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► **To cite this version:**

Camille Castera, Jérôme Bolte, Cédric Févotte, Edouard Pauwels. An Inertial Newton Algorithm for Deep Learning. 2020. hal-02140748v4

HAL Id: hal-02140748

<https://hal.science/hal-02140748v4>

Preprint submitted on 12 Oct 2020 (v4), last revised 20 Aug 2021 (v6)

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An Inertial Newton Algorithm for Deep Learning

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Keywords: Deep learning, nonconvex optimization, second-order methods, dynamical systems, stochastic optimization.

Abstract

We introduce a new second-order inertial optimization method for machine learning called INDIAN. It exploits the geometry of the loss function while only requiring stochastic approximations of the function values and the generalized gradients. This makes INDIAN fully implementable and adapted to large-scale optimization problems such as the training of deep neural networks. The algorithm combines both gradient-descent and Newton-like behaviors as well as inertia. We prove the convergence of INDIAN for most deep learning problems. To do so, we provide a well-suited framework to analyze deep learning loss functions involving tame optimization in which we study the continuous dynamical system together with the discrete stochastic approximations. We prove sublinear convergence for the continuous-time differential inclusion which underlies our algorithm. Besides, we also show how standard optimization mini-batch methods applied to nonsmooth nonconvex problems can yield a certain type of spurious stationary points never discussed before. We address this issue by providing a theoretical framework around the new idea of D -criticality; we then give a simple asymptotic analysis of INDIAN. Our algorithm allows for using an aggressive learning rate of $o(1/\log k)$. From an empirical viewpoint, we show that INDIAN returns competitive results with respect to state of the art (stochastic gradient descent, ADAGRAD, ADAM) on popular deep learning benchmark problems.

1. Introduction

Can we devise a learning algorithm for general deep neural networks (DNNs) featuring inertia and Newtonian directional intelligence only by means of backpropagation? In an optimization jargon: can we use second-order ideas in time and space for *nonsmooth nonconvex* optimization by only

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using a subgradient oracle? Before providing some answers to this question, let us have a glimpse at some of the fundamental optimization algorithms for training DNNs.

The backpropagation algorithm is, to this day, the fundamental block to compute gradients in deep learning (DL). It is used in most instances of the Stochastic Gradient Descent (SGD) algorithm (Robbins and Monro, 1951). The latter is powerful, flexible, capable of handling large-size problems, noise, and further comes with theoretical guarantees of many kinds. We refer to Bottou and Bousquet (2008), Moulines and Bach (2011) in a convex machine learning context and Bottou et al. (2018) for a recent account highlighting the importance of DL applications and their challenges. In the nonconvex setting, recent works of Adil (2018), Davis et al. (2019) follow the *Ordinary Differential Equations (ODE) approach* introduced in Ljung (1977), and further developed in Benaïm (1999), Kushner and Yin (2003), Benaïm et al. (2005), Borkar (2009). Two research directions have been explored in order to improve SGD’s training efficiency:

- using local geometry of empirical loss functions to obtain steeper descent directions,
- using past steps history to design more larger step sizes in the present.

The first approach is akin to quasi-Newton methods while the second revolves around Polyak’s inertial method (Polyak, 1964). The latter is inspired by the following appealing mechanical thought-experiment. Consider a heavy ball evolving on the graph of the loss (the loss *landscape*), subject to gravity and stabilized by some friction effects. Friction generates energy dissipation, so that the particle will eventually reach a steady state which one hopes to be a local minimum. These two approaches are already present in the DL literature: among the most popular algorithms for training DNNs, ADAGRAD (Duchi et al., 2011) features local geometrical aspects while ADAM (Kingma and Ba, 2014) combines inertial ideas with step sizes similar to the ones of ADAGRAD. Stochastic Newton and quasi-Newton algorithms have been considered by Martens (2010), Byrd et al. (2011, 2016) and recently reported to perform efficiently on several problems (Berahas et al., 2017, Xu et al., 2017). The work of Wilson et al. (2017) demonstrates that carefully tuned SGD and heavy-ball algorithms are competitive with concurrent methods.

However, deviating from the simplicity of SGD also comes with major challenges because of the high dimensionality of DL problems and the severe absence of regularity in DL (differential regularity is generally absent, but even weaker regularity such as semi-convexity or Clarke regularity are not always available). All sorts of practical and theoretical hardships are met: computing and even defining the Hessians is delicate, inverting them is unthinkable to this day, first and second-order Taylor approximations are unavailable due to nonsmoothness, and finally one has to deal with “shocks” which are inherent to inertial approaches in a nonsmooth context (“corners” and “walls” in the landscape of the loss generate velocity discontinuity). This makes in particular the study of the popular algorithms ADAGRAD and ADAM in full generality quite difficult, with recent progresses reported in Barakat and Bianchi (2018).

Our approach is inspired by the following continuous-time dynamical system introduced in Alvarez et al. (2002) and referred to as DIN (standing for “dynamical inertial Newton”):

$$\underbrace{\ddot{\theta}(t)}_{\text{Inertial term}} + \underbrace{\alpha \dot{\theta}(t)}_{\text{Friction term}} + \underbrace{\beta \nabla^2 \mathcal{J}(\theta(t)) \dot{\theta}(t)}_{\text{Newtonian effects}} + \underbrace{\nabla \mathcal{J}(\theta(t))}_{\text{Gravity effect}} = 0, \quad \text{for } t \in [0, +\infty), \quad (1)$$

where t is the time parameter which acts as a continuous epoch counter, \mathcal{J} is a given loss function (e.g., the empirical loss in DL applications), for now assumed C^2 (twice-differentiable), with gradient $\nabla \mathcal{J}$ and Hessian $\nabla^2 \mathcal{J}$. It can be shown that solutions to (1) converge to critical points of \mathcal{J} (Alvarez

et al., 2002). As such the discretization of (1) can support the design of algorithms that optimize \mathcal{J} and that leverage inertial properties with Newton’s method. To adapt this dynamics to DL we must first overcome the computational or conceptual difficulties raised by the second-order objects $\ddot{\theta}$ and $\nabla^2\mathcal{J}(\theta)$ appearing in (1). To do this, we propose in this paper to combine a phase-space lifting method introduced in Alvarez et al. (2002) with the use of Clarke subdifferential $\partial\mathcal{J}$. Clarke subdifferential defines a notion of differentiability for nonconvex and nonsmooth functions. This approach results in the study of a first-order differential inclusion in place of (1), namely,

$$\begin{cases} \dot{\theta}(t) + \beta\partial\mathcal{J}(\theta(t)) & +(\alpha - \frac{1}{\beta})\theta(t) + \frac{1}{\beta}\psi(t) \ni 0 \\ \dot{\psi}(t) & +(\alpha - \frac{1}{\beta})\theta(t) + \frac{1}{\beta}\psi(t) \ni 0 \end{cases}, \quad \text{for a.e. } t \in (0, +\infty). \quad (2)$$

This differential inclusion can then be discretized to obtain the practical algorithm INDIAN that is introduced in Section 2, together with a rigorous presentation of the concepts mentioned above.

Computation of the (sub)gradients and convergence proofs (in batch or mini-batch settings) typically rely on the sum-rule in smooth or convex settings, i.e., $\partial(\mathcal{J}_1 + \mathcal{J}_2) = \partial\mathcal{J}_1 + \partial\mathcal{J}_2$. Unfortunately this sum-rule does not hold in general in the nonconvex setting using the standard Clarke subdifferential. Yet, many DL studies ignore the failure of the sum rule: they use it in practice, but circumvent the theoretical problem by modeling their method through simple dynamics (e.g., smooth or convex). We tackle this difficulty as is, and show that such practice can create additional spurious stationary points that are not Clarke-critical. To address this question, we introduce the notion of D -criticality. It is less stringent than Clarke-criticality and it describes more accurately real-world implementation. We then show convergence of INDIAN to such D -critical points. Our theoretical results are general, simple and allow for aggressive step-sizes in $o(1/\log k)$. We first provide adequate calculus rules and tame nonsmooth Sard’s-like results for the new steady states we introduced. We then combine these results with a Lyapunov analysis from Alvarez et al. (2002) and the differential inclusion approximation method (Benaïm et al., 2005) to characterize the asymptotics of our algorithm similarly to Davis et al. (2019), Adil (2018). This provides a strong theoretical ground to our study since we can prove that our method converges to a connected component of the set of steady states even for networks with ReLU or other nonsmooth activation functions. For the smooth deterministic dynamics, we also show that convergence in values is of the form $O(1/t)$ where t is the running time. For doing so we provide a general result for the solutions of a family of differential inclusions having a certain type of favorable Lyapunov functions.

Our algorithm INDIAN shows great efficiency in practice. It has similar computational complexity to state-of-the-art methods SGD, ADAGRAD and ADAM, often achieves better training accuracy and shows good robustness to hyper-parameters selection. INDIAN can avoid parasitic oscillations and produce acceleration; a first illustration of the behavior of the induced dynamics is given in Figure 1 for a simple nonsmooth and nonconvex function in \mathbb{R}^2 .

The rest of the paper is organized as follows. INDIAN is introduced in details in Section 2 and its convergence is established in Section 3. Convergence rates of the underlying continuous time differential inclusion are obtained in Section 4. Section 5 describes experimental DL results on synthetic and real data sets (MNIST, CIFAR10, CIFAR100).

2. INDIAN: an Inertial Newton Algorithm for Deep Neural Networks

We first introduce our functional framework and then we describe step by step the process of building INDIAN from (1).

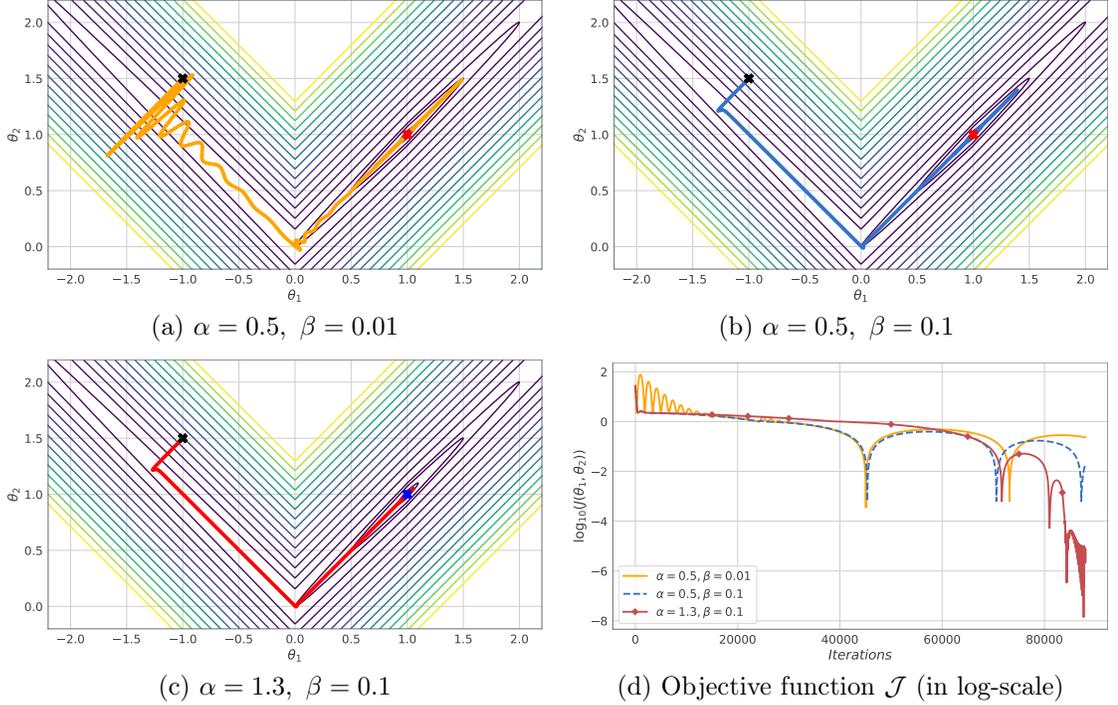


Figure 1: Illustration of INDIAN applied to the nonsmooth function $\mathcal{J}(\theta_1, \theta_2) = 100(\theta_2 - |\theta_1|)^2 + |1 - \theta_1|$. Subplots (a-c) represent the trajectories of the parameters θ_1 and θ_2 in \mathbb{R}^2 for three choices of hyper-parameters α and β , see (1) for an intuitive explanation. Subplot (d) displays the values of the objective function $\mathcal{J}(\theta_1, \theta_2)$ for the three settings considered.

