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# Circuits, Oscillations, and the Kuramoto Model as Visualized by Phasors (CDT-7)

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## Abstract

Several modeling approaches can benefit from data and dynamics transformations that allow simplification, generalization, and/or more conceptual treatment of the studied problem. In this work, we discuss the *phasor* representation, capable of mapping sinusoidal signals into values in the complex plane, which can be associated to vectors. As such, phasors allow visualizing and understanding amplitude and phase relationships, such as relative phase and phase coherence. The potential of phasors for diverse applications is illustrated with respect to AC linear circuits, filters, oscillators and the Kuramoto model.

‘...this brave o’erhanging firmament, this majestic roof  
fretted with golden fire.’

William Shakespeare.

## 1 Introduction

To a good extent, scientific modeling as well as technological developments often benefit from approaches capable of simplifying, and/or generalizing and/or obtaining more effective and approachable representations of aspects of the studied phenomena. This is frequently the case with *transformations* such as the Fourier series, in which a wide range of signals can be expressed in standardize way, more specifically in terms of a sum of sines and cosines with specific amplitudes, frequencies, and phases. Other examples include the Laplace transform, the Taylor series, the use logarithm to map products into additions, and the use of moments to express statistical properties of signals.

In the present text, we focus on *phasor* representations of signals. Basically, this approach allows a sinusoidal signal of time – with amplitude  $A$ , frequency  $f$  (or the associated angular velocity  $\omega = 2\pi f$ ) and phase  $\theta$  – to be mapped into a respective complex number, the *phasor*  $Ae^{i\theta}$ , where  $i = \sqrt{-1}$  (e.g. [1]). Recall that this complex number can also be associated to a vector in the plane. Phasor representations disregard explicit representation of time as well as the signal angular velocity. These two quantities can nevertheless be taken into account by as-

suming that the mapped phasors are rotating around the complex plane origin with angular velocity  $\omega$ . The concept of phasor is illustrated in Figure 1.

The advantage of the phasor approach derives mostly from its ability to represent in an intuitive and simple way the phases of the signals, as well as their interrelationships (e.g. relative phase, phase coherence, etc.). In addition, in cases where all signals have fixed frequency, it is possible to perform several operations between the involved signals in an objective and simplified geometrical way (e.g. vector addition).

As a consequence of these interesting properties, phasors are adopted and used in a wide range of scientific and technological areas, including but by no means being limited to: physics (e.g. oscillations), dynamical system, electrical and electronic engineering, control theory, complex systems, vibrations, acoustics, audio, neurosciences, among many others.

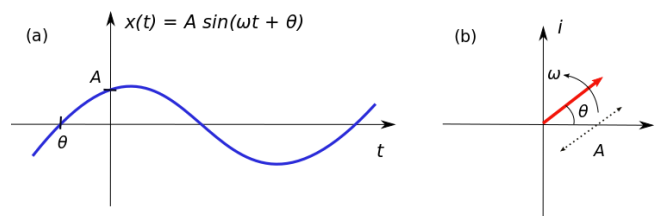


Figure 1: Plot of the signal  $x(t) = \omega t + \theta$  (a), and its respective *phasor* representation in the complex plane (b).

We shall start by revising some important properties of the sine and cosine functions, introducing the concept of

phasor representation, and then illustrating its potential with respect to some specific applications, namely AC linear circuits, linear filters, individual harmonic oscillators, systems of harmonic oscillators, and the Kuramoto model (e.g. [2, 3, 4]). The latter corresponds to a particularly interesting approach to modeling synchronization phenomena found in nature including fireflies flashing (e.g. [5]), heart and neuronal operation, clocks synchronization, and even aspects of economical and social dynamics.

## 2 The Sine and Cosine Functions

The sine and cosine trigonometric functions are particularly important in science, as they model several aspects of the real world, being also used in several technological applications. Though intrinsically simple, these functions exhibit some interesting, more subtle, features often arising from their periodical behavior. Some of these properties, namely those more closely related to phasors, are discussed in this section.

Both the sine and cosine functions take an angle (measured in radians or degrees) as argument, which we will call  $\Phi(t)$ , so that we have  $\sin(\Phi)$  and  $\cos(\Phi)$ , respectively. Their  $2\pi$ -periodicity can now be expressed as  $\sin(\Phi) = \sin(\Phi + 2\pi)$  and  $\cos(\Phi) = \cos(\Phi + 2\pi)$ . Though  $\Phi$  can take any real value, it is often interesting to consider that  $0 \leq \Phi < 2\pi$ .

