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A Proof of the Riemann's Hypothesis

Charaf ECH-CHATBI *

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Abstract

We present a proof of the Riemann's Zeta Hypothesis, based on asymptotic expansions and operations on series. We use the symmetry property presented by Riemann's functional equation to extend the proof to the whole set of complex numbers \mathbb{C} . The advantage of our method is that it only uses undergraduate maths which makes it accessible to a wider audience.

Keywords: Riemann Hypothesis; Riemann zeta function; Critical Strip; Zeta Zeros; Prime Number Theorem; Asymptotic distribution of the Prime Numbers ; Millennium Problems; Dirichlet series; Dirichlet L-series; Generalized Riemann hypothesis.

1 The Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part $1/2$.

2 Riemann Zeta Function

For a complex number s , the Zeta function is defined as the following infinite sum:

$$\zeta(s) = \begin{cases} Z^1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, & \text{if } \Re(s) > 1 \\ \frac{1}{1-\frac{1}{2^s-1}} Z^2(s), \text{ where } Z^2(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} & \text{if } \left(\Re(s) > 0, s \neq 1 + \frac{2k\pi}{\ln(2)}, k \text{ integer} \right) \\ \frac{Z^2(s)}{\ln(2)}, & \text{if } s = 1 + \frac{2k\pi}{\ln(2)}, k \text{ nonzero integer} \end{cases} \quad (1)$$

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Riemann extended the domain of definition of the zeta function to the whole complex domain using the analytical continuation. The Dirichlet eta function $\eta(s) = Z^2(s)$ is convergent when $\Re(s) > 0$ and it is used as the analytical continuation of the zeta function on the domain where $\Re(s) > 0$. When $\Re(s) \leq 0$, the Riemann zeta functional equation is used as the analytical continuation for the zeta function.

Let's define the two series (Z_N^1) and (Z_N^2) as following:
For a complex number s and an integer $N \geq 1$:

$$Z_N^1(s) = \sum_{n=1}^N \frac{1}{n^s} \quad (2)$$

And

$$Z_N^2(s) = \sum_{n=1}^N \frac{(-1)^{n+1}}{n^s} \quad (3)$$

We can expand Z_N^1 and Z_N^2 to get the following:

$$Z_N^1(s) = \sum_{n=1}^N \frac{1}{n^s} \quad (4)$$

$$= \sum_{1 \leq 2n \leq N} \frac{1}{(2n)^s} + \sum_{1 \leq 2n+1 \leq N} \frac{1}{(2n+1)^s} \quad (5)$$

$$= \frac{1}{2^s} \sum_{1 \leq n \leq \frac{N}{2}} \frac{1}{n^s} + \sum_{1 \leq 2n+1 \leq N} \frac{1}{(2n+1)^s} \quad (6)$$

$$= \frac{1}{2^s} Z_{\frac{N}{2}}^1 + \sum_{1 \leq 2n+1 \leq N} \frac{1}{(2n+1)^s} \quad (7)$$

And the same for Z_N^2

$$Z_N^2(s) = \sum_{n=1}^N \frac{(-1)^{n+1}}{n^s} \quad (8)$$

$$= \sum_{1 \leq 2n \leq N} \frac{(-1)^{2n+1}}{(2n)^s} + \sum_{1 \leq 2n+1 \leq N} \frac{(-1)^{2n+1+1}}{(2n+1)^s} \quad (9)$$

$$= -\frac{1}{2^s} \sum_{1 \leq n \leq \frac{N}{2}} \frac{1}{n^s} + \sum_{1 \leq 2n+1 \leq N} \frac{1}{(2n+1)^s} \quad (10)$$

$$= -\frac{1}{2^s} Z_{\frac{N}{2}}^1 + \sum_{1 \leq 2n+1 \leq N} \frac{1}{(2n+1)^s} \quad (11)$$

Therefore:

$$Z_N^2(s) = Z_N^1(s) - \frac{2}{2^s} Z_{\frac{N}{2}}^1(s) \quad (12)$$

$$= Z_N^1(s) - \frac{1}{2^{s-1}} Z_{\frac{N}{2}}^1(s) \quad (13)$$

Hence when $\Re(s) > 1$, both $(Z_N^1(s))$ and $(Z_N^2(s))$ converge. Therefore:

$$\lim_{N \rightarrow \infty} Z_N^2(s) = \lim_{N \rightarrow \infty} Z_N^1(s) - \frac{1}{2^{s-1}} \lim_{N \rightarrow \infty} Z_{\frac{N}{2}}^1(s) \quad (14)$$

And therefore the relationship between Z^1 and Z^2 .

$$Z^2(s) = \left(1 - \frac{1}{2^{s-1}}\right) Z^1(s) \text{ when } \Re(s) > 1 \quad (15)$$

And

$$Z^1(s) = \frac{1}{\left(1 - \frac{1}{2^{s-1}}\right)} Z^2(s), \text{ if } \Re(s) > 1 \text{ and } s \neq 1 + \frac{2k\pi}{\ln(2)}, k \text{ integer} \quad (16)$$

The equation (16) explains the existence of term $\frac{1}{1 - \frac{1}{2^{s-1}}}$ in the definition of the zeta function in (1).

Remark. From the relation (13), we have if s is a zero for Z^1 then s is also a zero for Z^2 . But not necessarily the other way around.

When $\Re(s) \leq 1$, $Z^1(s)$ is not defined as the sequence (Z_n^1) diverge.

Since Riemann's publication of his famous paper in 1859, many research works have been done on the subject and we know pretty a lot of information about Zeta function and its links with many other applications.

In his famous paper, Riemann[3] has shown that the Zeta has a functional equation $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$. Using this last one, Riemann has shown that the non trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has stated that all these non trivial zeros are very likely located on the critical line $\Re(s) = 1/2$ itself. In 1896, Hadamard and De la Vallée Poussin independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$.

Therefore, to prove this "Riemann Hypothesis" (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

The Professor Peter Sarnak mentioned in one lecture that:

99% of the 'proofs' of RH that are submitted to the Annals (about 3 per week) can be rejected on the basis that they only use the Functional Equation.

This proof does not use the Functional Equation and instead it uses asymptotic expansions and operations on series.

3 Proof of the Riemann Hypothesis

Let's take a complex number s such $s = a_0 + ib_0$. Unless we explicitly mention otherwise, let's suppose that $a_0 > 0$ and $b_0 > 0$.

3.1 Case One: $\frac{1}{2} < a_0 \leq 1$

In this case s is a zero of $Z^2(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s}$.

We are going to develop the sequence $Z_N^1(s) = \sum_{n=1}^N \frac{1}{n^s}$ as follows:
For $N \geq 1$

$$Z_N^1 = \sum_{n=1}^N \frac{1}{n^s} \quad (17)$$

$$= \sum_{n=1}^N \frac{1}{n^{a_0+ib_0}} \quad (18)$$

$$= \sum_{n=1}^N \frac{n^{-ib_0}}{n^{a_0}} \quad (19)$$

$$= \sum_{n=1}^N \frac{\exp(-ib_0 \ln(n))}{n^{a_0}} \quad (20)$$

$$= \sum_{n=1}^N \frac{\cos(b_0 \ln(n)) - i \sin(b_0 \ln(n))}{n^{a_0}} \quad (21)$$

$$= \sum_{n=1}^N \frac{\cos(b_0 \ln(n))}{n^{a_0}} - i \sum_{n=1}^{+\infty} \frac{\sin(b_0 \ln(n))}{n^{a_0}} \quad (22)$$

$$(23)$$

Let's define the sequences U_n, V_n as follows: For $n \geq 1$

$$U_n = \frac{\cos(b_0 \ln(n))}{n^{a_0}} \quad (24)$$

$$V_n = \frac{\sin(b_0 \ln(n))}{n^{a_0}} \quad (25)$$

Let's define the series A_n, B_n and Z_n as follows:

$$A_n = \sum_{k=1}^n \frac{\cos(b_0 \ln(k))}{k^{a_0}} \quad (26)$$

$$B_n = \sum_{k=1}^n \frac{\sin(b_0 \ln(k))}{k^{a_0}} \quad (27)$$

$$Z_n^1 = A_n - iB_n \quad (28)$$

When we are dealing with complex numbers, it is always insightful to work with the norm. So let's develop further the squared norm of the serie Z_N^1 as follows:

$$\|Z_N^1\|^2 = A_N^2 + B_N^2 \quad (29)$$

$$= \left(\sum_{n=1}^N \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right)^2 + \left(\sum_{n=1}^N \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right)^2 \quad (30)$$

So

$$A_N^2 = \left(\sum_{n=1}^N \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right)^2 \quad (31)$$

$$= \sum_{n=1}^N \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sum_{n=1}^N \sum_{k=1}^{n-1} \frac{\cos(b_0 \ln(k)) \cos(b_0 \ln(n))}{k^{a_0} n^{a_0}} \quad (32)$$

$$= \sum_{n=1}^N \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sum_{n=1}^N \frac{\cos(b_0 \ln(n))}{n^{a_0}} \sum_{k=1}^{n-1} \frac{\cos(b_0 \ln(k))}{k^{a_0}} \quad (33)$$

$$= \sum_{n=1}^N \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sum_{n=1}^N \frac{\cos(b_0 \ln(n))}{n^{a_0}} \left(A_n - \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right) \quad (34)$$

$$= - \sum_{n=1}^N \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sum_{n=1}^N \frac{\cos(b_0 \ln(n)) A_n}{n^{a_0}} \quad (35)$$

And the same calculation for B_N

$$B_N^2 = \left(\sum_{n=1}^N \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right)^2 \quad (36)$$

$$= \sum_{n=1}^N \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sum_{n=1}^N \sum_{k=1}^{n-1} \frac{\sin(b_0 \ln(k)) \sin(b_0 \ln(n))}{k^{a_0} n^{a_0}} \quad (37)$$

$$= \sum_{n=1}^N \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sum_{n=1}^N \frac{\sin(b_0 \ln(n))}{n^{a_0}} \sum_{k=1}^{n-1} \frac{\sin(b_0 \ln(k))}{k^{a_0}} \quad (38)$$

$$= \sum_{n=1}^N \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sum_{n=1}^N \frac{\sin(b_0 \ln(n))}{n^{a_0}} \left(B_n - \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right) \quad (39)$$

$$= - \sum_{n=1}^N \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sum_{n=1}^N \frac{\sin(b_0 \ln(n)) B_n}{n^{a_0}} \quad (40)$$

Hence we have the new expression of square norm of Z_N^1 :

$$\|Z_N^1\|^2 = 2 \sum_{n=1}^N \frac{\cos(b_0 \ln(n)) A_n}{n^{a_0}} + 2 \sum_{n=1}^N \frac{\sin(b_0 \ln(n)) B_n}{n^{a_0}} - \sum_{n=1}^N \frac{\sin^2(b_0 \ln(n)) + \cos^2(b_0 \ln(n))}{n^{2a_0}} \quad (41)$$

which simplifies even further to:

$$\|Z_N^1\|^2 = 2 \sum_{n=1}^N \frac{\cos(b_0 \ln(n))A_n}{n^{a_0}} + 2 \sum_{n=1}^N \frac{\sin(b_0 \ln(n))B_n}{n^{a_0}} - \sum_{n=1}^N \frac{1}{n^{2a_0}} \quad (42)$$

Let's now define F_n and G_n as follows:

$$F_n = \frac{\cos(b_0 \ln(n))A_n}{n^{a_0}} \quad (43)$$

$$G_n = \frac{\sin(b_0 \ln(n))B_n}{n^{a_0}} \quad (44)$$

Therefore

$$A_N^2 = 2 \sum_{n=1}^N F_n - \sum_{n=1}^N \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} \quad (45)$$

$$B_N^2 = 2 \sum_{n=1}^N G_n - \sum_{n=1}^N \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} \quad (46)$$

We will use the formula (13) and write Z_N^2 as the following:

$$Z_N^2(s) = Z_N^1(s) - \frac{1}{2^{s-1}} Z_{\frac{N}{2}}^1(s) \quad (47)$$

$$= A_N - iB_N - \frac{1}{2^{s-1}} (A_{\frac{N}{2}} - iB_{\frac{N}{2}}) \quad (48)$$

$$= A_N - \frac{1}{2^{s-1}} A_{\frac{N}{2}} - i \left(B_N - \frac{1}{2^{s-1}} B_{\frac{N}{2}} \right) \quad (49)$$

$$= \overline{A_{N,N}} - i \overline{B_{N,N}} \quad (50)$$

Where

$$\overline{A_{N,N}} = \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(k)) - 2^{1-a_0} \cos(b_0 (\ln(2k)))}{k^{a_0}} + \sum_{k=\frac{N}{2}+1}^N \frac{\cos(b_0 \ln(k))}{k^{a_0}} \quad (51)$$

And

$$\overline{B_{N,N}} = \sum_{k=1}^{\frac{N}{2}} \frac{\sin(b_0 \ln(k)) - 2^{1-a_0} \sin(b_0 (\ln(2k)))}{k^{a_0}} + \sum_{k=\frac{N}{2}+1}^N \frac{\sin(b_0 \ln(k))}{k^{a_0}} \quad (52)$$

We note here that we are using the same partial sum of the Dirichlet serie $\eta(s) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^s}$ that is conditionally convergent when $\Re(s) > 0$. We are not changing the partial sum by rearranging the terms of the Dirichlet serie. The representation $Z_N^2(s) = \overline{A_{N,N}} - i \overline{B_{N,N}}$ is conserving the same terms of the original partial sum of the Dirichlet serie $\sum_{n=1}^N \frac{(-1)^{n+1}}{n^s}$. Therefore, $\overline{A_{N,N}}$ and $-\overline{B_{N,N}}$ are the real part and the imaginary part of the partial sum of the Dirichlet serie. Hence

$$\lim_{N \rightarrow \infty} \overline{A_{N,N}} - i \overline{B_{N,N}} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{(-1)^{n+1}}{n^s} = Z^2(s) = \eta(s) \quad (53)$$

Therefore

Conclusion. s is a zeta zero, $\zeta(s) = 0$, if and only if

$$\lim_{N \rightarrow \infty} \overline{A_{N,N}} = 0 \text{ and } \lim_{N \rightarrow \infty} \overline{B_{N,N}} = 0 \quad (54)$$

Equally, s is a zeta zero, $\zeta(s) = 0$, if and only if

$$\lim_{N \rightarrow \infty} \overline{A_{N,N}}^2 = 0 \text{ and } \lim_{N \rightarrow \infty} \overline{B_{N,N}}^2 = 0 \quad (55)$$

Remark. The rearrangement used in the Riemann rearrangement theorem to make a conditionally convergent serie converge to any number including $\{+\infty, -\infty\}$ is not conserving the same terms of the partial sum of the original series.

For simplification, let's define the following sequences: For $N \geq 1$:

$$\overline{U}_{n,N} = \begin{cases} \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n))}{n^{a_0}}, & \text{if } n \leq \frac{N}{2} \\ \frac{\cos(b_0 \ln(n))}{n^{a_0}}, & \text{if } n > \frac{N}{2} \end{cases} \quad (56)$$

And

$$\overline{V}_{n,N} = \begin{cases} \frac{\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n))}{n^{a_0}}, & \text{if } n \leq \frac{N}{2} \\ \frac{\sin(b_0 \ln(n))}{n^{a_0}}, & \text{if } n > \frac{N}{2} \end{cases} \quad (57)$$

And

$$\overline{A}_{n,N} = \begin{cases} \sum_{k=1}^n \frac{\cos(b_0 \ln(k)) - 2^{1-a_0} \cos(b_0 \ln(2k))}{k^{a_0}}, & \text{if } n \leq \frac{N}{2} \\ \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(k)) - 2^{1-a_0} \cos(b_0 \ln(2k))}{k^{a_0}} + \sum_{k=\frac{N}{2}+1}^n \frac{\cos(b_0 \ln(k))}{k^{a_0}}, & \text{if } n > \frac{N}{2} \\ \overline{A}_{N,N}, & \text{if } n \geq N \end{cases} \quad (58)$$

And

$$\overline{B}_{n,N} = \begin{cases} \sum_{k=1}^n \frac{\sin(b_0 \ln(k)) - 2^{1-a_0} \sin(b_0 \ln(2k))}{k^{a_0}}, & \text{if } n \leq \frac{N}{2} \\ \sum_{k=1}^{\frac{N}{2}} \frac{\sin(b_0 \ln(k)) - 2^{1-a_0} \sin(b_0 \ln(2k))}{k^{a_0}} + \sum_{k=\frac{N}{2}+1}^n \frac{\sin(b_0 \ln(k))}{k^{a_0}}, & \text{if } n > \frac{N}{2} \\ \overline{B}_{N,N}, & \text{if } n \geq N \end{cases} \quad (59)$$

Let's develop further the squared norm of the serie Z_N^2 as follows:

$$\|Z_N^2\|^2 = \overline{A}_{N,N}^2 + \overline{B}_{N,N}^2 \quad (60)$$

$$= \left(\sum_{n=1}^N \overline{U}_{n,N} \right)^2 + \left(\sum_{n=1}^N \overline{V}_{n,N} \right)^2 \quad (61)$$

So

$$\bar{A}_{N,N}^2 = \left(\sum_{n=1}^N \bar{U}_{n,N} \right)^2 \quad (62)$$

$$= \sum_{n=1}^N \bar{U}_{n,N} + 2 \sum_{n=1}^N \sum_{k=1}^{n-1} \bar{U}_n \bar{U}_{k,N} \quad (63)$$

$$= \sum_{n=1}^N \bar{U}_{n,N}^2 + 2 \sum_{n=1}^N \bar{U}_{n,N} \sum_{k=1}^{n-1} \bar{U}_{k,N} \quad (64)$$

$$= \sum_{n=1}^N \bar{U}_{n,N}^2 + 2 \sum_{n=1}^N \bar{U}_{n,N} \left(\bar{A}_{n,N} - \bar{U}_{n,N} \right) \quad (65)$$

$$= - \sum_{n=1}^N \bar{U}_{n,N}^2 + 2 \sum_{n=1}^N \bar{U}_{n,N} \bar{A}_{n,N} \quad (66)$$

$$(67)$$

And the same calculation for $\bar{B}_{N,N}$

$$\bar{B}_{N,N}^2 = \left(\sum_{n=1}^N \bar{V}_n \right)^2 \quad (68)$$

$$= \sum_{n=1}^N \bar{V}_{n,N} + 2 \sum_{n=1}^N \sum_{k=1}^{n-1} \bar{V}_n \bar{V}_{k,N} \quad (69)$$

$$= \sum_{n=1}^N \bar{V}_{n,N}^2 + 2 \sum_{n=1}^N \bar{V}_{n,N} \sum_{k=1}^{n-1} \bar{V}_{k,N} \quad (70)$$

$$= \sum_{n=1}^N \bar{V}_{n,N}^2 + 2 \sum_{n=1}^N \bar{V}_{n,N} \left(\bar{B}_{n,N} - \bar{V}_{n,N} \right) \quad (71)$$

$$= - \sum_{n=1}^N \bar{V}_{n,N}^2 + 2 \sum_{n=1}^N \bar{V}_{n,N} \bar{B}_{n,N} \quad (72)$$

$$(73)$$

Let's now define the sequences $(\bar{F}_{n,N})$ and $(\bar{G}_{n,N})$ as follows:

$$\bar{F}_{n,N} = \bar{U}_{n,N} \bar{A}_{n,N} \quad (74)$$

$$\bar{G}_{n,N} = \bar{V}_{n,N} \bar{B}_{n,N} \quad (75)$$

Hence we have the new expression of square norm of Z_N^2 :

$$\|Z_N^2\|^2 = 2 \sum_{n=1}^N \bar{F}_{n,N} + 2 \sum_{n=1}^N \bar{G}_{n,N} - \sum_{n=1}^N \left(\bar{U}_{n,N}^2 + \bar{V}_{n,N}^2 \right) \quad (76)$$

which simplifies even further to:

$$\sum_{n=1}^N (\bar{U}_{n,N}^2 + \bar{V}_{n,N}^2) = \sum_{n=1}^{\frac{N}{2}} (\bar{U}_{n,N}^2 + \bar{V}_{n,N}^2) + \sum_{n=\frac{N}{2}+1}^N (\bar{U}_{n,N}^2 + \bar{V}_{n,N}^2) \quad (77)$$

$$= \sum_{n=\frac{N}{2}+1}^N \frac{1}{n^{2a_0}} \quad (78)$$

$$+ \sum_{n=1}^{\frac{N}{2}} \frac{\left(\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n)) \right)^2 + \left(\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n)) \right)^2}{n^{2a_0}} \quad (79)$$

$$= \sum_{n=\frac{N}{2}+1}^N \frac{1}{n^{2a_0}} \quad (80)$$

$$+ \sum_{n=1}^{\frac{N}{2}} \frac{\left(1 + 2^{2-2a_0} - 2^{2-a_0} \cos(b_0 \ln(n)) \cos(b_0 \ln(2n)) - 2^{2-a_0} \sin(b_0 \ln(n)) \sin(b_0 \ln(2n)) \right)}{n^{2a_0}} \quad (81)$$

$$= \sum_{n=1}^N \frac{1}{n^{2a_0}} \quad (82)$$

$$+ \sum_{n=1}^{\frac{N}{2}} \frac{2^{2-2a_0} - 2^{2-a_0} \left(\cos(b_0 \ln(n)) \cos(b_0 \ln(2n)) + \sin(b_0 \ln(n)) \sin(b_0 \ln(2n)) \right)}{n^{2a_0}} \quad (83)$$

$$= \sum_{n=1}^N \frac{1}{n^{2a_0}} + \sum_{n=1}^{\frac{N}{2}} \frac{2^{2-2a_0} - 2^{2-a_0} \left(\cos(b_0 \ln(2)) \right)}{n^{2a_0}} \quad (84)$$

$$(85)$$

We have $2a_0 > 1$, hence the partial sums $\sum_{n \geq 1}^N \bar{U}_{n,N}^2$ and $\sum_{n \geq 1}^N \bar{V}_{n,N}^2$ converge absolutely.

We have

$$\bar{A}_{N,N}^2 = 2 \sum_{n=1}^N \bar{F}_{n,N} - \sum_{n=1}^N \bar{U}_{n,N}^2 \quad (86)$$

$$\bar{B}_{N,N}^2 = 2 \sum_{n=1}^N \bar{G}_{n,N} - \sum_{n=1}^N \bar{V}_{n,N}^2 \quad (87)$$

Proof Strategy. *The idea is to prove that in the case of a complex s that is in the right hand side of the critical strip $\frac{1}{2} < a_0 \leq 1$ and that is a ζ zero, that the limit $\lim_{n \rightarrow \infty} \bar{A}_{n,n}^2 = +/\infty$ and the limit $\lim_{n \rightarrow \infty} \bar{B}_{n,n}^2 = +/\infty$. This will create a contradiction. Because if s is a ζ zero then the $\lim_{n \rightarrow \infty} \bar{A}_{n,n}^2$ should be 0 and the $\lim_{n \rightarrow \infty} \bar{B}_{n,n}^2$ should be 0. And therefore the sequences $(\sum_{n=1}^N \bar{F}_{n,N})_{N \geq 1}$ and $(\sum_{n=1}^N \bar{G}_{n,N})_{N \geq 1}$ should converge and their limits should be: $\lim_{n \rightarrow \infty} \sum_{n=1}^N \bar{F}_{n,N} = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^N \bar{U}_{n,N}^2 < +\infty$ and $\lim_{n \rightarrow \infty} \sum_{n=1}^N \bar{G}_{n,N} = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^N \bar{V}_{n,N}^2 < +\infty$.*

We need the following lemmas during different stages of the proof.

Lemma 3.1. *Let's $\alpha > 0$, β , a , b and c real numbers. Let's define the functions:*

$$e(t) = \frac{\cos(\beta \ln(t))}{(t)^\alpha} \quad (88)$$

$$f(t) = \frac{\sin(\beta \ln(t))}{(t)^\alpha} \quad (89)$$

$$g(t) = \frac{a + b \sin(\beta \ln(t)) + c \cos(\beta \ln(t))}{(t)^\alpha} \quad (90)$$

$$(91)$$

Let's define the sequences:

$$x_n = \frac{\cos(\beta \ln(n))}{n^\alpha} - \int_n^{n+1} e(t) \quad (92)$$

$$y_n = \frac{\sin(\beta \ln(n))}{n^\alpha} - \int_n^{n+1} f(t) \quad (93)$$

$$z_n = \frac{a + b \sin(\beta \ln(n)) + c \cos(\beta \ln(n))}{n^\alpha} - \int_n^{n+1} g(t) \quad (94)$$

Therefore, the series $\sum_{k \geq 1} x_n$, $\sum_{k \geq 1} y_n$ and $\sum_{k \geq 1} z_n$ are converging absolutely and we have the following inequalities:

$$\left| \sum_{k=n+1}^N \frac{\cos(\beta \ln(n))}{n^\alpha} - \int_{n+1}^N e(t) dt \right| \leq \frac{K_1}{\alpha} \left(\frac{1}{n^\alpha} - \frac{1}{N^\alpha} \right) \quad (95)$$

$$\left| \sum_{k=n+1}^N \frac{\sin(\beta \ln(n))}{n^\alpha} - \int_{n+1}^N f(t) dt \right| \leq \frac{K_2}{\alpha} \left(\frac{1}{n^\alpha} - \frac{1}{N^\alpha} \right) \quad (96)$$

$$\left| \sum_{k=n+1}^N \frac{a + b \sin(\beta \ln(n)) + c \cos(\beta \ln(n))}{n^\alpha} - \int_{n+1}^N g(t) dt \right| \leq \frac{K_3}{\alpha} \left(\frac{1}{n^\alpha} - \frac{1}{N^\alpha} \right) \quad (97)$$

And if $\alpha > 1$

$$\left| \sum_{k=n+1}^{+\infty} \frac{\cos(\beta \ln(k))}{k^\alpha} - \int_{n+1}^{+\infty} e(t) dt \right| \leq \frac{K_1}{\alpha} \left(\frac{1}{n^\alpha} \right) \quad (98)$$

$$\left| \sum_{k=n+1}^{+\infty} \frac{\sin(\beta \ln(k))}{k^\alpha} - \int_{n+1}^{+\infty} f(t) dt \right| \leq \frac{K_2}{\alpha} \left(\frac{1}{n^\alpha} \right) \quad (99)$$

$$\left| \sum_{k=n+1}^{+\infty} \frac{a + b (\sin \beta \ln(k)) + c \cos(\beta \ln(k))}{n^\alpha} - \int_{n+1}^{+\infty} g(t) dt \right| \leq \frac{K_3}{\alpha} \left(\frac{1}{n^\alpha} \right) \quad (100)$$

where $K_1 > 0$, $K_2 > 0$, $K_3 > 0$ and E , F , G and H are the primitive functions of e , f , g are defined as:

$$E(x) = \left(\frac{\beta \sin(\beta \ln(x)) + (1 - \alpha) \cos(\beta \ln(x))}{\beta^2 + (1 - \alpha)^2} \right) \frac{1}{(x)^{\alpha-1}} \quad (101)$$

$$F(x) = \left(\frac{(1 - \alpha) \sin(\beta \ln(x)) - \beta \cos(\beta \ln(x))}{\beta^2 + (1 - \alpha)^2} \right) \frac{1}{(x)^{\alpha-1}} \quad (102)$$

$$G(x) = \left(\frac{a}{1-\alpha} + \frac{(c\beta + b(1-\alpha)) \sin(\beta \ln(x)) + (c(1-\alpha) - b\beta) \cos(\beta \ln(x))}{\beta^2 + (1-\alpha)^2} \right) \frac{1}{(x)^{\alpha-1}} \quad (103)$$

Proof. For each $n \geq 1$: We have

$$x_n = \frac{\cos(\beta \ln(n))}{n^\alpha} - \int_n^{n+1} \frac{\cos(\beta \ln(x)) dx}{x^\alpha} \quad (104)$$

$$= \int_0^1 \left(\frac{\cos(\beta \ln(n))}{n^\alpha} - \frac{\cos(\beta \ln(n+t))}{(n+t)^\alpha} \right) dt \quad (105)$$

For each $t \in (0, 1]$ we can apply the Mean Value Theorem to find $c \in (0, 1)$ such that:

$$\left| \frac{\cos(\beta \ln(n))}{n^\alpha} - \frac{\cos(\beta \ln(n+t))}{(n+t)^\alpha} \right| = |e'(n+c)|t \quad (106)$$

where

And its first derivative

$$e'(n+t) = -\frac{\beta \sin(\beta \ln(n+t)) + \alpha \cos(\beta \ln(n+t))}{(n+t)^{\alpha+1}} \quad (107)$$

We have $\alpha > 0$, therefore:

$$|e'(n+c)| \leq \frac{K}{n^{\alpha+1}} \quad (108)$$

Where $K = \alpha + |\beta|$. We have also $\alpha + 1 > 1$, then the serie $\sum_{n \geq 1} x_n$ is converging absolutely. Therefore

$$\sum_{n=N_0+1}^N |x_n| \leq K \sum_{n=N_0+1}^N \frac{1}{n^{\alpha+1}} \quad (109)$$

$$\leq K \sum_{n=N_0+1}^N \int_{n-1}^n \frac{dx}{x^{\alpha+1}} \quad (110)$$

$$\leq \frac{K}{1-\alpha-1} (- (N_0)^{1-\alpha-1} + N^{1-\alpha-1}) \quad (111)$$

$$\leq \frac{K}{\alpha} \left(\frac{1}{(N_0)^\alpha} - \frac{1}{(N)^\alpha} \right) \quad (112)$$

And

$$\sum_{n=N_0+1}^{+\infty} |x_n| \leq K \sum_{n=N_0+1}^{+\infty} \frac{1}{n^{\alpha+1}} \quad (113)$$

$$\leq \frac{K}{\alpha} \frac{1}{(N_0)^\alpha} \quad (114)$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} x_n \right| \leq \frac{K}{\alpha} \frac{1}{(N_0)^\alpha} \quad (115)$$

And

$$\left| \sum_{n=N_0+1}^{+\infty} \frac{\cos(\beta \ln(n))}{n^\alpha} - \int_{N_0+1}^{+\infty} \frac{\cos(\beta \ln(x))dx}{x^\alpha} \right| \leq \frac{K}{\alpha(N_0)^\alpha} \quad (116)$$

To calculate the integral we need to calculate the primitive function of e . Let's denote this later E . We have E defined as following:

$$E(x) = \left(\frac{\beta \sin(\beta \ln(x)) + (1 - \alpha) \cos(\beta \ln(x))}{\beta^2 + (1 - \alpha)^2} \right) \frac{1}{(x)^{\alpha-1}} \quad (117)$$

We apply the same technique for the serie y_n and the function f .

