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SEMICLASSICAL BEHAVIOUR OF QUANTUM EIGENSTATES

GABRIEL RIVIÈRE

ABSTRACT. Given a quantum Hamiltonian, we explain how the dynamical properties of the underlying classical system affect the behaviour of quantum eigenstates in the semiclassical limit. We study this problem via the notion of semiclassical measures. We mostly focus on two opposite dynamical paradigms: completely integrable systems and chaotic ones. We recall standard tools from microlocal analysis and from dynamical systems. We show how to use them in order to illustrate the classical-quantum correspondance and to compare properties of completely integrable and chaotic systems.

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1. INTRODUCTION

The semiclassical regime of quantum mechanics is a regime where the Planck constant is negligible compared with the other physical actions of the system. In this limit, the physical system under consideration is governed by the equations of classical mechanics and one expects that the nature (e.g. chaotic versus integrable) of the corresponding classical system is reflected in the behaviour of the quantum system. From the mathematical point of view, this type of asymptotics is at the heart of microlocal analysis which has a wide range of applications: spectral theory, partial differential equations, symplectic topology, hyperbolic dynamics, random geometry, etc. This course will be mostly devoted to the application of microlocal analysis to the study of quantum eigenstates in the semiclassical limit which can also be viewed as a question of spectral geometry. Our main emphasis will be on the use of tools from semiclassical analysis in order to study the impact of classical dynamics on this problem. These lectures are complementary to the ones of Frédéric Faure which are devoted to the application of semiclassical methods to classical chaotic dynamics [54].

The plan of these lectures is as follows. In section 2, we collect some standard facts from dynamical systems and semiclassical analysis that will be used in the rest of these notes. The expert can easily skip (or read quickly) this part and we hope that the motivated non-expert will find enough material to understand what follows. In this section, we also point several classical books that can be used to find some complementary material and also the proofs of the results described in this part. In section 3, we discuss the question of observability and relate it to the quantum ergodicity property. The last three parts focus on three important examples: Zoll manifolds where the geodesic flow is periodic (section 4), flat tori which are the simplest examples of nondegenerate completely integrable systems (section 5) and negatively curved manifolds where the geodesic flow is strongly chaotic (section 6). For these three models, we try to give a review of what is currently known on the semiclassical behaviour of quantum eigenstates with a strong emphasis on the properties derived from semiclassical methods.

To end this introduction, we mention that several important and interesting aspects on the study of semiclassical quantum eigenstates will not be described in these lecture notes. In fact, we chose to focus on their description via the notion of semiclassical measures which is natural from the point of view of quantum mechanics. We think that this point of view already illustrates well the interplay between classical and quantum mechanics which is at the heart of this school. Among the subjects that we will not (or only very briefly)

discuss are the asymptotic distribution of eigenvalues, the relation with analytic number theory, the growth of L^p norms, the geometry of nodal sets, random waves models, graph models, etc.

2. BACKGROUND MATERIAL

In this preliminary section, we define Laplace eigenfunctions and discuss a couple of important examples. We also describe relevant tools from symplectic geometry and semiclassical analysis that we apply to define the so-called semiclassical measures and to derive some of their first properties. Details on this background material can for instance be found in Zworski's book on semiclassical analysis [145].

2.1. Laplace-Beltrami operators. In all these lectures, (M, g) denotes a compact, oriented, connected, smooth and Riemannian manifold which has no boundary and dimension $n \geq 2$. The Laplace-Beltrami operator is then defined as the unique operator Δ_g verifying

$$\forall \psi_1, \psi_2 \in C^\infty(M), \quad - \int_M \psi_1 \Delta_g \psi_2 d\text{Vol}_g = \int_M \langle d\psi_1, d\psi_2 \rangle_{g^*} d\text{Vol}_g,$$

where d is the usual differential, g^* is the induced metric on T^*M and Vol_g is the Riemannian volume on M . In local coordinates (x_1, \dots, x_n) , this operator reads

$$\Delta_g \psi = \frac{1}{\sqrt{|\det g|}} \sum_{i,j} \partial_{x_i} \left(g^{ij} \sqrt{|\det g|} \partial_{x_j} \psi \right).$$

In the following, we will focus our attention on solutions to the Laplace equation, i.e.

$$(1) \quad -\Delta_g \psi_\lambda = \lambda^2 \psi_\lambda, \quad \|\psi_\lambda\|_{L^2} = 1,$$

where $\lambda \geq 0$. Such solutions are called (normalized) Laplace eigenfunctions and they correspond to stationary solutions $u(t, x) = e^{-\frac{it\lambda^2}{2}} \psi_\lambda(x)$ of the Schrödinger equation

$$i\partial_t u = -\frac{1}{2} \Delta_g u.$$

The eigenvalues λ^2 represent the energy of the quantum eigenstate ψ_λ [56, §2.5]. Solutions to (1) are smooth by elliptic regularity [120, Prop. 2.2.4]. Following §3.1 from the same reference, one also knows that there exists a nondecreasing sequence

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow +\infty,$$

and an orthonormal basis $(\psi_{\lambda_j})_{j \geq 0}$ of $L^2(M)$ such that

$$\forall j \geq 0, \quad -\Delta_g \psi_{\lambda_j} = \lambda_j^2 \psi_{\lambda_j}.$$

The Hörmander-Weyl's law [120, Th. 3.3.1] also tells us:

$$(2) \quad N(\lambda) := |\{j \in \mathbb{Z}_+ : \lambda_j \leq \lambda\}| = \frac{\pi^{\frac{n}{2}}}{(2\pi)^n \Gamma(\frac{n}{2} + 1)} \text{Vol}_g(M) \lambda^n + \mathcal{O}(\lambda^{n-1}), \quad \text{as } \lambda \rightarrow +\infty.$$

The purpose of these notes is to give some insights on the properties of Laplace eigenfunctions in the high energy limit $\lambda \rightarrow +\infty$. This means that we are looking at quantum

eigenstates which are more and more “excited”. From the correspondence principle of quantum mechanics [56, §7.4], one expects that the properties of these eigenmodes are related to the properties of the underlying classical Hamiltonian, here the geodesic flow (see below). This regime is called the *semiclassical regime* of quantum mechanics.

As we shall see, except for certain specific examples¹, there is no explicit description of these eigenfunctions. Yet, we would like to measure if these quantum eigenstates can develop singularities in the large eigenvalue limit and to understand how this is related to the properties of the corresponding classical Hamiltonian. There are several natural ways to attack this question and let us list three of them:

- Given $\beta \in \mathbb{C}$, one can try to describe the level sets of these solutions:

$$\mathcal{Z}_{\psi_\lambda}(\beta) := \{x \in M : \psi_\lambda(x) = \beta\}.$$

This is related to the question of nodal sets that we will not discuss very precisely in these notes (except in section 7). The interested reader can have a look at the survey by Zelditch [142].

- Given $p \geq 2$, one can try to bound $\|\psi_\lambda\|_{L^p(M)}$ by a function depending only on λ and on (M, g) . In the case $p = +\infty$, it follows from the localized version of (2) that $\|\psi_\lambda\|_{L^\infty(M)} \leq C_{M,g}(1 + \lambda)^{\frac{n-1}{2}}$ – see Eq. (3.3.2) in [120]. This is a typical question in harmonic analysis which is related to the so-called Strichartz estimates for dispersive PDE. Again, we shall only describe very briefly this approach and, for a description of recent results in this direction, we refer to [121].
- Finally, one can try to study the probability measures $d\nu_\lambda(x) = |\psi_\lambda(x)|^2 d\text{Vol}_g(x)$. From the point of view of quantum mechanics, given a subset ω of M ,

$$\nu_\lambda(\omega) = \int_\omega |\psi_\lambda(x)|^2 d\text{Vol}_g(x)$$

represents the probability of finding a quantum particle in the state ψ_λ inside the set ω .

We will mostly focus our attention on the last problem and let us describe some natural questions that can be raised in this direction. First, given any solution ψ_λ to (1), it follows from [12, 44] that, for any *open* set ω ,

$$(3) \quad \omega \neq \emptyset \quad \implies \quad \nu_\lambda(\omega) > 0.$$

It is then natural to look for conditions on (M, g) and ω under which one can find $c_{\omega, M, g} > 0$ such that

$$(4) \quad \forall \psi_\lambda \text{ solution to (1),} \quad 0 < c_{\omega, M, g} \leq \int_\omega |\psi_\lambda(x)|^2 d\text{Vol}_g(x).$$

In that case, we say that Laplace eigenfunctions are uniformly *observable* in the subset ω . According to (3), it is sufficient to restrict ourselves to the case of large eigenvalues, i.e.

¹Even in these cases, the asymptotic description of these solutions turns out to be a subtle question.

Definition 2.1. Let ν be a probability measure on M . We say that ν is a quantum limit if there exists a sequence $(\tilde{\psi}_j)_{j \geq 1}$ of normalized Laplace eigenfunctions (not necessarily an orthonormal basis) with eigenvalues $\tilde{\lambda}_j \rightarrow +\infty$ such that

$$\forall a \in C^0(M), \quad \lim_{j \rightarrow +\infty} \int_M a(x) |\tilde{\psi}_j(x)|^2 d\text{Vol}_g(x) = \int_M a(x) d\nu(x).$$

We denote by $\mathcal{N}(\Delta_g)$ the set of quantum limits.

Remark 2.2. Recall that, as M is compact, the set of probability measures on M is compact for the weak- \star topology. In particular, given any sequence of probability measures on M , we can extract a converging subsequence.

By a contradiction argument, proving (4) follows then from showing that there exists a constant $\tilde{c}_{\omega, M, g} > 0$ such that for every $\nu \in \mathcal{N}(\Delta_g)$, $\nu(\omega) \geq \tilde{c}_{\omega, M, g}$. Now that we have defined $\mathcal{N}(\Delta_g)$, several other natural questions appear in order to keep track of the regularity of Laplace eigenfunctions as $\lambda \rightarrow +\infty$. How big is $\mathcal{N}(\Delta_g)$ inside the set of all probability measure on M ? What is the Sobolev regularity of an element inside $\mathcal{N}(\Delta_g)$? These lecture notes are an attempt to describe what can be said depending on the geometric framework.

As we shall use semiclassical conventions, we set $\hbar = \lambda^{-1}$ to be the effective Planck constant of our problem and, for $\hbar > 0$, we rewrite (1) as

$$(5) \quad -\hbar^2 \Delta_g \psi_\hbar = \psi_\hbar, \quad \|\psi_\hbar\|_{L^2} = 1.$$

Remark 2.3. We will often make the small (but standard) abuse of notations to write $\hbar \rightarrow 0^+$ when we mean that $\hbar_j \rightarrow 0^+$ as $j \rightarrow +\infty$. At some points, we will consider orthonormal basis $(\psi_j)_{j \geq 0}$ of $L^2(M)$ made of solutions to (5) with the small abuse of notations that $\psi_0 = 1$ corresponds formally to the parameter $\hbar_0 = +\infty$ and that $\psi_j = \psi_{\hbar_j}$. In that case, we will say that $(\psi_j)_{j \geq 1}$ is an orthonormal basis of Laplace eigenfunctions.

Remark 2.4. For the sake of simplicity, we will focus on the case of the (semiclassical) quantum Hamiltonian

$$\widehat{H}_\hbar := -\frac{1}{2} \hbar^2 \Delta_g.$$

Yet, some (but not all) of the results presented below can be extended to more general semiclassical operators of the form $-\frac{1}{2} \hbar^2 \Delta_g + \epsilon_\hbar V(x)$, where $0 \leq \epsilon_\hbar \leq 1$ and where V is a smooth and real-valued function on M .

Remark 2.5. Most of the results that we will present will also be valid in the more general setting of quasimodes, i.e. solutions to

$$-\hbar^2 \Delta_g \psi_\hbar = \psi_\hbar + \hbar r_\hbar,$$

where $\|r_\hbar\|_{L^2} \rightarrow 0$ at a convenient rate. In order to alleviate the presentation, we will not discuss the size of r_\hbar and we refer the reader to the corresponding references for more precise statements.

2.2. Some important examples. Before describing the analytical and dynamical tools needed for our study, let us describe three important examples.

2.2.1. The flat torus. The simplest example is given by the torus $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z})^n$ endowed with its canonical (Euclidean) metric. In that case, $\Delta_g = \Delta$ is the usual Laplacian on \mathbb{R}^n that we restrict to $2\pi\mathbb{Z}^n$ -periodic functions. For $\hbar > 0$, solutions to (5) are then given by

$$\psi_{\hbar}(x) = \sum_{k \in \mathbb{Z}^n : \|k\|_{\hbar} = 1} \hat{c}_k e^{ik \cdot x}, \quad (2\pi)^n \sum_{k \in \mathbb{Z}^n : \|k\|_{\hbar} = 1} |\hat{c}_k|^2 = 1.$$

Remark 2.6. In that case, the Hörmander-Weyl's law (2) has a simple proof by observing that

$$N(\hbar^{-1}) = |\{k \in \mathbb{Z}^n : \|k\| \leq \hbar^{-1}\}|,$$

and that this quantity can be compared with the volume of the ball of radius \hbar^{-1} centered at 0.

It is then easy to construct some elements inside $\mathcal{N}(\Delta)$. For instance, by taking $\psi_{\hbar}(x) = e^{ik \cdot x} / (2\pi)^{\frac{n}{2}}$ with $\hbar = \|k\|^{-1} \rightarrow 0^+$, one finds that $d\nu(x) = \frac{dx}{(2\pi)^n}$ belongs to $\mathcal{N}(\Delta)$. One can also construct more complicated examples. For instance, let p be an element in \mathbb{Z}^{n-1} and write $\psi_{\hbar}(x) = C \sin(p \cdot x') e^{in x_1}$ where $x = (x_1, x') \in \mathbb{T}^n$, C is a normalizing constant and $\hbar_n = (\|p\|^2 + n^2)^{-1} \rightarrow 0^+$. Then, $d\nu(x) = C^2 |\sin(p \cdot x')|^2 dx$ belongs to $\mathcal{N}(\Delta)$.

From these simple examples, one can already feel that, even if Laplace eigenfunctions have an explicit expression, their asymptotic description may lead to subtle arithmetic questions as their structure is related to the distribution of lattice points on circles of large radius. Exploiting this arithmetic structure, one can in fact show the following result on the regularity of elements inside $\mathcal{N}(\Delta)$ [43, 146, 74]:

Theorem 2.7. *Let \mathbb{T}^n be the torus endowed with its canonical metric. Then, every element ν of $\mathcal{N}(\Delta)$ is absolutely continuous with respect to the Lebesgue measure. Moreover, if $n = 2$, the density of ν belongs to $L^2(\mathbb{T}^2)$.*

Proof. Let us give the proof in the case $n = 2$ which follows from results due to Cooke [43] and Zygmund [146]. In higher dimension, this Theorem is due to Bourgain and this was reported in [74]. As the proof for $n \geq 3$ is more involved, we will not discuss it here. In dimension 2, it was shown in [43, 146] that, for any solution ψ_{\hbar} to (5) on \mathbb{T}^2 , one has $\|\psi_{\hbar}\|_{L^4(\mathbb{T}^2)} \leq 3^{\frac{1}{4}} / \sqrt{2\pi}$. Once we have this result, one can verify via the Cauchy-Schwarz inequality that, for any $\nu \in \mathcal{N}(\Delta)$,

$$\forall a \in C^0(\mathbb{T}^2), \quad \left| \int_{\mathbb{T}^2} a(x) d\nu(x) \right| \leq \sqrt{3} \|a\|_{L^2(\mathbb{T}^2)}.$$

In particular, $\nu \in L^2(\mathbb{T}^2)$ by Riesz representation Theorem. Hence, it remains to prove the upper bound on the L^4 norm of ψ_{\hbar} . For that purpose, we write

$$|\psi_{\hbar}(x)|^2 = \sum_{k, l : \|k\|_{\hbar} = \|l\|_{\hbar} = 1} \hat{c}_k \overline{\hat{c}_l} e^{i(k-l) \cdot x} = \sum_{p \in \mathbb{Z}^n} e^{ip \cdot x} \sum_{k : \|k\|_{\hbar} = \|k-p\|_{\hbar} = 1} \hat{c}_k \overline{\hat{c}_{k-p}}.$$

For $p = 0$, the sum over k is equal to $1/(2\pi)^2$ as ψ_h is normalized in L^2 . For $p \neq 0$, one can verify that there are at most two terms in the sum over k . Hence, by Plancherel Theorem and by Cauchy-Schwarz inequality, one finally has the expected inequality

$$\int_{\mathbb{T}^2} |\psi_h(x)|^4 dx \leq \frac{1}{(2\pi)^2} + 2(2\pi)^2 \sum_{p \in \mathbb{Z}^n \setminus \{0\}} \sum_{k: \|k\|_h = \|k-p\|_h = 1} |\hat{c}_k|^2 |\overline{\hat{c}_{k-p}}|^2 \leq \frac{3}{(2\pi)^2}.$$

□

The fact that quantum limits are absolutely continuous can be recovered by semiclassical methods as the ones described in paragraph 5.2. This semiclassical proof was first obtained by Macià [94] in dimension 2 and generalized later on to higher dimensions by Anantharaman and Macià [6]. Compared with the approach from [43, 146, 74], much less can be deduced on the regularity of elements inside $\mathcal{N}(\Delta)$. Yet, it has the advantage that it can be generalized to more general completely integrable systems [3, 5].

2.2.2. *The round sphere.* Consider now the 2-sphere

$$\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

In spherical coordinates, the canonical metric can be written as $g(\theta, \phi) = d\theta^2 + \sin^2 \theta d\phi^2$. Hence, one has

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin^2 \theta} \partial_\phi^2 + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta).$$

Solutions to (5) are linear combinations of the spherical harmonics [120, §3.4]. In that case, one has $\hbar^{-2} = l(l+1)$ for some integer $l \geq 0$, and for every such eigenvalue, the algebraic multiplicity is $2l+1$. Again, we have exact expressions which can be written in terms of Legendre polynomials. This allows us to construct explicit examples of elements lying in $\mathcal{N}(\Delta_g)$. For instance, given $l \in \mathbb{Z}_+$, one can consider the following example of Laplace eigenfunction

$$\psi_{\hbar_l}(\theta, \phi) = c_l e^{i l \phi} (\sin \theta)^l,$$

where c_l is some normalizing constant. A calculation shows that this constant is of order $l^{1/4} \asymp \hbar_l^{-1/4}$ as $l \rightarrow +\infty$. Similarly, one can verify that

$$\forall a \in \mathcal{C}^0(\mathbb{S}^2), \quad \lim_{l \rightarrow +\infty} \int_{\mathbb{S}^2} a |\psi_{\hbar_l}|^2 d\text{Vol}_g = \int_0^{2\pi} a\left(\frac{\pi}{2}, \phi\right) \frac{d\phi}{2\pi}.$$

Hence, the Lebesgue measure along the closed geodesic $\gamma_0 = \{(\frac{\pi}{2}, \phi) : 0 \leq \phi \leq 2\pi\}$ belongs to $\mathcal{N}(\Delta_{\mathbb{S}^2})$. In particular, we do not have any analogue of Theorem 2.7 in the case of the 2-sphere. By spherical symmetry, one can deduce that, given any closed geodesic γ on \mathbb{S}^2 , the Lebesgue measure δ_γ along it belongs to $\mathcal{N}(\Delta_{\mathbb{S}^2})$. More generally, a similar calculation shows that any finite convex combination of such measures is still inside $\mathcal{N}(\Delta_{\mathbb{S}^2})$. Hence, if we denote by $\mathcal{N}(\mathbb{S}^2)$ the closed convex hull of these measures, one has

$$(6) \quad \mathcal{N}(\mathbb{S}^2) \subset \mathcal{N}(\Delta_{\mathbb{S}^2}).$$

As a consequence of the results from paragraph 2.6, we shall see that one has in fact equality. Hence, in that case, we have a complete description of the set of quantum limits.

Again, compared with the case of the torus, it contains singular elements as eigenfunctions can concentrate along closed geodesics.

2.2.3. The hyperbolic plane. In the previous paragraphs, we saw two models corresponding respectively to manifolds with 0 curvature (the torus) and with constant positive curvature (the sphere). Let us now discuss the case of hyperbolic surfaces which are the simplest models of manifolds with negative curvature. We set

$$\mathbb{H} := \{x + iy \in \mathbb{C} : y > 0\},$$

that we endow with the Riemannian metric $\frac{dx^2 + dy^2}{y^2}$. One can verify that the sectional curvature is constant and equal to -1 . In this system of coordinates, the Laplace-Beltrami operator (or hyperbolic Laplacian) can be written as

$$\Delta_{\mathbb{H}} = y^2 (\partial_x^2 + \partial_y^2).$$

If we fix a subgroup Γ of $SL(2, \mathbb{R})$ acting by isometry on \mathbb{H} via the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}$, then we can consider the quotient $M = \mathbb{H}/\Gamma$ endowed with the induced Riemannian structure. If M is compact, we are in the framework described above and eigenfunctions are eigenfunctions of the hyperbolic Laplacian which are Γ -invariant. Compared with the previous examples, we do not have an explicit expression for such solutions, or at least solutions which are easy to manipulate. An important class of such manifolds are compact arithmetic surfaces which correspond to subgroups Γ that are derived from a quaternion division algebra \mathcal{A} [109]. In that case, one has more structure and one can expect to say more on the structures of elements inside $\mathcal{N}(\Delta_g)$ thanks to the arithmetic nature of the problem – see Section 6.

2.3. Hamiltonian flows. Given a smooth function $H : T^*M \rightarrow \mathbb{R}$, we recall how to define its corresponding Hamiltonian vector field X_H . To that aim, one needs to introduce the Liouville 1-form, which is an element θ of $\Omega^1(T^*M)$ defined as

$$\forall (x, \xi) \in T^*M, \quad \theta_{x, \xi} : V \in T_{x, \xi}T^*M \mapsto \xi(d_{x, \xi}\pi V),$$

where $\pi : (x, \xi) \in T^*M \mapsto x \in M$ is the canonical projection. In local coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, it reads

$$\theta_{x, \xi} = \sum_{i=1}^n \xi_i dx_i.$$

Then, we define the symplectic form on T^*M as $\omega = d\theta$, which can be written in local coordinates as

$$\omega_{x, \xi} = \sum_{i=1}^n d\xi_i \wedge dx_i.$$

The Hamiltonian vector field of H is then the unique vector field X_H such that

$$dH = \omega(\cdot, X_H).$$

If H is smooth, then X_H induces a smooth flow $\varphi_H^t : T^*M \rightarrow T^*M$. In local coordinates, one has the *Hamilton equations*:

$$\forall 1 \leq i \leq n, \quad \dot{x}_i = \partial_{\xi_i} H, \quad \dot{\xi}_i = -\partial_{x_i} H.$$

Given two smooth functions a and b on T^*M , we can define their Poisson bracket:

$$\{a, b\} = X_a(b) = db(X_a) = \omega(X_a, X_b) = \sum_{i=1}^n (\partial_{\xi_i} a \partial_{x_i} b - \partial_{x_i} a \partial_{\xi_i} b),$$

which will be useful when establishing the semiclassical correspondence for pseudodifferential operators in the following. By construction, the flow φ_H^t preserves H and ω along the evolution. In particular, it preserves the Liouville measure $L = \frac{|\omega^n|}{n!}$. Even if we will encounter other Hamiltonian functions at certain points, we will mostly focus on the one corresponding to the free evolution on (M, g) , i.e.

$$H_0(x, \xi) = \frac{\|\xi\|_{g^*(x)}^2}{2}.$$

In that case, we will write $\varphi^t = \varphi_{H_0}^t$ which is nothing else but the geodesic flow on T^*M . We will also denote by L_1 the desintegration of the Liouville measure along the energy layer $S^*M = H_0^{-1}(1/2)$ that we normalize to have $L_1(S^*M) = 1$. We still have that, for every $t \in \mathbb{R}$, $\varphi_*^t L_1 = L_1$. Equivalently, for every continuous function a on S^*M and for every t in \mathbb{R} ,

$$(7) \quad \int_{S^*M} a \circ \varphi^t dL_1 = \int_{S^*M} a dL_1.$$

For more details and background facts on symplectic geometry, we refer to [145, Ch. 2].

2.4. Birkhoff ergodic Theorem. We now collect a few basic facts from ergodic theory that will be needed in our analysis. We only state them for the geodesic flow $\varphi^t : S^*M \mapsto S^*M$ but they are in fact valid in the more general context of continuous flows acting on a compact metric space. Ergodic theory is concerned with the study of flows (or maps) which preserve some measure μ . We already encounter such an example of measure with the geodesic flow and the Liouville measure – see (7). More generally, we define

Definition 2.8. Let μ be a probability measure on S^*M . We say that μ is a φ^t -invariant probability measure on S^*M if, for any continuous function a on S^*M ,

$$\forall t \in \mathbb{R}, \quad \int_{S^*M} a \circ \varphi^t d\mu = \int_{S^*M} a d\mu.$$

We denote by $\mathcal{M}(\varphi^t)$ the set of all such measures.

As $L_1 \in \mathcal{M}(\varphi^t)$, this set is nonempty. Other examples of elements in $\mathcal{M}(\varphi^t)$ are given by measures carried by closed geodesics. More precisely, given a closed orbit γ of φ^t with minimal period T_γ and given a continuous function a on S^*M , we define

$$\int_{S^*M} a d\mu_\gamma = \frac{1}{T_\gamma} \int_0^{T_\gamma} a \circ \varphi^t(x, \xi) dt,$$

where (x, ξ) is any point on the closed orbit.

