A Proof of the Riemann's Hypothesis

Charaf Ech-Chatbi

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Abstract

We present a proof of the Riemann's Zeta Hypothesis, based on asymptotic expansions and operations on series. We use the symmetry property presented by Riemann's functional equation to extend the proof to the whole set of complex numbers $\mathbb{C}$. The advantage of our method is that it only uses undergraduate maths which makes it accessible to a wider audience.

Keywords: Riemann Hypothesis; Riemann zeta function; Zeta Zeros; Asymptotic distribution of the Prime Numbers; Millennium Problems.

1 The Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

2 Riemann Zeta Function

For a complex number s, the Zeta function is defined as the following infinite sum:

$$\zeta(s) = \begin{cases} Z^1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, & \text{if } \Re(s) > 1 \\ \frac{1}{1-\frac{1}{2^{s-1}}} Z^2(s), \text{where } Z^2(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} & \text{if } \left(\Re(s) > 0, s \neq 1 + \frac{2k\pi}{\ln(2)}, k \text{ integer} \right) \\ \frac{Z^{2\prime}(s)}{\ln(2)}, & \text{if } s = 1 + \frac{2k\pi}{\ln(2)}, k \text{ nonzero integer} \end{cases} \tag{1}$$

Riemann extended the domain of definition of the zeta function to the whole complex domain using the analytical continuation. The Dirichlet

*Maybank. One Raffles Quay, North Tower Level 35. 048583 Singapore. Email: charaf@melix.net or charaf.chatbi@gmail.com. Lnikedin/Twitter: @sharafmouzak. The opinions of this article are those of the author and do not reflect in any way the views or business of his employer.

eta function $\eta(s) = Z^2(s)$ is convergent when $\Re(s) > 0$ and it is used as the analytical continuation of the zeta function on the domain where $\Re(s) > 0$. When $\Re(s) \leq 0$, the Riemann zeta functional equation is used as the analytical continuation for the zeta function.

In his famous paper, Riemann[3] has shown that the Zeta has a functional equation $\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$. Using this last one, Riemann has shown that the non trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has stated that all these non trivial zeros are very likely located on the critical line $\Re(s) = 1/2$ itself. In 1896, Hadamard and De la Vallée Poussin independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$.

Therefore, to prove this "Riemann Hypothesis" (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

3 Proof of the Riemann Hypothesis

Let's take a complex number s such $s = a_0 + ib_0$. Unless we explicitly mention otherwise, let's suppose that $a_0 > 0$ and $b_0 > 0$.

3.1 Case One: $\frac{1}{2} < a_0 \leq 1$

In this case s is a zero of $Z^2(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s}$.

We are going to develop the sequence $Z_N^1(s) = \sum_{n=1}^{N} \frac{1}{n^s}$ as follows: For $N \geq 1$

$$
\begin{aligned}
Z_N^1 &= \sum_{n=1}^{N} \frac{1}{n^s} && (2)\\
&= \sum_{n=1}^{N} \frac{1}{n^{a_0+ib_0}} && (3)\\
&= \sum_{n=1}^{N} \frac{n^{-ib_0}}{n^{a_0}} && (4)\\
&= \sum_{n=1}^{N} \frac{\exp\big(-ib_0\ln(n)\big)}{n^{a_0}} && (5)\\
&= \sum_{n=1}^{N} \frac{\cos(b_0\ln(n)) - i\sin(b_0\ln(n))}{n^{a_0}} && (6)\\
&= \sum_{n=1}^{N} \frac{\cos(b_0\ln(n))}{n^{a_0}} - i\sum_{n=1}^{+\infty} \frac{\sin(b_0\ln(n))}{n^{a_0}} && (7)\\
& && (8)
\end{aligned}
$$

Let's define the sequences U_n, V_n as follows: For $n \geq 1$

$$U_n = \frac{\cos(b_0 \ln(n))}{n^{a_0}} \tag{9}$$

$$V_n = \frac{\sin(b_0 \ln(n))}{n^{a_0}} \tag{10}$$

Let's define the series A_n, B_n and Z_n as follows:

$$A_n = \sum_{k=1}^{n} \frac{\cos(b_0 \ln(k))}{k^{a_0}} \tag{11}$$

$$B_n = \sum_{k=1}^{n} \frac{\sin(b_0 \ln(k))}{k^{a_0}} \tag{12}$$

$$Z_n^1 = A_n - iB_n \tag{13}$$

When we are dealing with complex numbers, it is always insightful to work with the norm. So let's develop further the squared norm of the serie Z_N^1 as follows:

$$\left\|Z_N^1\right\|^2 = A_N^2 + B_N^2 \tag{14}$$

$$= \left(\sum_{n=1}^{N} \frac{\cos(b_0 \ln(n))}{n^{a_0}}\right)^2 + \left(\sum_{n=1}^{N} \frac{\sin(b_0 \ln(n))}{n^{a_0}}\right)^2 \tag{15}$$

So

$$A_N^2 = \left(\sum_{n=1}^{N} \frac{\cos(b_0 \ln(n))}{n^{a_0}}\right)^2 \tag{16}$$

$$= \sum_{n=1}^{N} \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + 2\sum_{n=1}^{N}\sum_{k=1}^{n-1} \frac{\cos(b_0 \ln(k))\cos(b_0 \ln(n))}{k^{a_0} n^{a_0}} \tag{17}$$

$$= \sum_{n=1}^{N} \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + 2\sum_{n=1}^{N} \frac{\cos(b_0 \ln(n))}{n^{a_0}} \sum_{k=1}^{n-1} \frac{\cos(b_0 \ln(k))}{k^{a_0}} \tag{18}$$

$$= \sum_{n=1}^{N} \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + 2\sum_{n=1}^{N} \frac{\cos(b_0 \ln(n))}{n^{a_0}} \left(A_n - \frac{\cos(b_0 \ln(n))}{n^{a_0}}\right) \tag{19}$$

$$= -\sum_{n=1}^{N} \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + 2\sum_{n=1}^{N} \frac{\cos(b_0 \ln(n))A_n}{n^{a_0}} \tag{20}$$

And the same calculation for B_N

$$B_N^2 = \left(\sum_{n=1}^{N} \frac{\sin(b_0 \ln(n))}{n^{a_0}}\right)^2 \tag{21}$$

$$= \sum_{n=1}^{N} \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2\sum_{n=1}^{N}\sum_{k=1}^{n-1} \frac{\sin(b_0 \ln(k))\sin(b_0 \ln(n))}{k^{a_0} n^{a_0}} \tag{22}$$

$$= \sum_{n=1}^{N} \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2\sum_{n=1}^{N} \frac{\sin(b_0 \ln(n))}{n^{a_0}} \sum_{k=1}^{n-1} \frac{\sin(b_0 \ln(k))}{k^{a_0}} \tag{23}$$

$$= \sum_{n=1}^{N} \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2\sum_{n=1}^{N} \frac{\sin(b_0 \ln(n))}{n^{a_0}} \left(B_n - \frac{\sin(b_0 \ln(n))}{n^{a_0}}\right) \tag{24}$$

$$= -\sum_{n=1}^{N} \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2\sum_{n=1}^{N} \frac{\sin(b_0 \ln(n))B_n}{n^{a_0}} \tag{25}$$

Hence we have the new expression of square norm of Z_N^1:

$$\left\|Z_N^1\right\|^2 = 2\sum_{n=1}^{N} \frac{\cos(b_0 \ln(n))A_n}{n^{a_0}} + 2\sum_{n=1}^{N} \frac{\sin(b_0 \ln(n))B_n}{n^{a_0}} - \sum_{n=1}^{N} \frac{\sin^2(b_0 \ln(n)) + \cos^2(b_0 \ln(n))}{n^{2a_0}} \tag{26}$$

which simplies even further to:

$$\left\|Z_N^1\right\|^2 = 2\sum_{n=1}^{N} \frac{\cos(b_0 \ln(n))A_n}{n^{a_0}} + 2\sum_{n=1}^{N} \frac{\sin(b_0 \ln(n))B_n}{n^{a_0}} - \sum_{n=1}^{N} \frac{1}{n^{2a_0}} \tag{27}$$

Let's now define F_n and G_n as follows:

$$F_n = \frac{\cos(b_0 \ln(n))A_n}{n^{a_0}} \tag{28}$$

$$G_n = \frac{\sin(b_0 \ln(n))B_n}{n^{a_0}} \tag{29}$$

Therefore

$$A_N^2 = 2\sum_{n=1}^{N} F_n - \sum_{n=1}^{N} \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} \tag{30}$$

$$B_N^2 = 2\sum_{n=1}^{N} G_n - \sum_{n=1}^{N} \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} \tag{31}$$

We will use the formula (13) and write Z_N^2 as the following:

$$Z_N^2(s) = Z_N^1(s) - \frac{1}{2^{s-1}} Z_{\frac{N}{2}}^1(s) \tag{32}$$

$$= A_N - iB_N - \frac{1}{2^{s-1}}\left(A_{\frac{N}{2}} - iB_{\frac{N}{2}}\right) \tag{33}$$

$$= A_N - \frac{1}{2^{s-1}} A_{\frac{N}{2}} - i\left(B_N - \frac{1}{2^{s-1}} B_{\frac{N}{2}}\right) \tag{34}$$

$$= A_{N,N} - iB_{N,N} \tag{35}$$

Where

$$A_{N,N}=\sum_{k=1}^{\frac{N}{2}}\frac{\cos(b_0\ln(k))-2^{1-a_0}\cos(b_0\big(\ln(2k)\big))}{k^{a_0}}+\sum_{k=\frac{N}{2}+1}^{N}\frac{\cos(b_0\ln(k))}{k^{a_0}}\quad(36)$$

And

$$B_{N,N}=\sum_{k=1}^{\frac{N}{2}}\frac{\sin(b_0\ln(k))-2^{1-a_0}\sin(b_0\big(\ln(2k)\big))}{k^{a_0}}+\sum_{k=\frac{N}{2}+1}^{N}\frac{\sin(b_0\ln(k))}{k^{a_0}}\quad(37)$$

We note here that we are using the same partial sum of the Dirichlet serie $\eta(s)=\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^s}$ that is conditionally convergent when $\Re(s)>0$. We are not changing the partial sum by rearraging the terms of the Dirichlet serie. The representation $Z_N^2(s)=A_{N,N}-iB_{N,N}$ is conserving the same terms of the original partial sum of the Dirichlet serie $\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n^s}$. Therefore, $A_{N,N}$ and $-B_{N,N}$ are the real part and the imaginary part of the partial sum of the Dirichlet serie. Hence

$$\lim_{N\to\infty}A_{N,N}-iB_{N,N}=\lim_{N\to\infty}\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n^s}=Z^2(s)=\eta(s)\qquad(38)$$

Therefore

Conclusion. *s is a zeta **zero**, $\zeta(s)=0$, if and only if*

$$\lim_{N\to\infty}A_{N,N}=0\ and\ \lim_{N\to\infty}B_{N,N}=0\qquad(39)$$

*Equally, s is a zeta **zero**, $\zeta(s)=0$, if and only if*

$$\lim_{N\to\infty}A_{N,N}^2=0\ and\ \lim_{N\to\infty}B_{N,N}^2=0\qquad(40)$$

Remark. *The rearragement used in the Riemann rearrangement theorem to make a conditionally convergent serie converge to any number including $\{+\infty,-\infty\}$ is not conserving the same terms of the partial sum of the original series.*

For simplification, let's define the following sequences: For $N\geq 1$:

$$U_{n,N}=\begin{cases}\frac{\cos(b_0\ln(n))-2^{1-a_0}\cos(b_0(\ln(2n)))}{n^{a_0}}, & \text{if } n\leq\frac{N}{2}\\ \frac{\cos(b_0\ln(n))}{n^{a_0}} & \text{if } n>\frac{N}{2}\end{cases}\qquad(41)$$

And

$$V_{n,N}=\begin{cases}\frac{\sin(b_0\ln(n))-2^{1-a_0}\sin(b_0(\ln(2n)))}{n^{a_0}}, & \text{if } n\leq\frac{N}{2}\\ \frac{\sin(b_0\ln(n))}{n^{a_0}} & \text{if } n>\frac{N}{2}\end{cases}\qquad(42)$$

And

$$A_{n,N}=\begin{cases}\sum_{k=1}^{n}\frac{\cos(b_0\ln(k))-2^{1-an0}\cos(b_0\ln(2k))}{k^{a_0}}, & \text{if } n\leq\frac{N}{2}\\ \sum_{k=1}^{\frac{N}{2}}\frac{\cos(b_0\ln(k))-2^{1-a_0}\cos(b_0\ln(2k))}{k^{a_0}}+\sum_{k=\frac{N}{2}+1}^{n}\frac{\cos(b_0\ln(k))}{k^{a_0}} & \text{if } n>\frac{N}{2}\\ A_{N,N} & \text{if } n\geq N\end{cases}\qquad(43)$$

