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# A STIFFLY STABLE SEMI-DISCRETE SCHEME FOR THE CHARACTERISTIC LINEAR HYPERBOLIC RELAXATION WITH BOUNDARY

BENJAMIN BOUTIN, THỊ HOÀI THƯƠNG NGUYỄN, AND NICOLAS SEGUIN

ABSTRACT. We study the stability of the semi-discrete central scheme for the linear damped wave equation with boundary. We exhibit a sufficient condition on the boundary to guarantee the uniform stability of the initial boundary value problem for relaxation system independent of stiffness of the source term and of the space step. The boundary is approximated using a summation-by-parts method and the stiff stability is proved by energy estimates and Laplace transform. We also investigate if the condition is also necessary, following the continuous case studied by Xin and Xu in [18].

**AMS classification:** 65M06, 65M12, 35F46, 35L50

**Keywords:** hyperbolic relaxation system, damped wave equation, summation by parts operators, central schemes, energy estimates.

## 1. INTRODUCTION

**1.1. Context and motivation.** In many physical situations, we are interested in hyperbolic systems of partial differential equations with relaxation terms [1]. Such systems are found in relaxing gas theory [3], water waves [14, 17] and reactive flows [4]. One of the main features of these models is related to the notion of dissipation, leading to smooth solutions and asymptotic stability. The study of the zero relaxation limit for such systems has caught much interest, both from a theoretical and numerical point of view, after the works of Liu [9], Chen, Levermore and Liu [2], Hanouzet and Natalini [6], Yong [20, 21]. In this article, we are concerned with numerical treatment of boundary for hyperbolic relaxation systems. Due to the effects of the boundary layer and the interactions of the boundary and initial layers, numerical schemes have to be properly designed in order to provide accurate approximations and consistent behaviors.

One of the simplest linear hyperbolic systems with relaxation is the linear damped wave equation

$$(1.1) \quad \begin{aligned} \partial_t u^\varepsilon + \partial_x v^\varepsilon &= 0, & u^\varepsilon, v^\varepsilon &\in \mathbb{R}, \\ \partial_t v^\varepsilon + a \partial_x u^\varepsilon &= -\frac{1}{\varepsilon} v^\varepsilon, \end{aligned}$$

where  $a > 0$  and the relaxation time  $\varepsilon > 0$  may be introduced to characterize the stiffness of the relaxation. When  $\varepsilon$  goes to zero, the model may be simplified. We expect indeed that for any position  $x$  and any time  $t$ , the solution  $(u^\varepsilon, v^\varepsilon)$  tends to  $(u(x), 0)$ , which is the solution of the corresponding equilibrium system [2, 18].

In order to determine a unique solution to the problem (1.1) in the quarter plane  $x > 0, t > 0$ , it is necessary to specify values of the solution at initial time

$$(1.2) \quad u^\varepsilon(x, 0) = u_0(x), \quad v^\varepsilon(x, 0) = v_0(x),$$

and to impose conditions on the solution at the boundary

$$(1.3) \quad B_u u^\varepsilon(0, t) + B_v v^\varepsilon(0, t) = b(t),$$

where  $B_u$  and  $B_v$  are constants. For simplicity, we also assume the initial data  $U_0(x) = (u_0(x), v_0(x))$  and the boundary data  $b(t)$  are sufficiently compatible at the space-time corner  $x = 0, t = 0$ , i.e.

$$U_0(0) = U_0'(0) = 0, \quad b(0) = b'(0) = 0.$$

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In some cases, the correct boundary conditions can be found easily from physical considerations. At a solid wall that bounds the flow of a fluid, for example, one sets the normal component of the fluid velocity equal to zero (if effects of viscosity are to be considered, the tangential component must also vanish). In other situations, the choice of boundary conditions is not as obvious. This is the case with artificial boundaries, which do not correspond to anything physical. In general, it is not possible to choose boundary conditions arbitrarily for a hyperbolic equation. This can be seen most easily in the case of the problem (1.1), which is a particular case of the Jin-Xin relaxation model in one space dimension [8].

It is easy to see that system (1.1) is diagonalizable with Riemann invariants  $\sqrt{a}u^\varepsilon \pm v^\varepsilon$  and eigenvalues  $\pm\sqrt{a}$ . Therefore the boundary condition (1.3) has to satisfy the Uniform Kreiss Condition (UKC)

$$(1.4) \quad B_u + \sqrt{a}B_v \neq 0$$

so that on the boundary  $x = 0$  the incoming flow  $\sqrt{a}u^\varepsilon + v^\varepsilon$  can be expressed in terms of the outgoing flow  $\sqrt{a}u^\varepsilon - v^\varepsilon$  and the data  $b(t)$ , and therefore the initial boundary value problem (IBVP) (1.1)-(1.3) is well-posed for each fixed  $\varepsilon$  (see [1, 18, 19]).

In [18], Z. Xin and W. Xu study the asymptotic equivalence of a general linear system of 1-dimensional conservation laws and the corresponding relaxation model proposed by S. Jin and Z. Xin [8] in the limit of small relaxation rate. The main interest is this asymptotic equivalence in the presence of physical boundaries. They identify and rigorously justify a necessary and sufficient condition (which they call the Stiff Kreiss Condition, or SKC in short) on the boundary condition to guarantee the uniform well-posedness of the IBVP for the relaxation system independent of the rate of relaxation in the particular case of system (1.1) with (1.3). In the case of system (1.1), the SKC simply reduces to

$$(1.5) \quad \frac{B_u}{B_v} \notin [-\sqrt{a}, 0].$$

The motivation of the present study is the difference approximation of the IBVP (1.1)-(1.3). The major issues in the theory of the relaxation approximations to equilibrium system of conservation laws is the appearance of stiff boundary layers in the presence of physical or numerical boundaries condition due to the additional characteristic speeds introduced in the relaxation systems. On the other hand, the stability estimate obtained for a certain approximation is the key to the proper error estimates. Thus, how to formulate boundary conditions for the relaxation systems to guarantee the uniform stability and to minimize the artificial boundary layer are crucial to the success of the schemes.

**1.2. Description of the semi-discrete numerical scheme.** Let  $\Delta x$  be the space step and  $U(x_j, t) = (u^\varepsilon, v^\varepsilon)^T(x_j, t)$  with  $x_j = j\Delta x$ ,  $j \in \mathbb{N}$ . The solution to (1.1)-(1.3) is approximated by a sequence  $U_j(t)$ . We focus in this paper on the central scheme for semi-discrete approximation of the IBVP and to derive a necessary condition for stiff stability. Let

$$A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = (B_u \quad B_v).$$

The semi-discrete approximation of the IBVP (1.1)-(1.3) can be reformulated as

$$(1.6) \quad \begin{aligned} \partial_t U_j(t) + (QU)_j(t) &= \frac{1}{\varepsilon} S U_j(t), & j \geq 1, \\ U_j(0) &= f_j, \\ B U_0(t) &= b(t), \end{aligned}$$

with the Cauchy data  $f_j = U(x_j, 0)$  and a difference operator  $(QU)_j(t)$  consistent with  $A\partial_x U(x_j, t)$  by  $(QU)(x_j, t) = A\partial_x U(x_j, t) + \mathcal{O}(\Delta x^p)$ ,  $p > 0$  defined everywhere including the boundary point. Let us note that the boundary condition is exactly satisfied at  $j = 0$ .

The summation by parts (SBP) finite difference operators were first derived in [11, 12]. In [15], the analysis was revisited and exact expressions for the finite difference coefficients were obtained. In the case of the central scheme, the modification of the difference operator at  $j = 0$  can also be expressed as an

addition of an extra boundary condition. It means that we use the centered approximation at  $j = 0$  but supply a boundary condition that determines  $U_{-1}$

$$U_1 - 2U_0 + U_{-1} = 0.$$

If we eliminate  $U_{-1}$ , we obtain a one-sided approximation. The extra boundary condition determines  $U_{-1}$  as a linear extrapolation of  $U_1$  and  $U_0$ . In [5], the energy estimate can be obtained if the scalar product and norm is defined by

$$(1.7) \quad \begin{aligned} \langle U, V \rangle_{\Delta x} &= \frac{\Delta x}{2} \langle U_0, V_0 \rangle + \Delta x \sum_{j=1}^{\infty} \langle U_j, V_j \rangle, \\ \|U\|_{\Delta x}^2 &= \langle U, U \rangle_{\Delta x}. \end{aligned}$$

with  $\langle \cdot, \cdot \rangle$  be the Euclidean product. We can write the difference operator

$$(1.8) \quad (\mathcal{Q}U)_j = \begin{cases} \frac{1}{2\Delta x} A (U_{j+1} - U_{j-1}), & j \geq 1, \\ \frac{1}{\Delta x} A (U_1 - U_0), & j = 0, \end{cases}$$

which uses a noncentered approximation at the boundary, so that the difference operator is defined at all gridpoints including the boundary.

Let us emphasize that the numerical scheme (1.6) needs one more equation at  $j = 0$  to be completely defined. This is a numerical degree of freedom since in the continuous case, only one equation is needed at the interface due to the number of incoming characteristics. We choose to define the remaining boundary value in agreement with the dissipativity of the source term. We then use a symmetric form of the problem, based on the matrices

$$P = \begin{pmatrix} B_u & B_v \\ 1 & 0 \end{pmatrix},$$

and a symmetric positive definite matrix

$$H_P = \begin{pmatrix} 1 & -B_u \\ -B_u & aB_v^2 + B_u^2 \end{pmatrix}$$

such that  $P^T H_P P$  is symmetric positive definite matrix,  $H_P P A P^{-1}$  is symmetric,  $H_P P S P^{-1}$  is negative definite. Since  $P^T H_P P A$  is also a symmetric matrix,

$$\langle U, P^T H_P P (\mathcal{Q}U) \rangle_{\Delta x} (t) = -\frac{1}{2} \langle U_0, P^T H_P P A U_0 \rangle (t),$$

which corresponds to the equality  $\int_0^{+\infty} \langle U, P^T H_P P A \partial_x U \rangle (x, t) dx = -\frac{1}{2} \langle U, P^T H_P P A U \rangle (0, t)$  of the continuous case. Moreover, at the boundary  $j = 0$ , we obtain

$$(1.9) \quad \langle \partial_t U_0, P^T H_P P U_0 \rangle (t) + \langle (\mathcal{Q}U)_0, P^T H_P P U_0 \rangle (t) = \frac{1}{\varepsilon} \langle S U_0, P^T H_P P U_0 \rangle (t).$$

Using homogeneous boundary condition  $B U_0(t) = 0$ , the previous equality (1.9) can be reformulated as

$$\partial_t (\Pi_2 H_P P U_0) (t) (\Pi_2 P U_0) (t) + (\Pi_2 H_P P (\mathcal{Q}U)_0) (t) (\Pi_2 P U_0) (t) = \frac{1}{\varepsilon} (\Pi_2 H_P P S U_0) (t) (\Pi_2 P U_0) (t),$$

where  $\Pi_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$ .

We propose the following numerical approximation at the boundary

$$\partial_t (\Pi_2 H_P P U_0) (t) + \Pi_2 H_P P (\mathcal{Q}U)_0 (t) = \frac{1}{\varepsilon} \Pi_2 H_P P S U_0 (t).$$

To sum up, the semi-discrete approximation of the IBVP (1.1)-(1.3) can be reformulated as

$$(1.10) \quad \begin{aligned} \partial_t U_j(t) + (\mathcal{Q}U)_j(t) &= \frac{1}{\varepsilon} S U_j(t), \quad j \geq 1, \\ U_j(0) &= f_j, \\ B U_0(t) &= b(t), \\ \partial_t (\Pi_2 H_P P U_0)(t) + \Pi_2 H_P P (\mathcal{Q}U)_0(t) &= \frac{1}{\varepsilon} \Pi_2 H_P P S U_0(t). \end{aligned}$$

where  $(\mathcal{Q}U)_{j \in \mathbb{N}}$  is given by (1.8).