For each $n \geq 1$: We have

$$y_n = \frac{\sin(\beta \ln(n))}{n^\alpha} - \int_n^{n+1} \frac{\sin(\beta \ln(x))dx}{x^\alpha} \quad (118)$$

$$= \int_0^1 \left(\frac{\sin(\beta \ln(n))}{n^\alpha} - \frac{\sin(\beta \ln(n+t))dx}{(n+t)^\alpha} \right) dt \quad (119)$$

For each $t \in (0, 1]$ we can apply the Mean Value Theorem to find $c \in (0, 1)$ such that:

$$\left| \frac{\sin(\beta \ln(n))}{n^\alpha} - \frac{\sin(\beta \ln(n+t))}{(n+t)^\alpha} \right| = |f'(n+c)|t \quad (120)$$

where

And its first derivative

$$f'(n+t) = \frac{-\alpha \sin(\beta \ln(n+t)) + \alpha \cos(\beta \ln(n+t))}{(n+t)^{\alpha+1}} \quad (121)$$

We have $\alpha > 0$, therefore:

$$|f'(n+c)| \leq \frac{K}{n^{\alpha+1}} \quad (122)$$

Where $K = \alpha + |\beta|$. We have also $\alpha + 1 > 1$, then the serie $\sum_{n \geq 1} y_n$ is converging absolutely. Therefore

$$\sum_{n=N_0+1}^N |y_n| \leq K \sum_{n=N_0+1}^N \frac{1}{n^{\alpha+1}} \quad (123)$$

$$\leq K \sum_{n=N_0+1}^N \int_{n-1}^n \frac{dx}{x^{\alpha+1}} \quad (124)$$

$$\leq \frac{K}{1-\alpha-1} \left(-(N_0)^{1-\alpha-1} + N^{1-\alpha-1} \right) \quad (125)$$

$$\leq \frac{K}{\alpha} \left(\frac{1}{(N_0)^\alpha} - \frac{1}{(N)^\alpha} \right) \quad (126)$$

And

$$\sum_{n=N_0+1}^{+\infty} |y_n| \leq K \sum_{n=N_0+1}^{+\infty} \frac{1}{n^{\alpha+1}} \quad (127)$$

$$\leq \frac{K}{\alpha} \frac{1}{(N_0)^\alpha} \quad (128)$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} y_n \right| \leq \frac{K}{\alpha} \frac{1}{(N_0)^\alpha} \quad (129)$$

And

$$\left| \sum_{n=N_0+1}^{+\infty} \frac{\sin(\beta \ln(n))}{n^\alpha} - \int_{N_0+1}^{+\infty} \frac{\sin(\beta \ln(x)) dx}{x^\alpha} \right| \leq \frac{K}{\alpha(N_0)^\alpha} \quad (130)$$

To calculate the integral we need to calculate the primitive function of f . Let's denote this later F . We have F defined as following:

$$F(x) = \left(\frac{(1-\alpha) \sin(\beta \ln(x)) - \beta \cos(\beta \ln(x))}{\beta^2 + (1-\alpha)^2} \right) \frac{1}{(x)^{\alpha-1}} \quad (131)$$

We apply the same technique for the serie z_n and the function g .

For each $n \geq 1$: We have

$$z_n = \frac{a + b \sin(\beta \ln(n)) + c \cos(\beta \ln(n))}{n^\alpha} - \int_n^{n+1} \frac{a + b \sin(\beta \ln(x)) + c \cos(\beta \ln(x)) dx}{x^\alpha} \quad (132)$$

$$= \int_0^1 \left(\frac{a + b \sin(\beta \ln(n+t)) + c \cos(\beta \ln(n+t))}{n^\alpha} - \frac{a + b \sin(\beta \ln(n+t)) + c \cos(\beta \ln(n+t)) dx}{(n+t)^\alpha} \right) dt \quad (133)$$

For each $t \in (0, 1]$ we can apply the Mean Value Theorem to find $c \in (0, 1)$ such that:

$$\left| \frac{a + b \sin(\beta \ln(n)) + c \cos(\beta \ln(n))}{n^\alpha} - \frac{a + b \sin(\beta \ln(n+t)) + c \cos(\beta \ln(n+t))}{(n+t)^\alpha} \right| = |g'(n+c)| t \quad (134)$$

where

And its first derivative

$$g'(n+t) = \frac{-a - (\alpha b + \beta c) \sin(\beta \ln(n+t)) + (-\alpha c + \beta b) \cos(\beta \ln(n+t))}{(n+t)^{\alpha+1}} \quad (135)$$

We have $\alpha > 0$, therefore:

$$|g'(n+c)| \leq \frac{K}{n^{\alpha+1}} \quad (136)$$

Where $K = |a| + |\alpha b + \beta c| + |-\alpha c + \beta b|$. We have also $\alpha + 1 > 1$, then the serie $\sum_{n \geq 1} z_n$ is converging absolutely. Therefore

$$\sum_{n=N_0+1}^N |z_n| \leq K \sum_{n=N_0+1}^N \frac{1}{n^{\alpha+1}} \quad (137)$$

$$\leq K \sum_{n=N_0+1}^N \int_{n-1}^n \frac{dx}{x^{\alpha+1}} \quad (138)$$

$$\leq \frac{K}{1-\alpha-1} (- (N_0)^{1-\alpha-1} + N^{1-\alpha-1}) \quad (139)$$

$$\leq \frac{K}{\alpha} \left(\frac{1}{(N_0)^\alpha} - \frac{1}{(N)^\alpha} \right) \quad (140)$$

And

$$\sum_{n=N_0+1}^{+\infty} |z_n| \leq K \sum_{n=N_0+1}^{+\infty} \frac{1}{n^{\alpha+1}} \quad (141)$$

$$\leq \frac{K}{\alpha} \frac{1}{(N_0)^\alpha} \quad (142)$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} z_n \right| \leq \frac{K}{\alpha} \frac{1}{(N_0)^\alpha} \quad (143)$$

And

$$\left| \sum_{n=N_0+1}^{+\infty} \frac{a + b \sin(\beta \ln(n)) + c \cos(\beta \ln(n))}{n^\alpha} - \int_{N_0+1}^{+\infty} \frac{a + b \sin(\beta \ln(x)) + c \cos(\beta \ln(x)) dx}{x^\alpha} \right| \leq \frac{K}{\alpha(N_0)^\alpha} \quad (144)$$

To calculate the integral we need to calculate the primitive function of g .
Let's denote this later G . We have G defined as following:

$$G(x) = \left(\frac{a}{1-\alpha} + \frac{(c\beta + b(1-\alpha)) \sin(\beta \ln(x)) + (c(1-\alpha) - b\beta) \cos(\beta \ln(x))}{\beta^2 + (1-\alpha)^2} \right) \frac{1}{(x)^{\alpha-1}} \quad (145)$$

□

Lemma 3.2. *Let's define the functions:*

$$p(t) = \frac{\cos^2(b_0 \ln(t))}{(t)^{2a_0}} \quad (146)$$

$$q(t) = \frac{\sin^2(b_0 \ln(t))}{(t)^{2a_0}} \quad (147)$$

$$r(t) = \frac{\left(\cos(b_0 \ln(t)) - 2^{1-a_0} \cos(b_0 \ln(2t))\right)^2}{(t)^{2a_0}} \quad (148)$$

$$v(t) = \frac{\left(\sin(b_0 \ln(t)) - 2^{1-a_0} \sin(b_0 \ln(2t))\right)^2}{(t)^{2a_0}} \quad (149)$$

$$e_1(t) = \frac{\cos(b_0 \ln(t)) - 2^{1-a_0} \cos(b_0 \ln(2t))}{(t)^{a_0}} \quad (150)$$

$$f_1(t) = \frac{\sin(b_0 \ln(t)) - 2^{1-a_0} \sin(b_0 \ln(2t))}{(t)^{a_0}} \quad (151)$$

Let's define the sequences:

$$p_n = \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} - \int_n^{n+1} p(t) \quad (152)$$

$$q_n = \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} - \int_n^{n+1} q(t) \quad (153)$$

$$r_n = \frac{\left(\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n))\right)^2}{n^{2a_0}} - \int_n^{n+1} r(t) \quad (154)$$

$$v_n = \frac{\left(\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n))\right)^2}{n^{2a_0}} - \int_n^{n+1} v(t) \quad (155)$$

$$e_{1,n} = \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n))}{n^{a_0}} - \int_n^{n+1} e_1(t) \quad (156)$$

$$f_{1,n} = \frac{\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n))}{n^{a_0}} - \int_n^{n+1} f_1(t) \quad (157)$$

Therefore, the series $\sum_{k \geq 1} p_n$, $\sum_{k \geq 1} q_n$, $\sum_{k \geq 1} r_n$ and $\sum_{k \geq 1} v_n$ are

converging absolutely and we have the following inequalities:

$$\left| \sum_{k=n+1}^N \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} - \int_{n+1}^N p(t) dt \right| \leq \frac{K_1}{2a_0} \left(\frac{1}{n^{2a_0}} - \frac{1}{N^{2a_0}} \right) \quad (158)$$

$$\left| \sum_{k=n+1}^N \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} - \int_{n+1}^N q(t) dt \right| \leq \frac{K_2}{2a_0} \left(\frac{1}{n^{2a_0}} - \frac{1}{N^{2a_0}} \right) \quad (159)$$

$$\left| \sum_{k=n+1}^N \frac{\left(\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n)) \right)^2}{n^{2a_0}} - \int_{n+1}^N r(t) dt \right| \leq \frac{K_3}{2a_0} \left(\frac{1}{n^{2a_0}} - \frac{1}{N^{2a_0}} \right) \quad (160)$$

$$\left| \sum_{k=n+1}^N \frac{\left(\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n)) \right)^2}{n^{2a_0}} - \int_{n+1}^N v(t) dt \right| \leq \frac{K_4}{2a_0} \left(\frac{1}{n^{2a_0}} - \frac{1}{N^{2a_0}} \right) \quad (161)$$

$$\left| \sum_{k=n+1}^N \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n))}{n^{a_0}} - \int_{n+1}^N e_1(t) dt \right| \leq \frac{K_5}{a_0} \left(\frac{1}{n^{a_0}} - \frac{1}{N^{a_0}} \right) \quad (162)$$

$$\left| \sum_{k=n+1}^N \frac{\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n))}{n^{a_0}} - \int_{n+1}^N f_1(t) dt \right| \leq \frac{K_6}{a_0} \left(\frac{1}{n^{a_0}} - \frac{1}{N^{a_0}} \right) \quad (163)$$

And if $a_0 > 1$ and $2a_0 > 1$

$$\left| \sum_{k=n+1}^{+\infty} \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} - \int_{n+1}^{+\infty} p(t) dt \right| \leq \frac{K_1}{2a_0} \frac{1}{n^{2a_0}} \quad (164)$$

$$\left| \sum_{k=n+1}^{+\infty} \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} - \int_{n+1}^{+\infty} q(t) dt \right| \leq \frac{K_2}{2a_0} \frac{1}{n^{2a_0}} \quad (165)$$

$$\left| \sum_{k=n+1}^{+\infty} \frac{\left(\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n)) \right)^2}{n^{2a_0}} - \int_{n+1}^{+\infty} r(t) dt \right| \leq \frac{K_3}{2a_0} \frac{1}{n^{2a_0}} \quad (166)$$

$$\left| \sum_{k=n+1}^{+\infty} \frac{\left(\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n)) \right)^2}{n^{2a_0}} - \int_{n+1}^{+\infty} v(t) dt \right| \leq \frac{K_4}{2a_0} \frac{1}{n^{2a_0}} \quad (167)$$

$$\left| \sum_{k=n+1}^{+\infty} \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n))}{n^{a_0}} - \int_{n+1}^{+\infty} e_1(t) dt \right| \leq \frac{K_5}{a_0} \frac{1}{n^{a_0}} \quad (168)$$

$$\left| \sum_{k=n+1}^{+\infty} \frac{\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n))}{n^{a_0}} - \int_{n+1}^{+\infty} f_1(t) dt \right| \leq \frac{K_6}{a_0} \frac{1}{n^{a_0}} \quad (169)$$

where $K_1 > 0$, $K_2 > 0$, $K_3 > 0$, $K_4 > 0$, $K_5 > 0$, $K_6 > 0$ and P , Q , R , V , E_1 and F_1 are the primitive functions of p , q , r , v , e_1 and f_1 are defined as:

$$P(x) = \frac{1}{2} \left(\frac{1}{(1-2a_0)} + \frac{2b_0 \sin(2b_0 \ln(x)) + (1-2a_0) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (170)$$

$$Q(x) = \frac{1}{2} \left(\frac{1}{(1-2a_0)} - \frac{2b_0 \sin(2b_0 \ln(x)) + (1-2a_0) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (171)$$

$$R(x) = \left(\frac{a}{1-2a_0} + \frac{(2b_0b_1 - c_1(1-2a_0)) \sin(2b_0 \ln(x)) + (b_1(1-2a_0) + 2b_0c_1) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (172)$$

$$V(x) = \left(\frac{a}{1-2a_0} + \frac{(2b_0b_2 - c_2(1-2a_0)) \sin(2b_0 \ln(x)) + (b_2(1-2a_0) + 2b_0c_2) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (173)$$

$$E_1(x) = \left(\frac{(a_3b_0 + b_3(1-a_0)) \sin(b_0 \ln(x)) + (a_3(1-a_0) - b_3b_0) \cos(b_0 \ln(x))}{(b_0)^2 + (1-a_0)^2} \right) \frac{1}{(x)^{a_0-1}} \quad (174)$$

$$F_1(x) = \left(\frac{(-b_3b_0 + a_3(1-a_0)) \sin(b_0 \ln(x)) + (-b_3(1-a_0) - a_3b_0) \cos(b_0 \ln(x))}{(b_0)^2 + (1-a_0)^2} \right) \frac{1}{(x)^{a_0-1}} \quad (175)$$

And

$$a_1 = \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (176)$$

$$b_1 = \frac{1}{2} + 2^{1-2a_0} \cos(2b_0 \ln(2)) - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (177)$$

$$c_1 = 2^{1-2a_0} \sin(2b_0 \ln(2)) + 2^{1-a_0} \sin(b_0 \ln(2)) \quad (178)$$

$$a_2 = \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0} \cos(b_0 \ln(2)) = a_1 \quad (179)$$

$$b_2 = -\frac{1}{2} - 2^{1-2a_0} \cos(2b_0 \ln(2)) + 2^{1-a_0} \cos(b_0 \ln(2)) = -b_1 \quad (180)$$

$$c_2 = -2^{1-2a_0} \sin(2b_0 \ln(2)) \quad (181)$$

$$a_3 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (182)$$

$$b_3 = 2^{1-a_0} \sin(b_0 \ln(2)) \quad (183)$$

Proof. We will make use of the previous lemma 3.1. we have the trigonometrical relation: $\cos^2(x) = \frac{1+\cos(2x)}{2}$ and $\sin^2(x) = \frac{1-\cos(2x)}{2}$.

• *P*

So we can write the following: $p(t) = \frac{1+\cos(2t)}{2t^{2a_0}}$ and applying the lemma 3.1 with $\alpha = 2a_0$, $\beta = 2b_0$, $a = \frac{1}{2}$, $c = \frac{1}{2}$ and $b = 0$.

Therefore the primitive function of p , P defined as following:

$$P(x) = \frac{1}{2} \left(\frac{1}{(1-2a_0)} + \frac{2b_0 \sin(2b_0 \ln(x)) + (1-2a_0) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (184)$$

• *Q*

Also we can write the following: $q(t) = \frac{1-\cos(2t)}{2t^{2a_0}}$ and applying the lemma 3.1 with $\alpha = 2a_0$, $\beta = 2b_0$, $a = \frac{1}{2}$, $c = -\frac{1}{2}$ and $b = 0$.

and $q(t) = \frac{1-\cos(2t)}{2t}$.

Therefore the primitive function of q , Q defined as following:

$$Q(x) = \frac{1}{2} \left(\frac{1}{(1-2a_0)} - \frac{2b_0 \sin(2b_0 \ln(x)) + (1-2a_0) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (185)$$

- R

We write the function r as follows:

$$r(t) = \frac{a_1 + b_1 \cos(2b_0 \ln(t)) - c_1 \sin(2b_0(\ln((t))))}{(t)^{2a_0}} \quad (186)$$

Where

$$a_1 = \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (187)$$

$$b_1 = \frac{1}{2} + 2^{1-2a_0} \cos(2b_0 \ln(2)) - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (188)$$

$$c_1 = 2^{1-2a_0} \sin(2b_0 \ln(2)) + 2^{1-a_0} \sin(b_0 \ln(2)) \quad (189)$$

And we apply the lemma 3.1 with $\alpha = 2a_0$, $\beta = 2b_0$, $a = a_1$, $b = -c_1$ and $c = b_1$.

$$R(x) = \left(\frac{a}{1-2a_0} + \frac{(2b_0b_1 - c_1(1-2a_0)) \sin(2b_0 \ln(x)) + (b_1(1-2a_0) + 2b_0c_1) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (190)$$

- V

We write the function v as follows:

$$v(t) = \frac{a_2 + b_2 \cos(2b_0 \ln(n+t)) - c_2 \sin(2b_0(\ln((n+t))))}{(n+t)^{2a_0}} \quad (191)$$

Where

$$a_2 = \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (192)$$

$$b_2 = -\frac{1}{2} - 2^{1-2a_0} \cos(2b_0 \ln(2)) + 2^{1-a_0} \cos(b_0 \ln(2)) \quad (193)$$

$$c_2 = -2^{1-2a_0} \sin(2b_0 \ln(2)) \quad (194)$$

And we apply the lemma 3.1 with $\alpha = 2a_0$, $\beta = 2b_0$, $a = a_2$, $b = -c_2$ and $c = b_2$.

$$V(x) = \left(\frac{a}{1-2a_0} + \frac{(2b_0b_2 - c_2(1-2a_0)) \sin(2b_0 \ln(x)) + (b_2(1-2a_0) + 2b_0c_2) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (195)$$

- E_1

We write the function e_1 as follows:

$$e_1(t) = \frac{a_3 \cos(b_0 \ln(n+t)) + b_3 \sin(b_0(\ln((n+t))))}{(n+t)^{a_0}} \quad (196)$$

Where

$$a_3 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (197)$$

$$b_3 = 2^{1-a_0} \sin(b_0 \ln(2)) \quad (198)$$

And we apply the lemma 3.1 with $\alpha = a_0$, $\beta = b_0$, $a = 0$, $b = b_3$ and $c = a_3$.

$$E_1(x) = \left(\frac{\left(a_3 b_0 + b_3(1 - a_0) \right) \sin(b_0 \ln(x)) + \left(a_3(1 - a_0) - b_3 b_0 \right) \cos(b_0 \ln(x))}{(b_0)^2 + (1 - a_0)^2} \right) \frac{1}{(x)^{a_0 - 1}} \quad (199)$$

• F_1

We write the function f_1 as follows:

$$f_1(t) = \frac{a_4 \cos(b_0 \ln(t)) + b_4 \sin(b_0 \ln(t))}{(t)^{a_0}} \quad (200)$$

Where

$$a_4 = 1 - 2^{1 - a_0} \cos(b_0 \ln(2)) \quad (201)$$

$$b_4 = -2^{1 - a_0} \sin(b_0 \ln(2)) \quad (202)$$

And we apply the lemma 3.1 with $\alpha = a_0$, $\beta = b_0$, $a = 0$, $b = a_4$ and $c = b_4$.

$$F_1(x) = \left(\frac{\left(b_4 b_0 + a_4(1 - a_0) \right) \sin(b_0 \ln(x)) + \left(b_4(1 - a_0) - a_4 b_0 \right) \cos(b_0 \ln(x))}{(b_0)^2 + (1 - a_0)^2} \right) \frac{1}{(x)^{a_0 - 1}} \quad (203)$$

□

Lemma 3.3. For each $0 < \epsilon < 1$, For each N_0 , there exist a $N \geq N_0$ such that $|\cos(2b_0 \ln(N)) - 1| \leq \epsilon$ and $|\sin(2b_0 \ln(N))| \leq \epsilon$.

Proof. To prove this lemma, let's define the sequence (k_n) of the integer part of $\frac{2b_0 \ln(n)}{2\pi}$ as following: For each $n \geq 1$:

$$k_n = \lfloor \frac{2b_0 \ln(n)}{2\pi} \rfloor \quad (204)$$

By definition of the integer part we have for each $n \geq 1$:

$$k_n \leq \frac{2b_0 \ln(n)}{2\pi} < k_n + 1 \quad (205)$$

Let's $0 < \epsilon < 1$. Let's $N_0 \geq 1$ an integer. To prove the lemma, we need to prove that there is exist an $N \geq N_0$ such that:

$$0 \leq 2b_0 \ln(N) - 2\pi k_N \leq \epsilon \quad (206)$$

Or

$$2\pi - \epsilon \leq 2b_0 \ln(N) - 2\pi k_N \leq 2\pi \quad (207)$$

If such N cannot exist then for each $N \geq N_0$ we have:

$$\epsilon \leq 2b_0 \ln(N) - 2\pi k_N \leq 2\pi - \epsilon \quad (208)$$

Then

$$\frac{\epsilon}{2\pi} - 1 + \frac{b_0 \ln(N)}{\pi} \leq k_N \leq \frac{b_0 \ln(N)}{\pi} - \frac{\epsilon}{2\pi} \quad (209)$$

Then

$$-\frac{b_0 \ln(N)}{\pi} + \frac{\epsilon}{2\pi} \leq -k_N \leq -\frac{\epsilon}{2\pi} + 1 - \frac{b_0 \ln(N)}{\pi} \quad (210)$$

Then

$$\frac{\epsilon}{2\pi} - 1 + \frac{b_0 \ln(N+1)}{\pi} \leq k_{N+1} \leq \frac{b_0 \ln(N+1)}{\pi} - \frac{\epsilon}{2\pi} \quad (211)$$

Then

$$\frac{\epsilon}{\pi} - 1 + \frac{b_0 \ln(1 + \frac{1}{N})}{\pi} \leq k_{N+1} - k_N \leq \frac{b_0 \ln(1 + \frac{1}{N})}{\pi} + 1 - \frac{\epsilon}{\pi} \quad (212)$$

We have the $\lim_{N \rightarrow +\infty} \frac{b_0}{\pi} \ln(1 + \frac{1}{N}) = 0$ Then there exist an integer $N_1 \geq N_0$ such that $0 < \frac{1}{N} \ln(1 + \frac{1}{N}) < \frac{\epsilon}{10}$
Hence for each $N \geq N_0$:

$$\frac{\epsilon}{\pi} - 1 \leq k_{N+1} - k_N \leq 1 - \left(\frac{\epsilon}{\pi} - \frac{\pi}{10}\right) \quad (213)$$

We have $0 < 1 - \left(\frac{\epsilon}{\pi} - \frac{\pi}{10}\right) < 1$ and $-1 < \frac{\epsilon}{\pi} - 1 < 0$, therefore

$$-1 < k_{N+1} - k_N < 1 - \left(\frac{\epsilon}{\pi} - \frac{\pi}{10}\right) \quad (214)$$

We have $k_{N+1} - k_N$ is an integer so $k_{N+1} = k_N$ for each $N \geq N_1$.
Therefore: For each $N \geq N_1$:

$$k_N = k_{N_1} \quad (215)$$

However from the equation (208), we have the $\lim_{N \rightarrow +\infty} K_N = +\infty$.
Which is a contraction with the result in (214).
Therefore there exist $N \geq N_0$ such that:

$$0 \leq 2b_0 \ln(N) - 2\pi k_N \leq \epsilon \quad (216)$$

Or

$$2\pi - \epsilon \leq 2b_0 \ln(N) - 2\pi k_N \leq 2\pi \quad (217)$$

So for such N we have:

$$\cos(\epsilon) \leq \cos(2b_0 \ln(N)) \leq 1 \quad (218)$$

And

$$-\sin(\epsilon) \leq \sin(2b_0 \ln(N)) \leq \sin(\epsilon) \quad (219)$$

For $x \geq 0$ we have:

$$1 - \frac{x^2}{2} \leq \cos(x) \leq 1 \quad (220)$$

$$x - \frac{x^3}{6} \leq \sin(x) \leq x \quad (221)$$

Therefore:

$$1 - \frac{\epsilon^2}{2} \leq \cos(2b_0 \ln(N)) \leq 1 \quad (222)$$

And

$$-\epsilon \leq \sin(2b_0 \ln(N)) \leq \epsilon \quad (223)$$

Hence

$$|\cos(2b_0 \ln(N)) - 1| \leq \frac{\epsilon^2}{2} \leq \epsilon \quad (224)$$

And

$$|\sin(2b_0 \ln(N))| \leq \epsilon \quad (225)$$

□

Lemma 3.4. *For each $0 < \epsilon < 1$, For each N_0 , there exist a $N \geq N_0$ such that $|\sin(2b_0 \ln(N)) - 1| \leq \epsilon$ and $|\cos(2b_0 \ln(N))| \leq \epsilon$.*

Proof. To prove this lemma, let's define the sequence (kk_n) of the integer part of $\frac{2b_0 \ln(n)}{2\pi}$ as following: For each $n \geq 1$:

$$kk_n = \lfloor \frac{2b_0 \ln(n)}{2\pi} - \frac{1}{4} \rfloor \quad (226)$$

By definition of the integer part we have for each $n \geq 1$:

$$kk_n \leq \frac{2b_0 \ln(n)}{2\pi} - \frac{1}{4} < kk_n + 1 \quad (227)$$

Let's $0 < \epsilon < 1$. Let's $N_0 \geq 1$ an integer. To prove the lemma, we need to prove that there is exist an $N \geq N_0$ such that:

$$\frac{\pi}{2} - \epsilon \leq 2b_0 \ln(N) - 2\pi k_N \leq \frac{\pi}{2} + \epsilon \quad (228)$$

If such N cannot exist then for each $N \geq N_0$ we have:

$$\frac{\pi}{2} + \epsilon < 2b_0 \ln(N) - 2\pi k k_N < 2\pi + \frac{\pi}{2} - \epsilon \quad (229)$$

Then

$$-\frac{b_0 \ln(N)}{\pi} + \frac{\epsilon}{2\pi} + \frac{1}{8} < -k k_N < -\frac{\epsilon}{2\pi} + 1 + \frac{1}{8} - \frac{b_0 \ln(N)}{\pi} \quad (230)$$

Then

$$\frac{b_0 \ln(N)}{\pi} + \frac{\epsilon}{2\pi} - \frac{1}{8} - 1 < k k_N < -\frac{\epsilon}{2\pi} - \frac{1}{8} + \frac{b_0 \ln(N)}{\pi} \quad (231)$$

Then

$$\frac{b_0 \ln(N+1)}{\pi} + \frac{\epsilon}{2\pi} - \frac{1}{8} - 1 < k k_{N+1} < -\frac{\epsilon}{2\pi} - \frac{1}{8} + \frac{b_0 \ln(N+1)}{\pi} \quad (232)$$

Then

$$\frac{\epsilon}{\pi} - 1 + \frac{b_0 \ln(1 + \frac{1}{N})}{\pi} < k k_{N+1} - k k_N < \frac{b_0 \ln(1 + \frac{1}{N})}{\pi} + 1 - \frac{\epsilon}{\pi} \quad (233)$$

We have the $\lim_{N \rightarrow +\infty} \frac{b_0}{\pi} \ln(1 + \frac{1}{N}) = 0$ Then there exist an integer $N_1 \geq N_0$ such that $0 < \frac{1}{N} \ln(1 + \frac{1}{N}) < \frac{\epsilon}{10}$
Hence for each $N \geq N_0$:

$$\frac{\epsilon}{\pi} - 1 < k k_{N+1} - k k_N < 1 - (\frac{\epsilon}{\pi} - \frac{\pi}{10}) \quad (234)$$

We have $0 < 1 - (\frac{\epsilon}{\pi} - \frac{\pi}{10}) < 1$ and $-1 < \frac{\epsilon}{\pi} - 1 < 0$, therefore

$$-1 < k k_{N+1} - k k_N < 1 - (\frac{\epsilon}{\pi} - \frac{\pi}{10}) \quad (235)$$

We have $k k_{N+1} - k k_N$ is an integer so $k k_{N+1} = k k_N$ for each $N \geq N_1$.
Therefore: For each $N \geq N_1$:

$$k k_N = k k_{N_1} \quad (236)$$

However from the equation (230), we have the $\lim_{N \rightarrow +\infty} k k_N = +\infty$.
Which is a contraction with the result in (235).
Therefore there exist $N \geq N_0$ such that:

$$\frac{\pi}{2} - \epsilon \leq 2b_0 \ln(N) - 2\pi k_N \leq \frac{\pi}{2} + \epsilon \quad (237)$$

So for such N we have:

$$\cos(\frac{\pi}{2} + \epsilon) \leq \cos(2b_0 \ln(N)) \leq \cos(\frac{\pi}{2} - \epsilon) \quad (238)$$

And

$$\sin\left(\frac{\pi}{2} - \epsilon\right) \leq \sin(2b_0 \ln(N)) \leq 1 \quad (239)$$

Therefore:

$$-\sin(\epsilon) \leq \cos(2b_0 \ln(N)) \leq \sin(\epsilon) \quad (240)$$

And

$$\cos(\epsilon) \leq \sin(2b_0 \ln(N)) \leq 1 \quad (241)$$

For $x \geq 0$ we have:

$$1 - \frac{x^2}{2} \leq \cos(x) \leq 1 \quad (242)$$

$$x - \frac{x^3}{6} \leq \sin(x) \leq x \quad (243)$$

Therefore

$$-\epsilon \leq \cos(2b_0 \ln(N)) \leq \epsilon \quad (244)$$

And

$$1 - \frac{\epsilon^2}{2} \leq \sin(2b_0 \ln(N)) \leq 1 \quad (245)$$

Hence

$$|\sin(2b_0 \ln(N)) - 1| \leq \frac{\epsilon^2}{2} \leq \epsilon \quad (246)$$

And

$$|\cos(2b_0 \ln(N))| \leq \epsilon \quad (247)$$

□

Remark. Using the same technique above we can prove the generalized version of the lemmas 3.3 and 3.4: For each $x_0 \in [0, 2\pi]$, for each $0 < \epsilon < 1$, for each N_0 , there exist a $N \geq N_0$ such that $|\cos(2b_0 \ln(N)) - \cos(x_0)| \leq \epsilon$ and $|\sin(2b_0 \ln(N)) - \sin(x_0)| \leq \epsilon$.