And

$$B_{n,N} = \begin{cases} \sum_{k=1}^{n} \frac{\sin(b_0 \ln(k)) - 2^{1-a_0}\sin(b_0 \ln(2k))}{k^{a_0}}, & \text{if } n \leq \frac{N}{2} \\ \sum_{k=1}^{\frac{N}{2}} \frac{\sin(b_0 \ln(k)) - 2^{1-a_0}\sin(b_0 \ln(2k))}{k^{a_0}} + \sum_{k=\frac{N}{2}+1}^{n} \frac{\sin(b_0 \ln(k))}{k^{a_0}} & \text{if } n > \frac{N}{2} \\ B_{N,N} & \text{if } n \geq N \end{cases} \tag{44}$$

Let's develop further the squared norm of the serie Z_N^2 as follows:

$$\left\| Z_N^2 \right\|^2 = A_{N,N}^2 + B_{N,N}^2 \tag{45}$$

$$= \left(\sum_{n=1}^{N} U_{n,N} \right)^2 + \left(\sum_{n=1}^{N} V_{n,N} \right)^2 \tag{46}$$

So

$$A_{N,N}^2 = \left(\sum_{n=1}^{N} U_{n,N} \right)^2 \tag{47}$$

$$= \sum_{n=1}^{N} U_{n,N} + 2 \sum_{n=1}^{N} \sum_{k=1}^{n-1} U_n U_{k,N} \tag{48}$$

$$= \sum_{n=1}^{N} U_{n,N}^2 + 2 \sum_{n=1}^{N} U_{n,N} \sum_{k=1}^{n-1} U_{k,N} \tag{49}$$

$$= \sum_{n=1}^{N} U_{n,N}^2 + 2 \sum_{n=1}^{N} U_{n,N} \left(A_{n,N} - U_{n,N} \right) \tag{50}$$

$$= -\sum_{n=1}^{N} U_{n,N}^2 + 2 \sum_{n=1}^{N} U_{n,N}\, A_{n,N} \tag{51}$$

$$\tag{52}$$

And the same calculation for $B_{N,N}$

$$B_{N,N}^2 = \left(\sum_{n=1}^{N} V_n \right)^2 \tag{53}$$

$$= \sum_{n=1}^{N} V_{n,N} + 2 \sum_{n=1}^{N} \sum_{k=1}^{n-1} V_n V_{k,N} \tag{54}$$

$$= \sum_{n=1}^{N} V_{n,N}^2 + 2 \sum_{n=1}^{N} V_{n,N} \sum_{k=1}^{n-1} V_{k,N} \tag{55}$$

$$= \sum_{n=1}^{N} V_{n,N}^2 + 2 \sum_{n=1}^{N} V_{n,N} \left(B_{n,N} - V_{n,N} \right) \tag{56}$$

$$= -\sum_{n=1}^{N} V_{n,N}^2 + 2 \sum_{n=1}^{N} V_{n,N}\, B_{n,N} \tag{57}$$

$$\tag{58}$$

Let's now define the sequences $(F_{n,N})$ and $(G_{n,N})$ as follows:

$$F_{n,N} = U_{n,N}A_{n,N} \tag{59}$$

$$G_{n,N} = V_{n,N}B_{n,N} \tag{60}$$

Hence we have the new expression of square norm of Z_N^2:

$$\left\|Z_N^2\right\|^2 = 2\sum_{n=1}^{N} F_{n,N} + 2\sum_{n=1}^{N} G_{n,N} - \sum_{n=1}^{N}\left(U_{n,N}^2 + V_{n,N}^2\right) \tag{61}$$

which simplies even further to:

$$\sum_{n=1}^{N}\left(U_{n,N}^2 + V_{n,N}^2\right) = \sum_{n=1}^{\frac{N}{2}}\left(U_{n,N}^2 + V_{n,N}^2\right) + \sum_{n=\frac{N}{2}+1}^{N}\left(U_{n,N}^2 + V_{n,N}^2\right) \tag{62}$$

$$= \sum_{n=\frac{N}{2}+1}^{N}\frac{1}{n^{2a_0}} \tag{63}$$

$$+\sum_{n=1}^{\frac{N}{2}}\frac{\left(\cos(b_0\ln(n)) - 2^{1-a_0}\cos(b_0(\ln(2n)))\right)^2 + \left(\sin(b_0\ln(n)) - 2^{1-a_0}\sin(b_0(\ln(2n)))\right)^2}{n^{2a_0}} \tag{64}$$

$$= \sum_{n=\frac{N}{2}+1}^{N}\frac{1}{n^{2a_0}} \tag{65}$$

$$+\sum_{n=1}^{\frac{N}{2}}\frac{\left(1 + 2^{2-2a_0} - 2^{2-a_0}\cos(b_0\ln(n))\cos(b_0(\ln(2n))) - 2^{2-a_0}\sin(b_0\ln(n))\sin(b_0(\ln(2n)))\right)}{n^{2a_0}} \tag{66}$$

$$= \sum_{n=1}^{N}\frac{1}{n^{2a_0}} \tag{67}$$

$$+\sum_{n=1}^{\frac{N}{2}}\frac{2^{2-2a_0} - 2^{2-a_0}\left(\cos(b_0\ln(n))\cos(b_0(\ln(2n))) + \sin(b_0\ln(n))\sin(b_0(\ln(2n)))\right)}{n^{2a_0}} \tag{68}$$

$$= \sum_{n=1}^{N}\frac{1}{n^{2a_0}} + \sum_{n=1}^{\frac{N}{2}}\frac{2^{2-2a_0} - 2^{2-a_0}\left(\cos(b_0\ln(2))\right)}{n^{2a_0}} \tag{69}$$

$$\tag{70}$$

We have $2a_0 > 1$, hence the partial sums $\sum_{n\geq 1}^{N} U_{n,N}^2$ and $\sum_{n\geq 1}^{N} V_{n,N}^2$ converge absolutely.

We have

$$A_{N,N}^2 = 2\sum_{n=1}^{N} F_{n,N} - \sum_{n=1}^{N} U_{n,N}^2 \tag{71}$$

$$B_{N,N}^2 = 2\sum_{n=1}^{N} G_{n,N} - \sum_{n=1}^{N} V_{n,N}^2 \tag{72}$$

Proof Strategy. *The idea is to prove that in the case of a complex s that is in the right hand side of the critical strip $\frac{1}{2} < a_0 \leq 1$ and that is a ζ zero, that the limit $\lim_{n\to\infty} A_{n,n}^2 = +/-\infty$* ***OR*** *the limit $\lim_{n\to\infty} B_{n,n}^2 = +/-\infty$. This will create a contradiction. Because if s is a ζ zero then the $\lim_{n\to\infty} A_{n,n}^2$ should be 0 and the $\lim_{n\to\infty} B_{n,n}^2$ should be 0. And therefore the sequences $(\sum_{n=1}^{N} F_{n,N})_{N\geq 1}$ and $(\sum_{n=1}^{N} G_{n,N})_{N\geq 1}$ should converge and their limits should be: $\lim_{n\to\infty} \sum_{n=1}^{N} F_{n,N} = \frac{1}{2}\lim_{N\to\infty} \sum_{n=1}^{N} U_{n,N}^2 < +\infty$ and $\lim_{n\to\infty} \sum_{n=1}^{N} G_{n,N} = \frac{1}{2}\lim_{N\to\infty} \sum_{n=1}^{N} V_{n,N}^2 < +\infty$.*

We need the following lemmas during different stages of the proof.

Lemma 3.1. *There is exist two constants $K_1 \geq 0$ and $K_2 \geq 0$ such that for $n \geq 1$ and $N \geq 1$ we have: $|A_{n,N}| \leq K_1 + K_2\, n^{1-a_0}$.*

Proof. We will make use of the lemma 3.5 in [1]. Let's $N \geq 1$. Let's $n \geq 1$.

Case: $n \leq \frac{N}{2}$

$$A_{n,N} = \sum_{k=1}^{n} \frac{\cos(b_0 \ln(n)) - 2^{1-a_0}\cos(b_0(\ln(2n)))}{k^{a_0}} \tag{73}$$

$$= \sum_{k=1}^{n} \frac{\left(1 - 2^{1-a_0}\cos(b_0\ln(2))\right)\cos(b_0\ln(k)) + \left(2^{1-a_0}\sin(b_0\ln(2))\right)\sin(b_0\ln(k))}{k^{a_0}} \tag{74}$$

$$= \left(1 - 2^{1-a_0}\cos(b_0\ln(2))\right)\sum_{k=1}^{n} \frac{\cos(b_0\ln(k))}{k^{a_0}} + \left(2^{1-a_0}\sin(b_0\ln(2))\right)\sum_{k=1}^{n} \frac{\sin(b_0\ln(k))}{k^{a_0}} \tag{75}$$

$$\tag{76}$$

Therefore using the lemma 3.5 in [1]:

$$|A_{n,N}| \leq \left|\left(1 - 2^{1-a_0}\cos(b_0\ln(2))\right)\right| \left|\sum_{k=1}^{n} \frac{\cos(b_0\ln(k))}{k^{a_0}}\right| \tag{77}$$

$$+ \left|\left(2^{1-a_0}\sin(b_0\ln(2))\right)\right| \left|\sum_{k=1}^{n} \frac{\sin(b_0\ln(k))}{k^{a_0}}\right| \tag{78}$$

$$\leq \left|\left(1 - 2^{1-a_0}\cos(b_0\ln(2))\right)\right| \left(K_1 + K_2\, n^{1-a_0}\right) + \left|\left(2^{1-a_0}\sin(b_0\ln(2))\right)\right| \left(K_1 + K_2\, n^{1-a_0}\right) \tag{79}$$

$$\leq \left(\left|\left(1 - 2^{1-a_0}\cos(b_0\ln(2))\right)\right| + \left|\left(2^{1-a_0}\sin(b_0\ln(2))\right)\right| \right)\left(K_1 + K_2\, n^{1-a_0}\right) \tag{80}$$

And this prove the case.

Case: $n > \frac{N}{2}$ We have:

$$\sum_{k=\frac{N}{2}+1}^{n} \frac{\cos(b_0\ln(n))}{k^{a_0}} = \sum_{k=1}^{n} \frac{\cos(b_0\ln(n))}{k^{a_0}} - \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0\ln(n))}{k^{a_0}} \tag{81}$$

$$\tag{82}$$

And

$$A_{n,N} = \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0(\ln(2n)))}{k^{a_0}} + \sum_{\frac{N}{2}+1}^{n} \frac{\cos(b_0 \ln(n))}{k^{a_0}} \quad (83)$$

$$(84)$$

Therefore using the first case of $n \leq \frac{N}{2}$ and the lemma 3.5 in [1]:

$$\left| \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0(\ln(2n)))}{k^{a_0}} \right| \leq \left(K_1 + K_2 \, (\frac{N}{2})^{1-a_0} \right) \quad (85)$$

$$\leq \left(K_1 + K_2 \, n^{1-a_0} \right), \text{ as } (\frac{N}{2})^{1-a_0} \leq n^{1-a_0} \quad (86)$$

And

$$\left| \sum_{k=1}^{n} \frac{\cos(b_0 \ln(n))}{k^{a_0}} - \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(n))}{k^{a_0}} \right| \leq \left| \sum_{k=1}^{n} \frac{\cos(b_0 \ln(n))}{k^{a_0}} \right| + \left| \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(n))}{k^{a_0}} \right| \quad (87)$$

$$\leq \left(\overline{\mathrm{K}}_1 + \overline{\mathrm{K}}_2 \, n^{1-a_0} \right) + \left(K_1 + K_2 \, (\frac{N}{2})^{1-a_0} \right) \quad (88)$$

$$\leq K_1 + \overline{\mathrm{K}}_1 + (K_2 + \overline{\mathrm{K}}_2) \, n^{1-a_0}, \text{ as } (\frac{N}{2})^{1-a_0} \leq n^{1-a_0} \quad (89)$$

Therefore

$$|A_{n,N}| \leq K_1 + \overline{\mathrm{K}}_1 + (K_2 + \overline{\mathrm{K}}_2) \, n^{1-a_0} \quad (90)$$

Which proves our case. □

Lemma 3.2. *For a large n, N and N_0 such $N_0 \leq n < n+1 \leq \frac{N}{2}$ We can write $A_{n,N} = E_1(n) + \gamma_n$ such that $\lim_{n \to +\infty} \gamma_n = 0$ and $E_1(n) = n^{1-a_0} \Big(X_1 \sin\big(b_0 \ln(n)\big) + X_2 \cos\big(b_0 \ln(n)\big) \Big)$ where X_1 and X_2 two real constants defined as follows:*

$$X_1 = \frac{\Big(a_3 b_0 + b_3(1-a_0)\Big)}{(b_0)^2 + (1-a_0)^2} \quad (91)$$

$$X_2 = \frac{\Big(a_3(1-a_0) - b_3 b_0\Big)}{(b_0)^2 + (1-a_0)^2} \quad (92)$$

$$a_3 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (93)$$

$$b_3 = 2^{1-a_0} \sin(b_0 \ln(2)) \quad (94)$$

Proof. We have $\lim_{N \to +\infty} A_{N,N} = 0$. Let's $\epsilon > 0$. Let's N_0 be such that for each $N \geq N_0$: $|A_{N,N}| \leq \epsilon$. Let's take n, N and N_0 such $N_0 \leq n < n+1 \leq \frac{N}{2}$. We have $|A_{N,N}| \leq \epsilon$.
Therefore:

We have the notation $dd_n = \big(\cos(b_0 \ln(n)) - 2^{1-a_0}\cos(b_0(\ln(2n)))\big)$.
We are going to use the lemma 3.2 in [1]. Let's develop further:

$$A_{N,N} - A_{n,N} = \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} + \sum_{k=\frac{N}{2}+1}^{N} \frac{\cos(b_0 \ln(k))}{k^{a_0}} \tag{95}$$

$$= \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} e_1(t)dt + \sum_{k=\frac{N}{2}+1}^{N} \frac{\cos(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^{N} \int_k^{k+1} e(t)dt \tag{96}$$

$$+ \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} e_1(t)dt + \sum_{k=\frac{N}{2}+1}^{N} \int_k^{k+1} e(t)dt \tag{97}$$

$$= \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} e_1(t)dt + \sum_{k=\frac{N}{2}+1}^{N} \frac{\cos(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^{N} \int_k^{k+1} e(t)dt \tag{98}$$

$$+ \int_{n+1}^{N} e_1(t)dt + \int_{\frac{N}{2}+1}^{N} e(t)dt \tag{99}$$

$$= -E_1(n+1) + \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} e_1(t)dt + \sum_{k=\frac{N}{2}+1}^{N} \frac{\cos(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^{N} \int_k^{k+1} e(t)dt \tag{100}$$

$$+ E_1(N) + E(N) - E(\frac{N}{2}+1) \tag{101}$$

Therefore from the lemma 3.2 in [1]:

$$|A_{n,N} - E_1(n+1)| \leq |A_{N,N}| + \left| \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} h(t)dt \right| \tag{102}$$

$$+ \left| \sum_{k=\frac{N}{2}+1}^{N} \frac{\cos(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^{N} \int_k^{k+1} g(t)dt \right| + \left| E_1(N) + E(N) - E(\frac{N}{2}+1) \right| \tag{103}$$

$$\leq \epsilon + \frac{K_5}{a_0}\Big(\frac{1}{n^{a_0}} - \frac{1}{(\frac{N}{2})^{a_0}}\Big) + \frac{K_1}{a_0}\Big(\frac{1}{(\frac{N}{2})^{a_0}} - \frac{1}{(N^{a_0})}\Big) + \left| E_1(N) + E(N) - E(\frac{N}{2}+1) \right| \tag{104}$$

$$\tag{105}$$

We have $N_0 \leq n < n+1 \leq \frac{N}{2}$. So

$$|A_{n,N} - E_1(n+1)| \leq \epsilon + \frac{K_5}{a_0}\Big(\frac{1}{n^{a_0}} + \frac{1}{n^{a_0}}\Big) + \frac{K_1}{a_0}\Big(\frac{1}{n^{a_0}} + \frac{1}{n^{a_0}}\Big) + \left| E_1(N) + E(N) - E(\frac{N}{2}+1) \right| \tag{106}$$

$$\leq \epsilon + 2\frac{K_1 + K_5}{a_0}\frac{1}{n^{a_0}} + \left| E_1(N) + E(N) - E(\frac{N}{2}+1) \right| \tag{107}$$

Quick calculation gives us: $E_1(N) + E(N) - E(\frac{N}{2}) = 0$. We apply the lemma 3.8 in [1] to the function E, we have $\lim_{N\to+\infty} E(\frac{N}{2}+1) - E(\frac{N}{2}) = 0$. Therefore: $\lim_{N\to+\infty} E_1(N) + E(N) - E(\frac{N}{2}) = 0$.

$$|A_{n,N} - E_1(n+1)| \leq \epsilon + 2\frac{K_1 + K_5}{a_0}\frac{1}{n^{a_0}} + \left| E_1(N) + E(N) - E(\frac{N}{2}+1) \right| \tag{108}$$

And this for every $\epsilon > 0$. $a_0 > 0$ so we also have the $\lim_{n\to+\infty} \frac{1}{n^{a_0}} = 0$. Therefore $\lim_{n\to+\infty} |A_{n,N} - E_1(n+1)| = 0$. Let's define the sequence γ_n as the following: $\gamma_n = A_{n,N} - E_1(n)$.
We apply the lemma 3.8 in [1] to the function E_1, we have the $\lim_{n\to+\infty} E_1(n+1) - E_1(n) = 0$. Therefore $\lim_{n\to+\infty} \gamma_n = 0$.
The expression of E_1 is also given by the lemma 3.2 in [1]. And this proves the lemma.

Remark. *There is a simple way of proving this lemma. As we have $A_{n,N} = A_{n,2n}$ for $n \leq \frac{N}{2}$, we can redo the steps above with $A_{n,2n}$ and develop $A_{2n,2n} - A_{n,2n}$. The steps above will be much easier and more straighforward.*

□

Lemma 3.3. *Let's define the functions:*

$$e_3(t) = \frac{\Big(\cos(b_0 \ln(t)) - 2^{1-a_0}\cos(b_0 \ln(2t))\Big)\Big(X_1 \sin\big(b_0 \ln(t)\big) + X_2 \cos\big(b_0 \ln(t)\big)\Big)}{(t)^{2a_0-1}} \tag{109}$$

Let's define the sequence:

$$e_{3,n} = e_3(n) - \int_n^{n+1} dt\, e_3(t) \tag{110}$$

$$\tag{111}$$

Therefore, the serie $\sum_{k\geq 1} e_{3,n}$ are converging absolutely and we have the following inequalities:

$$\left|\sum_{k=n+1}^{N} e_{3,n} - \int_{n+1}^{N} e_3(t)dt\right| \leq \frac{K}{2a_0}\Big(\frac{1}{n^{2a_0}} - \frac{1}{N^{2a_0}}\Big) \tag{112}$$

And if $a_0 > 1$

$$\left|\sum_{k=n+1}^{+\infty} e_{3,n} - \int_{n+1}^{+\infty} e_3(t)dt\right| \leq \frac{K}{2a_0}\frac{1}{n^{2a_0}} \tag{113}$$

where $K > 0$ and E_3 is the primitive function of e_3 is defined as:

$$E_3(x) = \Big(\frac{\alpha_3}{2-2a_0} + \frac{\Big(2b_0\beta_3 + \gamma_3(2-2a_0)\Big)\sin(2b_0\ln(x)) + \Big(\beta_3(2-2a_0) - 2b_0\gamma_3\Big)\cos(2b_0\ln(x))}{(2b_0)^2 + (2-2a_0)^2}\Big)\frac{1}{(x)^{2a_0-2}} \tag{114}$$

Proof. We are going to use the lemmas 3.1 and 3.2 defined in [1]. We will also use their notations. We can write the functions $e_3(t)$ as the following:

$$e_3(t)\,(t)^{2a_0-1} = \Big(\cos(b_0\ln(t)) - 2^{1-a_0}\cos(b_0\ln(2t))\Big)\Big(X_1\sin\big(b_0\ln(t)\big) + X_2\cos\big(b_0\ln(t)\big)\Big) \tag{115}$$

$$= \Big(Z_1\sin(b_0\ln t) + Z_2\cos(b_0\ln(t))\Big)\Big(X_1\sin(b_0\ln t) + X_2\cos(b_0\ln(t))\Big) \tag{116}$$

$$= \frac{X_1Z_1 + X_2Z_2}{2} + \frac{X_1Z_2 + X_2Z_1}{2}\sin(2b_0\ln(t)) + \frac{X_2Z_2 - X_1Z_1}{2}\cos(2b_0\ln t) \tag{117}$$

Where

$$Z_1 = 2^{1-a_0}\sin(b_0\ln(2)) = b_3 \quad (118)$$
$$Z_2 = 1 - 2^{1-a_0}\cos(b_0\ln(2)) = a_3 \quad (119)$$

Therefore we can write $e_3(t)$

$$e_3(t)\,(t)^{2a_0-1} = \Big(\frac{X_1Z_1 + X_2Z_2}{2}\Big) + \Big(\frac{X_1Z_2 + X_2Z_1}{2}\Big)\sin(2b_0\ln(t)) \quad (120)$$
$$+\Big(\frac{X_2Z_2 - X_1Z_1}{2}\Big)\cos(2b_0\ln t) \quad (121)$$
$$= \alpha_3 + \beta_3\sin(2b_0\ln(t)) + \gamma_3\cos(2b_0\ln t) \quad (122)$$

Where

$$\alpha_3 = \frac{X_1b_3 + X_2a_3}{2} \quad (123)$$
$$\beta_3 = \frac{X_1a_3 + X_2b_3}{2} \quad (124)$$
$$\gamma_3 = \frac{X_2a_3 - X_1b_3}{2} \quad (125)$$
$$X_1 = \frac{a_3b_0 + b_3(1-a_0)}{(b_0)^2 + (1-a_0)^2} \quad (126)$$
$$X_2 = \frac{a_3(1-a_0) - b_3b_0}{(b_0)^2 + (1-a_0)^2} \quad (127)$$

And we apply the lemma 3.1 in [1] with $\alpha = 2a_0 - 1$, $\beta = 2b_0$, $a = \alpha_3$, $b = \beta_3$ and $c = \gamma_3$.

$$E_3(x) = \Big(\frac{\alpha_3}{2-2a_0} + \frac{\Big(2b_0\beta_3 + \gamma_3(2-2a_0)\Big)\sin(2b_0\ln(x)) + \Big(\beta_3(2-2a_0) - 2b_0\gamma_3\Big)\cos(2b_0\ln(x))}{(2b_0)^2 + (2-2a_0)^2}\Big)\frac{1}{(x)^{2a_0-2}} \quad (128)$$

□

Lemma 3.4. *If $\frac{1}{2} < a_0 \leq 1$ the sequence $(\sum_{n=1}^{N} F_{n,N})_{N\geq 1}$ diverges.*

Proof. We want to prove that:

$$\lim_{N\to\infty}\sum_{n=1}^{N} F_{n,N} = -\infty \quad (129)$$

Let's n and N be integers:

Case One: $\frac{N}{2} < n \leq N$ We are going to use asymptotic expansions: We have

$$A_{n+1,N} = A_{n,N} + \frac{\cos(b_0\ln(n+1))}{(n+1)^{a_0}} \quad (130)$$

So $F_{n+1,N}$ can be written as follows:

$$F_{n+1,N} = \frac{\cos(b_0 \ln(n+1))A_{n,N}}{(n+1)^{a_0}} + \frac{\cos^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} \quad (131)$$

So $F_{n+1,N} - F_{n,N}$ can be written as

$$F_{n+1,N} - F_{n,N} = \left(\frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}}\right)A_{n,N} + \frac{\cos^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} \quad (132)$$

$$F_{n+1,N} - F_{n,N} = \left(U_{n+1,N} - U_{n,N}\right)A_{n,N} + \frac{\cos^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} \quad (133)$$

Let's do now the asymptotic expansion of $U_{n+1,N} - U_{n,N}$. For this we need the asymptotic expansion of $cos\big(b_0 \ln(n+1)\big)$.

$$\cos(b_0 \ln(n+1) = \cos\big(b_0 \ln(n) + b_0 \ln(1+\frac{1}{n})\big) \quad (134)$$

$$= \cos\big(b_0 \ln(n)\big)\cos\big(b_0 \ln(1+\frac{1}{n})\big) - \sin\big(b_0 \ln(n)\big)\sin\big(b_0 \ln(1+\frac{1}{n})\big) \quad (135)$$

we have the asymptotic expansion of $\ln(1+\frac{1}{n})$ in order two as follow:

$$\ln(1+\frac{1}{n}) = \frac{1}{n} + \mathcal{O}(\frac{1}{n^2}) \quad (136)$$

Using the asymptotic expansion of the functions sin and cos that I will spare you the details here, we have

$$\cos\left(b_0 \ln(1+\frac{1}{n})\right) = 1 + \mathcal{O}(\frac{1}{n^2}) \quad (137)$$

And

$$\sin\left(b_0 \ln(1+\frac{1}{n})\right) = \frac{b_0}{n} + \mathcal{O}(\frac{1}{n^2}) \quad (138)$$

Hence

$$\cos(b_0 \ln(n+1)) = \cos\big(b_0 \ln(n)\big) - \frac{b_0}{n}\sin\big(b_0 \ln(n)\big) + \mathcal{O}(\frac{1}{n^2}) \quad (139)$$

Also the Asymptotic expansion of $\frac{1}{(1+n)^{a_0}}$:

$$\frac{1}{(1+n)^{a_0}} = \frac{1}{n^{a_0}}\left(1 - \frac{a_0}{n} + \mathcal{O}(\frac{1}{n^2})\right) \quad (140)$$

Hence

$$\frac{\cos\big(b_0 \ln(n+1)\big)}{(1+n)^{a_0}} = \frac{\cos\big(b_0 \ln(n)\big)}{n^{a_0}} - \frac{b_0 \sin\big(b_0 \ln(n)\big) + a_0 \cos\big(b_0 \ln(n)\big)}{n^{a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \quad (141)$$