The set $\mathcal{M}(\varphi^t)$ is endowed with the weak topology induced by $\mathcal{C}^0(S^*M, \mathbb{R})$ on its dual. One can verify that it is a compact, convex and nonempty subset of $\mathcal{M}(\varphi^t)$. By definition, we say that a measure μ is ergodic if it is an extremal point of this convex set. Equivalently, it means that, if $\mu = t\mu_1 + (1-t)\mu_2$ with μ_1, μ_2 in $\mathcal{M}(\varphi^t)$ and $0 < t < 1$, then $\mu = \mu_1 = \mu_2$. Again, μ_γ is the simplest example of such a measure. In the case of negatively curved manifolds, the Liouville measure L_1 was shown to be ergodic by Anosov [10]. An equivalent manner to define ergodic measures is to require that, if A is a subset such that, for every t in \mathbb{R} , $\varphi^t(A) = A$, then $\mu(A) = 0$ or 1 . In the case of the geodesic flow on $S^*\mathbb{S}^d$ (for the round metric), one can then verify that the ergodic measures are exactly given by the measures μ_γ . A central result in ergodic theory is the Birkhoff ergodic Theorem² [52, §8.6]:

Theorem 2.9 (Birkhoff Ergodic Theorem). *Let μ be an element of $\mathcal{M}(\varphi^t)$. Then, for every continuous function $a : S^*M \rightarrow \mathbb{R}$ and for μ -a.e. (x, ξ) in S^*M ,*

$$(8) \quad \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T a \circ \varphi^t(x, \xi) dt = \mu_{x, \xi}(a),$$

where $(x, \xi) \mapsto \mu_{x, \xi}(a)$ belongs to $L^\infty(S^*M, \mu)$. Moreover, $(x, \xi) \mapsto \mu_{x, \xi}(a)$ is φ^t -invariant and one has

$$\int_{S^*M} \mu_{x, \xi}(a) d\mu(x, \xi) = \int_{S^*M} a d\mu,$$

Finally, if μ is ergodic then, for μ -a.e. (x, ξ) in S^*M ,

$$\mu_{x, \xi}(a) = \int_{S^*M} a d\mu.$$

Remark 2.10. Note that one can choose a full measure subset Ω (w.r.t. μ) such that (8) holds for a dense and countable subset of functions $(a_k)_{k \geq 1}$ inside $\mathcal{C}^0(S^*M, \mathbb{R})$. By construction, one has $|\mu_{x, \xi}(a_k)| \leq \|a_k\|_{\mathcal{C}^0}$ for every $k \geq 1$. Hence, by the Riesz representation Theorem, $\mu_{x, \xi}$ can be identified with a Radon measure on S^*M . One can also verify that $\mu_{x, \xi}$ is an ergodic measure in $\mathcal{M}(\varphi^t)$ for μ -a.e. (x, ξ) in S^*M . Thus, we can write the ergodic decomposition of any measure $\mu \in \mathcal{M}(\varphi^t)$:

$$\mu = \int_{S^*M} \mu_{x, \xi} d\mu(x, \xi).$$

Let us now illustrate this Theorem in the three geometries that we have encountered so far.

Example 2.11. In the case of a negatively curved manifold, Anosov proved that L_1 is an ergodic measure [10]. This tells us that, for L_1 -a.e. (x, ξ) in S^*M ,

$$\forall a \in \mathcal{C}^0(S^*M, \mathbb{R}), \quad \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T a \circ \varphi^t(x, \xi) dt = \int_{S^*M} a dL_1.$$

²We do not state this Theorem in its full generality and we only mention what will be needed in our analysis.

It means that if we look at the average of a along the orbit of a typical point of S^*M (w.r.t. the natural volume measure on S^*M), then it converges to the average of a over S^*M .

Example 2.12. Let (\mathbb{S}^n, g_0) be the n -dimensional sphere endowed with its canonical metric g_0 . In that case, the geodesic flow on $S^*\mathbb{S}^n$ is 2π -periodic, and one has, *for every* $(x, \xi) \in S^*\mathbb{S}^n$ and for every continuous function a ,

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T a \circ \varphi^t(x, \xi) dt = \mu_{\gamma_{x, \xi}}(a),$$

where $\gamma_{x, \xi} := \{\varphi^t(x, \xi) : 0 \leq t \leq 2\pi\}$ is the closed orbit issued from (x, ξ) . Observe that the map $(x, \xi) \mapsto \mu_{\gamma_{x, \xi}}(a)$ has the same regularity as the function a . Ergodic measures are exactly the measures μ_γ .

Example 2.13. Let (\mathbb{T}^n, g_0) be the n -dimensional torus endowed with its Euclidean metric. Given $\xi \in \mathbb{S}^{n-1}$, we define

$$\Lambda_\xi := \{k \in \mathbb{Z}^n : k \cdot \xi = 0\},$$

and Λ_ξ^\perp which is the orthonormal complement of the vector space generated by Λ_ξ . If the rank of Λ_ξ is equal to $n - j$, then the orbit of the point (x, ξ) under the geodesic flow φ^t fills a torus of dimension j . More precisely, one has, *for every* (x, ξ) in $S^*\mathbb{T}^n$ and for every continuous function a ,

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T a \circ \varphi^t(x, \xi) dt = \int_{\mathbb{T}(\xi)} a(x + y, \xi) d\mathfrak{h}_\xi(y),$$

where $\mathbb{T}(\xi) = \Lambda_\xi^\perp / (2\pi\mathbb{Z}^n \cap \Lambda_\xi^\perp)$ and \mathfrak{h}_ξ is the normalized Haar measure on $\mathbb{T}(\xi)$. These are exactly the ergodic measures of the geodesic flow on the flat torus.

From these three examples, we can remark that the dynamical properties of the geodesic flow are of very different nature depending on the choice of the manifold (M, g) and this will be responsible for the different behaviour of Laplace eigenfunctions that we will encounter.

2.5. Semiclassical calculus. So far, we have constructed two sets of probability measures. One set is built from the eigenfunctions of the Laplacian, and the other one from the geodesic flow. One of our objective is to compare these two sets of measures but one is made of measures on M and the other of measures on S^*M . In order to overcome this difference, we will explain how to lift elements of $\mathcal{N}(\Delta_g)$ into measures defined on S^*M in a natural manner from the point of view of quantum mechanics. This will also connect $\mathcal{N}(\Delta_g)$ to the dynamics on S^*M . To that aim, we need to introduce some tools from semiclassical analysis. We will do that in a somewhat condensed manner and we invite the reader to look at [145, Ch. 4 and 14] for a more detailed presentation. First of all, the goal is to associate to a function $a_\hbar(x, \xi)$ on T^*M (that may also depend on \hbar) an operator acting on $\mathcal{C}^\infty(M)$ (ideally on the Hilbert space $L^2(M)$). This is called a quantization procedure.

If we consider the case of $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$, then a natural way to proceed is to set, for every a in the Schwartz class $\mathcal{S}(\mathbb{R}^{2n})$ and for every $\hbar > 0$,

$$(9) \quad \forall u \in \mathcal{S}(\mathbb{R}^n), \quad \text{Op}_\hbar^w(a)u(x) := \frac{1}{(2\pi\hbar)^n} \iint_{\mathbb{R}^{2n}} a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{\hbar}\langle x-y, \xi \rangle} u(y) dy d\xi.$$

This is referred to the Weyl quantization of the observable a , and it can be extended to more general classes of observables. For instance, if $a(x, \xi) = a(x) \in \mathcal{S}(\mathbb{R}^n)$, then $\text{Op}_\hbar^w(a)u(x) = a(x)u(x)$. Similarly, if $a(x, \xi) = P(\xi)$ with P a polynomial, then $\text{Op}_\hbar^w(a) = P(-i\hbar\partial_x)$. Following [145, §14.2.2], this definition can be extended to manifolds. More precisely, we say that $A_\hbar : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ is an \hbar -pseudodifferential operator of order m if³ it can be written in local charts as $\text{Op}_\hbar^w(\tilde{a}_\hbar)$ where \tilde{a}_\hbar is the pullback on \mathbb{R}^{2n} of some function a_\hbar (that may depend on \hbar) belonging to the class of symbols

$$S^m(T^*M) := \left\{ a_\hbar(x, \xi) \in \mathcal{C}^\infty(T^*M) : \forall (\alpha, \beta) \in \mathbb{Z}_+^{2n}, |\partial_x^\alpha \partial_\xi^\beta \tilde{a}_\hbar(x, \xi)| \leq C_{\alpha, \beta} (1 + \|\xi\|_x^2)^{\frac{m-|\beta|}{2}} \right\},$$

with m belonging to \mathbb{R} . Here, the constant $C_{\alpha, \beta}$ is uniform for $0 < \hbar \leq 1$. The set of such operators is denoted by $\Psi^m(M)$ and one has for instance $-\frac{1}{2}\hbar^2\Delta_g \in \Psi^2(M)$. We also denote by $\Psi^{-\infty}(M)$ (resp. $S^{-\infty}(T^*M)$) the intersection of all the spaces $(\Psi^m(M))_{m \in \mathbb{R}}$ (resp. $(S^m(T^*M))_{m \in \mathbb{R}}$). Note that $\mathcal{C}_c^\infty(T^*M) \subset S^{-\infty}(T^*M)$. According to [145, Th. 14.1], there exist two maps

$$\sigma_\hbar : \Psi^m(M) \rightarrow S^m(T^*M)/\hbar S^{m-1}(T^*M), \quad \text{and} \quad \text{Op}_\hbar : S^m(T^*M) \rightarrow \Psi^m(T^*M)$$

such that $\sigma_\hbar(\text{Op}_\hbar(a_\hbar)) = a_\hbar \bmod \hbar S^{m-1}(T^*M)$. Note that this result says in particular that one can associate to each observable a_\hbar in $S^m(T^*M)$ an operator $\text{Op}_\hbar(a_\hbar)$ acting on smooth functions.

Remark 2.14. We say that $a_\hbar = \sigma_\hbar(A_\hbar)$ is the principal symbol of the operator A_\hbar . In the case of $-\frac{1}{2}\hbar^2\Delta_g$, the principal symbol is equal to $H_0(x, \xi) = \frac{1}{2}\|\xi\|_x^2$.

Let us review (without proof) some of the fundamental properties of these operators. First, we can choose Op_\hbar such that $\text{Op}_\hbar(a)u = a \times u$ if a depends only on the variable x and such that, for every u and v in $\mathcal{C}^\infty(M)$,

$$\langle \text{Op}_\hbar(a)u, v \rangle_{L^2(M)} = \langle u, \text{Op}_\hbar(\bar{a})v \rangle_{L^2(M)}.$$

These assumptions will simplify our exposition but they are not crucial and they can be weakened by requiring that there are only true modulo error terms of order $\mathcal{O}(\hbar)$. We now have the following properties that relate the properties of the operators obtained through the quantization Op_\hbar to the properties of the corresponding functions:

- (Composition formula) Let $a_\hbar \in S^{m_1}(T^*M)$ and let $b_\hbar \in S^{m_2}(T^*M)$. One has

$$\text{Op}_\hbar(a_\hbar) \circ \text{Op}_\hbar(b_\hbar) \in \Psi^{m_1+m_2}(M),$$

and its principal symbol is given by

$$(10) \quad \sigma_\hbar(\text{Op}_\hbar(a_\hbar) \circ \text{Op}_\hbar(b_\hbar)) = a_\hbar b_\hbar.$$

³One also needs to impose some regularizing assumption between distinct charts – see [145] for details.

- (Bracket properties) Let $a_{\hbar} \in S^{m_1}(T^*M)$ and let $b_{\hbar} \in S^{m_2}(T^*M)$. One has

$$[\text{Op}_{\hbar}(a_{\hbar}), \text{Op}_{\hbar}(b_{\hbar})] \in \hbar \Psi^{m_1+m_2-1}(M),$$

and its principal symbol is given by

$$(11) \quad \sigma_{\hbar}([\text{Op}_{\hbar}(a_{\hbar}), \text{Op}_{\hbar}(b_{\hbar})]) = -i\hbar\{a_{\hbar}, b_{\hbar}\}.$$

In the case of the flat torus (or of \mathbb{R}^n) and of the Weyl quantization, one has also an equality when $a = H_0$ in the sense that $[\text{Op}_{\hbar}^w(H_0), \text{Op}_{\hbar}^w(b_{\hbar})] = -i\hbar \text{Op}_{\hbar}^w(\{H_0, b_{\hbar}\})$.

- (Calderón-Vaillancourt Theorem) Let $a_{\hbar} \in S^0(T^*M)$. The operator

$$\text{Op}_{\hbar}(a_{\hbar}) : L^2(M) \rightarrow L^2(M)$$

is bounded and its norm is bounded by

$$(12) \quad \|\text{Op}_{\hbar}(a_{\hbar})\|_{L^2 \rightarrow L^2} \leq C_M \sum_{|\alpha| \leq D_M} \hbar^{\frac{|\alpha|}{2}} \|\partial^{\alpha} a_{\hbar}\|_{L^{\infty}},$$

where C_M and D_M are constants depending only on M .

- (Disjoint supports) Let $a_{\hbar} \in S^{m_1}(T^*M)$ and let $b_{\hbar} \in S^{m_2}(T^*M)$. Suppose that there exists two disjoint subsets Ω_1 and Ω_2 such that $\text{dist}(\Omega_1, \Omega_2) > 0$ and such that, for every $0 < \hbar \leq 1$, $\text{supp}(a_{\hbar}) \subset \Omega_1$ and $\text{supp}(b_{\hbar}) \subset \Omega_2$. Then,

$$(13) \quad \text{Op}_{\hbar}(a_{\hbar}) \text{Op}_{\hbar}(b_{\hbar}) = \mathcal{O}_{L^2 \rightarrow L^2}(\hbar^{\infty}),$$

where the convention $\mathcal{O}_{L^2 \rightarrow L^2}(\hbar^{\infty})$ means that, for every $N \geq 1$, there exists a constant $C_N > 0$ such that the norm of the operator is less than $C_N \hbar^N$.

- (Egorov Theorem) Let $T > 0$ and let $a_{\hbar} \in S^{-\infty}(T^*M)$ which is compactly supported. Then, there exists some constant $C_{T,a} > 0$ such that, for every $|t| \leq T$,

$$(14) \quad \left\| e^{-\frac{it\hbar\Delta_g}{2}} \text{Op}_{\hbar}(a_{\hbar}) e^{\frac{it\hbar\Delta_g}{2}} - \text{Op}_{\hbar}(a_{\hbar} \circ \varphi^t) \right\|_{L^2 \rightarrow L^2} \leq C_{a,T} \hbar.$$

This result establishes a correspondence between the classical dynamics induced by the geodesic flow and the quantum dynamics induced by the semiclassical Schrödinger equation

$$(15) \quad i\hbar \partial_t u_{\hbar} = -\frac{1}{2} \hbar^2 \Delta_g u_{\hbar}.$$

Determining the range of validity (in terms of T) of this kind of approximation is a key issue when studying the fine properties of Laplace eigenfunctions [16, 137, 2, 8, 6, 49]. In general, the best one can expect is that such a correspondence remains true up to times of order $c|\log \hbar|$ with $c > 0$ which can be expressed in terms of certain expansion rates of the geodesic flow [14, 27]. Yet, in the case of the torus (or of \mathbb{R}^n) and of the Weyl quantization, the exact bracket formula allows to prove that, for every $t \in \mathbb{R}$,

$$(16) \quad e^{-\frac{it\hbar\Delta}{2}} \text{Op}_{\hbar}^w(a_{\hbar}) e^{\frac{it\hbar\Delta}{2}} = \text{Op}_{\hbar}^w(a_{\hbar} \circ \varphi^t),$$

for every smooth function on $T^*\mathbb{T}^n$ (or $T^*\mathbb{R}^n$) all of whose derivatives are bounded.

- (Gårding inequality) Let $a_h \in S^0(T^*M)$ such that, for every $0 < \hbar \leq 1$, $a_h \geq 0$. Then, there exists a constant $C_a > 0$ such that

$$(17) \quad \text{Op}_{\hbar}(a_h) \geq -C_a \hbar.$$

These properties are the fundamental rules of our semiclassical quantization and they establish a connection between the operations we have on functions (product, Poisson bracket, pullback by the geodesic flow, etc.) and the ones on operators (composition, bracket, evolution by the Schrödinger flow, etc.).

2.6. Semiclassical measures. As a first application of these semiclassical rules, we will define the notion of semiclassical measures [58, 90, 145] and use it to relate elements of $\mathcal{N}(\Delta_g)$ with elements of $\mathcal{M}(\varphi^t)$. The idea is to lift the measure

$$\nu_h(a) = \int_M a(x) |\psi_h(x)|^2 d\text{Vol}_g(x)$$

into a distribution on T^*M by making use of the quantization we have just defined. Recall that ψ_h is a solution to (5), i.e. a Laplace eigenfunction. For such a sequence $(\psi_h)_{h \rightarrow 0^+}$, we set

$$w_h : a \in \mathcal{C}_c^\infty(T^*M) \mapsto \langle \psi_h, \text{Op}_{\hbar}(a) \psi_h \rangle_{L^2}.$$

Thanks to (12), this defines a *bounded sequence* in $\mathcal{D}'(T^*M)$ which can be extended to more general symbols belonging to $S^0(T^*M)$. In particular, when a is independent of the ξ variable, we recover the measure ν_h . This kind of quantity measures the distribution of the quantum state ψ_h in the phase space T^*M . It encodes more information than ν_h but it is not anymore a measure.

Example 2.15. Let us illustrate the content of this new distribution compared with ν_h . To that aim, consider a sequence of Lagrangian states on \mathbb{R}^n :

$$\forall 0 < \hbar \leq 1, \quad \psi_h(x) = \chi(x) e^{\frac{iS(x)}{\hbar}},$$

where χ and S are smooth functions and where χ is compactly supported on \mathbb{R}^n . The corresponding measure $d\nu_h(x) = |\chi(x)|^2 dx$ is constant and it does not capture the oscillations due to the phase factor. Now, if we consider the Wigner distribution

$$a \in \mathcal{C}_c^\infty(T^*\mathbb{R}^n) \mapsto \langle \psi_h, \text{Op}_{\hbar}^w(a) \psi_h \rangle_{L^2},$$

then an application of the stationary phase Lemma shows that the limit distribution is given by the measure

$$a \in \mathcal{C}_c^\infty(T^*\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} a(x, d_x S) |\chi(x)|^2 dx,$$

which is carried by the Lagrangian manifold $\mathcal{L}_S := \{(x, d_x S)\}$ – see [145, Chap. 5] for details.

The appearance of this quantity in the literature can at least be traced back to the works of Wigner [132]. In the mathematics literature, it implicitly appears many years later in the works on quantum ergodicity [117, 116, 135, 40, 67] and their systematic study was initiated by Tartar [124] and Gérard [59].

Definition 2.16. Let μ be an element of $\mathcal{D}'(T^*M)$. We say that μ is a semiclassical measure if there exists a sequence $(\psi_{\hbar_n})_{\hbar_n \rightarrow 0^+}$ satisfying

$$-\hbar^2 \Delta_g \psi_{\hbar_n} = \psi_{\hbar_n}, \quad \|\psi_{\hbar_n}\|_{L^2} = 1,$$

and such that, for every $a \in \mathcal{C}_c^\infty(T^*M)$,

$$\lim_{\hbar_n \rightarrow 0^+} \langle \psi_{\hbar_n}, \text{Op}_{\hbar_n}(a) \psi_{\hbar_n} \rangle_{L^2} = \langle \mu, a \rangle.$$

We denote by $\mathcal{M}(\Delta_g)$ the set of semiclassical measures.

Remark 2.17. Using the Calderón-Vaillancourt Theorem 12, we can verify that any sequence

$$w_{\hbar_n} : a \in \mathcal{C}_c^\infty(T^*M) \mapsto \langle \psi_{\hbar_n}, \text{Op}_{\hbar_n}(a) \psi_{\hbar_n} \rangle_{L^2}, \quad n \geq 1$$

is bounded in $\mathcal{D}'(T^*M)$. Hence, it admits converging subsequences.

We have then the following Theorem which among other things justifies the terminology measure:

Theorem 2.18. *One has*

$$\mathcal{M}(\Delta_g) \subset \mathcal{M}(\varphi^t),$$

and

$$\mathcal{N}(\Delta_g) := \left\{ \int_{S_x^*M} \mu(x, d\xi) : \mu \in \mathcal{M}(\Delta_g) \right\}.$$

Thanks to this Theorem and by passing to the semiclassical limit $\hbar \rightarrow 0^+$, we can identify the (Wigner) distributions $(w_\hbar)_{\hbar \rightarrow 0^+}$ of the quantum states $(\psi_\hbar)_{\hbar \rightarrow 0^+}$ with objects from classical mechanics, i.e. invariant measures of the geodesic flow. Moreover, the description of elements inside $\mathcal{N}(\Delta_g)$ can be recovered from the informations we have on the set $\mathcal{M}(\Delta_g)$. For instance, we can already deduce that $\nu \in \mathcal{N}(\Delta_g)$ cannot be the Dirac measure at some point as the geodesic flow has no fixed point on S^*M . An obvious question to ask is: how big is the set $\mathcal{M}(\Delta_g)$ inside $\mathcal{M}(\varphi^t)$? For instance, in the case of the 2-sphere, one can deduce from (6) that these two sets are equal. However, Theorem 2.7 together with example 2.13 show that this is no longer true in the case of the flat torus. We could have thought from the correspondance principle that all classical states can be recovered from quantum states in the semiclassical limit. Yet, this simple example shows us that this is not true in general and we will try to discuss what can be said under various geometric assumptions.

Proof. Let μ be an element of $\mathcal{M}(\Delta_g)$ issued from a sequence $(\psi_\hbar)_{\hbar \rightarrow 0^+}$ of Laplace eigenfunctions. First, from Gårding inequality (17), we can verify that μ is a positive distribution, hence a measure [112, Ch. 1]. Let now a be an element in $\mathcal{C}_c^\infty(T^*M)$ whose support does not intersect S^*M . Thanks to (10) and to (12), one has

$$\langle \psi_\hbar, \text{Op}_\hbar(a) \psi_\hbar \rangle = \langle \psi_\hbar, \text{Op}_\hbar(a(2H_0 - 1)^{-1}) (-\hbar^2 \Delta_g - 1) \psi_\hbar \rangle + \mathcal{O}(\hbar) = \mathcal{O}(\hbar),$$

where we also used (5). Hence, by letting $\hbar \rightarrow 0^+$, one finds that $\mu(a) = 0$. As this is valid for any a whose support does not intersect S^*M , this implies that the support of

μ is contained in S^*M . In order to show that μ is a probability measure, it is sufficient to verify that there is no escape of mass at infinity. To that aim, we reproduce the same argument as before except that we do not pick a to be compactly supported but rather in $S^0(T^*M)$. Take now a in $\mathcal{C}^\infty(M)$ and a smooth cutoff function $\chi \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$ which is equal to 1 on $[-1, 1]$ and to 0 outside $[-2, 2]$. One has then

$$\int_M a d\nu_h = \langle w_h, a\chi(H_0) \rangle + \langle \psi_h, \text{Op}_h(a(1 - \chi(H_0)))\psi_h \rangle,$$

where the second term on the right hand side tends to 0 from the previous discussion. Hence, by letting $\hbar \rightarrow 0^+$, the pushforward of μ on M is an element of $\mathcal{N}(\Delta_g)$.

Let now a be an element in $\mathcal{C}_c^\infty(T^*M)$ without any assumption on its support. From the eigenvalue equation (5), one knows that

$$\forall t \in \mathbb{R}, \quad \langle \psi_h, \text{Op}_h(a)\psi_h \rangle = \langle \psi_h, e^{-\frac{it\hbar\Delta_g}{2}} \text{Op}_h(a) e^{\frac{it\hbar\Delta_g}{2}} \psi_h \rangle.$$

Combined with (14), this yields

$$(18) \quad \forall t \in \mathbb{R}, \quad \langle \psi_h, \text{Op}_h(a)\psi_h \rangle = \langle \psi_h, \text{Op}_h(a \circ \varphi^t)\psi_h \rangle + \mathcal{O}_{t,a}(\hbar).$$

Letting $\hbar \rightarrow 0^+$, we finally get that $\varphi_*^t \mu = \mu$ for every t in \mathbb{R} . \square

We conclude this background section by mentioning without proof a version of the microlocal Weyl law [145, Th. 15.3]:

Theorem 2.19 (Microlocal Weyl law). *Let $(\psi_{\hbar_j})_{j \geq 0}$ be an orthonormal basis of Laplace eigenfunctions (i.e. solutions to (5)). Then, for every a in $\mathcal{C}_c^\infty(T^*M)$,*

$$\lim_{\hbar \rightarrow 0} \frac{1}{N_\hbar} \sum_{j: \hbar \leq \hbar_j < +\infty} \langle \psi_{\hbar_j}, \text{Op}_{\hbar_j}(a)\psi_{\hbar_j} \rangle = \int_{S^*M} a dL_1,$$

where

$$(19) \quad N_\hbar := |\{j \geq 0 : \hbar \leq \hbar_j < +\infty\}|.$$

Remark 2.20. When L_1 is an ergodic measure, then this Theorem tells us that we have a convex combination of distributions (which are asymptotically elements of $\mathcal{M}(\varphi^t)$) that converges to L_1 in $\mathcal{D}'(T^*M)$. Ergodicity exactly means that L_1 is an extreme point of $\mathcal{M}(\varphi^t)$. Thus, from an heuristic point of view, this microlocal Weyl law should imply that most of the elements in this convex sum converges to L_1 . This idea can be made rigorous and it is at the heart of the Quantum Ergodicity Theorem that we will state (and prove) in section 3.

