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## ON THE DENSITY OR MEASURE OF SETS AND THEIR SUMSETS IN THE INTEGERS OR THE CIRCLE

PIERRE-YVES BIENVENU AND FRANÇOIS HENNECART

ABSTRACT. Let  $d(A)$  be the asymptotic density (if it exists) of a sequence of integers  $A$ . For any real numbers  $0 \leq \alpha \leq \beta \leq 1$ , we solve the question of the existence of a sequence  $A$  of positive integers such that  $d(A) = \alpha$  and  $d(A + A) = \beta$ . More generally we study the set of  $k$ -tuples  $(d(iA))_{1 \leq i \leq k}$  for  $A \subset \mathbb{N}$ . This leads us to introduce subsets defined by diophantine constraints inside a random set of integers known as the set of “pseudo sth powers”. We consider similar problems for subsets of the circle  $\mathbb{R}/\mathbb{Z}$ , that is, we partially determine the set of  $k$ -tuples  $(\mu(iA))_{1 \leq i \leq k}$  for  $A \subset \mathbb{R}/\mathbb{Z}$ .

### 1. Introduction

We respectively denote by  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  the set of all natural integers, the set of all integers, the set of all rational numbers and the set of all real numbers. Let  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  be the torus (or the circle). For any finite set  $A$ , we denote its cardinality by  $|A|$ . For  $x \in \mathbb{R}$ , we denote by  $[x]$  its integral part and by  $\{x\} = x - [x]$  its fractional part. For  $A \subset \mathbb{N}$  and  $t > 1$ , we let  $A(t) = |A \cap [1, t]|$ . We define if it exists the so-called *asymptotic density* of  $A$  by

$$d(A) = \lim_{t \rightarrow \infty} \frac{A(t)}{t}.$$

Otherwise we define the *lower* and the *upper* asymptotic densities  $\underline{d}(A)$  and  $\overline{d}(A)$  using  $\liminf$  and  $\limsup$  instead of limits. More generally, if  $A \subset B \subset \mathbb{N}$ , we define if it exists the density of  $A$  inside  $B$  as

$$d_B(A) = \lim_{t \rightarrow \infty} \frac{A(t)}{B(t)}.$$

The density of  $A$  inside  $\mathbb{N}$  is therefore simply the density, and if  $B$  has a positive density, we have  $d_B(A) = d(A)/d(B)$ .

For a subset  $A$  of a semigroup  $G$ , let  $A + A = \{a + b : a, b \in A\}$ . For  $k \geq 1$ , we denote by  $kA$  its  $k$ -fold sumset. From Kneser’s Theorem [9], we know that for any subset  $A \subset \mathbb{N}$ , the inequality  $\underline{d}(2A) < 2\underline{d}(A)$  may only hold when  $\underline{d}(2A)$  is a rational number. Similarly, for any subset  $A$  of the circle  $\mathbb{T}$  equipped with its Haar probability measure  $\mu$ , a theorem of Raikov [14] implies that  $\underline{\mu}(2A) \geq \min(1, 2\underline{\mu}(A))$  where

$$\underline{\mu}(A) = \sup_{\substack{F \subset A \\ F \text{ closed}}} \mu(F).$$

In this paper, we firstly determine the possible values  $(\alpha, \beta)$  of pairs  $(\underline{d}(A), \underline{d}(2A))$  and  $(\underline{\mu}(A), \underline{\mu}(2A))$ . We first completely settle the case  $\beta \geq \min(2\alpha, 1)$ .

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**Theorem 1.1.** *Let  $(\alpha, \beta) \in [0, 1]^2$ . Suppose  $\beta \geq \min(2\alpha, 1)$ . Then the following statements both hold.*

- a) *There exists  $A \subset \mathbb{N}$  such that  $d(A)$  and  $d(2A)$  exist and equal  $\alpha$  and  $\beta$  respectively.*
- b) *There exists a measurable subset  $A \subset \mathbb{T}$  such that  $2A$  is measurable and  $\mu(A) = \alpha$  and  $\mu(2A) = \beta$ . Further, for  $\alpha > 0$ , we may take  $A$  to be open (in fact a finite union of open intervals).*

The case  $\beta = 2\alpha$  is obvious for the second item (with an interval  $A$ ), and is a special case of a theorem by Faisant *et al* [5] for the first item, whereas allowing different summands, Volkmann [16] proved that, given positive real numbers  $\alpha_1, \alpha_2$  and  $\gamma$  such that  $\alpha_1 + \alpha_2 \leq \gamma < 1$ , there exist<sup>1</sup>  $A_1, A_2$  such that  $d(A_i) = \alpha_i, i = 1, 2$ , and  $d(A_1 + A_2) = \gamma$ ; he actually proved the corresponding result for subsets of the circle too. A similar result was obtained by Nathanson [13], including a version for Schnirelmann's density.

More generally, we investigate the set  $\mathcal{D}_k$  of possible values of the tuple

$$(d(A), d(2A), \dots, d(kA))$$

when  $A$  ranges over the set of sequences for which all of these densities exist. For any real number  $\theta > 1$ , let

$$(1) \quad T_{k,\theta} = \left\{ n \geq 1 : 0 \leq \{\theta n\} < \frac{1}{k+1} \right\}.$$

Note that  $d(T_{k,\theta}) = 1/(k+1)$  if  $\theta$  is irrational by Weyl's criterion which we state in a moment (Theorem 1.3), while  $T_{k,\theta} = \mathbb{N}$  if  $\theta$  is an integer. In any case one has  $jT_{k,\theta} = \{n \geq 1 : 0 \leq \{\theta n\} < \frac{j}{k+1}\}$  for  $j \leq k+1$ , in particular.  $(k+1)T_{k,\theta} = \mathbb{N}$ . We now state the following key result which will be proved in Section 2 when  $k = 1$ . The general case  $k \geq 2$  will be tackled in Section 4.

**Proposition 1.2.** *Let  $\beta \in [0, 1]$  and  $k \geq 1$  be an integer. There exists a set  $A \subset T_{k,\theta}$  such that  $iA$  has density 0 for any  $i < k$ , whereas  $kA$  has density  $\beta$  inside  $kT_{k,\theta}$  and  $(k+1)A$  has density 1 inside  $\mathbb{N}$ .*

In particular, we have  $d(kA) = \beta k/(k+1)$  if  $\theta$  is irrational while  $d(kA) = \beta$  if  $\theta$  is an integer. Thus any  $(k+1)$ -tuple of the form  $(0, \dots, 0, \beta, 1)$  where  $\beta \leq 1$  belongs to  $\mathcal{D}_{k+1}$ .

In parallel, we consider the similar problem in the circle  $\mathbb{T}$  equipped with its Haar measure  $\mu$ . Thus let  $\mathcal{E}_k$  be the set of all the possible values of  $(\mu(A), \dots, \mu(kA))$  for  $A \subset \mathbb{T}$  for which these measures exist. We may sometimes need to work with the subset  $\mathcal{E}_k^o \subset \mathcal{E}_k$  of all the possible values of  $(\mu(A), \dots, \mu(kA))$  when  $A \subset \mathbb{T}$  is open and Riemann-measurable and similarly  $\mathcal{E}_k^c$ , where we consider closed sets  $A$ .

There is a close connection between  $\mathcal{E}_k$  and  $\mathcal{D}_k$  because of Weyl's criterion for equidistribution, of which we now state a direct consequence. For  $A \subset \mathbb{T}$  and  $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ , let

$$B_{\lambda,A} = \{n \in \mathbb{N} : \{\lambda n\} \in A\}.$$

**Theorem 1.3.** *For any irrational number  $\lambda$  and any Riemann-measurable function  $f : [0, 1] \rightarrow \mathbb{R}$ , we have*

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \sum_{n \leq x} f(\{\lambda n\}) = \int f.$$

<sup>1</sup>In Volkmann's construction, the sets of integers are sets of relative integers and not necessarily positive integers though.

In particular, for any Riemann-measurable subset  $A \subset \mathbb{T}$ , we have  $d(B_{\lambda,A}) = \mu(A)$ . The latter equality may be extended to open sets  $A$ .

The extension to open sets is [16, Lemma 4]. Further, Theorem 1.3 and a simple compactness argument shows that for any  $\epsilon > 0$  and irrational  $\lambda$ , there exists a constant  $C = C(\lambda, \epsilon)$  such that for any interval  $I$  of length at least  $\epsilon$  the set  $B_{\lambda,I}$  contains at least an integer less than  $C$ . Finally, the operation  $A \mapsto B_{\lambda,A}$  behaves well with respect to set addition.

**Lemma 1.4.** *Let  $k \geq 2$  and  $A_i \subset \mathbb{T}$  be open for  $i = 1, \dots, k$  and  $\lambda$  be irrational. Then  $d(\sum_{i=1}^k B_{\lambda,A_i}) = \mu(\sum_{i=1}^k A_i)$ .*

*Proof.* Let  $A \subset \mathbb{T}$  be open, and for any  $\epsilon > 0$  let  $A^\epsilon$  be the set of all  $a \in A$  whose distance to the boundary of  $A$  is at least  $\epsilon$ . Thus  $A = \bigcup_{\epsilon>0} A^\epsilon$  and  $\mu(A) = \lim_{\epsilon \rightarrow 0^+} \mu(A^\epsilon)$ . Further,  $\sum_{i=1}^k A_i = \bigcup_{\epsilon>0} \sum_{i=1}^k A_i^\epsilon$ . We observe that

$$(2) \quad B_{\lambda, \sum_{i=1}^k A_i^\epsilon} \subset \sum_{i=1}^k B_{\lambda, A_i} \subset B_{\lambda, \sum_{i=1}^k A_i}.$$

The rightmost inclusion is easy; for the leftmost one, let  $x \in B_{\lambda, \sum_{i=1}^k A_i^\epsilon}$ , thus  $x = \sum_{i=1}^k a_i$  where  $a_i \in A_i^\epsilon$ . Consequently,  $]a_i - \epsilon/k, a_i + \epsilon/k[ \subset A_i$  for  $i \in \{1, \dots, k-1\}$ . If  $n$  is large enough (larger than  $kC(\lambda, 2\epsilon/k)$ ), there exists  $n_1, \dots, n_{k-1} \leq n/k$  such that  $\{n_i \lambda\} \in ]a_i - \epsilon/k, a_i + \epsilon/k[$ . Let  $n_k = n - n_1 - \dots - n_{k-1} > 0$ . Then  $\{n_k \lambda\} = \{n \lambda\} - \{n_1 \lambda\} - \dots - \{n_{k-1} \lambda\} \pmod{1}$ , which implies  $\{n_k \lambda\} \pmod{1} \in ]a_k - \epsilon, a_k + \epsilon[ \subset A_k$ , in other words  $n_k \in B_{\lambda, A_k}$ . Thus  $n \in \sum_{i=1}^k B_{\lambda, A_i}$ .