2.1 Neural Networks with Lipschitz Continuous Prediction Function and loss functions

We consider DNNs of a general type represented by a function $f : (x, \theta) \in \mathbb{R}^M \times \mathbb{R}^P \mapsto y \in \mathbb{R}^D$ that is a *locally Lipschitz continuous* in θ . This includes for instance feed-forward, convolutional or residual networks used with ReLU, sigmoid, or tanh activation functions. Recall that a function $F : \mathbb{R}^P \rightarrow \mathbb{R}$ is locally Lipschitz continuous, if for any $\theta \in \mathbb{R}^P$, there exists a neighborhood V of θ and a constant $C > 0$ such that for any $\theta_1, \theta_2 \in V$,

$$|F(\theta_1) - F(\theta_2)| \leq C \|\theta_1 - \theta_2\|,$$

where $\|\cdot\|$ is any norm on \mathbb{R}^P . A function $F : \mathbb{R}^P \rightarrow \mathbb{R}^D$ is locally Lipschitz continuous if each of its coordinates is locally Lipschitz continuous. The variable $\theta \in \mathbb{R}^P$ is the parameter of the model (P can be very large), while $x \in \mathbb{R}^M$ and $y \in \mathbb{R}^D$ represent input and output data. For instance, the vector x may embody an image while y is a label explaining its content. Consider further a data set of N samples $(x_n, y_n)_{n=1, \dots, N}$. Training the network amounts to finding a value of the parameter θ such that, for each input data x_n of the data set, the output $f(x_n, \theta)$ of the model predicts the

real value y_n with good accuracy. To do so, we follow the traditional approach of minimizing an empirical risk loss function,

$$\mathbb{R}^P \ni \theta \mapsto \mathcal{J}(\theta) = \sum_{n=1}^N l(f(x_n, \theta), y_n), \quad (3)$$

where $l : \mathbb{R}^D \times \mathbb{R}^D \rightarrow \mathbb{R}$ is a locally Lipschitz continuous dissimilarity measure. In the sequel, for $n \geq 1$, we will sometimes denote by \mathcal{J}_n the n -th term of the sum: $\mathcal{J}_n(\theta) \triangleq l(f(x_n, \theta), y_n)$, so that $\mathcal{J} = \sum_{n=1}^N \mathcal{J}_n$. Despite the nonsmoothness and the nonconvexity of typical DL loss functions, they generally possess a very strong property sometimes called tameness. We now introduce this notion which is essential to obtain the convergence results of Section 3.

2.2 Neural Networks and Tameness in a Nutshell

Tameness refers to an ubiquitous geometrical property of loss functions and constraints encompassing most finite dimensional optimization problems met in practice. Prominent classes of tame objects are piecewise-linear or piecewise-polynomial objects (with finitely many pieces), and more generally, semi-algebraic objects. However, the notion is much more general, as we intend to convey below. The formal definition is given at the end of this subsection (Definition 1).

Informally, sets or functions are called tame when they can be described by a finite number of basic formulas, inequalities, or Boolean operations involving standard functions such as polynomial, exponential, or max functions. We refer to Attouch et al. (2010) for illustrations, recipes and examples within a general optimization setting or Davis et al. (2019) for illustrations in the context of neural networks. The reader is referred to Van den Dries (1998), Coste (2000), Shiota (2012) for foundational material. To apprehend the strength behind tameness it is convenient to remember that it models nonsmoothness by confining the study to sets and functions which are union of smooth pieces. This is the so-called *stratification* property of tame sets and functions. It was this property which motivated the term of *tame topology*,¹ see Van den Dries (1998). In a nonconvex optimization settings, the stratification property is crucial to generalize qualitative algorithmic results to nonsmooth objects.

All finite dimensional DL optimization models we are aware of yield tame loss functions \mathcal{J} . To understand this assertion and illustrate the wide scope of tameness assumptions, let us provide concrete examples (see also Davis et al. (2019)). Assume that the DNNs under consideration are built from the following traditional components:

- the network architecture describing f is fixed with an arbitrary number of layers of arbitrary dimensions and arbitrary Directed Acyclic Graph (DAG) representing computation,
- the activation functions are among classical ones: ReLU, sigmoid, SQNL, RReLU, tanh, APL, soft plus, soft clipping, and many others including multivariate activations (norm, sorting), or activations defined piecewise with polynomials, exponential and logarithm,
- the dissimilarity function l is a standard loss such as ℓ_p norms, logistic loss or cross-entropy or more generally a function defined piecewise using polynomials, exponentials and logarithms,

then one can easily show, by elementary quantifier elimination arguments (property (iii) below), that the corresponding loss, \mathcal{J} , is tame.

1. “*La topologie modérée*” wished for by Grothendieck.

For the sake of completeness, we provide below the formal definition of tameness and o-minimality.

Definition 1 [o-minimal structure] (Coste, 2000, Definition 1.5) An *o-minimal* structure on $(\mathbb{R}, +, \cdot)$ is a countable collection of sets $\mathcal{O} = \{\mathcal{O}_q\}_{q \geq 1}$ where each \mathcal{O}_q is itself a collection of subsets of \mathbb{R}^q , called *definable* subsets. They must have the following properties, for each $q \geq 1$:

- (i) (Boolean properties) \mathcal{O}_q contains the empty set, is stable by finite union, finite intersection and complementation;
- (ii) (Lifting property) if A belongs to \mathcal{O}_q , then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belong to \mathcal{O}_{q+1} .
- (iii) (Projection or quantifier elimination property) if $\Pi : \mathbb{R}^{q+1} \rightarrow \mathbb{R}^q$ is the canonical projection onto \mathbb{R}^q then for any A in \mathcal{O}_{q+1} , the set $\Pi(A)$ belongs to \mathcal{O}_q .
- (iv) (Semi-algebraicity) \mathcal{O}_q contains the family of algebraic subsets of \mathbb{R}^q , that is, every set of the form

$$\{\theta \in \mathbb{R}^q \mid \zeta(\theta) = 0\},$$

where $\zeta : \mathbb{R}^q \rightarrow \mathbb{R}$ is a polynomial function.

- (v) (Minimality property), the elements of \mathcal{O}_1 are exactly the finite unions of intervals and points.

A mapping $F : S \subset \mathbb{R}^m \rightarrow \mathbb{R}^q$ is said to be *definable in \mathcal{O}* if its graph is definable in \mathcal{O} as a subset of $\mathbb{R}^m \times \mathbb{R}^q$. For illustration of o-minimality in the context of optimization one is referred to Attouch et al. (2010), Davis et al. (2019).

From now on we fix an o-minimal structure \mathcal{O} and a set or a mapping definable in \mathcal{O} will be called *tame*.

2.3 From DIN to INDIAN

We describe in this section the construction of our proposed algorithm INDIAN from the discretization of the second-order ODE (1).

2.3.1 HANDLING NONSMOOTHNESS AND NONCONVEXITY

We first show how the formalism offered by Clarke's subdifferential can be applied to generalize (1) to the nonsmooth nonconvex setting. Recall that the dynamical system (1) is described by,

$$\ddot{\theta}(t) + \alpha \dot{\theta}(t) + \beta \nabla^2 \mathcal{J}(\theta(t)) \dot{\theta}(t) + \nabla \mathcal{J}(\theta(t)) = 0, \quad (4)$$

where \mathcal{J} is a twice-differentiable potential, $\alpha > 0$, $\beta > 0$ are two hyper-parameters and $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}^P$. We cannot exploit (4) directly since in most DL applications \mathcal{J} is not twice differentiable (and even not differentiable at all). We first overcome the explicit use of the Hessian matrix $\nabla^2 \mathcal{J}$ by introducing an auxiliary variable $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}^P$ like in Alvarez et al. (2002). Consider the following dynamical system (defined for \mathcal{J} merely differentiable),

$$\begin{cases} \dot{\theta}(t) + \beta \nabla \mathcal{J}(\theta(t)) & + (\alpha - \frac{1}{\beta})\theta(t) + \frac{1}{\beta}\psi(t) = 0 \\ \dot{\psi}(t) & + (\alpha - \frac{1}{\beta})\theta(t) + \frac{1}{\beta}\psi(t) = 0 \end{cases}, \quad \text{for a.e. } t \in (0, +\infty). \quad (5)$$

As explained in Alvarez et al. (2002), (4) is equivalent to (5) when \mathcal{J} is twice differentiable. Indeed, one can rewrite (4) into (5) by introducing $\psi = -\beta \dot{\theta} - \beta^2 \nabla \mathcal{J}(\theta) - (\alpha\beta - 1)\theta$. Conversely, one can

substitute the first line of (5) into the second one to retrieve (4). Note however that (5) does not require the existence of second-order derivatives.

Let us now introduce a new *nonconvex nondifferentiable* version of (5). By Rademacher's theorem, locally Lipschitz continuous functions $\mathcal{J} : \mathbb{R}^P \rightarrow \mathbb{R}$ are differentiable almost everywhere. Denote by \mathbb{R} the set of points where \mathcal{J} is differentiable. Then, $\mathbb{R}^P \setminus \mathbb{R}$ has zero Lebesgue measure. It follows that for any $\theta^* \in \mathbb{R}^P \setminus \mathbb{R}$, there exists a sequence of points in \mathbb{R} whose limit is this θ^* . This motivates the introduction of the subdifferential due to Clarke (1990), defined next.

Definition 2 (Clarke subdifferential of Lipschitz functions) *For any locally Lipschitz continuous function $F : \mathbb{R}^P \rightarrow \mathbb{R}$, the Clarke subdifferential of F at $\theta \in \mathbb{R}^P$, denoted $\partial F(\theta)$, is the set defined by,*

$$\partial F(\theta) = \text{conv} \left\{ v \in \mathbb{R}^P \mid \exists (\theta_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \text{ such that } \theta_k \xrightarrow[k \rightarrow \infty]{} \theta \text{ and } \nabla F(\theta_k) \xrightarrow[k \rightarrow \infty]{} v \right\}, \quad (6)$$

where conv denotes the convex hull operator. The elements of the Clarke subdifferential are called *Clarke subgradients*.

The Clarke subdifferential is a nonempty compact convex set. It coincides with the gradient for smooth functions and with the traditional subdifferential for nonsmooth convex functions. As already mentioned, and contrarily to the (sub)differential operator, it does not enjoy a sum rule.

Thanks to Definition 2, we can extend (5) to nondifferentiable functions. Since $\partial \mathcal{J}(\theta)$ is a set, we no longer study a differential equation but rather a *differential inclusion*, given by,

$$\begin{cases} \dot{\theta}(t) + \beta \partial \mathcal{J}(\theta(t)) & + (\alpha - \frac{1}{\beta})\theta(t) + \frac{1}{\beta}\psi(t) \ni 0 \\ \dot{\psi}(t) & + (\alpha - \frac{1}{\beta})\theta(t) + \frac{1}{\beta}\psi(t) \ni 0 \end{cases}, \quad \text{for a.e. } t \in (0, +\infty). \quad (7)$$

For a given initial condition $(\theta_0, \psi_0) \in \mathbb{R}^P \times \mathbb{R}^P$, we call *solution* (or *trajectory*) of this system any absolutely continuous curve (θ, ψ) from \mathbb{R}_+ to $\mathbb{R}^P \times \mathbb{R}^P$ for which $(\theta(0), \psi(0)) = (\theta_0, \psi_0)$ and (7) holds. We recall that absolute continuity amounts to the fact that θ is differentiable almost everywhere with integrable derivative and,

$$\theta(t) - \theta(0) = \int_0^t \dot{\theta}(s) \, ds, \quad \text{for } t \in [0, +\infty).$$

Due to the properties of the Clarke subdifferential, existence of a solution to differential inclusions such as (7) is ensured, see Aubin and Cellina (2012); note however that uniqueness of the solution does not hold in general. We will now use the structure of (7) to build a new algorithm to train DNNs.