**1.3. Main result.** For the continuous IBVP, the UKC (1.4) is not enough and a more stringent restriction has to be imposed on the structure of the boundary condition (1.3). Our aim is to study a necessary condition for stiff stability of the above semi-discrete IBVP, i.e. stability uniformly with respect to the stiffness of the relaxation term.

**Theorem 1.1.** *Under assumption*

$$(1.11) \quad B_u B_v > 0,$$

for any  $T > 0$  there exists a constant  $C_T > 0$  such that for any  $f \in l^2(\mathbb{N}, \mathbb{R}^2)$  and any  $b \in L^2(\mathbb{R}^+, \mathbb{R})$ , the solution  $U$  to (1.10) satisfies

$$(1.12) \quad \int_0^T |U_0(t)|^2 dt + \int_0^T \sum_{j \geq 0} \Delta x |U_j(t)|^2 dt \leq C_T \left( \sum_{j \geq 0} \Delta x |f_j|^2 + \int_0^T |b(t)|^2 dt \right),$$

where the constant  $C_T$  is independent of the data  $f$  and  $b$ , but most importantly of  $\varepsilon \in (0, +\infty)$  and  $\Delta x \in (0, 1]$ .

The proof of Theorem 1.1 is based on two ingredients, one with homogeneous boundary condition and the other with homogeneous initial condition.

**Proposition 1.2.** *(Homogeneous boundary condition) Let  $a > 0$  and  $B_u, B_v \in \mathbb{R}$ . Assume that  $\Delta x \in (0, 1]$  and  $\varepsilon > 0$  satisfies*

$$(1.13) \quad 2a \frac{B_u}{B_v} + \frac{\Delta x}{\varepsilon} \left( \frac{B_u}{B_v} \right)^2 > 0,$$

Then there exists a constant  $c > 0$  such that for any  $f \in l^2(\mathbb{N}, \mathbb{R}^2)$ ,

$$(1.14) \quad \langle U, P^T H_P P U \rangle_{\Delta x}(T) + c \int_0^T |U_0|^2(t) dt \leq \langle f, P^T H_P P f \rangle_{\Delta x},$$

holds.

If  $B_u B_v > 0$ , (1.14) holds uniformly.

If  $B_u B_v < 0$ , consider some  $\delta_0 > -2a \frac{B_v}{B_u}$ . Then there exists  $c(\delta_0) > 0$  such that (1.14) holds uniformly as soon as  $\frac{\Delta x}{\varepsilon} \geq \delta_0$  with  $c = c(\delta_0)$ .

**Proposition 1.3.** *(Homogeneous initial condition) Assume that  $B_u, B_v \in \mathbb{R}$  satisfies (1.11). Then, there exists a constant  $c > 0$  such that for any  $\alpha > 0$  there exists  $\Delta x_0 > 0$  with for any  $b \in L^2(\mathbb{R}^+, \mathbb{R})$  and  $\Delta x \leq \Delta x_0$*

$$(1.15) \quad \alpha \Delta x \int_0^\infty \sum_{j \geq 0} e^{-2\alpha t} |U_j(t)|^2 dt + \int_0^\infty e^{-2\alpha t} |U_0(t)|^2 dt \leq c \int_0^\infty e^{-2\alpha t} |b(t)|^2 dt.$$

holds.

*Remark 1.4.* Since  $P^T H_P P$  is symmetric positive definite matrix, the following inequality hold  $m, n > 0$

$$m \langle U, P^T H_P P U \rangle_{\Delta x}(t) \leq \langle U, U \rangle_{\Delta x}(t) \leq n \langle U, P^T H_P P U \rangle_{\Delta x}(t),$$

which will be useful to prove the estimate (1.12) with weighted in time from (1.14).

Xin and Xu in [18] considered the IBVP for the Jin-Xin relaxation model [8] and derived the SKC (1.5) for stiff well-posedness. They show in particular that the IBVP is well-posed if and only if (1.5) holds. In the discrete IBVP (1.10), it seems that the SKC is not sufficient to derive uniform stability estimates. In comparison, our stability condition (1.11) is more restrictive, but we are only able to prove that it is sufficient. We also perform a normal mode analysis to construct unstable solutions and, based on some numerical investigations, condition (1.11) would appear to be also necessary for the stiff stability. Let us mention that condition (1.13) is not implied by the SKC (1.5), probably due to some numerical diffusion at the boundary.

Proposition 1.2 is studied in Section 2.1 by discrete energy method. In order to illustrate the relevance of condition (1.13), in Section 2.2, we present numerical results, for different values of  $(B_u, B_v)$  and show that  $\langle U, P^T H_P P U \rangle_{\Delta x}(t)$  increase when (1.13) does not hold. In Section 3.1, we want to address the question of the existence of unstable solutions in order to derive necessary condition of stability by using the normal mode analysis. In Section 3.2, we present numerical results and show that  $B_u B_v > 0$  seems to be necessary to ensure the stiff stability of the discrete IBVP. Even if the boundary condition (1.3) satisfies the SKC, there exist unstable solutions of the discrete IBVP (1.10). To isolate the effects of possible boundary layer and avoid the complicated interactions of boundary and initial layer, in Section 4, we consider the IBVP (1.10) with homogeneous initial data and nonzero boundary data  $b(t)$ . In Section 4.1, the numerical solution  $U_j(t)$  can be constructed by Laplace transform. By using the Parseval's identity, under assumption  $B_u B_v > 0$ , Proposition 1.3 is proved in Section 4.2. In Appendix A, we study an example of how waves occur in model the action of an elastic string overtime, which is a particular case of linear damped wave equation. By using the Newton's Second Laws of Motion, we can derive the boundary condition  $B_u B_v > 0$  for this system.

## 2. STIFF STABILITY OF THE SEMI-DISCRETE IBVP WITH HOMOGENEOUS BOUNDARY CONDITION

In this section, we consider the IBVP (1.10) with homogeneous boundary condition  $b(t) = 0$ , nonzero Cauchy data  $(f_j) \in l^2(\mathbb{N}, \mathbb{R}^2)$  and prove Proposition 1.3 by using the energy method.

**2.1. The energy method.** In the continuous case, the energy estimates are obtained using integration by parts rules. Therefore, we need corresponding SBP rules for the discrete approximations of  $\partial/\partial x$  [5]. The sufficient stability condition (1.13) is deduced from discrete energy estimates.

Using the scalar product (1.7) with  $P^T H_P P$ , we obtain

$$(2.1) \quad \langle \partial_t U, P^T H_P P U \rangle_{\Delta x}(t) = \frac{\Delta x}{2} \langle \partial_t (H_P P U_0), P U_0 \rangle(t) + \Delta x \sum_{j=1}^{\infty} \langle \partial_t U_j, P^T H_P P U_j \rangle(t).$$

On the other hand, since  $P^T H_P P$  is a symmetric positive definite matrix and using the homogeneous boundary condition  $B U_0(t) = 0$  and thus  $P U_0 = (0, \Pi_2 P U_0)$ , the previous equality (2.1) can be reformulated as

$$(2.2) \quad \frac{1}{2} \partial_t \langle U, P^T H_P P U \rangle_{\Delta x}(t) = \frac{\Delta x}{2} \partial_t (\Pi_2 H_P P U_0)(t) (\Pi_2 P U_0)(t) + \Delta x \sum_{j=1}^{\infty} \langle \partial_t U_j, P^T H_P P U_j \rangle(t).$$

Now, we show how the difference operator  $(QU)_j$  can be applied for IBVP (1.10) with homogeneous boundary condition at all gridpoints including the boundary

$$\partial_t U_j(t) = \frac{1}{\varepsilon} S U_j(t) - \frac{1}{2\Delta x} A (U_{j+1}(t) - U_{j-1}(t)), \quad j \geq 1,$$

and

$$\partial_t (\Pi_2 H_P P U_0)(t) = \frac{1}{\varepsilon} \Pi_2 H_P P S U_0(t) - \frac{1}{\Delta x} \Pi_2 H_P P A (U_1 - U_0)(t).$$

Thus, (2.2) can be presented as

$$\begin{aligned}
\partial_t \langle U, P^T H_P P U \rangle_{\Delta x}(t) &= \frac{\Delta x}{\varepsilon} (\Pi_2 H_P P S U_0)(t) (\Pi_2 P U_0)(t) + \frac{2\Delta x}{\varepsilon} \sum_{j=1}^{\infty} \langle S U_j, P^T H_P P U_j \rangle(t) \\
(2.3) \quad &+ (\Pi_2 H_P P A U_0)(t) (\Pi_2 P U_0)(t) - (\Pi_2 H_P P A U_1)(t) (\Pi_2 P U_0)(t) \\
&- \sum_{j=1}^{\infty} \langle A U_{j+1}, P^T H_P P U_j \rangle(t) + \sum_{j=1}^{\infty} \langle A U_{j-1}, P^T H_P P U_j \rangle(t).
\end{aligned}$$

We observe that

$$\sum_{j=1}^{\infty} \langle A U_{j-1}, P^T H_P P U_j \rangle(t) = \langle A U_0, P^T H_P P U_1 \rangle(t) + \sum_{j=1}^{\infty} \langle A U_j, P^T H_P P U_{j+1} \rangle(t).$$

Since  $H_P P A P^{-1}$  is symmetric, one can proceed as above

$$\langle A U_0, P^T H_P P U_1 \rangle(t) = (\Pi_2 H_P P A U_1)(t) (\Pi_2 P U_0)(t).$$

Thus

$$(2.4) \quad \sum_{j=1}^{\infty} \langle A U_{j-1}, P^T H_P P U_j \rangle(t) = (\Pi_2 H_P P A U_1)(t) (\Pi_2 P U_0)(t) + \sum_{j=1}^{\infty} \langle A U_{j+1}, P^T H_P P U_j \rangle(t).$$

Substituting (2.4) into (2.3), we obtain

$$\partial_t \langle U, H U \rangle_{\Delta x}(t) + 2a \frac{B_u}{B_v} (u_0^\varepsilon)^2(t) + \frac{\Delta x}{\varepsilon} (v_0^\varepsilon)^2(t) = -\frac{2\Delta x}{\varepsilon} \sum_{j=1}^{\infty} (v_j^\varepsilon)^2(t),$$

where the symmetric (or Hermitian) positive definite matrix

$$H = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

is defined by

$$(2.5) \quad H = \frac{1}{B_v^2} P^T H_P P.$$

In order for the energy method to work, the boundary condition has to satisfy

$$2a \frac{B_u}{B_v} (u_0^\varepsilon)^2(t) + \frac{\Delta x}{\varepsilon} (v_0^\varepsilon)^2(t) \geq c |U_0(t)|^2$$

for some positive constant  $c$  whenever

$$B_u u_0^\varepsilon(t) + B_v v_0^\varepsilon(t) = 0.$$

This leads to the sufficient stability condition

$$2a \frac{B_u}{B_v} + \frac{\Delta x}{\varepsilon} \left( \frac{B_u}{B_v} \right)^2 > 0.$$

This implies directly

$$\partial_t \langle U, H U \rangle_{\Delta x}(t) + c |U_0|^2(t) \leq 0,$$

and thus

$$(2.6) \quad \langle U, H U \rangle_{\Delta x}(T) + c \int_0^T |U_0|^2(t) dt \leq \langle f, H f \rangle_{\Delta x}.$$

Moreover, we can see that

- If  $B_u B_v > 0$  then there exists  $c \leq \frac{2a B_u / B_v}{1 + (B_u / B_v)^2}$  such that (2.6) holds uniformly.