Taking densities and applying Theorem 1.3 in equation (2), we find that

$$\begin{aligned} \mu\left(\sum_{i=1}^k A_i^\epsilon\right) &= d(B_{\lambda, \sum_{i=1}^k A_i^\epsilon}) \leq d\left(\sum_{i=1}^k B_{\lambda, A_i}\right) \\ &\leq \bar{d}\left(\sum_{i=1}^k B_{\lambda, A_i}\right) \leq \bar{d}(B_{\lambda, \sum_{i=1}^k A_i}) = \mu\left(\sum_{i=1}^k A_i\right). \end{aligned}$$

Letting  $\epsilon \rightarrow 0^+$ , we conclude.  $\square$

Consequently,  $\mathcal{E}_k^o \subset \mathcal{D}_k$ ; in particular, the second item of Theorem 1.1 implies the first one when  $\alpha > 0$ , but we will provide another proof for it. Further, Raikov's theorem together with Theorem 1.1 means that  $\mathcal{E}_2 = \mathcal{E}_2^o = \mathcal{E}_2^c = \{(\alpha, \beta) \in [0, 1]^2 : \beta \geq \min(1, 2\alpha)\}$ .

To complete our description of  $\mathcal{D}_2$ , we need to understand which pairs  $(\alpha, \beta)$  satisfying  $\beta < 2\alpha$  belong to it, which we do in the next theorem. For an integer  $n$ , let  $v_2(n)$  be its dyadic valuation; we extend it to rational numbers by letting  $v_2(p/q) = v_2(p) - v_2(q)$ .

**Theorem 1.5.** *Let  $\beta \in \mathbb{Q} \cap ]0, 1[$  such that  $v_2(\beta) \leq 0$  and let  $\alpha \in ]0, 1[$  satisfy  $\beta < 2\alpha$ . We denote  $g_0 = \min\{g \geq 1 : g\beta \text{ is an odd integer}\}$ . Then there exists a sequence  $A \subset \mathbb{N}$  such that  $d(A) = \alpha$  and  $d(2A) = \beta$  if and only if*

$$\frac{\beta}{2} < \alpha \leq \frac{\beta}{2} + \frac{1}{2g_0}.$$

We briefly discuss iterated sumsets. It is not clear what constraints a tuple  $(\alpha_i)_{1 \leq i \leq k}$  must satisfy for a set  $A \subset \mathbb{T}$  satisfying  $\mu(iA) = \alpha_i$  to exist; we certainly need  $\alpha_i \geq \min(1, \alpha_j + \alpha_{i-j})$  for any  $j < i$  due to Raikov's theorem but it may not be sufficient. In particular, we will deduce the following constraint from a theorem of Gyarmati, Konyagin and Ruzsa [7].

**Theorem 1.6.** *There exists a constant  $c > 0$  such that the following holds. Let  $A \subset \mathbb{T}$  be closed, and suppose that  $\mu(2A) < c$ . Then  $\mu(3A) \geq \frac{3}{2}\mu(2A)$ .*

In view of Lev's result [11, Theorem 3] asserting that for any finite set of integers  $A$  one has

$$k(|(k+1)A| - 1) \geq (k+1)(|kA| - 1),$$

one may more generally imagine that  $\mu((k+1)A) \geq \frac{k+1}{k}\mu(kA)$  under certain restrictions on  $\mu(kA)$ . Note that another result from [7] implies that the constant  $c$  may not be taken to be 1. Gyarmati *et al.* conjecture that its optimal value is  $1/2$ . On the other hand, due to the Plünnecke-Ruzsa inequalities, we know that if  $d(2A) \leq Kd(A)$ , we must have  $d(3A) \ll K^3d(A)$ . Similarly, in the circle, if  $\mu(2A) < 3\mu(A)$  and  $\mu(A)$  is small enough, Moskvina *et al.* [12] showed that  $A$  must satisfy strict structural conditions that imply  $\mu(3A) \leq 3(\mu(2A) - \mu(A))$ .

We solve partially the problem with  $k = 3$ .

**Theorem 1.7.** *Let  $(\alpha, \beta, \gamma) \in ]0, 1]^3$ , and suppose that  $\beta < \min(1, 3\alpha)$  and  $\gamma \in [\min(1, 3\beta/2), \min(1, 2\beta - \alpha)]$  or that  $\beta = 3\alpha$  and  $\gamma \in [3\beta/2, 2\beta]$ . Then  $(\alpha, \beta, \gamma) \in \mathcal{E}_3$ .*

For general  $k$ , our understanding of  $\mathcal{E}_k$  and  $\mathcal{D}_k$  is yet poorer. Note that in general, our sets  $A \subset \mathbb{N}$  satisfy  $d((k+1)A) \geq \frac{k+1}{k}d(kA)$ , which, in view of the aforementioned result of Lev, may be inevitable. In the following statement we compile all the results that we can prove in this general context.

**Theorem 1.8.** *Let  $\alpha = (\alpha_1, \dots, \alpha_{k+1}) \in [0, 1]^{k+1}$ , where  $k \geq 1$ .*

- a) *If  $\alpha_1 = \dots = \alpha_{k-1} = 0$  and  $\alpha_{k+1} \geq \frac{k+1}{k}\alpha_k$ , or  $\alpha_{k+1} \geq \alpha_k$  and  $\alpha_{k+1}$  is the inverse of an integer, then  $\alpha \in \mathcal{D}_{k+1}$ .*
- b) *If  $\alpha_1 = \dots = \alpha_k = 0$ , then  $\alpha \in \mathcal{E}_{k+1}$ .*
- c) *If  $\alpha_i = i\alpha$  for each  $i$  and some  $\alpha \leq 1/(k+1)$ , then  $\alpha \in \mathcal{E}_{k+1}^0 \subset \mathcal{D}_{k+1}$ .*

The last item is obvious by taking an interval of length  $\alpha$ , and was also proved somewhat differently for  $\mathcal{D}_k$  in [5].

In the next section, we prove the complete description of  $\mathcal{D}_2$  and  $\mathcal{E}_2$  given in Theorems 1.1 and 1.5.

## 2. Sumsets in the integers

**2.1. A preliminary reduction.** We first show that Theorem 1.8 a) follows from the special case  $\alpha_{k+1} = 1$  stated in Proposition 1.2. Let  $\alpha \in [0, 1]^{k+1}$  satisfy the hypothesis of Theorem 1.8 a), and let  $\beta' = \alpha_k$  and  $\gamma' = \alpha_{k+1}$ . We distinguish several cases.

- a) We first assume that  $\gamma'$  is an irrational number. Let  $A$  be the set given in Proposition 1.2 with parameters  $\theta = \frac{1}{\gamma'}$  and  $\beta = \frac{\beta'}{\gamma'}$  and  $A'$  be defined by

$$A' = \{ \lfloor \theta a \rfloor : a \in A \}.$$

Since  $A \subset T_{k, \theta}$  we have for any  $a_1, \dots, a_{k+1} \in A$

$$\lfloor \theta a_1 \rfloor + \lfloor \theta a_2 \rfloor + \dots + \lfloor \theta a_{k+1} \rfloor = \lfloor \theta(a_1 + a_2 + \dots + a_{k+1}) \rfloor.$$

Since  $\theta > 1$ , we get  $d(jA') = \theta^{-1}d(jA)$ ,  $j = 1, 2, \dots, k+1$ .

This yields Theorem 1.8 a) when  $\gamma'$  is an irrational number.

- b) If  $\gamma'$  is the inverse of a positive integer  $q$ , we use again Proposition 1.2 with parameters  $\theta = \frac{1}{\gamma'}$  and  $\beta = \frac{\beta'}{\gamma'}$  to generate a set  $A$ . We define the set

$$(3) \quad A^{(q)} = \{ qa : a \in A \}$$

which satisfies

$$d((k-1)A^{(q)}) = 0 < d(kA^{(q)}) = \frac{\beta}{q} < d((k+1)A^{(q)}) = \frac{1}{q}.$$

- c) We finally assume that  $\gamma' = \frac{s}{q}$  is a rational number with  $2 \leq s < q$ . Upon multiplying numerator and denominator by appropriate numbers, we may assume that  $s = (k+1)r$  for some integer  $r$  satisfying  $3 \leq r < \frac{q}{k+1}$ . Let  $U = \{0, 1, \dots, r-2, r\}$ . Then  $|jU| = jr$  for any  $j$ . Letting  $A' = U + A^{(q)}$  where  $A^{(q)}$  is defined by (3), we thus obtain

$$\begin{cases} d((k-1)A') = |(k-1)U| \times d((k-1)A^{(q)}) = 0, \\ d(kA') = |kU| \times d(kA^{(q)}) = \frac{kr\beta}{q} = \frac{k}{k+1}\beta\gamma', \\ d((k+1)A') = |(k+1)U| \times d((k+1)A^{(q)}) = \frac{(k+1)r}{q} = \gamma'. \end{cases}$$

This concludes the proof of Theorem 1.8 a), assuming Proposition 1.2. We will now prove the latter, focussing first on the case  $k = 1$  (so concerning twofold sumsets, that is Theorem 1.1), since it is much more simple than, while retaining some important features of, the general case, which we handle later.

**2.2. Twofold sumsets.** Before embarking on the proof of Proposition 1.2 in the case  $k = 1$ , we need a quantitative version of Weyl's criterion (Theorem 1.3), due to Erdős and Turán [4, Theorem III].