2.3.2 DISCRETIZATION OF THE DIFFERENTIAL INCLUSION

To obtain the basic form of our algorithm, we discretize (7) according to the classical explicit Euler method. Given (θ, ψ) a solution of (7) and any time t_k , set $\theta_k = \theta(t_k)$ and $\psi_k = \psi(t_k)$. Then, at time $t_{k+1} = t_k + \gamma_k$ with γ_k positive small, one can approximate $\dot{\theta}(t_{k+1})$ and $\dot{\psi}(t_{k+1})$ by

$$\dot{\theta}(t_{k+1}) \simeq \frac{\theta_{k+1} - \theta_k}{\gamma_k}, \quad \dot{\psi}(t_{k+1}) \simeq \frac{\psi_{k+1} - \psi_k}{\gamma_k}.$$

This discretization yields the following algorithm,

$$\begin{cases} v_k & \in \partial \mathcal{J}(\theta_k) \\ \theta_{k+1} & = \theta_k + \gamma_k \left(\left(\frac{1}{\beta} - \alpha \right) \theta_k - \frac{1}{\beta} \psi_k - \beta v_k \right) \\ \psi_{k+1} & = \psi_k + \gamma_k \left(\left(\frac{1}{\beta} - \alpha \right) \theta_k - \frac{1}{\beta} \psi_k \right) \end{cases} \quad (8)$$

Although the algorithm above is well defined for our problem, it is not suited to DL. First, the computation of $\partial \mathcal{J}(\theta_k)$ is generally not possible since there is no general operational calculus for Clarke’s subdifferential; secondly, a mini-batch strategy must be designed to cope with the large dimensionality of DL problems which makes the absence of sum rule even more critical.

The next section is meant to address these issues and design a practical algorithm.

2.3.3 INDIAN ALGORITHM WITH A NEW NOTION OF STEADY STATES

In order to compute or approximate the subdifferential of \mathcal{J} at each iteration and to cope with large data sets, \mathcal{J} can be approximated by mini-batches, reducing the memory footprint and computational cost of evaluation. For any $\mathbf{B} \subset \{1, \dots, N\}$, let us define

$$\mathcal{J}_{\mathbf{B}}: \theta \mapsto \sum_{n \in \mathbf{B}} l(f(x_n, \theta), y_n). \quad (9)$$

Unlike in the differentiable case, subgradients do not in general sum up to a subgradient of the sum, that is $\partial \mathcal{J}_{\mathbf{B}}(\theta) \neq \sum_{n \in \mathbf{B}} \partial l(f(x_n, \theta), y_n)$ in general. To see this, take for example $0 = |\cdot| - |\cdot|$, the Clarke subgradient of this function at 0 is $\{0\}$, whereas $\partial(|0|) + \partial(-|0|) = [-1, 1] + [-1, 1] = [-2, 2]$. Standard DL solvers use backpropagation algorithms which implement smooth calculus on nonsmooth and nonconvex objects. Due to the absence of qualification conditions, *the resulting objects are not Clarke subgradients in general*. In order to match the real-world practice of DL, we introduce a notion of steady states that corresponds to the stationary points generated by a generic mini-batch approach. As we shall see, this allows both for practical applications and convergence analysis (despite the sum rule failure for Clarke subdifferential). We emphasize once more that our goal is to *capture the stationary points that are actually met in practice*.

For any $\mathbf{B} \subset \{1, \dots, N\}$, we introduce the following objects,

$$D\mathcal{J}_{\mathbf{B}} = \sum_{n \in \mathbf{B}} \partial [l(f(x_n, \cdot), y_n)], \quad D\mathcal{J} = \sum_{n=1}^N \partial [l(f(x_n, \cdot), y_n)]. \quad (10)$$

Observe that, for each \mathbf{B} , we have $D\mathcal{J}_{\mathbf{B}} \supset \partial \mathcal{J}_{\mathbf{B}}$ and that $\mathcal{J}_{\mathbf{B}}$ is differentiable almost everywhere with $D\mathcal{J}_{\mathbf{B}} = \partial \mathcal{J}_{\mathbf{B}} = \{\nabla \mathcal{J}_{\mathbf{B}}\}$, see Clarke (1990). In particular $D\mathcal{J} = \partial \mathcal{J}$ almost everywhere so that the potential differences with the Clarke subgradient occur on a negligible set. When \mathcal{J} is tame the equalities even hold on the complement of a finite union of manifolds of dimension strictly lower than P —use the classical stratification results for o-minimal structures, Coste (2000). A point satisfying $D\mathcal{J}(\theta) \ni 0$ will be called *D-critical*. Note that Clarke-critical points ($0 \in \partial \mathcal{J}$) are *D-critical* points but that the converse is not true. This terminology is motivated by favorable properties: sum and chain rules along curves (see Lemmas 4 and 5 below) and the existence of a tame Sard’s theorem (see Lemma 6). To our knowledge, this notion of a steady state has not previously been used in the literature.² To the best of our knowledge a direct approach modelling the

2. In a follow-up of this paper, Bolte and Pauwels (2020a) have further developed the present ideas and in particular the connection to the backpropagation algorithm.

mini-batch practice has never been considered before. While this notion is needed for the theoretical analysis, one should keep in mind that $D\mathcal{J}$ is actually what is computed numerically provided that the automatic differentiation library returns a Clarke subgradient. This computation is usually done with a backpropagation algorithm, similarly to the seminal method of Rumelhart and Hinton (1986).

Ultimately, one can rewrite (7) by replacing $\partial\mathcal{J}$ by $D\mathcal{J}$, which yields a differential inclusion adapted to study mini-batch approximations of nonsmooth loss functions \mathcal{J} . This reads,

$$\begin{cases} \dot{\theta}(t) + \beta D\mathcal{J}(\theta(t)) & +(\alpha - \frac{1}{\beta})\theta(t) + \frac{1}{\beta}\psi(t) \ni 0 \\ \dot{\psi}(t) & +(\alpha - \frac{1}{\beta})\theta(t) + \frac{1}{\beta}\psi(t) \ni 0 \end{cases}, \quad \text{for a.e. } t \in (0, +\infty). \quad (11)$$

Discretizing this system gives a workable version of INDIAN. Let us consider a sequence $(\mathbf{B}_k)_{k \in \mathbb{N}}$ of nonempty subsets of $\{1, \dots, N\}$, chosen independently and uniformly at random with replacement, and a sequence of positive step sizes $(\gamma_k)_{k \in \mathbb{N}}$. For a given initialization $(\theta_0, \psi_0) \in \mathbb{R}^P \times \mathbb{R}^P$, at iteration $k \geq 1$, our algorithm reads,

$$\text{(INDIAN)} \quad \begin{cases} v_k & \in D\mathcal{J}_{\mathbf{B}_k}(\theta_k) \\ \theta_{k+1} & = \theta_k + \gamma_k \left((\frac{1}{\beta} - \alpha)\theta_k - \frac{1}{\beta}\psi_k - \beta v_k \right) \\ \psi_{k+1} & = \psi_k + \gamma_k \left((\frac{1}{\beta} - \alpha)\theta_k - \frac{1}{\beta}\psi_k \right) \end{cases} \quad (12)$$

Here again $\alpha > 0$ and $\beta > 0$ are hyper-parameters of the algorithm. The mini-batch procedure forms a stochastic approximation of the deterministic dynamics obtained by choosing $\mathbf{B}_k = \{1, \dots, N\}$, i.e., when $\mathcal{J}_{\mathbf{B}_k} = \mathcal{J}$ (batch version). This can be seen by observing that the vectors v_k above may be written as $v_k = \tilde{v}_k + \eta_k$, where $\tilde{v}_k \in D\mathcal{J}(\theta_k)$ and η_k compensates for the missing subgradients and can be seen as a zero-mean noise. Hence, INDIAN admits the following general abstract stochastic formulation,

$$\begin{cases} w_k & \in D\mathcal{J}(\theta_k) \\ \theta_{k+1} & = \theta_k + \gamma_k \left((\frac{1}{\beta} - \alpha)\theta_k - \frac{1}{\beta}\psi_k - \beta w_k + \xi_k \right) \\ \psi_{k+1} & = \psi_k + \gamma_k \left((\frac{1}{\beta} - \alpha)\theta_k - \frac{1}{\beta}\psi_k \right) \end{cases} \quad (13)$$

where $(\xi_k)_{k \in \mathbb{N}}$ is a martingale difference noise sequence adapted to the filtration induced by (random) iterates up to k . While (12) is the version implemented in practice, its equivalent form (13) is more convenient for the convergence analysis of the next section. We stress that the equivalence between (12) and (13) relies on the use of $D\mathcal{J}$ and would not hold with the use of $\partial\mathcal{J}$ as in (8).

INDIAN in its general and practical form is summarized in Table 1.

3. Convergence Results for INDIAN

3.1 Main result: Accumulation Points of INDIAN are Critical

We now study the convergence of INDIAN. The main idea here is to prove that the discrete algorithm (12) asymptotically behaves like the solutions of the continuous differential inclusion (11). Besides tameness, our main assumption is the following:

Assumption 1 (Stochastic approximation) *The sets $(\mathbf{B}_k)_{k \in \mathbb{N}}$ are taken independently uniformly at random with replacement. The step-size sequence γ_k is positive with $\sum_k \gamma_k = +\infty$ and satisfies $\gamma_k = o\left(\frac{1}{\log k}\right)$, that is $\limsup_{k \rightarrow +\infty} |\gamma_k \log k| = 0$.*

Inertial Newton Algorithm for Deep Learning (INDIAN)

Objective function: $\mathcal{J} = \sum_{n=1}^N \mathcal{J}_n$, with $\mathcal{J}_n: \mathbb{R}^P \mapsto \mathbb{R}$ locally Lipschitz.

Hyper-parameters: (α, β) positive.

Mini-batches: $(\mathcal{B}_k)_{k \in \mathbb{N}}$, nonempty subsets of $\{1, \dots, N\}$.

Step sizes: $(\gamma_k)_{k \in \mathbb{N}}$ positive.

Initialization: $(\theta_0, \psi_0) \in \mathbb{R}^P \times \mathbb{R}^P$.

For $k \in \mathbb{N}$:

$$\begin{cases} v_k & \in \sum_{n \in \mathcal{B}_k} \partial[\mathcal{J}_n(\theta_k)] \\ \theta_{k+1} & = \theta_k + \gamma_k \left(\left(\frac{1}{\beta} - \alpha\right)\theta_k - \frac{1}{\beta}\psi_k - \beta v_k \right) \\ \psi_{k+1} & = \psi_k + \gamma_k \left(\left(\frac{1}{\beta} - \alpha\right)\theta_k - \frac{1}{\beta}\psi_k \right) \end{cases}$$

Table 1: INDIAN in a nutshell.

Typical admissible choices are $\gamma_k = C(k+1)^{-a}$ with $a \in (0, 1]$, $C > 0$. The main theoretical result of the paper follows.

Theorem 3 (INDIAN converges to the set of D -critical points of \mathcal{J}) *Assume that for $n \in \{1, \dots, N\}$, each \mathcal{J}_n is locally Lipschitz continuous, tame and that the step sizes satisfy Assumption 1. Set an initial condition (θ_0, ψ_0) and assume that there exists $M > 0$ such that $\sup_{k \geq 0} \|(\theta_k, \psi_k)\| \leq M$ almost surely, where $(\theta_k, \psi_k)_{k \in \mathbb{N}}$ are generated by INDIAN. Then, almost surely, any accumulation point $\bar{\theta}$ of the sequence $(\theta_k)_{k \in \mathbb{N}}$ satisfies $D\mathcal{J}(\bar{\theta}) \ni 0$. In addition $(\mathcal{J}(\theta_k))_{k \in \mathbb{N}}$ converges.*

Before proving Theorem 3, we will first make some comments and illustrate this result.