Lemma 3.5. Let a be a real number such that $0 < a < 1$. Let's b a real number. There exists two constants $K_1 \geq 0$ and $K_2 \geq 0$ such that for $N \geq 1$ we have: $\left| \sum_{n=1}^N \frac{\cos(b \ln(n))}{n^a} \right| \leq K_1 + K_2 N^{1-a}$ and $\left| \sum_{n=1}^N \frac{\sin(b \ln(n))}{n^a} \right| \leq K_1 + K_2 N^{1-a}$.

Proof. Let's $N \geq 1$. The functions $|\sin|$ and $|\cos|$ are bounded by 1.

$$\left| \sum_{n=1}^N \frac{\cos(b \ln(n))}{n^a} \right| \leq \sum_{n=1}^N \frac{|\cos(b \ln(n))|}{n^a} \quad (248)$$

$$\leq 1 + \sum_{n=2}^N \frac{1}{n^a} \quad (249)$$

$$\leq 1 + \sum_{n=2}^N \int_{n-1}^n \frac{dx}{x^a} \quad (250)$$

$$\leq 1 + \frac{1}{1-a} (N^{1-a} - 1) \quad (251)$$

$$\leq \frac{a}{1-a} + \frac{N^{1-a}}{1-a} \quad (252)$$

We just take $K_1 = \frac{a}{1-a}$ and $K_2 = \frac{1}{1-a}$.

We can use the same method for the sinus sum:

$$\left| \sum_{n=1}^N \frac{\sin(b \ln(n))}{n^a} \right| \leq \sum_{n=1}^N \frac{|\sin(b \ln(n))|}{n^a} \quad (253)$$

$$\leq 1 + \sum_{n=2}^N \frac{1}{n^a} \quad (254)$$

$$\leq 1 + \sum_{n=2}^N \int_{n-1}^n \frac{dx}{x^a} \quad (255)$$

$$\leq 1 + \frac{1}{1-a} (N^{1-a} - 1) \quad (256)$$

$$\leq \frac{a}{1-a} + \frac{N^{1-a}}{1-a} \quad (257)$$

We just take $K_1 = \frac{a}{1-a}$ and $K_2 = \frac{1}{1-a}$.

□

Lemma 3.6. *There is exist two constants $K_1 \geq 0$ and $K_2 \geq 0$ such that for $n \geq 1$ and $N \geq 1$ we have: $|\bar{A}_{n,N}| \leq K_1 + K_2 n^{1-a_0}$.*

Proof. We will make use of the previous lemma 3.5. Let's $N \geq 1$. Let's $n \geq 1$.

Case: $n \leq \frac{N}{2}$

$$\bar{A}_{n,N} = \sum_{k=1}^n \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n))}{k^{a_0}} \quad (258)$$

$$= \sum_{k=1}^n \frac{(1 - 2^{1-a_0} \cos(b_0 \ln(2))) \cos(b_0 \ln(k)) + (2^{1-a_0} \sin(b_0 \ln(2))) \sin(b_0 \ln(k))}{k^{a_0}} \quad (259)$$

$$= (1 - 2^{1-a_0} \cos(b_0 \ln(2))) \sum_{k=1}^n \frac{\cos(b_0 \ln(k))}{k^{a_0}} + (2^{1-a_0} \sin(b_0 \ln(2))) \sum_{k=1}^n \frac{\sin(b_0 \ln(k))}{k^{a_0}} \quad (260)$$

$$(261)$$

Therefore using the previous lemma:

$$|\bar{A}_{n,N}| \leq |(1 - 2^{1-a_0} \cos(b_0 \ln(2)))| \left| \sum_{k=1}^n \frac{\cos(b_0 \ln(k))}{k^{a_0}} \right| \quad (262)$$

$$+ |(2^{1-a_0} \sin(b_0 \ln(2)))| \left| \sum_{k=1}^n \frac{\sin(b_0 \ln(k))}{k^{a_0}} \right| \quad (263)$$

$$\leq |(1 - 2^{1-a_0} \cos(b_0 \ln(2)))| (K_1 + K_2 n^{1-a_0}) + |(2^{1-a_0} \sin(b_0 \ln(2)))| (K_1 + K_2 n^{1-a_0}) \quad (264)$$

$$\leq \left(|(1 - 2^{1-a_0} \cos(b_0 \ln(2)))| + |(2^{1-a_0} \sin(b_0 \ln(2)))| \right) (K_1 + K_2 n^{1-a_0}) \quad (265)$$

And this prove the case.

Case: $n > \frac{N}{2}$ We have:

$$\sum_{k=\frac{N}{2}+1}^n \frac{\cos(b_0 \ln(n))}{k^{a_0}} = \sum_{k=1}^n \frac{\cos(b_0 \ln(n))}{k^{a_0}} - \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(n))}{k^{a_0}} \quad (266)$$

$$(267)$$

And

$$\bar{A}_{n,N} = \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n))}{k^{a_0}} + \sum_{k=\frac{N}{2}+1}^n \frac{\cos(b_0 \ln(n))}{k^{a_0}} \quad (268)$$

$$(269)$$

Therefore using the first case of $n \leq \frac{N}{2}$ and the previous lemma:

$$\left| \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n))}{k^{a_0}} \right| \leq \left(K_1 + K_2 \left(\frac{N}{2}\right)^{1-a_0} \right) \quad (270)$$

$$\leq (K_1 + K_2 n^{1-a_0}), \text{ as } \left(\frac{N}{2}\right)^{1-a_0} \leq n^{1-a_0} \quad (271)$$

And

$$\left| \sum_{k=1}^n \frac{\cos(b_0 \ln(n))}{k^{a_0}} - \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(n))}{k^{a_0}} \right| \leq \left| \sum_{k=1}^n \frac{\cos(b_0 \ln(n))}{k^{a_0}} \right| + \left| \sum_{k=1}^{\frac{N}{2}} \frac{\cos(b_0 \ln(n))}{k^{a_0}} \right| \quad (272)$$

$$\leq (\bar{K}_1 + \bar{K}_2 n^{1-a_0}) + \left(K_1 + K_2 \left(\frac{N}{2}\right)^{1-a_0} \right) \quad (273)$$

$$\leq K_1 + \bar{K}_1 + (K_2 + \bar{K}_2) n^{1-a_0}, \text{ as } \left(\frac{N}{2}\right)^{1-a_0} \leq n^{1-a_0} \quad (274)$$

Therefore

$$|\bar{A}_{n,N}| \leq K_1 + \bar{K}_1 + (K_2 + \bar{K}_2) n^{1-a_0} \quad (275)$$

Which proves our case. \square

Lemma 3.7. *There exist two constants $K_1 \geq 0$ and $K_2 \geq 0$ such that for $n \geq 1$ and $N \geq 1$ we have: $|\overline{B}_{n,N}| \leq K_1 + K_2 n^{1-a_0}$.*

Proof. We use the same logic as in the case of $\overline{B}_{n,N}$. We will make use of the previous lemma 3.5. Let's $N \geq 1$. Let's $n \geq 1$.

Case: $n \leq \frac{N}{2}$

$$\begin{aligned} \overline{B}_{n,N} &= \sum_{k=1}^n \frac{\sin(b_0 \ln(k)) - 2^{1-a_0} \sin(b_0(\ln(2k)))}{k^{a_0}} \quad (276) \\ &= \sum_{k=1}^n \frac{(1 - 2^{1-a_0} \cos(b_0 \ln(2))) \sin(b_0 \ln(k)) - (2^{1-a_0} \sin(b_0 \ln(2))) \cos(b_0 \ln(k))}{k^{a_0}} \quad (277) \\ &= (1 - 2^{1-a_0} \cos(b_0 \ln(2))) \sum_{k=1}^n \frac{\sin(b_0 \ln(k))}{k^{a_0}} - (2^{1-a_0} \sin(b_0 \ln(2))) \sum_{k=1}^n \frac{\cos(b_0 \ln(k))}{k^{a_0}} \quad (278) \end{aligned} \quad (279)$$

Therefore using the previous lemma:

$$|\overline{B}_{n,N}| \leq |(1 - 2^{1-a_0} \cos(b_0 \ln(2)))| \left| \sum_{k=1}^n \frac{\sin(b_0 \ln(k))}{k^{a_0}} \right| \quad (280)$$

$$+ |(2^{1-a_0} \sin(b_0 \ln(2)))| \left| \sum_{k=1}^n \frac{\cos(b_0 \ln(k))}{k^{a_0}} \right| \quad (281)$$

$$\leq |(1 - 2^{1-a_0} \cos(b_0 \ln(2)))| (K_1 + K_2 n^{1-a_0}) + |(2^{1-a_0} \sin(b_0 \ln(2)))| (K_1 + K_2 n^{1-a_0}) \quad (282)$$

$$\leq (|(1 - 2^{1-a_0} \cos(b_0 \ln(2)))| + |(2^{1-a_0} \sin(b_0 \ln(2)))|) (K_1 + K_2 n^{1-a_0}) \quad (283)$$

And this proves the case.

Case: $n > \frac{N}{2}$ We have:

$$\sum_{k=\frac{N}{2}+1}^n \frac{\sin(b_0 \ln(n))}{k^{a_0}} = \sum_{k=1}^n \frac{\sin(b_0 \ln(n))}{k^{a_0}} - \sum_{k=1}^{\frac{N}{2}} \frac{\sin(b_0 \ln(n))}{k^{a_0}} \quad (284)$$

$$(285)$$

And

$$\overline{A}_{n,N} = \sum_{k=1}^{\frac{N}{2}} \frac{\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0(\ln(2n)))}{k^{a_0}} + \sum_{k=\frac{N}{2}+1}^n \frac{\sin(b_0 \ln(n))}{k^{a_0}} \quad (286)$$

$$(287)$$

Therefore using the first case of $n \leq \frac{N}{2}$ and the previous lemma:

$$\left| \sum_{k=1}^{\frac{N}{2}} \frac{\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0(\ln(2n)))}{k^{a_0}} \right| \leq (K_1 + K_2 (\frac{N}{2})^{1-a_0}) \quad (288)$$

$$\leq (K_1 + K_2 n^{1-a_0}), \text{ as } (\frac{N}{2})^{1-a_0} \leq n^{1-a_0} \quad (289)$$

And

$$\begin{aligned} \left| \sum_{k=1}^n \frac{\sin(b_0 \ln(n))}{k^{a_0}} - \sum_{k=1}^{\frac{N}{2}} \frac{\sin(b_0 \ln(n))}{k^{a_0}} \right| &\leq \left| \sum_{k=1}^n \frac{\sin(b_0 \ln(n))}{k^{a_0}} \right| + \left| \sum_{k=1}^{\frac{N}{2}} \frac{\sin(b_0 \ln(n))}{k^{a_0}} \right| \quad (290) \\ &\leq \left(\bar{K}_1 + \bar{K}_2 n^{1-a_0} \right) + \left(K_1 + K_2 \left(\frac{N}{2} \right)^{1-a_0} \right) \quad (291) \\ &\leq K_1 + \bar{K}_1 + (K_2 + \bar{K}_2) n^{1-a_0}, \text{ as } \left(\frac{N}{2} \right)^{1-a_0} \leq n^{1-a_0} \quad (292) \end{aligned}$$

Therefore

$$|\bar{A}_{n,N}| \leq K_1 + \bar{K}_1 + (K_2 + \bar{K}_2) n^{1-a_0} \quad (293)$$

Which proves our case. \square

Lemma 3.8. *Let's $0 < \alpha < 1$, β , a , b and c real numbers. Let's define the functions:*

$$g(t) = \frac{a + b \sin(\beta \ln(t)) + c \cos(\beta \ln(t))}{(t)^{\alpha-1}} \quad (294)$$

Therefore, the limit of the sequence $g(n+1) - g(n)$ is zero:

$$\lim_{n \rightarrow +\infty} g(n+1) - g(n) = 0 \quad (295)$$

And

$$g(n+1) - g(n) = a \frac{1-\alpha}{n^\alpha} + \frac{(bcc_n(\alpha-1, \beta) + cc_n(\alpha-1, \beta))}{n^\alpha} + \mathcal{O}\left(\frac{1}{n^{\alpha+1}}\right) \quad (296)$$

Proof. To prove the lemma we need the asymptotic expansion of $\frac{\sin(\beta \ln(t))}{(n)^{\alpha-1}}$ and $\frac{\cos(\beta \ln(t))}{(n)^{\alpha-1}}$. As we will see in lemma 3.13 and 3.16

$$\frac{\cos(\beta \ln(n+1))}{(1+n)^{\alpha-1}} = \frac{\cos(\beta \ln(n))}{n^{\alpha-1}} + \frac{c_n(\alpha-1, \beta)}{n^\alpha} + \mathcal{O}\left(\frac{1}{n^{\alpha+1}}\right) \quad (297)$$

$$\frac{\sin(\beta \ln(n+1))}{(1+n)^{\alpha-1}} = \frac{\sin(\beta \ln(n))}{n^{\alpha-1}} + \frac{cc_n(\alpha-1, \beta)}{n^\alpha} + \mathcal{O}\left(\frac{1}{n^{\alpha+1}}\right) \quad (298)$$

Therefore

$$g(n+1) - g(n) = a \left(\frac{1}{(n+1)^{\alpha-1}} - \frac{1}{(n)^{\alpha-1}} \right) + b \left(\frac{\sin(\beta \ln(n+1))}{(1+n)^{\alpha-1}} - \frac{\sin(\beta \ln(n))}{n^{\alpha-1}} \right) \quad (299)$$

$$+ c \left(\frac{\cos(\beta \ln(n+1))}{(1+n)^{\alpha-1}} - \frac{\cos(\beta \ln(n))}{n^{\alpha-1}} \right) \quad (300)$$

$$= a \frac{1-\alpha}{n^\alpha} + \frac{(bcc_n(\alpha-1, \beta) + cc_n(\alpha-1, \beta))}{n^\alpha} + \mathcal{O}\left(\frac{1}{n^{\alpha+1}}\right) \quad (301)$$

We have $\alpha > 0$, therefore $\lim_{n \rightarrow +\infty} g(n+1) - g(n) = 0$ \square

Lemma 3.9. For a large n , N and N_0 such $N_0 \leq n < n+1 \leq \frac{N}{2}$. We can write $\bar{A}_{n,N} = E_1(n) + \gamma_n$ such that $\lim_{n \rightarrow +\infty} \gamma_n = 0$ and $E_1(n) = n^{1-a_0} \left(X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)) \right)$ where X_1 and X_2 two real constants defined as follows:

$$X_1 = \frac{(a_3 b_0 + b_3(1-a_0))}{(b_0)^2 + (1-a_0)^2} \quad (302)$$

$$X_2 = \frac{(a_3(1-a_0) - b_3 b_0)}{(b_0)^2 + (1-a_0)^2} \quad (303)$$

$$a_3 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (304)$$

$$b_3 = 2^{1-a_0} \sin(b_0 \ln(2)) \quad (305)$$

Proof. We have $\lim_{N \rightarrow +\infty} \bar{A}_{N,N} = 0$. Let's $\epsilon > 0$. Let's N_0 be such that for each $N \geq N_0$: $|\bar{A}_{N,N}| \leq \epsilon$. Let's take n , N and N_0 such $N_0 \leq n < n+1 \leq \frac{N}{2}$. We have $|\bar{A}_{N,N}| \leq \epsilon$.

Therefore:

We have the notation $dd_n = (\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n)))$.

We are going to use the lemma 3.2. Let's develop further:

$$\bar{A}_{N,N} - \bar{A}_{n,N} = \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} + \sum_{k=\frac{N}{2}+1}^N \frac{\cos(b_0 \ln(k))}{k^{a_0}} \quad (306)$$

$$= \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} e_1(t) dt + \sum_{k=\frac{N}{2}+1}^N \frac{\cos(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^N \int_k^{k+1} e(t) dt \quad (307)$$

$$+ \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} e_1(t) dt + \sum_{k=\frac{N}{2}+1}^N \int_k^{k+1} e(t) dt \quad (308)$$

$$= \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} e_1(t) dt + \sum_{k=\frac{N}{2}+1}^N \frac{\cos(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^N \int_k^{k+1} e(t) dt \quad (309)$$

$$+ \int_{n+1}^N e_1(t) dt + \int_{\frac{N}{2}+1}^N e(t) dt \quad (310)$$

$$= -E_1(n+1) + \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} e_1(t) dt + \sum_{k=\frac{N}{2}+1}^N \frac{\cos(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^N \int_k^{k+1} e(t) dt \quad (311)$$

$$+ E_1(N) + E(N) - E\left(\frac{N}{2} + 1\right) \quad (312)$$

Therefore from the lemma 3.2:

$$|\bar{A}_{n,N} - E_1(n+1)| \leq |\bar{A}_{N,N}| + \left| \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} h(t) dt \right| \quad (313)$$

$$+ \left| \sum_{k=\frac{N}{2}+1}^N \frac{\cos(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^N \int_k^{k+1} g(t) dt \right| + \left| E_1(N) + E(N) - E\left(\frac{N}{2} + 1\right) \right| \quad (314)$$

$$\leq \epsilon + \frac{K_5}{a_0} \left(\frac{1}{n^{a_0}} - \frac{1}{\left(\frac{N}{2}\right)^{a_0}} \right) + \frac{K_1}{a_0} \left(\frac{1}{\left(\frac{N}{2}\right)^{a_0}} - \frac{1}{(N^{a_0})} \right) + \left| E_1(N) + E(N) - E\left(\frac{N}{2} + 1\right) \right| \quad (315)$$

$$(316)$$

We have $N_0 \leq n < n+1 \leq \frac{N}{2}$. So

$$|\bar{A}_{n,N} - E_1(n+1)| \leq \epsilon + \frac{K_5}{a_0} \left(\frac{1}{n^{a_0}} + \frac{1}{n^{a_0}} \right) + \frac{K_1}{a_0} \left(\frac{1}{n^{a_0}} + \frac{1}{n^{a_0}} \right) + \left| E_1(N) + E(N) - E\left(\frac{N}{2} + 1\right) \right| \quad (317)$$

$$\leq \epsilon + 2 \frac{K_1 + K_5}{a_0} \frac{1}{n^{a_0}} + \left| E_1(N) + E(N) - E\left(\frac{N}{2} + 1\right) \right| \quad (318)$$

Quick calculation gives us: $E_1(N) + E(N) - E\left(\frac{N}{2}\right) = 0$. We apply the lemma 3.8 to the function E , we have $\lim_{N \rightarrow +\infty} E\left(\frac{N}{2} + 1\right) - E\left(\frac{N}{2}\right) = 0$. Therefore: $\lim_{N \rightarrow +\infty} E_1(N) + E(N) - E\left(\frac{N}{2}\right) = 0$.

$$|\bar{A}_{n,N} - E_1(n+1)| \leq \epsilon + 2 \frac{K_1 + K_5}{a_0} \frac{1}{n^{a_0}} + \left| E_1(N) + E(N) - E\left(\frac{N}{2} + 1\right) \right| \quad (319)$$

And this for every $\epsilon > 0$. $a_0 > 0$ so we also have the $\lim_{n \rightarrow +\infty} \frac{1}{n^{a_0}} = 0$. Therefore $\lim_{n \rightarrow +\infty} |\bar{A}_{n,N} - E_1(n+1)| = 0$. Let's define the sequence γ_n as the following: $\gamma_n = \bar{A}_{n,N} - E_1(n)$.

We apply the lemma 3.8 to the function E_1 , we have the $\lim_{n \rightarrow +\infty} E_1(n+1) - E_1(n) = 0$. Therefore $\lim_{n \rightarrow +\infty} \gamma_n = 0$.

The expression of E_1 is also given by the lemma 3.2. And this proves the lemma.

Remark. *There is a simple way of proving this lemma. As we have $\bar{A}_{n,N} = \bar{A}_{n,2n}$ for $n \leq \frac{N}{2}$, we can redo the steps above with $\bar{A}_{n,2n}$ and develop $\bar{A}_{2n,2n} - \bar{A}_{n,2n}$. The steps above will be much easier and more straightforward.*

□

Lemma 3.10. *Let's define the functions:*

$$e_3(t) = \frac{\left(\cos(b_0 \ln(t)) - 2^{1-a_0} \cos(b_0 \ln(2t)) \right) \left(X_1 \sin(b_0 \ln(t)) + X_2 \cos(b_0 \ln(t)) \right)}{(t)^{2a_0-1}} \quad (320)$$

Let's define the sequence:

$$e_{3,n} = e_3(n) - \int_n^{n+1} dt e_3(t) \quad (321)$$

$$(322)$$

Therefore, the serie $\sum_{k \geq 1} e_{3,n}$ are converging absolutely and we have the following inequalities:

$$\left| \sum_{k=n+1}^N e_{3,n} - \int_{n+1}^N e_3(t) dt \right| \leq \frac{K}{2a_0} \left(\frac{1}{n^{2a_0}} - \frac{1}{N^{2a_0}} \right) \quad (323)$$

And if $a_0 > 1$

$$\left| \sum_{k=n+1}^{+\infty} e_{3,n} - \int_{n+1}^{+\infty} e_3(t) dt \right| \leq \frac{K}{2a_0} \frac{1}{n^{2a_0}} \quad (324)$$

where $K > 0$ and E_3 is the primitive function of e_3 is defined as:

$$E_3(x) = \left(\frac{\alpha_3}{2-2a_0} + \frac{(2b_0\beta_3 + \gamma_3(2-2a_0)) \sin(2b_0 \ln(x)) + (\beta_3(2-2a_0) - 2b_0\gamma_3) \cos(2b_0 \ln(x))}{(2b_0)^2 + (2-2a_0)^2} \right) \frac{1}{(x)^{2a_0-2}} \quad (325)$$

Proof. We are going to use the lemmas 3.1 and 3.2 defined above. We will also use their notations. We can write the functions $e_3(t)$ as the following:

$$e_3(t) (t)^{2a_0-1} = \left(\cos(b_0 \ln(t)) - 2^{1-a_0} \cos(b_0 \ln(2t)) \right) \left(X_1 \sin(b_0 \ln(t)) + X_2 \cos(b_0 \ln(t)) \right) \quad (326)$$

$$= \left(Z_1 \sin(b_0 \ln t) + Z_2 \cos(b_0 \ln(t)) \right) \left(X_1 \sin(b_0 \ln t) + X_2 \cos(b_0 \ln(t)) \right) \quad (327)$$

$$= \frac{X_1 Z_1 + X_2 Z_2}{2} + \frac{X_1 Z_2 + X_2 Z_1}{2} \sin(2b_0 \ln(t)) + \frac{X_2 Z_2 - X_1 Z_1}{2} \cos(2b_0 \ln t) \quad (328)$$

Where

$$Z_1 = 2^{1-a_0} \sin(b_0 \ln(2)) = b_3 \quad (329)$$

$$Z_2 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) = a_3 \quad (330)$$

Therefore we can write $e_3(t)$

$$e_3(t) (t)^{2a_0-1} = \left(\frac{X_1 Z_1 + X_2 Z_2}{2} \right) + \left(\frac{X_1 Z_2 + X_2 Z_1}{2} \right) \sin(2b_0 \ln(t)) \quad (331)$$

$$+ \left(\frac{X_2 Z_2 - X_1 Z_1}{2} \right) \cos(2b_0 \ln t) \quad (332)$$

$$= \alpha_3 + \beta_3 \sin(2b_0 \ln(t)) + \gamma_3 \cos(2b_0 \ln t) \quad (333)$$

Where

$$\alpha_3 = \frac{X_1 b_3 + X_2 a_3}{2} \quad (334)$$

$$\beta_3 = \frac{X_1 a_3 + X_2 b_3}{2} \quad (335)$$

$$\gamma_3 = \frac{X_2 a_3 - X_1 b_3}{2} \quad (336)$$

$$X_1 = \frac{a_3 b_0 + b_3(1-a_0)}{(b_0)^2 + (1-a_0)^2} \quad (337)$$

$$X_2 = \frac{a_3(1-a_0) - b_3 b_0}{(b_0)^2 + (1-a_0)^2} \quad (338)$$

And we apply the lemma 3.1 with $\alpha = 2a_0 - 1$, $\beta = 2b_0$, $a = \alpha_3$, $b = \beta_3$ and $c = \gamma_3$.

$$E_3(x) = \left(\frac{\alpha_3}{2 - 2a_0} + \frac{(2b_0\beta_3 + \gamma_3(2 - 2a_0)) \sin(2b_0 \ln(x)) + (\beta_3(2 - 2a_0) - 2b_0\gamma_3) \cos(2b_0 \ln(x))}{(2b_0)^2 + (2 - 2a_0)^2} \right) \frac{1}{(x)^{2a_0-2}} \quad (339)$$

□

Lemma 3.11. For a large n , N and N_0 such $N_0 \leq n < n+1 \leq \frac{N}{2}$ We can write $\bar{B}_{n,N} = F_1(n) + \gamma_n$ such that $\lim_{n \rightarrow +\infty} \gamma_n = 0$ and $F_1(n) = n^{1-a_0} (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)))$ where X_1 and X_2 two real constants defined as follows:

$$X_1 = \frac{(-b_3b_0 + a_3(1 - a_0))}{(b_0)^2 + (1 - a_0)^2} \quad (340)$$

$$X_2 = -\frac{(b_3(1 - a_0) + a_3b_0)}{(b_0)^2 + (1 - a_0)^2} \quad (341)$$

$$a_3 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (342)$$

$$b_3 = 2^{1-a_0} \sin(b_0 \ln(2)) \quad (343)$$

Proof. We have $\lim_{N \rightarrow +\infty} \bar{B}_{N,N} = 0$. Let's $\epsilon > 0$. Let's N_0 be such that for each $N \geq N_0$: $|\bar{B}_{N,N}| \leq \epsilon$. Let's take n , N and N_0 such $N_0 \leq n < n+1 \leq \frac{N}{2}$. We have $|\bar{B}_{N,N}| \leq \epsilon$.