- If  $B_u B_v < 0$ , consider some  $\delta_0 > -2a \frac{B_v}{B_u}$ . Then there exists  $c(\delta_0) > 0$  such that (2.6) holds uniformly as soon as  $\frac{\Delta x}{\varepsilon} \geq \delta_0$  with  $c = c(\delta_0)$ . For example, if we choose  $\delta_0 = -3a \frac{B_v}{B_u}$  then there exists  $c \leq \frac{-aB_u/B_v}{1 + (B_u/B_v)^2}$  such that (2.6) holds uniformly.

This ends the proof of Proposition 1.2.

Let us mention that condition (1.11) implies (1.13), for any  $T > 0$ , there exists a constant  $C_T > 0$  such that

$$(2.7) \quad \int_0^T \sum_{j \geq 0} \Delta x |U_j(t)|^2 dt + \int_0^T |U_0(t)|^2 dt \leq C_T \sum_{j \geq 0} \Delta x |f_j|^2.$$

This will be used to prove Theorem 1.1.

**2.2. Numerical experiments.** For fixed  $\Delta x$  and letting  $\varepsilon$  vary, we consider now the behavior of  $\langle U, HU \rangle_{\Delta x}$  with respect to time according to condition (1.13). We also look at the degenerate case when the UKC (1.4) does not hold (and then, none of the other stability conditions). Due to the previous analysis and the calculations of Xin and Xu in [18], we expect to see that  $\langle U, HU \rangle_{\Delta x}$  decreases when the boundary condition satisfies  $B_u B_v > 0$ . We also provide numerical results in the case when  $B_u B_v < 0$  while condition (1.13) still holds.

To isolate the effect of the possible boundary layer and avoid the complicated interactions of boundary and initial layer at  $x_j$  goes to infinity, we will consider the small values of  $T = 0.2$  and the initial data

$$f_j = \begin{cases} (15, 10)^T, & \text{if } 0 < x_j \leq 1/2, \\ 0, & \text{if } x_j > 1/2. \end{cases}$$

At the space-time corner  $x = 0, t = 0$ , the initial data satisfies

$$U_0(0) = U'_0(0) = 0.$$

Let  $a = 4$  and the space step  $\Delta x = 10^{-2}$ . The time discretization used for solving any ODE hereafter is the one given by the function ode45 in MATLAB. This function implements a Runge-Kutta method with a variable time step for efficient computation.

Firstly, we choose different values  $(B_u, B_v)$  such that the condition (1.13) is satisfied with  $\varepsilon = 10^{-2}$  and  $\varepsilon = 10^2$ . The following figures show the behavior of  $\langle U, HU \rangle_{\Delta x}(t)$  for all  $t < 0.2$ .

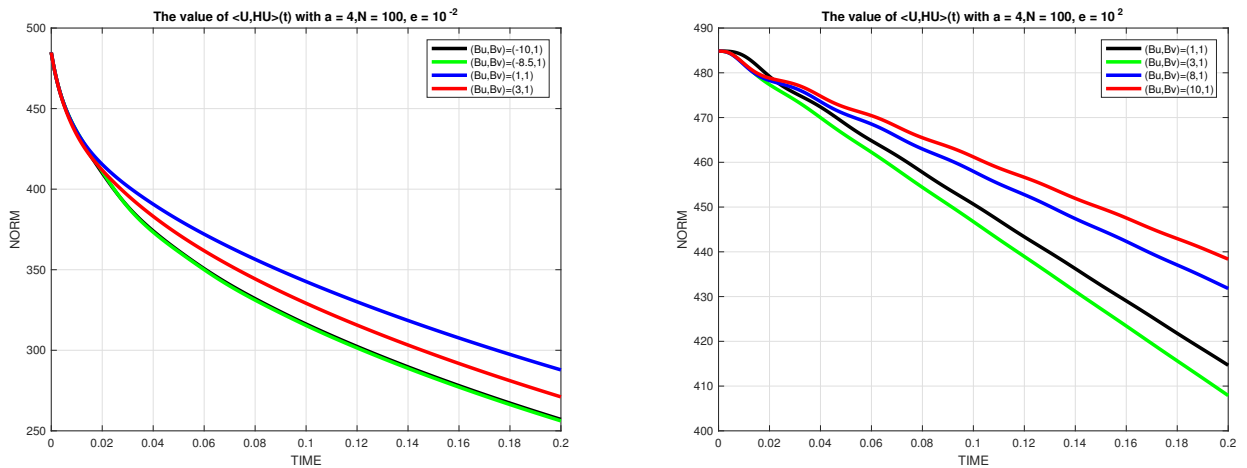


FIGURE 2.1. The values of  $\langle U, HU \rangle_{\Delta x}(t)$  with different values  $(B_u, B_v)$  satisfying condition (1.13) at  $\varepsilon = 10^{-2}$  (the left figure) and  $\varepsilon = 10^2$  (the right figure).

In Figure 2.1, we can see that



- Under assumption (1.13),  $\varepsilon \in (0, +\infty)$  and  $\Delta x = 10^{-2}$ , we have  $\langle U, HU \rangle_{\Delta x}(t) \leq \langle f, Hf \rangle_{\Delta x}$  with nonzero initial data  $f$  and homogeneous boundary condition  $b(t) \equiv 0$ . Even if  $B_u B_v < 0$  satisfies the stability condition (1.13), we get  $\langle U, HU \rangle_{\Delta x}(t)$  decrease for any  $t < T$ . For example, let  $\Delta x = 10^{-2}$ ,  $\varepsilon = 10^{-2}$ ,  $T = 0.2$  and  $(B_u, B_v) = (-8.5, 1)$ , for any  $t \in [0, T]$ ,  $\langle U, HU \rangle_{\Delta x}(t) \leq \langle f, Hf \rangle_{\Delta x}(t)$ .
- For  $\varepsilon = 10^{-2}$  the values of  $\langle U, HU \rangle_{\Delta x}(t)$  go down suddenly for small time  $t$ . In the case  $\varepsilon = 10^2$ , the decrease seems to be linear.

Secondly, we choose the values of  $(B_u, B_v)$  such that the condition (1.13) is not satisfied with  $\varepsilon = 10^{-2}$  nor with  $\varepsilon = 10^2$ . Besides, we present the values of  $\langle U, HU \rangle_{\Delta x}(t)$  if the boundary condition (1.3) with  $b(t) = 0$  does not satisfy the UKC (1.4). The following figures show the behavior of  $\langle U, HU \rangle_{\Delta x}(t)$  for all  $t < 0.2$ .

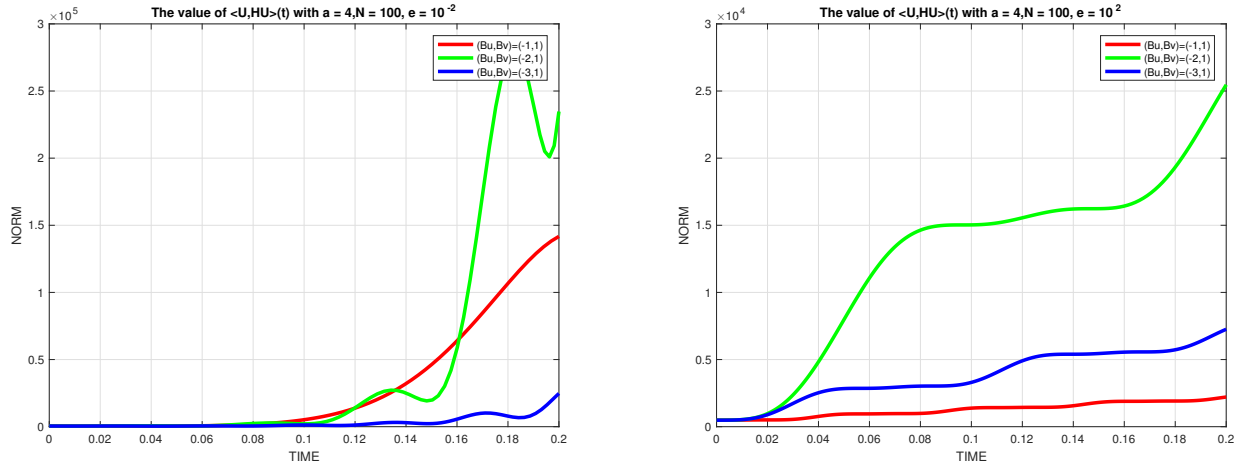


FIGURE 2.2. The values of  $\langle U, HU \rangle_{\Delta x}(t)$  with different values  $(B_u, B_v)$  not satisfying condition (1.13) at  $\varepsilon = 10^{-2}$  (the left figure) and  $\varepsilon = 10^2$  (the right figure).

In Figure 2.2, we can see that

- On the boundary  $x = 0$ , for all  $\varepsilon > 0$ , if the boundary condition (1.3) with homogeneous boundary condition  $b(t) \equiv 0$  does not satisfy the UKC, then  $v^\varepsilon = \sqrt{au}^\varepsilon$ . Therefore the numerical scheme of the IBVP is not stable for each fixed  $\varepsilon$ . For  $\varepsilon = 10^{-2}$  and  $\varepsilon = 10^2$ , if we choose  $(B_u, B_v) = (-2, 1)$  then the values of  $\langle U, HU \rangle_{\Delta x}(t)$  increase quickly.
- For all  $t < T$ , if the condition (1.13) fails, we get  $\langle U, HU \rangle_{\Delta x}(t) > \langle f, Hf \rangle_{\Delta x}$ . In the particular case  $\varepsilon = 10^{-2}$ , there exists  $0 < t_1 < t_2$  such that  $\langle U, HU \rangle_{\Delta x}(t_1) > \langle U, HU \rangle_{\Delta x}(t_2)$ . However, after that, by influence of the boundary condition, the values of  $\langle U, HU \rangle_{\Delta x}(t)$  increase rapidly. In the case  $\varepsilon = 10^2$ , the values of  $\langle U, HU \rangle_{\Delta x}(t)$  rise gradually.

Clearly, the numerical results show that the energy increases in time as soon as condition (1.13) does not hold. The behavior is even worse when the UKC (1.4) is not satisfied. It seems that condition (1.13) is also necessary to ensure the non-increase of the energy, but let us stress that an increasing energy with respect to time may not be in contradiction with the stiff stability.

### 3. STIFF STRONG STABILITY OF THE SEMI-DISCRETE IBVP

In the continuous case, the IBVP (1.1)-(1.3) is stiffly well-posed if and only if the boundary condition (1.3) satisfies the SKC (1.5). Now, we want to address the question of existence of unstable solution in order to derive necessary condition of stability for the discrete IBVP (1.10). Following Z. Xin and W. Xu in [18], we shall apply the normal mode analysis to derive the stiff boundary condition (1.11).

**3.1. Derivation of the stiff boundary condtion.** We look for (nontrivial) solutions of (1.10) satisfying the homogeneous boundary condition

$$BU_0(t) = 0$$

and of the form

$$(3.1) \quad U_j(t) = e^{\xi t/\varepsilon} \phi_j,$$

with  $\operatorname{Re} \xi > 0$ ,  $(\phi_j) \in l^2(\mathbb{N}, \mathbb{C}^2)$ . Such solutions, if they exist, clearly violate the  $\varepsilon$ -uniform  $l^2$  estimates in (1.12). Our goal is to find a sufficient condition to ensure that it does not exist.