3.2 Comments on the Results of Theorem 3

On the step sizes. First, Assumption 1 offers much more flexibility than the usual $O(1/\sqrt{k})$ assumption commonly used for SGD. We leverage the boundedness assumption on the norms of (θ_k, ψ_k) , the local Lipschitz continuity and finite-sum structure of \mathcal{J} , so that the noise is actually uniformly bounded and hence sub-Gaussian, allowing for much larger step sizes than in the more common bounded second moment setting. See Benaïm et al. (2005, Remark 1.5) and Benaïm (1999) for more details. The interest of this aggressive strategy is highlighted in Figure 4 of the experimental section.

On the scope of the theorem. Our result actually holds for general locally Lipschitz continuous tame functions with finite-sum structure and for the general stochastic process under uniformly bounded martingale increment noise. We do not use any other specific structure of DL loss functions. Other variants could be considered depending on the assumptions on the noise, see Benaïm et al. (2005).

On D -criticality. The result of Theorem 3 states that the bounded discrete trajectories of INDIAN are attracted by the D -critical points. Recall that D -critical points include local minimizers and thus our theoretical finding agrees with our empirical observations that most initializations lead to “valuable weights” θ and to efficient training. In particular for smooth networks where \mathcal{J} is differentiable, limit points of INDIAN are simply critical points of \mathcal{J} . The reader should however remember that when the algorithm is initialized on the D -critical set, the algorithm is stationary as well, *even when the initialization is non-Clarke critical*. This last point shows that D -points are not introduced to simplify the analysis but to *sharply model the use of mini-batch methods on nonconvex and nonsmooth problems*. Hopefully, in practice one can expect to avoid such points with overwhelming probability. Indeed, following our initial introduction of the D -points, Bolte and Pauwels (2020b) proved for SGD, under very mild conditions on the initialization, that D -critical points that are not Clarke-critical are not reached with probability one (see also Bianchi et al. (2020)).

On the boundedness assumption. The bounded assumption on the iterates is a classical assumption for first or second-order algorithms, see for instance Davis et al. (2019), Duchi and Ruan (2018). When using deterministic algorithms (i.e., without mini-batch approximations), properties such as the coercivity of \mathcal{J} can be sufficient to remove the boundedness assumption for descent algorithms. This does not remain true when dealing with mini-batch approximations, yet, in the case of INDIAN, the coercivity of \mathcal{J} would guarantee at least that the solutions of the continuous underlying differential inclusion (11) remain bounded. Indeed, we will prove in Section 3.4 that for any solution (θ, ψ) of (11), the function $E(\theta(t), \psi(t)) \triangleq 2(1 + \alpha\beta)\mathcal{J}(\theta(t)) + \left\| \left(\alpha - \frac{1}{\beta} \right) \theta(t) + \frac{1}{\beta} \psi(t) \right\|^2$ is decreasing in time (see Lemma 9 hereafter). As a consequence, we cannot have $\mathcal{J}(\theta(t)) \xrightarrow[t \rightarrow \infty]{} \infty$ so $\|\theta(t)\| \not\rightarrow \infty$ due to the coercivity of \mathcal{J} . In addition this guarantees $\|\psi(t)\| \not\rightarrow \infty$ as well. However DL loss functions are not coercive in general and studying the boundedness issue in DL or even for nonconvex semi-algebraic problems is far beyond the scope of this paper. Let us however mention that it is not uncommon to project the iterates on a given large ball to ensure boundedness; this is a matter for future research.

3.3 Preliminary Variational Results

Prior to proving Theorem 3, we extend some results known for the Clarke subdifferential of tame functions to the operator D that we previously introduced. First, we recall a useful result of Davis et al. (2019) which follows from the projection formula in Bolte et al. (2007b).

Lemma 4 (Chain rule for the Clarke subdifferential) *Let $\mathcal{J} : \mathbb{R}^P \rightarrow \mathbb{R}$ be a locally Lipschitz continuous tame function, then \mathcal{J} admits a chain rule, meaning that for all absolutely continuous curves $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}^P$, $\mathcal{J} \circ \theta$ is differentiable a.e. and for a.e. $t \geq 0$,*

$$\frac{d\mathcal{J}}{dt}(\theta(t)) = \langle \dot{\theta}(t), \partial\mathcal{J}(\theta(t)) \rangle = \langle \dot{\theta}(t), v \rangle, \quad \forall v \in \partial\mathcal{J}(\theta(t)). \quad (14)$$

Note that, even though \mathcal{J} is nondifferentiable on \mathbb{R}^P , the function $t \mapsto \mathcal{J}(\theta(t))$ is differentiable for a.e. $t > 0$. Indeed, as introduced in Section 2.3.1, an absolutely continuous curve from $t \geq 0$ to \mathbb{R}^P is differentiable for a.e. $t > 0$. This, combined with the chain-rule of Lemma 4 allows to differentiate $\mathcal{J} \circ \theta$ for a.e. $t > 0$ whenever \mathcal{J} is tame and locally Lipschitz continuous. Besides, notice that the

value of $\frac{d\mathcal{J}}{dt}(\theta(t))$ in (14) does not depend on the element v taken in $\partial\mathcal{J}(\theta(t))$ which justifies the notation $\langle \dot{\theta}(t), \partial\mathcal{J}(\theta(t)) \rangle$.

Consider now a function with an additive finite-sum structure (such as in DL):

$$\mathcal{J}: \mathbb{R}^P \ni \theta \mapsto \sum_{n=1}^N \mathcal{J}_n(\theta), \quad (15)$$

where each $\mathcal{J}_n: \mathbb{R}^P \mapsto \mathbb{R}$ is locally Lipschitz continuous and tame. We define for any $\theta \in \mathbb{R}^P$

$$D\mathcal{J}(\theta) = \sum_{n=1}^N \partial\mathcal{J}_n(\theta).$$

The following lemma is a direct generalization of the above chain rule.

Lemma 5 (Chain rule for $D\mathcal{J}$) *Let \mathcal{J} be a sum of tame functions like in (15). Let $c: [0, 1] \mapsto \mathbb{R}^P$ be an absolutely continuous curve so that $t \mapsto \mathcal{J}(c(t))$ is differentiable almost everywhere. For a.e. $t \in [0, 1]$, and for all $v \in D\mathcal{J}(c(t))$,*

$$\frac{d}{dt}\mathcal{J}(c(t)) = \langle v, \dot{c}(t) \rangle.$$

Proof By local Lipschitz continuity and absolute continuity, each \mathcal{J}_n is differentiable almost everywhere and Lemma 4 can be applied:

$$\frac{d}{dt}\mathcal{J}_n(c(t)) = \langle v_n, \dot{c}(t) \rangle, \text{ for all } v_n \in \partial\mathcal{J}_n(c(t)) \text{ and for a.e. } t \geq 0.$$

Thus

$$\frac{d}{dt}\mathcal{J}(c(t)) = \sum_{n=1}^N \frac{d}{dt}\mathcal{J}_n(c(t)) = \sum_{n=1}^N \langle v_n, \dot{c}(t) \rangle,$$

for any $v_n \in \partial\mathcal{J}_n(c(t))$, for all $n = \{1, \dots, N\}$, and for a.e. $t \geq 0$. This proves the desired result. ■

We finish this section with a Sard lemma for D -critical values, in the spirit of Bolte et al. (2007b).

Lemma 6 (A Sard's theorem for tame D -critical values) *Let,*

$$\mathbb{S} = D\text{-crit} \triangleq \{\theta \in \mathbb{R}^P \mid D\mathcal{J}(\theta) \ni 0\},$$

then $\mathcal{J}(\mathbb{S})$ is finite.

Proof The set \mathbb{S} is tame and hence it has a finite number of connected components. It is sufficient to prove that \mathcal{J} is constant on each connected component of \mathbb{S} . Without loss of generality, assume that \mathbb{S} is connected and consider $\theta_0, \theta_1 \in \mathbb{S}$. By Whitney regularity (Van den Dries, 1998, 4.15), there exists a tame continuous path Γ joining θ_0 to θ_1 . Because of the tame nature of the result, we should here conclude with only tame arguments and use the projection formula in Bolte et al. (2007b), but for convenience of readers who are not familiar with this result we use Lemma 4. Since Γ is tame, the monotonicity lemma (see for example Kurdyka (1998, Lemma 2)) gives the existence of a finite collection of real numbers $0 = a_0 < a_1 < \dots < a_q = 1$, such that Γ is C^1 on each segment (a_{j-1}, a_j) , $j = 1, \dots, q$. Applying Lemma 4 to each $\Gamma|_{(a_i, a_{i+1})}$, we see that \mathcal{J} is constant except perhaps on a finite number of points, thus \mathcal{J} is constant by continuity. ■

3.4 Proof of Convergence for INDIAN

Our approach follows the stochastic method for differential inclusions developed in Benaïm et al. (2005) for which the differential system (11) and its Lyapunov properties play fundamental roles.

The steady states of (11) are given by,

$$\mathsf{S} = \{(\theta, \psi) \in \mathbb{R}^P \times \mathbb{R}^P \mid 0 \in D\mathcal{J}(\theta), \psi = (1 - \alpha\beta)\theta\}. \quad (16)$$

These points are initialization values for which the system does not evolve and remains constant. Observe that the first coordinates of these points are D -critical for \mathcal{J} and that conversely any D -critical point of \mathcal{J} corresponds to a unique rest point in S .

Definition 7 (Lyapunov function) *Let A be a subset of $\mathbb{R}^P \times \mathbb{R}^P$, we say that $E : \mathbb{R}^P \times \mathbb{R}^P \rightarrow \mathbb{R}$ is a Lyapunov function for the set A and the dynamics (11) if*

(i) *for any solution (θ, ψ) of (11) with initial condition (θ_0, ψ_0) , we have:*

$$E(\theta(t), \psi(t)) \leq E(\theta_0, \psi_0) \text{ a.e. on } \mathbb{R}.$$

(ii) *for any solution (θ, ψ) of (11) with initial condition $(\theta_0, \psi_0) \notin A$, we have:*

$$E(\theta(t), \psi(t)) < E(\theta_0, \psi_0) \text{ a.e. on } \mathbb{R}.$$

In practice, to establish that a functional is Lyapunov, one can simply use differentiation through chain rule results, with in particular Lemma 4. In the context of INDIAN, we will use Lemma 5. To build a Lyapunov function for the dynamics (11) and the set S , consider the two following energy-like functions,

$$\begin{cases} E_{\min}(\theta(t), \psi(t)) &= (1 - \sqrt{\alpha\beta})^2 \mathcal{J}(\theta(t)) + \frac{1}{2} \left\| \left(\alpha - \frac{1}{\beta} \right) \theta(t) + \frac{1}{\beta} \psi(t) \right\|^2 \\ E_{\max}(\theta(t), \psi(t)) &= (1 + \sqrt{\alpha\beta})^2 \mathcal{J}(\theta(t)) + \frac{1}{2} \left\| \left(\alpha - \frac{1}{\beta} \right) \theta(t) + \frac{1}{\beta} \psi(t) \right\|^2. \end{cases} \quad (17)$$

Then the following lemma applies.