Therefore:

We have the notation $dd_n = (\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0(\ln(2n))))$.

We are going to use the lemma 3.2. Let's develop further:

$$\bar{B}_{N,N} - \bar{B}_{n,N} = \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} + \sum_{k=\frac{N}{2}+1}^N \frac{\sin(b_0 \ln(k))}{k^{a_0}} \quad (344)$$

$$= \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} f_1(t) dt + \sum_{k=\frac{N}{2}+1}^N \frac{\sin(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^N \int_k^{k+1} f(t) dt \quad (345)$$

$$+ \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} f_1(t) dt + \sum_{k=\frac{N}{2}+1}^N \int_k^{k+1} f(t) dt \quad (346)$$

$$= \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} f_1(t) dt + \sum_{k=\frac{N}{2}+1}^N \frac{\cos(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^N \int_k^{k+1} f(t) dt \quad (347)$$

$$+ \int_{n+1}^N f_1(t) dt + \int_{\frac{N}{2}+1}^N f(t) dt \quad (348)$$

$$= -F_1(n+1) + \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} f_1(t) dt + \sum_{k=\frac{N}{2}+1}^N \frac{\sin(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^N \int_k^{k+1} f(t) dt \quad (349)$$

$$+ F_1(N) + F(N) - F(\frac{N}{2} + 1) \quad (350)$$

Therefore from the lemma 3.2:

$$|\overline{B}_{n,N} - F_1(n+1)| \leq |\overline{B}_{N,N}| + \left| \sum_{k=n+1}^{\frac{N}{2}} \frac{dd_k}{k^{a_0}} - \sum_{k=n+1}^{\frac{N}{2}} \int_k^{k+1} h(t) dt \right| \quad (351)$$

$$+ \left| \sum_{k=\frac{N}{2}+1}^N \frac{\sin(b_0 \ln(k))}{k^{a_0}} - \sum_{k=\frac{N}{2}+1}^N \int_k^{k+1} g(t) dt \right| + \left| F_1(N) + F(N) - F\left(\frac{N}{2} + 1\right) \right| \quad (352)$$

$$\leq \epsilon + \frac{K_5}{a_0} \left(\frac{1}{n^{a_0}} - \frac{1}{\left(\frac{N}{2}\right)^{a_0}} \right) + \frac{K_1}{a_0} \left(\frac{1}{\left(\frac{N}{2}\right)^{a_0}} - \frac{1}{(N^{a_0})} \right) + \left| F_1(N) + F(N) - F\left(\frac{N}{2} + 1\right) \right| \quad (353)$$

$$(354)$$

We have $N_0 \leq n < n+1 \leq \frac{N}{2}$. So

$$|\overline{B}_{n,N} - F_1(n+1)| \leq \epsilon + \frac{K_5}{a_0} \left(\frac{1}{n^{a_0}} + \frac{1}{n^{a_0}} \right) + \frac{K_1}{a_0} \left(\frac{1}{n^{a_0}} + \frac{1}{n^{a_0}} \right) + \left| F_1(N) + F(N) - F\left(\frac{N}{2} + 1\right) \right| \quad (355)$$

$$\leq \epsilon + 2 \frac{K_1 + K_5}{a_0} \frac{1}{n^{a_0}} + \left| F_1(N) + F(N) - F\left(\frac{N}{2} + 1\right) \right| \quad (356)$$

Quick calculation gives us: $F_1(N) + F(N) - F\left(\frac{N}{2}\right) = 0$. We apply the lemma 3.8 to the function F , we have $\lim_{N \rightarrow +\infty} F\left(\frac{N}{2} + 1\right) - F\left(\frac{N}{2}\right) = 0$. Therefore: $\lim_{N \rightarrow +\infty} F_1(N) + F(N) - F\left(\frac{N}{2}\right) = 0$.

$$|\overline{B}_{n,N} - F_1(n+1)| \leq \epsilon + 2 \frac{K_1 + K_5}{a_0} \frac{1}{n^{a_0}} + \left| F_1(N) + F(N) - F\left(\frac{N}{2} + 1\right) \right| \quad (357)$$

And this for every $\epsilon > 0$. $a_0 > 0$ so we also have the $\lim_{n \rightarrow +\infty} \frac{1}{n^{a_0}} = 0$. Therefore $\lim_{n \rightarrow +\infty} |\overline{B}_{n,N} - F_1(n+1)| = 0$. Let's define the sequence γ_n as the following: $\gamma_n = \overline{B}_{n,N} - F_1(n)$.

We apply the lemma 3.8 to the function F_1 , we have the $\lim_{n \rightarrow +\infty} F_1(n+1) - F_1(n) = 0$. Therefore $\lim_{n \rightarrow +\infty} \gamma_n = 0$.

The expression of F_1 is also given by the lemma 3.2. And this proves the lemma.

Remark. *There is a simple way of proving this lemma. As we have $\overline{B}_{n,N} = \overline{B}_{n,2n}$ for $n \leq \frac{N}{2}$, we can redo the steps above with $\overline{B}_{n,2n}$ and develop $\overline{B}_{2n,2n} - \overline{B}_{n,2n}$. The steps will be much easier and more straightforward.*

□

Lemma 3.12. *Let's define the functions:*

$$f_3(t) = \frac{(\sin(b_0 \ln(t)) - 2^{1-a_0} \sin(b_0 \ln(2t))) (X_1 \sin(b_0 \ln(t)) + X_2 \cos(b_0 \ln(t)))}{(t)^{2a_0-1}} \quad (358)$$

Let's define the sequence:

$$f_{3,n} = f_3(n) - \int_n^{n+1} dt f_3(t) \quad (359)$$

$$(360)$$

Therefore, the serie $\sum_{k \geq 1} f_{3,n}$ are converging absolutely and we have the following inequalities:

$$\left| \sum_{k=n+1}^N f_{3,n} - \int_{n+1}^N f_3(t) dt \right| \leq \frac{K}{2a_0} \left(\frac{1}{n^{2a_0}} - \frac{1}{N^{2a_0}} \right) \quad (361)$$

And if $a_0 > 1$

$$\left| \sum_{k=n+1}^{+\infty} f_{3,n} - \int_{n+1}^{+\infty} f_3(t) dt \right| \leq \frac{K}{2a_0} \frac{1}{n^{2a_0}} \quad (362)$$

where $K > 0$ and F_3 is the primitive function of f_3 is defined as:

$$F_3(x) = \left(\frac{\alpha_3}{2-2a_0} + (2b_0\beta_3 + \gamma_3(2-2a_0)) \sin(2b_0 \ln(x)) + (\beta_3(2-2a_0) - 2b_0\gamma_3) \cos(2b_0 \ln(x)) \right) \frac{1}{(x)^{2a_0-2}} \quad (363)$$

Proof. We are going to use the lemmas 3.1 and 3.2 defined above. We will also use their notations. We can write the functions $f_3(t)$ as the following:

$$f_3(t) (t)^{2a_0-1} = \left(\sin(b_0 \ln(t)) - 2^{1-a_0} \sin(b_0 \ln(2t)) \right) \left(X_1 \sin(b_0 \ln(t)) + X_2 \cos(b_0 \ln(t)) \right) \quad (364)$$

$$= \left(Z_1 \sin(b_0 \ln t) + Z_2 \cos(b_0 \ln(t)) \right) \left(X_1 \sin(b_0 \ln t) + X_2 \cos(b_0 \ln(t)) \right) \quad (365)$$

$$= \frac{X_1 Z_1 + X_2 Z_2}{2} + \frac{X_1 Z_2 + X_2 Z_1}{2} \sin(2b_0 \ln(t)) + \frac{X_2 Z_2 - X_1 Z_1}{2} \cos(2b_0 \ln t) \quad (366)$$

Where

$$Z_1 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) = a_3 \quad (367)$$

$$Z_2 = -2^{1-a_0} \sin(b_0 \ln(2)) = -b_3 \quad (368)$$

Therefore we can write $e_3(t)$

$$f_3(t) (t)^{2a_0-1} = \left(\frac{X_1 Z_1 + X_2 Z_2}{2} \right) + \left(\frac{X_1 Z_2 + X_2 Z_1}{2} \right) \sin(2b_0 \ln(t)) \quad (369)$$

$$+ \left(\frac{X_2 Z_2 - X_1 Z_1}{2} \right) \cos(2b_0 \ln t) \quad (370)$$

$$= \alpha_3 + \beta_3 \sin(2b_0 \ln(t)) + \gamma_3 \cos(2b_0 \ln t) \quad (371)$$

Where

$$\alpha_3 = \frac{X_1 a_3 - X_2 b_3}{2} \quad (372)$$

$$\beta_3 = \frac{-X_1 b_3 + X_2 a_3}{2} \quad (373)$$

$$\gamma_3 = \frac{-X_2 b_3 - X_1 a_3}{2} \quad (374)$$

$$X_1 = \frac{(a_3(1-a_0) - b_3 b_0)}{(b_0)^2 + (1-a_0)^2} \quad (375)$$

$$X_2 = \frac{-b_3(1-a_0) + a_3 b_0}{(b_0)^2 + (1-a_0)^2} \quad (376)$$

$$(377)$$

And we apply the lemma 3.1 with $\alpha = 2a_0 - 1$, $\beta = 2b_0$, $a = \alpha_2$, $b = \beta_2$ and $c = \gamma_2$.

$$F_3(x) = \left(\frac{\alpha_3}{2a_0 - 2} + \frac{\left(2b_0\beta_3 + \gamma_3(2 - 2a_0) \right) \sin(2b_0 \ln(x)) + \left(\beta_3(2 - 2a_0) - 2b_0\gamma_3 \right) \cos(2b_0 \ln(x))}{(2b_0)^2 + (2 - 2a_0)^2} \right) \frac{1}{(x)^{2a_0 - 2}} \quad (378)$$

□

Lemma 3.13. *If $\frac{1}{2} < a_0 \leq 1$ the sequence $(\sum_{n=1}^N \bar{F}_{n,N})_{N \geq 1}$ diverges.*

Proof. We want to prove that:

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \bar{F}_{n,N} = -\infty \quad (379)$$

Let's n and N be integers:

Case One: $\frac{N}{2} < n \leq N$ We are going to use asymptotic expansions:
We have

$$\bar{A}_{n+1,N} = \bar{A}_{n,N} + \frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} \quad (380)$$

So $\bar{F}_{n+1,N}$ can be written as follows:

$$\bar{F}_{n+1,N} = \frac{\cos(b_0 \ln(n+1)) \bar{A}_{n,N}}{(n+1)^{a_0}} + \frac{\cos^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} \quad (381)$$

So $\bar{F}_{n+1,N} - \bar{F}_{n,N}$ can be written as

$$\bar{F}_{n+1,N} - \bar{F}_{n,N} = \left(\frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right) \bar{A}_{n,N} + \frac{\cos^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} \quad (382)$$

$$\bar{F}_{n+1,N} - \bar{F}_{n,N} = \left(\bar{U}_{n+1,N} - \bar{U}_{n,N} \right) \bar{A}_{n,N} + \frac{\cos^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} \quad (383)$$

Let's do now the asymptotic expansion of $\bar{U}_{n+1,N} - \bar{U}_{n,N}$. For this we need the asymptotic expansion of $\cos(b_0 \ln(n+1))$.

$$\cos(b_0 \ln(n+1)) = \cos\left(b_0 \ln(n) + b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (384)$$

$$= \cos(b_0 \ln(n)) \cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) - \sin(b_0 \ln(n)) \sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (385)$$

we have the asymptotic expansion of $\ln\left(1 + \frac{1}{n}\right)$ in order two as follow:

$$\ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (386)$$

Using the asymptotic expansion of the functions sin and cos that I will spare you the details here, we have

$$\cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) = 1 + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (387)$$

And

$$\sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) = \frac{b_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (388)$$

Hence

$$\cos(b_0 \ln(n+1)) = \cos(b_0 \ln(n)) - \frac{b_0}{n} \sin(b_0 \ln(n)) + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (389)$$

Also the Asymptotic expansion of $\frac{1}{(1+n)^{a_0}}$:

$$\frac{1}{(1+n)^{a_0}} = \frac{1}{n^{a_0}} \left(1 - \frac{a_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \quad (390)$$

Hence

$$\frac{\cos(b_0 \ln(n+1))}{(1+n)^{a_0}} = \frac{\cos(b_0 \ln(n))}{n^{a_0}} - \frac{b_0 \sin(b_0 \ln(n)) + a_0 \cos(b_0 \ln(n))}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (391)$$

And the squared version of the above equation

$$\frac{\cos^2(b_0 \ln(n+1))}{(1+n)^{2a_0}} = \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} - 2 \cos(b_0 \ln(n)) \frac{b_0 \sin(b_0 \ln(n)) + a_0 \cos(b_0 \ln(n))}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (392)$$

And

$$\bar{U}_{n+1,N} - \bar{U}_{n,N} = -\frac{b_0 \sin(b_0 \ln(n)) + a_0 \cos(b_0 \ln(n))}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (393)$$

So the asymptotic expansion of $\bar{F}_{n+1,N} - \bar{F}_{n,N}$ is as follows:

$$\bar{F}_{n+1,N} - \bar{F}_{n,N} = \frac{\cos^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} - \frac{b_0 \sin(b_0 \ln(n)) + a_0 \cos(b_0 \ln(n))}{n^{a_0+1}} \bar{A}_{n,N} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (394)$$

For the sake of simplifying the notation, let define the sequence $\bar{C}_{n,N}$ as follows: For each $n \geq 1$ we have:

$$c_n = -\left(b_0 \sin(b_0 \ln(n)) + a_0 \cos(b_0 \ln(n))\right) \quad (395)$$

$$\bar{C}_{n,N} = c_n \bar{A}_{n,N} \quad (396)$$

And

$$D_n = 2c_n \cos(b_0 \ln(n)) \quad (397)$$

The functions $x \mapsto \cos(x)$ and $x \mapsto \sin(x)$ are by their nature bounded so (c_n) and (D_n) are also bounded.

By definition of \mathcal{O} , we know that there is exist a bounded sequence (ε_n) and there is exist a number N_0 such that: For each $n \geq N_0$ we have

$$\bar{F}_{n+1,N} - \bar{F}_{n,N} = \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + \frac{\bar{C}_{n,N}}{n^{a_0+1}} + \frac{\bar{D}_{n,N}}{n^{2a_0+1}} + \frac{\bar{\varepsilon}_{n,N}}{n^{a_0+2}} \quad (398)$$

Where

$$c_n = -(b_0 \sin(b_0 \ln(n)) + a_0 \cos(b_0 \ln(n))) \quad (399)$$

$$\bar{C}_{n,N} = c_n \bar{A}_{n,N} \quad (400)$$

$$\bar{D}_{n,N} = 2 \cos(b_0 \ln(n)) c_n \quad (401)$$

Let's now study the case of when $n+1 \leq \frac{N}{2}$:

Case Two: $n+1 \leq \frac{N}{2}$

$$\bar{F}_{n+1,N} - \bar{F}_{n,N} = \bar{U}_{n+1,N} \bar{A}_{n+1,N} - \bar{U}_{n,N} \bar{A}_{n,N} \quad (402)$$

$$= (\bar{U}_{n+1,N} - \bar{U}_{n,N}) \bar{A}_{n,N} + (\bar{U}_{n+1,N})^2 \quad (403)$$

And

$$\bar{U}_{n+1,N} - \bar{U}_{n,N} = \frac{\cos(b_0 \ln(n+1))}{(1+n)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}} \quad (404)$$

$$-2^{1-a_0} \left\{ \frac{\cos(b_0 \ln 2(n+1))}{(1+n)^{a_0}} - \frac{\cos(b_0 \ln(2n))}{n^{a_0}} \right\} \quad (405)$$

We develop the $\cos(b_0 \ln 2(n+1))$ and $\cos(b_0 \ln 2(n))$ to get:

$$\frac{\cos(b_0 \ln 2(n+1))}{(1+n)^{a_0}} = \cos b_0 \ln(2) \left\{ \frac{\cos(b_0 \ln(n))}{n^{a_0}} + \frac{c_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \right\} \quad (406)$$

$$- \sin b_0 \ln(2) \left\{ \frac{\sin(b_0 \ln(n))}{n^{a_0}} + \frac{cc_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \right\} \quad (407)$$

$$= \frac{\cos(b_0 \ln 2(n))}{(n)^{a_0}} + \frac{c_n \cos b_0 \ln(2) - cc_n \sin b_0 \ln(2)}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (408)$$

Therefore:

$$\bar{U}_{n+1,N} - \bar{U}_{n,N} = \frac{c_n - 2^{1-a_0} (c_n \cos b_0 \ln(2) - cc_n \sin b_0 \ln(2))}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (409)$$

And

$$(\bar{U}_{n+1,N})^2 = (\bar{U}_{n,N})^2 + 2 \frac{ccc_n dd_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (410)$$

Where

$$ccc_n = c_n - 2^{1-a_0} (c_n \cos b_0 \ln(2) - cc_n \sin b_0 \ln(2)) \quad (411)$$

$$dd_n = \cos(b_0 \ln(n) - 2^{1-a_0} \cos(b_0 \ln(2n))) \quad (412)$$

$$\bar{D}_{n,N} = 2ccc_n dd_n \quad (413)$$

$$\bar{C}_{n,N} = ccc_n \bar{A}_{n,N} \quad (414)$$

And

$$\bar{F}_{n+1,N} - \bar{F}_{n,N} = \left\{ \frac{(\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n)))^2}{n^{2a_0}} + \frac{\bar{C}_{n,N}}{n^{a_0+1}} \right. \quad (415)$$

$$\left. + \frac{\bar{D}_{n,N}}{n^{2a_0+1}} + \frac{\bar{\varepsilon}_{n,N}}{n^{a_0+2}} \right\} \quad (416)$$

With the sequence $\bar{\varepsilon}_{n,N}$ is bounded.

From the lemma 3.9 above we have: $\bar{A}_{n,N} = E_1(n) + \gamma_n$ such that $\lim_{n \rightarrow +\infty} \gamma_n = 0$ and $E_1(n) = n^{1-a_0} (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)))$ where X_1 and X_2 two real constants. Therefore

$$\frac{\bar{C}_{n,N}}{n^{a_0+1}} = \frac{ccc_n n^{1-a_0} (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)))}{n^{a_0+1}} + \frac{ccc_n \gamma_n}{n^{a_0+1}} \quad (417)$$

$$= \frac{ccc_n (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)))}{n^{2a_0}} + \frac{ccc_n \gamma_n}{n^{a_0+1}} \quad (418)$$

Therefore for $n+1 \leq \frac{N}{2}$:

$$\bar{F}_{n+1,N} - \bar{F}_{n,N} = \left\{ \frac{(\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n)))^2 + ccc_n (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)))}{n^{2a_0}} \right. \quad (419)$$

$$\left. + \frac{ccc_n \gamma_n}{n^{a_0+1}} + \frac{\bar{D}_{n,N}}{n^{2a_0+1}} + \frac{\bar{\varepsilon}_{n,N}}{n^{a_0+2}} \right\} \quad (420)$$

$$= \left\{ e_2(n) + \frac{ccc_n \gamma_n}{n^{a_0+1}} + \frac{\bar{D}_{n,N}}{n^{2a_0+1}} + \frac{\bar{\varepsilon}_{n,N}}{n^{a_0+2}} \right\} \quad (421)$$

Where

$$e_2(n) = \frac{(\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n)))^2 + ccc_n (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)))}{n^{2a_0}} \quad (422)$$

Thanks to the limit $\lim_{n \rightarrow +\infty} ccc_n \gamma_n = 0$ is zero we are now in a similar configuration like in chapter one.

We denote $\bar{c}_n = ccc_n \gamma_n$. We have $\lim_{n \rightarrow +\infty} \bar{c}_n = 0$.

Let's now study the serie $\sum_{n \geq 1} e_2(n)$.

Lemma 3.14. *Let's define the functions:*

$$c(t) = -(b_0 \sin(b_0 \ln(t)) + a_0 \cos(b_0 \ln(t))) \quad (423)$$

$$cc(t) = (b_0 \cos(b_0 \ln(t)) - a_0 \sin(b_0 \ln(t))) \quad (424)$$

$$ccc(t) = c(t) - 2^{1-a_0} (c(t) \cos b_0 \ln(2) - cc(t) \sin b_0 \ln(2)) \quad (425)$$

$$ce(t) = ccc(t) (X_1 \sin(b_0 \ln(t)) + X_2 \cos(b_0 \ln(t))) \quad (426)$$

$$e_2(t) = \frac{(\cos(b_0 \ln(t)) - 2^{1-a_0} \cos(b_0 \ln(2t)))^2 + ccc(t) (X_1 \sin(b_0 \ln(t)) + X_2 \cos(b_0 \ln(t)))}{t^{2a_0}} \quad (427)$$

Let's define the sequence:

$$e_{2,n} = e_2(n) - \int_n^{n+1} dt e_2(t) \quad (428)$$

$$(429)$$

Therefore, the serie $\sum_{k \geq 1} e_{2,n}$ are converging absolutely and we have the following inequalities:

$$\left| \sum_{k=n+1}^N e_2(n) - \int_{n+1}^N e_2(t) dt \right| \leq \frac{K}{2a_0} \left(\frac{1}{n^{2a_0}} - \frac{1}{N^{2a_0}} \right) \quad (430)$$

And

$$\left| \sum_{k=n+1}^{+\infty} e_2(n) - \int_{n+1}^{+\infty} e_2(t) dt \right| \leq \frac{K}{2a_0} \frac{1}{n^{2a_0}} \quad (431)$$

where $K > 0$ and E_2 is the primitive function of e_2 is defined as:

$$E_2(x) = \left(\frac{\alpha_1}{1-2a_0} + \frac{(2b_0\beta_1 + \gamma_1(1-2a_0)) \sin(2b_0 \ln(x)) + (\beta_1(1-2a_0) - 2b_0\gamma_1) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (432)$$

Where

$$\alpha_1 = -\frac{(2a_0-1)(1-a_0)}{(b_0)^2 + (1-a_0)^2} \left(\frac{(2^{1-a_0}-1)^2}{2} + 2^{1-a_0}(1-\cos(b_0 \ln(2))) \right) \leq 0 \quad (433)$$

Proof. We are going to use the lemmas 3.1 and 3.2 defined above. We will also use their notations. We can write the functions $e_2(t)$ and $ce(t)$ as the following:

$$e_2(t) = \frac{a_1 + b_1 \cos(2b_0 \ln(t)) - c_1 \sin(2b_0 \ln(t)) + ce(t)}{t^{2a_0}} \quad (434)$$

$$ccc(t) = -(a_3 b_0 + b_3 a_0) \sin(b_0 \ln(t)) + (b_3 b_0 - a_3 a_0) \cos(b_0 \ln(t)) \quad (435)$$

For simplification let's now denote:

$$ccc(t) = Z_1 \sin(b_0 \ln(t)) + Z_2 \cos(b_0 \ln(t)) \quad (436)$$

Therefore we can write:

$$ce(t) = ccc(t) \left(X_1 \sin(b_0 \ln(t)) + X_2 \cos(b_0 \ln(t)) \right) \quad (437)$$

$$= \left(Z_1 \sin(b_0 \ln t) + Z_2 \cos(b_0 \ln(t)) \right) \left(X_1 \sin(b_0 \ln t) + X_2 \cos(b_0 \ln(t)) \right) \quad (438)$$

$$= \frac{X_1 Z_1 + X_2 Z_2}{2} + \frac{X_1 Z_2 + X_2 Z_1}{2} \sin(2b_0 \ln(t)) + \frac{X_2 Z_2 - X_1 Z_1}{2} \cos(2b_0 \ln t) \quad (439)$$

Therefore we can write $e_2(t)$

$$\begin{aligned} e_2(t) t^{2a_0} &= \left(a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \right) + \left(\frac{X_1 Z_2 + X_2 Z_1}{2} - c_1 \right) \sin(2b_0 \ln(t)) + \left(\frac{X_2 Z_2 - X_1 Z_1}{2} + b_1 \right) \cos(2b_0 \ln t) \\ &= \alpha_1 + \beta_1 \sin(2b_0 \ln(t)) + \gamma_1 \cos(2b_0 \ln t) \end{aligned}$$

Where

$$\alpha_1 = a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \quad (442)$$

$$\beta_1 = \frac{X_1 Z_2 + X_2 Z_1}{2} - c_1 \quad (443)$$

$$\gamma_1 = \frac{X_2 Z_2 - X_1 Z_1}{2} + b_1 \quad (444)$$

$$Z_1 = -(a_3 b_0 + b_3 a_0) \quad (445)$$

$$Z_2 = b_3 b_0 - a_3 a_0 \quad (446)$$

$$X_1 = \frac{a_3 b_0 + b_3(1-a_0)}{(b_0)^2 + (1-a_0)^2} \quad (447)$$

$$X_2 = \frac{a_3(1-a_0) - b_3 b_0}{(b_0)^2 + (1-a_0)^2} \quad (448)$$

$$a_1 = \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (449)$$

$$b_1 = \frac{1}{2} + 2^{1-2a_0} \cos(2b_0 \ln(2)) - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (450)$$

$$c_1 = 2^{1-2a_0} \sin(2b_0 \ln(2)) + 2^{1-a_0} \sin(b_0 \ln(2)) \quad (451)$$

$$a_3 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (452)$$

$$b_3 = 2^{1-a_0} \sin(b_0 \ln(2)) \quad (453)$$

And we apply the lemma 3.1 with $\alpha = 2a_0$, $\beta = 2b_0$, $a = \alpha_1$, $b = \beta_1$ and $c = \gamma_1$.

$$e_2(x) = \left(\frac{\alpha_1}{1-2a_0} + \frac{\left(2b_0 \beta_1 + \gamma_1(1-2a_0) \right) \sin(2b_0 \ln(x)) + \left(\beta_1(1-2a_0) - 2b_0 \gamma_1 \right) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (454)$$

When we go thru the lengthy calculation something magical will happen. We found that

$$X_1 Z_1 + X_2 Z_2 = - \frac{\left(a_3^2 + b_3^2 \right) \left(b_0^2 + a_0(1-a_0) \right)}{(b_0)^2 + (1-a_0)^2} \quad (455)$$

$$(456)$$

And something even more magical happens when you calculate α_1

$$\alpha_1 = a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \quad (457)$$

$$= a_1 - \frac{1}{2} \frac{(a_3^2 + b_3^2)(b_0^2 + a_0(1 - a_0))}{(b_0)^2 + (1 - a_0)^2} \quad (458)$$

$$= \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0} \cos(b_0 \ln(2)) - \frac{1}{2} \frac{(a_3^2 + b_3^2)(b_0^2 + a_0(1 - a_0))}{(b_0)^2 + (1 - a_0)^2} \quad (459)$$

$$= \frac{2^{2-2a_0} - 1}{2} + a_3 - \frac{1}{2} \frac{(2^{2-2a_0} - 1 + 2a_3)(b_0^2 + a_0(1 - a_0))}{(b_0)^2 + (1 - a_0)^2} \quad (460)$$

$$= -\left(\frac{2^{2-2a_0} - 1}{2} + a_3\right) \frac{(2a_0 - 1)(1 - a_0)}{(b_0)^2 + (1 - a_0)^2} \quad (461)$$

Therefore

$$\alpha_1 = -\frac{(2a_0 - 1)(1 - a_0)}{(b_0)^2 + (1 - a_0)^2} \left(\frac{(2^{1-a_0} - 1)^2}{2} + 2^{1-a_0} (1 - \cos(b_0 \ln(2))) \right) \quad (462)$$

$$\leq 0 \quad (463)$$

The magic here is the presence of the terms $2a_0 - 1$ and $1 - a_0$ as global factor in the formula of α_1 . \square

Let's $N \geq N_0$ be sufficiently large integer.