Substituting (3.1) into (1.10), we obtain the following problem

$$(3.2) \quad \phi_{j+1} - \phi_{j-1} = \frac{2\Delta x}{\varepsilon} M \phi_j, \quad j > 0$$

$$(3.3) \quad B\phi_0 = 0,$$

$$(3.4) \quad \Pi_2 P^{-T} H A \left( \phi_1 - \left( I + \frac{\Delta x}{\varepsilon} M \right) \phi_0 \right) = 0,$$

where

$$(3.5) \quad M = M(\xi) = A^{-1}(S - \xi I) = \frac{1}{a} \begin{pmatrix} 0 & -(1 + \xi) \\ -a\xi & 0 \end{pmatrix}.$$

The eigenvalues of  $M$  can be easily found to be

$$(3.6) \quad \mu_{\pm} = \mu_{\pm}(\xi) = \pm \sqrt{\frac{\xi(1 + \xi)}{a}},$$

with the corresponding eigenvectors

$$r_{\pm} = r_{\pm}(\xi) = \begin{pmatrix} 1 \\ \frac{a\mu_{\pm}}{1 + \xi} \end{pmatrix}.$$

In the half plane  $\{\xi \in \mathbb{C} : \operatorname{Re} \xi \geq 0\}$ , the complex function

$$h(\xi) = \sqrt{\xi(1 + \xi)}$$

is analytic. (As usual, we take  $\sqrt{z}$  to be the principal branch with the branch cut along the negative real axis.)

Let  $\xi = \alpha + i\beta$ ,  $\alpha \geq 0$ , and

$$p = \alpha(1 + \alpha) - \beta^2, \quad q = (1 + 2\alpha)\beta.$$

Then,

$$\operatorname{Re} h(\xi) = \operatorname{Re} \sqrt{p + iq} = \sqrt{\frac{p + \sqrt{p^2 + q^2}}{2}}.$$

Now we observe that

$$\begin{aligned} \sqrt{p^2 + q^2} &= \sqrt{(\alpha(1 + \alpha) - \beta^2)^2 + (1 + 2\alpha)^2 \beta^2} \\ &= \sqrt{(\alpha(1 + \alpha) + \beta^2)^2 + \beta^2} \\ &\geq \alpha(1 + \alpha) + \beta^2. \end{aligned}$$

Therefore,

$$(3.7) \quad \operatorname{Re} h(\xi) \geq \sqrt{\alpha(1 + \alpha)} \geq \alpha.$$

We further note that by using the basic inequality  $\sqrt{1+x} \leq 1+x/2$  ( $x \geq -1$ ), we can also obtain a close upper bound for  $\operatorname{Re} h(\xi)$

$$\operatorname{Re} h(\xi) \leq \frac{1+2\alpha}{2}.$$

In particular, we have from the above

$$\operatorname{Re} \mu_+(\xi) > 0, \quad \operatorname{Re} \mu_-(\xi) < 0, \quad \text{for } \operatorname{Re} \xi > 0.$$

To solve the recurrence equation (3.2) in  $l^2(\mathbb{N}, \mathbb{C}^2)$ , the following lemma will be useful.

**Lemma 3.1.** *Let  $v \in \mathbb{C}$ . If  $\operatorname{Re} v < 0$  then  $|v + \sqrt{v^2 + 1}| < 1$ .*

*Proof.* Assume that  $v = x + yi$  with  $x < 0$  and  $y \in \mathbb{R}$ .

- In the case  $y = 0$ ,

$$|v + \sqrt{v^2 + 1}| < 1 \Leftrightarrow x + \sqrt{x^2 + 1} > 0.$$

- In the case  $y \neq 0$ , we have

$$\sqrt{v^2 + 1} = a_1 + b_1 i,$$

where

$$(3.8) \quad \begin{aligned} a_1 &= \sqrt{\frac{p_1 + \sqrt{p_1^2 + q_1^2}}{2}}, & b_1 &= \operatorname{sgn}(q_1) \sqrt{\frac{-p_1 + \sqrt{p_1^2 + q_1^2}}{2}}, \\ p_1 &= x^2 - y^2 + 1, & q_1 &= 2xy. \end{aligned}$$

Now, we observe that

$$(3.9) \quad |v + \sqrt{v^2 + 1}| < 1 \Leftrightarrow x^2 + b_1^2 + y^2 + a_1^2 + 2a_1 x + 2b_1 y < 1.$$

On the other hand, we have

$$(3.10) \quad y^2 + a_1^2 = x^2 + b_1^2 + 1 \quad \text{and} \quad y = \frac{a_1 b_1}{x}.$$

Substituting (3.10) into (3.9), one obtains

$$|v + \sqrt{v^2 + 1}| < 1 \Leftrightarrow \sqrt{(x^2 - y^2 + 1)^2 + 4x^2 y^2} > x^2 + y^2 - 1.$$

Therefore,

$$|v + \sqrt{v^2 + 1}| < 1.$$

□

Let  $\bar{P}$  be the  $2 \times 2$  matrix whose columns are composed by the component of the vector  $r_{\pm}$

$$\bar{P} = \bar{P}(\xi) = \begin{pmatrix} 1 & 1 \\ \frac{a\mu_-(\xi)}{1+\xi} & \frac{a\mu_+(\xi)}{1+\xi} \end{pmatrix},$$

then

$$M = \bar{P} D \bar{P}^{-1}$$

with  $D = D(\xi) = \operatorname{diag}(\mu_+(\xi), \mu_-(\xi))$ .

Let us define

$$\psi_j = (\psi_j^I, \psi_j^{II}) = \bar{P}^{-1} \phi_j.$$

Now, (3.2) can be reformulated as

$$(3.11) \quad \begin{cases} \psi_{j+1}^I - \psi_{j-1}^I = \frac{2\mu_+(\xi)\Delta x}{\varepsilon} \psi_j^I, \\ \psi_{j+1}^{II} - \psi_{j-1}^{II} = \frac{2\mu_-(\xi)\Delta x}{\varepsilon} \psi_j^{II}. \end{cases}$$

Firstly, we find that  $\psi_j^I \in l^2(\mathbb{N}, \mathbb{C})$  satisfies

$$(3.12) \quad \psi_{j+1}^I - \psi_{j-1}^I = \frac{2\mu_+(\xi)\Delta x}{\varepsilon} \psi_j^I.$$

To solve the equation (3.12), we assume that

$$(3.13) \quad \psi_j^I = z^j R_1,$$

with  $|z| < 1$  and  $R_1 \in \mathbb{C}$ . Substituting (3.13) into (3.12), we obtain

$$(3.14) \quad z_{\pm} = z_{\pm}(\xi) = \frac{\mu_+(\xi)\Delta x}{\varepsilon} \pm \sqrt{\left(\frac{\mu_+(\xi)\Delta x}{\varepsilon}\right)^2 + 1}.$$

Applying Lemma 3.1 with  $\text{Re } \mu_+(\xi) > 0$  for  $\text{Re } \xi > 0$ , we can prove

$$|z_-| = \left| \frac{-\mu_+\Delta x}{\varepsilon} + \sqrt{\left(\frac{-\mu_+\Delta x}{\varepsilon}\right)^2 + 1} \right| < 1.$$

Thus, the solution of (3.12) satisfying  $\psi_j^I \in l^2(\mathbb{N}, \mathbb{C})$  can be represented as

$$\psi_j^I = z_-^j R_1.$$

Similarly, the solution of the following problem

$$\psi_{j+1}^{II} - \psi_{j-1}^{II} = \frac{2\mu_-(\xi)\Delta x}{\varepsilon} \psi_j^{II}$$

can be represented as

$$\psi_j^{II} = w_+^j R_2,$$

with  $R_2 \in \mathbb{C}$  and

$$(3.15) \quad w_+ = w_+(\xi) = \frac{\mu_-(\xi)\Delta x}{\varepsilon} + \sqrt{\left(\frac{\mu_-(\xi)\Delta x}{\varepsilon}\right)^2 + 1}$$

satisfies  $|w_+| < 1$  by using Lemma 3.1 with  $\text{Re } \mu_-(\xi) < 0$  for  $\text{Re } \xi > 0$ .

Therefore, the solution  $\phi_j \in l^2(\mathbb{N}, \mathbb{C}^2)$  of (3.2) can be presented as

$$(3.16) \quad \phi_j = c\bar{P}Z^j R,$$

with  $Z = Z(\xi) = \text{diag}(z_-(\xi), w_+(\xi))$ ,  $R = (R_1, R_2)^T \in \mathbb{C}^2$  and for any constant  $c$ .

Plugging (3.16) into the boundary condition (3.3) and (3.4), we get

$$(3.17) \quad \begin{cases} B\bar{P}R = 0, \\ \Pi_2 P^{-T} H A \bar{P} \left( Z - \left( I + \frac{\Delta x}{\varepsilon} D \right) \right) R = 0. \end{cases}$$

Let

$$(3.18) \quad \begin{aligned} g &= g(\xi) = \frac{a\mu_+(\xi)}{1+\xi}, & k &= k(\xi) = \frac{a\mu_-(\xi)}{1+\xi}, \\ \delta_1 &= \delta_1(\xi) = z_-(\xi) - \left( 1 + \frac{\mu_+(\xi)\Delta x}{\varepsilon} \right), \\ \delta_2 &= \delta_2(\xi) = w_+(\xi) - \left( 1 + \frac{\mu_-(\xi)\Delta x}{\varepsilon} \right). \end{aligned}$$

Thus, (3.17) can be reformulated as

$$NR = 0,$$

where

$$(3.19) \quad N = N(\xi) = \begin{pmatrix} B_u + k(\xi)B_v & B_u + g(\xi)B_v \\ -a\delta_1(\xi)(B_u - k(\xi)B_v) & -a\delta_2(\xi)(B_u - g(\xi)B_v) \end{pmatrix}.$$

**Proposition 3.2.** *Consider any  $\delta = \frac{\Delta x}{\varepsilon} > 0$  and any  $\xi \in \mathbb{C}$  with  $\operatorname{Re} \xi \geq 0$ . If  $B_u B_v > 0$  then  $\det(N(\xi)) \neq 0$  and therefore there is no unstable solution of (1.10) of the form (3.1).*

*Proof.* From (3.19), we can compute

$$\det(N) = a(\delta_1(B_u + gB_v)^2 - \delta_2(B_u - gB_v)^2),$$

with  $k(\xi) = -g(\xi)$ . On the other hand, we get

$$\det(N) \neq 0 \Leftrightarrow |\det(N)| \neq 0$$

and we can see that

$$(3.20) \quad |\det(N)| \neq 0 \Leftrightarrow \left| 1 - \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right| |\delta_1| \left| g + \frac{B_u}{B_v} \right|^2 \neq 0.$$

• Firstly, for any  $\delta > 0$ ,  $\xi \in \mathbb{C}$ ,  $\operatorname{Re} \xi \geq 0$  and  $B_u B_v > 0$ , we prove

$$\left| \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right| \neq 1.$$

Since  $\mu_-(\xi) = -\mu_+(\xi)$ , we have

$$(3.21) \quad \left| \frac{\delta_2}{\delta_1} \right| \leq 1 \Leftrightarrow \operatorname{Re} \sqrt{(\mu_+ \delta)^2 + 1} \geq 0.$$

Furthermore, the complex function  $g(\xi)$  is analytic and bounded in  $\operatorname{Re} \xi \geq 0$ . By the conformal mapping theorem,  $g(\xi)$  maps the half plane  $\operatorname{Re} \xi \geq 0$  to a simply connected closed bounded domain  $\Omega \subset \mathbb{C}$  whose boundary corresponds to the image of the imaginary axis  $\operatorname{Re} \xi = 0$  under  $g$ . The boundary curve

$$g(i\beta) = \frac{\sqrt{-a\beta^2 + a\beta i}}{1 + i\beta}, \quad -\infty \leq \beta \leq \infty$$

is a closed curve which intersects the real axis only at  $\beta = 0$  and at  $\beta = \pm\infty$  with  $g(0) = 0$ ,  $g(\pm i\infty) = \sqrt{a}$ . Besides, the curve is transversal to the real axis.