3. SEMICLASSICAL OBSERVABILITY

We are in position to discuss when (4) holds, i.e. we want to determine under which geometric (or dynamical) conditions, a sequence $(\psi_h)_{\hbar \rightarrow 0^+}$ verifying (5) puts some mass on a given open set of M (or S^*M). For that purpose, we can observe that a direct consequence of Theorem 2.18 is

Proposition 3.1. *Let a be an element in $C^0(S^*M, \mathbb{R})$ and set*

$$A_- := \lim_{T \rightarrow +\infty} \frac{1}{T} \inf \left\{ \int_0^T a \circ \varphi^t(x, \xi) dt : (x, \xi) \in S^*M \right\},$$

and

$$A_+ := \lim_{T \rightarrow +\infty} \frac{1}{T} \sup \left\{ \int_0^T a \circ \varphi^t(x, \xi) dt : (x, \xi) \in S^*M \right\}.$$

Then, for every $\mu \in \mathcal{M}(\Delta_g)$, one has

$$A_- \leq \mu(a) \leq A_+.$$

These bounds appear naturally when studying controllability or stabilization questions of certain partial differential equations [115]. As a Corollary of this Proposition, one gets

Theorem 3.2 (Uniform observability of Laplace eigenfunctions). *Let ω be an open set of M such that, for every geodesic γ of (M, g) ,*

$$\gamma \cap \omega \neq \emptyset.$$

Then, there exists some constant $c_{\omega, M, g} > 0$ such that, for every ψ_λ solution to

$$-\Delta_g \psi_\lambda = \lambda^2 \psi_\lambda, \quad \|\psi_\lambda\|_{L^2} = 1,$$

one has

$$\int_\omega |\psi_\lambda(x)|^2 d \text{Vol}_g(x) \geq c_{\omega, M, g} > 0.$$

The geometric assumption of this Theorem is in general sharp thanks to (6). Yet, we shall see later on that this result can be improved under appropriate assumptions on the manifold (M, g) – see Theorems 5.3 and 6.20. Note that the geometric condition appearing in this Theorem is often referred as a geometric control condition which is a terminology coming from control theory [15].

Proof. We proceed by contradiction and suppose that there exists a sequence of solutions $(\psi_{\hbar_n})_{\hbar_n > 0}$ to (5) such that the conclusion does not hold. Thanks to (3), one finds that $\hbar_n \rightarrow 0^+$. Up to an extraction, we denote by μ the corresponding semiclassical measure. We also set $\pi_1 : (x, \xi) \in S^*M \mapsto x \in M$ to be the canonical projection. Let now (x_0, ξ_0) be an element of S^*M . Thanks to our hypothesis on ω , there exists some $T_0 \in \mathbb{R}$ such that $\pi_1 \circ \varphi^{T_0}(x_0, \xi_0) \in \omega$. By continuity, this remains true on a small ball of radius $r_0 > 0$ centered at (x_0, ξ_0) . By compactness, we can then extract a finite number of balls $(B(x_i, \xi_i; r_i))_{i=1, \dots, N}$ such that this property holds true. Fix now a to be a continuous function on M which is compactly supported inside ω and which is equal to 1 on the union of all the open sets $\pi_1 \circ \varphi^{T_i}(B(x_i, \xi_i; r_i))$. One has

$$\int_\omega |\psi_{\hbar_n}(x)|^2 d \text{Vol}_g(x) \geq \int_M a(x) |\psi_{\hbar_n}(x)|^2 d \text{Vol}_g(x) = \langle \psi_{\hbar_n}, \text{Op}_{\hbar_n}(a) \psi_{\hbar_n} \rangle,$$

from which we can deduce the expected contradiction thanks to Proposition 3.1 as $\mu(a) \geq A_- > 0$. \square

3.1. Improving the observability bounds along density one subsequences. We will now show how to improve Proposition 3.1 along a typical subsequence of Laplace eigenfunctions. More precisely, following [107] – see also [115] for earlier related results of Sjöstrand in the case of the damped wave equation, one has

Theorem 3.3. *Let $(\psi_{h_j})_{j \geq 0}$ be an orthonormal basis of Laplace eigenfunctions (i.e. solutions to (5)). Then, for every a in $C_c^\infty(T^*M, \mathbb{R})$, there exists $S \subset \mathbb{Z}_+$ such that*

$$(20) \quad \lim_{h \rightarrow 0^+} \frac{1}{N_h} |\{j \in S : h \leq h_j < +\infty\}| = 1,$$

and

$$\text{essinf}\{L_{x,\xi}(a)\} \leq \liminf_{j \rightarrow +\infty, j \in S} \langle \psi_{h_j}, \text{Op}_{h_j}(a)\psi_{h_j} \rangle \leq \limsup_{j \rightarrow +\infty, j \in S} \langle \psi_{h_j}, \text{Op}_{h_j}(a)\psi_{h_j} \rangle \leq \text{esssup}\{L_{x,\xi}(a)\},$$

where

$$\text{esssup}\{L_{x,\xi}(a)\} := \inf \{C : L_{x,\xi}(a) \leq C \text{ for } L_1 - a.e. (x, \xi) \in S^*M\},$$

with $L_{x,\xi}$ being the ergodic component of (x, ξ) w.r.t. L_1 and where

$$\text{essinf}\{L_{x,\xi}(a)\} = -\text{esssup}\{-L_{x,\xi}(a)\}.$$

In the following, we will say that a subset S inside \mathbb{Z}_+ is of density 1 if it satisfies property (20). Using the conventions of Proposition 3.1, we note that

$$A_- \leq \text{essinf}\{L_{x,\xi}(a)\} \leq \text{esssup}\{L_{x,\xi}(a)\} \leq A_+,$$

and that these inequalities may be strict. Hence, this Theorem shows that the bounds of Proposition 3.1 can be improved along a typical subsequence of an orthonormal basis of Laplace eigenfunctions. Before giving some applications of this Theorem in the next section, let us give a proof of it. The key inputs compared with Proposition 3.1 are the Birkhoff ergodic Theorem and the microlocal Weyl law.

Proof. Up to replacing a by $-a$, it is sufficient to prove the upper bound. To that aim, we need to go through the proof of the upper bound of Proposition 3.1 which was a direct consequence of the invariance property proved in Theorem 2.18. This Theorem itself relied implicitly on (18) which itself followed from the Egorov Theorem (14). This tells us that, as $j \rightarrow +\infty$,

$$\forall T > 0, \quad \langle \psi_{h_j}, \text{Op}_{h_j}(a)\psi_{h_j} \rangle = \left\langle \psi_{h_j}, \text{Op}_{h_j} \left(\frac{1}{T} \int_0^T a \circ \varphi^t dt \right) \psi_{h_j} \right\rangle + o_{T,a}(1).$$

Arguing as in the proof of Theorem 2.18, we may also suppose without loss of generality that a is supported in a small neighborhood of S^*M and that it is 0-homogeneous in ξ near S^*M . We now fix a smooth cutoff function $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$ which is equal to 1 in a small neighborhood of 1/2. Fix also some $\varepsilon > 0$ and set $A_0 = \text{esssup}\{L_{x,\xi}(a)\}$. We define a smooth function $\tilde{a}_T(x, \xi) \in C_c^\infty(T^*M)$ such that

- $\tilde{a}_T(x, \xi) = \frac{\chi(H_0(x, \xi))}{T} \int_0^T a \circ \varphi^t(x, \xi) dt$ if $\chi(H_0(x, \xi)) \neq 0$ and

$$\frac{1}{T} \int_0^T a \circ \varphi^t(x, \xi) dt \leq A_0 + \sqrt{\varepsilon}.$$

- $\tilde{a}_T(x, \xi) \leq \chi(H_0(x, \xi))(A_0 + 2\sqrt{\varepsilon})$ otherwise.

Thanks to the Garding inequality (17) and for some fixed $T > 0$, we find that, as $j \rightarrow +\infty$,

$$\langle \psi_{\hbar_j}, \text{Op}_{\hbar_j}(a)\psi_{\hbar_j} \rangle \leq A_0 + 2\sqrt{\varepsilon} + \left\langle \psi_{\hbar_j}, \text{Op}_{\hbar_j} \left(\frac{\chi(H_0)}{T} \int_0^T a \circ \varphi^t dt - \tilde{a}_T \right) \psi_{\hbar_j} \right\rangle + o_{T,a}(1).$$

Then, from the Cauchy-Schwarz inequality, one gets

$$\langle \psi_{\hbar_j}, \text{Op}_{\hbar_j}(a)\psi_{\hbar_j} \rangle \leq A_0 + 2\sqrt{\varepsilon} + \left\| \text{Op}_{\hbar_j} \left(\frac{\chi(H_0)}{T} \int_0^T a \circ \varphi^t dt - \tilde{a}_T \right) \psi_{\hbar_j} \right\|_{L^2} + o_{T,a}(1).$$

In particular, one has

$$\begin{aligned} & \limsup_{\hbar \rightarrow 0^+} \frac{1}{N_{\hbar}} \left| \left\{ j : \hbar_j \geq \hbar \text{ and } \langle \psi_{\hbar_j}, \text{Op}_{\hbar_j}(a)\psi_{\hbar_j} \rangle \geq A_0 + 3\sqrt{\varepsilon} \right\} \right| \\ & \leq \limsup_{\hbar \rightarrow 0^+} \frac{1}{N_{\hbar}} \left| \left\{ j : \hbar_j \geq \hbar \text{ and } \left\| \text{Op}_{\hbar_j} \left(\frac{\chi(H_0)}{T} \int_0^T a \circ \varphi^t dt - \tilde{a}_T \right) \psi_{\hbar_j} \right\|_{L^2} \geq \sqrt{\varepsilon} \right\} \right|. \end{aligned}$$

By Bienaymé-Tchebychev inequality, we obtain that

$$\begin{aligned} & \frac{1}{N_{\hbar}} \left| \left\{ j : \hbar_j \geq \hbar \text{ and } \left\| \text{Op}_{\hbar_j} \left(\frac{\chi(H_0)}{T} \int_0^T a \circ \varphi^t dt - \tilde{a}_T \right) \psi_{\hbar_j} \right\|_{L^2} \geq \sqrt{\varepsilon} \right\} \right| \\ & \leq \frac{1}{\varepsilon N_{\hbar}} \sum_{j: \hbar_j \geq \hbar} \left\| \text{Op}_{\hbar_j} \left(\frac{\chi(H_0)}{T} \int_0^T a \circ \varphi^t dt - \tilde{a}_T \right) \psi_{\hbar_j} \right\|_{L^2}^2. \end{aligned}$$

Combining the microlocal Weyl law with the composition formula (10), these two bounds yields yields

$$\limsup_{\hbar \rightarrow 0^+} \frac{1}{N_{\hbar}} \left| \left\{ j : \hbar_j \geq \hbar \text{ and } \langle \psi_{\hbar_j}, \text{Op}_{\hbar_j}(a)\psi_{\hbar_j} \rangle \geq A_0 + 3\sqrt{\varepsilon} \right\} \right| \leq \frac{1}{\varepsilon} \int_{S^*M} \left(\frac{1}{T} \int_0^T a \circ \varphi^t dt - \tilde{a}_T \right)^2 dL_1.$$

From our construction of the function \tilde{a}_T , one can deduce that

$$\begin{aligned} & \limsup_{\hbar \rightarrow 0^+} \frac{1}{N_{\hbar}} \left| \left\{ j : \hbar_j \geq \hbar \text{ and } \langle \psi_{\hbar_j}, \text{Op}_{\hbar_j}(a)\psi_{\hbar_j} \rangle \geq A_0 + 3\sqrt{\varepsilon} \right\} \right| \\ & \leq \frac{4\|a\|_{\infty}}{\varepsilon} L_1 \left(\left\{ (x, \xi) : \frac{1}{T} \int_0^T a \circ \varphi^t(x, \xi) dt > A_0 + \sqrt{\varepsilon} \right\} \right). \end{aligned}$$

As this last inequality is valid for any $T > 0$, we can apply the Birkhoff ergodic Theorem to get

$$\lim_{\hbar \rightarrow 0^+} \frac{1}{N_{\hbar}} \left| \left\{ j : \hbar_j \geq \hbar \text{ and } \langle \psi_{\hbar_j}, \text{Op}_{\hbar_j}(a)\psi_{\hbar_j} \rangle \geq A_0 + 3\sqrt{\varepsilon} \right\} \right| = 0.$$

This gives us a subset S_{ε} of density 1 along which one has

$$\limsup_{j \rightarrow +\infty, j \in S} \langle \psi_{\hbar_j}, \text{Op}_{\hbar_j}(a)\psi_{\hbar_j} \rangle \leq \text{esssup}\{L_{x,\xi}(a)\} + 3\sqrt{\varepsilon}.$$

In order to construct a subset S of density 1 verifying the conclusion of the Theorem, one can for instance follow the argument of [40] to build S out of the subsets $S_{1/k}$ with $k \in \mathbb{Z}_+^*$ – see also [107, §4]. \square

As an application of this Theorem, we can mention the following Corollary [107]:

Corollary 3.4. *Let Λ be an invariant subset of S^*M such that*

$$L_1 = \int_{\Lambda} L_{x,\xi} dL_1(x, \xi).$$

*Let $(\psi_{\hbar_j})_{j \geq 0}$ be an orthonormal basis of Laplace eigenfunctions (i.e. solutions to (5)). Then, there exists $S \subset \mathbb{Z}_+$ such that any accumulation point (in $\mathcal{D}'(T^*M)$) of the sequence*

$$w_{\hbar_j} : a \in \mathcal{C}_c^\infty(T^*M) \mapsto \langle \psi_{\hbar_j}, \text{Op}_{\hbar_j}(a)\psi_{\hbar_j} \rangle, \quad j \in S,$$

belongs to the closure of the convex hull of $\{L_{x,\xi} : (x, \xi) \in \Lambda\}$.

3.2. Quantum ergodicity. We will now apply Theorem 3.3 in our three main geometric paradigms: round spheres, flat tori and negatively curved manifolds. In the case of the sphere \mathbb{S}^d endowed with its canonical metric, Theorem 3.3 is in fact weaker than Proposition 3.1 which holds without extracting a subsequence. A more interesting case is when the Liouville measure L_1 is ergodic, e.g. on negatively curved manifolds. In that case, $L_{x,\xi} = L_1$ for L_1 -almost every $(x, \xi) \in S^*M$ and we recover the Quantum Ergodicity Theorem [117, 116, 135, 40]:

Theorem 3.5 (Quantum Ergodicity). *Suppose that L_1 is ergodic and let $(\psi_{\hbar_j})_{j \geq 0}$ be an orthonormal basis of Laplace eigenfunctions (i.e. solutions to (5)). Then, there exists $S \subset \mathbb{Z}_+$ such that*

$$\lim_{\hbar \rightarrow 0^+} \frac{1}{N_{\hbar}} |\{j \in S : \hbar \leq \hbar_j < +\infty\}| = 1,$$

*and, for every a in $\mathcal{C}_c^\infty(T^*M)$,*

$$\lim_{j \rightarrow +\infty, j \in S} \langle \psi_{\hbar_j}, \text{Op}_{\hbar_j}(a)\psi_{\hbar_j} \rangle = \int_{S^*M} a dL_1.$$

In other words, this Theorem states that most eigenfunctions of an orthonormal basis become equidistributed on S^*M in the semiclassical limit. In particular, it says that L_1 is an element of $\mathcal{M}(\Delta_g)$ which may not be the case if L_1 is not ergodic. The fact that the density 1 subset can be chosen independently of a requires some additional density argument which is for instance explained in [40] – see also [145, Proof of Th. 15.5]. This Theorem raises several questions including the question of the size of S inside \mathbb{Z}_+ and we will discuss them more precisely in section 6. As above, we only stated this Theorem in the context of Laplace Beltrami operators on closed manifolds. Yet, it can be extended in many other contexts enjoying ergodic features. This includes semiclassical Hamiltonian operators [67], manifolds with boundary [60, 143], quantization of ergodic toral automorphisms [26], discontinuous metrics [75], sequences of compact hyperbolic surfaces [84], subriemannian Laplacians [41], etc.

In the case of flat tori \mathbb{T}^n , the pushforward of $L_{x,\xi}$ on \mathbb{T}^n is equal to $dx/(2\pi)^n$ for L_1 -almost every (x, ξ) in $S^*\mathbb{T}^n$. Hence, we can deduce from Theorem 3.3 the following Theorem which was initially proved⁴ in [97] – see also [125].

⁴The result in [97] also encompasses the case of rational polygons.

Theorem 3.6. *Suppose that \mathbb{T}^n is endowed with the Euclidean metric and let $(\psi_{\hbar_j})_{j \geq 0}$ be an orthonormal basis of Laplace eigenfunctions (i.e. solutions to (5)). Then, there exists $S \subset \mathbb{Z}_+$ such that*

$$\lim_{\hbar \rightarrow 0^+} \frac{1}{N_\hbar} |\{j \in S : \hbar \leq \hbar_j < +\infty\}| = 1,$$

and, for every a in $\mathcal{C}^0(\mathbb{T}^n)$,

$$\lim_{j \rightarrow +\infty, j \in S} \int_{\mathbb{T}^n} a(x) |\psi_{\hbar_j}(x)|^2 dx = \int_{\mathbb{T}^n} a(x) \frac{dx}{(2\pi)^n}.$$

Thanks to the examples of paragraph 2.2.1, one knows that it is in general necessary to extract a subsequence to get convergence. Compared with the case where the Liouville measure L_1 is ergodic, we emphasize that equidistribution only holds in the configuration space \mathbb{T}^n and not in the phase space $S^*\mathbb{T}^n$. In fact, if one considers the orthonormal basis given by $(e^{2i\pi k \cdot x} / (2\pi)^{\frac{n}{2}})_{k \in \mathbb{Z}^n}$, then an application of formula (9) and of the stationary phase Lemma allows to show that any semiclassical measure issued from this basis is of the form $\frac{1}{(2\pi)^n} \times \delta_0(\xi - \xi_0)$ where ξ_0 is an element of \mathbb{S}^{n-1} .

For more general geometric situations, it is not clear what the typical situation should be. In that direction, let us mention the following questions due to Šnirel'man [83, App.]. *If φ^t is not ergodic on S^*M , is it true that “almost all” eigenfunctions are asymptotically uniformly distributed on “almost all” of the ergodic components? What conditions are necessary or sufficient for such uniform distribution?* Very little is known on this natural question except for a recent work of Gomes [61]. In fact, these kind of questions also appear in the physics literature and it is sometimes referred as the Percival conjecture [101]. Following some ideas that go back to the works of Einstein [53], this conjecture roughly states that quantum eigenmodes split into two families [17]: regular states that concentrate in the part of phase space where the classical flow is regular and irregular ones concentrating where the classical flow is chaotic.

4. THE CASE OF ZOLL MANIFOLDS

In the case of the sphere \mathbb{S}^d endowed with its canonical metric, we already saw that

$$(21) \quad \mathcal{M}(\Delta_{\mathbb{S}^d}) = \mathcal{M}(\varphi^t).$$

In other words, the set of semiclassical measures is as big as it can be. This result was first proved by Jakobson and Zelditch [76] and extended to more general compact rank one symmetric spaces by Macià [93] – see also [72], and to quotients of \mathbb{S}^d by certain groups of isometries by Azagra and Macià [13]. The round sphere is in fact the simplest example of a Zoll manifold (M, g) , i.e. a manifold all of whose geodesics are closed [18]. In the general context of Zoll manifolds, thanks to a Theorem of Wadsley [18, §7.B], there exists some $l > 0$ such that $\varphi^l = \text{Id}_{S^*M}$, i.e. the geodesic flow is periodic. The existence of nontrivial manifolds of this type is due to Zoll [144]. Many years later, Guillemin showed that, near the round metric g_0 on \mathbb{S}^2 , there exists an infinite dimensional submanifold of Zoll metrics and he characterized its tangent space at g_0 [62].

A natural question to ask is whether (21) remains true on more general Zoll manifolds. The answer is no in general [95, 96]. Before stating a precise result, we need to introduce a couple of conventions. We will in fact study the slightly more general framework of Schrödinger eigenfunctions, i.e. solutions to

$$(22) \quad -\frac{1}{2}\hbar^2\Delta_g\psi_\hbar + \hbar^2V\psi_\hbar = \frac{1}{2}\psi_\hbar, \quad \|\psi_\hbar\|_{L^2} = 1,$$

where V is a smooth real valued function on (M, g) and \hbar is some positive parameter. As before, we can build from these sequences of eigenfunctions some sets of measures $\mathcal{M}(\Delta_g - 2V)$ and $\mathcal{N}(\Delta_g - 2V)$. Following the same proof⁵ as for Theorem 2.18, elements in $\mathcal{N}(\Delta_g - 2V)$ are exactly the pushforwards on M of elements in $\mathcal{M}(\Delta_g - 2V)$ and $\mathcal{M}(\Delta_g - 2V) \subset \mathcal{M}(\varphi^t)$. Moreover, given a smooth function b on $T^*M \setminus 0$, one can define its Radon transform

$$\forall (x, \xi) \in T^*M \setminus 0, \quad \mathcal{R}_g(b)(x, \xi) := \frac{\|\xi\|_x}{l} \int_0^{\frac{l}{\|\xi\|_x}} b \circ \varphi^s(x, \xi) ds.$$

Remark 4.1. With the conventions of the Birkhoff Ergodic Theorem 2.9, $\mathcal{R}_g(b)(x, \xi) = \mu_{x, \xi}(b)$ on S^*M where μ is any invariant measure on S^*M . In the following, we will make a small abuse of notations and denote by $\mathcal{R}_g(V)$ the function $\mathcal{R}_g(V \circ \pi)$ where $\pi : T^*M \mapsto M$ is the canonical projection.

Using these conventions, it was proved by Macià and the author [95] that semiclassical measures enjoy some additional regularity properties:

Theorem 4.2. *Let (M, g) be a Zoll manifold. Then, there exists a smooth function $q_0 : T^*M \setminus 0 \rightarrow \mathbb{R}$ depending only on (M, g) which is 0-homogeneous and φ^t -invariant, and such that, for any smooth function $V : M \rightarrow \mathbb{R}$, for any μ in $\mathcal{M}(\Delta_g - 2V)$ and for any $a \in C^0(S^*M)$, one has*

$$\forall t \in \mathbb{R}, \quad \int_{S^*M} a \circ \varphi_{q_0 + \mathcal{R}_g(V)}^t d\mu = \int_{S^*M} a d\mu.$$

In particular, for every $(x, \xi) \in S^*M$,

$$d_{x, \xi}(q_0 + \mathcal{R}_g(V)) \neq 0 \implies \mu(\{\varphi^t(x, \xi) : 0 \leq t \leq l\}) = 0.$$

In the case of \mathbb{S}^n , one has $q_0 = 0$. This result was recently extended to semiclassical harmonic oscillators by Arnaiz and Macià [11]. In terms of observability, the above Theorem has the following consequence:

Corollary 4.3 (Improved observability on Zoll manifolds). *Let (M, g) be a Zoll manifold, let $V : M \rightarrow \mathbb{R}$ be a smooth function and let ω be an open subset of M . Suppose that, for every geodesic γ of (M, g) , there exists some $t \in \mathbb{R}$ such that⁶*

$$\pi_1 \circ \varphi_{q_0 + \mathcal{R}_g(V)}^t(\gamma) \cap \omega \neq \emptyset.$$

⁵The term involving the potential only adds error terms which are of order $\mathcal{O}(\hbar^2)$.

⁶Here, γ is identified with its lift on S^*M and $\pi_1 : S^*M \rightarrow M$ is the restriction of π to S^*M .

Then, there exists some constant $c_{\omega,V,M,g} > 0$ such that, for every ψ_λ solution to⁷

$$(-\Delta_g + 2V)\psi_\lambda = \lambda^2\psi_\lambda, \quad \|\psi_\lambda\|_{L^2} = 1,$$

one has

$$\int_{\omega} |\psi_\lambda(x)|^2 d\text{Vol}_g(x) \geq c_{\omega,V,M,g} > 0.$$

The proof is similar to the one given for Theorem 3.2 and we omit it. Finally, in dimension 2, another application of Theorem 4.2 is the following [95]:

Theorem 4.4. *Suppose that $n = 2$ and that the assumptions of Theorem 4.2 are verified. Then, any ν in $\mathcal{N}(\Delta_g - 2V)$ can be decomposed as follows:*

$$\nu = f \text{Vol}_g + \alpha \nu_{\text{sing}}$$

where $f \in L^1(M, d\text{Vol}_g)$, $\alpha \in [0, 1]$ and ν_{sing} belongs to $\mathcal{N}_{\text{Crit}}(V + q_0)$ which is by definition the closed convex hull of the set of probability measures δ_γ , where $d_{x,\xi}(\mathcal{R}_g(V) + q_0) = 0$ for some (x, ξ) generating the closed geodesic γ .

This Theorem should be read in the light of the case where $V \equiv 0$ and where \mathbb{S}^2 is endowed with its canonical metric g_0 . In that case, we saw that eigenfunctions can concentrate on any closed geodesics and that $\mathcal{N}(\mathbb{S}^2) = \mathcal{N}(\Delta_{g_0})$. Here, this result states that quantum limits have in fact more regularity when V is nonconstant. We shall not prove this result and we refer to [95, Cor. 4.4] for details.

4.1. Examples. Before giving the proof of Theorem 4.2, let us give some examples of Zoll surfaces and of smooth potential where these results give nontrivial conclusion.