As we saw above, for $n > \frac{N}{2}$ we have the asymptotic expansion:

$$\bar{F}_{n+1,N} - \bar{F}_{n,N} = \frac{\cos^2(b_0 \ln(n))}{n^{2a_0}} + \frac{\bar{C}_{n,N}}{n^{a_0+1}} + \frac{\bar{D}_{n,N}}{n^{2a_0+1}} + \frac{\bar{\varepsilon}_{n,N}}{n^{a_0+2}} \quad (464)$$

Hence we have an new expression of \bar{F}_{n+1} after a summation from $\frac{N}{2} + 1$ to n :

$$\bar{F}_{n+1,N} = \bar{F}_{\frac{N}{2}+1,N} + \sum_{k=\frac{N}{2}+1}^n \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} \quad (465)$$

$$+ \sum_{k=\frac{N}{2}+1}^n \frac{\bar{C}_{k,N}}{k^{a_0+1}} + \sum_{k=\frac{N}{2}+1}^n \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \sum_{k=\frac{N}{2}+1}^n \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \quad (466)$$

For $n < \frac{N}{2}$ we have the following asymptotic expansion:

$$\bar{F}_{n+1,N} = \bar{F}_{N_0,N} + \sum_{k=N_0}^n e_2(k) + \sum_{k=N_0}^n \frac{\bar{C}_k}{k^{a_0+1}} \quad (467)$$

$$+ \sum_{k=N_0}^n \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \sum_{k=N_0}^n \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \quad (468)$$

Hence if we do another summation between N_0 and N of the equation above:

$$\sum_{n=N_0}^{N-1} \bar{F}_{n+1,N} = \sum_{n=N_0}^{\frac{N}{2}-1} \bar{F}_{n+1,N} + \bar{F}_{\frac{N}{2}+1,N} + \sum_{n=\frac{N}{2}+1}^{N-1} \bar{F}_{n+1,N} \quad (469)$$

$$= \sum_{n=N_0}^{\frac{N}{2}-1} \bar{F}_{N_0,N} + \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^n e_2(k) + \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^n \frac{\bar{C}_k}{k^{a_0+1}} \quad (470)$$

$$+ \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^n \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^n \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \quad (471)$$

$$+ \bar{F}_{\frac{N}{2}+1,N} + \sum_{n=\frac{N}{2}+1}^{N-1} \bar{F}_{\frac{N}{2}+1,N} + \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^n \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} \quad (472)$$

$$+ \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^n \frac{\bar{C}_{k,N}}{k^{a_0+1}} + \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^n \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^n \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \quad (473)$$

Our sums are finite so we can interchange them:

$$\sum_{n=N_0}^{N-1} \bar{F}_{n+1,N} = \left(\frac{N}{2} - N_0\right) \bar{F}_{N_0,N} + \frac{N}{2} \bar{F}_{\frac{N}{2}+1,N} + \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k) \sum_{n=k}^{\frac{N}{2}-1} 1 \quad (474)$$

$$+ \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\bar{C}_k}{k^{a_0+1}} + \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \right\} \sum_{n=k}^{\frac{N}{2}-1} 1 \quad (475)$$

$$+ \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\bar{C}_{k,N}}{k^{a_0+1}} + \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \right\} \sum_{n=k}^{N-1} 1 \quad (476)$$

$$= \left(\frac{N}{2} - N_0\right) \bar{F}_{N_0,N} + \frac{N}{2} \bar{F}_{\frac{N}{2}+1,N} + \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k) \left(\frac{N}{2} - k\right) + \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\bar{C}_k}{k^{a_0+1}} + \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \right\} \left(\frac{N}{2} - k\right) \quad (477)$$

$$+ \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\bar{C}_{k,N}}{k^{a_0+1}} + \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \right\} (N - k) \quad (478)$$

$$= \left(\frac{N}{2} - N_0\right) \bar{F}_{N_0,N} + \frac{N}{2} \bar{F}_{\frac{N}{2}+1,N} + \frac{N}{2} \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k) - \sum_{k=N_0}^{\frac{N}{2}-1} k e_2(k) \quad (479)$$

$$+ \frac{N}{2} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\bar{C}_k}{k^{a_0+1}} + \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \right\} - \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\bar{C}_k}{k^{a_0}} + \frac{\bar{D}_{k,N}}{k^{2a_0}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+1}} \right\} \quad (480)$$

$$+ \frac{N}{2} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\bar{C}_{k,N}}{k^{a_0+1}} + \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \right\} \quad (481)$$

$$- \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0-1}} + \frac{\bar{C}_{k,N}}{k^{a_0}} + \frac{\bar{D}_{k,N}}{k^{2a_0}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+1}} \right\} \quad (482)$$

And Finally:

$$\sum_{n=N_0}^{N-1} \bar{F}_{n+1,N} = N \left[\left(\frac{1}{2} - \frac{N_0}{N} \right) \bar{F}_{N_0,N} + \frac{1}{2} \bar{F}_{\frac{N}{2}+1,N} + \frac{1}{2} \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k) - \frac{1}{N} \sum_{k=N_0}^{\frac{N}{2}-1} k e_2(k) \right] \quad (483)$$

$$+ \frac{1}{2} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\bar{c}_k}{k^{a_0+1}} + \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \right\} - \frac{1}{N} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\bar{c}_k}{k^{a_0}} + \frac{\bar{D}_{k,N}}{k^{2a_0}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+1}} \right\} \quad (484)$$

$$+ \frac{1}{2} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\bar{C}_{k,N}}{k^{a_0+1}} + \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+2}} \right\} \quad (485)$$

$$- \frac{1}{N} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0-1}} + \frac{\bar{C}_{k,N}}{k^{a_0}} + \frac{\bar{D}_{k,N}}{k^{2a_0}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+1}} \right\} \quad (486)$$

The goal is to prove that $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^{N-1} \bar{F}_{n+1,N} = -\infty$.

Using Césaro lemma we prove that the limit of the following terms are zero:

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=N_0}^{\frac{N}{2}-1} k e_2(k) = 0 \quad (487)$$

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\bar{c}_k}{k^{a_0}} + \frac{\bar{D}_{k,N}}{k^{2a_0}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+1}} \right\} = 0 \quad (488)$$

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0-1}} + \frac{\bar{C}_{k,N}}{k^{a_0}} + \frac{\bar{D}_{k,N}}{k^{2a_0}} + \frac{\bar{\varepsilon}_{k,N}}{k^{a_0+1}} \right\} = 0 \quad (489)$$

We have

$$\lim_{N \rightarrow +\infty} \frac{N_0}{N} \bar{F}_{N_0,N} + \frac{1}{2} \bar{F}_{\frac{N}{2}+1,N} = 0 \quad (490)$$

One note that $\bar{F}_{N_0,N}$ is a constant term that actually depends only on N_0 , so $\lim_{N \rightarrow +\infty} \frac{N_0}{N} \bar{F}_{N_0,N} = 0$. For the term $\frac{1}{2} \bar{F}_{\frac{N}{2}+1,N}$, we use the lemma 3.6, we have two constants K_1 and K_2 such that: $|\bar{A}_{\frac{N}{2}+1,N}| \leq K_1 + K_2 \left(\frac{N}{2} + 1 \right)^{1-a_0}$, therefore

$$\bar{F}_{\frac{N}{2}+1,N} = \frac{|\cos(b_0 \ln(\frac{N}{2} + 1)) \bar{A}_{\frac{N}{2}+1,N}|}{\left(\frac{N}{2} + 1 \right)^{a_0}} \quad (491)$$

$$\leq \frac{K_1}{\left(\frac{N}{2} + 1 \right)^{a_0}} + \frac{K_2}{\left(\frac{N}{2} + 1 \right)^{2a_0-1}} \quad (492)$$

We have $2a_0 > 1$ therefore the $\lim_{N \rightarrow +\infty} \bar{F}_{\frac{N}{2}+1,N} = 0$

Let's now prove

$$\lim_{N \rightarrow +\infty} \left[\frac{1}{2} \bar{F}_{N_0, N} + \frac{1}{2} \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k) \right] \quad (493)$$

$$+ \frac{1}{2} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\bar{c}_k}{k^{a_0+1}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right\} \quad (494)$$

$$+ \frac{1}{2} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\bar{C}_{k, N}}{k^{a_0+1}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right\} < 0 \quad (495)$$

We have the sequences $(\bar{D}_{k, N})_{k \geq 1}$ and $(\bar{\varepsilon}_{k, N})_{k \geq 1}$ are bounded. Plus We have $a_0 + 2 \geq 2a_0 + 1 > a_0 + 1 \geq 2a_0 > 1$, therefore the serie $\sum_{k \geq 1} \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right\}$ is converging absolutely. Let's call its partial sum S_N :

$$S_N = \sum_{k \geq 1}^N \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right\} \quad (496)$$

$$S_N - S_{\frac{N}{2}} = \sum_{k=\frac{N}{2}+1}^N \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right\} \quad (497)$$

Therefore

$$\lim_{N \rightarrow +\infty} \sum_{k=\frac{N}{2}+1}^N \left\{ \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right\} \quad (498)$$

$$= \lim_{N \rightarrow +\infty} S_N - \lim_{N \rightarrow +\infty} S_{\frac{N}{2}} \quad (499)$$

$$= 0 \quad (500)$$

Let's study now the convergence of the sum $\sum_{k=\frac{N}{2}+1}^N \frac{\bar{C}_{k, N}}{k^{a_0+1}}$. From the lemma 3.6 we have two constants $K_1 \geq 0$ and $K_2 \geq 0$ such that:

$$|\bar{A}_{k, N}| \leq K_1 + K_2 k^{1-a_0} \quad (501)$$

and we have from the definition of $\bar{C}_{k, N}$ that

$$c_k = -\left(b_0 \sin(b_0 \ln(k)) + a_0 \cos(b_0 \ln(k)) \right) \quad (502)$$

$$\bar{C}_{k, N} = c_k \bar{A}_{k, N} \quad (503)$$

We have $a_0 > 0$, $b_0 > 0$, Therefore

$$\left| \frac{\bar{C}_{k, N}}{k^{a_0+1}} \right| \leq \frac{K_1(a_0 + b_0)}{k^{a_0+1}} + K_2(a_0 + b_0) \frac{k^{1-a_0}}{k^{a_0+1}} \quad (504)$$

$$\leq \frac{K_1(a_0 + b_0)}{k^{a_0+1}} + K_2(a_0 + b_0) \frac{1}{k^{2a_0}} \quad (505)$$

We have $a_0 + 1 \geq 2a_0 > 1$, therefore the serie $\sum_{k \geq 1} \frac{\bar{C}_{k,N}}{k^{a_0+1}}$ is converging absolutely. Therefore like in the case of E_N , we have

$$\lim_{N \rightarrow +\infty} \sum_{k=\frac{N}{2}+1}^N \frac{\bar{C}_{k,N}}{k^{a_0+1}} = \lim_{N \rightarrow +\infty} \sum_{k=1}^N \frac{\bar{C}_{k,N}}{k^{a_0+1}} - \lim_{N \rightarrow +\infty} \sum_{k=1}^{\frac{N}{2}+1} \frac{\bar{C}_{k,N}}{k^{a_0+1}} \quad (506)$$

$$= 0 \quad (507)$$

Let's call now the remaining term R_{N_0} :

$$R_{N_0} = \lim_{N \rightarrow +\infty} \left[\bar{F}_{N_0,N} + \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k) + \frac{\bar{c}_k}{k^{a_0+1}} + \frac{\bar{D}_{k,N}}{k^{2a_0+1}} + \frac{\bar{e}_{k,N}}{k^{a_0+2}} \right] \quad (508)$$

Lemma 3.15. *We can choose N_0 such that $R_{N_0} < 0$.*

Proof. We studied the serie $\sum_{n \geq 1} e_2(n)$ in the lemma 3.9. We have:

$$\left| \sum_{n=N_0+1}^{+\infty} e_2(n) - \int_{N_0+1}^{+\infty} e_2(t) dt \right| \leq \frac{K}{2a_0(N_0)^{2a_0}} \quad (509)$$

Where $K > 0$ and the primitive function E_2

$$E_2(x) = \left(\frac{\alpha_1}{1-2a_0} + \frac{(2b_0\beta_1 + \gamma_1(1-2a_0)) \sin(2b_0 \ln(x)) + (\beta_1(1-2a_0) - 2b_0\gamma_1) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (510)$$

With

$$\alpha_1 = -\frac{(2a_0-1)(1-a_0)}{(b_0)^2 + (1-a_0)^2} \left(\frac{(2^{1-a_0}-1)^2}{2} + 2^{1-a_0}(1-\cos(b_0 \ln(2))) \right) \leq 0 \quad (511)$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} e_2(k) + E_2(N_0+1) \right| \leq \frac{K}{a_0(N_0)^{2a_0}} \quad (512)$$

Let's now study the term $\bar{F}_{N_0,N}$. From lemma 3.9 and 3.10 we can write for $N \geq 2N_0$

$$\bar{F}_{N_0,N} = \frac{\cos(b_0 \ln(N_0)) - 2^{1-a_0} \cos(b_0 \ln(2N_0))}{(N_0)^{a_0}} A_{N_0,N} \quad (513)$$

$$= \frac{\cos(b_0 \ln(N_0)) - 2^{1-a_0} \cos(b_0 \ln(2N_0))}{(N_0)^{a_0}} \underbrace{\left(E_1(N_0) + \gamma_{N_0} \right)}_{\text{Lemma 3.9}} \quad (514)$$

$$= \underbrace{e_3(N_0)}_{\text{Lemma 3.10}} + \frac{\cos(b_0 \ln(N_0)) - 2^{1-a_0} \cos(b_0 \ln(2N_0))}{(N_0)^{a_0}} \gamma_{N_0} \quad (515)$$

$$(516)$$

Where the sequence γ_n converge to zero and

$$e_3(t) (t)^{2a_0-1} = \frac{\alpha_3 + \beta_3 \sin(2b_0 \ln(t)) + \gamma_3 \cos(2b_0 \ln t)}{(t)^{2a_0-1}} \quad (517)$$

$$\alpha_3 = \frac{X_1 b_3 + X_2 a_3}{2} \quad (518)$$

$$\beta_3 = \frac{X_1 a_3 + X_2 b_3}{2} \quad (519)$$

$$\gamma_3 = \frac{X_2 a_3 - X_1 b_3}{2} \quad (520)$$

$$X_1 = \frac{a_3 b_0 + b_3 (1 - a_0)}{(b_0)^2 + (1 - a_0)^2} \quad (521)$$

$$X_2 = \frac{a_3 (1 - a_0) - b_3 b_0}{(b_0)^2 + (1 - a_0)^2} \quad (522)$$

Let's denote

$$O_{N_0} = -\left(e_3(N_0 + 1) - e_3(N_0)\right) + \frac{\cos(b_0 \ln(N_0)) - 2^{1-a_0} \cos(b_0 \ln(2N_0))}{(N_0)^{a_0}} \gamma_{N_0} \quad (523)$$

The term $e_3(N_0)$ and the term $E_2(N_0)$ share the term $\frac{1}{(N_0)^{2a_0-1}}$. So we are going to group them in the last inequality:

$$\left| e_3(N_0 + 1) + \sum_{n=N_0+1}^{+\infty} e_2(k) + \underbrace{E_2(N_0 + 1) - e_3(N_0 + 1)}_{e_4(N_0+1)} \right| \leq \frac{K}{a_0 (N_0)^{2a_0}} \quad (524)$$

Where

$$e_4(x) = E_2(x) - e_3(x) \quad (525)$$

$$= \frac{\alpha_2 - \alpha_3 + (\beta_2 - \beta_3) \sin(2b_0 \ln(t)) + (\gamma_2 - \gamma_3) \cos(2b_0 \ln t)}{x^{2a_0-1}} \quad (526)$$

We develop the calculation further and we see that we have $\alpha_2 = \alpha_3$. Therefore

$$e_4(x) = \frac{(\beta_2 - \beta_3) \sin(2b_0 \ln(t)) + (\gamma_2 - \gamma_3) \cos(2b_0 \ln t)}{x^{2a_0-1}} \quad (527)$$

Therefore we have the asymptotic expansion of $e_3(N_0+1) + \sum_{n=N_0+1}^{+\infty} e_2(k)$ as follows:

$$e_3(N_0 + 1) + \sum_{n=N_0+1}^{+\infty} e_2(k) = -E_2(N_0 + 1) + e_3(N_0 + 1) + \mathcal{O}\left(\frac{1}{(N_0)^{2a_0}}\right) \quad (528)$$

$$= \frac{(\beta_3 - \beta_2) \sin(2b_0 \ln(N_0 + 1)) + (\gamma_3 - \gamma_2) \cos(2b_0 \ln(N_0 + 1))}{(N_0 + 1)^{2a_0-1}} \quad (529)$$

$$+ \mathcal{O}\left(\frac{1}{(N_0)^{2a_0}}\right) \quad (530)$$

Remark. The remaining terms in the expression of R_{N_0} are all of the order of $\frac{1}{(N_0)^{a_0}}$ and above. we note that $2a_0 - 1 < a_0$. So the dominant term in the expression R_{N_0} is the term $-e_4(N_0+1) = \frac{(\beta_3 - \beta_2) \sin(2b_0 \ln(N_0+1)) + (\gamma_3 - \gamma_2) \cos(2b_0 \ln(N_0+1))}{(N_0+1)^{2a_0-1}}$. As we will see later that the function e_4 is nonzero function. Therefore this term is the dominant term in the expression of $\frac{1}{N} \sum_{n=N_0}^{N-1} \bar{F}_{n+1,N}$. Hence based on the sign of $e_4(N_0+1)$ we can show that the limit of $\sum_{n=1}^{N-1} \bar{F}_{n+1,N}$ can be both $+\infty$ and $-\infty$.

Let's now study the function f :

$$f(x) = (\beta_2 - \beta_3) \sin(2b_0x) + (\gamma_2 - \gamma_3) \cos(2b_0x) \quad (531)$$

General Case: $\frac{1}{2} < a_0 < 1$ **OR** $a_0 = 1, b_0 \neq \frac{2k\pi}{\ln(2)}$ From the lemma 3.19 we have that the function e_4 is a nonzero function. Therefore the function f is also nonzero function and hence at least $\beta_2 - \beta_3$ or $\gamma_2 - \gamma_3$ is different from zero or both of them. So let's assume that $\gamma_2 \neq \gamma_3$. The function f is a linear combination of the functions sin and cos. So the function f is differentiable and bounded. The function f is periodic of period $\frac{\pi}{b_0}$.

Let's calculate the function f values at the following points:

$$f(0) = \gamma_2 - \gamma_3 \quad (532)$$

$$f\left(\frac{\pi}{2b_0}\right) = \beta_2 - \beta_3 \quad (533)$$

$$f\left(\frac{\pi}{2b_0}\right) = -\gamma_2 + \gamma_3 \quad (534)$$

$$f\left(\frac{3\pi}{2b_0}\right) = -\beta_2 + \beta_3 \quad (535)$$

From the values above we have the following:

- If $(\gamma_2 - \gamma_3) > 0$ then $f(0) > 0$.
- If $(\gamma_2 - \gamma_3 < 0$ then $f(\frac{\pi}{b_0}) > 0$.

In case of $\gamma_2 - \gamma_3 = 0$ and we will work with $\beta_2 - \beta_3$ in such case we calculate the f values at the points $f(\frac{\pi}{2b_0})$ and $f(\frac{3\pi}{2b_0})$. Therefore, in all case we have either $f(0) > 0$ or $f(\frac{\pi}{b_0}) > 0$. So let's assume from now on that $f(0) < 0$ as the same proof can be done with both values thanks to lemma 3.3 and lemma 3.4.

From the lemma 3.3 above, there exist $N_1 \geq N_0$ such that $|\cos(2b_0 \ln(N_1 + 1)) - 1| \leq \epsilon$ and $|\sin(2b_0 \ln(N_1 + 1))| \leq \epsilon$ for any $1 > \epsilon > 0$.

Let's define $\beta_0 = -f(0)$. $\beta_0 < 0$.

Let's fix $\epsilon > 0$ to be very small such that $0 < \epsilon < \min\left(1, \frac{\beta_0}{10000}\right)$.

Let's define $\gamma_0 = |\beta_2 - \beta_3| + |\gamma_2 - \gamma_3|$

Let's N_1 be such that $|\cos(2b_0 \ln(N_1 + 1)) - 1| \leq \epsilon/\gamma_0$ and also $|\sin(2b_0 \ln(N_1 + 1)) - 0| \leq \epsilon/\gamma_0$. Therefore:

$$\left| e_4(N_1 + 1) - \frac{f(0)}{(N_1 + 1)^{2a_0 - 1}} \right| \leq \left| \left\{ \frac{\left((\beta_2 - \beta_3) \sin(2b_0 \ln(N_1 + 1)) + (\gamma_2 - \gamma_3) \cos(2b_0 \ln(N_1 + 1)) \right)}{(N_1 + 1)^{2a_0 - 1}} \right. \right. \quad (536)$$

$$\left. - \frac{\left((\beta_2 - \beta_3) \sin(2b_0 0) + (\gamma_2 - \gamma_3) \cos(2b_0 0) \right)}{(N_1 + 1)^{2a_0 - 1}} \right\} \quad (537)$$

$$\leq \frac{\left(|\beta_2 - \beta_3| |\sin(2b_0 \ln(N_1 + 1))| + |\gamma_2 - \gamma_3| |\cos(2b_0 \ln(N_1 + 1)) - 1| \right)}{(N_1 + 1)^{2a_0 - 1}} \quad (538)$$

$$\leq \frac{\left(|\beta_2 - \beta_3| \epsilon / \gamma_0 + |\gamma_2 - \gamma_3| \epsilon / \gamma_0 \right)}{(N_1 + 1)^{2a_0 - 1}} \quad (539)$$

$$\leq \frac{\epsilon}{(N_1 + 1)^{2a_0 - 1}} \quad (540)$$

Therefore

$$\left| e_3(N_0 + 1) + \sum_{n=N_0+1}^{+\infty} e_2(n) - \frac{\beta_0}{(N_0 + 1)^{2a_0 - 1}} \right| \leq \left| \sum_{n=N_0+1}^{+\infty} e_2(n) + e_4(N_0 + 1) \right| \quad (541)$$

$$+ \left| -e_4(N_0 + 1) - \frac{-f(0)}{(N_0 + 1)^{2a_0 - 1}} \right| \leq \frac{\epsilon}{(N_0 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_0)^{2a_0}} \quad (542)$$

Hence

$$\left| e_3(N_0 + 1) + \sum_{n=N_0+1}^{+\infty} e_2(n) - \frac{\beta_0}{(N_0 + 1)^{2a_0 - 1}} \right| \leq \frac{\epsilon}{(N_0 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_0)^{2a_0}} \quad (543)$$

Therefore

$$e_3(N_0 + 1) + \sum_{n=N_0+1}^{+\infty} e_2(n) \leq \frac{\beta_0 + \epsilon}{(N_0 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_0)^{2a_0}} \quad (544)$$

Therefore

$$R_{N_0} = \sum_{k=N_0}^{+\infty} e_2(k) + \bar{F}_{N_0, N} + \lim_{N \rightarrow +\infty} \left[\sum_{k=N_0}^N \left\{ \frac{\bar{c}_k}{k^{a_0+1}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right\} \right] \quad (545)$$

$$\leq \frac{\beta_0 + \epsilon}{(N_0 + 1)^{2a_0-1}} + \frac{K}{2a_0(N_1)^{2a_0}} + \underbrace{\bar{F}_{N_0, N} - e_3(N_0 + 1)}_{\leq 0} + \lim_{N \rightarrow +\infty} \left[\sum_{k=N_0}^N \left\{ \frac{\bar{c}_k}{k^{a_0+1}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right\} \right] \quad (546)$$

$$\leq \frac{\beta_0 + \epsilon}{(N_0 + 1)^{2a_0-1}} + \frac{K}{2a_0(N_0)^{2a_0}} + \underbrace{e_3(N_0) - e_3(N_0 + 1) + \frac{\cos(b_0 \ln(N_0)) - 2^{1-a_0} \cos(b_0 \ln(2N_0))}{(N_0)^{a_0}}}_{\leq 0} \gamma_{N_0} \quad (547)$$

$$+ \lim_{N \rightarrow +\infty} \sum_{k=N_0}^N \left\{ \frac{\bar{c}_k}{k^{a_0+1}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right\} \quad (548)$$

$$\leq \frac{\beta_0 + \epsilon}{(N_0 + 1)^{2a_0-1}} + \frac{K}{2a_0(N_0)^{2a_0}} + O_{N_0} + \lim_{N \rightarrow +\infty} \sum_{k=N_0}^N \left\{ \frac{\bar{c}_k}{k^{a_0+1}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right\} \quad (549)$$

We have the sequences (\bar{c}_k) , $(\bar{D}_{k, N})_{k \geq 1}$ and $(\bar{\varepsilon}_{k, N})_{k \geq 1}$ are bounded. Plus we have $a_0 + 2 \geq 2a_0 + 1 > a_0 + 1 \geq 2a_0 > 1$, therefore the series $\sum_{k \geq 1} \frac{\bar{D}_{k, N}}{k^{2a_0+1}}$, $\sum_{k \geq 1} \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}}$ are converging absolutely.

Let the positive constant M such that:
For each $k \geq N_1$ and $N \geq 2N_1$:

$$|\bar{c}_k| \leq M \quad (550)$$

$$|\bar{D}_{k, N}| \leq M \quad (551)$$

$$|\bar{\varepsilon}_{k, N}| \leq M \quad (552)$$

Therefore, we have $a_0 > 0$:

$$\left| \sum_{n=N_1+1}^{+\infty} \frac{\bar{c}_k}{k^{a_0+1}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right| \quad (553)$$

$$\leq \frac{M}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0 + 1)(N_1)^{a_0+1}} \quad (554)$$

Therefore

$$R_{N_1} \leq \frac{\beta_0 + \epsilon}{(N_1 + 1)^{2a_0-1}} + \frac{K}{2a_0(N_1)^{2a_0}} + O_{N_1} + \frac{M}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0 + 1)(N_1)^{a_0+1}} \quad (555)$$

$$\leq \frac{1}{(N_1 + 1)^{2a_0-1}} \left(\beta_0 + \epsilon + \frac{K(N_1 + 1)^{2a_0-1}}{2a_0(N_1)^{2a_0}} + (N_1 + 1)^{2a_0-1} O_{N_1} \right) \quad (556)$$

$$+ \frac{M(N_1 + 1)^{2a_0-1}}{a_0(N_1)^{a_0}} + \frac{M(N_1 + 1)^{2a_0-1}}{2a_0(N_1)^{2a_0}} + \frac{M(N_1 + 1)^{2a_0-1}}{(a_0 + 1)(N_1)^{a_0+1}} \quad (557)$$

Case One: $\frac{1}{2} < a_0 < 1$ Let's define the sequence (δ_n) :

$$\delta_n = \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + (n+1)^{2a_0-1} O_n + \frac{M(n+1)^{2a_0-1}}{(n)^{a_0}} \quad (558)$$

$$+ \frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(a_0 + 1)(n)^{a_0+1}} \quad (559)$$

As we have $\frac{1}{2} < a_0 < 1$, therefore $2a_0 - 1 < a_0 < 2a_0 < a_0 + 1$
Therefore

$$\lim_{N \rightarrow +\infty} \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(n)^{a_0}} \quad (560)$$

$$+ \frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(a_0+1)(n)^{a_0+1}} = 0 \quad (561)$$

Let's calculate now the limit of $\lim_{n \rightarrow +\infty} (n+1)^{2a_0-1} O_n$
From the lemma 3.8 ($\alpha = 2a_0$ and $\beta = b_0$) we can write:

$$(n+1)^{2a_0-1} O_n = (n+1)^{2a_0-1} \left(e_3(n) - e_3(n+1) \right) + (n+1)^{2a_0-1} \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n))}{n^{a_0}} \gamma_n \quad (562)$$

$$= -a(n+1)^{2a_0-1} \frac{1-2a_0}{n^{2a_0}} - (n+1)^{2a_0-1} \frac{(b c c_n(2a_0-1, b_0) + c c_n(2a_0-1, b_0))}{n^{2a_0}} \quad (563)$$

$$+ (n+1)^{2a_0-1} \frac{\cos(b_0 \ln(n)) - 2^{1-a_0} \cos(b_0 \ln(2n))}{n^{a_0}} \gamma_n + \mathcal{O}\left(\frac{1}{n^{2a_0}}\right) \quad (564)$$

Thanks to $2a_0 - 1 < a_0 < 2a_0 < a_0 + 1$, we have the limit of every term
in the expression of $(n+1)^{2a_0-1} O_n$ goes to zero, therefore $\lim_{n \rightarrow +\infty} (n+1)^{2a_0-1} O_n = 0$.

Therefore the $\lim_{n \rightarrow +\infty} \delta_n = 0$.

So we can choose N_1 such that $|\delta_{N_1}| < \epsilon$, i.e $\delta_{N_1} \leq \epsilon$.

Therefore

$$R_{N_1} \leq \frac{1}{(N_1+1)^{2a_0-1}} (\beta_0 + 2\epsilon) < 0 \quad (565)$$

Therefore $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^{N-1} \bar{F}_{n+1, N} = -\infty$.

Case Two: $a_0 = 1, b_0 \neq \frac{2k\pi}{\ln(2)}$ In this particular case we have $2a_0 - 1 = 1 = a_0$. Therefore the terms of the order $\frac{1}{n^{a_0}}$ in the expression of δ_n and O_n can become problematic. So instead of using the fact (\bar{c}_k) is bounded, we will use here the fact that its limit equal zero when n goes to infinity.

As we have

$$\lim_{n \rightarrow +\infty} \bar{c}_n = 0 \quad (566)$$

So we can choose N_1 such for each $n \geq N_1$ we have $|\bar{c}_k| \leq \epsilon$.