Lemma 8 (Differentiation along DIN trajectories) *Let (θ, ψ) be a solution of (11) with initial condition (θ_0, ψ_0) . For a.e. $t > 0$, θ and ψ are differentiable at t , (11) holds, $\frac{\dot{\theta}(t) - \dot{\psi}(t)}{\beta} \in D\mathcal{J}(\theta(t))$ and*

$$\begin{aligned} \frac{dE_{\min}}{dt}(\theta(t), \psi(t)) &= - \left\| \sqrt{\alpha} \dot{\theta}(t) - \frac{1}{\sqrt{\beta}} \left(\dot{\psi}(t) - \dot{\theta}(t) \right) \right\|^2 \\ \frac{dE_{\max}}{dt}(\theta(t), \psi(t)) &= - \left\| \sqrt{\alpha} \dot{\theta}(t) + \frac{1}{\sqrt{\beta}} \left(\dot{\psi}(t) - \dot{\theta}(t) \right) \right\|^2 \end{aligned}$$

Proof Define $E_\lambda(\theta, \psi) = \lambda \mathcal{J}(\theta) + \frac{1}{2} \left\| \left(\alpha - \frac{1}{\beta} \right) \theta + \frac{1}{\beta} \psi \right\|^2$. We aim at choosing λ so that E_λ is a Lyapunov function. Because \mathcal{J} is tame and locally Lipschitz continuous, using Lemma 5 we know that for any absolutely continuous trajectory $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}^P$ and for a.e. $t > 0$,

$$\frac{d\mathcal{J}}{dt}(\theta(t)) = \langle \dot{\theta}(t), D\mathcal{J}(\theta(t)) \rangle = \langle \dot{\theta}(t), v(t) \rangle, \quad \forall v(t) \in D\mathcal{J}(\theta(t)). \quad (18)$$

Let θ and ψ be solutions of (DIN). For a.e. $t \geq 0$, we can differentiate $E_\lambda(\theta, \psi)$ to obtain

$$\begin{aligned} \frac{dE_\lambda}{dt}(\theta(t), \psi(t)) &= \lambda \langle \dot{\theta}(t), v(t) \rangle + \left(\alpha - \frac{1}{\beta}\right) \langle \dot{\theta}(t), \left(\alpha - \frac{1}{\beta}\right)\theta(t) + \frac{1}{\beta}\psi(t) \rangle \\ &\quad + \frac{1}{\beta} \langle \dot{\psi}(t), \left(\alpha - \frac{1}{\beta}\right)\theta(t) + \frac{1}{\beta}\psi(t) \rangle \end{aligned} \quad (19)$$

for all $v(t) \in D\mathcal{J}(\theta(t))$. Using (11), we get $\frac{1}{\beta}(\dot{\theta}(t) - \dot{\psi}(t)) \in D\mathcal{J}(\theta(t))$ and $-\dot{\psi}(t) = \left(\alpha - \frac{1}{\beta}\right)\theta(t) + \frac{1}{\beta}\psi(t)$ a.e. Choosing $v(t) = \frac{1}{\beta}(\dot{\theta}(t) - \dot{\psi}(t))$ yields:

$$\frac{dE_\lambda}{dt}(\theta(t), \psi(t)) = \lambda \left\langle \dot{\theta}(t), \frac{\dot{\theta}(t) - \dot{\psi}(t)}{\beta} \right\rangle - \left(\alpha - \frac{1}{\beta}\right) \langle \dot{\theta}(t), \dot{\psi}(t) \rangle - \frac{1}{\beta} \langle \dot{\psi}(t), \dot{\psi}(t) \rangle.$$

Then, expressing everything as a function of $\dot{\theta}$ and $\frac{1}{\beta}(\psi - \theta)$, one can show that a.e. on \mathbb{R}_+ :

$$\begin{aligned} \frac{dE_\lambda}{dt}(\theta, \psi)(t) &= -\alpha \|\dot{\theta}(t)\|^2 - \beta \left\| \frac{\dot{\theta}(t) - \dot{\psi}(t)}{\beta} \right\|^2 + (\lambda - \alpha\beta - 1) \langle \dot{\theta}(t), \frac{\dot{\theta}(t) - \dot{\psi}(t)}{\beta} \rangle \\ &= - \left\| \sqrt{\alpha}\dot{\theta}(t) + \frac{\alpha\beta + 1 - \lambda}{2\sqrt{\alpha}} \frac{\dot{\theta}(t) - \dot{\psi}(t)}{\beta} \right\|^2 - \left(\beta - \frac{(\alpha\beta + 1 - \lambda)^2}{4\alpha} \right) \left\| \frac{\dot{\theta}(t) - \dot{\psi}(t)}{\beta} \right\|^2. \end{aligned}$$

We aim at choosing λ so that E_λ is decreasing that is $\left(\beta - \frac{(\alpha\beta + 1 - \lambda)^2}{4\alpha}\right) > 0$. This holds whenever $\lambda \in [(1 - \sqrt{\alpha\beta})^2, (1 + \sqrt{\alpha\beta})^2]$. We choose $\lambda_{\min} = (1 - \sqrt{\alpha\beta})^2$, and $\lambda_{\max} = (1 + \sqrt{\alpha\beta})^2$, for these two values we obtain for a.e. $t > 0$,

$$\begin{cases} \dot{E}_{\lambda_{\min}}(\theta(t), \psi(t)) &= - \left\| \sqrt{\alpha}\dot{\theta}(t) + \frac{1}{\sqrt{\beta}} (\dot{\theta}(t) - \dot{\psi}(t)) \right\|^2 \\ \dot{E}_{\lambda_{\max}}(\theta(t), \psi(t)) &= - \left\| \sqrt{\alpha}\dot{\theta}(t) - \frac{1}{\sqrt{\beta}} (\dot{\theta}(t) - \dot{\psi}(t)) \right\|^2 \end{cases} \quad (20)$$

Remark finally that by definition $E_{\min} = E_{\lambda_{\min}}$ and $E_{\max} = E_{\lambda_{\max}}$. ■

Define $E = E_{\min} + E_{\max}$ and recall that $\mathbf{S} = \{(\theta, \psi) \in \mathbb{R}^P \times \mathbb{R}^P \mid 0 \in DJ(\theta), \psi = (1 - \alpha\beta)\theta\}$. By a direct integration argument, we obtain the following lemma.

Lemma 9 (*E is Lyapunov function for INDIAN with respect to \mathbf{S}*) *For any $(\theta_0, \psi_0) \notin \mathbf{S}$ and any solution (θ, ψ) with initial condition (θ_0, ψ_0) ,*

$$E(\theta(t), \psi(t)) < E(\theta_0, \psi_0), \text{ for a.e. } t > 0. \quad (21)$$

We are now in position to provide the desired proof.

Proof of Theorem 3 Lemmas 8 and 9 state that E is a Lyapunov function for the set \mathbf{S} and the dynamics (11). Let $\mathbf{C} = \{\theta \in \mathbb{R}^P \mid (\theta, \psi) \in \mathbf{S}\}$ which is actually the set of D -critical points of \mathcal{J} . Using Lemma 6 of Section 3.3, $\mathcal{J}(\mathbf{C})$ is finite. Moreover, since $E(\theta, \psi) = 2(1 + \alpha\beta)\mathcal{J}(\theta)$ for all $(\theta, \psi) \in \mathbf{S}$, E takes a finite number of values on \mathbf{S} , and in particular, $E(\mathbf{S})$ has empty interior.

Denote by L the set of accumulation points of the sequences $((\theta_k, \psi_k))_{k \in \mathbb{N}}$ produced by (12) starting at (θ_0, ψ_0) and L_1 its projection on $\mathbb{R}^P \times \{0\}$. We have the 3 following properties:

- By assumption, we have $\|(\theta_k, \psi_k)\| \leq M$ almost surely, for all $k \in \mathbb{N}$.
- By local Lipschitz continuity $\partial \mathcal{J}_B(\theta)$ is uniformly bounded for $\|\theta\| \leq M$ and any $B \subset \{1, \dots, N\}$, hence the centered noise $(\xi_k)_{k \in \mathbb{N}}$ is a uniformly bounded martingale difference sequence.
- By Assumption 1, the sequence $(\gamma_k)_{k \in \mathbb{N}}$ are chosen such that $\gamma_k = o(\frac{1}{\log k})$ (see Section 3.2).

Then the sufficient conditions of Remark 1.5 of Benaïm et al. (2005) state that the discrete process $(\theta_k, \psi_k)_{k \in \mathbb{N}}$ asymptotically behaves like the solutions of (11). We can then combine Proposition 3.27 and Theorem 3.6 of Benaïm et al. (2005), to obtain that the limit set L of the discrete process is contained in the set S where the Lyapunov has vanishing derivatives. Thus the set L_1 (the set of the first coordinates of all accumulation points) contains only D -critical points of \mathcal{J} . In addition, $E(L)$ is a singleton, and for all $(\theta, \psi) \in S$, we have $E(\theta, \psi) = \mathcal{J}(\theta)$, so $\mathcal{J}(L_1)$ is also a singleton and the theorem follows.

4. Towards Convergence Rates for INDIAN

In the previous section, connecting INDIAN to (11) was one of the keys to prove the convergence of the discrete dynamics. Let us now focus on the continuous dynamical system (7) in the deterministic case where \mathcal{J} and $\partial \mathcal{J}$ are not approximated anymore – we thus no longer use $D\mathcal{J}$ although this would be possible but would require more technical proofs. In this section and in this section only, we pertain to loss functions \mathcal{J} that are real semi-algebraic (a particular case of tame functions).³ Recall that a set is called semi-algebraic if it is a finite union of sets of the form,

$$\{\theta \in \mathbb{R}^P \mid \zeta(\theta) = 0, \zeta_i(\theta) < 0\}$$

where ζ, ζ_i are real polynomial functions. A function is called semi-algebraic if its graph is semi-algebraic.

We will prove that the continuous time system (7) is actually a quasi-gradient dynamic, and that we can characterize the convergence rate to critical points. Let us first introduce an essential mechanism to obtain such convergence rates: the Kurdyka-Łojasiewicz (KL) property.

4.1 The Nonsmooth Kurdyka-Łojasiewicz Property for the Clarke Subdifferential

The nonsmooth Kurdyka-Łojasiewicz (KL) property, as introduced in (Bolte et al., 2010), is a measure of “amenability to sharpness” (as illustrated at the end of Section 4.3). Here we provide a uniform version for the Clarke subdifferential of semi-algebraic functions as in Bolte et al. (2007b) and Bolte et al. (2014). In the sequel we denote by “dist” any given distance on \mathbb{R}^P .

Lemma 10 (Uniform Nonsmooth KL Property for the Clarke Subdifferential) *Let K be a nonempty compact set and let $L : \mathbb{R}^P \rightarrow \mathbb{R}$ be a semi-algebraic locally Lipschitz continuous function. Assume that L is constant on K , with value L^* . Then there exist $\varepsilon > 0$, $\delta > 0$, $a \in (0, 1)$ and $\rho > 0$ such that, for all*

$$v \in \{v \in \mathbb{R}^P \mid \text{dist}(v, K) < \varepsilon\} \cap \{v \in \mathbb{R}^P \mid L^* < L(v) < L^* + \delta\},$$

3. We could extend the results of this section to more general objects including analytic functions on bounded sets. The semi-algebraicity assumption is made here for the sake of clarity.

it holds that,

$$\rho(1-a)(L(v) - L(\bar{v}))^{-a} \text{dist}(0, \partial L(v)) > 1. \quad (22)$$

The proof directly follows from the general inequality provided in Bolte et al. (2007b) or the local result of Bolte et al. (2007b) with the compactness arguments of Bolte et al. (2014, Lemma 6). In the sequel, we make an abuse of notation by writing $\|\partial\mathcal{J}(\cdot)\| \triangleq \text{dist}(0, \partial\mathcal{J}(\cdot))$. To obtain a convergence rate we will use inequality (22) on the Lyapunov function E . But first we state a general result of convergence that is built around the KL property.

4.2 A General Asymptotic Rate Result

We state a general theorem that leads to the existence of a convergence rate. This theorem will hold in particular for (7). We start by stating the result.