One of the first interesting properties to be noticed about the sine and cosine functions is that they are closely interrelated, in the sense that any of them can be obtained by displacing the other along the angle argument. For instance, we have that

$$\begin{aligned} \sin(\Phi) &= \cos(\Phi - \pi/2), \\ \cos(\Phi) &= \sin(\Phi + \pi/2). \end{aligned} \quad (1)$$

It follows from this property that what we discuss and learn about one of these functions can be immediate and easily transferred for the other. In this work, we will focus on the sine function without loss of generality.

Several of the applications of the sine function involve expressing the angle argument as a function of time, i.e.  $\Phi(t)$ . Typically, this time-dependent argument is expressed in terms of a respective *frequency*  $f$  and a *phase*  $\theta$ . The *period* of this function is  $T = 1/f$ . Observe that the phase  $\theta$  corresponds to the value of the sine argument  $\Phi(t)$  at the initial time, i.e.  $\Phi(t = 0) = \theta$ . The *angular velocity* (or *frequency*)  $\omega = 2\pi f$  is also often considered. So, we can define a sinusoidal oscillating signal  $x(t)$  as the following function of  $t$ :

$$x(t) = A \sin(2\pi ft + \theta) = A \sin(\omega t + \theta). \quad (2)$$

One example of this type of signal  $x(t)$  is illustrated in Figure 1(a).

While the frequency  $f$  determines how fast the signal changes, the *phase*  $\theta$  can be understood as a consequence of displacements with respect to the reference sinusoidal signal  $\sin(2\pi f)$ . The frequency  $f$  (or the angular frequency  $\omega$ ) and the phase  $\theta$  can be understood as representing *parameters* of the function in Equation 2. Unlike *variables*, such as the independent-value  $t$  or the dependent-value  $x(t)$ , *parameters* remain constant during a given realization of the function. For instance, a linear pendulum will oscillate while keeping its parameters  $f$  and  $\theta$  constant.

We have that  $f$  is commonly measured in Hertz ( $Hz$ ),  $\omega$  in  $rad/s$  (radians per second), and  $\theta$  in  $rad$ . Note that degrees ( $^\circ$ ) can also be used instead of radians.

An interesting situation is defined when the phase is a function of time, i.e.

$$x(t) = A \sin(\omega t + \theta(t)). \quad (3)$$

In particular, if  $\theta(t)$  is an affine function of  $t$ , i.e.  $\theta(t) = \nu t + \theta_0$ , we have

$$x(t) = A \sin(\omega t + \nu t + \theta_0) = A \sin(\omega_e t + \theta_0), \quad (4)$$

with  $\omega_e = [\omega + \nu]$ . In other words, in this situation the phase  $\theta(t)$  behaves in an analogous way to an angular frequency. As an example, consider the situation in which we want to obtain  $x_2(t) = \sin(2\omega t)$  from a signal  $x_1 = \sin(\omega t + \theta(t))$ . This can be easily achieved by making  $\nu = \omega$ .

Recall that we have referred to the argument of the sine function as  $\Phi(t)$ . Now, the respective time derivative of this argument, given as

$$\frac{d\Phi(t)}{dt} = \dot{\Phi}(t) \quad (5)$$

has unit  $rad/s$  and can be effectively understood as the *instantaneous (angular) frequency* of the signal  $A \sin(\Phi(t))$ . For instance, if  $\theta(t) = \nu t + \theta_0$  we have, as expected, that  $\dot{\Phi}(t) = d(\omega t + \nu t + \theta_0)/dt = \omega + \nu = \omega_e$ .

## 3 Phasors

It is often interesting and useful to visualize signals such  $x(t)$  in Equation 18(a). We have already illustrated the Cartesian representation of the sine function in Figure 1(a). However, this type of representation can be highly redundant as a consequence of the  $2\pi$  periodicity of the sine. A more effective approach can be obtained by using the Euler equation to map the sine function in the complex plane (e.g. [6, 7, 8]), i.e.

$$x(t) = A \sin(\Phi(t)) = A \operatorname{Im} \left( e^{i\Phi(t)} \right), \quad (6)$$

which can be associated to the time-varying complex variable  $z(t) = A e^{i\Phi(t)}$ .