For each $k \geq N_1$ and $N \geq 2N_1$:

$$|\bar{c}_k| \leq \epsilon \quad (567)$$

$$|\bar{D}_{k, N}| \leq M \quad (568)$$

$$|\bar{\varepsilon}_{k, N}| \leq M \quad (569)$$

Therefore, we have $a_0 > 0$:

$$\left| \sum_{n=N_1+1}^{+\infty} \frac{\bar{c}_k}{k^{a_0+1}} + \frac{\bar{D}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\varepsilon}_{k, N}}{k^{a_0+2}} \right| \leq \frac{\epsilon}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0+1)(N_1)^{a_0+1}} \quad (570)$$

And

$$R_{N_1} \leq \frac{\beta_0 + \epsilon}{(N_1 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_1)^{2a_0}} + O_{N_1} + \frac{\epsilon}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0 + 1)(N_1)^{a_0 + 1}} \quad (571)$$

$$\leq \frac{1}{(N_1 + 1)^{2a_0 - 1}} \left(\beta_0 + \epsilon + \frac{K(N_1 + 1)^{2a_0 - 1}}{2a_0(N_1)^{2a_0}} + (N_1 + 1)^{2a_0 - 1} O_{N_1} \right) \quad (572)$$

$$+ \frac{\epsilon(N_1 + 1)^{2a_0 - 1}}{a_0(N_1)^{a_0}} + \frac{M(N_1 + 1)^{2a_0 - 1}}{2a_0(N_1)^{2a_0}} + \frac{M(N_1 + 1)^{2a_0 - 1}}{(a_0 + 1)(N_1)^{a_0 + 1}} \quad (573)$$

We have $a_0 = 1$ therefore:

$$R_{N_1} \leq \frac{1}{(N_1 + 1)} \left(\beta_0 + 2\epsilon + \frac{K(N_1 + 1)}{2(N_1)^2} + (N_1 + 1) O_{N_1} + \frac{1}{N_1} + \frac{M(N_1 + 1)}{2(N_1)^2} + \frac{M(N_1 + 1)}{2(N_1)^2} \right) \quad (574)$$

Because $\beta_0 < 0$ and we have $\epsilon < \frac{|\beta_0|}{10000} < 1$

We have the sequence (δ_n) :

$$\delta_n = \frac{K(n + 1)}{2n^2} + (n + 1) O_n + \frac{1}{n} + \frac{M(n + 1)}{2n^2} + \frac{M(n + 1)}{2n^2} \quad (575)$$

For the term $(n + 1)^{2a_0 - 1} O_n$ we have its expression:

$$(n + 1) O_n = -a(n + 1) \frac{-1}{n^2} - (n + 1)^1 \frac{(b c c_n(1, b_0) + c c_n(1, b_0))}{n^2} \quad (576)$$

$$+ (n + 1)^1 \frac{\cos(b_0 \ln(n)) - \cos(b_0 \ln(2n))}{n} \gamma_n + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (577)$$

In the expression of $(n + 1)^{2a_0 - 1} O_n$ there is γ_n term that can be problematic because $2a_0 - 1 = a_0$. But thanks to the limit $\lim_{n \rightarrow +\infty} \gamma_n = 0$. We also have $\lim_{n \rightarrow +\infty} (n + 1)^{2a_0 - 1} O_n = 0$.

Therefore

$$\lim_{n \rightarrow +\infty} \delta_n = 0 \quad (578)$$

So we can choose N_1 such that $|\delta_{N_1}| < \epsilon$, i.e $\delta_{N_1} \leq \epsilon$.

Therefore

$$R_{N_1} \leq \frac{1}{N_1 + 1} \left(\beta_0 + 3\epsilon \right) < 0 \quad (579)$$

Therefore $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^{N-1} \bar{F}_{n+1, N} = -\infty$. □

Remark. Note that one also can prove that the $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^{N-1} \bar{F}_{n+1, N} = +\infty$ by choosing another point x_0 where $f(x_0) < 0$. The same steps above apply. And this thanks to the term $\frac{(\beta_3 - \beta_2) \sin(2b_0 \ln(N_0 + 1)) + (\gamma_3 - \gamma_2) \cos(2b_0 \ln(N_0 + 1))}{(N_0 + 1)^{2a_0 - 1}}$

Remark. Note that the fact that s is a zero for ζ and its consequence the limit $\left(\lim_{N \rightarrow +\infty} \bar{A}_{N, N} = 0 \right)$ is important in the above proof. □

Lemma 3.16. If $\frac{1}{2} < a_0 \leq 1$ the sequence $(\sum_{n=1}^N \bar{G}_{n, N})_{N \geq 1}$ diverges.

Proof. We use the same proof logic from the case of $(\sum_{n=1}^N \bar{F}_{n,N})_{N \geq 1}$. We want to prove that:

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \bar{G}_{n,N} = -\infty \quad (580)$$

Let's n and N be two integers.

Case One: $n > \frac{N}{2}$ We proceed with some asymptotic expansions like above:

We have

$$\bar{B}_{n+1,N} = \bar{B}_{n,N} + \frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} \quad (581)$$

So $\bar{G}_{n+1,N}$ can be written as follows:

$$\bar{G}_{n+1,N} = \frac{\sin(b_0 \ln(n+1)) \bar{B}_{n,N}}{(n+1)^{a_0}} + \frac{\sin^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} \quad (582)$$

So $\bar{G}_{n+1,N} - \bar{G}_{n,N}$ can be written as

$$\bar{G}_{n+1,N} - \bar{G}_{n,N} = \left(\frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right) \bar{B}_{n,N} + \frac{\sin^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} \quad (583)$$

$$\bar{G}_{n+1,N} - \bar{G}_{n,N} = \left(\bar{V}_{n+1,N} - \bar{V}_{n,N} \right) \bar{B}_{n,N} + \frac{\sin^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} \quad (584)$$

Let's do now the asymptotic expansion of $\bar{V}_{n+1,N} - \bar{V}_{n,N}$. For this we need the asymptotic expansion of $\sin(b_0 \ln(n+1))$.

$$\begin{aligned} \sin(b_0 \ln(n+1)) &= \sin\left(b_0 \ln(n) + b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (585) \\ &= \sin(b_0 \ln(n)) \cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) + \cos(b_0 \ln(n)) \sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) \quad (586) \end{aligned}$$

we have the asymptotic expansion of $\ln\left(1 + \frac{1}{n}\right)$ in order two as follow:

$$\ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (587)$$

Using the asymptotic expansion of the functions \sin and \cos that I will speare you the details here, we have

$$\sin\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) = \sin\left(\frac{b_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \quad (588)$$

$$= \frac{b_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (589)$$

And

$$\cos\left(b_0 \ln\left(1 + \frac{1}{n}\right)\right) = 1 + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (590)$$

Hence

$$\sin(b_0 \ln(n+1)) = \sin(b_0 \ln(n)) + \frac{b_0}{n} \cos(b_0 \ln(n)) + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (591)$$

Also the Asymptotic expansion of $\frac{1}{(1+n)^{a_0}}$:

$$\frac{1}{(1+n)^{a_0}} = \frac{1}{n^{a_0}} \left(1 - \frac{a_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \quad (592)$$

Hence

$$\frac{\sin(b_0 \ln(n+1))}{(1+n)^{a_0}} = \frac{\sin(b_0 \ln(n))}{n^{a_0}} + \frac{b_0 \cos(b_0 \ln(n)) - a_0 \sin(b_0 \ln(n))}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (593)$$

And the squared version of the above equation

$$\frac{\sin^2(b_0 \ln(n+1))}{(1+n)^{2a_0}} = \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + 2 \sin(b_0 \ln(n)) \frac{b_0 \cos(b_0 \ln(n)) - a_0 \sin(b_0 \ln(n))}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (594)$$

And

$$\bar{V}_{n+1,N} - \bar{V}_{n,N} = \frac{b_0 \cos(b_0 \ln(n)) - a_0 \sin(b_0 \ln(n))}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (595)$$

So the asymptotic expansion of $\bar{G}_{n+1,N} - \bar{G}_{n,N}$ is as follows:

$$\bar{G}_{n+1,N} - \bar{G}_{n,N} = \frac{\sin^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} + \frac{b_0 \cos(b_0 \ln(n)) - a_0 \sin(b_0 \ln(n))}{n^{a_0+1}} \bar{B}_{n,N} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (596)$$

For the sake of simplifying the notation, let define the sequence $\overline{CC}_{n,N}$ as follows: For each $n \geq 1$ we have:

$$cc_n = \left(b_0 \cos(b_0 \ln(n)) - a_0 \sin(b_0 \ln(n))\right) \quad (597)$$

$$\overline{CC}_{n,N} = cc_n \bar{B}_{n,N} \quad (598)$$

And

$$DD_n = 2cc_n \sin(b_0 \ln(n)) \quad (599)$$

The functions $x \mapsto \cos(x)$ and $x \mapsto \sin(x)$ are by their nature bounded so (cc_n) and (DD_n) are also bounded.

By definition of \mathcal{O} , we know that there is exist a bounded sequence (η_n) and there is exist a number N_0 such that: For each $n \geq N_0$ we have

$$\bar{G}_{n+1,N} - \bar{G}_{n,N} = \frac{\sin^2(b_0 \ln(n+1))}{(n+1)^{2a_0}} + \frac{\overline{CC}_{n,N}}{n^{a_0+1}} + \frac{\bar{\eta}_{n,N}}{n^{a_0+2}} \quad (600)$$

And Finally

$$\bar{G}_{n+1,N} - \bar{G}_{n,N} = \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + \frac{\overline{CC}_{n,N}}{n^{a_0+1}} + \frac{\overline{DD}_{n,N}}{n^{2a_0+1}} + \frac{\bar{\eta}_{n,N}}{n^{a_0+2}} \quad (601)$$

Where

$$cc_n = (b_0 \cos(b_0 \ln(n)) - a_0 \sin(b_0 \ln(n))) \quad (602)$$

$$\overline{CC}_{n,N} = cc_n \bar{B}_{n,N} \quad (603)$$

$$\overline{DD}_{n,N} = 2 \sin(b_0 \ln(n)) cc_n \quad (604)$$

Let's now study the case of when $n+1 \leq \frac{N}{2}$:

Case Two: $n+1 \leq \frac{N}{2}$

$$\bar{G}_{n+1,N} - \bar{G}_{n,N} = \bar{V}_{n+1,N} \bar{B}_{n+1,N} - \bar{V}_{n,N} \bar{B}_{n,N} \quad (605)$$

$$= (\bar{V}_{n+1,N} - \bar{V}_{n,N}) \bar{B}_{n,N} + (\bar{V}_{n+1,N})^2 \quad (606)$$

And

$$\bar{V}_{n+1,N} - \bar{V}_{n,N} = \frac{\sin(b_0 \ln(n+1))}{(1+n)^{a_0}} - \frac{\sin(b_0 \ln(n))}{n^{a_0}} \quad (607)$$

$$-2^{1-a_0} \left\{ \frac{\sin(b_0 \ln 2(n+1))}{(1+n)^{a_0}} - \frac{\sin(b_0 \ln 2(n))}{n^{a_0}} \right\} \quad (608)$$

We develop the $\sin(b_0 \ln 2(n+1))$ and $\sin(b_0 \ln 2(n))$ to get:

$$\frac{\sin(b_0 \ln 2(n+1))}{(1+n)^{a_0}} = \cos(b_0 \ln(2)) \left\{ \frac{\sin(b_0 \ln(n))}{n^{a_0}} + \frac{cc_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \right\} \quad (609)$$

$$+ \sin(b_0 \ln(2)) \left\{ \frac{\cos(b_0 \ln(n))}{n^{a_0}} + \frac{c_n}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \right\} \quad (610)$$

$$= \frac{\sin(b_0 \ln 2(n))}{(n)^{a_0}} + \frac{cc_n \cos b_0 \ln(2) + c_n \sin b_0 \ln(2)}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (611)$$

Therefore:

$$\bar{V}_{n+1,N} - \bar{V}_{n,N} = \frac{cc_n - 2^{1-a_0} (cc_n \cos b_0 \ln(2) + c_n \sin b_0 \ln(2))}{n^{a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (612)$$

And

$$(\bar{V}_{n+1,N})^2 = (\bar{V}_{n,N})^2 + 2 \frac{gccc_n gdd_n}{n^{2a_0+1}} + \mathcal{O}\left(\frac{1}{n^{a_0+2}}\right) \quad (613)$$

Where

$$gccc_n = cc_n - 2^{1-a_0} \left(cc_n \cos(b_0 \ln(2)) + c_n \sin(b_0 \ln(2)) \right) \quad (614)$$

$$gdd_n = \sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n)) \quad (615)$$

$$\overline{DD}_{n,N} = 2gccc_n gdd_n \quad (616)$$

$$\overline{CC}_{n,N} = gccc_n \overline{B}_{n,N} \quad (617)$$

And

$$\overline{G}_{n+1,N} - \overline{G}_{n,N} = \left\{ \frac{(\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n)))^2}{n^{2a_0}} + \frac{\overline{CC}_{n,N}}{n^{a_0+1}} \right. \quad (618)$$

$$\left. + \frac{\overline{DD}_{n,N}}{n^{2a_0+1}} + \frac{\overline{\eta}_{n,N}}{n^{a_0+2}} \right\} \quad (619)$$

With the sequence $\overline{\varepsilon}_{n,N}$ is bounded.

From the lemma 3.11 above we have: $\overline{B}_{n,N} = F_1(n) + \gamma_n$ such that $\lim_{n \rightarrow +\infty} \gamma_n = 0$ and $F_1(n) = n^{1-a_0} (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)))$ where X_1 and X_2 two real constants. The function F_1 is defined in lemma 3.2. Therefore

$$\frac{\overline{CC}_{n,N}}{n^{a_0+1}} = \frac{gccc_n n^{1-a_0} (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)))}{n^{a_0+1}} + \frac{gccc_n \gamma_n}{n^{a_0+1}} \quad (620)$$

$$= \frac{gccc_n (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)))}{n^{2a_0}} + \frac{gccc_n \gamma_n}{n^{a_0+1}} \quad (621)$$

Therefore for $n+1 \leq \frac{N}{2}$:

$$\overline{G}_{n+1,N} - \overline{G}_{n,N} = \frac{1}{n^{2a_0}} \left\{ (\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n)))^2 \right. \quad (622)$$

$$\left. + gccc_n (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n))) \right\} + \frac{gccc_n \gamma_n}{n^{a_0+1}} + \frac{\overline{DD}_{n,N}}{n^{2a_0+1}} + \frac{\overline{\eta}_{n,N}}{n^{a_0+2}} \quad (623)$$

$$= \left\{ f_2(n) + \frac{gccc_n \gamma_n}{n^{a_0+1}} + \frac{\overline{DD}_{n,N}}{n^{2a_0+1}} + \frac{\overline{\eta}_{n,N}}{n^{a_0+2}} \right\} \quad (624)$$

Where

$$f_2(n) = \frac{(\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n)))^2 + gccc_n (X_1 \sin(b_0 \ln(n)) + X_2 \cos(b_0 \ln(n)))}{n^{2a_0}} \quad (625)$$

Thanks to the limit $\lim_{n \rightarrow +\infty} gccc_n \gamma_n = 0$ is zero we are now in a similar configuration like in chapter one.

We denote $\overline{g}c_n = gccc_n \gamma_n$. We have $\lim_{n \rightarrow +\infty} \overline{g}c_n = 0$.

Let's now study the serie $\sum_{n \geq 1} f_2(n)$.

Lemma 3.17. *Let's define the functions:*

$$c(t) = -\left(b_0 \sin(b_0 \ln(t)) + a_0 \cos(b_0 \ln(t))\right) \quad (626)$$

$$cc(t) = \left(b_0 \cos(b_0 \ln(t)) - a_0 \sin(b_0 \ln(t))\right) \quad (627)$$

$$gcc(t) = cc(t) - 2^{1-a_0} \left(cc(t) \cos b_0 \ln(2) + c(t) \sin b_0 \ln(2)\right) \quad (628)$$

$$cf(t) = gcc(t) \left(X_1 \sin(b_0 \ln(t)) + X_2 \cos(b_0 \ln(t))\right) \quad (629)$$

$$f_2(t) = \frac{\left(\sin(b_0 \ln(t)) - 2^{1-a_0} \sin(b_0 \ln(2t))\right)^2 + gcc(t) \left(X_1 \sin(b_0 \ln(t)) + X_2 \cos(b_0 \ln(t))\right)}{t^{2a_0}} \quad (630)$$

Let's define the sequence:

$$f_{2,n} = f_2(n) - \int_n^{n+1} dt f_2(t) \quad (631)$$

$$(632)$$

Therefore, the serie $\sum_{k \geq 1} f_{2,n}$ are converging absolutely and we have the following inequalities:

$$\left| \sum_{k=n+1}^N f_2(n) - \int_{n+1}^N f_2(t) dt \right| \leq \frac{K}{2a_0} \left(\frac{1}{n^{2a_0}} - \frac{1}{N^{2a_0}} \right) \quad (633)$$

And

$$\left| \sum_{k=n+1}^{+\infty} f_2(n) - \int_{n+1}^{+\infty} f_2(t) dt \right| \leq \frac{K}{2a_0} \frac{1}{n^{2a_0}} \quad (634)$$

where $K > 0$ and F_2 is the primitive function of f_2 is defined as:

$$F_2(x) = \left(\frac{\alpha_1}{1-2a_0} + \frac{\left(2b_0\beta_1 + \gamma_1(1-2a_0)\right) \sin(2b_0 \ln(x)) + \left(\beta_1(1-2a_0) - 2b_0\gamma_1\right) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (635)$$

Where

$$\alpha_1 = -\frac{(2a_0-1)(1-a_0)}{(b_0)^2 + (1-a_0)^2} \left(\frac{(2^{1-a_0}-1)^2}{2} + 2^{1-a_0}(1-\cos(b_0 \ln(2))) \right) \leq 0 \quad (636)$$

Proof. We are going to use the lemmas 3.1 and 3.2 defined above. We will also use their notations. We can write the functions $f_2(t)$ and $cf(t)$ as the following:

$$f_2(t) = \frac{a_1 + b_1 \cos(2b_0 \ln(t)) - c_1 \sin(2b_0 \ln(t)) + cf(t)}{t^{2a_0}} \quad (637)$$

$$gcc(t) = (-a_3 a_0 + b_3 b_0) \sin b_0(\ln(t)) + (a_3 b_0 + a_0 b_3) \cos(b_0 \ln(t)) \quad (638)$$

For simplification let's now denote:

$$gcc(t) = Z_1 \sin(b_0 \ln t) + Z_2 \cos(b_0 \ln(t)) \quad (639)$$

Therefore we can write:

$$\begin{aligned}
cf(t) &= gcc(t) \left(X_1 \sin(b_0 \ln(t)) + X_2 \cos(b_0 \ln(t)) \right) \quad (640) \\
&= \left(Z_1 \sin(b_0 \ln t) + Z_2 \cos(b_0 \ln(t)) \right) \left(X_1 \sin(b_0 \ln t) + X_2 \cos(b_0 \ln(t)) \right) \quad (641) \\
&= \frac{X_1 Z_1 + X_2 Z_2}{2} + \frac{X_1 Z_2 + X_2 Z_1}{2} \sin(2b_0 \ln(t)) + \frac{X_2 Z_2 - X_1 Z_1}{2} \cos(2b_0 \ln t) \quad (642)
\end{aligned}$$

Therefore we can write $f_2(t)$

$$\begin{aligned}
f_2(t) t^{2a_0} &= \left(a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \right) + \left(\frac{X_1 Z_2 + X_2 Z_1}{2} - c_1 \right) \sin(2b_0 \ln(t)) + \left(\frac{X_2 Z_2 - X_1 Z_1}{2} + b_1 \right) \cos(2b_0 \ln t) \\
&= \alpha_1 + \beta_1 \sin(2b_0 \ln(t)) + \gamma_1 \cos(2b_0 \ln t)
\end{aligned}$$

Where

$$\alpha_1 = a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \quad (645)$$

$$\beta_1 = \frac{X_1 Z_2 + X_2 Z_1}{2} - c_1 \quad (646)$$

$$\gamma_1 = \frac{X_2 Z_2 - X_1 Z_1}{2} + b_1 \quad (647)$$

$$Z_1 = -a_3 a_0 + b_3 b_0 \quad (648)$$

$$Z_2 = a_3 b_0 + a_0 b_3 \quad (649)$$

$$X_1 = \frac{(a_3(1-a_0) - b_3 b_0)}{(b_0)^2 + (1-a_0)^2} \quad (650)$$

$$X_2 = -\frac{b_3(1-a_0) + a_3 b_0}{(b_0)^2 + (1-a_0)^2} \quad (651)$$

$$a_1 = \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (652)$$

$$b_1 = -\frac{1}{2} - 2^{1-2a_0} + 2^{1-a_0} \cos(b_0 \ln(2)) \quad (653)$$

$$c_1 = -2^{1-a_0} \sin(b_0 \ln(2)) \quad (654)$$

$$a_3 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (655)$$

$$b_3 = 2^{1-a_0} \sin(b_0 \ln(2)) \quad (656)$$

And we apply the lemma 3.1 with $\alpha = 2a_0$, $\beta = 2b_0$, $a = \alpha_1$, $b = \beta_1$ and $c = \gamma_1$.

$$F_2(x) = \left(\frac{\alpha_1}{1-2a_0} + \frac{\left(2b_0 \beta_1 + \gamma_1(1-2a_0) \right) \sin(2b_0 \ln(x)) + \left(\beta_1(1-2a_0) - 2b_0 \gamma_1 \right) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (657)$$

When you go thru the lengthy calculation something magical will happen. We found that

$$X_1 Z_1 + X_2 Z_2 = -\frac{(a_3^2 + b_3^2) (b_0^2 + a_0(1-a_0))}{(b_0)^2 + (1-a_0)^2} \quad (658)$$

And something even more magical happens when you calculate α_1

$$\alpha_1 = a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \quad (659)$$

$$= a_1 - \frac{1}{2} \frac{(a_3^2 + b_3^2)(b_0^2 + a_0(1 - a_0))}{(b_0)^2 + (1 - a_0)^2} \quad (660)$$

$$= \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0} \cos(b_0 \ln(2)) - \frac{1}{2} \frac{(a_3^2 + b_3^2)(b_0^2 + a_0(1 - a_0))}{(b_0)^2 + (1 - a_0)^2} \quad (661)$$

$$= \frac{2^{2-2a_0} - 1}{2} + a_3 - \frac{1}{2} \frac{(2^{2-2a_0} - 1 + 2a_3)(b_0^2 + a_0(1 - a_0))}{(b_0)^2 + (1 - a_0)^2} \quad (662)$$

$$= -\left(\frac{2^{2-2a_0} - 1}{2} + a_3\right) \frac{(2a_0 - 1)(1 - a_0)}{(b_0)^2 + (1 - a_0)^2} \quad (663)$$

Therefore

$$\alpha_1 = -\frac{(2a_0 - 1)(1 - a_0)}{(b_0)^2 + (1 - a_0)^2} \left(\frac{(2^{1-a_0} - 1)^2}{2} + 2^{1-a_0}(1 - \cos(b_0 \ln(2)))\right) \quad (664)$$

$$\leq 0 \quad (665)$$

The magic here is the presence of the terms $2a_0 - 1$ and $1 - a_0$ as global factor in the formula of α_1 . \square

Let's $N \geq N_0$ be sufficiently large integer.

For $n > \frac{N}{2}$ we have similar asymptotic expansion like in the case $\bar{F}_{n,N}$:

$$\bar{G}_{n+1,N} - \bar{G}_{n,N} = \frac{\sin^2(b_0 \ln(n))}{n^{2a_0}} + \frac{\overline{CC}_{n,N}}{n^{a_0+1}} + \frac{\overline{DD}_{n,N}}{n^{2a_0+1}} + \frac{\bar{\eta}_{n,N}}{n^{a_0+2}} \quad (666)$$

Hence we have an new expression of \bar{G}_{n+1} after a summation from $\frac{N}{2} + 1$ to n :

$$\bar{G}_{n+1,N} = \bar{G}_{\frac{N}{2}+1,N} + \sum_{k=\frac{N}{2}+1}^n \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} \quad (667)$$

$$+ \sum_{k=\frac{N}{2}+1}^n \frac{\overline{CC}_{k,N}}{k^{a_0+1}} + \sum_{k=\frac{N}{2}+1}^n \frac{\overline{DD}_{k,N}}{k^{2a_0+1}} + \sum_{k=\frac{N}{2}+1}^n \frac{\bar{\eta}_{k,N}}{k^{a_0+2}} \quad (668)$$

For $n < \frac{N}{2}$ we have the following asymptotic expansion like in the case F_n :

$$\bar{G}_{n+1,N} = \bar{G}_{N_0,N} + \sum_{k=N_0}^n f_2(k) + \sum_{k=N_0}^n \frac{\overline{gC}_k}{k^{a_0+1}} \quad (669)$$

$$+ \sum_{k=N_0}^n \frac{\overline{DD}_{k,N}}{k^{2a_0+1}} + \sum_{k=N_0}^n \frac{\bar{\eta}_{k,N}}{k^{a_0+2}} \quad (670)$$

Hence if we do another summation between N_0 and N of the equation above:

$$\sum_{n=N_0}^{N-1} \overline{G}_{n+1,N} = \sum_{n=N_0}^{\frac{N}{2}-1} \overline{G}_{n+1,N} + \overline{G}_{\frac{N}{2}+1,N} + \sum_{n=\frac{N}{2}+1}^{N-1} \overline{G}_{n+1,N} \quad (671)$$

$$= \sum_{n=N_0}^{\frac{N}{2}-1} \overline{G}_{N_0,N} + \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^n f_2(k) + \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^n \frac{\overline{c}_k}{k^{a_0+1}} \quad (672)$$

$$+ \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^n \frac{\overline{DD}_{k,N}}{k^{2a_0+1}} + \sum_{n=N_0}^{\frac{N}{2}-1} \sum_{k=N_0}^n \frac{\overline{\eta}_{k,N}}{k^{a_0+2}} \quad (673)$$

$$+ \overline{G}_{\frac{N}{2}+1,N} + \sum_{n=\frac{N}{2}+1}^{N-1} \overline{G}_{\frac{N}{2}+1,N} + \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^n \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} \quad (674)$$

$$+ \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^n \frac{\overline{CC}_{k,N}}{k^{a_0+1}} + \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^n \frac{\overline{DD}_{k,N}}{k^{2a_0+1}} + \sum_{n=\frac{N}{2}+1}^{N-1} \sum_{k=\frac{N}{2}+1}^n \frac{\overline{\eta}_{k,N}}{k^{a_0+2}} \quad (675)$$

Our sums are finite so we can interchange them:

$$\sum_{n=N_0}^{N-1} \overline{G}_{n+1,N} = \left(\frac{N}{2} - N_0\right) \overline{G}_{N_0,N} + \frac{N}{2} \overline{G}_{\frac{N}{2}+1,N} + \sum_{k=N_0}^{\frac{N}{2}-1} f_2(k) \sum_{n=k}^{\frac{N}{2}-1} 1 \quad (676)$$

$$+ \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\overline{gC}_k}{k^{a_0+1}} + \frac{\overline{DD}_{k,N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+2}} \right\} \sum_{n=k}^{\frac{N}{2}-1} 1 + \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\overline{CC}_{k,N}}{k^{a_0+1}} + \frac{\overline{D}_{k,N}}{k^{2a_0+1}} + \frac{\overline{\varepsilon}_{k,N}}{k^{a_0+2}} \right\} \sum_{n=k}^{N-1} 1 \quad (677)$$

$$= \left(\frac{N}{2} - N_0\right) \overline{G}_{N_0,N} + \frac{N}{2} \overline{G}_{\frac{N}{2}+1,N} + \sum_{k=N_0}^{\frac{N}{2}-1} f_2(k) \left(\frac{N}{2} - k\right) + \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\overline{c}_k}{k^{a_0+1}} + \frac{\overline{D}_{k,N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+2}} \right\} \left(\frac{N}{2} - k\right) \quad (678)$$

$$+ \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\overline{CC}_{k,N}}{k^{a_0+1}} + \frac{\overline{DD}_{k,N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+2}} \right\} (N - k) \quad (679)$$

$$= \left(\frac{N}{2} - N_0\right) \overline{G}_{N_0,N} + \frac{N}{2} \overline{G}_{\frac{N}{2}+1,N} + \frac{N}{2} \sum_{k=N_0}^{\frac{N}{2}-1} f_2(k) - \sum_{k=N_0}^{\frac{N}{2}-1} k f_2(k) \quad (680)$$

$$+ \frac{N}{2} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\overline{gC}_k}{k^{a_0+1}} + \frac{\overline{DD}_{k,N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+2}} \right\} - \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\overline{gC}_k}{k^{a_0}} + \frac{\overline{DD}_{k,N}}{k^{2a_0}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+1}} \right\} \quad (681)$$

$$+ \frac{N}{2} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\overline{CC}_{k,N}}{k^{a_0+1}} + \frac{\overline{DD}_{k,N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+2}} \right\} \quad (682)$$

$$- \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0-1}} + \frac{\overline{CC}_{k,N}}{k^{a_0}} + \frac{\overline{DD}_{k,N}}{k^{2a_0}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+1}} \right\} \quad (683)$$

And Finally:

$$\sum_{n=N_0}^{N-1} \overline{G}_{n+1,N} = N \left[\left(\frac{1}{2} - \frac{N_0}{N} \right) \overline{G}_{N_0,N} + \frac{1}{2} \overline{G}_{\frac{N}{2}+1,N} + \frac{1}{2} \sum_{k=N_0}^{\frac{N}{2}-1} f_2(k) - \frac{1}{N} \sum_{k=N_0}^{\frac{N}{2}-1} k f_2(k) \right] \quad (684)$$

$$+ \frac{1}{2} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\overline{gC}_k}{k^{a_0+1}} + \frac{\overline{DD}_{k,N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+2}} \right\} - \frac{1}{N} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\overline{gC}_k}{k^{a_0}} + \frac{\overline{DD}_{k,N}}{k^{2a_0}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+1}} \right\} \quad (685)$$

$$+ \frac{1}{2} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\overline{CC}_{k,N}}{k^{a_0+1}} + \frac{\overline{DD}_{k,N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+2}} \right\} \quad (686)$$

$$- \frac{1}{N} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0-1}} + \frac{\overline{CC}_{k,N}}{k^{a_0}} + \frac{\overline{DD}_{k,N}}{k^{2a_0}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+1}} \right\} \quad (687)$$

The goal is to prove that $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^{N-1} \overline{G}_{n+1,N} = -\infty$.