Theorem 11 *Let $X : [0, +\infty) \rightarrow \mathbb{R}^P$ be a bounded absolutely continuous trajectory and let $L : \mathbb{R}^P \rightarrow \mathbb{R}$ be a semi-algebraic locally Lipschitz continuous function. If there exists $c_1 > 0$ such that for a.e. $t > 0$,*

$$\frac{dL}{dt}(X(t)) \leq -c_1 \|(\partial L)(X(t))\|^2, \quad (i)$$

then $L(X(t))$ converges to a limit value L^* and,

$$|L(X(t)) - L^*| = O\left(\frac{1}{t}\right).$$

If in addition there exists $c_2 > 0$ such that for a.e. $t > 0$,

$$c_2 \|\dot{X}(t)\| \leq \|(\partial L)(X(t))\|, \quad (ii)$$

then, X converges to a critical point of L with a rate of the form $O(1/t^b)$ with $b > 0$.⁴

Proof We first prove the convergence of $L(X(\cdot))$. Suppose that (i) holds. Since X is bounded and L is continuous, $L(X(\cdot))$ is bounded. Moreover from (i), $L(X(\cdot))$ is decreasing, so it converges to some value L^* . To simplify suppose $L \geq 0$ and $L^* = 0$. Define,

$$\mathsf{I} = \{x \in \mathbb{R}^P \mid L(x) = 0\}.$$

Suppose first that there exists $s \geq 0$, such that $X(s) \in \mathsf{I}$. Since $L(X(\cdot))$ is decreasing with limit 0, then for all $t \geq s$, $L(X(t)) = 0$ and the convergence rate holds true.

Let us thus assume that for all $t \geq 0$, $L(X(t)) > 0$. The trajectory X is bounded in \mathbb{R}^P , hence there exists a compact set $\mathsf{C} \subset \mathbb{R}^P$ such that $X(t) \in \mathsf{C}$ for all $t \geq 0$. Define $\mathsf{K} = \mathsf{I} \cap \mathsf{C}$. It is a compact set since I is closed (by continuity of L) and C is compact. Moreover, L is constant on K . As such by Lemma 10, there exist $\varepsilon > 0$, $\delta > 0$, $a \in (0, 1)$ and a constant $\rho > 0$ such that for all

$$v \in \{v \in \mathbb{R}^P, \text{dist}(v, \mathsf{K}) < \varepsilon\} \cap \{0 < L(v) < \delta\},$$

it holds that

$$\rho(1-a)(L(v))^{-a} \text{dist}(0, \partial L(v)) > 1.$$

4. In some cases we even have linear rates or finite convergence as detailed in the proof.

We have $L(X(t)) \rightarrow 0$ so there exists $t_0 \geq 0$ such that for all $t \geq t_0$, $0 < L(X(t)) < \delta$. Without loss of generality, we assume $t_0 = 0$. Similarly, we have $\text{dist}(X(t), K) \rightarrow 0$, so we may assume that for all $t \geq 0$, $\text{dist}(X(t), K) < \varepsilon$. Thus, for all $t \geq 0$,

$$\rho(1-a)L(X(t))^{-a}\|\partial L(X(t))\| > 1.$$

Going back to assumption (i), for a.e. $t > 0$, one has

$$\frac{dL}{dt}(X(t)) \leq -c_1\|(\partial L)(X(t))\|^2,$$

but the KL property implies that for a.e. $t > 0$,

$$-\|\partial L(X(t))\|^2 < -\frac{1}{\rho^2(1-a)^2}L(X(t))^{2a}.$$

Therefore,

$$\frac{dL}{dt}(X(t)) < -\frac{c_1}{\rho^2(1-a)^2}L(X(t))^{2a}.$$

We consider two cases depending on the value of a . If $0 < a \leq 1/2$, then for t large, $L(X(t)) < 1$ so $-L(X(t))^{2a} < -L(X(t))$ and hence,

$$\frac{dL}{dt}(X(t)) < -\frac{c_1}{\rho^2(1-a)^2}L(X(t)),$$

so we obtain a linear rate. When $1/2 < a < 1$, we have for a.e. $t > 0$,

$$L(X(t))^{-2a}\frac{d}{dt}L(X(t)) = \frac{1}{1-2a}\frac{d}{dt}L(X(t))^{1-2a} < -\frac{c_1}{(\rho^2(1-a)^2)}, \quad (23)$$

with $1-2a < 0$. We can integrate from 0 to $t > 0$:

$$L(X(t))^{1-2a} > \frac{(2a-1)c_1}{\rho^2(1-a)^2}t + L(X(0))^{1-2a} > \frac{(2a-1)c_1}{\rho^2(1-a)^2}t.$$

Since $\frac{1}{1-2a} < -1$, one obtains a convergence rate of the form $O\left(t^{\frac{1}{1-2a}}\right)$. In both cases the rate is at least $O\left(\frac{1}{t}\right)$.

We assume now that both (i) and (ii) holds and prove convergence of the trajectory with a convergence rate. Let $t > s > 0$, by the fundamental theorem of calculus and the triangular inequality,

$$\|X(t) - X(s)\| \leq \left\| \int_s^t \dot{X}(\tau) d\tau \right\| \leq \int_s^t \|\dot{X}(\tau)\| d\tau. \quad (24)$$

We wish to bound $\|\dot{X}\|$ using L . Using the chain rule (Lemma 4 of Section 3.3), for a.e. $\tau > 0$,

$$\frac{d}{d\tau}L(X(\tau))^{1-a} = (1-a)L(X(\tau))^{-a}\langle \dot{X}(\tau), (\partial L)(X(\tau)) \rangle. \quad (25)$$

Then, from (i), we deduce that for a.e. $\tau > 0$,

$$\langle \dot{X}(\tau), (\partial L)(X(\tau)) \rangle = \frac{dL}{d\tau}(X(\tau)) \leq -c_1\|(\partial L)(X(\tau))\|^2, \quad (26)$$

so

$$\frac{d}{d\tau} L(X(\tau))^{1-a} \leq -c_1(1-a)L(X(\tau))^{-a} \|(\partial L)(X(\tau))\|^2. \quad (27)$$

The KL property (22) implies that for a.e. $\tau > 0$,

$$-(1-a)L(X(\tau))^{-a} \|(\partial L)(X(\tau))\| < -\frac{1}{\rho}. \quad (28)$$

Putting this in (27) and using assumption (ii) we finally obtain

$$\frac{d}{dt} L(X(\tau))^{1-a} < -\frac{c_1}{\rho} \|(\partial L)(X(\tau))\| \leq -\frac{c_1 c_2}{\rho} \|\dot{X}(\tau)\|. \quad (29)$$

We can use that in (24),

$$\begin{aligned} \|X(t) - X(s)\| &\leq -\frac{\rho}{c_1 c_2} \int_s^t \frac{d}{dt} L(X(\tau))^{1-a} d\tau \\ &= \frac{\rho}{c_1 c_2} (L(X(s))^{1-a} - L(X(t))^{1-a}). \end{aligned} \quad (30)$$

Then, using the convergence rate we already proved for L , we deduce that the Cauchy criterion holds for X inside the compact (hence complete) subset $\mathbf{C} \subset \mathbb{R}^P$ containing the trajectory. Thus, X converges, and from (i) we have that $\liminf_{t \rightarrow +\infty} \|\partial L(X(t))\| = 0$ because ∂L has closed graph. This shows that the limit is a critical points of L . Finally, taking the limit in (30) and using the convergence rate of L we obtain a rate for X as well. \blacksquare

Remark 12 Theorem 11 takes the form of a general recipe to obtain a convergence rate since it may be applied in many cases, to curves or flows, provided that a convenient Lyapunov function is given. Note also that it is sufficient for assumptions (i) and (ii) to hold only after some time $t_0 > 0$ as in such case, one could simply do a time shift to use the theorem.

4.3 Application to INDIAN

We now apply Theorem 11 to the deterministic continuous dynamical model of INDIAN (7).

Theorem 13 (Convergence rates) *Suppose that \mathcal{J} is semi-algebraic locally Lipschitz continuous and lower bounded. Then, any bounded trajectory (θ, ψ) that solves (7) converges to a point $(\bar{\theta}, \bar{\psi}) \in \mathbf{S}$, with a convergence rate of the form $O(t^{-b})$ with $b > 0$. Moreover $\mathcal{J}(\theta(t))$ converges to its limit $\bar{\mathcal{J}}$ with rate $|\mathcal{J}(\theta(t)) - \bar{\mathcal{J}}| = O(\frac{1}{t})$.*

Proof Let (θ, ψ) be a bounded solution of (7). We would like to use Theorem 11 with $X = (\theta, \psi)$, and a well chosen function. In the proof of Theorem 3 we proved a descent property along the trajectory for the function $E(\theta, \psi) = 2(1 + \alpha\beta)\mathcal{J}(\theta) + \left\| \left(\alpha - \frac{1}{\beta} \right) \theta + \frac{1}{\beta} \psi \right\|^2$. This function is semi-algebraic, locally Lipschitz continuous, so it remains to prove that (i) and (ii) hold for E along (θ, ψ) .

For $t \geq 0$, denote $w(t) = (\alpha - \frac{1}{\beta})\theta(t) + \frac{1}{\beta}\psi(t)$, then according to Lemma 8 for a.e. $t > 0$,

$$\begin{aligned} \frac{dE}{dt}(\theta(t), \psi(t)) &= -\|\sqrt{\alpha}\dot{\theta}(t) - \frac{1}{\sqrt{\beta}}(\dot{\psi}(t) - \dot{\theta}(t))\|^2 - \|\sqrt{\alpha}\dot{\theta}(t) + \frac{1}{\sqrt{\beta}}(\dot{\psi}(t) - \dot{\theta}(t))\|^2 \\ &= -2\alpha\|\dot{\theta}(t)\|^2 - \frac{2}{\beta}\|\dot{\psi}(t) - \dot{\theta}(t)\|^2 = -2\alpha\|\dot{\theta}(t)\|^2 - \frac{2}{\beta}\|\beta\partial\mathcal{J}(\theta(t))\|^2 \\ &= -2\alpha\|-\beta\partial\mathcal{J}(\theta(t)) - w(t)\|^2 - 2\beta\|\partial\mathcal{J}(\theta(t))\|^2. \end{aligned} \quad (31)$$

On the other hand, by standard results on the sum rule, we have for all $(\theta, \psi) \in \mathbb{R}^P \times \mathbb{R}^P$,

$$\partial E(\theta, \psi) = 2 \begin{pmatrix} (1 + \alpha\beta)\partial\mathcal{J}(\theta) + (\alpha - \frac{1}{\beta}) \left((\alpha - \frac{1}{\beta})\theta + \frac{1}{\beta}\psi \right) \\ \frac{1}{\beta} \left((\alpha - \frac{1}{\beta})\theta + \frac{1}{\beta}\psi \right) \end{pmatrix}, \quad (32)$$

so for a.e. $t > 0$,

$$\frac{\|\partial E(\theta(t), \psi(t))\|^2}{4} = \left\| (1 + \alpha\beta)\partial\mathcal{J}(\theta(t)) + (\alpha - \frac{1}{\beta})w(t) \right\|^2 + \left\| \frac{1}{\beta}w(t) \right\|^2. \quad (33)$$

We wish to find $c_1 > 0$, such that $\frac{1}{2}\frac{dE}{dt} + \frac{c_1}{4}\|\partial E\|^2 < 0$. This follows from the following claim.

Claim: let $r_1 > 0$, $r_2 \in \mathbb{R}$, $r_3 > 0$, then there exist C_1 and C_2 two positive constants such that for any $a, b \in \mathbb{R}$,

$$C_1(a^2 + b^2) \leq (r_1a + r_2b)^2 + r_3b^2 \leq C_2(a^2 + b^2). \quad (34)$$

Indeed, the function $Q : (a, b) \mapsto (r_1a + r_2b)^2 + r_3b^2$ is a positive definite quadratic form, C_1 and C_2 can be taken to be two eigenvalues of the positive definite matrix which represents Q . Hence (34) holds for all a and b .