4.1.1. Surfaces of revolution with $V \equiv 0$. Theorem 4.2 shows that Laplace eigenfunctions cannot concentrate on closed geodesics that are not critical points of the function q_0 . In particular, one needs q_0 to be nonconstant on S^*M for the Theorem (and its Corollary) not being empty when $V \equiv 0$. To that aim, we need to explain where the function q_0 comes from. As shown in [95], it can be expressed in terms of the subprincipal symbol that appears when the Laplace-Beltrami is put under a quantum normal form as in the works of Duistermaat-Guillemin [45] and Colin de Verdière [39]. The dependence of q_0 in terms of the metric is a little bit intricate. It was computed explicitly by Zelditch [138, 139] in the case of Zoll metrics on \mathbb{S}^2 all of whose geodesics are of length 2π . For instance, examples of such metrics can be obtained in the following manner. Let σ be a smooth odd function on $[-1, 1]$ satisfying $\sigma(1) = 0$ and such that

$$g_\sigma(\theta, \phi) = (1 + \sigma(\cos \theta))^2 d\theta^2 + \sin^2 \theta d\phi^2$$

is a smooth metric on \mathbb{S}^2 . In [18], these manifolds are referred as Tannery surfaces and are shown to verify the Zoll property. In [95], we implemented this metric in the formulas obtained by Zelditch and, together with Theorem 4.2, we were able to deduce that

⁷Note that λ may be complex for finitely many eigenvalues.

Proposition 4.5. *Suppose that $\sigma'(0) \neq 0$. Then, q_0 is nonconstant on $S^*\mathbb{S}^2$ and one has*

$$\mathcal{M}(\Delta_{g_\sigma}) \neq \mathcal{M}(\varphi^t).$$

This result contrasts with (21). In particular, it shows that the fact that all invariant measures are semiclassical measures for Δ_g is not due to the periodicity of the flow. It is rather a consequence of the specific structure of the spectrum of the round sphere, e.g. high multiplicity of the eigenvalues.

4.1.2. *Schrödinger operators on the round sphere.* Let us now consider the sphere \mathbb{S}^2 endowed with its canonical metric g_0 . In that case, q_0 is constant according to (21) and we can discuss what happens depending on the properties of V . Denote by $G(\mathbb{S}^2) \simeq S^*\mathbb{S}^2/\mathbb{S}^1$ the space of closed geodesics on \mathbb{S}^2 , which is a smooth symplectic manifold [18]. This space $G(\mathbb{S}^2)$ can be identified with \mathbb{S}^2 [18, p. 54]. This can be easily seen as follows. Take an oriented closed geodesic γ . It belongs to a unique 2-plane in \mathbb{R}^3 which can be oriented via the orientation of the geodesic, and γ can be identified with the unit vector in \mathbb{S}^2 which is directly orthogonal to this oriented 2-plane. Hence, the Radon transform $\mathcal{R}_{g_0}(V)$ can be identified with a function of $G(\mathbb{S}^2) \simeq \mathbb{S}^2$, and the flow $\varphi_{\mathcal{R}_{g_0}(V)}^t$ can be viewed as an Hamiltonian flow on the symplectic manifold $G(\mathbb{S}^2)$. In that framework, one finds that, if $\gamma \in G(\mathbb{S}^2)$ verifies $d_\gamma \mathcal{R}_{g_0}(V) \neq 0$, then, for every $\nu \in \mathcal{N}(\Delta_{g_0} - 2V)$, $\nu(\gamma) = 0$. In other words, eigenfunctions of Schrödinger operators can only concentrate on closed geodesics which are critical points of $\mathcal{R}_{g_0}(V)$. Recall from the works of Guillemin that $\mathcal{R}_{g_0} : V \in \mathcal{C}_{\text{even}}^\infty(\mathbb{S}^2) \mapsto \mathcal{R}_{g_0}(V) \in \mathcal{C}_{\text{even}}^\infty(\mathbb{S}^2)$ is an isomorphism [62]. Note also that, if V is an odd function, then its Radon transform vanishes but we can still formulate a statement by replacing $\mathcal{R}_{g_0}(V)$ with some appropriate function of V [96].

Corollary 4.3 can also be rewritten as follows. Suppose that

$$K_{\omega,V} := \left\{ \gamma \in G(\mathbb{S}^2) : \forall t \in \mathbb{R}, \varphi_{\mathcal{R}_{g_0}(V)}^t(\gamma) \cap \omega = \emptyset \right\} = \emptyset.$$

Then, there exists some constant $c_{\omega,V} > 0$ such that, for every ψ_λ solution to

$$(-\Delta_{g_0} + 2V)\psi_\lambda = \lambda^2\psi_\lambda, \quad \|\psi_\lambda\|_{L^2} = 1,$$

one has

$$\int_\omega |\psi_\lambda(x)|^2 d\text{Vol}_g(x) \geq c_{\omega,V} > 0.$$

Let us explain how to construct ω and V such that $K_{\omega,V} = \emptyset$ while $K_{\bar{\omega}} \neq \emptyset$. Write $\mathbb{S}^2 := \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and set

$$\omega = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z > \epsilon\},$$

with $\epsilon > 0$ small enough. In particular, there are infinitely many geodesics which belong to $K_{\bar{\omega}} \subset K_\omega$, i.e. the geometric control condition $K_\omega = \emptyset$ fails. In the space of geodesics $G(\mathbb{S}^2) \simeq \mathbb{S}^2$, the geodesics belonging to $K_{\bar{\omega}}$ correspond to a small neighborhood of the two poles $(0, 0, -1)$ and $(0, 0, 1)$ of \mathbb{S}^2 . Hence, if one chooses $V \in \mathcal{C}_{\text{even}}^\infty(\mathbb{S}^2)$ in such a way that $\mathcal{R}_{g_0}(V)$ has no critical points in a slightly bigger neighborhood⁸, then one finds

⁸This is possible thanks to Guillemin's result.

that $K_{\omega, V} = \emptyset$. Indeed, in that case, the Hamiltonian flow $\varphi_{\mathcal{R}_{g_0}(V)}^t$ has no critical point inside K_{ω} . Thus, it transports the uncontrolled geodesics of K_{ω} to geodesics which are geometrically controlled by the geodesic flow. Note that the condition of having no critical points inside K_{ω} is a priori nongeneric among smooth functions.

Finally, in order to illustrate Theorem 4.4, we note that $\mathcal{R}_{g_0}(V)$ can always be identified with a smooth function on the real projective plane $\mathbb{R}P^2$. Hence, for a generic choice of potential V , the set $\mathcal{N}_{\text{Crit}}(V)$ is the convex hull of finitely many measures carried by closed geodesics depending only on V .

4.2. Proof of Theorem 4.2. For the sake of simplicity, we will only give the proof in the case of \mathbb{S}^d endowed with its canonical metric. The argument is based on the quantum averaging method that goes back to the works of Weinstein on spectral asymptotics of Schrödinger operators on the sphere [131] – see also [45, 39] for more general geometric frameworks.

4.2.1. Weinstein's averaging method. First, we need to fix some conventions and to collect some well-known facts on the spectral properties of the Laplace-Beltrami operator on \mathbb{S}^d . Recall that the eigenvalues of $-\Delta_{g_0}$ are of the form

$$E_k = \left(k + \frac{d-1}{2}\right)^2 - \frac{(d-1)^2}{4},$$

where k runs over the set of nonnegative integer. In particular, we can write

$$(23) \quad -\Delta_{g_0} = A^2 - \left(\frac{d-1}{2}\right)^2,$$

where A is a selfadjoint pseudodifferential operator of order 1 with principal symbol $\|\xi\|_x$ and satisfying

$$(24) \quad e^{2i\pi A} = e^{i\pi(d-1)} \text{Id}.$$

Given a in $\mathcal{C}_c^\infty(T^*\mathbb{S}^d \setminus 0)$, we then set, by analogy with the Radon transform of a ,

$$\mathcal{R}_{\text{qu}}(\text{Op}_{\hbar}(a)) := \frac{1}{2\pi} \int_0^{2\pi} e^{-isA} \text{Op}_{\hbar}(a) e^{isA} ds.$$

An important observation which is due to Weinstein [131] is that the following exact commutation relation holds:

$$[\mathcal{R}_{\text{qu}}(\text{Op}_{\hbar}(a)), A] = 0.$$

In particular, from (23), one has

$$(25) \quad [\mathcal{R}_{\text{qu}}(\text{Op}_{\hbar}(a)), \Delta_{g_0}] = 0.$$

Finally, a variant⁹ of the Egorov Theorem (14) allows to relate the operator $\mathcal{R}_{\text{qu}}(\text{Op}_{\hbar}(a))$ to the classical Radon transform as follows:

$$(26) \quad \mathcal{R}_{\text{qu}}(\text{Op}_{\hbar}(a)) = \text{Op}_{\hbar}(\mathcal{R}_{g_0}(a)) + \hbar R,$$

⁹The principal symbol of the Hamiltonian is now $\|\xi\|$ instead of $\|\xi\|^2/2$.

where R is a pseudodifferential operator in $\Psi^{-\infty}(\mathbb{S}^d)$.

4.2.2. *Extra invariance properties on \mathbb{S}^d .* Let us now apply these properties to derive some invariance properties of the elements in $\mathcal{M}(\Delta_{g_0} - 2V)$. We fix μ in $\mathcal{M}(\Delta_{g_0} - 2V)$ which is generated by a sequence $(\psi_h)_{h \rightarrow 0^+}$ and a in $\mathcal{C}_c^\infty(T^*\mathbb{S}^d \setminus 0)$. We use the eigenvalue equation to write

$$\left\langle \psi_h, \left[-\frac{1}{2}\hbar^2 \Delta_{g_0} + \hbar^2 V, \mathcal{R}_{\text{qu}}(\text{Op}_\hbar(a)) \right] \psi_h \right\rangle = 0.$$

According to (25), this implies that

$$\langle \psi_h, [V, \mathcal{R}_{\text{qu}}(\text{Op}_\hbar(a))] \psi_h \rangle = 0.$$

Combining (26) with the commutation formula for pseudodifferential operators (25) and with the Calderón-Vaillancourt Theorem (12), we get

$$\frac{\hbar}{i} \langle \psi_h, \text{Op}_\hbar(\{V, \mathcal{R}_{g_0}(a)\}) \psi_h \rangle = \mathcal{O}(\hbar^2).$$

Hence, after letting \hbar go to 0, one finds that

$$\mu(\{V, \mathcal{R}_{g_0}(a)\}) = 0.$$

Applying the invariance by the geodesic flow twice, one finally has

$$(27) \quad \mu(\{\mathcal{R}_{g_0}(V), a\}) = \mu(\{\mathcal{R}_{g_0}(V), \mathcal{R}_{g_0}(a)\}) = 0.$$

This is valid for any smooth test function a in $\mathcal{C}_c^\infty(T^*\mathbb{S}^d \setminus 0)$. Thus, we have just proved that any μ in $\mathcal{M}(\Delta_{g_0} - 2V)$ is invariant by the Hamiltonian flow $\varphi_{\mathcal{R}_{g_0}(V)}^t$ of $\mathcal{R}_{g_0}(V)$ which is well defined on $S^*\mathbb{S}^d \subset T^*\mathbb{S}^d \setminus 0$.

4.3. **Equidistribution on the round sphere.** We saw in the previous sections that $\mathcal{M}(\Delta_g) = \mathcal{M}(\varphi^t)$ when $(M, g) = (\mathbb{S}^2, g_0)$ is the 2-sphere endowed with its canonical metric. Equivalently, there exists sequences of eigenfunctions that concentrate along closed geodesic. In this last paragraph related to Zoll manifolds, we would like to see that one can still obtain equidistribution as in Theorem 3.5 provided we make appropriate assumptions on the sequence of eigenfunctions we consider.

4.3.1. *Equidistribution by probabilistic averaging.* In relation with the random waves model of Berry, one can in fact show that typical orthonormal basis of Laplace eigenfunctions are equidistributed on the 2-sphere following the works of Zelditch [136], Van der Kam [127] and Burq-Lebeau [33]. To that aim, recall that

$$L^2(\mathbb{S}^2) = \bigoplus_{l=0}^{+\infty} \text{Ker}(\Delta_{g_0} + l(l+1)),$$

and that each eigenspace $E_l := \text{Ker}(\Delta_{g_0} + l(l+1))$ is of dimension $2l+1$. We denote by S_l the unit sphere inside E_l . The set \mathcal{B}_l of orthonormal basis inside E_l can be identified with the unitary group $U(2l+1)$ which is endowed with a natural probability measure given by the Haar measure that we denote here by \mathbb{P}_l . Given an orthonormal basis $b_l = (e_{-l}, \dots, e_0, \dots, e_l)$ in \mathcal{B}_l and $-l \leq m \leq l$, we define the map $T_m : b_l \mapsto e_m$. These maps

induce measures on S_l that can be identified with the uniform measure $\tilde{\mathbb{P}}_l$ on S_l . The next Theorem shows that a typical orthonormal basis on S_l is equidistributed in the semiclassical limit $l \rightarrow +\infty$ [136, 127, 33]:

Theorem 4.6. *Let a be a continuous function on \mathbb{S}^2 . Then, there exists some constant $C_a > 0$ such that, for every $l \geq 0$,*

$$\mathbb{P}_l \left(\left\{ b_l \in \mathcal{B}_l : \sup_{-l \leq m \leq l} \left| \int_{\mathbb{S}^2} a |e_m|^2 d \text{Vol}_{g_0} - \int_{\mathbb{S}^2} a \frac{d \text{Vol}_{g_0}}{\text{Vol}_{g_0}(\mathbb{S}^2)} \right| \geq l^{-\frac{1}{8}} \right\} \right) \leq C_a e^{-C_a \sqrt{l}}.$$

Hence, for a generic orthonormal basis inside \mathcal{B}_l , eigenfunctions will tend to be equidistributed as $l \rightarrow +\infty$. More precisely, if we endow the set of Laplace orthonormal basis with the product measure $\mathbb{P} = \prod_{l \geq 0} \mathbb{P}_l$, then this Theorem implies that, for \mathbb{P} -a.e. orthonormal basis of Laplace eigenfunctions $(\psi_j)_{j \geq 0}$, one has

$$\forall a \in \mathcal{C}^0(\mathbb{S}^2), \quad \lim_{j \rightarrow +\infty} \int_{\mathbb{S}^2} a |\psi_j|^2 d \text{Vol}_{g_0} = \int_{\mathbb{S}^2} a \frac{d \text{Vol}_{g_0}}{\text{Vol}_{g_0}(\mathbb{S}^2)}.$$

In particular, compared with the Quantum Ergodicity results of Section 3 (e.g. Theorem 3.5), one does not need to extract a density 1 subsequence to obtain equidistribution of Laplace eigenfunctions on \mathbb{S}^2 (for a generic choice of orthonormal basis). We also emphasize that no ergodic behaviour is available on \mathbb{S}^2 . Instead of ergodicity, the averaging property comes from a probabilistic phenomenon called concentration of measure [86].

Remark 4.7. For the sake of simplicity, we stated the Theorem in the case of observables depending only on the configuration space but the argument can be extended to more general observables using the notion of Wigner distributions as we did before – see [136, 33] for more details. Note that, in some sense, this result gives a kind of negative answer on \mathbb{S}^2 to Šnirel'man's question raised at the end of Section 3.

Proof. First, without loss of generality, we can suppose that a is real valued and that $\int_{\mathbb{S}^2} a d \text{Vol}_{g_0} = 0$. We have then

$$\begin{aligned} & \mathbb{P}_l \left(\left\{ b_l \in \mathcal{B}_l : \sup_{-l \leq m \leq l} \left| \int_{\mathbb{S}^2} a |e_m|^2 d \text{Vol}_{g_0} \right| \geq l^{-\frac{1}{8}} \right\} \right) \\ & \leq (2l + 1) \tilde{\mathbb{P}}_l \left(\left\{ e \in S_l : \left| \int_{\mathbb{S}^2} a |e|^2 d \text{Vol}_{g_0} \right| \geq l^{-\frac{1}{8}} \right\} \right) \end{aligned}$$

Hence, it is sufficient to show that there exists some constant $C_a > 0$ such that, for every $l \geq 1$,

$$(28) \quad \tilde{\mathbb{P}}_l \left(\left\{ e \in S_l : \left| \int_{\mathbb{S}^2} a |e|^2 d \text{Vol}_{g_0} \right| \geq l^{-\frac{1}{8}} \right\} \right) \leq C_a e^{-C_a l^{\frac{3}{4}}}.$$

To that aim, we introduce the random variable

$$Y_l : e \in S_l \mapsto \int_{\mathbb{S}^2} a |e|^2 d \text{Vol}_{g_0} \in \mathbb{R},$$

which is a Lipschitz map with constant $2\|a\|_\infty$. We can now use the main probabilistic argument needed for our proof. This is a property of the Lebesgue measure on spheres of large dimension which is referred as concentration of the measure [86, Ch. 1]. This property [86, Eq. (1.13), Th. (2.3)] implies that, for every $r > 0$

$$\tilde{\mathbb{P}}_l(\{e \in S_l : |Y_l - m_l| \geq r\}) \leq 2e^{-\frac{2lr^2}{(2\|a\|_\infty)^2}},$$

where m_l is the median of the random variable Y_l . Letting $3r = l^{-\frac{1}{8}}$ yields an upper of the form $2e^{-\frac{l^{\frac{3}{4}}}{18\|a\|_\infty^2}}$. Using this inequality, we find that

$$\left| \int_{S_l} Y_l d\tilde{\mathbb{P}}_l - m_l \right| \leq \frac{l^{-\frac{1}{8}}}{3} + 4\|a\|_\infty e^{-\frac{l^{\frac{3}{4}}}{18\|a\|_\infty^2}}.$$

Hence, there exists some constant $C > 0$ (depending only on a) such that

$$\tilde{\mathbb{P}}_l \left(\left\{ e \in S_l : \left| Y_l - \int_{S_l} Y_l d\tilde{\mathbb{P}}_l \right| \geq l^{-\frac{1}{8}} \right\} \right) \leq Ce^{-Cl^{\frac{3}{4}}}.$$

It now remains to verify that the expectation of Y_l with respect to $\tilde{\mathbb{P}}_l$ is equal to 0 and we will be done. To see this, we recall that the dimension of E_l is equal to $2l + 1$ and we write

$$\begin{aligned} \int_{S_l} Y_l d\tilde{\mathbb{P}}_l &= \int_{\mathbb{S}^2} a(x) \left(\int_{S_l} |e(x)|^2 d\tilde{\mathbb{P}}_l(e) \right) d\text{Vol}_{g_0}(x) \\ &= \int_{\mathbb{S}^2} a(x) \left(\frac{1}{2l+1} \int_{\mathcal{B}_l} \sum_{e \in b} |e(x)|^2 d\mathbb{P}_l(b) \right) d\text{Vol}_{g_0}(x). \end{aligned}$$

Observe now that $K_l(x, y) := \sum_{e \in b} e(x)\overline{e(y)}$ is the kernel of the orthogonal projector on the space E_l , hence independent of the choice of b . Invariance by rotation also tells us that $K_l(Rx, Ry) = K_l(x, y)$ for any element $R \in SO(3)$. Thus, $K_l(x, x)$ is constant and one finds that this constant is equal to $\frac{2l+1}{\text{Vol}_{g_0}(\mathbb{S}^2)}$ as $\dim(E_l) = 2l + 1$. From this, we can deduce that $\int_{S_l} Y_l d\tilde{\mathbb{P}}_l = 0$ as we supposed that the average of a is equal to 0. This concludes the proof of Theorem 4.6. \square

4.3.2. Equidistribution by Hecke averaging. There is still another way to obtain equidistribution of Laplace eigenfunctions on the canonical 2-sphere \mathbb{S}^2 by introducing operators commuting with the Laplacian. This approach was initiated by Böcherer, Sarnak and Schulze-Pillot [20] who conjectured that joint eigenfunctions of the Laplace operator on the 2-sphere and of certain Hecke type operators must be equidistributed as $\hbar \rightarrow 0^+$. Hecke type operators on the 2-sphere are defined as follows. For $N \geq 2$, consider a finite set of rotations R_1, \dots, R_N in $SO(3)$ and define the operator

$$T_N \psi(x) = \frac{1}{\sqrt{2N-1}} \sum_{j=1}^N (\psi(R_j x) + \psi(R_j^{-1} x)).$$

This operator commutes with Δ_{g_0} . Hence, it makes sense to consider joint eigenfunctions of these two operators. When the rotations correspond to certain special elements in an order of a quaternion division algebra, then Böcherer, Sarnak and Schulze-Pillot [20] conjectured that joint eigenfunctions of T_N and Δ_{g_0} must be equidistributed as in Theorem 3.5 without extracting a subsequence. This conjecture remains open. In a recent work [30], Brooks, Le Masson and Lindenstrauss showed that, as soon as (R_1, \dots, R_N) generate a free subgroup, joint eigenfunctions of T_N and Δ_{g_0} verify the property of Theorem 3.6, i.e. they equidistribute on \mathbb{S}^2 provided that we extract a density 1 subsequence.

5. THE CASE OF THE TORUS

We move on to a case where the geodesic flow is not periodic but where the flow is still quite regular, namely the case of flat tori. In that case, we completely described the dynamics of the geodesic flow in paragraph 2.2.1. Geodesic flows on flat tori are in some sense the simplest examples of nondegenerate completely integrable systems. In fact, H_0 can be written as $\frac{1}{2} \sum_{j=1}^n f_j^2$ where $f_j(x, \xi) = \xi_j$ and one has $\{f_i, f_j\} = 0$ for every (i, j) and $df_1 \wedge df_2 \dots \wedge df_n \neq 0$. This exactly means that $F = (f_1, f_2, \dots, f_n)$ is a nondegenerate completely integrable system on $T^*\mathbb{T}^n \setminus 0$. The geodesic flow preserves the level sets of the moment map F , and we would like to analyze how this regular dynamics influences the structure of the set $\mathcal{M}(\Delta)$. Another important feature of this example is that the quantum Hamiltonian $-\hbar^2 \Delta$ can be written as

$$-\hbar^2 \Delta = \hat{F}_1^2 + \hat{F}_2^2 + \dots + \hat{F}_n^2,$$

where $\hat{F}_j = \text{Op}_\hbar^w(f_j) = -i\hbar \partial_{x_j}$. In particular, $[\hat{F}_i, \hat{F}_j] = 0$ for every (i, j) and we say that $(\hat{F}_1, \hat{F}_2, \dots, \hat{F}_n)$ is a quantum completely integrable system. For an introduction on these topics, we refer the reader to the book of Vũ Ngọc [129]. Another crucial property of the torus is its arithmetic structure that we already encountered in the proof of Theorem 2.7 for instance. In this section, we will try to illustrate both aspects of this problem. First, we will show how an arithmetic approach allows to improve Theorem 3.6. Then, we will make use of semiclassical methods and of the complete integrability to improve (in the case of the torus) the observability estimate from Theorem 3.2.

5.1. Improving Theorem 3.6 by arithmetic means. We begin with the following improvement of Theorem 3.6 which was proved in [70]:

Theorem 5.1. *Suppose that \mathbb{T}^n is endowed with the Euclidean metric. Then, there exists some constant $C_n > 0$ such that for every orthonormal basis of Laplace eigenfunctions $(\psi_{h_j})_{j \geq 0}$ (i.e. solutions to (5)) and, for every a in $L^2(\mathbb{T}^n)$,*

$$\forall \hbar > 0, \quad \frac{1}{N_\hbar} \sum_{j: h_j \geq \hbar} \left| \int_{\mathbb{T}^n} a(x) |\psi_{h_j}(x)|^2 dx - \int_{\mathbb{T}^n} a(x) \frac{dx}{(2\pi)^n} \right|^2 \leq C_n \|a\|_{L^2(\mathbb{T}^n)}^2 \hbar.$$

The fact that this quantity goes to 0 follows from Theorem 3.6 and the improvement here is that we can provide an explicit rate of convergence. If we had followed a semiclassical strategy, then the best bound that can be obtained seems to be of order $\hbar^{\frac{2}{3}}$ [69]. In [88],

Lester and Rudnick showed how to make use of arithmetic results to improve¹⁰ this up to the threshold \hbar modulo some constant that depends on a certain number of derivatives of a . Then, in [70], together with Hezari, we showed how to combine both approach to get the above result. The proof being rather elementary, we will present it as an illustration of what can be gained from these arithmetic methods.