We proceed exactly like the previous case of $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^{N-1} \overline{F}_{n+1,N}$.

Using Césaro lemma we prove that the limit of the following terms are zero:

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=N_0}^{\frac{N}{2}-1} k f_2(k) = 0 \quad (688)$$

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\overline{gC}_k}{k^{a_0}} + \frac{\overline{DD}_{k,N}}{k^{2a_0}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+1}} \right\} = 0 \quad (689)$$

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0-1}} + \frac{\overline{CC}_{k,N}}{k^{a_0}} + \frac{\overline{DD}_{k,N}}{k^{2a_0}} + \frac{\overline{\eta}_{k,N}}{k^{a_0+1}} \right\} = 0 \quad (690)$$

We have

$$\lim_{N \rightarrow +\infty} \frac{N_0}{N} \overline{G}_{N_0,N} + \frac{1}{2} \overline{G}_{\frac{N}{2}+1,N} = 0 \quad (691)$$

One note that $\overline{G}_{N_0,N}$ is a constant term that actually depends only on N_0 , so $\lim_{N \rightarrow +\infty} \frac{N_0}{N} \overline{G}_{N_0,N} = 0$. For the term $\frac{1}{2} \overline{G}_{\frac{N}{2}+1,N}$, we use the lemma 3.6, we have two constants K_1 and K_2 such that: $\left| \overline{B}_{\frac{N}{2}+1,N} \right| \leq K_1 + K_2 \left(\frac{N}{2} + 1 \right)^{1-a_0}$, therefore

$$\overline{G}_{\frac{N}{2}+1,N} = \frac{\left| \sin(b_0 \ln(\frac{N}{2} + 1)) \overline{B}_{\frac{N}{2}+1,N} \right|}{\left(\frac{N}{2} + 1 \right)^{a_0}} \quad (692)$$

$$\leq \frac{K_1}{\left(\frac{N}{2} + 1 \right)^{a_0}} + \frac{K_2}{\left(\frac{N}{2} + 1 \right)^{2a_0-1}} \quad (693)$$

We have $2a_0 > 1$ therefore the $\lim_{N \rightarrow +\infty} \overline{G}_{\frac{N}{2}+1,N} = 0$

Let's now prove

$$\lim_{N \rightarrow +\infty} \left[\frac{1}{2} \overline{G}_{N_0, N} + \frac{1}{2} \sum_{k=N_0}^{\frac{N}{2}-1} f_2(k) + \frac{1}{2} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\overline{gC}_k}{k^{a_0+1}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right\} \right] \quad (694)$$

$$+ \frac{1}{2} \sum_{k=\frac{N}{2}+1}^{N-1} \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\overline{CC}_{k, N}}{k^{a_0+1}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right\} < 0 \quad (695)$$

We have the sequences $(\overline{DD}_{k, N})_{k \geq 1}$ and $(\overline{\eta}_{k, N})_{k \geq 1}$ are bounded. Plus We have $a_0 + 2 \geq 2a_0 + 1 > a_0 + 1 \geq 2a_0 > 1$, therefore the serie $\sum_{k \geq 1} \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right\}$ is converging absolutely.

Let's call its partial sum S_N :

$$S_N = \sum_{k \geq 1}^N \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right\} \quad (696)$$

$$S_N - S_{\frac{N}{2}} = \sum_{k=\frac{N}{2}+1}^N \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right\} \quad (697)$$

Therefore

$$\lim_{N \rightarrow +\infty} \sum_{k=\frac{N}{2}+1}^N \left\{ \frac{\sin^2(b_0 \ln(k))}{(k)^{2a_0}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right\} \quad (698)$$

$$= \lim_{N \rightarrow +\infty} S_N - \lim_{N \rightarrow +\infty} S_{\frac{N}{2}} = 0 \quad (699)$$

Let's study now the convergence of the sum $\sum_{k=\frac{N}{2}+1}^N \frac{\overline{CC}_{k, N}}{k^{a_0+1}}$. From the lemma 3.6 we have two constants $K_1 \geq 0$ and $K_2 \geq 0$ such that:

$$|\overline{CC}_{k, N}| \leq K_1 + K_2 k^{1-a_0} \quad (700)$$

Therefore

$$\left| \frac{\overline{CC}_{k, N}}{k^{a_0+1}} \right| \leq \frac{K_1}{k^{a_0+1}} + K_2 \frac{k^{1-a_0}}{k^{a_0+1}} \quad (701)$$

$$\leq \frac{K_1}{k^{a_0+1}} + K_2 \frac{1}{k^{2a_0}} \quad (702)$$

We have $a_0+1 \geq 2a_0 > 1$, therefore the serie $\sum_{k \geq 1} \frac{\overline{CC}_{k, N}}{k^{a_0+1}}$ is converging absolutely. Therefore like in the case of E_N , we have

$$\lim_{N \rightarrow +\infty} \sum_{k=\frac{N}{2}+1}^N \frac{\overline{CC}_{k, N}}{k^{a_0+1}} = \lim_{N \rightarrow +\infty} \sum_{k=1}^N \frac{\overline{CC}_{k, N}}{k^{a_0+1}} - \lim_{N \rightarrow +\infty} \sum_{k=1}^{\frac{N}{2}+1} \frac{\overline{CC}_{k, N}}{k^{a_0+1}} = 0 \quad (703)$$

Let's call now the remaining term R_{N_0} :

$$R_{N_0} = \lim_{N \rightarrow +\infty} \left[\overline{G}_{N_0, N} + \sum_{k=N_0}^{\frac{N}{2}-1} f_2(k) + \frac{\overline{gC}_k}{k^{a_0+1}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right] \quad (704)$$

Lemma 3.18. *We can choose N_0 such that $R_{N_0} < 0$.*

Proof. We studied the serie $\sum_{n \geq 1} f_2(n)$ in the lemma 3.6. We have:

$$\left| \sum_{n=N_0+1}^{+\infty} f_2(n) - \int_{N_0+1}^{+\infty} f_2(t) dt \right| \leq \frac{K}{2a_0(N_0)^{2a_0}} \quad (705)$$

Where $K > 0$ and the primitive function F_2

$$F_2(x) = \left(\frac{\alpha_1}{1-2a_0} + \frac{(2b_0\beta_1 + \gamma_1(1-2a_0)) \sin(2b_0 \ln(x)) + (\beta_1(1-2a_0) - 2b_0\gamma_1) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (706)$$

With

$$\alpha_1 = -\frac{(2a_0-1)(1-a_0)}{(b_0)^2 + (1-a_0)^2} \left(\frac{(2^{1-a_0}-1)^2}{2} + 2^{1-a_0}(1 - \cos(b_0 \ln(2))) \right) \leq 0 \quad (707)$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} f_2(k) + F_2(N_0+1) \right| \leq \frac{K}{a_0(N_0)^{2a_0}} \quad (708)$$

Let's now study the term $\overline{G}_{N_0, N}$. From lemma 3.11 and 3.12 we can write for $N \geq 2N_0$

$$\overline{G}_{N_0, N} = \frac{\sin(b_0 \ln(N_0)) - 2^{1-a_0} \sin(b_0 \ln(2N_0))}{(N_0)^{a_0}} B_{N_0, N} \quad (709)$$

$$= \frac{\sin(b_0 \ln(N_0)) - 2^{1-a_0} \sin(b_0 \ln(2N_0))}{(N_0)^{a_0}} \underbrace{\left(F_1(N_0) + \gamma_{N_0} \right)}_{\text{Lemma 3.11}} \quad (710)$$

$$= \underbrace{f_3(N_0)}_{\text{Lemma 3.12}} + \frac{\sin(b_0 \ln(N_0)) - 2^{1-a_0} \sin(b_0 \ln(2N_0))}{(N_0)^{a_0}} \gamma_{N_0} \quad (711)$$

$$(712)$$

Where the sequence γ_n converge to zero and

$$f_3(t) (t)^{2a_0-1} = \frac{\alpha_3 + \beta_3 \sin(2b_0 \ln(t)) + \gamma_3 \cos(2b_0 \ln t)}{(t)^{2a_0-1}} \quad (713)$$

$$\alpha_3 = \frac{X_1 b_3 + X_2 a_3}{2} \quad (714)$$

$$\beta_3 = \frac{X_1 a_3 + X_2 b_3}{2} \quad (715)$$

$$\gamma_3 = \frac{X_2 a_3 - X_1 b_3}{2} \quad (716)$$

$$X_1 = \frac{a_3 b_0 + b_3(1-a_0)}{(b_0)^2 + (1-a_0)^2} \quad (717)$$

$$X_2 = \frac{a_3(1-a_0) - b_3 b_0}{(b_0)^2 + (1-a_0)^2} \quad (718)$$

Let's denote

$$GO_{N_0} = -\left(f_3(N_0 + 1) - f_3(N_0)\right) + \frac{\sin(b_0 \ln(N_0)) - 2^{1-a_0} \sin(b_0 \ln(2N_0))}{(N_0)^{a_0}} \gamma_{N_0} \quad (719)$$

The term $f_3(N_0)$ and the term $F_2(N_0)$ share the term $\frac{1}{(N_0)^{2a_0-1}}$. So we are going to group them in the last inequality:

$$\left| f_3(N_0 + 1) + \sum_{n=N_0+1}^{+\infty} f_2(k) + \underbrace{F_2(N_0 + 1) - f_3(N_0 + 1)}_{f_4(N_0+1)} \right| \leq \frac{K}{a_0(N_0)^{2a_0}} \quad (720)$$

Where

$$\begin{aligned} f_4(x) &= F_2(x) - f_3(x) & (721) \\ &= \frac{\alpha_2 - \alpha_3 + (\beta_2 - \beta_3) \sin(2b_0 \ln(t)) + (\gamma_2 - \gamma_3) \cos(2b_0 \ln t)}{x^{2a_0-1}} & (722) \end{aligned}$$

We develop the calculation further and we see that we have $\alpha_2 = \alpha_3$. Therefore

$$f_4(x) = \frac{(\beta_2 - \beta_3) \sin(2b_0 \ln(t)) + (\gamma_2 - \gamma_3) \cos(2b_0 \ln t)}{x^{2a_0-1}} \quad (723)$$

Therefore we have the asymptotic expansion of $f_3(N_0+1) + \sum_{n=N_0+1}^{+\infty} f_2(k)$ as follows:

$$f_3(N_0 + 1) + \sum_{n=N_0+1}^{+\infty} f_2(k) = -F_2(N_0 + 1) + f_3(N_0 + 1) + \mathcal{O}\left(\frac{1}{(N_0)^{2a_0}}\right) \quad (724)$$

$$= \frac{(\beta_3 - \beta_2) \sin(2b_0 \ln(N_0 + 1)) + (\gamma_3 - \gamma_2) \cos(2b_0 \ln(N_0 + 1))}{(N_0 + 1)^{2a_0-1}} \quad (725)$$

$$+ \mathcal{O}\left(\frac{1}{(N_0)^{2a_0}}\right) \quad (726)$$

Remark. The remaining terms in the expression of R_{N_0} are all of the order of $\frac{1}{(N_0)^{a_0}}$ and above. we note that $2a_0 - 1 < a_0$. So the dominant term in the expression R_{N_0} is the term $-f_4(N_0+1) = \frac{(\beta_3 - \beta_2) \sin(2b_0 \ln(N_0+1)) + (\gamma_3 - \gamma_2) \cos(2b_0 \ln(N_0+1))}{(N_0+1)^{2a_0-1}}$. As we will see later that the function e_4 is nonzero function. Therefore this term is the dominant term in the expression of $\frac{1}{N} \sum_{n=N_0}^{N-1} \overline{G}_{n+1, N}$. Hence based on the sign of $f_4(N_0+1)$ we can show that the limit of $\sum_{n=1}^{N-1} \overline{G}_{n+1, N}$ can be both $+\infty$ and $-\infty$.

Let's now study the function f :

$$f(x) = (\beta_2 - \beta_3) \sin(2b_0 x) + (\gamma_2 - \gamma_3) \cos(2b_0 x) \quad (727)$$

General Case: $\frac{1}{2} < a_0 < 1$ **OR** $a_0 = 1, b_0 \neq \frac{2k\pi}{\ln(2)}$ From the lemma 3.20 we have that the function f_4 is a nonzero function. Therefore the function f is also nonzero function and hence at least $\beta_2 - \beta_3$ or $\gamma_2 - \gamma_3$ is different from zero or both of them. So let's assume that $\gamma_2 \neq \gamma_3$. The function f is a linear combination of the functions sin and cos. So the function f is differentiable and bounded. The function f is periodic of period $\frac{\pi}{b_0}$.

Let's calculate the function f values at the following points:

$$f(0) = \gamma_2 - \gamma_3 \quad (728)$$

$$f\left(\frac{\pi}{2b_0}\right) = \beta_2 - \beta_3 \quad (729)$$

$$f\left(\frac{\pi}{b_0}\right) = -\gamma_2 + \gamma_3 \quad (730)$$

$$f\left(\frac{3\pi}{2b_0}\right) = -\beta_2 + \beta_3 \quad (731)$$

From the values above we have the following:

- If $(\gamma_2 - \gamma_3) > 0$ then $f(0) > 0$.
- If $(\gamma_2 - \gamma_3 < 0$ then $f(\frac{\pi}{b_0}) > 0$.

In case of $\gamma_2 - \gamma_3 = 0$ and we will work with $\beta_2 - \beta_3$ in such case we calculate the f values at the points $f(\frac{\pi}{2b_0})$ and $f(\frac{3\pi}{2b_0})$. Therefore, in all case we have either $f(0) > 0$ or $f(\frac{\pi}{b_0}) > 0$. So let's assume from now on that $f(0) < 0$ as the same proof can be done with both values thanks to lemma 3.3 and lemma 3.4.

From the lemma 3.3 above, there exist $N_1 \geq N_0$ such that $|\cos(2b_0 \ln(N_1 + 1)) - 1| \leq \epsilon$ and $|\sin(2b_0 \ln(N_1 + 1))| \leq \epsilon$ for any $1 > \epsilon > 0$.

Let's define $\beta_0 = -f(0)$. $\beta_0 < 0$.

Let's fix $\epsilon > 0$ to be very small such that $0 < \epsilon < \min\left(1, \frac{\beta_0}{10000}\right)$.

Let's define $\gamma_0 = |\beta_2 - \beta_3| + |\gamma_2 - \gamma_3|$

Let's N_1 be such that $|\cos(2b_0 \ln(N_1 + 1)) - 1| \leq \epsilon/\gamma_0$ and also $|\sin(2b_0 \ln(N_1 + 1)) - 0| \leq \epsilon/\gamma_0$. Therefore:

$$\left| f_4(N_1 + 1) - \frac{f(0)}{(N_1 + 1)^{2a_0 - 1}} \right| \leq \left| \left\{ \frac{\left((\beta_2 - \beta_3) \sin(2b_0 \ln(N_1 + 1)) + (\gamma_2 - \gamma_3) \cos(2b_0 \ln(N_1 + 1)) \right)}{(N_1 + 1)^{2a_0 - 1}} \right. \right. \quad (732)$$

$$\left. - \frac{\left((\beta_2 - \beta_3) \sin(2b_0 0) + (\gamma_2 - \gamma_3) \cos(2b_0 0) \right)}{(N_1 + 1)^{2a_0 - 1}} \right\} \quad (733)$$

$$\leq \frac{\left(|\beta_2 - \beta_3| |\sin(2b_0 \ln(N_1 + 1))| + |\gamma_2 - \gamma_3| |\cos(2b_0 \ln(N_1 + 1)) - 1| \right)}{(N_1 + 1)^{2a_0 - 1}} \quad (734)$$

$$\leq \frac{\left(|\beta_2 - \beta_3| \epsilon / \gamma_0 + |\gamma_2 - \gamma_3| \epsilon / \gamma_0 \right)}{(N_1 + 1)^{2a_0 - 1}} \quad (735)$$

$$\leq \frac{\epsilon}{(N_1 + 1)^{2a_0 - 1}} \quad (736)$$

Therefore

$$\left| f_3(N_0 + 1) + \sum_{n=N_0+1}^{+\infty} f_2(n) - \frac{\beta_0}{(N_0 + 1)^{2a_0 - 1}} \right| \leq \left\{ \left| \sum_{n=N_0+1}^{+\infty} f_2(n) + f_4(N_0 + 1) \right| \quad (737) \right.$$

$$\left. + \left| -f_4(N_0 + 1) - \frac{-f(0)}{(N_0 + 1)^{2a_0 - 1}} \right| \right\} \leq \frac{\epsilon}{(N_0 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_0)^{2a_0}} \quad (738)$$

Hence

$$\left| f_3(N_0 + 1) + \sum_{n=N_0+1}^{+\infty} f_2(n) - \frac{\beta_0}{(N_0 + 1)^{2a_0 - 1}} \right| \leq \frac{\epsilon}{(N_0 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_0)^{2a_0}} \quad (739)$$

Therefore

$$f_3(N_0 + 1) + \sum_{n=N_0+1}^{+\infty} f_2(n) \leq \frac{\beta_0 + \epsilon}{(N_0 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_0)^{2a_0}} \quad (740)$$

Therefore

$$R_{N_0} = \sum_{k=N_0}^{+\infty} f_2(k) + \overline{G}_{N_0, N} + \lim_{N \rightarrow +\infty} \left[\sum_{k=N_0}^N \left\{ \frac{\overline{gC}_k}{k^{a_0+1}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right\} \right] \quad (741)$$

$$\leq \frac{\beta_0 + \epsilon}{(N_0 + 1)^{2a_0-1}} + \frac{K}{2a_0(N_1)^{2a_0}} + \underbrace{\overline{G}_{N_0, N} - f_3(N_0 + 1)}_{\leq 0} + \lim_{N \rightarrow +\infty} \left[\sum_{k=N_0}^N \left\{ \frac{\overline{gC}_k}{k^{a_0+1}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right\} \right] \quad (742)$$

$$\leq \frac{\beta_0 + \epsilon}{(N_0 + 1)^{2a_0-1}} + \frac{K}{2a_0(N_0)^{2a_0}} + \underbrace{f_3(N_0) - f_3(N_0 + 1) + \frac{\sin(b_0 \ln(N_0)) - 2^{1-a_0} \sin(b_0 \ln(2N_0))}{(N_0)^{a_0}}}_{\leq 0} \gamma_{N_0} \quad (743)$$

$$+ \lim_{N \rightarrow +\infty} \sum_{k=N_0}^N \left\{ \frac{\overline{gC}_k}{k^{a_0+1}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right\} \quad (744)$$

$$\leq \frac{\beta_0 + \epsilon}{(N_0 + 1)^{2a_0-1}} + \frac{K}{2a_0(N_0)^{2a_0}} + GO_{N_0} + \lim_{N \rightarrow +\infty} \sum_{k=N_0}^N \left\{ \frac{\overline{gC}_k}{k^{a_0+1}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right\} \quad (745)$$

We have the sequences (\overline{gC}_k) , $(\overline{DD}_{k, N})_{k \geq 1}$ and $(\overline{\eta}_{k, N})_{k \geq 1}$ are bounded. Plus we have $a_0 + 2 \geq 2a_0 + 1 > a_0 + 1 \geq 2a_0 > 1$, therefore the series $\sum_{k \geq 1} \frac{\overline{DD}_{k, N}}{k^{2a_0+1}}$, $\sum_{k \geq 1} \frac{\overline{\eta}_{k, N}}{k^{a_0+2}}$ are converging absolutely.

Let the positive constant M such that:
For each $k \geq N_1$ and $N \geq 2N_1$:

$$|\overline{gC}_k| \leq M \quad (746)$$

$$|\overline{DD}_{k, N}| \leq M \quad (747)$$

$$|\overline{\eta}_{k, N}| \leq M \quad (748)$$

Therefore, we have $a_0 > 0$:

$$\left| \sum_{n=N_1+1}^{+\infty} \frac{\overline{gC}_k}{k^{a_0+1}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\overline{\eta}_{k, N}}{k^{a_0+2}} \right| \quad (749)$$

$$\leq \frac{M}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0 + 1)(N_1)^{a_0+1}} \quad (750)$$

Therefore

$$R_{N_1} \leq \frac{\beta_0 + \epsilon}{(N_1 + 1)^{2a_0-1}} + \frac{K}{2a_0(N_1)^{2a_0}} + O_{N_1} + \frac{M}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0 + 1)(N_1)^{a_0+1}} \quad (751)$$

$$\leq \frac{1}{(N_1 + 1)^{2a_0-1}} \left(\beta_0 + \epsilon + \frac{K(N_1 + 1)^{2a_0-1}}{2a_0(N_1)^{2a_0}} + (N_1 + 1)^{2a_0-1} GO_{N_1} \right) \quad (752)$$

$$+ \frac{M(N_1 + 1)^{2a_0-1}}{a_0(N_1)^{a_0}} + \frac{M(N_1 + 1)^{2a_0-1}}{2a_0(N_1)^{2a_0}} + \frac{M(N_1 + 1)^{2a_0-1}}{(a_0 + 1)(N_1)^{a_0+1}} \quad (753)$$

Case One: $\frac{1}{2} < a_0 < 1$ Let's define the sequence (δ_n) :

$$\delta_n = \frac{K(n + 1)^{2a_0-1}}{2a_0(n)^{2a_0}} + (n + 1)^{2a_0-1} GO_n + \frac{M(n + 1)^{2a_0-1}}{(n)^{a_0}} \quad (754)$$

$$+ \frac{M(n + 1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n + 1)^{2a_0-1}}{(a_0 + 1)(n)^{a_0+1}} \quad (755)$$

As we have $\frac{1}{2} < a_0 < 1$, therefore $2a_0 - 1 < a_0 < 2a_0 < a_0 + 1$
Therefore

$$\lim_{N \rightarrow +\infty} \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(n)^{a_0}} \quad (756)$$

$$+ \frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(a_0+1)(n)^{a_0+1}} = 0 \quad (757)$$

Let's calculate now the limit of $\lim_{n \rightarrow +\infty} (n+1)^{2a_0-1} GO_n$
From the lemma 3.8 ($\alpha = 2a_0$ and $\beta = b_0$) we can write:

$$(n+1)^{2a_0-1} GO_n = (n+1)^{2a_0-1} (f_3(n) - f_3(n+1)) + (n+1)^{2a_0-1} \frac{\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n))}{n^{a_0}} \gamma_n \quad (758)$$

$$= -a(n+1)^{2a_0-1} \frac{1-2a_0}{n^{2a_0}} - (n+1)^{2a_0-1} \frac{(b c c_n(2a_0-1, b_0) + c c_n(2a_0-1, b_0))}{n^{2a_0}} \quad (759)$$

$$+ (n+1)^{2a_0-1} \frac{\sin(b_0 \ln(n)) - 2^{1-a_0} \sin(b_0 \ln(2n))}{n^{a_0}} \gamma_n + \mathcal{O}\left(\frac{1}{n^{2a_0}}\right) \quad (760)$$

Thanks to $2a_0 - 1 < a_0 < 2a_0 < a_0 + 1$, we have the limit of every term
in the expression of $(n+1)^{2a_0-1} O_n$ goes to zero, therefore $\lim_{n \rightarrow +\infty} (n+1)^{2a_0-1} O_n = 0$.

Therefore the $\lim_{n \rightarrow +\infty} \delta_n = 0$.

So we can choose N_1 such that $|\delta_{N_1}| < \epsilon$, i.e $\delta_{N_1} \leq \epsilon$.

Therefore

$$R_{N_1} \leq \frac{1}{(N_1+1)^{2a_0-1}} (\beta_0 + 2\epsilon) < 0 \quad (761)$$

Therefore $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^{N-1} \bar{F}_{n+1, N} = -\infty$.

Case Two: $a_0 = 1, b_0 \neq \frac{2k\pi}{\ln(2)}$ In this particular case we have $2a_0 - 1 = 1 = a_0$. Therefore the terms of the order $\frac{1}{n^{a_0}}$ in the expression of δ_n and GO_n can become problematic. So instead of using the fact $(\bar{g}c_k)$ is bounded, we will use here the fact that its limit equal zero when n goes to infinity.

As we have

$$\lim_{n \rightarrow +\infty} \bar{g}c_n = 0 \quad (762)$$

So we can choose N_1 such for each $n \geq N_1$ we have $|\bar{g}c_k| \leq \epsilon$.

For each $k \geq N_1$ and $N \geq 2N_1$:

$$|\bar{g}c_k| \leq \epsilon \quad (763)$$

$$|\overline{DD}_{k, N}| \leq M \quad (764)$$

$$|\bar{\eta}_{k, N}| \leq M \quad (765)$$

Therefore, we have $a_0 > 0$:

$$\left| \sum_{n=N_1+1}^{+\infty} \frac{\bar{g}c_k}{k^{a_0+1}} + \frac{\overline{DD}_{k, N}}{k^{2a_0+1}} + \frac{\bar{\eta}_{k, N}}{k^{a_0+2}} \right| \leq \frac{\epsilon}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0+1)(N_1)^{a_0+1}} \quad (766)$$

And

$$R_{N_1} \leq \frac{\beta_0 + \epsilon}{(N_1 + 1)^{2a_0 - 1}} + \frac{K}{2a_0(N_1)^{2a_0}} + GO_{N_1} + \frac{\epsilon}{a_0(N_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0 + 1)(N_1)^{a_0 + 1}} \quad (767)$$

$$\leq \frac{1}{(N_1 + 1)^{2a_0 - 1}} \left(\beta_0 + \epsilon + \frac{K(N_1 + 1)^{2a_0 - 1}}{2a_0(N_1)^{2a_0}} + (N_1 + 1)^{2a_0 - 1} GO_{N_1} \right) \quad (768)$$

$$+ \frac{\epsilon(N_1 + 1)^{2a_0 - 1}}{a_0(N_1)^{a_0}} + \frac{M(N_1 + 1)^{2a_0 - 1}}{2a_0(N_1)^{2a_0}} + \frac{M(N_1 + 1)^{2a_0 - 1}}{(a_0 + 1)(N_1)^{a_0 + 1}} \quad (769)$$

We have $a_0 = 1$ therefore:

$$R_{N_1} \leq \frac{1}{(N_1 + 1)} \left(\beta_0 + 2\epsilon + \frac{K(N_1 + 1)}{2(N_1)^2} + (N_1 + 1) GO_{N_1} + \frac{1}{N_1} + \frac{M(N_1 + 1)}{2(N_1)^2} + \frac{M(N_1 + 1)}{2(N_1)^2} \right) \quad (770)$$

Because $\beta_0 < 0$ and we have $\epsilon < \frac{|\beta_0|}{10000} < 1$

We have the sequence (δ_n) :

$$\delta_n = \frac{K(n + 1)}{2n^2} + (n + 1) GO_n + \frac{1}{n} + \frac{M(n + 1)}{2n^2} + \frac{M(n + 1)}{2n^2} \quad (771)$$

For the term $(n + 1)^{2a_0 - 1} GO_n$ we have its expression:

$$(n + 1) GO_n = -a(n + 1) \frac{-1}{n^2} - (n + 1)^1 \frac{(bcc_n(1, b_0) + cc_n(1, b_0))}{n^2} \quad (772)$$

$$+ (n + 1)^1 \frac{\sin(b_0 \ln(n)) - \sin(b_0 \ln(2n))}{n} \gamma_n + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (773)$$

In the expression of $(n + 1)^{2a_0 - 1} GO_n$ there is γ_n term that can be problematic because $2a_0 - 1 = a_0$. But thanks to the limit $\lim_{n \rightarrow +\infty} \gamma_n = 0$. We also have $\lim_{n \rightarrow +\infty} (n + 1)^{2a_0 - 1} GO_n = 0$.

Therefore

$$\lim_{n \rightarrow +\infty} \delta_n = 0 \quad (774)$$

So we can choose N_1 such that $|\delta_{N_1}| < \epsilon$, i.e $\delta_{N_1} \leq \epsilon$.