**Theorem 2.1.** *For any sequence  $s_j$  of elements of the torus  $\mathbb{T}$  and any interval  $A$ , we have for any integers  $n$  and  $m$  the bound*

$$\left| \frac{1}{n} |\{1 \leq j \leq n : s_j \in A\}| - \mu(A) \right| \ll \frac{1}{m} + \frac{1}{n} \sum_{k=1}^m \frac{1}{k} \left| \sum_{j=1}^n e^{2i\pi s_j k} \right|,$$

where the implied constant is absolute.

Applying this with  $s_j = \{\theta j\}$  for some irrational number  $\theta$  and using the standard exponential sum bound

$$\left| \sum_{j=1}^m e^{2i\pi j\theta} \right| \leq \frac{1}{2\|\theta\|},$$

where  $\|\theta\| = \min_{k \in \mathbb{Z}} |\theta - k|$ , we obtain

$$\left| \frac{1}{n} B_{\theta, A}(n) - \mu(A) \right| \ll \frac{1}{m} + \frac{1}{n} \sum_{k=1}^m \frac{1}{k\|\theta k\|},$$

where

$$(4) \quad B_{\theta, A}(n) = \{1 \leq j \leq n : \{\theta j\} \in A\}.$$

The series  $\sum_{k=1}^m \frac{1}{k\|\theta k\|}$  diverges as  $m$  tends to infinity, but selecting  $m = m(n)$  as a sufficiently slowly increasing function of  $n$ , one may achieve

$$\frac{1}{n} \sum_{k=1}^{m(n)} \frac{1}{k\|\theta k\|} \asymp \frac{1}{m(n)} \rightarrow 0$$

as  $n$  tends to infinity, and thus there exists a non increasing function  $\eta : \mathbb{N} \rightarrow \mathbb{R}_+$  (depending on  $\theta$  only) which tends to zero such that

$$(5) \quad \left| \frac{1}{n} B_{\theta, A}(n) - \mu(A) \right| \leq \eta(n).$$

Note that the bound (5) is uniform in  $A$ ; in particular, it is still valuable if  $A$  is replaced by a sequence  $A_n$  of intervals of sufficiently slowly decaying measure (e.g.  $\mu(A_n) \geq 2\eta(n)$ ). Also we note that using the sequence  $s_j = \{\theta(j + X)\}$ , we may obtain the more general bound

$$(6) \quad \left| \frac{1}{n} \left( B_{\theta,A}(n + X) - B_{\theta,A}(X) \right) - \mu(A) \right| \leq \eta(n)$$

for any integers  $X$  and  $n$ .

We now start the proof of Proposition 1.2 in the case  $k = 1$ . We will adopt a probabilistic construction. The setting corresponds to that given in [8, Chapter III, Theorem 13]. We shall use the notation  $\mathbb{P}(\mathcal{A})$  for the probability measure of an event  $\mathcal{A}$  and  $\mathbb{E}(Z)$  for the expectation of a random variable  $Z$ .

Let  $\theta$  be an irrational number,  $\eta$  be a non increasing function tending to 0 such that (5) holds.

Our aim is to study the number of representations of  $n$  under the form  $n = k_1 + k_2$  with  $k_1 < n/2 < k_2$  and  $k_1, k_2 \in T_{1,\theta}$  where  $T_{1,\theta}$  is defined by (1). When  $\{\theta n\} < 1/2$  we shall deduce from (5) the lower bound  $B_{\theta,I}(n/2) > \frac{n}{2} (\{\theta n\} - \eta(n/2))$  with  $I = ]0, \{\theta n\}[$ . We thus need to assume that  $\{\theta n\}$  is sufficiently larger than  $\eta(n/2)$ . Similarly when  $\{\theta n\} > 1/2$  we need to assume  $\{\theta n\}$  is sufficiently smaller than  $1 - \eta(n/2)$ . So we are led to restrict our attention to those integers  $n$  that belong to

$$X_\theta = \{n \in \mathbb{N} : 2\eta(n/2) < \{\theta n\} < 1 - 2\eta(n/2)\}.$$

By (5) and the ensuing remarks we have

$$(7) \quad d(X_\theta) = 1.$$

We now define our desired random sequence  $A$ . Let  $(\xi_k)_{k \geq 1}$  be a sequence of mutually independent Boolean random variables such that

$$\mathbb{P}(\xi_k = 1) = \beta_k, \quad k \geq 1,$$

where  $\beta_k$  is the constant sequence equal to  $\beta$  if  $\beta > 0$  and the decaying sequence  $k^{-1/5}$  if  $\beta = 0$ . Let  $A$  be the random sequence consisting of the integers  $k \in T_{1,\theta}$  such that  $\xi_k = 1$ . Then the expectation of its counting function  $A(x) = \sum_{0 < k \leq x} \xi_k$  satisfies

$$\mathbb{E}(A(x)) = \sum_{\substack{k \leq x \\ \{\theta k\} < 1/2}} \beta_k = \begin{cases} O(x^{4/5}) & \text{if } \beta = 0, \\ \beta T_{1,\theta}(x) & \text{if } \beta > 0. \end{cases}$$

By the strong law of large numbers for the mean of mutually independent random variables (cf. [8, chapter III, Theorem 11]) we infer that almost surely  $\frac{A(x)}{T_{1,\theta}(x)} \sim \beta$  as  $x \rightarrow \infty$  if  $\beta > 0$ , and  $A(x) \ll x^{4/5}$  if  $\beta = 0$ . It follows that in both cases the density of  $A$  inside  $T_{1,\theta}$  satisfies  $d_{T_{1,\theta}}(A) = \beta$  almost surely, as required.

Now we prove that  $2A \supset X_\theta \setminus F$ , where  $F$  is almost surely a finite set. This would imply that  $d(2A) = 1$ . Let  $n \in X_\theta$ . We define

$$K_n = \{0 < k < n/2 : k \in T_{1,\theta} \cap (n - T_{1,\theta})\},$$

and

$$R(n) = \sum_{k \in K_n} \xi_k \xi_{n-k}.$$

Then by the independence of the  $\xi_k$ 's

$$(8) \quad \mathbb{P}(R(n) = 0) = \prod_{k \in K_n} \mathbb{P}(\xi_k \xi_{n-k} = 0) \leq (1 - \beta_n^2)^{|K_n|} \leq \exp(-|K_n| \beta_n^2).$$

Now we need a lower bound for  $|K_n|$ . By definition  $k < n/2$  belongs to  $K_n$  if and only if  $\{\theta k\} < 1/2$  and  $\{\theta(n - k)\} < 1/2$ .

Let  $I = ]2\eta(n/2), 1/2[$ . Since  $n \in X_\theta$ , we have  $\{\theta n\} \in I \cup (1 - I)$ . Suppose for instance  $\{\theta n\} \in I$ , the case  $\{\theta n\} \in 1 - I$  is similar. Then for any  $k$  such that  $\{\theta k\} < \{\theta n\} < 1/2$ , we have  $\{\theta(n - k)\} = \{\theta n\} - \{\theta k\} < 1/2$ . Thus  $k \in K_n$ . This means that

$$K_n \supset \{0 < k < n/2 : \{\theta k\} < \{\theta n\}\},$$

whence  $|K_n| \geq \frac{n}{2}(\{\theta n\} - \eta(n/2)) \geq \frac{n}{2}\eta(n/2)$  by (5). If  $\{\theta n\} \in 1 - I = [1/2, 1 - 2\eta(n)[$  instead, it suffices to replace the condition  $\{\theta k\} < \{\theta n\}$  by  $\frac{1}{2} - \{\theta k\} < 1 - \{\theta n\}$  to obtain the same result.

One can choose  $\eta(n)$  to be arbitrarily slowly decaying, say  $\eta(n) \geq n^{-1/2}$ . This way  $|K_n| \gg n^{1/2}$ , so that  $\beta_n^2 |K_n| \gg n^{1/10}$  and from (8) we get

$$\sum_{n \in X_\theta} \mathbb{P}(R(n) = 0) < \infty.$$

We conclude by the Borel-Cantelli lemma (cf. [15, Lemma 1.2]) that almost surely, all but finitely many integers in  $X_\theta$  are sums of 2 terms from the random sequence  $A$ . The result follows from (7). This finishes the proof of Proposition 1.2 in the case where  $k = 1$ , and thus of Theorem 1.1.

We now determine which pairs  $(\alpha, \beta) \in \mathbb{R}^2$  with  $0 < \alpha \leq \beta < 2\alpha$  belong to  $\mathcal{D}_2$ , that is, we prove Theorem 1.5.

Let  $A \subset \mathbb{N}$  such that  $\beta = d(2A) < 2d(A) = 2\alpha$ . By Kneser's theorem [9] there exists a (minimal) positive integer  $g$  such that  $x + g \in 2A$  for all but finitely many  $x \in 2A$ , and

$$d(2A) \geq 2d(A) - \frac{1}{g}.$$

Let  $G = \mathbb{Z}/g\mathbb{Z}$  be the group of the residue classes modulo  $g$  and

$$\begin{aligned} \bar{A} &= \{\bar{x} \in G : \bar{x} \cap A \neq \emptyset\}, \\ \bar{B} &= \{\bar{x} \in \bar{A} : |\bar{x} \cap A| = \infty\}, \\ \bar{C} &= \{\bar{x} \in \bar{A} : 0 < |\bar{x} \cap A| < \infty\} = \bar{A} \setminus \bar{B}. \end{aligned}$$

Let

$$\tilde{A} = \bigcup_{\bar{x} \in \bar{B}} (\bar{x} \cap \mathbb{N}) \cup \{x \in \{0, 1, \dots, g-1\} : \bar{x} \in \bar{C}\}.$$

Then

$$d(A) \leq d(\tilde{A}) = \frac{|\bar{B}|}{g}, \quad d(2A) = \frac{|\bar{A} + \bar{B}|}{g}.$$

If  $|\bar{A} + \bar{B}| < |\bar{A}| + |\bar{B}| - 1$  then by Kneser's theorem (cf. [9]; see equivalently [15, Theorem 5.5])  $\bar{A} + \bar{B}$  would have a non trivial *period* in  $G$

$$\{\bar{x} \in G : \bar{x} + \bar{A} + \bar{B} = \bar{A} + \bar{B}\} \neq \{\bar{0}\},$$

in contradiction with the minimality of  $g$ . We thus have  $|\bar{A} + \bar{B}| \geq |\bar{A}| + |\bar{B}| - 1$ . On the other hand our hypothesis  $d(2A) < 2d(A)$  implies  $|\bar{A}| + |\bar{B}| - 1 < 2|\bar{B}|$ . We deduce  $\bar{B} = \bar{A}$  and finally  $|2\bar{A}| = 2|\bar{A}| - 1$ .

