent between them and independent of the variables above. Their variance are fixed.

It remains to prove that for every element of K_1 and K_2 the Gaussian distribution satisfies

$$\mathbb{P}\{\det(X) = 0\} = 0 \text{ and } \mathbb{P}\{\det(Y) = 0\} = 0,$$

where X, Y is a representation of the conditional distribution of $\tilde{X}'(0), \tilde{X}'(t)$. It is sufficient to study the case of $\det(X)$. Recall that we have proved above that all the coordinates of X are independent. The only difficulty is that the variance of the first column may vanish.

Let us consider the $d \times (d-1)$ matrix $X'_{:,-1}$ that consists of columns $2, \ldots, d$ of X. Because all the entries of $X_{:,-1}$ are independent they span a subspace of dimension $d-1$ a.s.

Then the rank of $X'_{:,-1}$ is almost surely $(d-1)$ or $\mathrm{Im}(X'_{:,-1})$ is a $(d-1)$ space. The distribution of the random matrix $Y := X'_{:,-1}$ has a density that can be written

$$f_Y[dY] = (Const)e^{-\frac{1}{2}\mathrm{Trace}(Y^\top \Sigma^{-1} Y)}[dY],$$

where as before we have denoted Σ the covariance matrix of each column vector, which is diagonal. This density function is translated on the Grassmannian giving a bounded density with respect to the Haar measure.

Recall that we have proved above that all the coordinates of X are independent. Let $X_{:,1}$ be the first column of X. Conditioning on $X_{:,-1}$, by independence, the distribution of $X_{:,1}$ remains unchanged. A representation for this random variable is

$$X_{:,1} = \mu + \xi, \text{ with } \mu = \mathbb{E}(X_{:,1}),$$

where ξ (because some $\tilde{\sigma}_i$ can vanish) has an absolute continuous distribution on the space

$$S_I = (\xi_i = 0, \text{ for } i \in I), \text{ with } I = \{i : \tilde{\sigma}_i = 0\}.$$

But since at least one $\tilde{\sigma}_i = 1$, we have $S_I \neq \{1, \ldots, d\}$.

Because of its absolute continuity, almost surely, ξ can not be included in a given subspace E that does not contain S_I. In conclusion, given its absolutely continuous distribution over the Grassmannian, with probability one, $\mathrm{Im}(X_{:,-1})$ cannot contain any fixed affine space.

As a consequence we can apply Lemma 9 (b) for getting the inequality in the other direction.

It remains to give a lower bound to the density.

$$p_{X(0),X(t)}(u,u) = \prod_{i=1}^{d} \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho_i^2(r^2)}} e^{-\frac{u^2}{1-\rho_i(r^2)}}.$$

Since $\rho_i(r^2) \to 1$ as $r \to 0$ the term $e^{-\frac{u^2}{1-\rho_i(r^2)}}$ is lower bounded. Then is suffices to use (8).

□

3.4 Critical points

Let $Y : Y(t)$ be a random field from $\mathbb{R}^d \to \mathbb{R}$. Critical points points of Y are in fact zeros of $X = X'(t)$. Strictly speaking this process does not satisfies the hypotheses of Theorem 11 because $X'(t)$ is the Hessian of $Y(t)$ so it is symmetric and its distribution in $\mathbb{R}^{d^2}$ is not N.D. But the result holds true with a very similar proof mutatis mutandis.

Theorem 11. *Suppose that*

- *Y is Gaussian stationary, centred and has C^2 sample paths.*
- *$Y'(t)$ is N.D., $Y''(t)$ has a non degenerated distribution in the space of symmetric matrices of dimension $d \times d$.*

Define

$$\bar{S}^2_{max}(r) := \max_{i=1,\dots,d} \max_{\lambda \in \mathbb{S}^{d-1}} \mathrm{Var}\big(Y''_{i\lambda} \big| \, Y'(0), Y''(\lambda r)\big),$$

if

$$\int \frac{\bar{S}^2_{max}(r)}{r} dr \text{ converges at } 0,$$

then for all compact $S \subset \mathbb{R}^d$: the second moment of the number of critical points of Y included in S is finite.