And the squared version of the above equation

$$\frac{\cos^2\big(b_0\ln(n+1)\big)}{(1+n)^{2a_0}}=\frac{\cos^2\big(b_0\ln(n)\big)}{n^{2a_0}}-2\cos\big(b_0\ln(n)\big)\frac{b_0\sin\big(b_0\ln(n)\big)+a_0\cos\big(b_0\ln(n)\big)}{n^{2a_0+1}}+\mathcal{O}(\frac{1}{n^{a_0+2}}) \quad (142)$$

And

$$U_{n+1,N}-U_{n,N}=-\frac{b_0\sin\big(b_0\ln(n)\big)+a_0\cos\big(b_0\ln(n)\big)}{n^{a_0+1}}+\mathcal{O}(\frac{1}{n^{a_0+2}}) \quad (143)$$

So the asymptotic expansion of $F_{n+1,N}-F_{n,N}$ is as follows:

$$F_{n+1,N}-F_{n,N}=\frac{\cos^2(b_0\ln(n+1))}{(n+1)^{2a_0}}-\frac{b_0\sin\big(b_0\ln(n)\big)+a_0\cos\big(b_0\ln(n)\big)}{n^{a_0+1}}A_{n,N}+\mathcal{O}(\frac{1}{n^{a_0+2}}) \quad (144)$$

For the sake of simplifying the notation, let define the sequence $C_{n,N}$ as follows: For each $n\geq 1$ we have:

$$\begin{aligned} c_n &= -\Big(b_0\sin\big(b_0\ln(n)\big)+a_0\cos\big(b_0\ln(n)\big)\Big) && (145)\\ C_{n,N} &= c_nA_{n,N} && (146)\end{aligned}$$

And

$$D_n = 2c_n\cos\big(b_0\ln(n)\big) \quad (147)$$

The functions $x\longmapsto\cos(x)$ and $x\longmapsto\sin(x)$ are by their nature bounded so (c_n) and (D_n) are also bounded.

By definition of $\mathcal{O}$, we know that there is exist a bounded sequence (ε_n) and there is exist a number N_0 such that: For each $n\geq N_0$ we have

$$F_{n+1,N}-F_{n,N}=\frac{\cos^2(b_0\ln(n))}{n^{2a_0}}+\frac{C_{n,N}}{n^{a_0+1}}+\frac{D_{n,N}}{n^{2a_0+1}}+\frac{\varepsilon_{n,N}}{n^{a_0+2}} \quad (148)$$

Where

$$\begin{aligned} c_n &= -\Big(b_0\sin\big(b_0\ln(n)\big)+a_0\cos\big(b_0\ln(n)\big)\Big) && (149)\\ C_{n,N} &= c_nA_{n,N} && (150)\\ D_{n,N} &= 2\cos\big(b_0\ln(n)\big)c_n && (151)\end{aligned}$$

Let's now study the case of when $n+1\leq\frac{N}{2}$:

Case Two: $n+1\leq\frac{N}{2}$

$$\begin{aligned} F_{n+1,N}-F_{n,N} &= U_{n+1,N}A_{n+1,N}-U_{n,N}A_{n,N} && (152)\\ &= \Big(U_{n+1,N}-U_{n,N}\Big)A_{n,N}+\big(U_{n+1,N}\big)^2 && (153)\end{aligned}$$

And

$$U_{n+1,N} - U_{n,N} = \frac{\cos\big(b_0 \ln(n+1)\big)}{(1+n)^{a_0}} - \frac{\cos\big(b_0 \ln(n)\big)}{n^{a_0}} \tag{154}$$

$$-2^{1-a_0}\Big\{\frac{\cos\big(b_0 \ln 2(n+1)\big)}{(1+n)^{a_0}} - \frac{\cos\big(b_0 \ln(2n)\big)}{n^{a_0}}\Big\} \tag{155}$$

We develop the $\cos\big(b_0 \ln 2(n+1)\big)$ and $\cos\big(b_0 \ln 2(n)\big)$ to get:

$$\frac{\cos\big(b_0 \ln 2(n+1)\big)}{(1+n)^{a_0}} = \cos b_0 \ln(2)\Big\{\frac{\cos\big(b_0 \ln(n)\big)}{n^{a_0}} + \frac{c_n}{n^{a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}})\Big\} \tag{156}$$

$$-\sin b_0 \ln(2)\Big\{\frac{\sin\big(b_0 \ln(n)\big)}{n^{a_0}} + \frac{cc_n}{n^{a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}})\Big\} \tag{157}$$

$$= \frac{\cos\big(b_0 \ln 2(n)\big)}{(n)^{a_0}} + \frac{c_n \cos b_0 \ln(2) - cc_n \sin b_0 \ln(2)}{n^{a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{158}$$

Therefore:

$$U_{n+1,N} - U_{n,N} = \frac{c_n - 2^{1-a_0}\Big(c_n \cos b_0 \ln(2) - cc_n \sin b_0 \ln(2)\Big)}{n^{a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{159}$$

And

$$\Big(U_{n+1,N}\Big)^2 = \Big(U_{n,N}\Big)^2 + 2\frac{ccc_n\, dd_n}{n^{2a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{160}$$

Where

$$ccc_n = c_n - 2^{1-a_0}\Big(c_n \cos b_0 \ln(2) - cc_n \sin b_0 \ln(2)\Big) \tag{161}$$

$$dd_n = \cos\big(b_0 \ln(n) - 2^{1-a_0} \cos\big(b_0 \ln(2n) \tag{162}$$

$$D_{n,N} = 2\, ccc_n\, dd_n \tag{163}$$

$$C_{n,N} = ccc_n A_{n,N} \tag{164}$$

And

$$F_{n+1,N} - F_{n,N} = \Bigg\{\frac{\big(\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0(\ln(2n)))\big)^2}{n^{2a_0}} + \frac{C_{n,N}}{n^{a_0+1}} \tag{165}$$

$$+\frac{D_{n,N}}{n^{2a_0+1}} + \frac{\varepsilon_{n,N}}{n^{a_0+2}}\Bigg\} \tag{166}$$

With the sequence $\varepsilon_{n,N}$ is bounded.

From the lemma 3.2 above we have: $A_{n,N} = E_1(n) + \gamma_n$ such that $\lim_{n\to+\infty} \gamma_n = 0$ and $E_1(n) = n^{1-a_0}\Big(X_1 \sin\big(b_0 \ln(n)\big) + X_2 \cos\big(b_0 \ln(n)\big)\Big)$ where X_1 and X_2 two real constants. Therefore

$$\frac{C_{n,N}}{n^{a_0+1}} = \frac{ccc_n n^{1-a_0}\Big(X_1\sin\big(b_0\ln(n)\big)+X_2\cos\big(b_0\ln(n)\big)\Big)}{n^{a_0+1}} + \frac{ccc_n\gamma_n}{n^{a_0+1}} \quad (167)$$

$$= \frac{ccc_n\Big(X_1\sin\big(b_0\ln(n)\big)+X_2\cos\big(b_0\ln(n)\big)\Big)}{n^{2a_0}} + \frac{ccc_n\gamma_n}{n^{a_0+1}} \quad (168)$$

Therefore for $n+1 \leq \frac{N}{2}$:

$$F_{n+1,N}-F_{n,N} = \Bigg\{ \frac{\big(\cos(b_0\ln(n)) - 2^{1-a_0}\cos(b_0(\ln(2n)))\big)^2 + ccc_n\Big(X_1\sin\big(b_0\ln(n)\big)+X_2\cos\big(b_0\ln(n)\big)\Big)}{n^{2a_0}} \quad (169)$$

$$+\frac{ccc_n\gamma_n}{n^{a_0+1}} + \frac{D_{n,N}}{n^{2a_0+1}} + \frac{\varepsilon_{n,N}}{n^{a_0+2}}\Bigg\} \quad (170)$$

$$= \Bigg\{ e_2(n) + \frac{ccc_n\gamma_n}{n^{a_0+1}} + \frac{D_{n,N}}{n^{2a_0+1}} + \frac{\varepsilon_{n,N}}{n^{a_0+2}}\Bigg\} \quad (171)$$

Where

$$e_2(n) = \frac{\big(\cos(b_0\ln(n)) - 2^{1-a_0}\cos(b_0(\ln(2n)))\big)^2 + ccc_n\Big(X_1\sin\big(b_0\ln(n)\big)+X_2\cos\big(b_0\ln(n)\big)\Big)}{n^{2a_0}} \quad (172)$$

Thanks to the limit $\lim_{n\to+\infty} ccc_n\gamma_n = 0$ is zero we are now in a similar configuration like in chapter one.
We denote $c_n = ccc_n\gamma_n$. We have $\lim_{n\to+\infty} c_n = 0$.

Let's now study the serie $\sum_{n\geq 1} e_2(n)$.

Lemma 3.5. *Let's define the functions:*

$$c(t) = -\Big(b_0\sin\big(b_0\ln(t)\big)+a_0\cos\big(b_0\ln(t)\big)\Big) \quad (173)$$

$$cc(t) = \Big(b_0\cos\big(b_0\ln(t)\big)-a_0\sin\big(b_0\ln(t)\big)\Big) \quad (174)$$

$$ccc(t) = c(t) - 2^{1-a_0}\Big(c(t)\cos b_0\ln(2) - cc(t)\sin b_0\ln(2)\Big) \quad (175)$$

$$ce(t) = ccc(t)\Big(X_1\sin\big(b_0\ln(t)\big)+X_2\cos\big(b_0\ln(t)\big)\Big) \quad (176)$$

$$e_2(t) = \frac{\big(\cos(b_0\ln(t)) - 2^{1-a_0}\cos(b_0(\ln(2t)))\big)^2 + ccc(t)\Big(X_1\sin\big(b_0\ln(t)\big)+X_2\cos\big(b_0\ln(t)\big)\Big)}{t^{2a_0}} \quad (177)$$

Let's define the sequence:

$$e_{2,n} = e_2(n) - \int_n^{n+1} dt\, e_2(t) \quad (178)$$

$$(179)$$

Therefore, the serie $\sum_{k\geq 1} e_{2,n}$ are converging absolutely and we have the following inequalities:

$$\left|\sum_{k=n+1}^{N} e_2(n) - \int_{n+1}^{N} e_2(t)dt\right| \leq \frac{K}{2a_0}\Big(\frac{1}{n^{2a_0}} - \frac{1}{N^{2a_0}}\Big) \quad (180)$$

And

$$\left| \sum_{k=n+1}^{+\infty} e_2(n) - \int_{n+1}^{+\infty} e_2(t)dt \right| \leq \frac{K}{2a_0} \frac{1}{n^{2a_0}} \tag{181}$$

where $K > 0$ and E_2 is the primitive function of e_2 is defined as:

$$E_2(x) = \Big(\frac{\alpha_1}{1-2a_0} + \frac{\Big(2b_0\beta_1 + \gamma_1(1-2a_0)\Big)\sin(2b_0 \ln(x)) + \Big(\beta_1(1-2a_0) - 2b_0\gamma_1\Big)\cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \Big) \frac{1}{(x)^{2a_0-1}} \tag{182}$$

Where

$$\alpha_1 = -\frac{(2a_0-1)(1-a_0)}{(b_0)^2 + (1-a_0)^2} \Big(\frac{\left(2^{1-a_0}-1\right)^2}{2} + 2^{1-a_0}\big(1 - \cos(b_0 \ln(2))\big) \Big) \leq 0 \tag{183}$$

Proof. We are going to use the lemmas 3.1 and 3.2 defined in [1]. We will also use their notations. We can write the functions $e_2(t)$ and $ce(t)$ as the following:

$$e_2(t) = \frac{a_1 + b_1 \cos(2b_0 \ln(t)) - c_1 \sin(2b_0(\ln((t)))) + ce(t)}{t^{2a_0}} \tag{184}$$

$$ccc(t) = -\big(a_3b_0 + b_3a_0\big)\sin(b_0(\ln((t)))) + \big(b_3b_0 - a_3a_0\big)\cos(b_0 \ln(t)) \tag{185}$$

For simplification let's now denote:

$$ccc(t) = Z_1 \sin(b_0 \ln t) + Z_2 \cos(b_0 \ln(t)) \tag{186}$$

Therefore we can write:

$$ce(t) = ccc(t)\Big(X_1 \sin\big(b_0 \ln(t)\big) + X_2 \cos\big(b_0 \ln(t)\big)\Big) \tag{187}$$

$$= \Big(Z_1 \sin(b_0 \ln t) + Z_2 \cos(b_0 \ln(t))\Big)\Big(X_1 \sin(b_0 \ln t) + X_2 \cos(b_0 \ln(t))\Big) \tag{188}$$

$$= \frac{X_1Z_1 + X_2Z_2}{2} + \frac{X_1Z_2 + X_2Z_1}{2}\sin(2b_0 \ln(t)) + \frac{X_2Z_2 - X_1Z_1}{2}\cos(2b_0 \ln t) \tag{189}$$