Figure 1(b) shows the phasor representation of the signal  $x(t) = A \sin(2\pi ft + \theta)$  in Figure 1(a) into the complex plane. Observe that the sine values are mapped into a circle of radius  $A$ , while its argument  $0 \leq \Phi(t) < 2\pi$  corresponds to the angle between each of the function points and the Real axis. It is also possible to understand the respective signal  $x(t)$  as being obtained by rotating the phasors counterclockwise with constant angular velocity  $\omega$  while taking the respective projections of the phasor onto the imaginary axis.

For the very same reason that the phasor representation allows the elimination of the redundancy derived from the sine periodicity, it also implies that we are no longer able to geometrically discriminate between values of  $\Phi(t)$  (i.e. angles) that are larger than  $2\pi$  or smaller than 0. In other words, we can only visualize the function  $x(t)$  as defined for  $0 \leq \Phi(t) < 2\pi$ . The discussed Cartesian and complex representations of  $x(t)$  can be understood as being *unwrapped* and *wrapped* around the period  $0 \leq \Phi(t) < 2\pi$ , respectively.

Now, the *smallest difference* between the two phases, which provides a measurement of the similarity between the two respective phases, can be immediately appreciated in terms of the smallest angle between the two phasors in the respective complex-plane representation. Note that, in the general case, this angle cannot be always obtained mathematically by making  $\delta = \theta_2 - \theta_1$ , as a consequence of the periodicity of the phasor representation. For instance, in case  $\theta_1 = 5^\circ$  and  $\theta_2 = 355^\circ$ , we would have  $\delta = 350^\circ$  instead of the expected smallest angle difference of just  $10^\circ$ . For such reasons, it becomes interesting to use other ways of obtaining measurements expressing angle differences while disregarding the possible discontinuities implied by the angle periodicity. An interesting possibility is to adopt  $\Delta\theta = |\sin(\theta_2 - \theta_1)|$ .

## 4 AC Linear Circuits

A powerful application of phasors regards the time-varying analysis of linear electric circuits, which is usually called *alternated current* (AC) analysis (e.g. [1, 9]), as a counterpoint to *direct current* (DC) analysis. The AC regime involves a *transient* period followed by an *steady-state* stage. Here we focus attention on the latter situation, characterized by the almost complete elimination of the effects of the initial charges and fields in the capacitors and inductors. The complete behavior of the circuit can be obtained by linear superimposing the transient and

Table 1: The reactances and impedances of the passive components  $R$ ,  $C$  and  $L$ . Observe that  $i = e^{i(\pi/2)}$  and  $-i = e^{i(-\pi/2)}$ .

| component | reactance                   | impedance                          |
|-----------|-----------------------------|------------------------------------|
| $R$       | $R$                         | $R$                                |
| $C$       | $X_C = -\frac{1}{\omega C}$ | $Z_C = iX_C = -\frac{i}{\omega C}$ |
| $L$       | $X_L = \omega L$            | $Z_L = iX_L = i\omega L$           |

steady-state solutions.

Interestingly, the linear circuit framework is not only very useful in electricity and electronics, but it can also be related and applied to other concepts and areas such as acoustics, vibrations, temperature, flow, hydrodynamics, magnetism, among others.

AC linear circuits involve combinations of some of the following possible components: voltage sources ( $v(t)$ , in Volts), current sources ( $i(t)$ , in ampères,  $A$ ), resistors ( $R$ , in ohms,  $\Omega$ ), capacitors ( $C$ , in farads,  $F$ ), and inductors ( $L$ , in henries,  $H$ ). Sources are understood as *active* components, while resistors, capacitors and inductors are said to be *passive*. Note that, in addition to at least one time-varying voltage or current source, AC analysis may involve also other time-varying or time-constant voltage and/or current sources. Circuits incorporating more than one source can be analyzed through *linear superimposition* of the solutions found considering each of them separately.

The time-varying signals are assumed to be in the harmonic form. For instance, we can have a voltage source  $v(t) = A \sin(\omega t + \theta)$ , where  $A \in \Re$  is the signal *amplitude*. This signal can be represented in the complex plane as

$$v(t) = \operatorname{Im} \left( A \cdot e^{i(\omega t + \theta)} \right) = A \operatorname{Im} \left( e^{i\omega t} \cdot e^{i\theta} \right). \quad (7)$$

So, the original voltage signal  $v(t)$  is transformed into the real number  $A \cdot e^{i\omega t} \cdot e^{i\theta}$ . Because a linear circuit with a source operating at  $\omega$  will have currents and voltages at all passive components oscillating with that same angular frequency, we can take into account only the amplitude and phase of these signals, transforming  $v(t)$  into the respective phasor  $V = A e^{i\theta}$ .