Proof. For the sake of simplicity, let a be a smooth real valued function. First, it can be remarked that, the functions $\psi_{\hbar_j}(x)$ being trigonometric polynomials, we can write

$$S(a, \hbar) := \sum_{j:\hbar_j=\hbar} \left| \int_{\mathbb{T}^n} a(x) |\psi_{\hbar_j}(x)|^2 dx - \int_{\mathbb{T}^n} a(x) \frac{dx}{(2\pi)^n} \right|^2 = \sum_{j:\hbar_j=\hbar} \left| \int_{\mathbb{T}^n} a_{\hbar}(x) |\psi_{\hbar_j}(x)|^2 dx \right|^2,$$

where

$$a_{\hbar}(x) = \sum_{p:1 \leq \|p\| \leq 2\hbar^{-1}} \widehat{a}_p e^{ip \cdot x}.$$

In some sense, we are microlocalizing in a compact part of phase space as we did before. Then, we can perform the same trick as in the proof of Theorem 3.3, i.e. average by the Schrödinger flow $e^{it\Delta}$. For every $T > 0$, it gives us

$$S(a, \hbar) = \sum_{j:\hbar_j=\hbar} \left| \left\langle \psi_{\hbar_j}, \left(\frac{1}{T} \int_0^T e^{-it\Delta} a_{\hbar} e^{it\Delta} dt \right) \psi_{\hbar_j} \right\rangle \right|^2 \leq \sum_{j:\hbar_j=\hbar} \left\| \left(\frac{1}{T} \int_0^T e^{-it\Delta} a_{\hbar} e^{it\Delta} dt \right) \psi_{\hbar_j} \right\|^2,$$

where the second inequality follows from the Cauchy-Schwarz inequality. As the trace is independent of the choice of basis, this leads to

$$S(a, \hbar) \leq \sum_{k:\|k\|=\hbar=1} \left\| \left(\frac{1}{T} \int_0^T e^{-it\Delta} a_{\hbar} e^{it\Delta} dt \right) e_k \right\|^2,$$

where $e_k(x) = e^{2i\pi k \cdot x}$ for $k \in \mathbb{Z}^n$. We can now write

$$\left(\frac{1}{T} \int_0^T e^{-it\Delta} a_{\hbar} e^{it\Delta} dt \right) e_k = \sum_{p:1 \leq \|p\| \leq 2\hbar^{-1}} \widehat{a}_p \left(\frac{1}{T} \int_0^T e^{it(\|k+p\|^2 - \|k\|^2)} dt \right) e_{k+n}.$$

Then, from Plancherel's inequality, we find

$$(29) \quad S(a, \hbar) \leq (2\pi)^n \sum_{k:\|k\|=\hbar=1} \sum_{p:1 \leq \|p\| \leq 2\hbar^{-1}} |\widehat{a}_p|^2 \left| \frac{1}{T} \int_0^T e^{it(\|k+p\|^2 - \|k\|^2)} dt \right|^2.$$

Up to this point we followed exactly the steps of our proof of Theorem 3.3 and we made everything explicit thanks to the structure of the torus. In the proof of Theorem 3.3, we then took the limits $\hbar \rightarrow 0^+$ and $T \rightarrow +\infty$ (in this order). Here, we proceed differently and we first take the limit $T \rightarrow +\infty$ which is possible due to the specific structure of our problem. The drawback of doing this is that there will not be any dynamical interpretation

¹⁰They also prove such a bound for moments of order 1.

of this limit. Yet, the advantage is that it will reduce the problem to a diophantine problem that can be handled. More precisely, after letting $T \rightarrow +\infty$, one finds

$$(30) \quad S(a, \hbar) \leq \sum_{p: 1 \leq \|p\| \leq 2\hbar^{-1}} |\widehat{a}_p|^2 |\{k \in \mathbb{Z}^n : \|k\|\hbar = 1 \text{ and } \|k\| = \|k+p\|\}|.$$

This inequality holds for eigenfunctions corresponding to the eigenvalue \hbar^{-2} . If we sum over all eigenvalues $\leq \hbar^{-2}$, then we find

$$\begin{aligned} V(a, \hbar) &:= \frac{1}{N_\hbar} \sum_{j: \hbar_j \geq \hbar} \left| \int_{\mathbb{T}^n} a(x) |\psi_{\hbar_j}(x)|^2 dx - \int_{\mathbb{T}^n} a(x) \frac{dx}{(2\pi)^n} \right|^2 \\ &\leq \frac{1}{N_\hbar} \|a\|_{L^2}^2 \sup_{1 \leq \|p\|\hbar \leq 2} |\{k \in \mathbb{Z}^n : \|k\|\hbar \leq 1 \text{ and } \|k\| = \|k+p\|\}|. \end{aligned}$$

Recall from the Weyl law (2) that N_\hbar is of order \hbar^{-n} . Moreover, for every $1 \leq \|p\|\hbar \leq 2$, the number of lattice points we want to count is the number of lattice points $k \in \mathbb{Z}^n$ such that $\|k\|\hbar \leq 1$ and $\langle p, p - 2k \rangle = 0$. This means that $p - 2k$ lies on some hyperplane orthogonal to p . The number of lattice points with $\|k\|\hbar \leq 1$ in this hyperplane is of order $\mathcal{O}(\hbar^{1-n})$ for some constant that can be made uniform in p and this concludes the proof of the Theorem. \square

Remark 5.2. When $n = 2$, the reader can check that we can recover Theorem 2.7 from (30) – see [70] for details.

5.2. Observability on flat tori by semiclassical means. Theorem 2.7 told us that quantum limits are absolutely continuous with respect to the Lebesgue measure. Another important property is that they are uniformly observable on any nonempty open set ω [73, 74, 34, 94, 6, 35, 21]:

Theorem 5.3 (Unconditional observability on \mathbb{T}^n). *Suppose that \mathbb{T}^n is endowed with the Euclidean metric and that ω is a nonempty open set. Then, there exists some constant $c_\omega > 0$ such that, for every ψ_λ solution to*

$$-\Delta \psi_\lambda = \lambda^2 \psi_\lambda, \quad \|\psi_\lambda\|_{L^2} = 1,$$

one has

$$\int_\omega |\psi_\lambda(x)|^2 dx \geq c_\omega > 0.$$

In dimension 2, this Theorem was first proved by Jaffard [73] using the theory of lacunary Fourier series of Kahane and it can also be recovered by Jakobson's result on the structure of quantum limits in dimension 2 [74]. Still in dimension 2, Burq and Zworski recovered this result by more semiclassical methods [34, 35]. Then, Anantharaman and Macià extended this result in higher dimension [94, 6] by making a crucial use of the complete integrability of the system. Their proof also allowed to recover the absolute continuity of quantum limits on the torus (see Theorem 2.7) and to encompass the case of Schrödinger operators $-\frac{1}{2}\Delta + V$ with V that may be only in L^∞ – see also [21] for potentials with low regularity in dimension 2. A key ingredient of Anantharaman-Macià's proof is a second microlocalization

procedure that was subsequently used for more general completely integrable systems [3, 5]. Note that second microlocalization procedure were also used in previous works of Vasy and Wunsch to study the wavefront set properties of eigenfunctions for general completely integrable systems in dimension 2 [133, 128, 134].

The end of this section will be devoted to the proof of Theorem 5.3 following the strategy of Anantharaman and Macià [94, 6]. In dimension 2, it is not the fastest way to prove Theorem 5.3. For instance, a simple arithmetic proof follows from the works of Bourgain and Rudnick [24, Eq. (1.13-18)] by replacing in that reference the normalized arc-length measure $d\sigma(x)$ by any smooth (normalized) density $\rho(x)dx$. Yet our semiclassical approach has the advantage that it can be generalized in higher dimensions and that it illustrates the classical-quantum correspondence in a rather explicit manner. Compared with [94, 6], we make use of an (elementary) arithmetic argument in the final step in order to simplify the exposition but this is not crucial – see [6, §7] for a more analytical approach.

In order to prove this Theorem, we already observed that it is sufficient to prove the following proposition:

Proposition 5.4. *Suppose that \mathbb{T}^2 is endowed with the Euclidean metric and that ω is a nonempty open set. Then, there exists some constant $c_\omega > 0$ such that, for every $\mu \in \mathcal{M}(\Delta)$,*

$$\mu(\omega \times \mathbb{S}^1) \geq c_\omega.$$

Proof. We proceed by contradiction. By compactness and up to a diagonal argument, it is sufficient to suppose that there exists some $\mu \in \mathcal{M}(\Delta)$ such that $\mu(\omega \times \mathbb{S}^1) = 0$ and to obtain a contradiction. We denote by $(\psi_h)_{h \rightarrow 0^+}$ the sequence of solutions to (5) used to construct μ .

Using the conventions of example 2.13, recall that

$$\Lambda_\xi := \{p \in \mathbb{Z}^2 : p \cdot \xi = 0\}.$$

Hence, we can split \mathbb{S}^1 in two parts by setting

$$\Omega := \{\xi \in \mathbb{S}^1 : \Lambda_\xi = \{0\}\}.$$

One has then the following decomposition of μ into two invariant measures:

$$\mu = \mu|_{\mathbb{T}^2 \times \Omega} + \mu|_{\mathbb{T}^2 \times \mathbb{S}^1 \setminus \Omega}.$$

From the ergodic properties of the geodesic flow on $\mathbb{T}^2 \times \Omega$ described in example 2.13, one knows that $\mu(\omega \times \mathbb{S}^1) = 0$ implies that $\mu|_{\mathbb{T}^2 \times \Omega} = 0$. Hence, the semiclassical measure is concentrated in the rational direction in momentum space:

$$\mu = \sum_{\xi \in \mathbb{S}^1 : \text{rk} \Lambda_\xi = 1} \mu|_{\mathbb{T}^2 \times \{\xi\}},$$

which is a countable sum. Now, on the one hand, as μ is a probability measure on $\mathbb{T}^2 \times \mathbb{S}^1$, there exists some $\xi_0 \in \mathbb{S}^1$ such that $\text{rk} \Lambda_{\xi_0} = 1$ and such that $\mu|_{\mathbb{T}^2 \times \{\pm \xi_0\}}$ is nontrivial. On the other hand, one knows that $\mu(\omega \times \mathbb{S}^1) = 0$. Still using the conventions of example 2.13, this implies that

$$(31) \quad \mu((\omega + \mathbb{T}(\xi_0)) \times \{\pm \xi_0\}) = 0.$$

Now, we can fix some smooth function $b_{\xi_0}(x)$ such that

- it is compactly supported inside the open set $\omega + \mathbb{T}(\xi_0)$,
- its Fourier decomposition is of the form

$$(32) \quad b_{\xi_0}(x) = \sum_{k \in \Lambda_{\xi_0}} \widehat{b}_k e^{ik \cdot x},$$

- it takes values in $[0, 1]$ and it does not identically vanish.

In particular, one has

$$(33) \quad \int_{\mathbb{T}^2 \times \{\pm \xi_0\}} b_{\xi_0}(x) d\mu(x, \xi) = 0.$$

In order to get our contradiction, we only need to analyze the semiclassical measure in the direction of ξ_0 and to test it against functions whose Fourier coefficients are in Λ_{ξ_0} . To that aim, we need to come back at the quantum level $\hbar > 0$ and to analyze the semiclassical (Wigner) distribution more precisely:

$$\langle w_{\hbar}, b_{\xi_0} \rangle = \langle \psi_{\hbar}, \text{Op}_{\hbar}^w(b_{\xi_0}) \psi_{\hbar} \rangle.$$

We introduce a smooth cutoff function $\chi : \mathbb{R} \rightarrow [0, 1]$ which is equal to 1 near 0 and to 0 outside a slightly larger neighborhood. We also fix some $R > 0$ (that will tend to $+\infty$ in the end) and we rewrite

$$\begin{aligned} \langle w_{\hbar}, b_{\xi_0} \rangle &= \left\langle \psi_{\hbar}, \text{Op}_{\hbar}^w \left(b_{\xi_0}(x) \chi^2 \left(\frac{\xi \cdot \xi_0^{\perp}}{R\hbar} \right) \right) \psi_{\hbar} \right\rangle \\ &+ \left\langle \psi_{\hbar}, \text{Op}_{\hbar}^w \left(b_{\xi_0}(x) \left(1 - \chi^2 \left(\frac{\xi \cdot \xi_0^{\perp}}{R\hbar} \right) \right) \right) \psi_{\hbar} \right\rangle, \end{aligned}$$

where ξ_0^{\perp} is the unit vector which is directly orthogonal to ξ_0 . This operation splits the Wigner distribution w_{\hbar} into two parts. The first part allows to analyze the mass of the eigenfunction in a neighborhood of size $R\hbar$ of the \mathbb{R} -vector space $\langle \Lambda_{\xi_0} \rangle$ generated by Λ_{ξ_0} . We say that we are performing a second microlocalization on the submanifold $\langle \Lambda_{\xi_0} \rangle$. The second term describes what happens away from this submanifold. Again, we will apply the Egorov Theorem which is exact in that case – see (16). Let us start with the first term

$$\langle w_{\hbar}^{\xi_0}, b_{\xi_0} \rangle := \left\langle \psi_{\hbar}, \text{Op}_{\hbar}^w \left(b_{\xi_0}(x) \left(1 - \chi^2 \left(\frac{\xi \cdot \xi_0^{\perp}}{R\hbar} \right) \right) \right) \psi_{\hbar} \right\rangle,$$

that we rewrite in the following manner

$$\begin{aligned}
\langle w_h^{\xi_0}, b_{\xi_0} \rangle &= \sum_{k \in \Lambda_{\xi_0}} \widehat{b}_k \left\langle \psi_h, \text{Op}_h^w \left(e^{ik \cdot x} \left(1 - \chi^2 \left(\frac{\xi \cdot \xi_0^\perp}{R\hbar} \right) \right) \right) \psi_h \right\rangle \\
&= \sum_{k \in \Lambda_{\xi_0}} \widehat{b}_k \left\langle \psi_h, \text{Op}_{R^{-1}}^w \left(e^{ik \cdot x} \left(1 - \chi^2 \left(\xi \cdot \xi_0^\perp \right) \right) \right) \psi_h \right\rangle \\
&= \sum_{k \in \Lambda_{\xi_0}} \widehat{b}_k \left\langle \psi_h, \text{Op}_{R^{-1}}^w \left(\left(\frac{1}{R} \int_0^R e^{ik \cdot (x+t\xi)} dt \right) \left(1 - \chi^2 \left(\xi \cdot \xi_0^\perp \right) \right) \right) \psi_h \right\rangle,
\end{aligned}$$

where the last equality follows from the exact Egorov formula (16) applied with $\hbar = R^{-1}$ and $t = R$. For $k \neq 0$ and ξ in the support of $(1 - \chi(\xi \cdot \xi_0^\perp))$, there exists some constant $c > 0$ such that $|k \cdot \xi| \geq c\|k\|$. Hence, from the Calderón-Vaillancourt Theorem (12), one has

$$\left\| \text{Op}_{R^{-1}}^w \left(\left(\frac{1}{R} \int_0^R e^{ik \cdot (x+t\xi)} dt \right) \left(1 - \chi \left(\xi \cdot \xi_0^\perp \right) \right) \right) \right\|_{L^2 \rightarrow L^2} \leq C \sum_{|\alpha| \leq D} R^{-\frac{|\alpha|}{2}} \frac{2\|k\|^{|\alpha|}}{cR\|k\|} C_\chi,$$

where $C_\chi > 0$ depends only on a finite number of derivatives of χ . Implementing this in our computation of $\langle w_h^{\xi_0}, b_{\xi_0} \rangle$ and using that $|\widehat{b}_k| = \mathcal{O}(\|k\|^{-\infty})$, we find that

$$(34) \quad \langle w_h, b_{\xi_0} \rangle = \widehat{b}_0 + \left\langle \psi_h, \text{Op}_h^w \left(\left(b_{\xi_0}(x) - \widehat{b}_0 \right) \chi^2 \left(\frac{\xi \cdot \xi_0^\perp}{R\hbar} \right) \right) \psi_h \right\rangle + \mathcal{O}(R^{-1}).$$

In some sense, we have been able to use one more time the averaging by the geodesic flow as, thanks to our truncation, the momentum variable ξ remained far from the direction ξ_0^\perp where b_{ξ_0} has all its Fourier coefficients. We now set

$$\langle b_{\xi_0} \rangle(x) = b_{\xi_0}(x) - \widehat{b}_0 = \sum_{k \in \Lambda_{\xi_0} \setminus 0} \widehat{b}_k e^{ik \cdot x},$$

and we want to study the properties of

$$\begin{aligned}
\langle w_{h, \xi_0}, b_{\xi_0} \rangle &:= \left\langle \psi_h, \text{Op}_h^w \left(\langle b_{\xi_0} \rangle(x) \chi^2 \left(\frac{\xi \cdot \xi_0^\perp}{R\hbar} \right) \right) \psi_h \right\rangle \\
&= \left\langle \psi_h, \text{Op}_{R^{-1}}^w \left(\langle b_{\xi_0} \rangle(x) \chi^2 \left(\xi \cdot \xi_0^\perp \right) \right) \psi_h \right\rangle.
\end{aligned}$$

In other words, we study the properties of the Wigner distribution w_h in a neighborhood of size $\sim R\hbar$ of $\langle \Lambda_{\xi_0} \rangle$. Here, we will not be able to use invariance by the geodesic flow: we have to proceed differently. Thanks to the composition rule (10) for pseudodifferential operators, one first observes that

$$\langle w_{h, \xi_0}, b_{\xi_0} \rangle = \int_{\mathbb{T}^2} \langle b_{\xi_0} \rangle(x) \left| \chi \left(-\frac{i\xi_0^\perp \cdot \partial_x}{R} \right) \psi_h(x) \right|^2 dx + \mathcal{O}(R^{-1}).$$

Let us now decompose $\chi\left(\frac{-i\xi_0^\perp \cdot \partial_x}{R}\right)\psi_h(x)$, i.e.

$$(35) \quad \chi\left(-\frac{i\xi_0^\perp \cdot \partial_x}{R}\right)\psi_h(x) = \sum_{k: \|k\|_h=1} \chi\left(\frac{k \cdot \xi_0^\perp}{R}\right) \widehat{\psi}_h(k) e^{ik \cdot x}.$$

This means that we have a trigonometric polynomial whose coefficients lie on two arcs of length $\asymp R$ (as $\hbar \rightarrow 0^+$) on the circle of radius \hbar^{-1} centered at 0. There are in fact at most 4 lattice points having this property.

Remark 5.5. This last observation follows from a classical geometric fact due to Jarnik [77]. Let k_1, k_2 and k_3 be three distinct elements of \mathbb{Z}^2 lying on a circle of radius T . They form a triangle whose area \mathcal{A} is at least equal to $1/2$. Then, one has the classical geometric formula relating the area of a triangle with the radius T of its circumscribed triangle:

$$2T \leq 4AT = \|k_1 - k_2\| \|k_2 - k_3\| \|k_3 - k_1\| \leq \max\{\|k_i - k_j\|\}^3.$$

In particular, there are at most two lattice points on an arc of length $(2T)^{1/3}$.

Hence, there are at most four terms in the sum (35). Thus, using Riemann-Lebesgue lemma and up to an extraction, we can suppose that $\left|\chi\left(\frac{\langle -i\partial_x, \xi_0^\perp \rangle}{R}\right)\psi_h(x)\right|^2 dx$ converges weakly (as $\hbar \rightarrow 0^+$) to some finite (nonnegative) measure of the form

$$d\nu_{\xi_0, R}(x) = (\widehat{\nu}_0(\xi_0, R) + \widehat{\nu}_l^+(\xi_0, R) \cos(l \cdot x) + \widehat{\nu}_l^-(\xi_0, R) \sin(l \cdot x)) dx,$$

where $\|l\| \neq 0$ is a lattice point of length $\mathcal{O}(R)$ and $\widehat{\nu}_l^*(\xi_0, R) \in \mathbb{R}$ for every l . Then, up to another extraction, this measure converges (as $R \rightarrow +\infty$) to some finite and nonnegative measure of the form

$$d\nu_{\xi_0}(x) = (\widehat{\nu}_0(\xi_0) + \widehat{\nu}_l^+(\xi_0) \cos(l \cdot x) + \widehat{\nu}_l^-(\xi_0) \sin(l \cdot x)) dx,$$

with l that may be different from before. We now pass to the limits $\hbar \rightarrow 0^+$ and $R \rightarrow +\infty$ (in this order) in (34):

$$\int_{\mathbb{T}^2 \times \mathbb{S}^1} b_{\xi_0}(x) d\mu(x, \xi) = \widehat{b}_0(1 - \nu_{\xi_0}(\mathbb{T}^2)) + \int_{\mathbb{T}^2} b_{\xi_0}(x) d\nu_{\xi_0}(x).$$

As we already observed it, μ is carried by the rational directions in the ξ -variable. Hence,

$$\sum_{\xi: \text{rk}\Lambda_\xi=1} \int_{\mathbb{T}^2 \times \{\xi\}} b_{\xi_0}(x) d\mu(x, \xi) = \widehat{b}_0(1 - \nu_{\xi_0}(\mathbb{T}^2)) + \int_{\mathbb{T}^2} b_{\xi_0}(x) d\nu_{\xi_0}(x).$$

Using the dynamical properties of the geodesic flow described in example 2.13 and the φ^t -invariance of $\mu|_{\mathbb{T}^2 \times \{\xi\}}$, we can deduce that there are only two terms in the sum on the left hand side, i.e.

$$0 = \int_{\mathbb{T}^2 \times \{\pm\xi_0\}} b_{\xi_0}(x) d\mu(x, \xi) = \widehat{b}_0(1 - \nu_{\xi_0}(\mathbb{T}^2)) + \int_{\mathbb{T}^2} b_{\xi_0}(x) d\nu_{\xi_0}(x).$$

As ν_{ξ_0} is a nonnegative measure, as $b_{\xi_0} \geq 0$ and as $\widehat{b}_0 \neq 0$, we deduce that ν_{ξ_0} is a probability measure. Moreover, we know that its density is a trigonometric polynomial

which contradicts the fact that $\int_{\mathbb{T}^2} b_{\xi_0}(x) d\nu_{\xi_0}(x) = 0$. Hence, one cannot have $\mu(\omega \times \mathbb{S}^1) = 0$ which concludes the proof of the Proposition. \square

6. THE ERGODIC CASE

We will now consider the ergodic case which has attracted the more attention since the seminal works of Šnirel'man [117, 116], Zelditch [135] and Colin de Verdière [40] notably through the Quantum Unique Ergodicity Conjecture of Rudnick and Sarnak [109]. Questions in that direction are part of the more general question of Quantum Chaos which tries to describe the manifestation of the chaotic features of a system from classical mechanics on its quantum counterpart. This kind of problems can be traced back to the early days of quantum mechanics where the question of the quantization of nonintegrable systems was already raised [53].

6.1. Quantum (unique) ergodicity. Recall from Theorem 3.5 that, if the Liouville measure is ergodic, then, for any orthonormal basis of Laplace eigenfunctions, most of the eigenfunctions are equidistributed inside S^*M in the semiclassical limit. This raises naturally the question of determining if there could be other accumulation points. In [40], Colin de Verdière states the following:

Le problème le plus intéressant est de savoir si on peut s'affranchir de la condition d'extraire une sous-suite de densité 1: quelles sont les valeurs d'adhérence de la suite des w_{\hbar_j} pour la convergence vague? Ces mesures sont nécessairement invariantes par le flot géodésique mais il y a une foule de telles mesures singulières (par exemple, si γ est une géodésique fermée). Il me semble que ces mesures pour des géodésiques fermées des surfaces de Riemann à courbure -1 ne peuvent pas être des limites vagues d'une suite ψ_{\hbar_j} . De toutes les façons, il y a d'autres mesures portées par des ensembles de dimension de Hausdorff > 1 .

A few years later, Colin de Verdière and Parisse showed that Laplace eigenfunctions cannot concentrate too fast on a closed hyperbolic geodesic of φ^t [42]. Recall that a closed hyperbolic geodesic γ of period T_γ is a closed orbit of the geodesic flow $\varphi^t : S^*M \rightarrow S^*M$ such that $d\varphi^{T_\gamma}|_\gamma$ has 1 as an eigenvalue of multiplicity 1 and such that all its other eigenvalues are of modulus $\neq 1$. This result was generalized by Toth-Zelditch [126] and Christianson [37, 38]. More precisely, following these references, one has

Theorem 6.1. *Let γ be closed hyperbolic geodesic of φ^t and let $a \in \mathcal{C}_c^\infty(T^*M, [0, 1])$ which is equal to 0 near γ and to 1 outside a slightly bigger neighborhood (depending on γ). Then, there exists some $c_{a,\gamma} > 0$ such that, for any sequence $(\psi_{\hbar})_{\hbar \rightarrow 0^+}$ of solutions to (5),*

$$\liminf_{\hbar \rightarrow 0^+} |\log \hbar| \langle \psi_{\hbar}, \text{Op}_{\hbar}(a)\psi_{\hbar} \rangle \geq c_{a,\gamma} > 0.$$

Colin de Verdière and Parisse also showed that this Theorem is sharp by constructing an explicit example where a sequence of eigenfunctions concentrate on a closed hyperbolic geodesic at this rate [42]. Note that this Theorem does not require anything on the global properties of the flow. In particular, the Liouville measure is not necessarily ergodic and it may happen that the Quantum Ergodicity property is not satisfied on the manifold. Below,

we will give a proof of this result which is valid for more general invariant hyperbolic subsets satisfying some appropriate smallness conditions that can be expressed in terms of classical dynamical quantities.

At the same period, this kind of questions was also addressed by Rudnick and Sarnak in the context of arithmetic surfaces [109]. In that case, one can consider a subset $\mathcal{M}_{\text{Hecke}}(\Delta_g)$ of $\mathcal{M}(\Delta_g)$ which is made of semiclassical measures obtained through a sequence of joint eigenfunctions of Δ_g and of the Hecke operators. Recall that, on an arithmetic surface, one can define a family of operators $(T_m)_{m \in \mathbb{Z}_+}$ acting on $L^2(M)$ and commuting with the Laplacian. They are called Hecke operators and, as they commute with the Laplacian, it is then natural to consider a joint basis of eigenfunctions and to expect that they have stronger properties. In that direction, Rudnick and Sarnak proved [109]:

Theorem 6.2. *Let M be a compact arithmetic surface and let μ be an element of $\mathcal{M}_{\text{Hecke}}(\Delta_g)$. Then, for any closed orbit γ of the geodesic flow, one has*

$$\mu(\gamma) = 0.$$

This result relies on arithmetic methods and it shows that joint eigenfunctions of Δ_g and of the Hecke operators cannot concentrate on closed orbits of the geodesic flow. In the same article, Rudnick and Sarnak also conjectured that a much stronger property holds:

Conjecture 6.3 (Quantum Unique Ergodicity). *Let (M, g) be a negatively curved manifold. Then,*

$$(36) \quad \mathcal{M}(\Delta_g) = \{L\}.$$

This conjecture remains widely open even if important progress have been made over the last twenty years. The end of these notes is devoted to a description of these results. The level of technicality being much more important than in the previous sections, we will mostly (but not always) give sketch of proofs in order to illustrate some of the main ideas behind these results. We refer the reader to the literature for more details.