4 Random fields from $\mathbb{R}^D$ to $\mathbb{R}^d$, $d < D$

In this section we study the level sets of a random field $\mathbb{R}^D$ to $\mathbb{R}^d$. Of course the case $d > D$ has no interest because almost surely the level set is empty. The case $d = D$ has been considered in the preceding sections so we assume $d < D$. The result presented here is, in some sense, a by-product of Theorem 8, but by its simplicity it is the most surprising result and one of the main results of this paper.

Theorem 12. *Let $X : X(t)$ a stationary random field $\mathbb{R}^D$ to $\mathbb{R}^d$, $d < D$ with C^1 paths. By the implicit function theorem, a.s. for every u the level set C_u is a manifold and its $D-d$ dimensional measure σ_{D-d} is well defined. Let $C(u, K)$ be the restriction of C_u to a compact set $K \subset \mathbb{R}^D$. Assume that the distributions of*

$X(t)$ and $X'(t)$ are N.D.
Then for every u and K

$$\mathbb{E}\big(\sigma^2_{D-d} C(u,K)\big) < +\infty. \tag{11}$$

Proof. The Kac-Rice formula reads

$$\begin{aligned}\mathbb{E}\big(&\sigma^2_{D-d} C(u,K)\big)\\ &= \int_{K^2} \mathbb{E}_{\mathcal{C}}\Big(\big(\det(X'(s)X'(s)^\top)\det(X'(t)X'(t)^\top)\big)^{\frac{1}{2}}|\,|\Big)\\ &\qquad\qquad p_{X(s),X(t)}(u,u)dsdt,\end{aligned}$$

where again $\mathbb{E}_{\mathcal{C}}$ denotes the expectation conditional to $\mathcal{C} = \{X(0) = X(t) = u\}$. Using the arguments in the proof of Theorem 8, we have

$$p_{X(0),X(t)}(u,u) \le (Const)\|t\|^{-d}.$$

By Cauchy-Schwarz inequality and symmetry

$$\begin{aligned}A(t,u) &:= \mathbb{E}_{\mathcal{C}}\left(\big(\det(X'(0)X'(0)^\top)\det(X'(t)X'(t)^\top)\big)^{\frac{1}{2}}|\right)\\ &\le \mathbb{E}_{\mathcal{C}}\left(\big(\det(X'(0)X'(0)^\top)|\big)\right).\end{aligned}$$

Using (4) we have to bound

$$\mathbb{E}_{\mathcal{C}} \prod_{i=1}^{d} \|\nabla X_i(0)\|^2.$$

Now we borrow results from the proof of Theorem 8, to get that for every i:

$$\begin{aligned}&\mathbb{E}_{\mathcal{C}}(X'_{i,1}(0)) \to 0\\ &\mathbb{E}_{\mathcal{C}}(X'_{i,j}(0)) \text{ is bounded} \quad j \ne 1,\end{aligned}$$

Because of the contracting property of the conditional expectation $\mathrm{Var}_{\mathcal{C}}(X'_{i,j}(0))$ is bounded. So, it follows directly that $\mathcal{A}(t,u)$ is upper-bounded. The integrability of t^{-d} in $\mathbb{R}^D$ gives the result. □

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5 Appendix

Proof of lemma 2. Let us consider the integral

$$\int_0^\delta \frac{\sigma^2(\tau)}{\tau} d\tau.$$

For τ small enough

$$\frac{\sigma^2(\tau)}{\tau} \sim (\frac{1}{\lambda_2})^{\frac{3}{2}} \frac{\lambda_2(1-r^2(\tau)) - (r'(\tau))^2}{\tau^3}.$$