Since  $B_u B_v > 0$ ,  $\operatorname{Re} g(\xi) \geq 0$  in  $\operatorname{Re} \xi \geq 0$ , we observe that

$$(3.22) \quad \left| \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right|^2 \leq 1 \Leftrightarrow \operatorname{Re} g \geq 0.$$

According to (3.21) and (3.22), we obtain

$$(3.23) \quad \left| \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right| \leq 1.$$

From (3.23), we assume that

$$\left| \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right| = 1 \Leftrightarrow \begin{cases} \left| \frac{\delta_2}{\delta_1} \right| = 1, \\ \left| \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right|^2 = 1 \end{cases} \Leftrightarrow \begin{cases} \operatorname{Re} \sqrt{(\mu+\delta)^2 + 1} = 0, \\ \xi = 0. \end{cases}$$

Since  $\xi = 0$ , we have  $\operatorname{Re} \sqrt{(\mu+\delta)^2 + 1} = 1$ . Thus,

$$(3.24) \quad \left| \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right| \neq 1.$$

According to (3.23) and (3.24), for any  $\delta > 0$ ,  $\xi \in \mathbb{C}$ ,  $\operatorname{Re} \xi > 0$  and  $B_u B_v > 0$ , we have

$$\left| 1 - \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right| \neq 0.$$

• Secondly, for all  $\xi \in \mathbb{C}$ ,  $\operatorname{Re} \xi > 0$ , we get

$$(3.25) \quad |\delta_1| = |1 + \sqrt{(\mu+\delta)^2 + 1}| \geq 1, \quad \text{for all } \delta > 0$$

and

$$(3.26) \quad \left| g + \frac{B_u}{B_v} \right|^2 \geq \left( \frac{B_u}{B_v} \right)^2 > 0,$$

with  $B_u B_v > 0$ ,  $\operatorname{Re} g(\xi) \geq 0$  in  $\operatorname{Re} \xi \geq 0$ .

Therefore, we have

$$|\det(N)| \neq 0.$$

□

Actually, we will need in the Section 4.2 a more stringent version of the estimate (3.24) uniform in  $\delta > 0$  and  $\xi \in \mathbb{C}$  with  $\operatorname{Re} \xi \geq 0$ . This is the object of the next lemma

**Lemma 3.3.** *Assume  $B_u B_v > 0$ . There exists  $c \in (0, 1)$  such that for any  $\delta = \frac{\Delta x}{\varepsilon} > 0$  and  $\xi \in \mathbb{C}$  with  $\operatorname{Re} \xi \geq 0$*

$$(3.27) \quad \left| \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right| \leq 1 - c.$$

where  $g$ ,  $\delta_1$  and  $\delta_2$  are the same as in (3.18).

*Proof.* Firstly, from (3.21) and (3.22) we can see that for any  $\delta > 0$ ,  $\xi \in \mathbb{C}$  with  $\operatorname{Re} \xi \geq 0$ ,  $B_u B_v > 0$

$$\left| \frac{\delta_2}{\delta_1} \right| \leq 1 \quad \text{and} \quad |\tau(g(\xi))| \leq 1,$$

where

$$\tau(g(\xi)) = \frac{\frac{B_u}{B_v} - g(\xi)}{\frac{B_u}{B_v} + g(\xi)}.$$

Furthermore, we get  $g(\xi)$  maps the half plane  $\text{Re } \xi \geq 0$  to a simply connected closed bounded domain  $\Omega$ . Thus,  $|\tau(g(\xi))|$  tends to 1 only if  $\text{Re } g(\xi)$  goes to 0.

Secondly, let  $\xi = \alpha + i\beta$  with  $\alpha \geq 0$  and  $\beta \in \mathbb{R}$ , after some calculations, one obtains

$$\text{Re } g(\xi) = \sqrt{\frac{a \left( p + \sqrt{p^2 + q^2} \right)}{(1 + \alpha)^2 + \beta^2}},$$

where

$$p = \alpha(1 + \alpha) + \beta^2, \quad q = \beta.$$

Thus, for all  $\alpha \geq 0$ ,  $\beta \in \mathbb{R}$ ,  $\text{Re } g(\xi)$  goes to 0 only if  $\xi$  tends to 0. Therefore, outside a neighborhood of 0 in  $\text{Re } \xi \geq 0$

$$|\tau(g(\xi))| \leq c < 1.$$

Moreover, for any  $\delta > 0$ ,  $\xi \in \mathbb{C}$  with  $\text{Re } \xi \geq 0$ ,  $\left| \frac{\delta_2}{\delta_1} \right|$  tends to 1 only if  $\text{Re } \sqrt{(\mu_+(\xi)\delta)^2 + 1}$  goes to 0. In a neighborhood of 0 in  $\text{Re } \xi \geq 0$ , for any  $\delta > 0$ ,  $\text{Re } \sqrt{(\mu_+(\xi)\delta)^2 + 1} \geq c_1 > 0$  (for more details, see Lemma B.1). Thus,

$$\left| \frac{\delta_2}{\delta_1} \right| \leq c_2 < 1.$$

Therefore, (3.27) is true.  $\square$

**3.2. Numerical experiments for the necessity of the boundary condition.** Using the normal mode analysis, we exhibit condition (1.11) which ensures the non existence of unstable solutions of the form (3.1). However, we are not able to prove that this condition is also necessary, i.e. that there exists an unstable solution to (1.10) with homogeneous boundary condition as soon as  $B_u B_v < 0$ . After presenting numerical results in the favorable case (1.11), we study numerically  $|\det(N)|$  when  $B_u B_v < 0$ , and more importantly, when  $B_u/B_v < -\sqrt{a}$ , which is a sub-case of the SKC (1.5).

Let

$$F \left( \alpha, \beta, \frac{B_u}{B_v} \right) = \delta_1 \left( \frac{B_u}{B_v} + g \right)^2 - \delta_2 \left( \frac{B_u}{B_v} - g \right)^2,$$

we have

$$|\det(N)| \neq 0 \Leftrightarrow \left| F \left( \alpha, \beta, \frac{B_u}{B_v} \right) \right| \neq 0.$$

The following figures display three-dimensional data in two dimensions using contours or color-coded regions. It takes three arguments: a grid of  $\alpha$  values, a grid of  $\beta$  values and a grid of  $|F(\alpha, \beta, B_u/B_v)|$ . We choose the vertical axis as the set of values  $\beta$  and the horizontal axis as the set of values  $\alpha$ . For each of the values  $B_u/B_v$ , a contour line or isoline of a function  $|F(\alpha, \beta, B_u/B_v)|$  is a curve along which the function has a constant value, so that the curve joins points of equal value.

Firstly, for all  $(\alpha, \beta) \in \mathbb{R}^2$ , we consider the values of  $|F(\alpha, \beta, 1/40)|$  with  $\delta = 10$  (see Figure 3.1) and  $|F(\alpha, \beta, 1)|$  with  $\delta = 10^{-2}$  (see Figure 3.2) satisfies in this cases,  $B_u/B_v > 0$ .

In Figure 3.1, we can see that in the case  $\delta = 10$ , in the half plane  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ , there exists  $c > 0$  such that  $|F(\alpha, \beta, 1/40)| \geq c$ . For example, for all  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ ,  $|F(\alpha, \beta, 1/40)| \geq 0.001$ .

In Figure 3.2, we can see that in the case  $\delta = 10^{-2}$ , in the half plane  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ , there exists  $c > 0$  such that  $|F(\alpha, \beta, 1)| \geq c$ . For example, for all  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ ,  $|F(\alpha, \beta, 1/40)| \geq 1.04$ . Secondly, we consider for any  $\delta > 0$ ,  $\xi \in \mathbb{R}$ ,  $\text{Re } \xi > 0$ ,  $|F(\alpha, \beta, B_u/B_v)| \neq 0$  if  $B_u/B_v < 0$ ? Even if  $B_u/B_v < -\sqrt{a}$  satisfies the SKC (1.5), there exists a nontrivial solution of (1.10) of the form (3.1)?

The following figures show the behavior of  $|F(\alpha, \beta, B_u/B_v)| \neq 0$  with  $a = 4$ ,  $B_u/B_v = -1$ ,  $\delta = 1$  (see Figure 3.3) and  $\delta = 10$  (see Figure 3.4)

In Figure 3.3, we can see that in the case  $\delta = 1$ , in the half plane  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ , there exists  $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}$  such that  $|F(\alpha, \beta, -1)| \ll 1$ . For example, we have  $|F(0.2027, 0.1471, -1)| \approx 10^{-4}$ . However, in





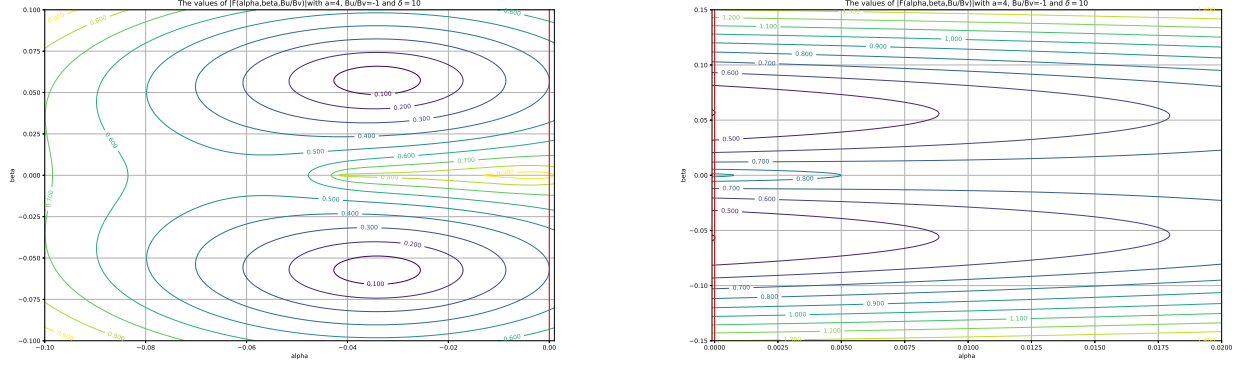


FIGURE 3.4. The values of  $|F(\alpha, \beta, -1)|$  with  $a = 4$ ,  $\delta = 10$ ,  $\alpha < 0$  (the left figure) and  $\alpha > 0$  (the right figure) and  $\beta \in \mathbb{R}$  case  $B_u/B_v < 0$ .

In the continuous case, Z. Xin and W. Xu [18] show that there is no unstable solution if  $B_u/B_v < -\sqrt{a}$ . For discrete IBVP (1.10), these numerical experiments suggest the following conjecture to be true.

*Conjecture 3.4.* Consider the case  $B_u/B_v < -\sqrt{a}$ . For any  $\delta > 0$ , there exists  $\xi \in \mathbb{C}$  with  $\text{Re } \xi > 0$  such that  $|\det(N(\xi))| = 0$ . In other words, there exists an unstable solution of (1.10) of the form (3.1).

For all  $(\alpha, \beta) \in \mathbb{R}^2$ , we choose  $a = 4$  and  $B_u/B_v = -3.5$  satisfy  $B_u/B_v < -\sqrt{a}$ . The following figures show the behavior of  $|F(\alpha, \beta, -3.5)|$  with  $\delta = 10$  and  $\delta = 10^{-2}$ .