6.2. Exceptional subsequences. We start with negative results which illustrate the importance of the geometric assumptions for the result to hold. First, we will begin with simple toy models of quantum mechanics which indicate that (36) cannot be only a consequence of the hyperbolic properties of the geodesic flow on negatively curved manifolds. Then, we will show that ergodicity of L_1 is not enough to get (36).

6.2.1. Quantum maps. A popular model in Quantum Chaos is the one of quantum maps. The idea is to consider a compact symplectic manifold (M_0, ω_0) and a (quantizable) symplectomorphism T_0 acting on it. The difference with the geodesic flow acting on T^*M is that now the phase space is compact which makes the analysis in some sense easier even if Laplace eigenfunctions were asymptotically concentrated on a compact part of phase space. The simplest example is the one given by a symplectic linear map A in $SL_{2n}(\mathbb{Z})$ acting on the torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ endowed with its canonical symplectic structure [26]. Examples of such maps are

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A := \begin{pmatrix} B & 0 \\ 0 & (B^T)^{-1} \end{pmatrix},$$

where $B \in SL_n(\mathbb{Z})$. As before, we shall denote by $\mathcal{M}(A)$ the set of A -invariant measures, e.g. the Lebesgue on $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ or the Dirac measure at a fixed point of A . If 1 is not an eigenvalue of A , there is a natural way to quantize the symplectic dynamical system $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}, A)$ [26]. For every positive integer $N = (2\pi\hbar)^{-1}$, this gives rise to a finite dimensional Hilbert space \mathcal{H}_N of dimension N^n on which one can define a quantization procedure

$$a \in \mathcal{C}^\infty(\mathbb{R}^{2n}/\mathbb{Z}^{2n}) \mapsto \text{Op}_\hbar^w(a) \in \mathcal{L}(\mathcal{H}_N)$$

satisfying the same properties as in paragraph 2.5. Instead of quantizing the geodesic flow, we are now quantizing A which yields some unitary matrix \widehat{A}_N acting on \mathcal{H}_N and verifying an Egorov Theorem which is exact, i.e.

$$\forall a \in \mathcal{C}^\infty(\mathbb{R}^{2n}/\mathbb{Z}^{2n}), \quad \widehat{A}_N^{-1} \text{Op}_\hbar^w(a) \widehat{A}_N = \text{Op}_\hbar^w(a \circ A).$$

Eigenfunctions of the Laplacian are replaced by the eigenvectors of \widehat{A}_N and, if A is ergodic (e.g. if A is an hyperbolic matrix), one can prove an analogue of the Quantum Ergodicity Theorem [26] and raise similar questions. As above, we can define

- the set $\mathcal{M}(\widehat{A})$ of accumulation points of the Wigner distributions as $N \rightarrow +\infty$. Again, one has $\mathcal{M}(\widehat{A}) \subset \mathcal{M}(A)$.
- the subset $\mathcal{M}_{\text{Hecke}}(\widehat{A})$ of (Hecke) semiclassical measures which are obtained through a sequence of joint eigenfunctions of \widehat{A}_N and of a certain group of isometries associated with A [82].

If the matrix A is hyperbolic, the dynamical situation is close to the one of negatively curved manifolds (ergodicity, strong mixing properties) and one can ask if the Quantum Unique Ergodicity property is true in that framework. In dimension 2, Kurlberg and Rudnick proved that Hecke eigenfunctions satisfy this property [82]. On the other hand, De Bièvre, Faure and Nonnenmacher constructed explicit examples of non-Hecke eigenfunctions for which this property fails [55]. More precisely, these two results read

Theorem 6.4. *Suppose that*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then, one has

- (1) $\mathcal{M}_{\text{Hecke}}(\widehat{A}) = \{\text{Leb}\}$,
- (2) $\frac{1}{2}\delta_0 + \frac{1}{2}\text{Leb} \in \mathcal{M}(\widehat{A})$. *In particular,*

$$\mathcal{M}(\widehat{A}) \neq \{\text{Leb}\}.$$

This Theorem shows that the chaotic properties of A (e.g. its hyperbolicity) are not enough to obtain Quantum Unique Ergodicity. In particular, if Conjecture 6.3 is true, then it has to use other features of geodesic flow and of the Laplace eigenfunctions.

Remark 6.5. The counterexample obtained in [55] is constructed through a subsequence of semiclassical parameters \hbar_N for which \widehat{A}_N has large spectral multiplicity in its spectrum. More precisely, $\widehat{A}_N^{T_N} = \text{Id}_{\mathcal{H}_N}$ for some $T_N \simeq 2 \ln N / \chi_{\max} \ll \dim \mathcal{H}_N$ where χ_{\max} is the

positive Lyapunov exponent of A . This periodicity is exploited by the authors to propagate a coherent state $|0, N\rangle$ microlocalized in a small ball of radius $N^{-1/2}$ centered at the fixed point 0 of A . Then, they set

$$\psi_N := \sum_{k=-T_N/2}^{T_N/2} \widehat{A}_N^k |0, N\rangle.$$

The time $T_N/4$ is referred as the Ehrenfest time in the physics literature and it corresponds to the scale of times for which the semiclassical approximation remains valid. Here, it means that, on the interval $[-T_N/4, T_N/4]$, the propagated coherent state remains microlocalized near 0 which is a fixed point of A . This part of the coherent state will contribute to the part $\frac{1}{2}\delta_0$ of the semiclassical measure. On the interval $[T_N/4, T_N/2]$ (similarly on $[-T_N/2, -T_N/4]$), the coherent state will begin to spread out over the torus and this will lead to the part $\frac{1}{2}\text{Leb}$ of the measure. Note that the choice of the fixed point 0 is rather arbitrary and we would choose a point x_0 belonging to some periodic orbit of A . In that case, the singular part of the measure will be the Dirac measure along this orbit.

Regarding Remark 6.5, it could be thought that spectral degeneracies would be the reason for the failure of Quantum Ergodicity Theorem. In fact, the Hecke operators of Kurlberg and Rudnick allow to reduce the multiplicity of the eigenvalues by considering a joint spectrum and this leads to an unique semiclassical measure. However, the situation is not as simple as was shown by Kelmer in the case of higher dimensional tori [79]:

Theorem 6.6. *Suppose that*

$$A = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

is an hyperbolic and symplectic matrix with distinct eigenvalues. Suppose also that $EF^T = GH^T = 0 \pmod{2}$. Then, a necessary and sufficient condition for having

$$\mathcal{M}_{\text{Hecke}}(\widehat{A}) = \{\text{Leb}\}$$

is that there is no A -invariant subspace $\{0\} \neq E_0 \subset \mathbb{Q}^{2n}$ which is isotropic¹¹ with respect to the symplectic form. Moreover, for any A -invariant subspace $E_0 \subset \mathbb{Q}^{2n}$ which is isotropic with respect to the symplectic form,

$$\text{Leb}_{X_0} \in \mathcal{M}_{\text{Hecke}}(\widehat{A}),$$

where

$$X_0 := \{x \in \mathbb{R}^{2n}/\mathbb{Z}^{2n} : \forall p \in E_0 \cap \mathbb{Z}^{2n}, p \cdot x \in \mathbb{Z}\}.$$

Remark 6.7. In order to illustrate this Theorem, one can take the example of

$$A = \begin{pmatrix} B & 0 \\ 0 & (B^T)^{-1} \end{pmatrix},$$

¹¹It means that the restriction of the symplectic form to E_0 vanishes.

where B is an hyperbolic matrix in $SL_n(\mathbb{Z})$. In that case, the Theorem does not exactly apply as eigenvalues may have multiplicity ≥ 2 . Yet, by a direct calculation, one can still verify that $\mathcal{M}_{\text{Hecke}}(\widehat{A}) \neq \{\text{Leb}\}$ and that $\text{Leb}_{\mathbb{R}^n/\mathbb{Z}^n \times \{0\}} \in \mathcal{M}_{\text{Hecke}}(\widehat{A})$.

This Theorem shows that Hecke eigenfunctions can also fail to equidistribute. Moreover, it gives examples of semiclassical measures that are concentrated on a submanifold of $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$. In particular, in the case of (chaotic) quantum maps, one cannot obtain an improvement of Proposition 3.1 that would state observability for any open set $\omega \neq \emptyset$.

6.2.2. Bunimovich stadium. In their Quantum Unique Ergodicity conjecture 6.3, Rudnick and Sarnak carefully stated that it should hold on negatively curved manifolds. This allows to exclude the case of certain nonpositively curved manifolds where the Liouville measure is ergodic for the geodesic flow. In that case, motivated by numerical experiments on the related model of the Bunimovich stadium, it was in fact widely believe that Quantum Unique Ergodicity should fail. Let us recall this model. For later purpose, we fix some parameter¹² $\omega \in [1, 2]$ and we define the Bunimovich stadium as the following domain of \mathbb{R}^2 :

$$D_\omega := ([-\omega, \omega] \times [-\pi/2, \pi/2]) \cup \{(x, y) \in \mathbb{R}^2 : (x \pm \omega)^2 + y^2 \leq \pi^2/4\}.$$

The analogue of the geodesic flow is the billiard flow φ_ω^t acting on $D_\omega \times \mathbb{S}^1$ with Descartes' laws for the reflection at the boundary. This flow preserves the Lebesgue measure L_ω on $D_\omega \times \mathbb{S}^1$ which is ergodic thanks to the work of Bunimovich [32]. In fact, such a flow falls in the category of nonuniformly hyperbolic flows (as the geodesic flow on nonpositively curved surfaces). One can then consider the eigenvalue problem

$$(37) \quad -\hbar^2 \Delta \psi_\hbar = \psi_\hbar, \quad \psi_\hbar|_{\partial D_\omega} = 0, \quad \|\psi_\hbar\|_{L^2} = 1,$$

and define the set of semiclassical measures $\mathcal{M}(\Delta_{D_\omega}) \subset \mathcal{M}(\varphi_\omega^t)$ [60]. In that case, Gérard-Leichtnam and Zelditch-Zworski proved that the Quantum Ergodicity property holds [60, 143]. Moreover, it was proved in [60, 66] that, for a sequence of eigenfunctions $(\psi_\hbar)_{\hbar \rightarrow 0^+}$ verifying the Quantum Ergodicity property, then the boundary values $|\hbar \partial_n \psi_\hbar(s)|^2 d\sigma(s)$ become equidistributed on ∂D_ω , where σ is the Lebesgue measure on ∂D_ω . In that framework, Hassell proved the following [65]:

Theorem 6.8. *For a.e. $\omega \in [1, 2]$, one has*

$$\mathcal{M}(\Delta_{D_\omega}) \neq \{L_\omega\}.$$

In particular, this Theorem shows that ergodicity is not enough to expect Quantum Unique Ergodicity to hold. This result was generalized by Hassell and Hillairet to other contexts where the manifold with (or without) boundary has a rectangular (or cylindrical) part with various boundary conditions [65].

Sketch of Proof. The idea is to apply arguments from analytic perturbation theory. More precisely, one can order the eigenvalues $\hbar_j(\omega)^{-2}$ in a nondecreasing order and verify that

¹²The choice of the interval $[1, 2]$ is arbitrary.

they are piecewise smooth. Then, one of the key observation is the Hadamard variational formula which states that, for every ω where it makes sense,

$$\hbar_j(\omega)^2 \frac{d}{d\omega} (\hbar_j(\omega)^{-2}) = - \int_{\partial D_\omega} \rho_\omega |\hbar_j(\omega) \partial_n \psi_{\hbar_j}(\omega)|^2 d\sigma$$

where $(\psi_{\hbar_j}(\omega))_{j \geq 1}$ is the corresponding orthonormal basis and where

$$\rho_\omega(x, y) = \text{sgn}(x) n_\omega^1(x, y),$$

with $n_\omega(x, y) = n_\omega^1(x, y) \partial_x + n_\omega^2(x, y) \partial_y$ being the normal to ∂D_ω at (x, y) . Hence, according to [60, 66], if we suppose that $\mathcal{M}(\Delta_{D_\omega}) = \{L_\omega\}$ for some $\omega \in [1, 2]$, then the boundary values are equidistributed and we can deduce that

$$\limsup_{j \rightarrow +\infty} \hbar_j(\omega)^2 \frac{d}{d\omega} (\hbar_j(\omega)^{-2}) \leq -c' < 0,$$

for some c' independent of ω . Now, from the Weyl law (2), one knows that $h_j(\omega) \approx c(\omega)j^{-1/2}$ for some positive $c(\omega) > 0$. This implies that, for some uniform constant $c > 0$ and for j large enough,

$$(38) \quad \frac{d}{d\omega} \hbar_j(\omega)^{-2} \leq -cj.$$

Introduce now

$$G := \{\omega \in [1, 2] : \mathcal{M}(\Delta_{D_\omega}) = \{L_\omega\}\},$$

and, for $n \in \mathbb{Z}_+^*$ and $\alpha > 0$

$$N_\omega(\alpha, n) := |\{j : \hbar_j(\omega)^{-2} \in [n^2 - \alpha, n^2 + \alpha]\}|.$$

Writing things a little bit informally, one has

$$\int_G N_\omega(\alpha, n) d\omega \lesssim \sum_{c_1 n^2 \leq j \leq c_2 n^2} \text{Leb}(\{\omega \in G : \hbar_j(\omega)^{-2} \in [n^2 - \alpha, n^2 + \alpha]\}),$$

for some positive constant c_1, c_2 related to the constant appearing in the Weyl law. Now from (38), one knows that $\hbar_j(\omega_2)^{-2} - \hbar_j(\omega_1)^{-2} \gtrsim n^2(\omega_1 - \omega_2)$ which yields

$$\int_G N_\omega(\alpha, n) d\omega \lesssim \sum_{c_1 n^2 \leq j \leq c_2 n^2} \frac{2\alpha}{n^2} = \mathcal{O}_\alpha(1),$$

where the involved constant is uniform in n even if we can note that it depends on α . If the Lebesgue measure of G was positive, then it would mean that, on a set of positive measure inside G , $N_\omega(\alpha, n_k)$ is uniformly bounded along a certain subsequence of integer $n_k \rightarrow +\infty$ – see [65, p. 613] for details. In other words, for any α , one can find some $\omega \in G$ and some sequence of integers $n_k \rightarrow +\infty$ such that the number of eigenvalues of Δ_{D_ω} inside $[n_k^2 - \alpha, n_k^2 + \alpha]$ is uniformly bounded as $k \rightarrow +\infty$ by some constant M_α .

Following earlier arguments of Heller-O'Connor [100] and Zelditch [140] – see also [65, §1], this leads to a contradiction with the fact that $\mathcal{M}(\Delta_{D_\omega}) = \{L_\omega\}$. To see this, we define

$$\psi_{\hbar_k}(x, y) := \sin(n_k y)\chi(x) \text{ if } n_k \text{ is even and } \psi_{\hbar_k}(x) := \cos(n_k y)\chi(x) \text{ otherwise,}$$

where χ is a smooth function which is compactly supported in $[-1, 1]$ and which is chosen such that $\|\psi_{\hbar_k}\|_L^2 = 1$. This does not define a sequence of solutions to (37) but, letting $\hbar_k = n_k^{-1}$, one has the quasimode equation

$$\| -\hbar_k^2 \Delta \psi_{\hbar_k} - \psi_{\hbar_k} \| \leq \frac{\alpha \hbar_k^2}{2}.$$

Even if it is not an eigenmode, one can write the spectral decomposition of the Laplacian to get

$$\left\| \mathbf{1}_{[1-\alpha\hbar_k^2, 1+\alpha\hbar_k^2]}(-\hbar_k^2 \Delta_{D_\omega}) \psi_{\hbar_k} \right\|_{L^2}^2 \geq \frac{3}{4}.$$

In particular, there exist some exact eigenmodes $\tilde{\psi}_k$ such that $|\langle \psi_{\hbar_k}, \tilde{\psi}_k \rangle| \geq (3/4M_\alpha)^{1/2}$ for every $k \geq 1$. We denote by h_k^{-2} the corresponding eigenvalue. Following example 2.15, one can now compute the semiclassical measure of the quasimodes $(\psi_{\hbar_k})_{k \geq 1}$. Their limit density is given by

$$\mu(x, y, \xi, \eta) = c |\chi(x)|^2 \delta_0(\xi, \eta \pm 1),$$

for some positive constant $c > 0$. Such states are called *bouncing ball modes*, in the sense that they are localized on the closed orbits of the billiard flow located in the rectangular part of the domain. Now take a symbol a which is supported in a neighborhood of the support of μ (meaning near the vertical direction) and which is equal to 1 near this support. Hence, for an appropriate choice of such a , one has, for k large enough,

$$\langle \psi_{\hbar_k}, \text{Op}_{\hbar_k}(|1-a|^2) \psi_{\hbar_k} \rangle \leq \frac{3}{16M_\alpha}.$$

We now write

$$\begin{aligned} |\langle \tilde{\psi}_k, \text{Op}_{\hbar_k}(|a|^2) \tilde{\psi}_k \rangle| &= \| \text{Op}_{\hbar_k}(a) \tilde{\psi}_k \|_{L^2}^2 + o(1) \\ &\geq |\langle \text{Op}_{\hbar_k}(a) \tilde{\psi}_k, \psi_{\hbar_k} \rangle|^2 + o(1) \\ &\geq |\langle \tilde{\psi}_k, \psi_{\hbar_k} \rangle - \langle \tilde{\psi}_k, (\text{Id} - \text{Op}_{\hbar_k}(a)) \psi_{\hbar_k} \rangle|^2 + o(1) \\ &\geq \left(\sqrt{\frac{3}{4M_\alpha}} - |\langle \psi_{\hbar_k}, \text{Op}_{\hbar_k}(|1-a|^2) \psi_{\hbar_k} \rangle|^{\frac{1}{2}} \right)^2 + o(1) \geq \frac{3}{16M_\alpha} + o(1), \end{aligned}$$

where the first and the last line are consequences of the composition rule (10) for pseudo-differential operators and where we use the Cauchy-Schwarz inequality on the second and the last line. This shows that the sequence $(\tilde{\psi}_k)_{k \geq 1}$ puts asymptotically some mass in the vertical direction and this yields the expected contradiction as D_ω is supposed to satisfy the Quantum Unique Ergodicity property. \square

6.3. Towards quantum unique ergodicity. Now that we have described these several obstructions to Quantum Unique Ergodicity, we can come back to the case of negatively curved manifolds where this conjecture was initially stated. Before discussing the current state of the art, we need to begin with the introduction of tools from hyperbolic dynamical systems.

6.3.1. Hyperbolic dynamics, entropy and pressure. In the following, we fix a Riemannian metric on S^*M , e.g. the Sasaki metric g_S induced by g [111]. One can define the notion of hyperbolic subset [1]:

Definition 6.9. Let Λ be a compact subset of S^*M which is invariant by the geodesic flow φ^t . We say that Λ is **hyperbolic** (for the flow φ^t) if there exist some $C > 0$, $\chi > 0$ and a family of subspaces $E_u(\rho), E_s(\rho) \subset T_\rho S^*M$ (for every $\rho = (x, \xi) \in \Lambda$) satisfying the following properties, for every ρ in Λ and for every $t \geq 0$,

- (1) $T_\rho S^*M = \mathbb{R}X_{H_0}(\rho) \oplus E_u(\rho) \oplus E_s(\rho)$ with $X_{H_0}(\rho) = \frac{d}{dt}(\varphi^t(\rho))|_{t=0}$,
- (2) $d_\rho \varphi^t E_{u/s}(\rho) = E_{u/s}(\varphi^t(\rho))$,
- (3) for every v in $E_u(\rho)$, $\|d_\rho \varphi^{-t} v\| \leq C e^{-\chi t} \|v\|$,
- (4) for every v in $E_s(\rho)$, $\|d_\rho \varphi^t v\| \leq C e^{-\chi t} \|v\|$.

If $\Lambda = S^*M$, we say that the flow φ^t has the **Anosov property**.

These assumptions roughly tell us that two points which are close will tend to separate from each other at an exponential rate. In that sense, the dynamical system $\varphi^t|_\Lambda$ is very sensitive to perturbations and it is a good model of a classical chaotic system. Note that the simplest example of such an hyperbolic subset is given by an hyperbolic closed geodesic as in Theorem 6.1. The main example where $\Lambda = S^*M$ is the geodesic flow on a negatively curved manifold. In that case, the flow has strong chaotic properties: ergodicity and mixing of the Liouville measure [10]. In that framework, the Liouville measure was also shown to be exponentially mixing [91].

We can define the **unstable Jacobian** at a point $(x, \xi) \in S^*M$ as follows¹³:

$$J^u(x, \xi) := \left| \det \left(d_{\varphi^1(x, \xi)} \varphi^{-1} |_{E^u(\varphi^1(x, \xi))} \right) \right|,$$

where we endow the unstable spaces $E^u(x, \xi)$ and $E^u(\varphi^1(x, \xi))$ with the Sasaki metric in order to defined the Jacobian. It defines a continuous function on Λ [78]. For every $\epsilon > 0$ and every $T \geq 0$, we also define the Bowen ball centered at $(x, \xi) \in S^*M$

$$B_g(x, \xi; \epsilon, T) := \{(x', \xi') \in S^*M : \forall 0 \leq t \leq T, d_g(\varphi^t(x, \xi), \varphi^t(x', \xi')) < \epsilon\},$$

where d_g is the Riemannian distance induced by g on S^*M . These are exactly the points of S^*M whose trajectories remain close to the one of (x, ξ) up to time T . A subset $F \subset \Lambda$ is said to be (ϵ, T) -separated if for every (x, ξ) and (x', ξ') in F , $(x', \xi') \in B_g(x, \xi; \epsilon, T)$ implies $(x, \xi) = (x', \xi')$. Our goal is now to measure the size of an hyperbolic set Λ by taking into

¹³The choice of the time 1 and of the metric are arbitrary and the reader can check that it will not change the following.

account the dynamical properties of the flow. More precisely, for every $0 \leq s \leq 1$, we define the **topological pressure** of the set Λ (with respect to $-s \log J^u$) as follows [130]:

$$P_{\text{top}}(\Lambda, s) := \lim_{\epsilon \rightarrow 0} \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \sup_F \left\{ \sum_{\rho \in F} \exp \left(s \int_0^T \log J^u \circ \varphi^t(\rho) dt \right) \right\},$$

where the supremum is taken over all the (ϵ, T) -separated sets F . In order to understand the meaning of this function, we can note that the quantity $\exp \left(s \int_0^T \log J^u \circ \varphi^t(\rho) dt \right)$ roughly measures the Riemannian volume (to the power s) of the Bowen ball centered at (x, ξ) . Hence, this quantity looks a little bit like the ones used to define the Hausdorff dimension of a set [102] except that we consider dynamical balls rather than classical balls.

Example 6.10. A simple example of calculation is given by a closed orbit γ of the geodesic flow of minimal period T_γ . In that case, one can verify that $P_{\text{top}}(\Lambda, s) = s \log |\det d\varphi^{-T_\gamma}|_{E^u(\gamma)}$.

One can show that the map $s \mapsto P_{\text{top}}(\Lambda, s)$ is continuous and convex [110]. In the case $s = 0$, this defines a quantity which is referred as the **topological entropy** of Λ that is denoted by $h_{\text{top}}(\Lambda) \geq 0$. In the case of a manifold with constant negative (sectional) curvatures equal to -1 , one has $P_{\text{top}}(\Lambda, s) = h_{\text{top}}(\Lambda) - s(n - 1)$.

If we now fix an invariant probability measure μ on Λ , we can also define a related quantity which is called the **Kolmogorov-Sinaï entropy** [130]. This quantity was originally defined by using partition of the space [80, 81, 114] and ideas coming from information theory [113]. We will introduce it via an equivalent definition due to Brin and Katok [29]. Precisely, they showed that, if μ is ergodic, the following limit exists for μ -a.e. (x, ξ) on Λ :

$$h_{KS}(\mu) = \lim_{\epsilon \rightarrow 0} \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \mu(B(x, \xi; \epsilon, T)),$$

and it is equal to the entropy as originally defined by Kolmogorov and Sinaï. It measures in some sense the complexity of the geodesic flow from the point of view of the ergodic measure μ . More generally, if μ is not ergodic, the Kolmogorov-Sinaï entropy is defined as

$$h_{KS}(\mu) := \int_{\Lambda} h_{KS}(\mu_{x,\xi}) d\mu(x, \xi),$$

where $\mu = \int_{\Lambda} \mu_{x,\xi} d\mu(x, \xi)$ is the ergodic decomposition of μ – see Remark 2.10. The entropy map $\mu \mapsto h_{KS}(\mu)$ is an affine function, i.e. $h_{KS}(t\mu_1 + (1-t)\mu_2) = th_{KS}(\mu_1) + (1-t)h_{KS}(\mu_2)$ for every $t \in [0, 1]$ and every $(\mu_1, \mu_2) \in \mathcal{M}(\varphi^t)$.