Then integrating by parts

$$\int_0^\delta \frac{\lambda_2(1-r^2(\tau)) - (r'(\tau))^2}{\tau^3} d\tau$$

$$= \frac{\lambda_2(1-r^2(\delta)) - (r'(\delta))^2}{2\delta^2} + \int_0^\delta \frac{r'(\tau)}{\tau}(\frac{-\lambda_2 r(\tau) - r''(\tau)}{\tau}) d\tau.$$

Hence we need to consider the second term that is equal to

$$-\lambda_2 \int_0^\delta \frac{r'(\tau)}{\tau}(\frac{r(\tau)-1}{\tau}) d\tau - \int_0^\delta \frac{r'(\tau)}{\tau}(\frac{r''(\tau)+\lambda_2}{\tau}) d\tau,$$

as the first term is evidently convergent, the above sum is convergent if and only if

$$\int_0^\delta \frac{r''(\tau)+\lambda_2}{\tau} d\tau < \infty.$$

□

Proof of lemma 3. For short we will write r, r', r'' instead of $r(\tau), r'(\tau), r''(\tau)$. Items $(a) - (d)$ are easy consequences of regression formulas (see [5] page 100 for example).

To prove (e) we study first the behavior of $\frac{r'}{\sigma(\tau)}$ near to zero. We need consider two cases.

The first one is when the fourth spectral moment λ_4 is finite: we have $r(t) = 1 - \lambda_2 t^2/2 + \lambda_4 t^4/(4!) + o(t^4)$. By using a Taylor expansion of fourth order on the numerator and the denominator of the fraction $(\frac{r'u}{(1+r)\sigma(\tau)})^2$, we obtain

$$(\frac{r'u}{(1+r)\sigma(\tau)})^2 \to \frac{\lambda_2^2 u^2}{\lambda_4 - \lambda_2^2} \leq (Const)\, u^2, \quad \text{giving } (e).$$

Consider now the second case : $\lambda_4 = +\infty$.
Given that $r''(\tau) - r''(0) = \int_0^\infty \frac{1-\cos(\tau\lambda)}{\tau^2\lambda^2/2} d\mu(\lambda)$, we have by Fatou's lemma

$$\begin{aligned}\liminf_{\tau\to 0} \frac{r''(\tau) - r''(0))}{\tau^2} &\geq \int_0^{+\infty} \liminf_{\tau\to 0} \frac{1-\cos(\tau\lambda)}{\tau^2\lambda^2/2} \lambda^4 d\mu(\lambda) \\ &= \int_0^\infty \lambda^4 d\mu(\lambda) = +\infty. \end{aligned} \tag{12}$$

Since $1 + r$ tends to 2, it suffices to study the behaviour of

$$\frac{r'^2}{\sigma^2(\tau)} \simeq \frac{\lambda_2^3}{\frac{\lambda_2(1-r^2)-r'^2}{\tau^4}},$$

Note that $\lambda_2(1 - r^2(\tau)) - r'^2(\tau) = 2\,\lambda_2(1 - r(\tau)) - r'^2(\tau) + O(\tau^4)$. Furthermore by using the l'Hospital rule

$$\lim_{\tau\to 0} \frac{2\,\lambda_2(1 - r(\tau)) - r'^2(\tau)}{\tau^4} = \lim_{\tau\to 0} \left(\frac{-r'(\tau)}{2\tau}\right) \left(\frac{r"(\tau) - r"(0))}{\tau^2}\right) = +\infty,$$

because of (12) and since we know that $\frac{-r'(\tau)}{2\tau} \to \frac{\lambda_2}{2}$. Thus $\frac{r'(\tau)}{\sigma(\tau)} \to 0$. These two results imply that *(e)* holds. □