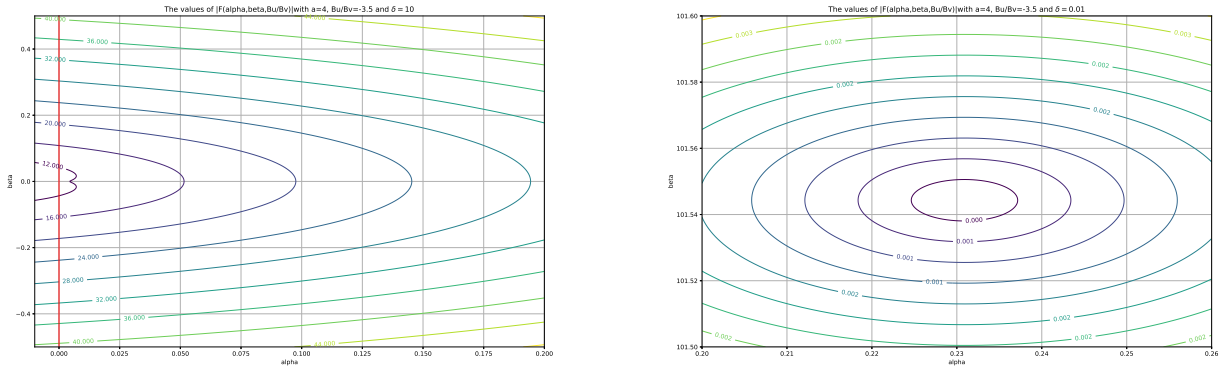


FIGURE 3.5. The values of  $|F(\alpha, \beta, -3.5)|$  with  $\delta = 10$ ,  $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}$  (the left figure) and the values of  $|F(\alpha, \beta, -3.5)|$  with  $\delta = 10^{-2}$ ,  $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$  (the right figure) case  $B_u/B_v < -\sqrt{a}$ .

In Figure 3.5, we can see that in the case  $\delta = 10$ , for all  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ ,  $|F(\alpha, \beta, -3.5)| \neq 0$ . In the other case  $\delta = 10^{-2}$ , there exists  $(\alpha_0, \beta_0) \in \mathbb{R}^+ \times \mathbb{R}^+$  such that  $|F(\alpha_0, \beta_0, -3.5)| \ll 1$ . For example, we have  $F(0.23, 101.55, -3.5) \approx 10^{-4}$ .

## 4. STIFF STABILITY OF THE SEMI-DISCRETE IBVP WITH HOMOGENEOUS INITIAL CONDITION

The semi-discrete approximation of the IBVP (1.10) with homogeneous initial condition can be presented as

$$(4.1) \quad \begin{aligned} \partial_t U_j(t) + A \frac{U_{j+1}(t) - U_{j-1}(t)}{2\Delta x} &= \frac{1}{\varepsilon} S U_j(t), \quad j > 0, \\ U_j(0) &= 0, \\ B U_0(t) &= b(t), \end{aligned}$$

$$\partial_t (\Pi_2 P^{-T} H U_0)(t) + \Pi_2 P^{-T} H A \frac{U_1(t) - U_0(t)}{\Delta x} = \frac{1}{\varepsilon} \Pi_2 P^{-T} H S U_0(t).$$

For difference approximations, the Laplace transform is already more powerful tool for problems in one space dimension. It is used to determine stability features when the energy method is not sufficient. Under assumption  $B_u B_v > 0$ , the numerical solution  $U_j(t)$  can be constructed by the method of Laplace transform. By using the Parseval's identity, we get the expected result of Proposition 1.3.

**4.1. Solution by Laplace transform.** The numerical solution  $U_j(t)$  of the IBVP (4.1) can be constructed by the method of Laplace transform. Let

$$\tilde{U}_j(\xi) = \mathcal{L}U_j = \int_0^\infty e^{-\xi t} U_j(t) dt, \quad \text{Re } \xi > 0.$$

With  $U_j(0) \equiv 0$ , we have

$$\mathcal{L}(\partial_t U_j) = \xi \tilde{U}_j(\xi) - U_j(0) = \xi \tilde{U}_j(\xi)$$

and therefore (4.1) become

$$(4.2) \quad \tilde{U}_{j+1}(\xi) - \tilde{U}_{j-1}(\xi) = \frac{2\Delta x}{\varepsilon} M(\varepsilon\xi) \tilde{U}_j(\xi), \quad j > 0,$$

$$(4.3) \quad B \tilde{U}_0(\xi) = \tilde{b}(\xi),$$

$$(4.4) \quad \Pi_2 P^{-T} H A \left( \tilde{U}_1(\xi) - \left( I + \frac{\Delta x}{\varepsilon} M(\varepsilon\xi) \right) \tilde{U}_0(\xi) \right) = 0,$$

where

$$\tilde{b}(\xi) = \mathcal{L}b = \int_0^\infty e^{-\xi t} b(t) dt$$

and the matrix  $M$  is the same as in (3.5).

Note that the eigenvalues  $\mu_\pm(\xi)$  of the matrix  $M(\xi)$  satisfy

$$\text{Re } \mu_-(\xi) < 0, \quad \text{Re } \mu_+(\xi) > 0, \quad \text{for } \text{Re } \xi > 0.$$

The same as in (3.11)-(3.16), the solution  $\tilde{U}_j(\xi)$  of (4.2) is given by

$$\tilde{U}_j(\xi) = \bar{P}(\varepsilon\xi) Z^j(\varepsilon\xi) R.$$

The value of vector  $R$  can be determined easily from the boundary condition (4.3) and (4.4)

$$R = \frac{\tilde{b}(\xi)}{\det(N(\varepsilon\xi))} N_1(\varepsilon\xi),$$

where

$$(4.5) \quad N_1 = N_1(\xi) = a \begin{pmatrix} -\delta_2(\xi)(B_u - g(\xi)B_v) \\ \delta_1(\xi)(B_u + g(\xi)B_v) \end{pmatrix}$$

and the matrix  $N$  is the same as in (3.19). Therefore,

$$\tilde{U}_j(\xi) = \frac{\tilde{b}(\xi)}{\det(N(\varepsilon\xi))} \bar{P}(\varepsilon\xi) Z^j(\varepsilon\xi) N_1(\varepsilon\xi).$$

With  $\tilde{U}_j(\xi)$  found, the numerical solution  $U_j(t)$  of (4.1) can be obtained by inverting the Laplace transform

$$U_j(t) = \mathcal{L}^{-1}\tilde{U}_j(\xi) = \frac{1}{2\pi} \int_{-\infty}^{-\infty} e^{(\alpha+i\beta)t} \tilde{U}_j(\alpha+i\beta) d\beta, \quad \alpha > 0.$$

**4.2. Stiff stability analysis.** Under condition  $B_u B_v > 0$ , we consider Proposition 1.3 with homogeneous initial condition  $(f_j)_{j \in \mathbb{N}} \equiv 0$  and nonzero boundary data  $b(t)$ .

**Lemma 4.1.** *Assume  $B_u B_v > 0$ . There exists  $C > 0$  such that for any  $\delta = \frac{\Delta x}{\varepsilon} > 0$  and  $\xi \in \mathbb{C}$  with  $\operatorname{Re} \xi \geq 0$*

$$\frac{|\det(N(\xi))|^2}{\|N_1(\xi)\|^2} \geq C B_u^2.$$

*Proof.* From (3.19) and (4.5), we can compute

$$|\det(N)|^2 = a^2 \left| 1 - \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right|^2 |\delta_1|^2 \left| \frac{B_u}{B_v} + g \right|^4 B_v^4$$

and

$$\|N_1\|^2 = a^2 \left( 1 + \left| \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right| \right) |\delta_1|^2 \left| \frac{B_u}{B_v} + g \right|^2 B_v^2.$$

Thus,

$$\frac{|\det(N)|^2}{\|N_1\|^2} = \left| 1 - \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right|^2 \left( 1 + \left| \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right| \right)^{-1} \left| \frac{B_u}{B_v} + g \right|^2 B_v^2.$$

Using Lemma 3.3, for any  $\xi \in \mathbb{C}$ ,  $\operatorname{Re} \xi \geq 0$  and  $\delta > 0$ , there exists  $c > 0$  such that

$$\left| 1 - \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right|^2 \geq c$$

and from (3.23), we have

$$\left( 1 + \left| \frac{\delta_2}{\delta_1} \left( \frac{\frac{B_u}{B_v} - g}{\frac{B_u}{B_v} + g} \right)^2 \right| \right)^{-1} \geq 1/2.$$

Since  $B_u B_v > 0$ ,  $\operatorname{Re} g(\xi) \geq 0$  in  $\operatorname{Re} \xi \geq 0$ , we get

$$\left| g + \frac{B_u}{B_v} \right|^2 \geq \left( \frac{B_u}{B_v} \right)^2.$$

Therefore, there exists  $C > 0$  such that

$$\frac{|\det(N)|^2}{\|N_1\|^2} \geq C B_u^2.$$

□

Now, we prove the uniform  $l^2$  estimate (1.15). By an application of the following Parseval's identity [5][16]

$$\int_0^\infty e^{-2\alpha t} |U_j(t)|^2 dt = \frac{1}{2\pi} \oint_{-\infty}^\infty |\tilde{U}_j(\alpha + i\beta)|^2 d\beta, \quad \alpha > 0,$$

we have

$$\begin{aligned} \int_0^\infty e^{-2\alpha t} |U_0(t)|^2 dt &= \frac{1}{2\pi} \oint_{-\infty}^\infty |\tilde{U}_0(\alpha + i\beta)|^2 d\beta \\ &= \frac{1}{2\pi} \oint_{-\infty}^\infty |\tilde{b}(\xi)|^2 \frac{\|N_1\|^2}{|\det(N)|^2} |\bar{P}|^2 d\beta. \end{aligned}$$

where  $\xi = \alpha + i\beta$ . We fix  $\alpha > 0$  from now on.

According to Lemma 4.1, there exists  $C_1 > 0$  such that for any  $\delta > 0$ ,  $\xi \in \mathbb{C}$ ,  $\operatorname{Re} \xi \geq 0$ ,

$$\frac{\|N_1\|^2}{|\det(N)|^2} \leq C_1.$$

On the other hand, since  $k(\xi) = -g(\xi)$  is uniformly bounded in  $\operatorname{Re} \xi \geq 0$ , we obtain

$$(4.6) \quad \begin{aligned} \int_0^\infty e^{-2\alpha t} |U_0(t)|^2 dt &\leq O(1) \frac{1}{2\pi} \oint_{-\infty}^\infty |\tilde{b}(\alpha + i\beta)|^2 d\beta \\ &\leq O(1) \int_0^\infty e^{-2\alpha t} |b(t)|^2 dt. \end{aligned}$$

This, together with a consequence of the hyperbolicity of (1.1) by using the classical argument of changing the data  $b$  to zero after time  $T$  and unchanged before time  $T$ , we obtain the desired boundary estimate

$$(4.7) \quad \int_0^T |U_0(t)|^2 dt \leq K_T \int_0^T |b(t)|^2 dt.$$

**Lemma 4.2.** *Let  $v \in \mathbb{C}$ . If  $\operatorname{Re} v < 0$  then*

$$\left| v + \sqrt{v^2 + 1} \right| \leq \operatorname{Re} v + \sqrt{(\operatorname{Re} v)^2 + 1}.$$

*Proof.* Assume that  $v = x + yi$  with  $x < 0$  and  $y \in \mathbb{R}$ .