Let  $r = |\bar{A}|$ . Then  $\beta = \frac{2r-1}{g}$  with  $1 \leq r \leq \frac{g+1}{2}$ . We get

$$\frac{\beta}{2} < \alpha \leq \frac{r}{g} = \frac{\beta}{2} + \frac{1}{2g}.$$

We proved the following.



**Proposition 2.2.** *Let  $A$  such that  $d(2A) < 2d(A)$ . Then there exist two positive integers  $g$  and  $r \leq \frac{g+1}{2}$  such that*

$$d(2A) = \frac{2r-1}{g} \quad \text{and} \quad \frac{d(2A)}{2} < d(A) \leq \frac{d(2A)}{2} + \frac{1}{2g}.$$

Conversely, let  $\beta \in ]0, 1[ \cap \mathbb{Q}$  have non positive dyadic valuation, and  $g$  be the smallest positive integer for which  $g\beta$  is odd, thus  $\beta = \frac{2r-1}{g}$  for some integer  $r \geq 1$ . Let  $\alpha$  satisfy

$$\frac{\beta}{2} < \alpha \leq \frac{\beta}{2} + \frac{1}{2g} = \frac{r}{g}.$$

Then let  $V = \{0, \dots, r-1\}$ , so that  $|2V| = 2r-1 \leq g$ . Let  $\beta_0 = \alpha g/r$ , thus  $\beta_0 \in ]0, 1[$ . We apply Proposition 1.2 with  $k=1$  to get a set  $A_0$  such that  $d(A_0) = \beta_0$  and  $d(2A_0) = 1$ . The associated rescaled set  $A_0^{(g)} = \{ga : a \in A_0\}$  satisfies  $d(A_0^{(g)}) = \beta_0/g = \alpha/r$  and  $d(2A_0^{(g)}) = 1/g$ . Finally let  $A$  be the set

$$A = V + A_0^{(g)}.$$

Then  $A$  clearly admits the required density  $\alpha$ . Finally we have  $2A = 2V + 2A_0^{(g)}$  giving  $d(2A) = \frac{2r-1}{g} = \beta$  as desired.

This completes the proof of Theorem 1.5.  $\square$

**Example 2.3.** Assume that there exists a set  $A$  such that  $\alpha = d(A) = 4/9$  and  $\beta = d(2A) = 5/9$ . Applying Proposition 2.2 we get  $g \leq 3$  and  $r \in \{1, 2\}$ , whence  $\beta = 1, 1/2$  or  $1/3$ , a contradiction.

**Example 2.4.** Fixing  $\alpha = 1/5$  and  $\beta = 3/10$  in Theorem 1.5 yields  $g_0 = 10$ . Further the bounds  $\beta < 2\alpha \leq \beta + 1/g_0$  are satisfied hence there exists a set  $A$  such that  $d(A) = 1/5$  and  $d(2A) = 3/10$ . The only admissible choice for  $g, r$  in Proposition 2.2 is  $g = 10$  and  $r = 2$ .

### 3. Measures of sumsets in the circle

**3.1. Twofold sumsets.** To start with, we show that in order to achieve a large ratio  $\mu(2A)/\mu(A)$ , a large number of connected components will be necessary.

**Lemma 3.1.** *Let  $A$  be a disjoint union of  $k$  intervals. Then  $\mu(2A) \leq (k+1)\mu(A)$ . If the intervals are open, the equality case happens when all the  $\binom{k+1}{2}$  intervals of the sum are pairwise disjoint.*

*Proof.* Let  $A = \bigcup_{j=1}^k I_j$ . So  $2A = \bigcup_{i \leq j} (I_i + I_j)$ . Let  $\mu(I_i) = m_i$ , so  $\mu(I_i + I_j) = m_i + m_j$  and  $\mu(2A) \leq \sum_{i \leq j} (m_i + m_j) = (k+1) \sum_{i=1}^k m_i$ . The equality case is clear.  $\square$

We now attempt to prove the first item of Theorem 1.1 in the case  $\alpha > 0$ . Let  $(\alpha, \beta) \in ]0, 1]^2$  satisfy  $\beta \geq \min(2\alpha, 1)$ . If  $\beta = \min(2\alpha, 1)$ , the interval  $A = ]0, \alpha[$  satisfies  $\mu(A) = \alpha, \mu(2A) = \beta$ . So we now suppose  $0 < \alpha < 1/2$  and  $\beta > 2\alpha$ .

First, note that for any  $k$ , if  $A = [0, \ell] \cup \{2\ell\} \cup \dots \cup \{(k-1)\ell\}$ , then  $2A = [0, k\ell]$  so we can achieve a duplication ratio  $\mu(2A)/\mu(A) = k$ . The idea is then to somewhat “thicken” the singletons, in order to reduce the duplication ratio of the set.

Let  $k = \lfloor \beta/\alpha \rfloor$ , thus  $k \leq \beta/\alpha < k+1$  and  $k \geq 2$ .

Then let  $A = ]0, x[ \cup (\{2x, \dots, kx\} + ] - \epsilon, 0[)$ , for some  $x \leq \alpha$  and  $\epsilon \leq x/2$  to be determined later. Note that

$$2A = ]0, (k+1)x[ \cup (\{(k+2)x, \dots, 2kx\} + ] - 2\epsilon, 0[).$$

Thus  $\mu(A) = x + (k-1)\epsilon$  and  $\mu(2A) = (k+1)x + 2(k-1)\epsilon$ . The doubling ratio is therefore

$$f(\epsilon/x) = \frac{(k+1)x + 2(k-1)\epsilon}{x + (k-1)\epsilon} = (k+1) \frac{1 + 2\frac{k-1}{k+1}\frac{\epsilon}{x}}{1 + (k-1)\frac{\epsilon}{x}}.$$

We have  $f(0) = k+1$  and while  $f(1/2) = 4k/(k+1) \leq k$ . Therefore by continuity of  $f$ , there is a value of the ratio  $y = \epsilon/x$  for which the doubling ratio is the desired  $\beta/\alpha$ .

Then there remains to pick  $x$  such that  $\alpha = x + (k-1)\epsilon = x(1 + (k-1)y)$ , namely  $x = \frac{\alpha}{1+(k-1)y}$ , and then the corresponding  $\epsilon$ .

In the case  $\alpha = 0$ , a radically different construction will be necessary. Let  $C \subset [0, 1]$  be the classical ternary Cantor set. It is well known that  $C + C = [0, 2]$  (cf. [1, Corollary 2.3]). Scaling  $C$  by a factor  $\beta/2$  and projecting it to the circle, we obtain the set  $A = \{\beta c/2 : c \in C\}$  of measure 0 such that  $2A = [0, \beta]$ , thus  $\mu(2A) = \beta$ .

**3.2. Threefold sumsets.** First we prove Theorem 1.6. We will derive it from the following theorem of Gyarmati, Konyagin and Ruzsa [7].

**Proposition 3.2.** *There exists an absolute constant  $c > 0$  such that the following holds. Let  $p \geq 29$  be a prime. Let  $A \subset \mathbb{Z}/p\mathbb{Z}$  and let  $(n, s) = (|2A|, |3A|)$ . If  $n < cp$ , then  $s \geq \frac{3n-1}{2}$ .*

We derive the analogous result for measures in the circle  $\mathbb{T}$  by a standard method. We first prove Theorem 1.6 for simple sets, that is, the union of finitely many closed intervals. Let  $A \subset \mathbb{T}$  be a simple set. Let  $c$  be the constant given by Proposition 3.2 and suppose that  $\mu(2A) < c$ . Let  $p \geq 29$  be a prime, that we will let tend to infinity ultimately. Let

$$A_p = \{j \in \mathbb{Z}/p\mathbb{Z} : j/p \in A\}.$$

One may check that  $|A_p| = p\mu(A) + O(1)$  as  $p$  tends to infinity. Further note that  $(kA)_p = kA_p$  for any  $k \in \mathbb{N}$ . Since  $2A$  and  $3A$  are simple, one has  $|(kA)_p| = p\mu(kA) + O(1)$  for  $k = 2, 3$ ; thus we have  $|(2A)_p| < cp$  for  $p$  sufficiently large, so we can apply Proposition 3.2 and conclude in the case of simple sets.

Now if  $A$  is closed (that is, compact), writing  $I_\delta = ]-\delta, \delta[$ , we have  $A = \bigcap_{\delta > 0} (A + I_\delta)$ , in fact  $kA = \bigcap_{\delta > 0} (kA + I_{k\delta})$  for any integer  $k \geq 1$ . So for any fixed  $\epsilon > 0$ , we can choose  $\delta$  such that  $\mu(kA + I_{k\delta}) \leq \mu(kA) + \epsilon$ . Further, by compactity, there exists a simple set  $A'$  (the union of finitely many translates of  $I_\delta$ ) such that  $A \subset A' \subset A + I_\delta$ . We have

$$\mu(3A) \geq \mu(3A') - \epsilon \geq \frac{3}{2}\mu(2A) - \epsilon.$$

Letting  $\epsilon$  tend to zero, we conclude the proof of Theorem 1.6.  $\square$

We prove Theorem 1.7. If  $\alpha \geq 1/3$ , the triplets  $(\alpha, \beta, \gamma)$  that belong to  $\mathcal{E}_k$  are the ones for which  $\beta \geq \min(1, 2\alpha)$  and  $\gamma = 1$ .

We now consider triplets where  $\alpha < 1/3$ ; we prove the following proposition, which implies Theorem 1.7.