Proof of the lemma 4. We can write

$$Y_2 - m_2 = \rho(Y_1 - m_1) + \sqrt{1-\rho^2} Z_1,$$

where Z_1 is a standard Gaussian independent of Y_1. Thus

$$\begin{aligned} Y_1Y_2 &= (m_1m_2 + \rho) + (m_2 + \rho m_1)(Y_1 - m_1) \\ &\quad + m_1\sqrt{1-\rho^2} Z_1 + \rho((Y_1 - m_1)^2 - 1) + \sqrt{1-\rho^2}(Y_1 - m_1)Z_1. \end{aligned}$$

This formula yields that $\mathbb{E}|Y_1Y_2|$ is a continuous function of (m_1, m_2, ρ) and by compactness is upper bounded.

In the other direction, setting $Y_i = m_i + \bar{Y}_i$, $i = 1, 2$ we have

$$Y_1Y_2 = [m_1m_2] + [m_1\bar{Y}_2 + m_2\bar{Y}_1] + [\bar{Y}_1\bar{Y}_2],$$

Where the three brackets are in different Itô-Wiener chaos and thus the joint variance is the sum of the variance of each term. This implies that

$$\mathrm{Var}(Y_1Y_2) \geq \mathrm{Var}(\bar{Y}_1\bar{Y}_2) = \mathbb{E}(\bar{Y}_1^2\bar{Y}_2^2) = \mathbb{E}\Big(\big(H_2(\bar{Y}_1) + 1\big)\big(H_2(\bar{Y}_2) + 1\big)\Big) = 1 + 2\rho^2,$$

the function $H_2(y) = y^2 - 1$ is the second Hermite polynomial and also we have used the Mehler formula: $\mathbb{E}(H_i(\bar{Y}_1)H_j(\bar{Y}_2) = \delta_{ij}\rho^i i!$. Note that $H_0(y) = 1$.

As a consequence

$$\mathrm{Var}(Y_1Y_2) \geq 1.$$

and Y_1Y_2 is never a.s. constant. Thus $\mathbb{E}|Y_1Y_2| > 0$ and by compactness is lower bounded. □

Proof of lemma 7. Note that for n large enough, the distribution of Z_n is N.D. As a consequence

$$\begin{aligned}\mathbb{E}(T|\, Z_n = z) &= \mathrm{Cov}(T, Z_n)(\mathrm{Var}(Z_n))^{-1}z \\ \mathrm{Var}(T|Z_n) &= \mathrm{Var}(T) - \mathrm{Cov}(T, Z_n)(\mathrm{Var}(Z_n))^{-1}\mathrm{Cov}(Z_n, T)\end{aligned}$$

but

$$\mathrm{Var}(Z_n) \to \mathrm{Var}(Z),\ \text{N.D.}$$

This implies $(\mathrm{Var}(Z_n))^{-1} \to (\mathrm{Var}(Z))^{-1})$. The rest is plain. □

Proof of lemma 9. Let Σ be the variance-covariance matrix of Z and let μ be its expectation. Both vary in a compact sets K_1, K_2. Let $\Sigma^{\frac{1}{2}}$ be the square root of Σ defined in the spectral way. Using operator norm [9], it is easy to prove that $\Sigma^{\frac{1}{2}}$ is a (uniformly) continuous function of Σ. The random vector Z admits the following représentation

$$Z = \mu + \Sigma^{\frac{1}{2}}\xi \quad \xi \sim N(0, I_{2d^2}).$$

The function $\det(X)\det(Y)$, as a polynomial, is a continuous function of Z and by consequence $\mathbb{E}\big(|\det(X)\det(Y)|\big)$ is a continuous function of μ, Σ. The first assertion follows by compactness.

In the other direction we have by additivity

$$\mathbb{P}\{\det(X) = 0\} = 0, \quad \mathbb{P}\{\det(Y) = 0\} = 0.$$

This implies that

$$\mathbb{E}(|\det(X)\det(Y)|) > 0.$$

Again the second inequality is obtained by compactness. □