Therefore we can write $e_2(t)$

$$e_2(t)\, t^{2a_0} = \Big(a_1 + \frac{X_1Z_1 + X_2Z_2}{2}\Big) + \Big(\frac{X_1Z_2 + X_2Z_1}{2} - c1\Big)\sin(2b_0 \ln(t)) + \Big(\frac{X_2Z_2 - X_1Z_1}{2} + b_1\Big)\cos(2b_0 \ln t)$$
$$= \alpha_1 + \beta_1 \sin(2b_0 \ln(t)) + \gamma_1 \cos(2b_0 \ln t)$$

Where

$$\alpha_1 = a_1 + \frac{X_1Z_1 + X_2Z_2}{2} \tag{192}$$

$$\beta_1 = \frac{X_1Z_2 + X_2Z_1}{2} - c1 \tag{193}$$

$$\gamma_1 = \frac{X_2Z_2 - X_1Z_1}{2} + b_1 \tag{194}$$

$$Z_1 = -\big(a_3b_0 + b_3a_0\big) \tag{195}$$

$$Z_2 = b_3b_0 - a_3a_0 \tag{196}$$

$$X_1 = \frac{a_3b_0 + b_3(1-a_0)}{(b_0)^2 + (1-a_0)^2} \tag{197}$$

$$X_2 = \frac{a_3(1-a_0) - b_3b_0}{(b_0)^2 + (1-a_0)^2} \tag{198}$$

$$a_1 = \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0}\cos(b_0\ln(2)) \tag{199}$$

$$b_1 = \frac{1}{2} + 2^{1-2a_0}\cos(2b_0\ln(2)) - 2^{1-a_0}\cos(b_0\ln(2)) \tag{200}$$

$$c_1 = 2^{1-2a_0}\sin(2b_0\ln(2)) + 2^{1-a_0}\sin(b_0\ln(2)) \tag{201}$$

$$a_3 = 1 - 2^{1-a_0}\cos(b_0\ln(2)) \tag{202}$$

$$b_3 = 2^{1-a_0}\sin(b_0\ln(2)) \tag{203}$$

And we apply the lemma 3.1 in [1] with $\alpha = 2a_0$, $\beta = 2b_0$, $a = \alpha_1$, $b = \beta_1$ and $c = \gamma_1$.

$$e_2(x) = \Big(\frac{\alpha_1}{1-2a_0} + \frac{\Big(2b_0\beta_1 + \gamma_1(1-2a_0)\Big)\sin(2b_0\ln(x)) + \Big(\beta_1(1-2a_0) - 2b_0\gamma_1\Big)\cos(2b_0\ln(x))}{(2b_0)^2 + (1-2a_0)^2}\Big)\frac{1}{(x)^{2a_0-1}} \tag{204}$$

When we go thru the lengthy calculation something magical will happen. We found that

$$X_1Z_1 + X_2Z_2 = -\frac{\Big(a_3^2 + b_3^2\Big)\Big(b_0^2 + a_0(1-a_0)\Big)}{(b_0)^2 + (1-a_0)^2} \tag{205}$$

$$\tag{206}$$

And something even more magical happens when you calculate α_1

$$\alpha_1 = a_1 + \frac{X_1Z_1 + X_2Z_2}{2} \tag{207}$$

$$= a_1 - \frac{1}{2}\frac{\Big(a_3^2 + b_3^2\Big)\Big(b_0^2 + a_0(1-a_0)\Big)}{(b_0)^2 + (1-a_0)^2} \tag{208}$$

$$= \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0}\cos(b_0\ln(2)) - \frac{1}{2}\frac{\Big(a_3^2 + b_3^2\Big)\Big(b_0^2 + a_0(1-a_0)\Big)}{(b_0)^2 + (1-a_0)^2} \tag{209}$$

$$= \frac{2^{2-2a_0} - 1}{2} + a_3 - \frac{1}{2}\frac{\Big(2^{2-2a_0} - 1 + 2a_3\Big)\Big(b_0^2 + a_0(1-a_0)\Big)}{(b_0)^2 + (1-a_0)^2} \tag{210}$$

$$= -\Big(\frac{2^{2-2a_0} - 1}{2} + a_3\Big)\frac{(2a_0-1)(1-a_0)}{(b_0)^2 + (1-a_0)^2} \tag{211}$$

Therefore

$$\alpha_1 = -\frac{(2a_0-1)(1-a_0)}{(b_0)^2+(1-a_0)^2}\Big(\frac{\left(2^{1-a_0}-1\right)^2}{2}+2^{1-a_0}\left(1-\cos(b_0\ln(2))\right)\Big) \quad (212)$$

$$\leq 0 \quad (213)$$

The magic here is the presence of the terms $2a_0 - 1$ and $1 - a_0$ as global factor in the formula of α_1. □

Let's $N \geq N_0$ be sufficiently large integer.

As we saw above, for $n > \frac{N}{2}$ we have the asymptotic expansion:

$$F_{n+1,N} - F_{n,N} = \frac{\cos^2(b_0\ln(n))}{n^{2a_0}} + \frac{C_{n,N}}{n^{a_0+1}} + \frac{D_{n,N}}{n^{2a_0+1}} + \frac{\varepsilon_{n,N}}{n^{a_0+2}} \quad (214)$$

Hence we have an new expression of F_{n+1} after a summation from $\frac{N}{2}+1$ to n:

$$F_{n+1,N} = F_{\frac{N}{2}+1,N} + \sum_{k=\frac{N}{2}+1}^{n} \frac{\cos^2(b_0\ln(k))}{(k)^{2a_0}} \quad (215)$$

$$+ \sum_{k=\frac{N}{2}+1}^{n} \frac{C_{k,N}}{k^{a_0+1}} + \sum_{k=\frac{N}{2}+1}^{n} \frac{D_{k,N}}{k^{2a_0+1}} + \sum_{k=\frac{N}{2}+1}^{n} \frac{\varepsilon_{k,N}}{k^{a_0+2}} \quad (216)$$

For $n < \frac{N}{2}$ we have the following asymptotic expansion:

$$F_{n+1,N} = F_{N_0,N} + \sum_{k=N_0}^{n} e_2(k) + \sum_{k=N_0}^{n} \frac{c_k}{k^{a_0+1}} \quad (217)$$

$$+ \sum_{k=N_0}^{n} \frac{D_{k,N}}{k^{2a_0+1}} + \sum_{k=N_0}^{n} \frac{\varepsilon_{k,N}}{k^{a_0+2}} \quad (218)$$

Hence if we do another summation between N_0 and N of the equation above:

$$\sum_{n=N_0}^{N-1} F_{n+1,N} = \sum_{n=N_0}^{\frac{N}{2}-1} F_{n+1,N} + F_{\frac{N}{2}+1,N} + \sum_{n=\frac{N}{2}+1}^{N-1} F_{n+1,N} \tag{219}$$

$$= \sum_{n=N_0}^{\frac{N}{2}-1} F_{N_0,N} + \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^{n} e_2(k) + \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^{n} \frac{c_k}{k^{a_0+1}} \tag{220}$$

$$+ \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^{n} \frac{D_{k,N}}{k^{2a_0+1}} + \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^{n} \frac{\varepsilon_{k,N}}{k^{a_0+2}} \tag{221}$$

$$+F_{\frac{N}{2}+1,N} + \sum_{n=\frac{N}{2}+1}^{N-1} F_{\frac{N}{2}+1,N} + \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^{n} \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} \tag{222}$$

$$+ \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^{n} \frac{C_{k,N}}{k^{a_0+1}} + \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^{n} \frac{D_{k,N}}{k^{2a_0+1}} + \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^{n} \frac{\varepsilon_{k,N}}{k^{a_0+2}} \tag{223}$$

Our sums are finite so we can interchange them:

$$\sum_{n=N_0}^{N-1} F_{n+1,N} = (\frac{N}{2} - N_0) F_{N_0,N} + \frac{N}{2} F_{\frac{N}{2}+1,N} + \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k) \sum_{n=k}^{\frac{N}{2}-1} 1 \tag{224}$$

$$+ \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{c_k}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}} \right\} \sum_{n=k}^{\frac{N}{2}-1} 1 \tag{225}$$

$$+ \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{C_{k,N}}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}} \right\} \sum_{n=k}^{N-1} 1 \tag{226}$$

$$= (\frac{N}{2} - N_0) F_{N_0,N} + \frac{N}{2} F_{\frac{N}{2}+1,N} + \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k) (\frac{N}{2} - k) + \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{c_k}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}} \right\} (\frac{N}{2} - k) \tag{227}$$

$$+ \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{C_{k,N}}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}} \right\} (N - k) \tag{228}$$

$$= (\frac{N}{2} - N_0) F_{N_0,N} + \frac{N}{2} F_{\frac{N}{2}+1,N} + \frac{N}{2} \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k) - \sum_{k=N_0}^{\frac{N}{2}-1} k\, e_2(k) \tag{229}$$

$$+ \frac{N}{2} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{c_k}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}} \right\} - \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{c_k}{k^{a_0}} + \frac{D_{k,N}}{k^{2a_0}} + \frac{\varepsilon_{k,N}}{k^{a_0+1}} \right\} \tag{230}$$

$$+ \frac{N}{2} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{C_{k,N}}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}} \right\} \tag{231}$$

$$- \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0-1}} + \frac{C_{k,N}}{k^{a_0}} + \frac{D_{k,N}}{k^{2a_0}} + \frac{\varepsilon_{k,N}}{k^{a_0+1}} \right\} \tag{232}$$

And Finally:

$$\sum_{n=N_0}^{N-1} F_{n+1,N} = N\Bigg[(\frac{1}{2}-\frac{N_0}{N})F_{N_0,N}+\frac{1}{2}F_{\frac{N}{2}+1,N}+\frac{1}{2}\sum_{k=N_0}^{\frac{N}{2}-1} e_2(k)-\frac{1}{N}\sum_{k=N_0}^{\frac{N}{2}-1} k\, e_2(k) \tag{233}$$

$$+\frac{1}{2}\sum_{k=N_0}^{\frac{N}{2}-1}\left\{\frac{c_k}{k^{a_0+1}}+\frac{D_{k,N}}{k^{2a_0+1}}+\frac{\varepsilon_{k,N}}{k^{a_0+2}}\right\}-\frac{1}{N}\sum_{k=N_0}^{\frac{N}{2}-1}\left\{\frac{c_k}{k^{a_0}}+\frac{D_{k,N}}{k^{2a_0}}+\frac{\varepsilon_{k,N}}{k^{a_0+1}}\right\} \tag{234}$$

$$+\frac{1}{2}\sum_{k=\frac{N}{2}+1}^{N-1}\left\{\frac{\cos^2(b_0\ln(k))}{(k)^{2a_0}}+\frac{C_{k,N}}{k^{a_0+1}}+\frac{D_{k,N}}{k^{2a_0+1}}+\frac{\varepsilon_{k,N}}{k^{a_0+2}}\right\} \tag{235}$$

$$-\frac{1}{N}\sum_{k=\frac{N}{2}+1}^{N-1}\left\{\frac{\cos^2(b_0\ln(k))}{(k)^{2a_0-1}}+\frac{C_{k,N}}{k^{a_0}}+\frac{D_{k,N}}{k^{2a_0}}+\frac{\varepsilon_{k,N}}{k^{a_0+1}}\right\}\Bigg] \tag{236}$$

The goal is to prove that $\lim_{N\to+\infty}\sum_{n=N_0}^{N-1} F_{n+1,N} = -\infty$.