**Proposition 3.3.** *The set of triplets  $(\mu(A), \mu(2A), \mu(3A))$  for sets  $A \subset [0, 1/3] \subset \mathbb{T}$  having at most two connected components is*

$$\{(\alpha, \beta, \gamma) \in [0, 1]^3 : \beta \in [2\alpha, 3\alpha], \gamma \in [3\beta/2, 2\beta - \alpha[ \text{ or } \beta = 3\alpha, \gamma \in [3\beta/2, 2\beta]\}.$$

*Proof.* We may take  $A$  of the form  $]0, x[ \cup ]y, z[$  for some  $0 \leq x \leq y \leq z \leq 1/3$ . So  $2A = ]0, 2x[ \cup ]y, x+z[ \cup ]2y, 2z[$  and  $3A = ]0, 3x[ \cup ]y, 2x+z[ \cup ]2y, 2z+x[ \cup ]3y, 3z[$ .

We are seeking for which triplets  $(\alpha, \beta, \gamma)$  the system

$$\begin{cases} \alpha = & x + z - y \\ \beta = & 3\alpha - \max(0, 2x - y) - \max(0, x + z - 2y) \\ \gamma = & 6\alpha - \max(0, 3x - y) - \max(0, 2x + z - 2y) - \max(2z + x - 3y, 0) \end{cases}$$

admits solutions. We now discuss the existence of solutions according to the number of connected components of  $2A$  and  $3A$ , that is, for each max above, whether it is positive or not. In the following discussion, the necessary conditions we provide may always easily be seen to be sufficient, although we do not always explicitly state it.

- 1) If  $2A$  is an interval, then so is  $3A$  so  $\gamma = 3\alpha = 3\beta/2$ .
- 2) If  $2A$  has two connected components, so there is exactly one overlap between the intervals of  $2A$ , we distinguish.
  - a) If  $2x > y$  and  $x + z < 2y$ , so  $2A = ]0, x + z[ \cup ]2y, 2z[$ , we have  $\beta = x - 2y + 3z$ . We have necessarily  $3x > y$  and  $2x + z > 2y$ , so  $3A = ]0, 2z + x[ \cup ]3y, 3z[$  where the last two intervals may overlap or not.
    - i) If they do, so  $2z + x > 3y$ , we have  $\gamma = 3z$ . So  $\beta = x - 2y + \gamma$  and  $\alpha = x - y + \gamma/3$ . Get  $\alpha - \beta = y - 2\gamma/3$  so  $y = \alpha - \beta + 2\gamma/3$  while  $x = 2\alpha - \beta + \gamma/3$ . We check that the inequalities are satisfied:  $2x - y = 3\alpha - \beta > 0$  so  $\beta < 3\alpha$ ,  $2y - x - z = -\beta + 2\gamma/3 > 0$  implies  $\gamma > 3\beta/2$ . Further, we need  $2z + x - 3y = -\alpha + 2\beta - \gamma > 0$  which amounts to  $3\beta/2 < \gamma < 2\beta - \alpha < 5\alpha$ . Conversely, whenever these conditions are satisfied, the system has solutions.
    - ii) Otherwise we have  $2z + x < 3y$ , hence  $\gamma = 3(z - y) + 2z + x$ . Thus a solution exists if and only if  $\gamma = 2\beta - \alpha$ .
  - b) Now if  $2x < y$  and  $x + z > 2y$ , so  $2A = ]0, 2x[ \cup ]y, 2z[$ , we have  $\beta = 2x - y + 2z$ . We have necessarily  $2x + z > 2y$  and  $2z + x > 3y$ , so  $3A = ]0, 3x[ \cup ]y, 3z[$ , where the two intervals may or not overlap.
    - i) If they do, so  $2x < y < 3x$ , we have  $\gamma = 3z$ . Further we find  $y = \beta - 2\alpha$ , and  $x = \beta - \alpha - \gamma/3$ . So  $y - 2x = -\beta + 2\gamma/3 > 0$  implies yet again  $\gamma > 3\beta/2$ . Further  $y - 3x = -2\beta + \alpha + \gamma < 0$  implies  $\gamma < 2\beta - \alpha$ . Also  $x + z - 2y = 3\alpha - \beta > 0$  amounts to  $\beta < 3\alpha$ .
    - ii) Otherwise, so  $y > 3x$ , we find  $\gamma = 3z - y + 3x = 3\alpha + 2y$  and again  $\gamma = 2\beta - \alpha$ .
- 3) If  $2A$  has three connected components (no overlap), then  $\beta = 3\alpha$ . We have  $2x < y$  and  $x + z < 2y$ . We distinguish according to the presence of overlaps or not in  $3A$ .
  - a) If there is no overlap, we have  $\gamma = 6\alpha$ . It is realisable, just take  $x$ , then  $y > 3x$ , then  $y < z < \min((3y - x)/2, 1/3)$ , then all constraints are realised. We can achieve that for any value of  $\alpha \leq 1/6$ .
  - b) If  $3A$  is connected,  $\gamma = 3z$ . Now the conditions  $2x < y$  and  $x + z < 2y$  imply  $z < 3(y - x)$ , which is equivalent to  $2z > 3(x + z - y)$ , and finally  $\gamma > 3\beta/2$ .
  - c) If there is exactly one overlap, that is, if  $3A$  has three connected components, we distinguish.
    - i) Suppose  $3x > y$ . So  $2x + z < 2y$  and  $2z + x < 3y$ . Then  $\gamma = 6\alpha - 3x + y$ . This imposes  $\gamma \in ]5\alpha, 6\alpha[ = ]5\beta/3, 2\beta[$ .
    - ii) Now suppose  $2x + z > 2y$ . So  $y > 3x$  and  $2z + x < 3y$ . Then  $\gamma = 6\alpha - 2x - z + 2y = 5\alpha - x + y > 5\alpha$ .
    - iii) If only the last gap is overcome,  $\gamma = 6\alpha - 2z - x + 3y = 5\alpha - z + 2y > 5\alpha$ .
  - d) If  $3A$  has two connected components, we distinguish.

- i) If all but the last gap are overcome,  $\gamma = 6\alpha - 3x + y - 2x - z + 2y = 5\alpha - 4x + 2y > 5\alpha$ .
- ii) If all but the middle gap are overcome,  $\gamma = 6\alpha - 3x + y - 2z - x + 3y = 5\alpha - 3x - z + 3y > 5\alpha$ .
- iii) If all but the first gap are overcome,  $\gamma = 6\alpha - 2x - z + 2y - 2z - x + 3y > 5\alpha$ .  $\square$

Regarding sets with  $k$  connected components when  $k \geq 3$ , the determination of the possible triplets  $(\alpha, \beta, \gamma)$  becomes untractable by this method. Nevertheless, we can easily see that the structure of the set of the possible triplets remains similar, that is, a connected union of finitely many (in fact  $O_k(1)$  many) polytopes, where a polytope is the intersection of finitely many half-spaces.

**3.3. Further iterated sumsets.** We now prove Theorem 1.8 b). Let  $\beta \in (0, 1]$  and  $k \geq 1$  an be integer. We argue as in the proof of Theorem 1.1 b) in the particular case  $\alpha = 0$ . Let  $C_{k+2}$  be the Cantor set of initial segment  $[0, 1] \subset \mathbb{R}$  and ratio of dissection  $1/(k+2)$ . It is known [1, Corollary 2.3] that  $\mu(kC_{k+2})$  has measure 0 whereas  $(k+1)C_{k+2} = [0, k+1]$ . The suitable re= of  $C_{k+2}$  by the factor  $\beta/(k+1)$  provides the desired construction.

Note that this does not imply Theorem 1.8 a) in the particular case  $\alpha_k = 0$  since the openness condition of Lemma 1.4 may not be removed. Indeed, if  $A \subset \mathbb{T}$  has measure zero, one may see that  $B_{\lambda, A}$  is empty for almost all  $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ , since the map  $\lambda \mapsto n\lambda$  on the circle is measure-preserving for any integer  $n$ . So we need the specific argument based on Proposition 1.2 we gave in Section 2 when  $k = 1$ . The general case  $k \geq 2$  will follow from Proposition 4.6.

#### 4. Iterated sumsets in the integers

We now prove Proposition 1.2 for  $k \geq 2$ . The (probabilistic) argument we will use subsumes, but is significantly more complicated than, the one used in Section 2, which is why we preferred to present it separately. First of all we collect a number of useful but technical results.

**4.1. Preliminary lemmas.** First we need to somewhat generalise the bound (6) obtained via the Erdős-Turán theorem.

**Proposition 4.1.** *Let  $k, D, M, X$  be integers. Let  $f = \sum_{i=1}^k P_i \mathbf{1}_{I_i}$  where  $(I_i)_{i \leq k}$  is a family of pairwise disjoint intervals in  $[0, 1[$  and  $P_i$  a polynomial of degree less than  $D$  whose coefficients are all at most  $M$ . Then*

$$\left| \frac{1}{N} \sum_{X < n \leq N+X} f(\{ \theta n \}) - \int f \right| = O(MDk\sqrt{\eta(N)}).$$

A function  $f$  satisfying the above hypothesis will naturally be referred to as *piecewise polynomial*.

*Proof.* It suffices to prove it for monomials and for  $k = 1$ , the general case following by linear combinations (incurring an extra factor  $Mk$ ). Thus let  $a < b$  be in  $[0, 1[$ , and let  $d \leq D$  and  $f$  be defined by  $f(x) = x^d \mathbf{1}_{(a,b)}$ . Using the bound (6), we note that

$$a^d((b-a) - O(\eta(N))) \leq \frac{1}{N} \sum_{X < n \leq N+X} \{ \theta n \}^d \mathbf{1}_{(a,b)}(\{ \theta n \}) \leq b^d((b-a) + O(\eta(N)))$$

Further, observe that

$$a^d(b-a) \leq \int_a^b x^d dx \leq b^d(b-a).$$

Hence

$$(a^d - b^d)(b - a) - O(\eta(N)) \leq \frac{1}{N} \sum_{X < n \leq N+X} \{\theta n\}^d \mathbf{1}_{(a,b)}(\{\theta n\}) - \int_a^b x^d dx$$

$$\leq (b^d - a^d)(b - a) + O(\eta(N)).$$

Given that  $b^d - a^d \leq d(b - a)$ , we find that

$$\left| \sum_{X < n \leq N+X} \{\theta n\}^d \mathbf{1}_{(a,b)}(\{\theta n\}) - \int_a^b x^d dx \right| \leq d(b - a)^2 + O(\eta(N)).$$

Then splitting the interval  $[a, b]$  into  $O(\sqrt{\eta(N)}^{-1})$  consecutive intervals of size  $[\sqrt{\eta(N)}]$ , we obtain, for each of these intervals, an error term of size  $O(d\eta(N))$ , and so in total, an error term of size  $O(D\sqrt{\eta(N)})$ .  $\square$

A certain type of sums will appear in the sequel, for which we now give an asymptotic.