Applying the previous claim to (33) and (31) leads to the existence of $c_1 > 0$ such that for a.e. $t > 0$,

$$\frac{dE}{dt}(\theta(t), \psi(t)) \leq -c_1\|\partial E(\theta(t), \psi(t))\|^2,$$

so assumption (i) holds for INDIAN.

It now remains to show that (ii) of Theorem 11 holds i.e. that there exists $c_2 > 0$ such that for (θ, ψ) solution of (7) and for a.e. $t > 0$, $\|\partial E(\theta(t), \psi(t))\|^2 \geq c_2 \left(\|\dot{\theta}(t)\|^2 + \|\dot{\psi}(t)\|^2 \right)$. Using (7) and (33) we obtain:

$$\frac{\|\partial E(\theta(t), \psi(t))\|^2}{4} = \left\| \frac{1}{\beta}(1 + \alpha\beta)\dot{\theta}(t) + \left[(\alpha - \frac{1}{\beta}) - \frac{1}{\beta}(1 + \alpha\beta) \right] \dot{\psi}(t) \right\|^2 + \frac{1}{\beta^2}\|\dot{\psi}(t)\|^2, \quad (35)$$

and applying the claim (34) again to (35) one can show that there exist $c_2 > 0$, such that for a.e. $t > 0$,

$$\|\partial E(\theta(t), \psi(t))\|^2 \geq c_2 \left(\|\dot{\theta}(t)\|^2 + \|\dot{\psi}(t)\|^2 \right).$$

So assumption (ii) holds for (7). To conclude, we can apply Theorem 11 to (7) and the proof is complete. \blacksquare

Remark 14 (a) Since the discrete algorithm INDIAN asymptotically resembles to its continuous time version (see the proof of Theorem 3), the results above suggest that similar behaviors and rates could be hoped for INDIAN itself. Yet, these results remain difficult to obtain in the case of DL, in particular in the mini-batch setting because of the noise $(\xi_k)_{k \in \mathbb{N}}$.

(b) The proof above is significantly simpler when $\alpha\beta > 1$ since Alvarez et al. (2002) proved that in this case, (7) is equivalent to a gradient system, thus assumptions (i) and (ii) of Theorem 11 instantly holds.

(c) Theorems 11 and 13 can be adapted to the case when the Clarke subdifferential is replaced by $D\mathcal{J}$, but we do not state it here for the sake of simplicity.

(d) Theorems 11 and 13 are actually valid by assuming that \mathcal{J} belongs to a polynomially bounded o-minimal structure. One of the most common instance of such structures is the one given by globally subanalytic sets (as illustrated in a example below). We refer to Bolte et al. (2007a) for a definition and further references.

Let us now comment the results of Theorem 13. First, we restrained here to semi-algebraic loss functions \mathcal{J} , which are a subset of tame loss functions. Most networks, activations and dissimilarity measures mentioned in Section 2.2 fall into this category. Nonetheless, the loss functions of the DL experiments of Section 5.2 are not semi-algebraic. Indeed, the dissimilarity measure l used is the cross-entropy: $l(f(x_n, \theta), y_n) = -\sum_{d=1}^D \mathbf{1}_{[y_n]_d=1} \log([f(x_n, \theta)]_d)$. Such a function cannot be described by polynomials and presents a singularity whenever $[f(x_n, \theta)]_d = 0$. Fortunately, due to the numerical precision but also to the “soft-max” functions often used in classification experiments, the outputs of the network f , for inputs restricted to a compact set, have values in $[\varepsilon, 1]$ for some small $\varepsilon > 0$. Therefore, the singularity at 0 is harmless and the cross-entropy acts as a globally subanalytic function. As a consequence the nonsmooth Łojasiewicz inequality holds, and the theorems apply (see also numerical experiments).

The rate of convergence of the trajectory in Theorem 13 is non-explicit in the sense that the exponent $b > 0$ is unknown in general. In the light of the proof of Theorem 11, this exponent depends on the KL exponent a of the Lyapunov function, which is itself hard to determine in practice. However the intuition is that small exponents a may yield faster convergence rates (indeed, when $a \in (0, 1/2)$ we actually have a linear rate). As an example, for the function: $t \in \mathbb{R} \mapsto |t|^c$ with $c > 1$, the exponent at 0 is $a = 1 - \frac{1}{c}$ and thus, the closer c is to 1, the smaller a is, and the faster the convergence becomes.

5. Experiments

In this section we first discuss the role and influence of the hyper-parameters of INDIAN using the 2D example given in Figure 1. We then compare INDIAN with SGD, ADAGRAD and ADAM on deep learning problems for image recognition.

5.1 Understanding the Role of the Hyper-parameters of INDIAN

Both hyper-parameters α and β can be seen as damping coefficients from the viewpoint of mechanics as discussed by Alvarez et al. (2002) and sketched in the introduction. Recall the second-order time continuous dynamics which served as a model to the design INDIAN:

$$\ddot{\theta}(t) + \alpha \dot{\theta}(t) + \beta \nabla^2 \mathcal{J}(\theta(t)) \dot{\theta}(t) + \nabla \mathcal{J}(\theta(t)) = 0.$$

This differential equation was inspired by Newton’s Second Law of dynamics asserting that the acceleration of a material point coincides with the sum of forces applied to the particle. As recalled in the introduction three forces are at stake: the gravity and two friction terms. The parameter α calibrates the *viscous damping* intensity as in the Heavy Ball friction method of Polyak (1964). It acts as a dissipation term but it can also be seen as a proximity parameter of the system with the usual gradient descent: the higher α is, the more DIN behaves like a pure gradient descent.⁵ On the other hand the parameter β can be seen as a *Newton damping* which takes into account the geometry of the landscape to brake or accelerate the dynamics in an adaptive anisotropic fashion, see Alvarez and Pérez (1998), Alvarez et al. (2002) for further insights.

We now turn our attention to INDIAN, and illustrate the versatility of the hyper-parameters α and β in this case. We proceed on a 2D visual nonsmooth ill-conditioned example à la Rosenbrock, see Figure 1. For this example, we aim at finding the minimum of the function $\mathcal{J}(\theta_1, \theta_2) = 100(\theta_2 - |\theta_1|)^2 + |1 - \theta_1|$. This function has a V-shaped valley, and a unique critical point at $(1, 1)$ which is also the global minimum. Starting from the point $(-1, 1.5)$ (the black cross), we apply INDIAN with constant steps $\gamma_k = 10^{-4}$. Figure 1 shows that when β is too small, the trajectory presents many transverse oscillations as well as longitudinal ones close to the critical point (subplot (a)). Then, increasing β significantly reduces transverse oscillations (subplot (b)). Finally, the longitudinal oscillations are reduced by choosing a higher α (subplot (c)). In addition, these behaviors are also reflected in the values of the objective function (subplot (d)). The orange curve (first setting) presents large oscillations. Moreover, looking at the red curve, corresponding to plot (c), there is a short period between 20,000 and 60,000 iterations when the decrease is slower than for the other values of α and β , but still it presents fewer oscillations. In the longer term, the third choice ($\alpha = 1.3$, $\beta = 0.1$) provides remarkably good performance.

The choice of these hyper-parameters may come with rates of convergence for convex and strongly convex smooth functions (Attouch et al., 2019). Following this work, one may also consider to make α and β vary in time (for example like the famous Nesterov damping $\frac{\alpha}{t}$). In our DL experiments we will however keep these parameters constant so that our theorems still hold. Yet, different behaviors depending on (α, β) can also be observed for DL problems as illustrated on Figure 2 and described next.

5.2 Training a DNN with INDIAN

Before comparing INDIAN to state-of-the-art algorithms in DL, we first describe the methodology we followed.

5.2.1 METHODOLOGY

- We train a DNN for classification using the three most common image data sets (MNIST, CIFAR10, CIFAR100) (LeCun et al., 1998, Krizhevsky, 2009). These data sets are composed of 60,000 small images associated with a label (numbers, objects, animals, etc.). We split the data sets into 50,000 images for training and 10,000 for testing.
- Regarding the network, we use a slightly modified version of the popular Network in Network (NiN) (Lin et al., 2013). It is a reasonably large convolutional network with $P \sim 10^6$ parameters to optimize. We use ReLU activation functions.

5. This is easier to see when one rescales \mathcal{J} by α .

- The dissimilarity measure l that is used in the empirical loss \mathcal{J} given by (3) is set to the cross-entropy. The loss \mathcal{J} is optimised with respect to θ (the weights of the DNN) on the training data. The classification accuracy of the trained DNN is measured using the test data of 10,000 images. Measuring the accuracy boils down to counting how many of the 10,000 were correctly classified (in percentage).
- Based on the results of Section 5.1, we run INDIAN for four different values of (α, β) :

$$(\alpha, \beta) \in \{(0.1, 0.1), (0.5, 0.1), (0.5, 0.5), (0.5, 1)\}.$$

Given an initialisation of the weights θ_0 , we initialize ψ_0 such that the initial velocity is in the direction of $-\nabla\mathcal{J}(\theta_0)$. More precisely, we use $\psi_0 = (1 - \alpha\beta)\theta_0 - (\beta^2 - \beta)\nabla\mathcal{J}(\theta_0)$.

- We compare our algorithm INDIAN with the classical SGD algorithm and the popular ADAGRAD (Duchi et al., 2011) and ADAM(Kingma and Ba, 2014) algorithms. At each iteration k , we compute the approximation of $\partial\mathcal{J}(\theta)$ on a subset $\mathbf{B}_k \subset \{1, \dots, 50,000\}$ of size 32. The algorithms are initialized with the same random weights (drawn from a normal distribution). Five random initializations are considered for each experiment.
- Regarding the selection of step sizes, ADAGRAD and ADAM both use an adaptive procedure based on past gradients, see Duchi et al. (2011), Kingma and Ba (2014). For the other two algorithms (INDIAN and SGD), we use the classical step size schedule $\gamma_k = \frac{\gamma_0}{\sqrt{k+1}}$, which meets Assumption 1. For all four algorithms, choosing the right initial step length γ_0 is often critical in terms of efficiency. We choose this γ_0 using a grid-search: for each algorithm we select the initial step size that most decreases the training error \mathcal{J} after fifteen epochs (one epoch consisting in a complete pass over the data). The test data is not used to chose the initial step size nor other hyper-parameters. Note that we could use more flexible step size schedules but chose a standard schedule for simplicity. Other decay schemes are considered in Figure 4.

For these experiments, we used `Keras 2.2.4` (Chollet, 2015) with `Tensorflow 1.13.1` (Abadi et al., 2016) as backend. The INDIAN algorithm is available in Pytorch, Keras and Tensorflow: <https://github.com/camcastera/Indian-for-DeepLearning/> (Castera, 2019).

5.2.2 RESULTS

Figure 2 displays the training loss \mathcal{J} and test accuracy with respect to epochs for INDIAN in its four considered hyper-parameter configurations and for the three data sets considered. Figure 3 displays the performance of INDIAN with the hyper-parameter configuration that led to smallest average training error in Figure 2, with comparison to SGD, ADAGRAD and ADAM. In these two figures (and also in subsequent Figure 4), solid lines represent mean values and pale surfaces represent the best and worst runs in terms of training loss and validation accuracy over five random initializations.

Figure 2 suggests that the tuning of the hyper-parameters α and β is not crucial to obtain satisfactory results both for training and testing. It mostly affects the training speed. Thus, INDIAN looks quite stable with respect to these hyper-parameters. Setting $(\alpha, \beta) = (0.5, 0.1)$ appears to be a good default choice when necessary. Nevertheless, tuning these hyper-parameters is of course advised to get the most from INDIAN.