Example 6.11. In the case of the Lebesgue measure along a closed orbit, one can verify that the entropy is equal to 0. From the volume Lemma of Bowen and Ruelle [28], one also knows that, for an Anosov geodesic flow, $h_{KS}(L_1) = - \int_{S^*M} \log J^u dL_1$.

More generally Ruelle proved that, for any invariant measure on Λ , one has [110]

$$(39) \quad h_{KS}(\mu) \leq - \int_{\Lambda} \log J^u d\mu,$$

with equality in the case of Anosov flow if and only if $\mu = L_1$ [87].

Remark 6.12. The topological pressure is related to the Kolmogorov-Sinaï entropy via the variational principle [130]:

$$P_{\text{top}}(\Lambda, s) = \sup \left\{ h_{KS}(\mu) + s \int_{\Lambda} \log J^u d\mu : \mu \in \mathcal{M}(\varphi^t) \text{ and } \mu(\Lambda) = 1 \right\}.$$

In particular, the topological entropy is the supremum of the Kolmogorov-Sinaï entropy over all invariant measures on Λ . Combined with Ruelle's inequality (39), this variational principle shows that $PP_{\text{top}}(\Lambda, 1) \leq 0$. One can in fact prove that the Bowen's equation

$$P_{\text{top}}(\Lambda, s) = 0$$

admits an unique solution $s_{\Lambda} \in [0, 1]$. In analogy with the definition of the Hausdorff dimension, this number s_{Λ} measures in some sense the size of Λ along the unstable direction.

6.3.2. The arithmetic case. Using this notion of Kolmogorov-Sinaï entropy, Bourgain and Lindenstrauss refined Theorem 6.2 in the following manner [23]:

Theorem 6.13. *Let M be a compact arithmetic surface. Then, there exists some constant $h_0 > 0$ such that, for every $\mu \in \mathcal{M}_{\text{Hecke}}(\Delta_g)$ and for μ -a.e. $(x, \xi) \in S^*M$, one has*

$$h_{KS}(\mu_{x,\xi}) \geq h_0 > 0.$$

In particular, if Λ is an hyperbolic subset such that $h_{\text{top}}(\Lambda) < h_0$, then

$$\mu(\Lambda) = 0.$$

Hence, not only Hecke eigenfunctions cannot concentrate on closed geodesics, but also they cannot concentrate on subsets with small topological entropy. In [89], Lindenstrauss proved that elements $\mu \in \mathcal{M}_{\text{Hecke}}(\Delta_g)$ verify an extra invariance property that he calls *Hecke-recurrent* and that follows from the fact that they are Hecke eigenfunctions. We denote by $\mathcal{M}_{\text{Hecke}}(\varphi^t)$ the subset of Hecke-recurrent measure in $\mathcal{M}(\varphi^t)$. Lindenstrauss proved the following general classification result for measures in $\mathcal{M}_{\text{Hecke}}(\varphi^t)$ which yields an answer to the Quantum Unique Ergodicity conjecture for *Hecke* Laplace eigenfunctions [89]:

Theorem 6.14. *Let M be a compact arithmetic surface and let $\mu \in \mathcal{M}_{\text{Hecke}}(\varphi^t)$. If $h_{KS}(\mu_{x,\xi}) > 0$ for μ -a.e. $(x, \xi) \in S^*M$, then $\mu = L_1$. In particular, from Theorem 6.13,*

$$\mathcal{M}_{\text{Hecke}}(\Delta_g) = \{L_1\}.$$

This Theorem can be generalized to arithmetic surfaces of finite volume that may be noncompact like the modular surface $M = \mathbb{H}^2/PSL_2(\mathbb{Z})$ [89, 123]. More recently, it was also extended to the case where we consider Laplace eigenfunctions which are eigenfunctions of only *one* (and not all) Hecke operator [31]. We will not give proofs of these results which rely on techniques rather different from the semiclassical ones presented in these lectures. We refer the interested reader to the above references for more details on these arithmetic aspects of the Quantum Unique Ergodicity conjecture.

6.3.3. *Entropic results in variable curvature.* In the general framework, progress have also been made using the notion of Kolmogorov-Sinaï entropy. This was pioneered by Anantharaman and Nonnenmacher [2, 8] and it lead to quantitative dynamical constraints on the set of semiclassical measures. More precisely, Anantharaman proved [2]

Theorem 6.15. *Suppose that φ^t has the Anosov property. Then, for every*

$$h_0 < \frac{1}{2} \max_{S^*M} |\log J^u|,$$

there exists some constant $c(h_0) > 0$ such that, for every $\mu \in \mathcal{M}(\Delta_g)$, one has

$$\mu(\{(x, \xi) \in S^*M : h_{KS}(\mu_{x, \xi}) \geq h_0\}) \geq c(h_0) > 0.$$

In particular, one has

$$h_{KS}(\mu) > 0,$$

and, for any invariant subset Λ ,

$$h_{\text{top}}(\Lambda) \leq h_0 \implies \mu(\Lambda) \leq 1 - c(h_0) < 1.$$

This Theorem says that Laplace eigenfunctions cannot concentrate on hyperbolic subsets which are too small in terms of topological entropy. For instance, it tells us that, if γ is a closed orbit of φ^t , then $\mu(\gamma) < 1$ for every semiclassical measure μ . Compared with Theorem 6.13, this result is valid in full generality for Anosov geodesic flows without any constraints on the sequence of Laplace eigenfunctions that one chooses. Of course, it leads to a weaker conclusion. We shall give some ideas of this proof in paragraph 6.4.

In a subsequent work with Nonnenmacher [8], they developped a slightly different approach based on an entropic uncertainty principle [92] and they obtained more explicit lower bounds on $h_{KS}(\mu)$. For instance, in the case of *constant negative curvature* equal to -1 , they obtained the following explicit lower bound

$$\forall \mu \in \mathcal{M}(\Delta_g), \quad h_{KS}(\mu) \geq \frac{n-1}{2},$$

which is half of Ruelle's upper bound (39). Showing the Quantum Unique Ergodicity conjecture would then amount to remove the $1/2$ in this lower bound thanks to the works of Ledrappier and Young [87]. This argument was optimized by Anantharaman, Koch and Nonnenmacher and it lead to the following general bound [4]:

Theorem 6.16. *Suppose that φ^t has the Anosov property. Set*

$$\chi_{\max} := \lim_{t \rightarrow +\infty} \frac{1}{t} \log \sup \{ \|d_{x, \xi} \varphi^t\| : (x, \xi) \in S^*M \}.$$

Then, for every $\mu \in \mathcal{M}(\Delta_g)$, one has

$$h_{KS}(\mu) \geq - \int_{S^*M} \log J^u d\mu - \frac{n-1}{2} \chi_{\max}.$$

Theorem 6.15 gave a rather unexplicit lower bound $h_0 c(h_0)$ on the Kolmogorov-Sinai entropy. Here, the lower bound is more explicit but the drawback of the proof is that this lower bound may be negative (and thus the result empty). This only happens when the expansion rate χ_{\max} is too large compared with the averaged Lyapunov exponents with respect to μ . This term χ_{\max} comes out because their proof relies on an application of the Egorov Theorem (14) for range of times that tend to $+\infty$ as $\hbar \rightarrow 0$. In [8, 48], it is shown that the optimal range of times for which the Egorov property (14) remains true (up to a $o(1)$ remainder) is

$$T_E(\hbar) = \frac{|\log \hbar|}{2\chi_{\max}},$$

which is known as the Ehrenfest time. This is the time scale until which the semiclassical approximation remains valid. Understanding the semiclassical properties of a quantum system beyond this time scale is a difficult and subtle question that one has to face in order to make improvements on our understanding of the set $\mathcal{M}(\Delta_g)$.

Motivated by this Theorem and by the counterexamples of Theorems 6.4 and 6.6 – see also [7], Anantharaman and Nonnenmacher also formulated that the following general result should hold [8]:

Conjecture 6.17. *Suppose that φ^t has the Anosov property. Then, for every $\mu \in \mathcal{M}(\Delta_g)$, one has*

$$h_{KS}(\mu) \geq -\frac{1}{2} \int_{S^*M} \log J^u d\mu.$$

This lower bound is in fact sharp in the counterexamples that we have described for quantum maps, and proving the Quantum Unique Ergodicity conjecture would amount to remove the $1/2$ in this inequality. In the case of certain quantum maps with variable Lyapunov exponents in dimension 2, this conjecture was proved by Gutkin [63]. Then, in the case of Anosov geodesic flows, one gets [104, 106]

Theorem 6.18. *Suppose that φ^t has the Anosov property. Then, for every $P_0 > 0$, there exists some constant $c(P_0) > 0$ such that, for every $\mu \in \mathcal{M}(\Delta_g)$, one has*

$$\mu \left(\left\{ (x, \xi) : h_{KS}(\mu_{x,\xi}) \geq -\frac{1}{2} \int_{S^*M} \log J^u d\mu_{x,\xi} - P_0 \right\} \right) \geq c(P_0) > 0.$$

In particular, one has, for any invariant subset Λ ,

$$P_{\text{top}} \left(\Lambda, \frac{1}{2} \right) \leq -P_0 \implies \mu(\Lambda) \leq 1 - c(P_0) < 1.$$

Moreover, if $\dim M = 2$, then, one has, for every $\mu \in \mathcal{M}(\Delta_g)$,

$$h_{KS}(\mu) \geq -\frac{1}{2} \int_{S^*M} \log J^u d\mu.$$

This Theorem answers Anantharaman-Nonnenmacher's question in dimension 2 and, in that case, it can also be extended to nonpositively curved manifold provided that we choose an appropriate definition for the unstable Jacobian [103]. In higher dimensions,

this conjecture remains open and, besides Theorem 6.16, the best explicit bounds we are aware of are for locally symmetric spaces [9] and for symplectic linear maps of the multidimensional torus [105].

The proof of these different results is slightly technical and it involves tools from ergodic theory, hyperbolic dynamical systems and semiclassical analysis. Instead of doing that, we will explain how these methods can be used to extend Theorem 6.1 to more general hyperbolic subsets Λ . More precisely, together with Nonnenmacher, we proved [108]:

Theorem 6.19. *Let Λ be an hyperbolic subset for φ^t such that*

$$P_{\text{top}}\left(\Lambda, \frac{1}{2}\right) < 0.$$

*Let $a \in \mathcal{C}_c^\infty(T^*M, [0, 1])$ which is equal to 0 near Λ and to 1 outside a slightly bigger neighborhood (depending on Λ). Then, there exists some $c_{a, \Lambda} > 0$ such that, for any sequence $(\psi_h)_{h \rightarrow 0^+}$ of solutions to (5),*

$$\liminf_{h \rightarrow 0^+} |\log h| \langle \psi_h, \text{Op}_h(a) \psi_h \rangle \geq c_{a, \Lambda} > 0.$$

We emphasize that, compared with Theorems 6.15, 6.16 and 6.18, no global assumption is made on the manifold. We will give a sketch of proof in paragraph 6.4 and we will also briefly explain which steps can be optimized when we make some global assumptions on the manifold (e.g. Anosov property of the geodesic flow).

6.3.4. Unconditional observability in dimension 2. We can now come back to the question of observability in the case of an Anosov geodesic flow. Given an open set ω of S^*M (not necessarily of M), one can define the hyperbolic subset

$$\Lambda_\omega := \bigcap_{t \in \mathbb{R}} \varphi^t(S^*M \setminus \omega).$$

The entropic Theorems from paragraph 6.3.3 imply that if

$$P_{\text{top}}\left(\Lambda_\omega, \frac{1}{2}\right) < 0,$$

then $\mu(\omega) \geq c_\omega > 0$ for any $\mu \in \mathcal{M}(\Delta_g)$. Hence, if ω is an open set of S^*M such that Λ_ω is not too big in a dynamical sense, then eigenfunctions are uniformly observables on ω . In the case where ω is an open set of M , these results implies an observability Theorem like Theorem 5.3 provided that $P_{\text{top}}\left(\Lambda_{S^*\omega}, \frac{1}{2}\right) < 0$. In the case of hyperbolic surfaces, this condition was removed by Dyatlov and Jin [49] using a fractal uncertainty principle due to Bourgain and Dyatlov [22] – see also [51]:

Theorem 6.20. *Suppose that M is an hyperbolic surface. Then, for any open set $\omega \neq \emptyset$ inside S^*M , there exists some constant $c_{\omega, M, g} > 0$ such that, for any $\mu \in \mathcal{M}(\Delta_g)$,*

$$\mu(\omega) \geq c_{\omega, M, g} > 0.$$

Compared with the case of flat tori, we emphasize that this Theorem is valid for any nonempty open set inside S^*M (and not only on M). An extension of this result to variable negative curvature was recently announced by Dyatlov, Jin and Nonnenmacher [50]. A natural question is to understand what can happen in higher dimensions. In that case, one has to keep in mind Kelmer's construction (see Theorem 6.6) which shows that such a result cannot be a consequence of hyperbolicity only.

Remark 6.21. If we fix an invariant subset $\Lambda \neq S^*M$, then Theorem 6.20 tells us that

$$\mu(\Lambda) \leq 1 - c(\Lambda) < 1,$$

for some constant $c(\Lambda) > 0$ depending on Λ . Compared with Theorems 6.15 and 6.18, this is valid for *any* subset Λ . Yet, the constant depends on Λ while the constant appearing in Theorems 6.15 and 6.18 depends only on the topological pressure of Λ provided it is small enough.

For the sake of completeness, let us finally record the following remarkable Corollary of Theorem 6.20:

Corollary 6.22 (Unconditional observability on negatively curved surfaces). *Suppose that M is an hyperbolic surface and let ω be a nonempty open set of M . Then, there exists some constant $c_{\omega,g} > 0$ such that, for every ψ_λ solution to*

$$-\Delta\psi_\lambda = \lambda^2\psi_\lambda, \quad \|\psi_\lambda\|_{L^2} = 1,$$

one has

$$\int_\omega |\psi_\lambda(x)|^2 d\text{Vol}_g(x) \geq c_{\omega,g} > 0.$$

6.4. Sketch of proof of Theorem 6.19. The proof of Theorem 6.19 being slightly simpler than the ones of Theorem 6.15 or 6.18, we will focus on this case without getting too much into the details. In the end, we will explain how a global assumption on the dynamics allows to improve the argument.

6.4.1. Cover of the hyperbolic subset Λ . Let $(V_a)_{a \in W}$ be an open cover of Λ by some open subsets of T^*M adapted to the dynamics. In order to avoid complications, we remain vague on this aspect and we refer to [108, Par. 2.2] for more details. We will just say that they are constructed by mimicking the construction of Bowen balls. Instead of considering balls of radius ϵ to construct dynamical balls, we consider a fixed open cover that we refine by the dynamics to obtain a cover adapted to the dynamics. In that manner, we obtain the following estimate

$$(40) \quad \sum_{a \in W} \sup_{\rho \in V_a} \exp\left(\frac{1}{2} \int_0^{T_0} \log J^u \circ \varphi^t(\rho) dt\right) \leq e^{T_0(P_{\text{top}}(\Lambda, 1/2) + \epsilon_0)},$$

where $\epsilon_0 > 0$ is some small parameter which is fixed in advance and $T_0 > 0$ is large enough. We complete this cover by setting V_∞ to be an open set not intersecting Λ (or its image in the dilates of Λ near S^*M). In the following, we set $\overline{W} = W \cup \{\infty\}$. Our goal is to prove that at least a fraction $|\log \hbar|^{-1}$ of the Wigner distribution associated to the sequence

$(\psi_h)_{0 < h \leq 1}$ is contained inside V_∞ . To that aim, we introduce a partition of unity $(P_a)_{a \in \overline{W}}$ associated to the open cover $(V_a)_{a \in \overline{W}}$, i.e.

$$\forall a \in \overline{W}, P_a \in \mathcal{C}_c^\infty(V_a, [0, 1]) \text{ and } \sum_{a \in \overline{W}} P_a(\rho) = 1,$$

for every ρ in a small neighborhood S^*M . Again, the reason for working near S^*M is that eigenfunctions are asymptotically concentrated on S^*M .

6.4.2. *Quantum partitions of identity.* We now quantize this smooth partition into a family of pseudodifferential operators $(\pi_a)_{a \in \overline{W}}$ so that $\pi_a^* = \pi_a$, $\sigma(\pi_a) = P_a$ and

$$\sum_{a \in \overline{W}} \pi_a = \text{Id}_{L^2(M)} + \mathcal{O}(\hbar^\infty),$$

for data microlocalized near S^*M , e.g. for our sequence $(\psi_h)_{0 < h \leq 1}$ of solutions to (5). We say that we have a quantum partition of identity and these play a central role in the above entropic results. We can now construct a quantum analogue of the Bowen balls, and to that aim we set

$$\pi_a(t) = e^{-\frac{ith\Delta_g}{2}} \pi_a e^{\frac{ith\Delta_g}{2}},$$

We still have the relation

$$\sum_{a \in \overline{W}} \pi_a(t) = \text{Id}_{L^2(M)} + \mathcal{O}(\hbar^\infty),$$

for data microlocalized near S^*M . Hence, for $N \geq 0$, one can write

$$(41) \quad \sum_{\alpha \in \overline{W}^N} \langle \psi_h, \pi_{\alpha_{N-1}}((N-1)T_0) \dots \pi_{\alpha_1}(T_0) \pi_{\alpha_0}(0) \psi_h \rangle = 1 + \mathcal{O}(\hbar^\infty).$$

Each of these operators can be thought as a semiclassical analogue of the characteristic function of a Bowen ball. In fact, for N fixed and using (10) and (14), the principal symbol of this operator is $P_{\alpha_{N-1}} \circ \varphi^{(N-1)T_0} \times \dots \times P_{\alpha_1} \circ \varphi^{T_0} \times P_{\alpha_0}$. The points inside the support of this symbol are exactly the points ρ which at time 0 are inside V_{α_0} , at time T_0 inside V_{α_1} , etc. The Egorov Theorem (14) remains true for times of order $N = \kappa |\log \hbar|$ with $\kappa > 0$ small enough up to some small error of order $\mathcal{O}(\hbar^\delta)$ with $\delta > 0$ depending on κ [145, Th. 11.12]. Hence, the symbol remains of this form for this range of N and we will use this observation several times in the following.

6.4.3. *Hyperbolic dispersion estimates.* The first key ingredient is an hyperbolic dispersion estimate which, in the present case, is due to Nonnenmacher and Zworski [99]. This estimate will allow us to control the terms in the sum (41) which correspond to words α in W^N , i.e. points whose trajectory remains close to Λ . More precisely, for every $\mathcal{K} > 0$, they show the existence of $C_{\mathcal{K}} > 0$ and $\hbar_{\mathcal{K}} > 0$ such that, for every $0 < \hbar \leq \hbar_{\mathcal{K}}$, for every $N \leq \mathcal{K} |\log \hbar|$ and for every α in W^N ,

$$(42) \quad \left\| \pi_{\alpha_{N-1}}((N-1)T_0) \dots \pi_{\alpha_1}(T_0) \pi_{\alpha_0} \right\|_{L^2 \rightarrow L^2} \leq C_{\mathcal{K}} \hbar^{-\frac{N}{2}} (1 + \mathcal{O}(\epsilon))^{NT_0} \prod_{k=0}^{N-1} \sup_{\rho \in V_{\alpha_k}} \exp \left(\frac{1}{2} \int_0^{T_0} \log J^u \circ \varphi^t(\rho) dt \right).$$

where $\epsilon > 0$ is an upper bound on the diameter of the open sets $(V_\alpha)_{\alpha \in W}$. This is in fact the only place of the proof where the hyperbolicity is used. Proving this upper bound is quite subtle and it is one of the major steps in the proofs of [2, 7]. We will skip the argument that would require to introduce more background material related to Fourier integral operators and we will just explain how it can be used here.

If we implement this upper bound and the estimate (40) in (41), we can deduce that

$$(43) \quad \sum_{\alpha \in \overline{W}^N - W^N} \langle \psi_h, \pi_{\alpha_{N-1}}((N-1)T_0) \dots \pi_{\alpha_1}(T_0) \pi_{\alpha_0}(0) \psi_h \rangle = 1 + \mathcal{O}\left(\hbar^{-\frac{n}{2}} e^{NT_0(P_{\text{top}}(\Lambda, 1/2) + \epsilon_0)}\right).$$

Now, using the fact that $P_{\text{top}}(\Lambda, 1/2) < 0$ and taking $N = [\mathcal{K} \log \hbar]$ with $\mathcal{K} > 0$ large enough, one finds that

$$(44) \quad \sum_{\alpha \in \overline{W}^N - W^N} \langle \psi_h, \pi_{\alpha_{N-1}}((N-1)T_0) \dots \pi_{\alpha_1}(T_0) \pi_{\alpha_0}(0) \psi_h \rangle = 1 + o(1).$$

Hence, we have shown that those elements α of the partition that correspond to trajectories that spend some time away from Λ have a total mass which is asymptotically of order 1. The difficulty is now to show that there is indeed some fraction of the mass outside Λ . This will follow from a combinatorial work whose ideas go back to the works of Anantharaman [2] who showed that these quantities verify some subadditive properties.

Before explaining that, let us observe that (44) can be rewritten as

$$(45) \quad \left\langle \psi_h, \sum_{\alpha \in \overline{W}^N - W^N} \pi_{\alpha_{N-1}}(T_0(N-1)) \dots \pi_{\alpha_1}(T_0) \pi_{\alpha_0}(0) \psi_h \right\rangle = 1 + o(1).$$

Thus, by Cauchy-Schwarz inequality,

$$(46) \quad \left\| \sum_{\alpha \in \overline{W}^N - W^N} \pi_{\alpha_{N-1}}((N-1)T_0) \dots \pi_{\alpha_1}(T_0) \pi_{\alpha_0}(0) \psi_h \right\| \geq 1 + o(1).$$

6.4.4. Subadditivity property: from long logarithmic times to short ones. Let us write $N = kN_1 \simeq \mathcal{K} \log \hbar$ with $N_1 \simeq [\kappa \log \hbar]$ so that the semiclassical rules of paragraph 2.5 applies up to small error terms (e.g. the Egorov Theorem or the composition rule) at the scales N_1 . We can already note that k is bounded in terms of \mathcal{K} and κ . Hence, using the semiclassical rules of paragraph 2.5, one finds

$$\left\| \sum_{\gamma \in W^{N_1}} \pi_{\gamma_{N_1-1}}((N_1-1)T_0) \dots \pi_{\gamma_1}(T_0) \pi_{\gamma_0} \right\|_{L^2 \rightarrow L^2} = \mathcal{O}(1).$$

Observe now that

$$\overline{W}^N - W^N = \bigsqcup_{p=1}^k \left\{ \alpha = (\bar{\gamma}, \gamma, \tilde{\gamma}) : \bar{\gamma} \in \overline{W}^{N_1(p-1)}, \gamma \in \overline{W}^{N_1} - W^{N_1}, \tilde{\gamma} \in W^{N_1(k-p)} \right\}.$$

This allows us to rewrite (46) under the form

$$(47) \quad 1 + o(1) \leq \mathcal{O}(1) \sum_{p=1}^k \left\| \sum_{\gamma \in \overline{W}^{N_1} - W^{N_1}} \pi_{\gamma_{N_1-1}} e^{\frac{i\hbar T_0 \Delta_g}{2}} \dots \pi_{\gamma_1} e^{\frac{i\hbar T_0 \Delta_g}{2}} \pi_{\gamma_0} e^{\frac{i\hbar T_0 N_1 p \Delta_g}{2}} \psi_h \right\|.$$

Then, using that $e^{\frac{i\hbar T_0 \Delta_g}{2}}$ is unitary and that the $(\psi_h)_{0 < h \leq 1}$ are solutions to (5),

$$(48) \quad 1 + o(1) \leq \mathcal{O}(1) \left\| \sum_{\gamma \in \overline{W}^{N_1} - W^{N_1}} \pi_{\gamma_{N_1-1}} ((N_1 - 1)T_0) \dots \pi_{\gamma_1}(T_0) \pi_{\gamma_0}(0) \psi_h \right\|.$$

Taking the square of this inequality, we obtain the existence of a constant $C > 0$ such that

$$(49) \quad C \leq \left\| \sum_{\gamma \in \overline{W}^{N_1} - W^{N_1}} \pi_{\gamma_{N_1-1}} ((N_1 - 1)T_0) \dots \pi_{\gamma_1}(T_0) \pi_{\gamma_0}(0) \psi_h \right\|^2.$$

Using Egorov property (14) for short logarithmic times (recall that $N_1 \simeq [\kappa |\log \hbar|]$ with $\kappa > 0$ small enough) and the composition rule, this inequality can be rewritten as

$$(50) \quad C \leq \left\langle \psi_h, \text{Op}_\hbar \left(\left(\sum_{\gamma \in \overline{W}^{N_1} - W^{N_1}} P_{\gamma_{N_1-1}} \circ \varphi^{(N_1-1)T_0} \times \dots \times P_{\gamma_1} \circ \varphi^{T_0} \times P_{\gamma_0} \right)^2 \right) \psi_h \right\rangle + o(1).$$

Now we can make use of the Gårding inequality (17) and of the fact that the principal symbol is ≤ 1 to deduce:

$$(51) \quad C \leq \left\langle \psi_h, \text{Op}_\hbar \left(\sum_{\gamma \in \overline{W}^{N_1} - W^{N_1}} P_{\gamma_{N_1-1}} \circ \varphi^{(N_1-1)T_0} \times \dots \times P_{\gamma_1} \circ \varphi^{T_0} \times P_{\gamma_0} \right) \psi_h \right\rangle + o(1).$$

6.4.5. Subadditivity again: from short logarithmic times to finite times. We can now reiterate the same procedure to go down to finite times. We note that

$$\overline{W}^{N_1} - W^{N_1} = \bigsqcup_{p=1}^{N_1-1} \left\{ \alpha = (\overline{\gamma}, \infty, \tilde{\gamma}) : \overline{\gamma} \in \overline{W}^{p-1}, \tilde{\gamma} \in W^{N_1-p} \right\}.$$

Using the fact that we have a partition of unity near S^*M (and that ψ_h is microlocalized near S^*M) and the Gårding inequality, we can deduce from (51) that

$$(52) \quad C \leq \sum_{p=1}^{N_1-1} \left\langle \psi_h, \text{Op}_\hbar (P_\infty \circ \varphi^{pT_0}) \psi_h \right\rangle + o(1).$$

Using one last time Egorov theorem for short logarithmic times and the fact that the $(\psi_h)_{0 < h \leq 1}$ are Laplace eigenfunctions, we can conclude that

$$C \leq N_1 \left\langle \psi_h, \text{Op}_\hbar (P_\infty) \psi_h \right\rangle + o(1).$$

Recall now that N_1 is of order $|\log \hbar|$ which concludes the proof of Theorem 6.19.