**Lemma 4.2.** *Let  $0 < \alpha < 1$ ,  $0 \leq \beta < \infty$  and*

$$(9) \quad J_N(\alpha, \beta) := \sum_{0 < x < N} \frac{1}{x^\alpha (N - x)^\beta}.$$

Then

$$J_N(\alpha, \beta) = \begin{cases} B(1 - \alpha, 1 - \beta) N^{1 - \alpha - \beta} + O(N^{-\min(\alpha, \beta)}) & \text{if } \beta < 1, \\ N^{-\alpha} \log N + O(N^{-\alpha}) & \text{if } \beta = 1, \\ \zeta(\beta) N^{-\alpha} + O(N^{-\alpha - 1 + 1/\beta}) & \text{if } \beta > 1, \end{cases}$$

where  $B(\cdot, \cdot)$  denotes the Euler beta function defined by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

and  $\zeta(\cdot)$  is the Riemann zeta function.

This can be proved by considering Riemann sums; we omit the standard details. The beta function satisfies the following functional equation involving Euler's gamma function:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

By induction, we may achieve the following simple lemma.

**Lemma 4.3.** *Let  $(\alpha_1, \dots, \alpha_s) \in ]0, 1[^s$ . Then*

$$\sum_{\substack{1 \leq u_1, \dots, u_s \leq n \\ \sum_i u_i = n}} \prod_i u_i^{-\alpha_i} = O(n^{s-1 - \sum_i \alpha_i}).$$

Further, let  $\epsilon : \mathbb{N} \rightarrow \mathbb{R}_+$  tend to 0. Then there exists a sequence  $\epsilon'$  depending only on  $\epsilon$  that tends to zero such that

$$\sum_{\substack{1 \leq u_1, \dots, u_s \leq n \\ \sum_i u_i = n}} \epsilon(u_1) \prod_i u_i^{-\alpha_i} = \epsilon'(n) n^{s-1 - \sum_i \alpha_i}.$$

*Proof.* We prove the second part for  $s = 2$ , the rest following by a simple induction. Let  $K_\delta$  be such that for all  $k \geq K_\delta$ , we have  $\epsilon(k) \leq \delta$ . Further let  $M$  be an upper bound for  $\epsilon$ . Then

$$\sum_{k < n} \epsilon(k) k^{-\alpha_1} (n - k)^{-\alpha_2} \leq M \sum_{k < K_\delta} k^{-\alpha_1} (n - k)^{-\alpha_2} + \delta \sum_{k < n} k^{-\alpha_1} (n - k)^{-\alpha_2}$$

The right-hand side is  $O(K_\delta^{1-\alpha_1-\alpha_2} + \delta n^{1-\alpha_1-\alpha_2})$  by Lemma 4.2. We have  $K_\delta \rightarrow \infty$  (unless  $\epsilon(k) = 0$  eventually) as  $\delta \rightarrow 0$ , but choosing  $\delta$  as a sufficiently slowly decaying function of  $n$ , we can make the error term as small as  $o(n^{1-\alpha_1-\alpha_2})$  as desired.  $\square$

For any real number  $0 \leq x \leq 1$  and any integer  $1 \leq j \leq k-1$ , let

$$a_j(x) = \max\left(0, x - \frac{j}{k+1}\right), \quad b_j(x) = \min\left(x, \frac{j}{k+1}\right)$$

and  $I_j(x)$  be the open interval

$$I_j(x) = ]a_j(x), b_1(x)[.$$

Let  $f_1 = \mathbf{1}_{]0,1[}$  and

$$f_{j+1}(x) = \int_{a_j(x)}^{b_1(x)} f_j(x-y)dy = \int_{a_1(x)}^{b_j(x)} f_j(y)dy, \quad 1 \leq j \leq k-1.$$

Then for any  $1 \leq j \leq k-1$

- i)  $a_j$  and  $b_j$  are piecewise affine. Further  $a_j(x) + b_j(x) = x$ .
- ii)  $\mu(I_j(x)) = b_1(x) - a_j(x) = \max\left(0, \min\left(x, \frac{1}{k+1}, \frac{j+1}{k+1} - x\right)\right)$ . As a result,  $f_j$  is supported on  $]0, \frac{j}{k+1}[$ .
- iii)  $f_j$  is a non negative, nonzero piecewise polynomial function. In fact  $f_j$  has only finitely many zeros on  $]0, \frac{j}{k+1}[$ .

We will need the following estimate.

**Lemma 4.4.** *Let  $(\alpha, \beta) \in ]0, 1[^2$ . Let  $\theta > 1$  be irrational and  $x \in ]0, 1[$ . Then for any  $j$ , we have*

$$\sum_{\substack{0 < u < N \\ \{\theta u\} \in I_j(x)}} f_j(x - \{\theta u\}) \frac{1}{u^\alpha} \frac{1}{(N-u)^\beta} = J_N(\alpha, \beta)(f_{j+1}(x) + O(\eta'(N))),$$

where  $\eta'$  is a function  $\mathbb{N} \rightarrow \mathbb{R}_+$  which tends to zero and depends only on  $\theta$ .

*Proof.* We decompose the interval of summation  $[1, N[$  into subintervals of some length  $m = f(N)$  tending to infinity rather slowly,  $m = o(N)$  at any rate, even  $m \ll N^{o(1)}$  but not too slowly either; we fix  $m = \lfloor \eta(N)^{-1/2} \rfloor$  for definiteness. We write

$$[1, N[ = \bigcup_{0 \leq k < \lfloor \frac{N}{m} \rfloor} ]km, (k+1)m] \cup \left] \left\lfloor \frac{N}{m} \right\rfloor m, N \right[$$

where the last interval has at most  $m$  elements.

Let  $K = \lfloor \frac{N}{m} \rfloor$ . Let  $a = -\alpha$  and  $b = -\beta$ . We note that

$$\sum_{n \in ]\lfloor \frac{N}{m} \rfloor m, N[} n^a (N-n)^b \leq m(Km)^a.$$

Denoting by  $S$  the sum to estimate, this implies that

$$S = \sum_{0 \leq k < K} \sum_{\substack{n \in ]km, (k+1)m \\ \{\theta n\} \in I_j(x)}} f_j(x - \{\theta n\}) n^a (N-n)^b + O(N^{a+o(1)}).$$

Also we note that when  $n \in ]km, (k+1)m]$ , the expression  $n^a (N-n)^b$  may be regarded as approximately constant, more precisely

$$n^a (N-n)^b = m^{a+b} k^a (K-k)^b (1 + O(1/k))(1 + O(1/(K-k))).$$

We may restrict the sum over  $k$  to reasonably large  $k$ , like between  $\sqrt{K}$  and  $K - \sqrt{K}$ ; indeed, we have

$$\sum_{0 < u \leq m\sqrt{K}} u^a (N - u)^b \leq (N - N^{1/2+o(1)})^b N^{a/2+1/2+o(1)} = N^{b+a/2+1/2+o(1)}$$

which is negligible to  $N^{a+b+1}$ . We may argue analogously to discard the sum over  $k \geq K - \sqrt{K}$ . This way  $(1 + O(1/k))(1 + O(1/(K - k))) = 1 + O(1/\sqrt{K})$  for any  $k$  considered. Thus  $S$ , up to an error  $O(N^{a+b+1}/\sqrt{K})$ , equals

$$(10) \quad m^{a+b}(1 + O(1/\sqrt{K})) \sum_{\sqrt{K} \leq k < K - \sqrt{K}} k^a (K - k)^b \sum_{\substack{n \in ]km, (k+1)m] \\ \{\theta n\} \in I_j(x)}} f_j(x - \{\theta n\}).$$

We now apply Proposition 4.1 to the inner sum. By definition of  $f_{j+1}$ , we obtain

$$\sum_{\substack{n \in ]km, (k+1)m] \\ \{\theta n\} \in I_j(x)}} f_j(x - \{\theta n\}) = m(f_{j+1}(x) + O(\eta(\sqrt{m}))).$$

Injecting that in (10), we find that

$$S = m^{a+b+1}(1 + O(1/\sqrt{K}))(f_{j+1}(x) + O(\eta(\sqrt{m}))) \sum_{\sqrt{K} \leq k < K - \sqrt{K}} k^a (K - k)^b + O(N^c)$$

for some  $c < a + b + 1$ . Now we have from (9)

$$m^{a+b+1} \sum_{\sqrt{K} \leq k < K - \sqrt{K}} k^a (K - k)^b = J_N(-a, -b) + O(N^c)$$

by the same arguments as above. Finally, upon gathering all error terms together (whereby the term in  $O(\eta(\sqrt{m}))$  provides the largest one), we obtain the desired conclusion.  $\square$

We are now ready to show the following key estimate.

**Lemma 4.5.** *There exists a positive function  $\eta''$  depending only on  $\theta$  and  $k$ , and which tends to 0 at infinity, such that for any integer  $n$  we have*

$$(11) \quad S_k(n) := \sum_{\substack{0 < u_1 < \dots < u_k < n \\ \forall i, u_i \in T_{k, \theta} \\ n = u_1 + \dots + u_k}} (u_1 \dots u_k)^{-1+1/k} = \lambda_k f_k(\{\theta n\}) + O(\eta''(n)),$$

where  $\lambda_k = \frac{\Gamma(\frac{1}{k})^k}{k!}$ .