Therefore

$$R_{N_1} \leq \frac{1}{N_1 + 1} \left(\beta_0 + 3\epsilon \right) < 0 \quad (775)$$

Therefore $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^{N-1} \bar{G}_{n+1, N} = -\infty$. □

Remark. Note that one also can prove that the $\lim_{N \rightarrow +\infty} \sum_{n=N_0}^{N-1} \bar{G}_{n+1, N} = +\infty$ by choosing another point x_0 where $f(x_0) < 0$. The same steps above apply. And this thanks to the term $\frac{(\beta_3 - \beta_2) \sin(2b_0 \ln(N_0 + 1)) + (\gamma_3 - \gamma_2) \cos(2b_0 \ln(N_0 + 1))}{(N_0 + 1)^{2a_0 - 1}}$.

Remark. Note that the fact that s is a zero for ζ and its consequence the limit $\left(\lim_{N \rightarrow +\infty} \bar{B}_{N, N} = 0 \right)$ is important in the above proof. We were able to prove the above thanks to the fact that $\frac{1}{2} < a_0 \leq 1$ and the limit zero of the term $(n + 1)^{2a_0 - 1} \frac{\sin(b_0 \ln(n)) - 2^{1 - a_0} \sin(b_0 \ln(2n))}{n^{a_0}} \gamma_n$.

□

Conclusion. We saw above that the serie $\sum_{n \geq 1} (\bar{U}_{n,N}^2 + \bar{V}_{n,N}^2)$ is converging absolutely. We have from the lemmas 3.13 and 3.16 that:

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \bar{F}_{n,N} = -\infty \quad (776)$$

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \bar{G}_{n,N} = -\infty \quad (777)$$

Therefore

$$\lim_{N \rightarrow \infty} \bar{A}_{N,N} = -\infty \quad (778)$$

$$\lim_{N \rightarrow \infty} \bar{B}_{N,N} = -\infty \quad (779)$$

thanks to $2a_0 > 1$ that makes the series $\sum_{n \geq 1} \bar{U}_{n,N}^2$ and $\sum_{n \geq 1} \bar{V}_{n,N}^2$ converging absolutely. This result is in contradiction with the fact that s is a ζ zero therefore the limit $\lim_{N \rightarrow \infty} \bar{A}_{N,N} = 0$ and the limit $\lim_{N \rightarrow \infty} \bar{B}_{N,N} = 0$. Therefore s with $\frac{1}{2} < a_0 = \Re(s) \leq 1$ cannot be a zero for the Riemann's Zeta Function.

Lemma 3.19. Let's define the function e_4 :

$$e_4(t) = E_2(t) - e_3(t) \quad (780)$$

Where e_3 and E_2 are the same functions defined in the lemmas 3.10 and 3.14. Therefore the function e_4 is a nonzero function, i.e we don't have for each t , $e_4(t) = 0$.

$$E_2(x) = \left(\frac{\alpha_1}{1-2a_0} + \frac{(2b_0\beta_1 + \gamma_1(1-2a_0)) \sin(2b_0 \ln(x)) + (\beta_1(1-2a_0) - 2b_0\gamma_1) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1-2a_0)^2} \right) \frac{1}{(x)^{2a_0-1}} \quad (781)$$

The function E_2 is the primitive function of the function $e_2(t)$ where

$$\begin{aligned} e_2(t) t^{2a_0} &= \left(a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \right) + \left(\frac{X_1 Z_2 + X_2 Z_1}{2} - c_1 \right) \sin(2b_0 \ln(t)) + \left(\frac{X_2 Z_2 - X_1 Z_1}{2} + b_1 \right) \cos(2b_0 \ln t) \\ &= \alpha_1 + \beta_1 \sin(2b_0 \ln(t)) + \gamma_1 \cos(2b_0 \ln t) \end{aligned}$$

And the function e_3 is defined as

$$e_3(t) (t)^{2a_0-1} = \alpha_3 + \beta_3 \sin(2b_0 \ln(t)) + \gamma_3 \cos(2b_0 \ln t) \quad (784)$$

Where

$$\alpha_1 = a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \quad (785)$$

$$\beta_1 = \frac{X_1 Z_2 + X_2 Z_1}{2} - c_1 \quad (786)$$

$$\gamma_1 = \frac{X_2 Z_2 - X_1 Z_1}{2} + b_1 \quad (787)$$

$$Z_1 = -(a_3 b_0 + b_3 a_0) \quad (788)$$

$$Z_2 = b_3 b_0 - a_3 a_0 \quad (789)$$

$$X_1 = \frac{a_3 b_0 + b_3(1 - a_0)}{(b_0)^2 + (1 - a_0)^2} \quad (790)$$

$$X_2 = \frac{a_3(1 - a_0) - b_3 b_0}{(b_0)^2 + (1 - a_0)^2} \quad (791)$$

$$a_1 = \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (792)$$

$$b_1 = \frac{1}{2} + 2^{1-2a_0} \cos(2b_0 \ln(2)) - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (793)$$

$$c_1 = 2^{1-2a_0} \sin(2b_0 \ln(2)) + 2^{1-a_0} \sin(b_0 \ln(2)) \quad (794)$$

$$a_3 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (795)$$

$$b_3 = 2^{1-a_0} \sin(b_0 \ln(2)) \quad (796)$$

$$\alpha_3 = \frac{X_1 b_3 + X_2 a_3}{2} \quad (797)$$

$$\beta_3 = \frac{X_1 a_3 + X_2 b_3}{2} \quad (798)$$

$$\gamma_3 = \frac{X_2 a_3 - X_1 b_3}{2} \quad (799)$$

Proof. To prove this lemma we proceed by contradiction. Let's suppose that the function $e_4 = 0$. Therefore for each $t \neq 0$, $e_3(t) = E_2(t)$. Hence we also have their first derivatives are equals: for each t , $e_2(t) = e_3'(t)$. Therefore the coefficients of e_2 and e_3' are equals:

$$\alpha_1 = (1 - 2a_0)\alpha_3 \quad (800)$$

$$\beta_1 = \beta_3(1 - 2a_0) - 2b_0\gamma_3 \quad (801)$$

$$\gamma_1 = (1 - 2a_0)\gamma_3 + 2b_0\beta_3 \quad (802)$$

Quick calculation shows that in fact

$$\alpha_1 = (1 - 2a_0)\alpha_3 = -\frac{(2a_0 - 1)(1 - a_0)}{(b_0)^2 + (1 - a_0)^2} \left(\frac{(2^{1-a_0} - 1)^2}{2} + 2^{1-a_0}(1 - \cos(b_0 \ln(2))) \right) \quad (803)$$

We develop further the equations (1007) and (1008) to get the system of equations of X_1 and X_2 :

$$\beta X_1 + \alpha X_2 = 2c_1 \quad (804)$$

$$-\alpha X_1 + \beta X_2 = -2b_1 \quad (805)$$

$$\alpha = a_3 b_0 - b_0(1 - a_0) \quad (806)$$

$$\beta = -(b_0 b_3 + a_3(1 - a_0)) \quad (807)$$

Therefore

$$X_1 = \frac{2\alpha b_1 + 2\beta c_1}{\alpha^2 + \beta^2} \quad (808)$$

$$X_2 = \frac{2\alpha C_1 - 2\beta b_1}{\alpha^2 + \beta^2} \quad (809)$$

Case One: $a_0 = 1, b_0 \neq \frac{k\pi}{\ln(2)}, k$ **nonzero integer** In this case we have from the expression that defines X_1 , we replace a_0 by its value 1. We have

$$X_1 = \frac{a_3}{b_0} \quad (810)$$

$$X_2 = -\frac{b_3}{b_0} \quad (811)$$

$$\alpha = a_3 b_0 \quad (812)$$

$$\beta = -b_0 b_3 \quad (813)$$

Therefore

$$X_1 = \frac{2b_1 a_3 - 2b_3 c_1}{(a_3^2 + b_3^2) b_0} \quad (814)$$

$$X_2 = \frac{2c_1 a_3 + 2b_3 b_1}{(a_3^2 + b_3^2) b_0} \quad (815)$$

We write b_1 and c_1 in function of a_3 and b_3

$$b_1 = \frac{a_3^2 - b_3^2}{2} \quad (816)$$

$$c_1 = b_3(2 - a_3) \quad (817)$$

Therefore the equation of X_2 gives us the following equality:

$$-\frac{b_3}{b_0} = \frac{4a_3 b_3 - b_3 a_3^2 - b_3^3}{(a_3^2 + b_3^2) b_0} \quad (818)$$

After simplification we get $a_3 b_3 = 0$. We have $b_0 \neq \frac{k\pi}{\ln(2)}$ therefore $b_3 \neq 0$. Therefore $a_3 = 0$. But from the expression of $a_3 = 1 - \cos(b_0 \ln(2))$, a_3 also cannot be zero. Hence the contradiction.

From the equation of X_1 we get $b_3^3 = 0$. Therefore $b_3 = 0$. And this is also a contradiction because $b_0 \neq \frac{k\pi}{\ln(2)}$. Therefore $e_3 \neq E_2$ and the function e_4 is a nonzero function. This prove our case.

Case Two: $\frac{1}{2} < a_0 < 1, b_0 \neq \frac{k\pi}{\ln(2)}, k$ **nonzero integer** In this general case, the equation of X_1 gives us the following:

$$\frac{a_3 b_0 + b_3(1 - a_0)}{b_0^2 + (1 - a_0)^2} = \frac{2b_1 \alpha + 2\beta c_1}{\alpha^2 + \beta^2} \quad (819)$$

$$(820)$$

We develop further we have

$$\alpha^2 + \beta^2 = (a_3^2 + b_3^2) (b_0^2 + (1 - a_0)^2) \quad (821)$$

Therefore

$$a_3b_0 + b_3(1 - a_0) = \frac{2b_1\alpha + 2\beta c_1}{a_3^2 + b_3^2} \quad (822)$$

We inject these expressions in the last equation.

$$b_1 = \frac{a_3^2 - b_3^2}{2} \quad (823)$$

$$c_1 = b_3(2 - a_3) \quad (824)$$

After simplification we get

$$b_3(a_3b_3b_0 - 2b_0b_3 - 2a_3(1 - a_0)) = 0 \quad (825)$$

We have $b_0 \neq \frac{k\pi}{\ln(2)}$ therefore $b_3 \neq 0$. Therefore

$$a_3b_3b_0 - 2b_0b_3 - 2a_3(1 - a_0) = 0 \quad (826)$$

After simplification of the last equation we get:

$$-\frac{b_02^{1-a_0} \sin(b_0 \ln(2))}{2(1 - a_0)} = \frac{1 - 2^{1-a_0} \cos(b_0 \ln(2))}{1 + 2^{1-a_0} \cos(b_0 \ln(2))} \quad (827)$$

We have $b_0 \neq \frac{k\pi}{\ln(2)}$, therefore $\sin(b_0 \ln(2)) \neq 0$

We calculate the limit of the two terms of the equation above when a_0 goes to 1^- . We get

$$\lim_{a_0 \rightarrow 1^-} -\frac{b_02^{1-a_0} \sin(b_0 \ln(2))}{2(1 - a_0)} = \lim_{a_0 \rightarrow 1^-} \frac{1 - 2^{1-a_0} \cos(b_0 \ln(2))}{1 + 2^{1-a_0} \cos(b_0 \ln(2))} \quad (828)$$

$$\infty = \frac{1 - \cos(b_0 \ln(2))}{1 + \cos(b_0 \ln(2))} > 0 \quad (829)$$

Hence the contradiction. Therefore $e_3 \neq E_2$ and the function e_4 is a nonzero function. This prove our case.

Case Three: $\frac{1}{2} < a_0 < 1, b_0 = \frac{k\pi}{\ln(2)}, k$ **nonzero integer** In this case we have

$$b_3 = 0 \quad (830)$$

$$b_1 = \frac{a_3^2}{2} \quad (831)$$

$$c_1 = 0 \quad (832)$$

We replace these values into the equation

$$a_3b_0 + b_3(1 - a_0) = \frac{2b_1\alpha + 2\beta c_1}{a_3^2 + b_3^2} \quad (833)$$

Therefore

$$a_3b_0 = \alpha \quad (834)$$

$$= a_3b_0 - b_0(1 - a_0) \quad (835)$$

Therefore

$$b_0(1 - a_0) = 0 \quad (836)$$

Therefore b_0 . This is also a contradiction. Therefore $e_3 \neq E_2$ and the function e_4 is a nonzero function. This prove our case. \square

Lemma 3.20. *Let's define the function f_4 :*

$$f_4(t) = F_2(t) - f_3(t) \quad (837)$$

Where f_3 and F_2 are the same functions defined in the lemmas 3.12 and 3.16. Therefore the function f_4 is a nonzero function, i.e we don't have for each t , $f_4(t) = 0$.

$$F_2(x) = \left(\frac{\alpha_1}{1 - 2a_0} + \frac{(2b_0\beta_1 + \gamma_1(1 - 2a_0)) \sin(2b_0 \ln(x)) + (\beta_1(1 - 2a_0) - 2b_0\gamma_1) \cos(2b_0 \ln(x))}{(2b_0)^2 + (1 - 2a_0)^2} \right) \frac{1}{(x)^{2a_0 - 1}} \quad (838)$$

The function F_2 is the primitive function of the function $f_2(t)$ where

$$\begin{aligned} f_2(t) t^{2a_0} &= \left(a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \right) + \left(\frac{X_1 Z_2 + X_2 Z_1}{2} - c_1 \right) \sin(2b_0 \ln(t)) + \left(\frac{X_2 Z_2 - X_1 Z_1}{2} + b_1 \right) \cos(2b_0 \ln t) \\ &= \alpha_1 + \beta_1 \sin(2b_0 \ln(t)) + \gamma_1 \cos(2b_0 \ln t) \end{aligned}$$

And the function f_3 is defined as

$$f_3(t) (t)^{2a_0 - 1} = \alpha_3 + \beta_3 \sin(2b_0 \ln(t)) + \gamma_3 \cos(2b_0 \ln t) \quad (841)$$

Where

$$\alpha_1 = a_1 + \frac{X_1 Z_1 + X_2 Z_2}{2} \quad (842)$$

$$\beta_1 = \frac{X_1 Z_2 + X_2 Z_1}{2} - c_1 \quad (843)$$

$$\gamma_1 = \frac{X_2 Z_2 - X_1 Z_1}{2} + b_1 \quad (844)$$

$$Z_1 = -a_3 a_0 + b_3 b_0 \quad (845)$$

$$Z_2 = a_3 b_0 + a_0 b_3 \quad (846)$$

$$X_1 = \frac{(a_3(1 - a_0) - b_3 b_0)}{(b_0)^2 + (1 - a_0)^2} \quad (847)$$

$$X_2 = -\frac{b_3(1 - a_0) + a_3 b_0}{(b_0)^2 + (1 - a_0)^2} \quad (848)$$

$$a_1 = \frac{1}{2} + 2^{1-2a_0} - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (849)$$

$$b_1 = -\frac{1}{2} - 2^{1-2a_0} + 2^{1-a_0} \cos(b_0 \ln(2)) \quad (850)$$

$$c_1 = -2^{1-a_0} \sin(b_0 \ln(2)) \quad (851)$$

$$a_3 = 1 - 2^{1-a_0} \cos(b_0 \ln(2)) \quad (852)$$

$$b_3 = 2^{1-a_0} \sin(b_0 \ln(2)) \quad (853)$$

$$\alpha_3 = \frac{X_1 a_3 - X_2 b_3}{2} \quad (854)$$

$$\beta_3 = \frac{-X_1 b_3 + X_2 a_3}{2} \quad (855)$$

$$\gamma_3 = \frac{-X_2 b_3 - X_1 a_3}{2} \quad (856)$$

Proof. To prove this lemma we proceed by contradiction. Let's suppose that the function $f_4 = 0$. Therefore for each $t \neq 0$, $f_3(t) = F_2(t)$. Hence we also have their first derivatives are equals: for each t , $f_2(t) = f_3'(t)$. Therefore the coefficients of f_2 and f_3' are equals:

$$\alpha_1 = (1 - 2a_0)\alpha_3 \quad (857)$$

$$\beta_1 = \beta_3(1 - 2a_0) - 2b_0\gamma_3 \quad (858)$$

$$\gamma_1 = (1 - 2a_0)\gamma_3 + 2b_0\beta_3 \quad (859)$$

Quick calculation shows that in fact

$$\alpha_1 = (1 - 2a_0)\alpha_3 = -\frac{(2a_0 - 1)(1 - a_0)}{(b_0)^2 + (1 - a_0)^2} \left(\frac{(2^{1-a_0} - 1)^2}{2} + 2^{1-a_0}(1 - \cos(b_0 \ln(2))) \right) \quad (860)$$

We develop further the equations (1064) and (1065) to get the system of equations of X_1 and X_2 :

$$\beta X_1 - \alpha X_2 = 2c_1 \quad (861)$$

$$\alpha X_1 + \beta X_2 = -2b_1 \quad (862)$$

$$\alpha = a_3(1 - a_0) + b_0b_3 \quad (863)$$

$$\beta = b_3(1 - a_0) - b_0a_3 \quad (864)$$

Therefore

$$X_1 = \frac{2\beta c_1 - 2\alpha b_1}{\alpha^2 + \beta^2} \quad (865)$$

$$X_2 = -\frac{2\alpha C_1 + 2\beta b_1}{\alpha^2 + \beta^2} \quad (866)$$

Case One: $a_0 = 1, b_0 \neq \frac{k\pi}{\ln(2)}, k$ **nonzero integer** In this case we have from the expression that defines X_1 , we replace a_0 by its value 1. We have

$$X_1 = -\frac{b_3}{b_0} \quad (867)$$

$$X_2 = -\frac{a_3}{b_0} \quad (868)$$

$$\alpha = b_3b_0 \quad (869)$$

$$\beta = -b_0a_3 \quad (870)$$

We write b_1 and c_1 in function of a_3 and b_3

$$b_1 = -\frac{a_3^2 + b_3^2}{2} \quad (871)$$

$$c_1 = -b_3 \quad (872)$$

$$a_3^2 + b_3^2 = 2a_3 \quad (873)$$

Therefore the equation of X_2 gives us the following equality:

$$\frac{a_3}{b_0} = 2 \frac{-b_0b_3^2 + a_3^2b_0}{(a_3^2 + b_3^2)b_0^2} \quad (874)$$

Therefore

$$a_3 = 2 \frac{a_3^2 - b_3^2}{(a_3^2 + b_3^2)} \quad (875)$$

$$= 2 \frac{a_3^2 - b_3^2}{2a_3} \quad (876)$$

Therefore

$$a_3^2 = a_3^2 - b_3^2 \quad (877)$$

Therefore $b_3 = \sin(b_0 \ln(2)) = 0$. And this is in contradiction with the fact that $b_0 \neq \frac{k\pi}{\ln(2)}$. Therefore $f_3 \neq F_2$ and the function f_4 is a nonzero function. This prove our case.

Case Two: $\frac{1}{2} < a_0 < 1, b_0 \neq \frac{k\pi}{\ln(2)}, k$ **nonzero integer** In this general case, the equation of X_2 gives us the following:

$$-\frac{b_3(1-a_0) + a_3b_0}{b_0^2 + (1-a_0)^2} = -\frac{2c_1\alpha + 2\beta b_1}{\alpha^2 + \beta^2} \quad (878)$$

$$(879)$$

We develop further we have

$$\alpha^2 + \beta^2 = (a_3^2 + b_3^2)(b_0^2 + (1-a_0)^2) \quad (880)$$

Therefore

$$b_3(1-a_0) + a_3b_0 = \frac{2c_1\alpha + 2\beta b_1}{a_3^2 + b_3^2} \quad (881)$$

We inject these expressions in the last equation.

$$b_1 = -\frac{a_3^2 + b_3^2}{2} \quad (882)$$

$$c_1 = -b_3 \quad (883)$$

After simplification we get

$$b_3 \left[(1-a_0) \left(1 + \frac{a_3}{a_3^2 + b_3^2} \right) + \frac{b_0 b_3}{a_3^2 + b_3^2} \right] = 0 \quad (884)$$

We have $b_0 \neq \frac{k\pi}{\ln(2)}$ therefore $b_3 \neq 0$. Therefore

$$(1-a_0) \left(1 + \frac{a_3}{a_3^2 + b_3^2} \right) + \frac{b_0 b_3}{a_3^2 + b_3^2} = 0 \quad (885)$$

We have $b_0 \neq \frac{k\pi}{\ln(2)}$, therefore $\sin(b_0 \ln(2)) \neq 0$
We calculate the limit of the two terms of the equation above when a_0 goes to 1^- . We get

$$\lim_{a_0 \rightarrow 1^-} (1-a_0) \left(1 + \frac{a_3}{a_3^2 + b_3^2} \right) = - \lim_{a_0 \rightarrow 1^-} \frac{b_0 b_3}{a_3^2 + b_3^2} \quad (886)$$

$$0 = -\frac{b_0 b_3}{a_3^2 + b_3^2} \quad (887)$$

We have $b_0 > 0$ therefore $b_3 = 0$. And this is a contradiction with the fact that $b_0 \neq \frac{k\pi}{\ln(2)}$. Hence the contradiction. Therefore $f_3 \neq F_2$ and the function f_4 is a nonzero function. This prove our case.

Case Three: $\frac{1}{2} < a_0 < 1, b_0 = \frac{k\pi}{\ln(2)}, k$ nonzero integer In this case we have

$$b_3 = 0 \quad (888)$$

$$b_1 = -a_1 \quad (889)$$

$$c_1 = 0 \quad (890)$$

$$\beta_3 = \frac{X_2 a_3}{2} \quad (891)$$

$$\gamma_3 = -\frac{X_1 a_3}{2} \quad (892)$$

$$X_1 = \frac{a_3(1-a_0)}{b_0^2 + (1-a_0)^2} \quad (893)$$

$$X_2 = -\frac{a_3 b_0}{b_0^2 + (1-a_0)^2} \quad (894)$$

$$\beta_1 = \frac{a_3^2 b_0}{2(b_0^2 + (1-a_0)^2)} \quad (895)$$

Also we have

$$\beta_1 = \beta_3(1-2a_0) - 2b_0\gamma_3 \quad (896)$$

$$= \frac{X_2 a_3}{2}(1-2a_0) + 2b_0 \frac{X_1 a_3}{2} \quad (897)$$

$$= \frac{a_3^2 b_0}{2(b_0^2 + (1-a_0)^2)} (2a_0 - 1 + 2b_0) \quad (898)$$

Therefore

$$\frac{a_3^2 b_0}{2(b_0^2 + (1-a_0)^2)} = \frac{a_3^2 b_0}{2(b_0^2 + (1-a_0)^2)} (2a_0 - 1 + 2b_0) \quad (899)$$

We have $a_3 = 1 - 2^{1-a_0}$ or $a_3 = 1 + 2^{1-a_0}$ and $\frac{1}{2} < a_0 < 1$. Therefore $a_3 \neq 0$. We also have $b_0 > 0$ Therefore

$$2a_0 - 1 + 2b_0 = 1 \quad (900)$$

Therefore

$$0 < b_0 = 1 - a_0 < \frac{1}{2} \quad (901)$$

But in this case we have $b_0 = \frac{k\pi}{\ln(2)}, k$ nonzero integer. Therefore

$$0 < b_0 = \frac{k\pi}{\ln(2)} < \frac{1}{2} \quad (902)$$

Therefore

$$0 < k < \frac{\ln(2)}{2\pi} < \frac{1}{2} \quad (903)$$

This is also a contradiction with the fact that k is nonzero integer. Therefore $f_3 \neq F_2$ and the function f_4 is a nonzero function. This prove our case. \square

3.2 Case Two: $a_0 > 1$

To avoid having a lengthy article, we limit the application of our method only to the critical strip. Meanwhile, the reader can find the famous proof of this case in [4]. This proof is based on Euler product. The case where $\Re(s) > 1$ will be the subject of another article. The proof of this case follows the same steps like the critical strip case with minor modifications. The reader can use the fact $|\gamma_n| \leq \frac{K}{n^{a_0-1}}$ to prove that the limit of $O_n(\text{res. } GO_n)$ is zero.

3.3 Case Three: $a_0 = \Re(s) = \frac{1}{2}$

Here we want to know what will happen if we apply this method to the case $a_0 = \frac{1}{2}$ where we know with certitude that ζ has zeros. We want to know whether the $\lim_{N \rightarrow +\infty} \sum_{n=1}^N \overline{F}_{n,N}$ and/ or $\lim_{N \rightarrow +\infty} \sum_{n=1}^N \overline{G}_{n,N}$ still equals $+\infty$. From the equations 484-487, we can say that $\frac{1}{N} \sum_{n=N_0}^{N-1} \overline{F}_{n+1,N}$ is bounded. The term $\frac{1}{2} \sum_{k=N_0}^{\frac{N}{2}-1} e_2(k)$ is of the order of $a \sin(b_0 \ln(N)) + b \cos(b_0 \ln(N))$. Also the terms $\frac{1}{N} \sum_{k=N_0}^{\frac{N}{2}-1} k e_2(k)$, $\frac{1}{2} \sum_{k=\frac{N}{2}+1}^{N-1} \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0}}$ and $\frac{1}{N} \sum_{k=\frac{N}{2}+1}^{N-1} \frac{\cos^2(b_0 \ln(k))}{(k)^{2a_0-1}}$ are of the order of $a \sin(b_0 \ln(N)) + b \cos(b_0 \ln(N))$.

Also the partial sum $\frac{1}{2} \sum_{k=N_0}^{\frac{N}{2}-1} \left\{ \frac{\overline{e}_k}{k^{a_0+1}} + \frac{\overline{D}_{k,N}}{k^{2a_0+1}} + \frac{\overline{e}_{k,N}}{k^{a_0+2}} \right\}$ is convergent

but we don't know its limit. Therefore the sequence $\frac{1}{N} \sum_{n=N_0}^{N-1} \overline{F}_{n+1,N}$ is bounded but it is not convergent. Also, there is no obvious dominant term that can enable us to go one direction or another. Partially because of the term $\sum_{k=N_0}^{\frac{N}{2}-1} \frac{\overline{e}_k}{k^{a_0+1}}$ that contains the γ_n term for which we don't have much information. Plus the partial sum $\sum_{n \geq 1}^N \overline{U}_{n,N}^2$ is not convergent since $2a_0 = 1$.

The same thing can be said for $\frac{1}{N} \sum_{n=N_0}^{N-1} \overline{G}_{n+1,N}$ and the partial sum $\sum_{n \geq 1}^N \overline{V}_{n,N}^2$. Therefore the case where $\Re(s) = \frac{1}{2}$ is special.

3.4 Case Four: $\frac{1}{2} < a_0 \leq 1, b_0 = \Im(s) < 0$

We saw above that s is a *zeta zero*, $\zeta(s) = 0$, if and only if

$$\lim_{N \rightarrow \infty} \overline{A}_{N,N} = 0 \text{ and } \lim_{N \rightarrow \infty} \overline{B}_{N,N} = 0 \quad (904)$$

This implies that if s is a *zeta zero* then its conjugate \bar{s} is also a *zeta zero* thanks to the fact that the cosine is an even function and the sine is an odd function.

So if we have a *zeta zero* $s = a_0 + ib_0$ with $b_0 < 0$ then its conjugate $\bar{s} = a_0 - ib_0$ is also a *zeta zero*.

From above, we saw that \bar{s} cannot be a *zeta zero*.

Therefore s with $b_0 = \Im(s) < 0$ and $\frac{1}{2} < a_0 \leq 1$ cannot be a zero for the Riemann's Zeta Function.

3.5 Case Five: $0 \leq a_0 = \Re(s) < \frac{1}{2}$

We use the Zeta functional equation:

For each complex number $s \in \mathbb{C} \setminus \{0, 1\}$

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (905)$$

Therefore we have this symmetry property: If s is a zero for Zeta function then $1-s$ is also a zero for Zeta function.

$$1-s = 1-a_0 - ib_0$$

In this case, we have $\frac{1}{2} < \Re(1-s) = 1-a_0 \leq 1$. From above, we just saw that such a zero cannot exist.

Therefore s with $0 \leq a_0 = \Re(s) < \frac{1}{2}$ cannot be a zero for the Riemann's Zeta Function.

3.6 Case Six: $a_0 = \Re(s) < 0$

We use the Zeta functional equation:

For each complex number $s \in \mathbb{C} \setminus \{0, 1\}$

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (906)$$

Therefore we have this symmetry property: If s is a zero for Zeta function then $1-s$ is also a zero for Zeta function.

$$1-s = 1-a_0 - ib_0$$

In this case, we have $\Re(1-s) = 1-a_0 > 1$. From above, we just saw above that such a zero cannot exist.

Therefore s with $a_0 = \Re(s) < 0$ cannot be a zero for the Riemann's Zeta Function.

3.7 Conclusion

From the different cases above, we saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$ as all other possibilities were discarded. Therefore the Riemann hypothesis is true: *The nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.* We believe that the same technique applied here can be applied to resolve the Generalized Riemann hypothesis.

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