6.4.6. Adding a global assumption on the dynamics. As the reader can see, the proof of Theorem 6.19 is already quite involved but it illustrates the general scheme that was initiated by Anantharaman in [2]. In order to obtain the results for Anosov flows (e.g. Theorems 6.15), one needs to use an earlier version of the hyperbolic dispersion estimate (42) due to Anantharaman and Nonnenmacher [2, 8] and valid for any sequence γ associated with a partition of unity. In particular, it still holds for sequences which correspond to trajectories that are away from Λ : this is where one uses the global assumption on the dynamics (i.e. the Anosov property). Implementing this, we can be less restrictive when we cut the sum in two pieces at the beginning of the proof. In particular, we can allow sequences γ corresponding to trajectories that are away from Λ for a small fraction of times. Proceeding like this, we have less terms in the remaining sums and we can proceed to a better subadditive argument that will not end with this $|\log \hbar|$ factor – see [2, 106] for more details.

6.5. Sketch of proof of Theorem 6.20. To end this section on Laplace eigenfunctions for chaotic geodesic flows, we give a sketch of the proof of Theorem 6.20. This is based on a fractal uncertainty principle [51, 22] that can be used to replace the hyperbolic dispersion estimate (42) used in the previous argument. We follow the presentation of [46] and we refer the reader to this reference for an account on these recent developments. We fix a sequence of Laplace eigenfunctions $(\psi_\hbar)_{\hbar \rightarrow 0^+}$ (i.e. solutions to (5)) having a single semiclassical measure. We let ω be a nonempty open set of S^*M and $\omega_1 \neq \emptyset$ such that $\overline{\omega_1} \subset \omega$. As before, we introduce a partition (π_1, π_2) of identity, i.e

$$\pi_1 + \pi_2 = \text{Id}_{L^2(M)} + \mathcal{O}(\hbar),$$

for sequences of initial data which are microlocalized on S^*M . We also make the assumption that the principal symbol P_2 of π_2 does not intersect ω_1 (and its dilation) in a small neighborhood of S^*M while the operator π_1 is microlocalized in ω near S^*M . We now fix some logarithmic time

$$N = [\mathcal{K} |\log \hbar|],$$

where $0 < \mathcal{K} < 1$ is a parameter which is close to 1. In that geometric context, the Egorov Theorem is only valid up to times of order $\frac{1}{2} |\log \hbar|$ [27, 8, 48]. Thus, N_1 is larger than the scale where we can expect the semiclassical rules of paragraph 2.5 to work. In order to overcome this difficulty, Dyatlov and Jin make use of two quantization procedures that are adapted respectively to symbols whose derivatives are bounded in \hbar along the stable manifold (resp. along the unstable manifold) but may blow in $\hbar^{-\kappa}$ when one differentiates in other directions. These kind of quantization procedures adapted to a Lagrangian foliation were constructed by Dyatlov and Zahl [51, §3] and they are denoted respectively by $\text{Op}_\hbar^{L_s}$ and $\text{Op}_\hbar^{L_u}$ in the present case. They correspond to different class of symbols and they are a priori not compatible with each other. In particular, $\text{Op}_\hbar^{L_s}(b_1) \text{Op}_\hbar^{L_u}(b_2)$ is not necessarily a pseudodifferential operator. Yet, it has the advantage that, up to the time scale N , we

can apply semiclassical rules provided we consider the appropriate sense of times. More precisely, one has among other things

$$(53) \quad \Pi_+ := \pi_2(-1)\pi_2(-2)\dots\pi_2(-N) = \text{Op}_h^{L_u} \left(\prod_{j=1}^N P_2 \circ \varphi^{-j} \right) + \mathcal{O}(\hbar^{1-\kappa}),$$

and

$$(54) \quad \Pi_- := \pi_2(N-1)\dots\pi_2(1)\pi_2(0) = \text{Op}_h^{L_s} \left(\prod_{j=0}^{N-1} P_2 \circ \varphi^j \right) + \mathcal{O}(\hbar^{1-\kappa}),$$

as we had in the previous proof but for much smaller scales of times, i.e. $\kappa|\log \hbar|$ with some fixed $\kappa \ll 1$. One can then use a version of the fractal uncertainty principle as it appears in the work of Bourgain and Dyatlov [22] in order to show that

$$(55) \quad \left\| \text{Op}_h^{L_s} \left(\prod_{j=0}^{N-1} P_2 \circ \varphi^j \right) \text{Op}_h^{L_u} \left(\prod_{j=1}^N P_2 \circ \varphi^{-j} \right) \right\|_{L^2 \rightarrow L^2} \leq C\hbar^\beta,$$

for some $\beta > 0$ depending only on the open set ω . This is exactly the estimate that will replace the hyperbolic dispersion estimate (42) that we used in our previous proof. Again these last steps are the ones that crucially use the hyperbolic structure. Let us now try to implement this new information. As before, we use the quantum partition of unity to write

$$\sum_{\gamma \in \{1,2\}^{2N}} \langle \psi_\hbar, \pi_{\gamma_{N-1}}(N-1)\dots\pi_{\gamma_1}(1)\pi_{\gamma_0}(0)\pi_{\gamma_{-1}}(-1)\dots\pi_{\gamma_{-N}}(-N)\psi_\hbar \rangle = 1 + o(1).$$

Implementing (55), we find that

$$\sum_{\gamma \in \{1,2\}^{2N} \setminus \{2\}^{2N}} \langle \psi_\hbar, \pi_{\gamma_{N-1}}(N-1)\dots\pi_{\gamma_1}(1)\pi_{\gamma_0}(0)\pi_{\gamma_{-1}}(-1)\dots\pi_{\gamma_{-N}}(-N)\psi_\hbar \rangle = 1 + o(1).$$

We now proceed to a subadditive argument as before. First, we set $N_1 = [\kappa|\log \hbar|]$ with $\kappa \ll 1$ Reorganizing the sums as we did before and applying the Cauchy-Schwarz inequality, we find that there exists some constant $C > 0$ such that

$$C \leq \left\| \sum_{\gamma \in \{1,2\}^{N_1} \setminus \{2\}^{N_1}} \pi_{\gamma_{N_1-1}}(N_1-1)\dots\pi_{\gamma_1}(1)\pi_{\gamma_0}(0)\psi_\hbar \right\|.$$

As we are now handling much smaller time scales, we can proceed as in the previous proof to find that

$$C^2 \leq \sum_{p=1}^{N_1-1} \langle \psi_\hbar, \text{Op}_h(P_1 \circ \varphi^p)\psi_\hbar \rangle + o(1).$$

Applying one last time the Egorov Theorem and using the fact that ψ_h is a normalized Laplace eigenfunction, we find that

$$\frac{C^2 + o(1)}{N_1} \leq \langle \psi_h, \pi_1 \psi_h \rangle.$$

Recall now that the pseudodifferential operator π_1 is microlocalized inside the open set ω we are interested in. Hence, one has a fraction $|\log \hbar|^{-1}$ which is contained inside this open set. As before, in order to remove this $|\log \hbar|^{-1}$ factor, we have to be slightly more careful in the definition of Π_{\pm} and to allow that there is a small fraction of terms with $\gamma = 1$, meaning that it corresponds to trajectories that send a small amount of times away from the open set ω_1 . If this fraction is small enough in terms of the exponent $\beta > 0$ appearing in the fractal uncertainty principle (55), then one can still apply the estimate to these extra terms. The remaining sums are again much smaller and performing an argument similar to the one from [2] allows to remove the logarithmic factor – see [49] for details.

6.6. Some comments on the key estimates (42) and (55). The two proofs that we have described follows a similar scheme that was initially introduced in [2]. In particular, both proofs require at some point an estimate on the L^2 -norm of elements inside our quantum partition associated with classical trajectories that avoid the region where we want to show that the quantum state is observable. Hence, one needs to show that this norm is asymptotically small and this is where the chaotic features of the flow are crucially used in both proofs. Yet, the proofs of these two estimates are of rather different nature even if they both consist in estimating some oscillatory integral. Observe that an important difference between the two estimates is that (42) yields a nontrivial information for symbols P_i supported in a subset of small diameter while (55) holds for *any* symbol P_2 . This subtle difference between the two statements is responsible for the fact that the entropic results yields observability for big enough subsets of S^*M while Theorem 6.20 gives it for any subset (under more restrictive assumptions on M).

The hyperbolic dispersion estimates of Anantharaman, Nonnenmacher and Zworski is based on WKB type asymptotics. More precisely, after some Fourier decomposition which is responsible for the loss $\hbar^{-\frac{\alpha}{2}}$ in (42), the proof amounts to a precise description of the action of elements inside the quantum partition on Lagrangian states $(a(x)e^{\frac{iS(x)}{\hbar}})_{\hbar>0}$ (as the ones from example (2.15)). Then, the decay follows from the exact expression obtained in the WKB asymptotics and from the fact that the Lagrangian submanifolds $\mathcal{L}_S = \{(x, d_x S)\}$ become closer to the unstable direction under the classical evolution.

On the other hand, using the hyperbolic structure, Dyatlov and Jin show how to reduce the crucial upper bound (55) to some question from harmonic analysis. This question is referred as a fractal uncertainty principle in the sense that it gives an estimate on the L^2 mass of a function inside a subset having a “fractal” structure when its Fourier transform is itself supported in a subset with similar properties. More precisely, relying on the unique ergodicity of the horocycle flow¹⁴, they show that the supports of the symbols involved

¹⁴This is the flow generated by the vector field in the unstable direction.

in (55) have holes of uniform size¹⁵ $0 < \delta < 1$ along the stable (resp. unstable) direction at all scales between \hbar^K and 1 – see [46, Lemma 3.2] for a precise statement. The supports are said to be δ -porous at these scales. Then, one can reduce the problem to a one dimensional problem where the stable and unstable directions are in some sense dual to each other in terms of Fourier transform. Thus, proving the upper bound (55) amounts to prove some subtle estimate on the properties of the one-dimensional Fourier transform [22]. Roughly speaking, if the Fourier transform \widehat{f} of $f \in L^2(\mathbb{R})$ is supported in a set Y having holes of size $0 < \delta < 1$ at each scale between \hbar^{K-1} and \hbar^{-1} , then the L^2 -mass of f inside a subset having holes of size δ at each scale between \hbar^K and 1 is of order $\hbar^{\beta-2(1-K)}\|f\|_{L^2}$ for some $\beta > 0$ (depending only on δ thus on ω). We refer the reader to the recent review of Dyatlov [47] for background and ideas behind these fractal uncertainty principles.

7. APPLICATION TO NODAL SETS AND L^p NORMS

We mostly focused on the study of eigenfunctions via their semiclassical measures which are natural from the point of view of quantum mechanics. Yet, from the mathematical perspective, there are several other interesting quantities that can be used to describe the singularities and the concentration properties of Laplace eigenfunctions. To end up these lectures, we would like to mention at least two of them and to explain how they are related to the questions discussed in these notes.

7.1. Bound on L^p norms. Without any assumption on the Riemannian manifold (M, g) , the best bounds that one can expect on the L^p norms of Laplace eigenfunctions are due to Hörmander for $p = +\infty$ [71] and to Sogge [118] for $p < +\infty$:

Theorem 7.1. *Let $2 \leq p \leq +\infty$. There exists $C_{M,g,p} > 0$ such that, for every ψ_λ solution to*

$$-\Delta_g \psi_\lambda = \lambda^2 \psi_\lambda,$$

one has

$$\|\psi_\lambda\|_{L^p(M)} \leq C_{M,g,p} (1 + \lambda)^{\sigma(p)} \|\psi_\lambda\|_{L^2(M)},$$

où

$$\sigma(p) := \max \left\{ \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{p} \right), \frac{n-1}{2} - \frac{n}{p} \right\}.$$

The exponent $p_c = \frac{2(n+1)}{n-1}$ plays an important role as it corresponds to the value where the functions defining $\sigma(p)$ coincide. Any improvement on the upper bound of the $L^{p_c}(M)$ -norm will then give by interpolation improvements for every $2 < p < +\infty$. This result is optimal if we do not make any extra assumptions on (M, g) as these upper bounds are saturated on the sphere endowed with its canonical metric [121]. Yet, it is natural to wonder if appropriate geometric assumptions allow to improve this result. We will not try to review the vast literature on the subject and we will just make a simple heuristic calculation that allows to understand the relation with the Wigner distributions we have

¹⁵The size depends only on ω .

encountered in the previous sections. This computation is at the origin of a joint work with Hezari [68] and it was improved by Sogge in [122].

Let us consider an orthonormal sequence of Laplace eigenfunctions $(\psi_k)_{k \in S}$ and a sequence of radii $r_k > 0$ such that one can show

$$(56) \quad \forall x_0 \in M, \forall k \in S, \quad \frac{1}{\text{Vol}_g(B(x_0, r_k))} \int_{B(x_0, r_k)} |e_k(x)|^2 d\text{Vol}_g(x) \leq c_0,$$

for some uniform constant $0 < c_0$. This kind of behaviour typically occurs when one has a sequence of eigenfunctions verifying the Quantum Ergodicity property. Let us now try to explain some heuristic idea to improve Sogge's estimate. We fix x_0 in M and we work in a small geodesic chart centered at x_0 . We rescale the function ψ_k near x_0 by setting $\tilde{\psi}_k(y) = \psi_k(r_k y)$. This new function verifies near 0:

$$\tilde{\Delta}_g \tilde{\psi}_k \approx r_k^2 \lambda_k^2 \tilde{\psi}_k$$

and, thanks to (56),

$$\int_{B(0,1)} |\tilde{\psi}_k(y)|^2 dy \lesssim c_0.$$

Hence, locally near x_0 , one has a quasimode of the Laplacian to which we can apply Sogge's estimates¹⁶. This implies that for $p = p_c$,

$$\int_{B(0,1)} |\tilde{\psi}_k(y)|^{p_c} dy \lesssim r_k \lambda_k \left(\int_{B(0,1)} |\tilde{\psi}_k(y)|^2 dy \right)^{\frac{p_c}{2}},$$

where we recall that $\sigma(p_c) = 1/p_c$. We now make the change of variables in the other direction, i.e. $x = r_k y$, and we get

$$\frac{1}{\text{Vol}_g(B(x_0, r_k))} \int_{B(x_0, r_k)} |\psi_k(x)|^{p_c} d\text{Vol}_g(x) \lesssim r_k \lambda_k \left(\frac{1}{\text{Vol}_g(B(x_0, r_k))} \int_{B(x_0, r_k)} |\psi_k(y)|^2 d\text{Vol}_g(x) \right)^{\frac{p_c}{2}}.$$

Using property (56), we obtain

$$\int_{B(x_0, r_k)} |\psi_k(x)|^{p_c} d\text{Vol}_g(x) \lesssim r_k \lambda_k c_0^{\frac{p_c}{2}} \text{Vol}_g(B(x_0, r_k)).$$

Let us now cover M by a "minimal" family of balls of radius r_k . Summing these inequalities, we find the following upper bound:

$$\left(\int_M |\psi_k(x)|^{p_c} d\text{Vol}_g(x) \right)^{\frac{1}{p_c}} \lesssim (r_k \lambda_k)^{\frac{1}{p_c}} c_2^{\frac{1}{2}},$$

which improves Sogge's upper bound by a factor $r_k^{\sigma(p_c)}$. This heuristic argument shows that an upper bound like (56) allows to improve Theorem 7.1. Hence, it transfers the problem of estimating L^p -norms into a problem of controlling the Wigner distribution inside small balls of the configuration space. This formal argument can be made rigorous [68, 122]:

¹⁶These estimates are valid for not too bad quasimodes.

Theorem 7.2. *There exists some $C_{M,g} > 0$ such that, for every ψ_λ solution to*

$$-\Delta_g \psi_\lambda = \lambda^2 \psi_\lambda, \quad \|\psi_\lambda\|_{L^2} = 1,$$

with $\lambda > 0$, one has, for every $\lambda^{-1} \leq r \leq \text{Inj}(M, g)$,

$$\|\psi_\lambda\|_{L^{p_c}(M)} \leq C_{M,g,p} \lambda^{\sigma(p_c)} \left(r^{-\frac{n+1}{2}} \sup_{x \in M} \left\{ \int_{B(x,r)} |\psi_\lambda(x)|^2 d\text{Vol}_g(x) \right\} \right)^{\frac{1}{n+1}}.$$

In particular, one can apply the Quantum Ergodicity Theorem 3.5 to show the following. If L_1 is ergodic, then one can extract a density one subsequence of an orthonormal basis along which one has the upper bound $o(\lambda^{\sigma(p_c)})$ [122]. In the case of negatively curved manifolds, we showed with Hezari how to refine a quantitative version of the Quantum Ergodicity Theorem [137] and how to derive from it an upper bound of the type (56) for radius of logarithmic size in λ [68] – see also [64]. As a direct Corollary, this gives a logarithmic improvement of Sogge’s upper bound at $p = p_c$ along a density one subsequence.

This is not only the manner to relate L^p -norms with Wigner distributions and let us single out two other approaches that appeared recently in the literature. First, for $p \leq p_c$, Blair and Sogge showed in a series of works how to control L^p -norms by Keckya-Nikodym norms – see [119, 19] and the references therein. More precisely, Keckya-Nikodym norms can be expressed in terms of the Wigner distribution of Laplace eigenfunctions in small tubes of size $\lambda^{-1/2} = \sqrt{\hbar}$ around pieces of geodesics. So, one more time, the problem is transferred to a question on Wigner distributions. For the case $p = +\infty$, Galkowski-Toth [57] and then Canzani-Galkowski [36] developed some new approach to control the L^∞ -norm of Laplace eigenfunctions in terms of their semiclassical measures. Roughly speaking, they showed how to control the value of ψ_λ at a point x_0 in terms of the mass that the Wigner distribution puts on the (n -dimensional) submanifold $\cup_{t=-T}^T \varphi^t(S_{x_0}^* M)$ for some large enough $T > 0$. In particular, if one can exhibit some situations where this mass goes to 0 as $\lambda \rightarrow +\infty$ (e.g. if L_1 is ergodic), then it yields an improvement on the growth of the L^∞ -norm.

7.2. Complex zeroes. We conclude with the question of nodal sets which are the vanishing locus of Laplace eigenfunctions. Contrary to the case of L^p -norms, there is in general no clear relation between the geometry of these nodal sets and the dynamical properties of the geodesic flow. Yet, we would like to mention a result due to Zelditch [141] who showed how to relate complex zeroes of Laplace eigenfunctions to Wigner distributions and thus to the dynamics of the geodesic flow. We also refer to [98] for earlier related results in the case of quantum maps.

In [141], Zelditch proposed that, on a real analytic manifold (M, g) , one should consider the vanishing locus of the complexified eigenfunctions rather than the one of real eigenfunctions. In some very vague sense, one can expect that things become simpler as when one goes from real roots of a polynomial to complex ones. Let us be a little bit more precise. If (M, g) is a real analytic compact manifold, there is a natural complexification of the manifold which can be identified with some ball bundle $B_{\epsilon_0}^* M := \{(x, \xi) \in T^* M : \|\xi\|_x < \epsilon_0\}$ [141]. This complexification is known as the Grauert tube of (M, g) . Building on a

seminal work of Boutet de Monvel [25] – see also [85], Zelditch explained how a solution ψ_λ to (1) can be analytically continued to the Grauert tube into a function $\psi_\lambda^{\mathbb{C}}$.

Example 7.3. In order to understand this procedure, one can take the example of the flat torus \mathbb{T}^n . In that case, the Grauert tube is given by

$$\{z = x + iy : x \in \mathbb{T}^n, y \in \mathbb{R}^n\},$$

and an eigenfunction like $\cos(n.x)$ naturally extends as $\cos(n.z)$.

Once one has defined these analytic continuations of eigenfunctions, it is natural to consider their “complex zeros”:

$$\mathcal{Z}_{\psi_\lambda^{\mathbb{C}}} := \{(x, \xi) \in B_{\epsilon_0}^* M : \psi_\lambda^{\mathbb{C}}(x, \xi) = 0\}.$$

With these conventions, Zelditch proved [141]:

Theorem 7.4. *Let (M, g) be a real analytic manifold and let $(\psi_\lambda)_{\lambda \rightarrow +\infty}$ be a sequence of solutions to (1) generating a single semiclassical measure μ . Suppose that, for any non empty open subset ω of S^*M , $\mu(\omega) > 0$. Then, for any $\theta \in \Omega_c^{n-1, n-1}(B_{\epsilon_0}^* M \setminus 0)$, one has*

$$\lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} \int_{\mathcal{Z}_{\psi_\lambda^{\mathbb{C}}}} \theta = \frac{i}{\pi} \int_{B_{\epsilon_0}^* M} \partial \bar{\partial} \|\xi\| \wedge \theta.$$

In other words, this Theorem shows that the complex zeros of a sequence of Laplace eigenfunctions become equidistributed inside the Grauert tube under a certain condition on their semiclassical measure. The relation with Wigner distribution becomes slightly more clear when one applies Lelong-Poincaré formula:

$$\int_{\mathcal{Z}_{\psi_\lambda^{\mathbb{C}}}} \theta = \frac{i}{2\pi} \int_{B_{\epsilon_0}^* M} \partial \bar{\partial} \log |\psi_\lambda^{\mathbb{C}}|^2 \wedge \theta.$$

Hence one can reduce the problem (up to an integration by part) to the description of the limit of $\frac{1}{\lambda} \log |\psi_\lambda^{\mathbb{C}}|^2$ in the space of distributions (or currents). To that aim, one can define

$$U_\lambda^{\mathbb{C}}(x, \xi) := \frac{\psi_\lambda^{\mathbb{C}}(x, \xi)}{\|\psi_\lambda^{\mathbb{C}}\|_{L^2(\partial B_{\|\xi\|}^* M)}}.$$

Hence, the problem amounts to compute the weak limit of

$$\frac{1}{\lambda} \log |\psi_\lambda^{\mathbb{C}}|^2 = \frac{1}{\lambda} \log |U_\lambda^{\mathbb{C}}|^2 + \frac{2}{\lambda} \log \|\psi_\lambda^{\mathbb{C}}\|_{L^2(\partial B_{\|\xi\|}^* M)}.$$

Going through the analytic continuation procedure, one can verify that the second term converges weakly to $2\|\xi\|$ [141, §4.2] and this does not depend on the properties of the semiclassical measure. Then, Zelditch shows that the weak limits of $|U_\lambda^{\mathbb{C}}|^2|_{\partial B_{\|\xi\|}^* M}$ are in fact given by the semiclassical measure μ of our sequence of eigenfunctions and he proves that $\frac{1}{\lambda} \log |U_\lambda^{\mathbb{C}}|^2$ is a bounded sequence of subharmonic functions [141, §5.1]. In particular, either the sequence tends uniformly to $-\infty$ on every compact, or it converges to some $v \leq 0$ in L_{loc}^1 . The first case cannot occur as μ is a probability measure. Now, one has to verify

that v is equal to 0. If not, this would mean that one can find a nonempty open subset ω where the semiclassical measure is 0 and thus contradict our assumption on μ – see [141, §5.1] for details.

As stated in [141], one has the following consequence of Theorem 3.5:

Corollary 7.5. *Let (M, g) be a real analytic manifold such that L_1 is ergodic. Then, for any orthonormal basis $(\psi_j)_{j \geq 1}$ of Laplace eigenfunctions, one can find a density 1 subset S inside \mathbb{Z}_+^* such that, for any $\theta \in \Omega_c^{n-1, n-1}(B_{\epsilon_0}^* M \setminus 0)$, one has*

$$\lim_{j \rightarrow +\infty, j \in S} \frac{1}{\lambda_j} \int_{\mathcal{Z}_{\psi_j^c}} \theta = \frac{i}{\pi} \int_{B_{\epsilon_0}^* M} \partial \bar{\partial} \|\xi\| \wedge \theta.$$

Now regarding the recent developments of Dyatlov and Jin (see Theorem 6.20), one has also

Corollary 7.6. *Let (M, g) be a real analytic surface with constant negative curvature and let $(\psi_\lambda)_{\lambda \rightarrow +\infty}$ be a sequence of solutions to (1). Then, for any $\theta \in \Omega_c^{n-1, n-1}(B_{\epsilon_0}^* M \setminus 0)$, one has*

$$\lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} \int_{\mathcal{Z}_{\psi_\lambda^c}} \theta = \frac{i}{\pi} \int_{B_{\epsilon_0}^* M} \partial \bar{\partial} \|\xi\| \wedge \theta.$$

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