*Proof.* Let

$$E_k(n) := \sum_{\substack{0 < u_1, \dots, u_k < n \\ \exists i \neq j : u_i = u_j \\ n = u_1 + \dots + u_k}} (u_1 \dots u_k)^{-1+1/k}$$

and

$$S'_k(n) := \sum_{\substack{0 < u_1, \dots, u_k < n \\ \forall i, u_i \in T_{k, \theta} \\ n = u_1 + \dots + u_k}} (u_1 \dots u_k)^{-1+1/k},$$

so that  $S'_k(n) = O(E_k(n)) + k!S_k(n)$ . We observe that  $E_k(n) = O(n^{-1/k})$ . Further, reformulating the diophantine constraints using the intervals  $I_j$ , we have the

decomposition

$$(12) \quad S'_k(n) = \sum_{\substack{u_1 < n \\ \{\theta u_1\} \in I_{k-1}(\{\theta n\})}} u_1^{-1+1/k} \sum_{\substack{u_2 < n-u_1 \\ \{\theta u_2\} \in I_{k-2}(\{\theta(n-u_1)\})}} u_2^{-1+1/k} \dots \\ \dots \sum_{\substack{u_{k-1} < n-u_1-\dots-u_{k-2} \\ \{\theta u_{k-1}\} \in I_1(\{\theta(n-u_1-\dots-u_{k-2})\})}} (u_{k-1}(n-u_1-\dots-u_{k-1}))^{-1+1/k}.$$

To simplify the notation, let us denote  $n_j = n - u_1 - \dots - u_{k-j}$ , thus  $n_k = n$  and  $n_j = n_{j+1} - u_{k-j}$ . We shall prove by induction on  $j \leq k$  that

$$(13) \quad S'_k(n) = C_j \sum_{\substack{0 < u_1 < n \\ \{\theta u_1\} \in I_{k-1}(\{\theta n\})}} u_1^{-1+1/k} \sum_{\substack{0 < u_2 < n_1 \\ \{\theta u_2\} \in I_{k-2}(\{\theta n_1\})}} u_2^{-1+1/k} \dots \\ \dots \sum_{\substack{0 < u_{k-j} < n_{j+1} \\ \{\theta u_{k-j}\} \in I_j(\{\theta n_{j+1}\})}} u_{k-j}^{-1+1/k} (n_{j+1} - u_{k-j})^{-1+j/k} f_j(\{\theta n_j\}) + \epsilon_j(n)$$

where  $C_j = \prod_{i=1}^{j-1} B\left(\frac{1}{k}, \frac{i}{k}\right)$  and  $\epsilon_j$  tends to 0. When  $j = k$ , there is no more summation at all and (13) boils down to  $C_k f_k(\{\theta n\}) + \epsilon_k(n)$ , which is the desired result since

$$C_k = \prod_{j=1}^{k-1} B\left(\frac{1}{k}, \frac{j}{k}\right) = \prod_{j=1}^{k-1} \frac{\Gamma\left(\frac{1}{k}\right)\Gamma\left(\frac{j}{k}\right)}{\Gamma\left(\frac{j+1}{k}\right)} = \Gamma\left(\frac{1}{k}\right)^k.$$

Equation (12) is the  $j = 1$  case. We now suppose that (13) holds for some  $j \leq k-1$ . Let  $A_j(n)$  be the main-term of the right-hand side of (13). Using Lemma 4.4 on the innermost sum, and reparametrising by writing  $n_{j+1} = v_1$  and  $u_i = v_{i+1}$  in the error term, we find

$$A_j(n) = A_{j+1}(n) + O\left(\sum_{\substack{v_1, \dots, v_{k-j} \leq n \\ \sum v_i = n}} \eta'(v_1) v_1^{-1+\frac{j+1}{k}} \prod_{i=2}^{k-j} v_i^{-1+1/k}\right).$$

By Lemma 4.3 and since  $\eta'$  tends to 0, the expression between brackets in the above equation is certainly  $o(1)$ . This concludes the induction step and therefore the proof of the lemma.  $\square$

**4.2. The construction.** We argue as in Section 2 by the probabilistic method (see [8, Chapter III]). Let  $c > 0$  and  $\xi_m$ ,  $m \geq 1$ , be a sequence of independent Boolean random variables such that

$$\mathbb{P}(\xi_m = 1) = \frac{c}{m^{1-1/k}}.$$

Let  $S$  be the random increasing sequence of the  $m$ 's such that  $\xi_m = 1$ . This is essentially a sequence of pseudo  $k$ -th powers. These objects have been well studied since their introduction by Erdős and Renyi [3]. In particular Goguel [6] computed the (almost sure) density of  $kS$  and Deshouillers and Iosifescu [2] found that the density of  $(k+1)S$  is almost surely 1. Now we let  $A = S \cap T_{k,\theta}$ , where  $T_{k,\theta}$  was defined by equation (1). From now on we will suppose  $\theta$  is irrational; if  $\theta$  is an integer,  $T_{k,\theta} = \mathbb{N}$  so  $A = S$  and the previous references apply. The treatment of this simpler case may still be read out from our proofs by discarding all the (then vacuous) diophantine conditions. The next proposition implies Proposition 1.2.

**Proposition 4.6.** *Almost surely we have*

- a)  $d(jA) = 0$ , for any  $j = 1, \dots, k-1$ ,
- b)  $d((k+1)A) = 1$ ,



c)  $d(kA) = \frac{k}{k+1} - F_k(c)$  where  $F_k(c)$  is a continuous function and increasing from 0 to  $k/(k+1)$  when  $c$  is decreasing from  $\infty$  to 0.

*Proof.* a) By the appropriate version of the strong law of large numbers (cf. [8, chapter III, Theorem 11]) we infer that with probability 1,  $A(x) \sim x^{1/k}$  when  $x \rightarrow \infty$ , thus for any  $1 \leq j \leq k-1$

$$(jA)(x) \ll x^{j/k}, \text{ as } x \text{ tends to infinity.}$$

It follows that  $d(jA) = 0$  almost surely.

b) Let  $n$  be a positive integer and observe that  $0 < \{\theta n\} < 1$ . We denote  $I(t, k)$  the open interval

$$I(t, k) = \left] \max \left( 0, \frac{t}{k} - \frac{1}{k(k+1)} \right), \min \left( \frac{t}{k}, \frac{1}{k+1} \right) \right[.$$

and

$$R_{k+1}(n) = \sum_{\substack{0 < u_1 < \dots < u_k < u_{k+1} < n \\ n = u_1 + \dots + u_{k+1} \\ \{\theta u_i\} \in I(\{\theta n\}, k), (1 \leq i \leq k)}} \xi_{u_1} \cdots \xi_{u_k} \xi_{u_{k+1}}.$$

Then  $R_{k+1}(n) > 0$  implies that  $n \in (k+1)A$ . Moreover

$$\{R_{k+1}(n) = 0\} = \bigcap_{\substack{0 < u_1 < \dots < u_k < u_{k+1} < n \\ n = u_1 + \dots + u_{k+1} \\ \{\theta u_i\} \in I(\{\theta n\}, k), (1 \leq i \leq k)}} \{\xi_{u_1} \cdots \xi_{u_k} \xi_{u_{k+1}} = 0\}.$$

We denote by  $\mathcal{U}(n)$  the set of the ordered  $(k+1)$ -uples  $\underline{u}$  such that  $n = \sum_{i=1}^{k+1} u_i$  and  $\{\theta u_i\} \in I(\{\theta n\}, k)$ ,  $i = 1, \dots, k$ .

The events  $\mathcal{A}(\underline{u}) = \{\xi_{u_1} \cdots \xi_{u_k} \xi_{u_{k+1}} = 1\}$ ,  $\underline{u} \in \mathcal{U}(n)$ , are not necessarily pairwise independent: for distinct  $(k+1)$ -tuples  $\underline{u}, \underline{v}$ , the events  $\mathcal{A}(\underline{u})$  and  $\mathcal{A}(\underline{v})$  are not independent if and only if  $\underline{u} \sim \underline{v}$ , where the notation  $\sim$  means  $u_i = v_j$  for some  $i, j$ . Let

$$\mu_n = \sum_{\underline{u} \in \mathcal{U}(n)} \mathbb{P}(\mathcal{A}(\underline{u})), \quad \Delta_n = \sum_{\substack{\underline{u} \neq \underline{v} \in \mathcal{U}(n) \\ \underline{u} \sim \underline{v}}} \mathbb{P}(\mathcal{A}(\underline{u}) \cap \mathcal{A}(\underline{v})).$$

By Janson's inequality [15, Theorem 1.28]

$$(14) \quad \mathbb{P}(R_{k+1}(n) = 0) \leq \exp \left( -\frac{\mu_n^2}{2(\mu_n + \Delta_n)} \right).$$

We firstly have

$$\mu_n = c^{k+1} \sum_{\substack{0 < u_1 < \dots < u_k < n \\ \{\theta u_i\} \in I(\{\theta n\}, k)}} (u_1 \cdots u_k (n - u_1 - \dots - u_k))^{-1+1/k}.$$

The summand in the inner-sum is at least  $\left(\frac{n}{k+1}\right)^{-k+1/k}$ , hence

$$\begin{aligned} \mu_n &\geq c^{k+1} \frac{B_{\theta, I}(n)^k - \binom{k}{2} B_{\theta, I}(n)^{k-1}}{k!} \left(\frac{n}{k+1}\right)^{-k+1/k} \\ &\geq c^{k+1} \left( B_{\theta, I}(n)^k - \binom{k}{2} B_{\theta, I}(n)^{k-1} \right) n^{-k+1/k} \end{aligned}$$

where  $I = I(\{\theta n\}, k)$  and  $B_{\theta, I}(n)$  is defined by (4). By equation (5),

$$\frac{B_{\theta, I}(n)}{n} \geq \min \left( \frac{\{\theta n\}}{k}, \frac{1}{k(k+1)}, \frac{1 - \{\theta n\}}{k} \right) - \eta(n).$$

Hence if  $2k\eta(n) < \{\theta n\} < 1 - 2k\eta(n)$ , we have

$$(15) \quad \mu_n \geq (1 - o(1)) c^{k+1} n^{1/k} \eta(n)^k.$$

Now we examine  $\Delta_n$ . By a discussion according to the number  $s \leq k-1$  of positions where two distinct  $(k+1)$ -tuples in  $\mathcal{U}(n)$  agree, and ignoring the diophantine conditions, we get

$$(16) \quad \Delta_n \leq \sum_{s=1}^{k-1} c^{s+2(k+1-s)} \Delta_n(s, k+1-s),$$

where

$$\Delta_n(s, r) := \sum_{\substack{0 < u_1, \dots, u_s < n \\ \sum_{i=1}^s u_i < n}} (u_1 \cdots u_s)^{-1+1/k} \left( \sum_{\substack{0 < v_1, \dots, v_r < n \\ n = u_1 + \dots + u_s + v_1 + \dots + v_r}} (v_1 \cdots v_r)^{-1+1/k} \right)^2.$$