- In the case  $y = 0$ , we have

$$\left| v + \sqrt{v^2 + 1} \right| = x + \sqrt{x^2 + 1}.$$

- In the case  $y \neq 0$ , firstly, we prove

$$(4.8) \quad a_1 x^2 + b_1^2 x + a_1 b_1^2 \geq 0,$$

where  $a_1$  and  $b_1$  are the same as in (3.8).

Now, (4.8) can be reformulated as

$$2x^2 \sqrt{p_1^2 + q_1^2} + 2x^2 y^2 \geq p_1 - \sqrt{p_1^2 + q_1^2},$$

where  $p_1$  and  $q_1$  are the same as in (3.8).

Therefore,

$$a_1 x^2 + b_1^2 x + a_1 b_1^2 \geq 0.$$

Secondly, we prove

$$\begin{aligned} \left| v + \sqrt{v^2 + 1} \right| &\leq x + \sqrt{x^2 + 1} \\ \Leftrightarrow (x + a_1)^2 + (y + b_1)^2 &\leq \left( x + \sqrt{x^2 + 1} \right)^2. \end{aligned}$$

By using (4.8) and

$$x^2 + b_1^2 + 1 = a_1^2 \quad \text{and} \quad y = \frac{a_1 b_1}{x},$$

we can also obtain

$$\begin{aligned} (x + a_1)^2 + (y + b_1)^2 &\leq \left(x + \sqrt{x^2 + 1}\right)^2 \\ \Leftrightarrow (x + a_1)^2(x^2 + b_1^2) &\geq 0. \end{aligned}$$

Therefore

$$\left|v + \sqrt{v^2 + 1}\right| \leq \operatorname{Re} v + \sqrt{(\operatorname{Re} v)^2 + 1}.$$

□

Similarly, by an application of the Parseval's identity, we have

$$\begin{aligned} \int_0^\infty \sum_{j \geq 0} e^{-2\alpha t} |U_j(t)|^2 dt &= \frac{1}{2\pi} \oint_0^\infty \sum_{j \geq 0} |\tilde{U}_j(\alpha + i\beta)|^2 d\beta \\ &= \frac{1}{2\pi} \oint_0^\infty \sum_{j \geq 0} |\tilde{b}(\xi)|^2 \frac{\|N_1(\varepsilon\xi)\|^2}{|\det(N(\varepsilon\xi))|^2} |\bar{P}(\varepsilon\xi)|^2 (|z_-(\varepsilon\xi)|^{2j} + |w_+(\varepsilon\xi)|^{2j}) d\beta, \end{aligned}$$

where  $z_-(\varepsilon\xi)$  and  $w_+(\varepsilon\xi)$  are the same as in (3.14) and (3.15).

Since  $k(\varepsilon\xi) = -g(\varepsilon\xi)$  is uniformly bounded in  $\operatorname{Re} \xi \geq 0$ ,  $\varepsilon > 0$  and using Lemma 4.1, we obtain

$$\int_0^\infty \sum_{j \geq 0} e^{-2\alpha t} |U_j(t)|^2 dt \leq O(1) \frac{1}{2\pi} \int_0^\infty \sum_{j \geq 0} |\tilde{b}(\xi)|^2 (|z_-(\varepsilon\xi)|^{2j} + |w_+(\varepsilon\xi)|^{2j}) d\beta.$$

On the other hand, since  $\mu_-(\xi) = -\mu_+(\xi)$  in  $\operatorname{Re} \xi \geq 0$ , we get

$$|z_-(\varepsilon\xi)| = |w_+(\varepsilon\xi)|.$$

Thus,

$$\int_0^\infty \sum_{j \geq 0} e^{-2\alpha t} |U_j(t)|^2 dt \leq O(1) \frac{1}{2\pi} \int_0^\infty \sum_{j \geq 0} |w_+(\varepsilon\xi)|^{2j} |\tilde{b}(\xi)|^2 d\beta.$$

By using (3.6) and (3.7), for all  $\varepsilon > 0$ ,  $\xi \in \mathbb{C}$ ,  $\operatorname{Re} \xi > 0$ , we have

$$\operatorname{Re} \mu_-(\varepsilon\xi) \leq -\frac{\varepsilon \operatorname{Re} \xi}{\sqrt{a}} < 0.$$

Furthermore, we can prove

$$(4.9) \quad \left( \frac{\operatorname{Re} \mu_-(\varepsilon\xi)\Delta x}{\varepsilon} + \sqrt{\left( \frac{\operatorname{Re} \mu_-(\varepsilon\xi)\Delta x}{\varepsilon} \right)^2 + 1} \right)^2 \leq \left( \eta\Delta x + \sqrt{\eta^2\Delta x^2 + 1} \right)^2,$$

where  $\eta = -\frac{\operatorname{Re} \xi}{\sqrt{a}}$ .

By using Lemma 4.2 and (4.9), we have

$$\begin{aligned} \sum_{j \geq 0} |w_+(\varepsilon\xi)|^{2j} &= \left( 1 - \left| \frac{\mu_-(\varepsilon\xi)\Delta x}{\varepsilon} + \sqrt{\left( \frac{\mu_-(\varepsilon\xi)\Delta x}{\varepsilon} \right)^2 + 1} \right|^2 \right)^{-1} \\ &\leq \left( 1 - \left( \eta\Delta x + \sqrt{\eta^2\Delta x^2 + 1} \right)^2 \right)^{-1}. \end{aligned}$$

If we assume that  $\Delta x \leq -\frac{3}{4\eta}$  then

$$\sum_{j \geq 0} |w_+(\varepsilon\xi)|^{2j} \leq -\eta^{-1}\Delta x^{-1}.$$

Therefore,

$$\alpha \Delta x \int_0^\infty \sum_{j \geq 0} e^{-2\alpha t} |U_j(t)|^2 dt \leq O(1) \frac{1}{2\pi} \int_0^\infty |\tilde{b}(\xi)|^2 d\xi.$$

By an application of the Parseval's identity

$$(4.10) \quad \alpha \Delta x \int_0^\infty \sum_{j \geq 0} e^{-2\alpha t} |U_j(t)|^2 dt \leq O(1) \int_0^\infty e^{-2\alpha t} |b(t)|^2 dt.$$

According to (4.6) and (4.10), there exists  $c > 0$  such that

$$(4.11) \quad \alpha \Delta x \int_0^\infty \sum_{j \geq 0} e^{-2\alpha t} |U_j(t)|^2 dt + \int_0^\infty e^{-2\alpha t} |U_0(t)|^2 dt \leq c \int_0^\infty e^{-2\alpha t} |b(t)|^2 dt.$$

This ends the proof of proposition 1.3.

From (4.7), (4.10) and the hyperbolicity of (1.1), for any  $T > 0$ , there exists a constant  $C_T > 0$  such that

$$(4.12) \quad \int_0^T \sum_{j \geq 0} \Delta x |U_j(t)|^2 dt + \int_0^T |U_0(t)|^2 dt \leq C_T \int_0^T |b(t)|^2 dt.$$

By linearity, we can break up the IBVP (1.10) into two simpler problems, one with homogeneous initial condition and the other with homogeneous boundary condition. From (2.7) and (4.12), we get the expected result of Theorem 1.1.

## APPENDIX A. MODELING AN ELASTIC STRING

**A.1. Derivation of the damped wave equation.** The damped wave equation in one space dimension can be derived in a variety of different physical settings. As an example of how waves occur in physical systems, we now derive the damped wave equation for a stretched string. Other physical systems, such as sound waves in air, can be analyzed in a similar way (see [10, 13]). We start by considering model the action of an elastic string over time. Consider a tiny element of the string between  $x$  and  $x + \Delta x$

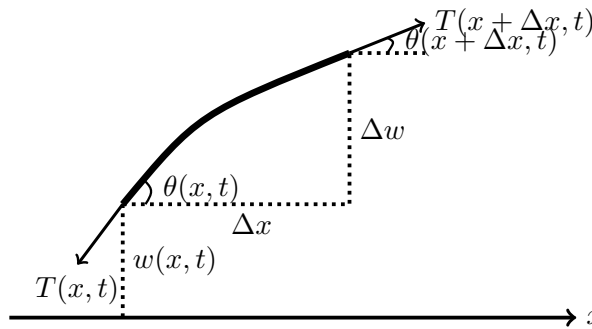


Figure A.1: Modeling an Elastic String.

The following quantities are needed in our derivation

- $w(x, t)$  denotes vertical displacement of the string from the  $x$ -axis at position  $x$  and time  $t$ .
- $\theta(x, t)$  is an angle between the string and a horizontal line at position  $x$  and time  $t$ .
- $T(x, t)$  is a tension in the string at position  $x$ .

We can dispose of all the  $\theta$ 's observing from the figure that

$$(A.1) \quad \tan \theta(x, t) = \lim_{\Delta x \rightarrow 0} \frac{\Delta w}{\Delta x} = \frac{\partial w}{\partial x} : \text{slope of tangent at } (x, t) \text{ in } wx - \text{plane}$$

The Newton's Second Law of Motion ( $F = ma$ ) states that

$$(A.2) \quad F = (\rho\Delta x) \frac{\partial^2 w}{\partial t^2}$$

where  $\rho$  is the linear density of the string and  $\Delta x$  is the length of the segment.

The force  $F$  comes from the tension in the string and also the damping force. The damping force acts in the opposite direction to the motion and is denoted by  $-c \frac{\partial w}{\partial t}$  with  $c > 0$ . We assume for our model that there are only transverse vibrations, and so the string does not move horizontally, but only vertically. So, we know that the total horizontal force must be zero. Balancing the forces in the horizontal direction gives

$$(A.3) \quad T(x + \Delta x, t) \cos \theta(x + \Delta x, t) = T(x, t) \cos \theta(x, t) = \tau = \text{const},$$

where  $\tau$  is the constant horizontal tension. Balancing the forces in the vertical direction yields

$$(A.4) \quad \begin{aligned} F &= T(x + \Delta x, t) \sin \theta(x + \Delta x, t) - T(x, t) \sin \theta(x, t) - c \frac{\partial w}{\partial t} \Delta x \\ &= T(x + \Delta x, t) \cos \theta(x + \Delta x, t) \tan \theta(x + \Delta x, t) - T(x, t) \cos \theta(x, t) \tan \theta(x, t) - c \frac{\partial w}{\partial t} \Delta x. \end{aligned}$$

Substituting (A.3) into (A.4) yields,

$$(A.5) \quad \begin{aligned} F &= \tau (\tan \theta(x + \Delta x, t) - \tan \theta(x, t)) - c \frac{\partial w}{\partial t} \Delta x \\ &= \tau \left( \frac{\partial w}{\partial x}(x + \Delta x, t) - \frac{\partial w}{\partial x}(x, t) \right) - c \frac{\partial w}{\partial t} \Delta x. \end{aligned}$$

So, the vertical component of Newton's Law becomes

$$\rho \frac{\partial^2 w}{\partial t^2}(\xi, t) = \tau \frac{\frac{\partial w}{\partial x}(x + \Delta x, t) - \frac{\partial w}{\partial x}(x, t)}{\Delta x} - c \frac{\partial w}{\partial t}$$

for  $\xi \in [x, x + \Delta x]$ . Dividing by  $\rho$  and letting  $\Delta x$  tends to 0 gives

$$(A.6) \quad \frac{\partial^2 w}{\partial t^2} = \frac{\tau}{\rho} \frac{\partial^2 w}{\partial x^2} - \frac{c}{\rho} \frac{\partial w}{\partial t}.$$

In order to guarantee that Eq.(A.6) has unique solution, we need initial and boundary condition on the displacement  $w(x, t)$ . There are now 2 initial conditions and boundary condition (see [10, 13]).