Using Césaro lemma we prove that the limit of the following terms are zero:

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{k=N_0}^{\frac{N}{2}-1} k\, e_2(k) = 0 \tag{237}$$

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{k=N_0}^{\frac{N}{2}-1}\left\{\frac{c_k}{k^{a_0}}+\frac{D_{k,N}}{k^{2a_0}}+\frac{\varepsilon_{k,N}}{k^{a_0+1}}\right\}=0 \tag{238}$$

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{k=\frac{N}{2}+1}^{N-1}\left\{\frac{\cos^2(b_0\ln(k))}{(k)^{2a_0-1}}+\frac{C_{k,N}}{k^{a_0}}+\frac{D_{k,N}}{k^{2a_0}}+\frac{\varepsilon_{k,N}}{k^{a_0+1}}\right\}=0 \tag{239}$$

We have

$$\lim_{N\to+\infty}\frac{N_0}{N}F_{N_0,N}+\frac{1}{2}F_{\frac{N}{2}+1,N}=0 \tag{240}$$

One note that $F_{N_0,N}$ is a constant term that actually depends only on N_0, so $\lim_{N\to+\infty}\frac{N_0}{N}F_{N_0,N}=0$. For the term $\frac{1}{2}F_{\frac{N}{2}+1,N}$, we use the lemma 3.1, we have two constants K_1 and K_2 such that: $\left|A_{\frac{N}{2}+1,N}\right| \leq K_1+K_2\,(\frac{N}{2}+1)^{1-a_0}$, therefore

$$F_{\frac{N}{2}+1,N} = \frac{\left|\cos(b_0\ln(\frac{N}{2}+1))A_{\frac{N}{2}+1,N}\right|}{(\frac{N}{2}+1)^{a_0}} \tag{241}$$

$$\leq \frac{K_1}{(\frac{N}{2}+1)^{a_0}}+\frac{K_2}{(\frac{N}{2}+1)^{2a_0-1}} \tag{242}$$

We have $2a_0>1$ therefore the $\lim_{N\to+\infty}F_{\frac{N}{2}+1,N}=0$

Let's now prove

$$\lim_{N\to+\infty}\left[\frac{1}{2}F_{N_0,N}+\frac{1}{2}\sum_{k=N_0}^{\frac{N}{2}-1}e_2(k)\right. \tag{243}$$

$$+\frac{1}{2}\sum_{k=N_0}^{\frac{N}{2}-1}\left\{\frac{c_k}{k^{a_0+1}}+\frac{D_{k,N}}{k^{2a_0+1}}+\frac{\varepsilon_{k,N}}{k^{a_0+2}}\right\} \tag{244}$$

$$\left.+\frac{1}{2}\sum_{k=\frac{N}{2}+1}^{N-1}\left\{\frac{\cos^2(b_0\ln(k))}{(k)^{2a_0}}+\frac{C_{k,N}}{k^{a_0+1}}+\frac{D_{k,N}}{k^{2a_0+1}}+\frac{\varepsilon_{k,N}}{k^{a_0+2}}\right\}\right]<0 \tag{245}$$

We have the sequences $(D_{k,N})_{k\geq 1}$ and $(\varepsilon_{k,N})_{k\geq 1}$ are bounded. Plus We have $a_0+2\geq 2a_0+1>a_0+1\geq 2a_0>1$, therefore the serie $\sum_{k\geq 1}\left\{\frac{\cos^2(b_0\ln(k))}{(k)^{2a_0}}+\frac{D_{k,N}}{k^{2a_0+1}}+\frac{\varepsilon_{k,N}}{k^{a_0+2}}\right\}$ is converging absolutely.
Let's call its partial sum S_N:

$$S_N=\sum_{k\geq 1}^{N}\left\{\frac{\cos^2(b_0\ln(k))}{(k)^{2a_0}}+\frac{D_{k,N}}{k^{2a_0+1}}+\frac{\varepsilon_{k,N}}{k^{a_0+2}}\right\} \tag{246}$$

$$S_N-S_{\frac{N}{2}}=\sum_{k=\frac{N}{2}+1}^{N}\left\{\frac{\cos^2(b_0\ln(k))}{(k)^{2a_0}}+\frac{D_{k,N}}{k^{2a_0+1}}+\frac{\varepsilon_{k,N}}{k^{a_0+2}}\right\} \tag{247}$$

Therefore

$$\lim_{N\to+\infty}\sum_{k=\frac{N}{2}+1}^{N}\left\{\frac{\cos^2(b_0\ln(k))}{(k)^{2a_0}}+\frac{D_{k,N}}{k^{2a_0+1}}+\frac{\varepsilon_{k,N}}{k^{a_0+2}}\right\} \tag{248}$$

$$=\lim_{N\to+\infty}S_N-\lim_{N\to+\infty}S_{\frac{N}{2}} \tag{249}$$

$$=0 \tag{250}$$

Let's study now the convergence of the sum $\sum_{k=\frac{N}{2}+1}^{N}\frac{C_{k,N}}{k^{a_0+1}}$. From the lemma 3.1 we have two constants $K_1\geq 0$ and $K_2\geq 0$ such that:

$$|A_{k,N}|\leq K_1+K_2\,k^{1-a_0} \tag{251}$$

and we have from the definition of $C_{k,N}$ that

$$c_k = -\Big(b_0\sin\big(b_0\ln(k)\big)+a_0\cos\big(b_0\ln(k)\big)\Big) \tag{252}$$

$$C_{k,N} = c_k A_{k,N} \tag{253}$$

We have $a_0>0$, $b_0>0$, Therefore

$$\left|\frac{C_{k,N}}{k^{a_0+1}}\right| \leq \frac{K_1(a_0+b_0)}{k^{a_0+1}}+K_2(a_0+b_0)\,\frac{k^{1-a_0}}{k^{a_0+1}} \tag{254}$$

$$\leq \frac{K_1(a_0+b_0)}{k^{a_0+1}}+K_2(a_0+b_0)\,\frac{1}{k^{2a_0}} \tag{255}$$

We have $a_0+1 \geq 2a_0 > 1$, therefore the serie $\sum_{k\geq 1} \frac{C_{k,N}}{k^{a_0+1}}$ is converging absolutely. Therefore like in the case of E_N, we have

$$\lim_{N\to+\infty} \sum_{k=\frac{N}{2}+1}^{N} \frac{C_{k,N}}{k^{a_0+1}} = \lim_{N\to+\infty} \sum_{k=1}^{N} \frac{C_{k,N}}{k^{a_0+1}} - \lim_{N\to+\infty} \sum_{k=1}^{\frac{N}{2}+1} \frac{C_{k,N}}{k^{a_0+1}} \tag{256}$$

$$= 0 \tag{257}$$

Let's call now the remaining term R_{N_0}:

$$R_{N_0} = \lim_{N\to+\infty} \left[F_{N_0,N} + \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k) + \frac{c_k}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}} \right\} \right] \tag{258}$$

Lemma 3.6. *We can choose N_0 such that $R_{N_0} < 0$.*

Proof. We studied the serie $\sum_{n\geq 1} e_2(n)$ in the lemma 3.5. We have:

$$\left| \sum_{n=N_0+1}^{+\infty} e_2(n) - \int_{N_0+1}^{+\infty} e_2(t)dt \right| \leq \frac{K}{2a_0(N_0)^{2a_0}} \tag{259}$$

Where $K > 0$ and the primitive function E_2

$$E_2(x) = \left(\frac{\alpha_1}{1-2a_0} + \frac{\Big(2b_0\beta_1 + \gamma_1(1-2a_0)\Big)\sin(2b_0\ln(x)) + \Big(\beta_1(1-2a_0) - 2b_0\gamma_1\Big)\cos(2b_0\ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \tag{260}$$

With

$$\alpha_1 = -\frac{(2a_0-1)(1-a_0)}{(b_0)^2+(1-a_0)^2} \Big(\frac{\left(2^{1-a_0}-1\right)^2}{2} + 2^{1-a_0}\big(1-\cos(b_0\ln(2))\big) \Big) \leq 0 \tag{261}$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} e_2(k) + E_2(N_0+1) \right| \leq \frac{K}{a_0(N_0)^{2a_0}} \tag{262}$$

Let's now study the term $F_{N_0,N}$. From lemma 3.2 and 3.3 we can write for $N \geq 2N_0$

$$F_{N_0,N} = \frac{\cos(b_0\ln(N_0)) - 2^{1-a_0}\cos(b_0\ln(2N_0))}{(N_0)^{a_0}} A_{N_0,N} \tag{263}$$

$$= \frac{\cos(b_0\ln(N_0)) - 2^{1-a_0}\cos(b_0\ln(2N_0))}{(N_0)^{a_0}} \Big(\underbrace{E_1(N_0) + \gamma_{N_0}}_{Lemma\ 3.2} \Big) \tag{264}$$

$$= \underbrace{e_3(N_0)}_{Lemma\ 3.3} + \frac{\cos(b_0\ln(N_0)) - 2^{1-a_0}\cos(b_0\ln(2N_0))}{(N_0)^{a_0}} \gamma_{N_0} \tag{265}$$

$$\tag{266}$$

Where the sequence γ_n converge to zero and

$$
\begin{aligned}
e_3(t)\,(t)^{2a_0-1} &= \frac{\alpha_3+\beta_3\sin(2b_0\ln(t))+\gamma_3\cos(2b_0\ln t)}{(t)^{2a_0-1}} && (267)\\
\alpha_3 &= \frac{X_1b_3+X_2a_3}{2} && (268)\\
\beta_3 &= \frac{X_1a_3+X_2b_3}{2} && (269)\\
\gamma_3 &= \frac{X_2a_3-X_1b_3}{2} && (270)\\
X_1 &= \frac{a_3b_0+b_3(1-a_0)}{(b_0)^2+(1-a_0)^2} && (271)\\
X_2 &= \frac{a_3(1-a_0)-b_3b_0}{(b_0)^2+(1-a_0)^2} && (272)
\end{aligned}
$$

Let's denote

$$
O_{N_0} = -\Big(e_3(N_0+1)-e_3(N_0)\Big)+\frac{\cos(b_0\ln(N_0))-2^{1-a_0}\cos(b_0\ln(2N_0))}{(N_0)^{a_0}}\gamma_{N_0} \qquad (273)
$$

The term $e_3(N_0)$ and the term $E_2(N_0)$ share the term $\frac{1}{(N_0)^{2a_0-1}}$. So we are going to group them in the last inequality:

$$
\left|e_3(N_0+1)+\sum_{n=N_0+1}^{+\infty}e_2(k)+\underbrace{E_2(N_0+1)-e_3(N_0+1)}_{e_4(N_0+1)}\right| \leq \frac{K}{a_0(N_0)^{2a_0}} \qquad (274)
$$

Where

$$
\begin{aligned}
e_4(x) &= E_2(x)-e_3(x) && (275)\\
&= \frac{\alpha_2-\alpha_3+(\beta_2-\beta_3)\sin(2b_0\ln(t))+(\gamma_2-\gamma_3)\cos(2b_0\ln t)}{x^{2a_0-1}} && (276)
\end{aligned}
$$

We develop the calculation further and we see that we have $\alpha_2=\alpha_3$. Therefore

$$
e_4(x) = \frac{(\beta_2-\beta_3)\sin(2b_0\ln(t))+(\gamma_2-\gamma_3)\cos(2b_0\ln t)}{x^{2a_0-1}} \qquad (277)
$$

Therefore we have the asymptotic expansion of $e_3(N_0+1)+\sum_{n=N_0+1}^{+\infty}e_2(k)$ as follows:

$$
\begin{aligned}
e_3(N_0+1)+\sum_{n=N_0+1}^{+\infty}e_2(k) &= -E_2(N_0+1)+e_3(N_0+1)+\mathcal{O}\Big(\frac{1}{(N_0)^{2a_0}}\Big) && (278)\\
&= \frac{(\beta_3-\beta_2)\sin(2b_0\ln(N_0+1))+(\gamma_3-\gamma_2)\cos(2b_0\ln(N_0+1))}{(N_0+1)^{2a_0-1}} && (279)\\
&+ \mathcal{O}\Big(\frac{1}{(N_0)^{2a_0}}\Big) && (280)
\end{aligned}
$$

Remark. *The remaining terms in the expression of R_{N_0} are all of the order of $\frac{1}{(N_0)^{a_0}}$ and above. we note that $2a_0-1 < a_0$. So the dominant term in the expression R_{N_0} is the term $-e_4(N_0+1) = \frac{(\beta_3-\beta_2)\sin(2b_0\ln(N_0+1))+(\gamma_3-\gamma_2)\cos(2b_0\ln(N_0+1))}{(N_0+1)^{2a_0-1}}$. As we will see later that the function e_4 is nonzero function. Therefore this term is the dominant term in the expression of $\frac{1}{N}\sum_{n=N_0}^{N-1} F_{n+1,N}$. Hence based on the sign of $e_4(N_0+1)$ we can show that the limit of $\sum_{n=1}^{N-1} F_{n+1,N}$ can be both $+\infty$ and $-\infty$.*

Let's now study the function f:

$$f(x) = (\beta_2-\beta_3)\sin(2b_0x) + (\gamma_2-\gamma_3)\cos(2b_0x) \tag{281}$$

General Case: $\frac{1}{2} < a_0 < 1$ **OR** $a_0 = 1, b_0 \neq \frac{2k\pi}{\ln(2)}$ From the lemma 3.7 below we have that the function e_4 is a nonzero function. Therefore the function f is also nonzero function and hence at least $\beta_2-\beta_3$ or $\gamma_2-\gamma_3$ is different from zero or both of them. So let's assume that $\gamma_2 \neq \gamma_3$.
The function f is a linear combination of the functions sin and cos. So the function f is differentiable and bounded. The function f is periodic of period $\frac{\pi}{b_0}$.
Let's calculate the function f values at the following points:

$$f(0) = \gamma_2-\gamma_3 \tag{282}$$
$$f(\frac{\pi}{2b_0}) = \beta_2-\beta_3 \tag{283}$$
$$f(\frac{\pi}{2b_0}) = -\gamma_2+\gamma_3 \tag{284}$$
$$f(\frac{3\pi}{2b_0}) = -\beta_2+\beta_3 \tag{285}$$

From the values above we have the following:

- If $\Big(\gamma_2-\gamma_3\Big) > 0$ then $f(0) > 0$.
- If $\Big(\gamma_2-\gamma_3 < 0$ then $f(\frac{\pi}{b_0}) > 0$.