Applying Lemma 4.3, we see that the inner sum is  $\ll (n - u_1 - \dots - u_s)^{-1+r/k}$ . For every fixed tuple  $(u_1, \dots, u_{s-1})$  in the sum above, we now apply Lemma 4.2 to the sum

$$\sum_{u_s < n - \sum_{i=1}^{s-1} u_i} u_s^{-1+1/k} (n - u_1 - \dots - u_s)^{-1+r/k}.$$

If  $1 - r/k \geq 1/2$ , we obtain

$$\Delta_n(s, r) \ll \log n \sum_{\substack{0 < u_1, \dots, u_s < n \\ n = u_1 + \dots + u_s}} (u_1 \cdots u_s)^{-1+1/k} \ll \frac{\log n}{n^{1-s/k}}$$

where we used Lemma 4.3 for the second inequality. If  $1 - r/k < 1/2$  then by Lemmas 4.2 and 4.3 again

$$\begin{aligned} \Delta_n(s, r) &\ll \sum_{\substack{0 < u_1, \dots, u_s < n \\ u_1 + \dots + u_s < n}} (u_1 \cdots u_s)^{-1+1/k} (n - u_1 - u_2 - \dots - u_s)^{-1+(2r/k-1)} \\ &\ll \begin{cases} n^{-1+(2r/k-1)+s/k} \ll 1 & \text{if } s \leq 2(k-r), \\ \sum_{\substack{0 < u_1, \dots, u_t < n \\ u_1 + \dots + u_t < n}} (u_1 \cdots u_t)^{-1+1/k} \ll n^{t/k} & \text{if } t := s - 2(k-r) > 0. \end{cases} \end{aligned}$$

Notice that if  $s+r = k+1$  with  $s > 0$ , then  $s - 2(k-r) > 0$  implies  $t = 2 - s = 1$  and  $s = 1$ . We can now inject our upper bounds for  $\Delta_n(s, r)$  in equation (16), in which the main contribution is given by  $s = 1$ , from the above discussion. We get

$$\Delta_n \ll_k c^{2k+1} n^{1/k} + O_{k,c}(1).$$

By (14) and (15) with the Borel-Cantelli lemma, we infer that almost surely, all but finitely many integers  $n$  such that  $2k\eta(n) < \{\theta n\} < 1 - 2k\eta(n)$  are sums of  $k+1$  members of  $A$  and that  $d((k+1)A) = 1$  since their complementary set in  $\mathbb{N}$ , namely

$$\{n \in \mathbb{N} : 0 \leq \{\theta n\} \leq 2k\eta(n)\} \cup \{n \in \mathbb{N} : 1 - 2k\eta(n) \leq \{\theta n\} < 1\}$$

has density 0.

c) Let  $n$  such that  $0 < \{\theta n\} < k/(k+1)$ . We consider

$$(17) \quad R_k(n) := k! \sum_{\substack{0 < u_1 < \dots < u_k < n \\ u_i \in \mathcal{T}_{k,\theta} \\ n = u_1 + \dots + u_k}} \xi_{u_1} \cdots \xi_{u_k}$$

that is the random variable counting the representations of  $n$  as a sum of  $k$  distinct members of  $A$ . The key result is Lemma 4.5.

As in the study of  $R_{k+1}(n)$  in the previous paragraph we need to show that the probabilistic dependence between the events  $\{\xi_{u_1} \cdots \xi_{u_k} = 1\}$  is not too large. We

shall use Landreau's work on sums of  $k$  pseudo  $k$ -th powers (cf. [10, Lemma 1 (i) and Lemma 5 (iii)]):

$$\begin{aligned} \mathbb{P}(R_k(n) = 0) &= \exp \left\{ - \sum_{\substack{0 < u_1 < \dots < u_k < n \\ u_i \in T_{k,\theta} \\ n = u_1 + \dots + u_k}} \mathbb{E}(\xi_{u_1} \dots \xi_{u_k}) \right\} + O_k\left(\frac{1}{n^{1/k}}\right) \\ &= e^{-c^k S_k(n)} + O_k\left(\frac{1}{n^{1/k}}\right). \end{aligned}$$

Since  $\eta''(t) \rightarrow 0$  when  $t \rightarrow \infty$ , we deduce from Lemma 4.5 that

$$(18) \quad \mathbb{P}(R_k(n) = 0) = e^{-c^k \lambda_k f_k(\{\theta n\})} + o(1).$$

When  $k/(k+1) \leq \{\theta n\} < 1$  we clearly have  $R_k(n) = 0$ , hence  $\mathbb{P}(R_k(n) = 0) = 1$ .

Let  $\zeta_n$ ,  $n \geq 1$ , be the sequence of Boolean random variables defined by

$$\mathbb{P}(\zeta_n = 1) = \mathbb{P}(R_k(n) = 0),$$

and

$$X_N = \frac{1}{N} \sum_{n=1}^N \zeta_n.$$

By (18) we have

$$\sum_{n=1}^N \mathbb{P}(R_k(n) = 0) = \sum_{n=1}^N e^{-c^k \lambda_k f_k(\{\theta n\})} + o(N).$$

Hence the expectation of  $X_N$  satisfies

$$\mathbb{E}(X_N) = \frac{1}{N} \sum_{n=1}^N \mathbb{P}(R_k(n) = 0) = \sum_{n=1}^N e^{-c^k \lambda_k f_k(\{\theta n\})} + o(1).$$

We get by Theorem 1.3 and the fact that  $f_k$  is supported on  $]0, k/(k+1)[$  the asymptotic

$$(19) \quad \mathbb{E}(X_N) \sim \frac{1}{k+1} + \int_0^{k/(k+1)} e^{-c^k \lambda_k f_k(t)} dt.$$

We denote by  $F_k(c)$  the above integral.

We follow the arguments used in the proof of [8, chapter III, Theorem 4' (iii)] or alternatively [10, Section 4] to estimate the variance  $\mathbb{V}(X_N)$ . We may ignore the diophantine conditions in (17), the only resulting effect being to increase the related variance. We finally get  $\mathbb{V}(X_N) = O(N^{-1/k})$  and consequently by [8, chapter III, lemma 34] that

$$\text{with probability 1, } \lim_{N \rightarrow \infty} X_N = \frac{1}{k+1} + F_k(c).$$

Hence almost surely  $d(kA) = \frac{k}{k+1} - F_k(c)$ . Observing that  $f_k$  is a non negative piecewise polynomial function that has finitely many zeros on  $]0, k/(k+1)[$ , we see that  $F_k(c)$  is a decreasing continuous function satisfying  $\lim_{c \rightarrow 0^+} F_k(c) = k/(k+1)$  and  $\lim_{c \rightarrow \infty} F_k(c) = 0$ ; this ends the proof of Proposition 4.6.  $\square$

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## REFERENCES

- [1] Cabrelli, C.A.; Hare, K.E.; Molter, U.M.; Sums of Cantor sets. *Ergodic Theory Dynam. Systems* **17** (1997), no. 6, 1299–1313.
- [2] Deshouillers, J.-M.; Iosifescu, M.; Sommes de  $s + 1$  pseudo-puissances  $s$ -ièmes. *Rev. Roumaine Math. Pures Appl.* **45** (2000), no. 3, 427–435.
- [3] Erdős, P.; Rényi, A.; Additive properties of random sequences of positive integers. *Acta Arith.* **6** (1960), 83–110.
- [4] Erdős, P.; Turán, P.; On a problem in the theory of uniform distribution. I. *Nederl. Akad. Wetensch. Proc.* **51** (1948), 1146–1154.
- [5] Faisant, A.; Grekos G.; Pandey R.K.; Somu, S.T; Additive Complements for a given Asymptotic Density, arXiv:1809.07584 [math.NT].
- [6] Goguel, J.H.; Über Summen von zufälligen Folgen natürlichen Zahlen, *J. Reine Angew. Math.* **278/279** (1975), 63–77.
- [7] Gyarmati, K.; Konyagin, S; Ruzsa, I.Z.; Double and triple sums modulo a prime. Additive combinatorics, 271–277, CRM Proc. Lecture Notes, 43, Amer. Math. Soc., Providence, RI, 2007
- [8] Halberstam, H.; Roth, K.F.; Sequences. Second edition. Springer-Verlag, New York-Berlin, 1983. xviii+292 pp.
- [9] Kneser, M.; Abschätzung der asymptotischen Dichte von Summenmengen, *Math. Zeit.* **58** (1953), 459–484.
- [10] Landreau, B.; Etude probabiliste des sommes de  $s$  puissances  $s$ -ièmes, *Compositio Math.* **99** (1995), 1–31.
- [11] Lev, V.F.; Structure theorem for multiple addition and the Frobenius problem, *J. Number Theory* **58** (1996), 79–88.
- [12] Moskvin, D.A.; Freiman, G.A.; Judin, A.A.; Inverse problems of additive number theory and local limit theorems for lattice random variables. (Russian) Number-theoretic studies in the Markov spectrum and in the structural theory of set addition,. Kalinin. Gos. Univ., Moscow (1973), pp. 148–162.
- [13] Nathanson, M.B.; Best possible results on the density of sumsets. Analytic number theory (Allerton Park, IL), 395–403, Progr. Math. 85, Birkhäuser, Boston, MA, 1990.
- [14] Raikov, D.A; On the addition of point-sets in the sense of Schnirelmann, *Mat. Sb.* **5** (1939), 425–440.
- [15] Tao, T., Vu, V.H.; Additive combinatorics. Cambridge Studies in Advanced Mathematics, 105. Cambridge University Press, Cambridge, 2006. xviii+512 pp.
- [16] Volkmann, B.; On uniform distribution and the density of sum sets, *Proc. Amer. Math. Soc.* **8** (1957), 130–136.

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