**A.2. Initial conditions.** The initial position of string and initial velocity can be written as follow

$$(A.7) \quad w(x, 0) = f(x) \quad \text{and} \quad w_t(x, 0) = h(x).$$

To see why we need 2 initial condition, note that the Taylor series of  $w(x, t)$  about  $t = 0$  is

$$w(x, t) = w(x, 0) + w_t(x, 0)t + w_{tt}(x, 0)\frac{t^2}{2} + w_{ttt}(x, 0)\frac{t^3}{3!} + \dots$$

From the initial condition (A.7) and the PDE (A.6) give

$$\begin{aligned} w_{tt}(x, 0) &= (\tau/\rho)w_{xx}(x, 0) - (c/\rho)w_t(x, 0) = (\tau/\rho)f''(x) - (c/\rho)h(x), \\ w_{ttt}(x, 0) &= (\tau/\rho)w_{txx}(x, 0) - (c/\rho)w_{tt}(x, 0) = (\tau/\rho)h''(x) - (c\tau/\rho^2)f''(x) + (c/\rho)^2h(x). \end{aligned}$$

Higher order terms can be found similarly. Therefore, the two initial conditions for  $w(x, 0)$  and  $w_t(x, 0)$  are sufficient to determine  $w(x, t)$  near  $t = 0$ .

**A.3. Boundary condition.** We assumed the string is connected to frictionless cylinders of mass  $m_1$  that move vertically on tracks at  $x = 0$  with an acceleration  $g(t)$ .

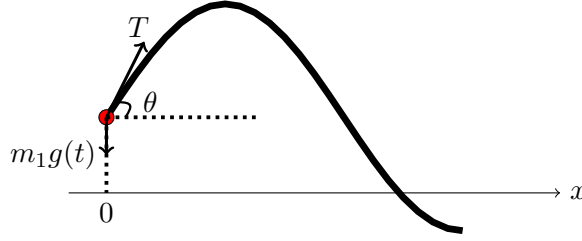


Figure A.2: Boundary condition at  $x = 0$ .

**Lemma A.1.** For any  $c > 0$ ,  $\rho > 0$  and  $m_1 > 0$ , the boundary condition can be written as follow

$$B_u w_x(0, t) - B_v w_t(0, t) = g(t).$$

with  $B_u B_v > 0$ .

*Proof.* Performing a force balance at  $x = 0$  gives

$$(A.8) \quad T \sin \theta - c \frac{\partial w}{\partial t} = m_1 g(t).$$

In other words, the vertical tension in the string balances the mass of the cylinder. However,  $\tau = T \cos \theta = \text{const}$  and  $\tan \theta = w_x$ , so that (A.8) becomes

$$T \cos \theta \tan \theta - c \frac{\partial w}{\partial t} = m_1 g(t),$$

or

$$(A.9) \quad B_u w_x(0, t) - B_v w_t(0, t) = g(t),$$

where  $B_u = \tau/m_1$  and  $B_v = c/m_1$ . □

To summarize, the IBVP of the linear damped wave equation in one space dimension is

$$(A.10) \quad \begin{aligned} \text{PDE} : & \frac{\partial^2 w}{\partial t^2}(x, t) = a \frac{\partial^2 w}{\partial x^2}(x, t) - \frac{1}{\varepsilon} \frac{\partial w}{\partial t}(x, t), \quad x > 0, t > 0, \\ \text{BC} : & B_u w_x(0, t) - B_v w_t(0, t) = g(t), \\ \text{IC} : & w(x, 0) = f(x), \quad w_t(x, 0) = h(x) \end{aligned}$$

where

$$(A.11) \quad a = \tau/\rho, \quad \varepsilon = \rho/c.$$

Let

$$u^\varepsilon(x, t) = w(x, t) \quad \text{and} \quad v^\varepsilon(x, t) = -w_t(x, t).$$

The IBVP (A.10) can be presented

$$(A.12) \quad \begin{aligned} \partial_t u^\varepsilon(x, t) + \partial_x v^\varepsilon(x, t) &= 0, \quad u^\varepsilon(x, t), v^\varepsilon(x, t) \in \mathbb{R}, \\ \partial_t v^\varepsilon(x, t) + a \partial_x u^\varepsilon(x, t) &= -\frac{1}{\varepsilon} v^\varepsilon(x, t), \end{aligned}$$

with the initial data

$$(A.13) \quad u^\varepsilon(x, 0) = f(x), \quad v^\varepsilon(x, 0) = -h(x),$$

and the linear boundary condition

$$(A.14) \quad B_u u^\varepsilon(0, t) + B_v v^\varepsilon(0, t) = g(t).$$

*Remark A.2.* The boundary condition  $B_u B_v > 0$  corresponds to stability condition (1.11) of the linear damped wave equation (A.12).



## APPENDIX B.

**Lemma B.1.** *There exists  $c > 0$  such that for any  $\delta > 0$ ,  $\xi \in \mathbb{C}$  with  $\operatorname{Re} \xi \in [0, 1]$  and  $\operatorname{Im} \xi \in [-1, 1]$*

$$\operatorname{Re} \sqrt{1 + (\mu_+(\xi)\delta)^2} \geq c,$$

where  $\mu_+(\xi) = \sqrt{\frac{\xi(1+\xi)}{a}}$  and  $a > 0$ .

*Proof.* Assume that  $\xi = \alpha + i\beta$  with  $\alpha \in [0, 1]$  and  $\beta \in [-1, 1]$ . Let  $\bar{\delta} = \delta^2/a$ . After some calculations, one obtains

$$\operatorname{Re} \sqrt{1 + (\mu_+(\xi)\delta)^2} = \frac{1}{\sqrt{2}} \sqrt{h(\alpha, \beta, \bar{\delta})},$$

where

$$h(\alpha, \beta, \bar{\delta}) = 1 + \bar{\delta} (\alpha(1+\alpha) - \beta^2) + \sqrt{(1 + \bar{\delta} (\alpha(1+\alpha) - \beta^2))^2 + \bar{\delta}^2 \beta^2 (1 + 2\alpha)^2}.$$

Now, we consider the behavior of  $h(\alpha, \beta, \bar{\delta})$  in the following cases

Case 1: For any  $\bar{\delta} > 0$ ,  $\alpha \in [0, 1]$  and  $\beta = 0$ , we have

$$h(\alpha, 0, \bar{\delta}) = 2(1 + \bar{\delta}\alpha(1+\alpha)) \geq 2.$$

Thus,

$$\operatorname{Re} \sqrt{1 + (\mu_+(\xi)\delta)^2} \geq 1.$$

Case 2: For any  $\bar{\delta} > 0$ ,  $\alpha \in [0, 1]$  and  $\beta \in [-1, 0) \cup (0, 1]$  such that  $\beta^2 \leq \alpha(1+\alpha)$ , we get

$$h(\alpha, \beta, \bar{\delta}) \geq 1 + \sqrt{1 + \bar{\delta}^2 \beta^2 (1 + 2\alpha)^2} \geq 2.$$

Thus,

$$\operatorname{Re} \sqrt{1 + (\mu_+(\xi)\delta)^2} \geq 1.$$

Case 3: For any  $\bar{\delta} > 0$ ,  $\alpha \in [0, 1]$  and  $\beta \in [-1, 0) \cup (0, 1]$  such that  $\beta^2 > \alpha(1+\alpha)$ , we denote

$$h(\alpha, \beta, \bar{\delta}) = 1 - \tau\bar{\delta} + \sqrt{(1 - \tau\bar{\delta})^2 + \bar{\delta}^2 \beta^2 (1 + 2\alpha)^2},$$

where

$$\tau = \beta^2 - \alpha(1+\alpha).$$

Note that

$$(B.1) \quad 0 < \tau \leq \beta^2 \leq 1.$$

• Firstly, we consider

$$\bar{\delta} \leq \frac{1}{\tau}.$$

Thus,

$$h(\alpha, \beta, \bar{\delta}) \geq \sqrt{(1 - \tau\bar{\delta})^2 + \bar{\delta}^2 \beta^2 (1 + 2\alpha)^2}$$

or

$$\begin{aligned} h^2(\alpha, \beta, \bar{\delta}) &\geq (1 - \tau\bar{\delta})^2 + \bar{\delta}^2 \beta^2 (1 + 2\alpha)^2 \\ &\geq (1 - \tau\bar{\delta})^2 + \bar{\delta}^2 \beta^2. \end{aligned}$$

From (B.1), we have

$$\begin{aligned} h^2(\alpha, \beta, \bar{\delta}) &\geq 1 - 2\tau\bar{\delta} + \tau^2\bar{\delta}^2 + \bar{\delta}^2\tau \\ &\geq 1 - 2\tau\bar{\delta} + \tau^2\bar{\delta}^2 \left(1 + \frac{1}{\tau}\right) \\ &\geq 1 - 2\tau\bar{\delta} + 2\tau^2\bar{\delta}^2. \end{aligned}$$

Since  $\tau\bar{\delta} \in (0, 1]$ , we have

$$1 - 2\tau\bar{\delta} + 2\tau^2\bar{\delta}^2 \geq \frac{1}{2}.$$

Thus,

$$h^2(\alpha, \beta, \bar{\delta}) \geq \frac{1}{2},$$

and therefore,

$$\operatorname{Re}\sqrt{1 + (\mu_+(\xi)\delta)^2} \geq 2^{-3/4}.$$

• Secondly, we consider

$$(B.2) \quad \bar{\delta} > \frac{1}{\tau}.$$

Then

$$\begin{aligned} h(\alpha, \beta, \bar{\delta}) &= -\sqrt{(1 - \tau\bar{\delta})^2} + \sqrt{(1 - \tau\bar{\delta})^2 + \bar{\delta}^2\beta^2(1 + 2\alpha)^2} \\ &\geq -\sqrt{(1 - \tau\bar{\delta})^2} + \sqrt{(1 - \tau\bar{\delta})^2 + \bar{\delta}^2\beta^2}. \end{aligned}$$

From (B.1) and (B.2), we get

$$\bar{\delta}\beta^2 > \frac{\beta^2}{\tau} \geq 1.$$

Thus,

$$\begin{aligned} h(\alpha, \beta, \bar{\delta}) &\geq -\sqrt{(1 - \tau\bar{\delta})^2} + \sqrt{(1 - \tau\bar{\delta})^2 + \bar{\delta}} \\ &\geq \frac{\bar{\delta}}{\sqrt{(1 - \tau\bar{\delta})^2} + \sqrt{(1 - \tau\bar{\delta})^2 + \bar{\delta}}} \\ &\geq \frac{\bar{\delta}}{2\sqrt{(1 - \tau\bar{\delta})^2 + \bar{\delta}}}. \end{aligned}$$

On the other hand, from (B.1) and (B.2), we have

$$\begin{aligned} (1 - \tau\bar{\delta})^2 + \bar{\delta} &\leq 1 + \tau^2\bar{\delta}^2 + \bar{\delta} \\ &\leq 1 + (\tau^2 + 1)\bar{\delta}^2 \quad (\text{since } \bar{\delta}^2 \geq \bar{\delta} \geq 1) \\ &\leq 1 + 2\bar{\delta}^2. \end{aligned}$$

Thus,

$$h(\alpha, \beta, \bar{\delta}) \geq \frac{\bar{\delta}}{2\sqrt{1 + 2\bar{\delta}^2}}.$$

Since  $\bar{\delta} > 1$ , we obtain

$$h(\alpha, \beta, \bar{\delta}) \geq \frac{1}{2\sqrt{3}}.$$

Therefore,

$$\operatorname{Re}\sqrt{1 + (\mu_+(\xi)\delta)^2} \geq \frac{1}{2\sqrt[4]{3}}.$$

□

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