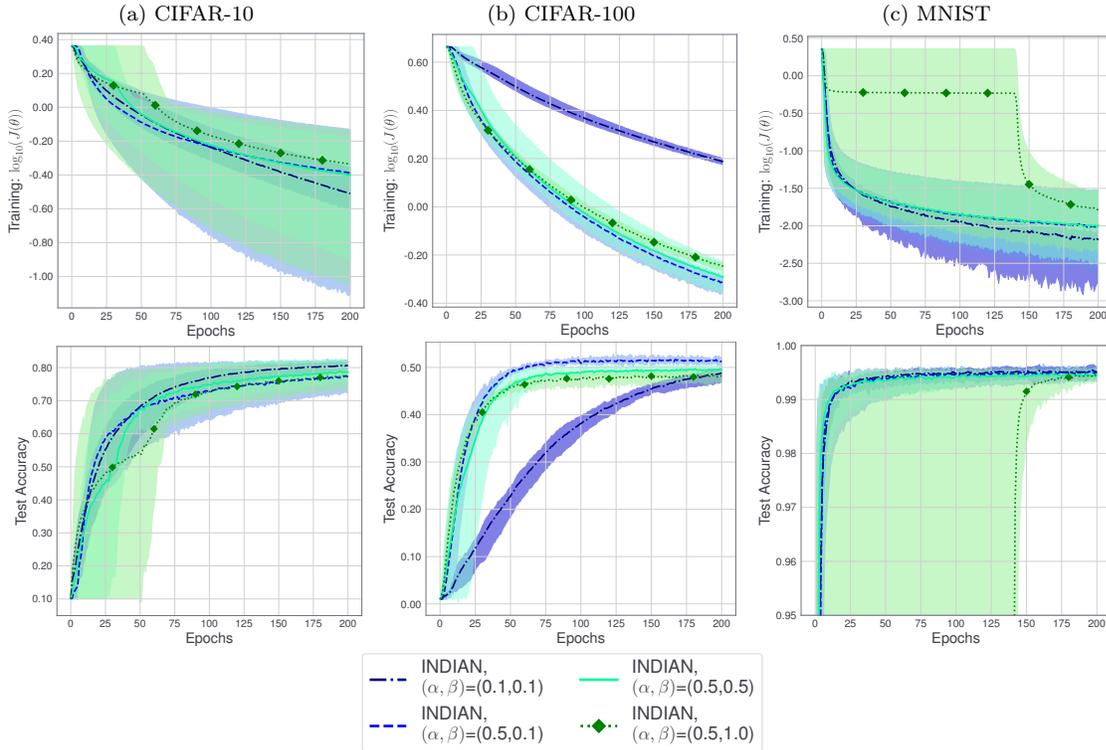


Figure 2: Analysis of the sensibility of INDIAN to the choice of α and β using NiN for three different image classification problems. Top: logarithm of the loss function $\mathcal{J}(\theta)$ during the training. Bottom: classification accuracy on the test set.

Figure 3 shows that best performing methods achieve state-of-the-art accuracy using NiN and represent what can be achieved with a moderately large network and coarse grid-search tuning of the initial step size. In our comparison, INDIAN and ADAM outperform SGD and ADAGRAD for training. While ADAM seems to be faster in the early training phase, INDIAN achieves the best accuracy almost every time especially on CIFAR-100 (Figure 3(b)). Thus INDIAN appears to be competitive in comparison to the other algorithms with the advantage of having solid theoretical foundations and a simple step-size rule as compared to ADAM and ADAGRAD.

Finally let us point out that although ADAM was faster in the experiments of Figure 3, INDIAN can outperform ADAM using the slow step size decay discussed in Section 3.2. Indeed, in the previous experiments we used a standard decreasing step size of the form $\gamma_0/\sqrt{k+1}$ for simplicity, but Assumption 1 allows for step sizes decreasing much slower. As such, we also considered decays of the form $\gamma_0(k+1)^{-q}$ with $q \leq 1/2$. The results are displayed on top of Figure 4. Except when q is too small (too slow decay, e.g., $q = 1/16$), these results show that some decays slower than $q = 1/2$ make INDIAN a little bit faster than any of the other algorithms we tried. In particular,

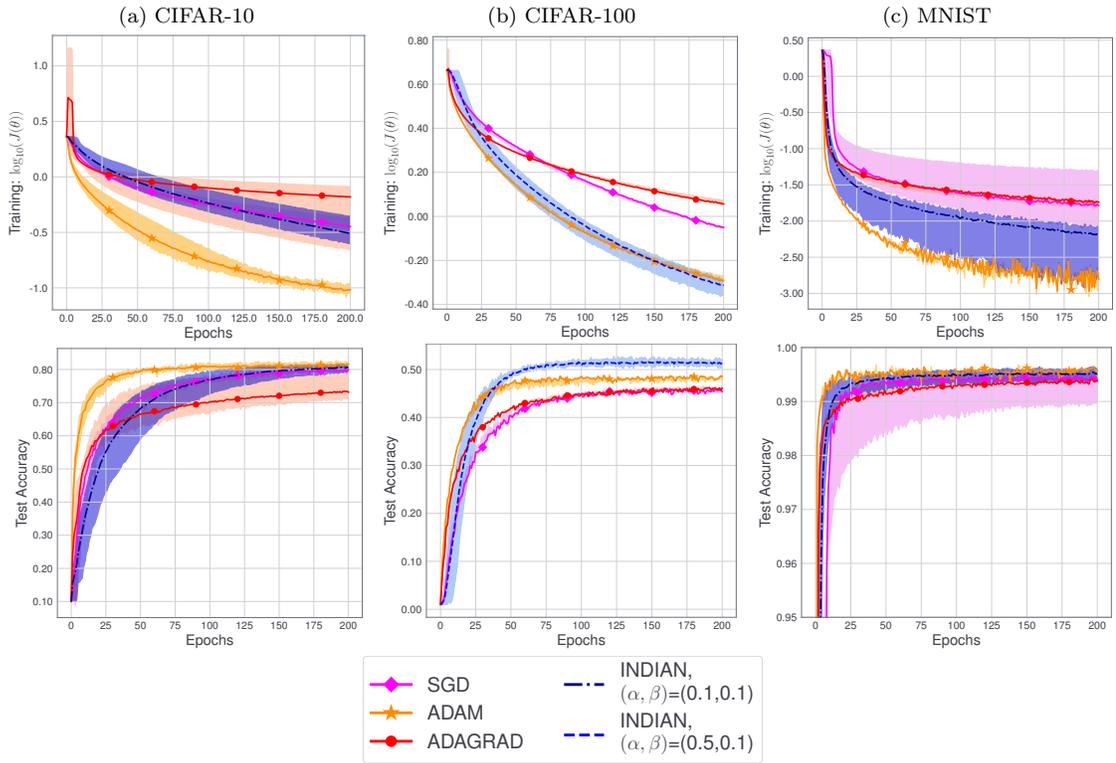


Figure 3: Comparison of INDIAN with state-of-the-art algorithms SGD, ADAM and ADAGRAD. Top: logarithm of the loss function $\mathcal{J}(\theta)$ during the training. Bottom: classification accuracy on the test set.

with a step size decay proportional to $k^{-1/4}$, INDIAN outperforms ADAM (bottom of Figure 4). This suggests that tuning q can also significantly accelerate the training process.

6. Conclusion

We introduced a novel stochastic optimization algorithm featuring inertial and Newtonian behavior motivated by applications to deep learning. We provided a powerful algorithmic convergence analysis under weak hypotheses applicable to most DL problems. We also provided new general results to study differential inclusions on Clarke subdifferential and obtain convergence rates for the continuous time counterpart of our algorithm. We would like to point out that, apart from SGD (Davis et al., 2019), the convergence of concurrent methods in such a general setting is still an open question. Our result seems moreover to be the first one to be able to rigorously handle the analysis of mini-batch subsampling for ReLU DNNs via the introduction of the D -critical points. Our experiments show that INDIAN is very competitive with state-of-the-art algorithms for DL but also very malleable.

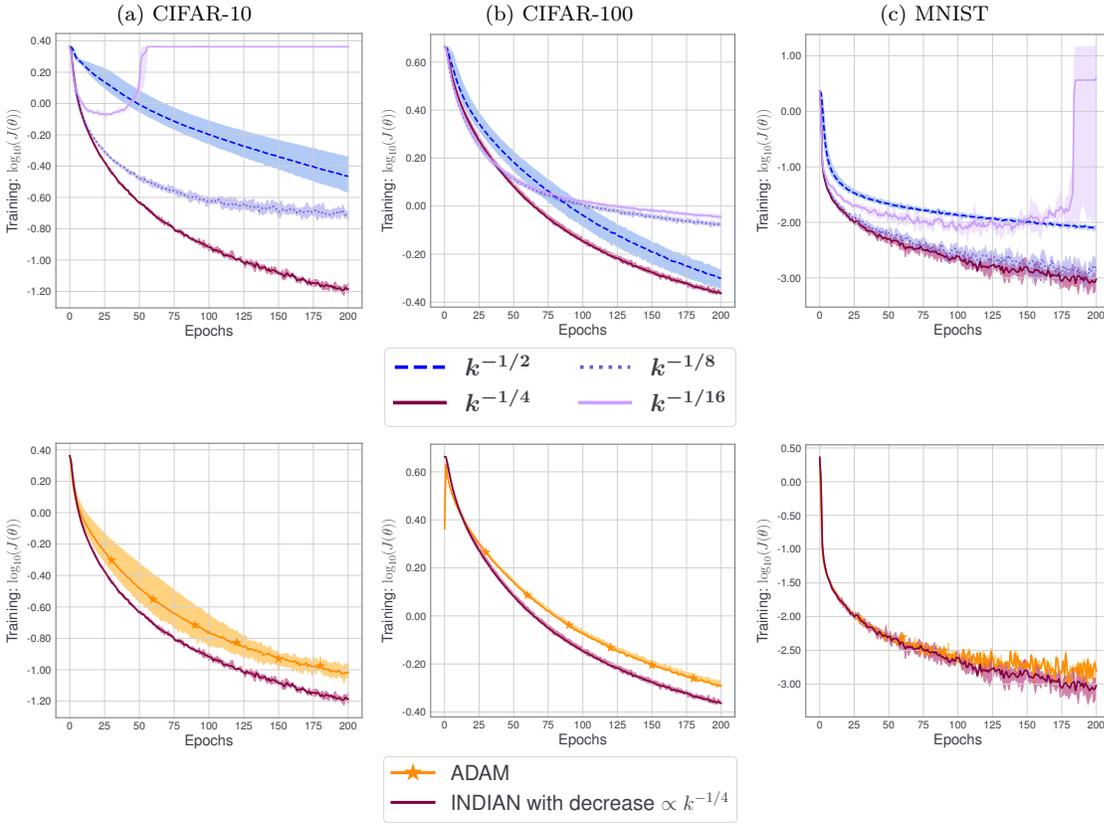


Figure 4: On top: Training loss of INDIAN on three image classification problems with various step size decays. In the legend, k^{-q} means a step size decay at iteration k of the form $\gamma_k = \gamma_0 k^{-q}$. The bottom row show the comparison between INDIAN with a well-chosen step size decay and ADAM.

We stress that these numerical manipulations were performed on substantial DL benchmarks with only limited algorithm tuning.

This facilitates reproducibility and allows to stay as close as possible to the reality of DL applications in machine learning.

Acknowledgments

The authors acknowledge the support of the European Research Council (ERC FACTORY-CoG-6681839), the Agence Nationale de la Recherche (ANR 3IA-ANITI, ANR-17-EURE-0010 CHESS, ANR-19-CE23-0017 MASDOL) and the Air Force Office of Scientific Research (FA9550-18-1-0226).

Part of the numerical experiments were done using the OSIRIM platform of IRIT, supported by the CNRS, the FEDER, Région Occitanie and the French government (<http://osirim.irit.fr/site/en>). We thank the development team of the following libraries that were used in the experiments: Python (Rossum, 1995), Numpy (Walt et al., 2011), Matplotlib (Hunter, 2007), Pytorch (Paszke et al., 2019), Tensorflow and Keras (Abadi et al., 2016, Chollet, 2015).

We thank Hedy Attouch and Sixin Zhang for useful discussions.

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