In case of $\gamma_2-\gamma_3 = 0$ and we will work with $\beta_2-\beta_3$ in such case we calculate the f values at the points $f(\frac{\pi}{2b_0})$ and $f(\frac{3\pi}{2b_0})$. Therefore, in all case we have either $f(0) > 0$ or $f(\frac{\pi}{b_0}) > 0$. So let's assume from now on that $f(0) < 0$ as the same proof can be done with both values thanks to lemma 3.3 and lemma 3.4 in [1].
From the lemma 3.3 in [1], there exist $N_1 \geq N_0$ such that $|\cos(2b_0\ln(N_1+1))-1| \leq \epsilon$ and $|\sin(2b_0\ln(N_1+1))| \leq \epsilon$ for any $1 > \epsilon > 0$.
Let's define $\beta_0 = -f(0)$. $\beta_0 < 0$.
Let's fix $\epsilon > 0$ to be very small such that $0 < \epsilon < \min\Big(1, \frac{\beta_0}{10000}\Big)$.

Let's define $\gamma_0 = |\beta_2-\beta_3| + |\gamma_2-\gamma_3|$

Let's N_1 be such that $|\cos(2b_0\ln(N_1+1))-1| \leq \epsilon/\gamma_0$ and also $|\sin(2b_0\ln(N_1+1))-0| \leq \epsilon/\gamma_0$. Therefore:

$$\left|e_4(N_1+1)-\frac{f(0)}{(N_1+1)^{2a_0-1}}\right| \leq \left|\left\{\frac{\left((\beta_2-\beta_3)\sin(2b_0\ln(N_1+1))+(\gamma_2-\gamma_3)\cos(2b_0\ln(N_1+1))\right)}{(N_1+1)^{2a_0-1}}\right.\right. \tag{286}$$

$$\left.\left.-\frac{\left((\beta_2-\beta_3)\sin(2b_00)+(\gamma_2-\gamma_3)\cos(2b_00)\right)}{(N_1+1)^{2a_0-1}}\right\}\right| \tag{287}$$

$$\leq \frac{\left(|\beta_2-\beta_3|\,|\sin(2b_0\ln(N_1+1))|+|\gamma_2-\gamma_3|\,|\cos(2b_0\ln(N_1+1))-1|\right)}{(N_1+1)^{2a_0-1}} \tag{288}$$

$$\leq \frac{\left(|\beta_2-\beta_3|\,\epsilon/\gamma_0+|\gamma_2-\gamma_3|\,\epsilon/\gamma_0\right)}{(N_1+1)^{2a_0-1}} \tag{289}$$

$$\leq \frac{\epsilon}{(N_1+1)^{2a_0-1}} \tag{290}$$

Therefore

$$\left|e_3(N_0+1)+\sum_{n=N_0+1}^{+\infty}e_2(n)-\frac{\beta_0}{(N_0+1)^{2a_0-1}}\right| \leq \left\{\left|\sum_{n=N_0+1}^{+\infty}e_2(n)+e_4(N_0+1)\right|\right. \tag{291}$$

$$\left.+\left|-e_4(N_0+1)-\frac{-f(0)}{(N_0+1)^{2a_0-1}}\right|\right\} \leq \frac{\epsilon}{(N_0+1)^{2a_0-1}}+\frac{K}{2a_0(N_0)^{2a_0}} \tag{292}$$

Hence

$$\left|e_3(N_0+1)+\sum_{n=N_0+1}^{+\infty}e_2(n)-\frac{\beta_0}{(N_0+1)^{2a_0-1}}\right| \leq \frac{\epsilon}{(N_0+1)^{2a_0-1}}+\frac{K}{2a_0(N_0)^{2a_0}} \tag{293}$$

Therefore

$$e_3(N_0+1)+\sum_{n=N_0+1}^{+\infty}e_2(n) \leq \frac{\beta_0+\epsilon}{(N_0+1)^{2a_0-1}}+\frac{K}{2a_0(N_0)^{2a_0}} \tag{294}$$

Therefore

$$R_{N_0} = \sum_{k=N_0}^{+\infty} e_2(k) + F_{N_0,N} + \lim_{N\to+\infty}\left[\sum_{k=N_0}^{N}\left\{\frac{c_k}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}}\right\}\right] \tag{295}$$

$$\leq \frac{\beta_0+\epsilon}{(N_0+1)^{2a_0-1}} + \frac{K}{2a_0(N_1)^{2a_0}} + \underbrace{F_{N_0,N} - e_3(N_0+1)} + \lim_{N\to+\infty}\left[\sum_{k=N_0}^{N}\left\{\frac{c_k}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}}\right\}\right] \tag{296}$$

$$\leq \frac{\beta_0+\epsilon}{(N_0+1)^{2a_0-1}} + \frac{K}{2a_0(N_0)^{2a_0}} + \underbrace{e_3(N_0) - e_3(N_0+1) + \frac{\cos(b_0\ln(N_0)) - 2^{1-a_0}\cos(b_0\ln(2N_0))}{(N_0)^{a_0}}\gamma_{N_0}} \tag{297}$$

$$+ \lim_{N\to+\infty}\sum_{k=N_0}^{N}\left\{\frac{c_k}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}}\right) \tag{298}$$

$$\leq \frac{\beta_0+\epsilon}{(N_0+1)^{2a_0-1}} + \frac{K}{2a_0(N_0)^{2a_0}} + O_{N_0} + \lim_{N\to+\infty}\sum_{k=N_0}^{N}\left\{\frac{c_k}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}}\right) \tag{299}$$

We have the sequences (c_k), $(D_{k,N})_{k\geq 1}$ and $(\varepsilon_{k,N})_{k\geq 1}$ are bounded. Plus we have $a_0+2 \geq 2a_0+1 > a_0+1 \geq 2a_0 > 1$, therefore the series $\sum_{k\geq 1}\frac{D_{k,N}}{k^{2a_0+1}}$, $\sum_{k\geq 1}\frac{\varepsilon_{k,N}}{k^{a_0+2}}$ are converging absolutely.

Let the positive constant M such that:
For each $k \geq N_1$ and $N \geq 2N_1$:

$$|c_k| \leq M \tag{300}$$

$$|D_{k,N}| \leq M \tag{301}$$

$$|\varepsilon_{k,N}| \leq M \tag{302}$$

Therefore, we have $a_0 > 0$:

$$\left|\sum_{n=N_1+1}^{+\infty}\frac{c_k}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}}\right| \tag{303}$$

$$\leq \frac{M}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0+1)(N_1)^{a_0+1}} \tag{304}$$

Therefore

$$R_{N_1} \leq \frac{\beta_0+\epsilon}{(N_1+1)^{2a_0-1}} + \frac{K}{2a_0(N_1)^{2a_0}} + O_{N_1} + \frac{M}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0+1)(N_1)^{a_0+1}} \tag{305}$$

$$\leq \frac{1}{(N_1+1)^{2a_0-1}}\left(\beta_0+\epsilon + \frac{K(N_1+1)^{2a_0-1}}{2a_0(N_1)^{2a_0}} + (N_1+1)^{2a_0-1}O_{N_1}\right. \tag{306}$$

$$\left. + \frac{M(N_1+1)^{2a_0-1}}{a_0(N_1)^{a_0}} + \frac{M(N_1+1)^{2a_0-1}}{2a_0(N_1)^{2a_0}} + \frac{M(N_1+1)^{2a_0-1}}{(a_0+1)(N_1)^{a_0+1}}\right) \tag{307}$$

Case One: $\frac{1}{2} < a_0 < 1$ Let's define the sequence (δ_n):

$$\delta_n = \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + (n+1)^{2a_0-1}O_n + \frac{M(n+1)^{2a_0-1}}{(n)^{a_0}} \tag{308}$$

$$+ \frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(a_0+1)(n)^{a_0+1}} \tag{309}$$

As we have $\frac{1}{2} < a_0 < 1$, therefore $2a_0 - 1 < a_0 < 2a_0 < a_0 + 1$
Therefore

$$\lim_{N\to+\infty} \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(n)^{a_0}} \tag{310}$$

$$+\frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(a_0+1)(n)^{a_0+1}} = 0 \tag{311}$$

Let's calculate now the limit of $\lim_{n\to+\infty}(n+1)^{2a_0-1}\, O_n$
From the lemma 3.8 in [1] ($\alpha = 2a_0$ and $\beta = b_0$) we can write:

$$(n+1)^{2a_0-1}O_n = (n+1)^{2a_0-1}\Big(e_3(n) - e_3(n+1)\Big) + (n+1)^{2a_0-1}\frac{\cos(b_0\ln(n)) - 2^{1-a_0}\cos(b_0\ln(2n))}{n^{a_0}}\gamma_n \tag{312}$$

$$= -a(n+1)^{2a_0-1}\frac{1-2a_0}{n^{2a_0}} - (n+1)^{2a_0-1}\frac{(b\,cc_n(2a_0-1,b_0) + c\,c_n(2a_0-1,b_0))}{n^{2a_0}} \tag{313}$$

$$+(n+1)^{2a_0-1}\frac{\cos(b_0\ln(n)) - 2^{1-a_0}\cos(b_0\ln(2n))}{n^{a_0}}\gamma_n + \mathcal{O}(\frac{1}{n^{2a_0}}) \tag{314}$$

Thanks to $2a_0 - 1 < a_0 < 2a_0 < a_0 + 1$, we have the limit of every term in the expression of $(n+1)^{2a_0-1}O_n$ goes to zero, therefore $\lim_{n\to+\infty}(n+1)^{2a_0-1}\, O_n = 0$.
Therefore the $\lim_{n\to+\infty}\delta_n = 0$.
So we can choose N_1 such that $|\delta_{N_1}| < \epsilon$, i.e $\delta_{N_1} \leq \epsilon$.
Therefore

$$R_{N_1} \leq \frac{1}{(N_1+1)^{2a_0-1}}\Big(\beta_0 + 2\epsilon\Big) < 0 \tag{315}$$

Therefore $\lim_{N\to+\infty}\sum_{n=N_0}^{N-1} F_{n+1,N} = -\infty$.

Case Two: $a_0 = 1, b_0 \neq \frac{2k\pi}{\ln(2)}$ In this particular case we have $2a_0 - 1 = 1 = a_0$. Therefore the terms of the order $\frac{1}{n^{a_0}}$ in the expression of δ_n and O_n can become problematic. So instead of using the fact (c_k) is bounded, we will use here the fact that its limit equal zero when n goes to infinity.
As we have

$$\lim_{n\to+\infty} c_n = 0 \tag{316}$$

So we can choose N_1 such for each $n \geq N_1$ we have $|c_k| \leq \epsilon$.
For each $k \geq N_1$ and $N \geq 2N_1$:

$$|c_k| \leq \epsilon \tag{317}$$

$$|D_{k,N}| \leq M \tag{318}$$

$$|\varepsilon_{k,N}| \leq M \tag{319}$$

Therefore, we have $a_0 > 0$:

$$\left|\sum_{n=N_1+1}^{+\infty}\frac{c_k}{k^{a_0+1}} + \frac{D_{k,N}}{k^{2a_0+1}} + \frac{\varepsilon_{k,N}}{k^{a_0+2}}\right| \leq \frac{\epsilon}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0+1)(N_1)^{a_0+1}} \tag{320}$$

And

$$R_{N_1} \leq \frac{\beta_0+\epsilon}{(N_1+1)^{2a_0-1}} + \frac{K}{2a_0(N_1)^{2a_0}} + O_{N_1} + \frac{\epsilon}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0+1)(N_1)^{a_0+1}} \tag{321}$$

$$\leq \frac{1}{(N_1+1)^{2a_0-1}}\left(\beta_0+\epsilon+\frac{K(N_1+1)^{2a_0-1}}{2a_0(N_1)^{2a_0}} + (N_1+1)^{2a_0-1}\,O_{N_1}\right. \tag{322}$$

$$\left. + \frac{\epsilon(N_1+1)^{2a_0-1}}{a_0(N_1)^{a_0}} + \frac{M(N_1+1)^{2a_0-1}}{2a_0(N_1)^{2a_0}} + \frac{M(N_1+1)^{2a_0-1}}{(a_0+1)(N_1)^{a_0+1}}\right) \tag{323}$$

We have $a_0 = 1$ therefore:

$$R_{N_1} \leq \frac{1}{(N_1+1)}\left(\beta_0+2\epsilon+\frac{K(N_1+1)}{2(N_1)^2} + (N_1+1)\,O_{N_1} + \frac{1}{N_1} + \frac{M(N_1+1)}{2(N_1)^2} + \frac{M(N_1+1)}{2(N_1)^2}\right) \tag{324}$$

Because $\beta_0 < 0$ and we have $\epsilon < \frac{|\beta_0|}{10000} < 1$
We have the sequence (δ_n):

$$\delta_n = \frac{K(n+1)}{2n^2} + (n+1)\,O_n + \frac{1}{n} + \frac{M(n+1)}{2n^2} + \frac{M(n+1)}{2n^2} \tag{325}$$

For the term $(n+1)^{2a_0-1}O_n$ we have its expression:

$$(n+1)O_n = -a(n+1)\frac{-1}{n^2} - (n+1)^1\frac{(b\,cc_n(1,b_0) + c\,c_n(1,b_0))}{n^2} \tag{326}$$

$$+(n+1)^1\frac{\cos(b_0\ln(n)) - \cos(b_0\ln(2n))}{n}\gamma_n + \mathcal{O}(\frac{1}{n^2}) \tag{327}$$

In the expression of $(n+1)^{2a_0-1}O_n$ there is γ_n term that can be problematic because $2a_0 - 1 = a_0$. But thanks to the limit $\lim_{n\to+\infty}\gamma_n = 0$.
We also have $\lim_{n\to+\infty}(n+1)^{2a_0-1}O_n = 0$.
Therefore

$$\lim_{n\to+\infty}\delta_n = 0 \tag{328}$$

So we can choose N_1 such that $|\delta_{N_1}| < \epsilon$, i.e $\delta_{N_1} \leq \epsilon$.
Therefore

$$R_{N_1} \leq \frac{1}{N_1+1}\left(\beta_0+3\epsilon\right) < 0 \tag{329}$$

Therefore $\lim_{N\to+\infty}\sum_{n=N_0}^{N-1}F_{n+1,N} = -\infty$.

□

Remark. *Note that one also can prove that the* $\lim_{N\to+\infty}\sum_{n=N_0}^{N-1}F_{n+1,N} = +\infty$ *by chosing another point* x_0 *where* $f(x_0) < 0$. *The same steps above apply. And this thanks to the term* $\frac{(\beta_3-\beta_2)\sin(2b_0\ln(N_0+1))+(\gamma_3-\gamma_2)\cos(2b_0\ln(N_0+1))}{(N_0+1)^{2a_0-1}}$

□

Conclusion. *We saw above that the serie* $\sum_{n\geq 1} U_{n,N}^2$ *is converging absolutely thanks to* $2a_0 > 1$. *We have from the lemma 3.4 that:*

$$\lim_{N\to\infty} \sum_{n=1}^{N} F_{n,N} \quad = \quad -\infty \tag{330}$$

Therefore

$$\lim_{N\to\infty} A_{N,N} \quad = \quad -\infty \tag{331}$$

This result is in contradiction with the fact that s *is a* ζ *zero therefore the limit* $\lim_{N\to\infty} A_{N,N} = 0$ *should be zero. Therefore* s *with* $\frac{1}{2} < a_0 = \Re(s) \leq 1$ *cannot be a zero for the Riemann's Zeta Function.*

Lemma 3.7. *Let's define the function* e_4*:*

$$e_4(t) \quad = \quad E_2(t) - e_3(t) \tag{332}$$

Where e_3 *and* E_2 *are the same functions defined in the lemmas 3.3 and 3.5. Therefore the function* e_4 *is a nonzero function, i.e we don't have for each* t, $e_4(t) = 0$.

$$E_2(x) = \left(\frac{\alpha_1}{1-2a_0} + \frac{\Big(2b_0\beta_1 + \gamma_1(1-2a_0)\Big)\sin(2b_0\ln(x)) + \Big(\beta_1(1-2a_0) - 2b_0\gamma_1\Big)\cos(2b_0\ln(x))}{(2b_0)^2 + (1-2a_0)^2}\right)\frac{1}{(x)^{2a_0-1}} \tag{333}$$

The function E_2 *is the primitive function of the function* $e_2(t)$ *where*

$$\begin{aligned} e_2(t)\, t^{2a_0} = \Big(a_1 + \frac{X_1Z_1 + X_2Z_2}{2}\Big) + \Big(\frac{X_1Z_2 + X_2Z_1}{2} - c1\Big)\sin(2b_0\ln(t)) + \Big(\frac{X_2Z_2 - X_1Z_1}{2} + b_1\Big)\cos(2b_0\ln t) \\ = \alpha_1 + \beta_1\sin(2b_0\ln(t)) + \gamma_1\cos(2b_0\ln t) \end{aligned}$$

And the function e_3 *is defined as*

$$e_3(t)\,(t)^{2a_0-1} = \alpha_3 + \beta_3\sin(2b_0\ln(t)) + \gamma_3\cos(2b_0\ln t) \tag{336}$$

Where

$$\alpha_1 = a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \tag{337}$$

$$\beta_1 = \frac{X_1 Z_2 + X_2 Z_1}{2} - c1 \tag{338}$$

$$\gamma_1 = \frac{X_2 Z_2 - X_1 Z_1}{2} + b_1 \tag{339}$$

$$Z_1 = -\big(a_3 b_0 + b_3 a_0\big) \tag{340}$$

$$Z_2 = b_3 b_0 - a_3 a_0 \tag{341}$$

$$X_1 = \frac{a_3 b_0 + b_3(1-a_0)}{(b_0)^2 + (1-a_0)^2} \tag{342}$$

$$X_2 = \frac{a_3(1-a_0) - b_3 b_0}{(b_0)^2 + (1-a_0)^2} \tag{343}$$

$$a_1 = \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0}\cos(b_0 \ln(2)) \tag{344}$$

$$b_1 = \frac{1}{2} + 2^{1-2a_0}\cos(2b_0 \ln(2)) - 2^{1-a_0}\cos(b_0 \ln(2)) \tag{345}$$

$$c_1 = 2^{1-2a_0}\sin(2b_0 \ln(2)) + 2^{1-a_0}\sin(b_0 \ln(2)) \tag{346}$$

$$a_3 = 1 - 2^{1-a_0}\cos(b_0 \ln(2)) \tag{347}$$

$$b_3 = 2^{1-a_0}\sin(b_0 \ln(2)) \tag{348}$$

$$\alpha_3 = \frac{X_1 b_3 + X_2 a_3}{2} \tag{349}$$

$$\beta_3 = \frac{X_1 a_3 + X_2 b_3}{2} \tag{350}$$

$$\gamma_3 = \frac{X_2 a_3 - X_1 b_3}{2} \tag{351}$$

Proof. To prove this lemma we proceed by contradiction. Let's suppose that the function $e_4 = 0$. Thererfore for each $t \neq 0$, $e_3(t) = E_2(t)$. Hence we also have their first derivatives are equals: for each t, $e_2(t) = e_3'(t)$. Therefore the coeffeicients of e_2 and e_3' are equals:

$$\alpha_1 = (1-2a_0)\alpha_3 \tag{352}$$

$$\beta_1 = \beta_3(1-2a_0) - 2b_0\gamma_3 \tag{353}$$

$$\gamma_1 = (1-2a_0)\gamma_3 + 2b_0\beta_3 \tag{354}$$

Quick calculation shows that in fact

$$\alpha_1 = (1-2a_0)\alpha_3 = -\frac{(2a_0-1)(1-a_0)}{(b_0)^2 + (1-a_0)^2}\Big(\frac{\big(2^{1-a_0}-1\big)^2}{2} + 2^{1-a_0}\big(1-\cos(b_0\ln(2))\big)\Big) \tag{355}$$

We develop further the equations (353) and (354) to get the system of equations of X_1 and X_2:

$$\beta X_1 + \alpha X_2 = 2c_1 \tag{356}$$

$$-\alpha X_1 + \beta X_2 = -2b_1 \tag{357}$$

$$\alpha = a_3 b_0 - b_0(1-a_0) \tag{358}$$

$$\beta = -\Big(b_0 b_3 + a_3(1-a_0)\Big) \tag{359}$$

Therefore

$$X_1 = \frac{2\alpha b_1 + 2\beta c_1}{\alpha^2 + \beta^2} \tag{360}$$

$$X_2 = \frac{2\alpha C_1 - 2\beta b_1}{\alpha^2 + \beta^2} \tag{361}$$

Case One: $a_0 = 1, b_0 \neq \frac{k\pi}{\ln(2)}, k$ **nonzero integer** In this case we have from the expression that defines X_1, we repace a_0 by its value 1. We have

$$X_1 = \frac{a_3}{b_0} \tag{362}$$

$$X_2 = -\frac{b_3}{b_0} \tag{363}$$

$$\alpha = a_3 b_0 \tag{364}$$

$$\beta = -b_0 b_3 \tag{365}$$

Therefore

$$X_1 = \frac{2b_1 a_3 - 2b_3 c_1}{(a_3^2 + b_3^2) b_0} \tag{366}$$

$$X_2 = \frac{2c_1 a_3 + 2b_3 b_1}{(a_3^2 + b_3^2) b_0} \tag{367}$$

We write b_1 and c_1 in function of a_3 and b_3

$$b_1 = \frac{a_3^2 - b_3^2}{2} \tag{368}$$

$$c_1 = b_3(2 - a_3) \tag{369}$$

Therefore the equation of X_2 gives us the following equality:

$$-\frac{b_3}{b_0} = \frac{4a_3 b_3 - b_3 a_3^2 - b_3^3}{(a_3^2 + b_3^2) b_0} \tag{370}$$

After simplification we get $a_3 b_3 = 0$. We have $b_0 \neq \frac{k\pi}{\ln(2)}$ therefore $b_3 \neq 0$. Therefore $a_3 = 0$. But from the expression of $a_3 = 1 - \cos(b_0 \ln(2))$, a_3 also cannot be zero. Hence the contradiction.
From the equation of X_1 we get $b_3^3 = 0$. Therefore $b_3 = 0$. And this is also a contradiction because $b_0 \neq \frac{k\pi}{\ln(2)}$ Therefore $e_3 \neq E_2$ and the function e_4 is a nonzero function. This prove our case.

Case Two: $\frac{1}{2} < a_0 < 1, b_0 \neq \frac{k\pi}{\ln(2)}, k$ **nonzero integer** In this general case, the equation of X_1 gives us the following:

$$\frac{a_3 b_0 + b_3(1 - a_0)}{b_0^2 + (1 - a_0)^2} = \frac{2b_1\alpha + 2\beta c_1}{\alpha^2 + \beta^2} \tag{371}$$

$$\tag{372}$$

We develop further we have

$$\alpha^2 + \beta^2 = \left(a_3^2 + b_3^2\right)\left(b_0^2 + (1 - a_0)^2\right) \tag{373}$$

Therefore

$$a_3 b_0 + b_3(1 - a_0) = \frac{2b_1\alpha + 2\beta c_1}{a_3^2 + b_3^2} \tag{374}$$

We inject these expressions in the last equation.

$$b_1 = \frac{a_3^2 - b_3^2}{2} \tag{375}$$

$$c_1 = b_3(2 - a_3) \tag{376}$$

After simplification we get

$$b_3\Big(a_3 b_3 b_0 - 2b_0 b_3 - 2a_3(1 - a_0)\Big) \quad = \quad 0 \tag{377}$$

We have $b_0 \neq \frac{k\pi}{\ln(2)}$ therefore $b_3 \neq 0$. Therefore

$$a_3 b_3 b_0 - 2b_0 b_3 - 2a_3(1 - a_0) \quad = \quad 0 \tag{378}$$

After simplification of the last equation we get:

$$-\frac{b_0 2^{1-a_0}\sin(b_0 \ln(2))}{2(1 - a_0)} = \frac{1 - 2^{1-a_0}\cos(b_0 \ln(2))}{1 + 2^{1-a_0}\cos(b_0 \ln(2))} \tag{379}$$

We have $b_0 \neq \frac{k\pi}{\ln(2)}$, therefore $\sin(b_0 \ln(2)) \neq 0$
We calculate the limit of the two terms of the equation above when a_0 goes to 1^-. We get

$$\lim_{a_0 \to 1^-} -\frac{b_0 2^{1-a_0}\sin(b_0 \ln(2))}{2(1 - a_0)} = \lim_{a_0 \to 1^-} \frac{1 - 2^{1-a_0}\cos(b_0 \ln(2))}{1 + 2^{1-a_0}\cos(b_0 \ln(2))} \tag{380}$$

$$\infty = \frac{1 - \cos(b_0 \ln(2))}{1 + \cos(b_0 \ln(2))} > 0 \tag{381}$$

Hence the contradiction. Therefore $e_3 \neq E_2$ and the function e_4 is a nonzero function. This prove our case.

Case Three: $\frac{1}{2} < a_0 < 1, b_0 = \frac{k\pi}{\ln(2)}, k$ **nonzero integer** In this case we have

$$b_3 \quad = \quad 0 \tag{382}$$

$$b_1 \quad = \quad \frac{a_3^2}{2} \tag{383}$$

$$c_1 \quad = \quad 0 \tag{384}$$

We replace these values into the equation

$$a_3 b_0 + b_3(1 - a_0) = \frac{2b_1\alpha + 2\beta c_1}{a_3^2 + b_3^2} \tag{385}$$

Therefore

$$a_3 b_0 \quad = \quad \alpha \tag{386}$$

$$\quad = \quad a_3 b_0 - b_0(1 - a_0) \tag{387}$$

Therefore

$$b_0(1 - a_0) = 0 \tag{388}$$

Therefore b_0. This is also a contradiction. Therefore $e_3 \neq E_2$ and the function e_4 is a nonzero function. This prove our case. □

3.2 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$ as all other possibilities can be discarded using the functional equation like in [1]. Therefore the Riemann hypothesis is true: *The nontrivial zeros of* $\zeta(s)$ *have real part equal to* $\frac{1}{2}$. We believe that the same technique applied here can be applied to resolve the Generalized Riemann hypothesis.

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