In a purely resistive resistor  $R$ , the voltage and current across it can be related by Ohm's law  $v(t) = R i(t)$ . This law can be immediately extended to capacitor and inductors by defining the respective *reactances* and *impedances*, as given in Table 1. As with resistance, the reactances and impedances are measure in Ohms.

Observe that reactances assume real values, while impedances are purely imaginary. Let's consider a component consisting of a resistor  $R$  in series with a capacitor  $C$  and an inductor  $L$ . The *total reactance* is defined as

$$X = X_C + X_L \quad (8)$$

and the *total impedance* is given as

$$Z = R + iX = R + i(X_C + X_L) = R + Z_C + Z_L. \quad (9)$$

This composed impedance device represents a general case of impedance in a linear AC circuit. Interestingly, impedances  $Z$  can be associated in a similar way to finding the equivalent resistance of a network of resistors.

The above definitions allows the Ohm's law to be rewritten as:

$$V = ZI, \quad (10)$$

in which  $I$  is oriented in opposition to  $V$  along the passive component  $Z$ .

Let's consider a single capacitor  $C$  connected to a voltage source  $V = Ae^{i\theta}$  with a generic phase  $\theta$  and arbitrary amplitude  $A$ . For this case, we have that  $I = V/Z = V/Z_C = V/(-i/\omega C) = iV\omega C = e^{i(\pi/2)}e^{i\theta}A\omega C = \omega ACe^{i(\pi/2+\theta)}$ , and consequently the voltage  $V$  lags (in the sense of being delayed in time) the current  $I$  by a relative phase  $\pi/2$ , irrespectively of  $\omega$ ,  $A$ , and  $C$ . The respective phasors are illustrated in Figure 2(a).

Now, if instead of a capacitor, we attach an inductor  $L$  to the source  $V$ , we have  $I = V/Z_L = V/(i\omega L) = -iV/(\omega L) = Ae^{i(-\pi/2)}e^{i\theta}/(\omega L) = A/(\omega L)e^{i(\theta-\pi/2)}$ , and now it is the current that lags the voltage by a relative phase  $\pi/2$ . The respective phasor diagram is given in Figure 2(b).

As a more general example, consider the circuit containing a resistor  $R = 1000\Omega$  and a capacitor  $C = 0.01\mu F$  in series with a voltage source  $v(t) = 2\sin(2\pi ft + \pi/4)$ , i.e.  $V = 2e^{i(\pi/4)}$  Volts, as shown in Figure 2. We adopt  $f = 1/(2\pi RC) \approx 15915.49 Hz$ , so that  $\omega = 10^5 rad/s$ ,

It follows that  $X_C = -1/(10^{-3})\Omega = -10^3\Omega$  and  $Z_C = -i10^3\Omega$ , so that  $Z = 1000 - i10^3\Omega = 1000 + 10^3e^{i(-\pi/2)}\Omega$ . Now, we can determine the current  $I$  by using the complex Ohm's law as:

$$I = \frac{V}{Z} = \frac{2e^{i(\pi/4)}}{1000 + 10^3e^{i(-\pi/2)}} = \sqrt{2}e^{i(\pi/2)} = i\sqrt{2} mA. \quad (11)$$

By considering  $R$  and  $C$  in terms of their separated impedances  $Z_R = R = 1000\Omega$  and  $Z_C = -i/(\omega C) = -i10^3\Omega$ , we can now obtain the voltages across these components as  $V_R = RI = \sqrt{2}e^{i(\pi/2)}$  volts and  $V_C = Z_C I = 10^3\sqrt{2}10^{-3} = \sqrt{2}$  volts. Observe that we could also have calculated  $V_C$  from  $V$  and  $V_R$  by using Kirchhoff's voltage law, i.e.  $V = V_C + V_R \Rightarrow V_C = V - V_R = 2e^{i(\pi/4)} - i\sqrt{2} = \sqrt{2}$  volts.

Note that Kirchhoff's voltage law can be suitably represented as the complex (or vector) sum in Figure 2(c). Observe also that the phasor  $I$  defined by the current  $i(t)$  is parallel to the phasor associated to  $V_R$ .

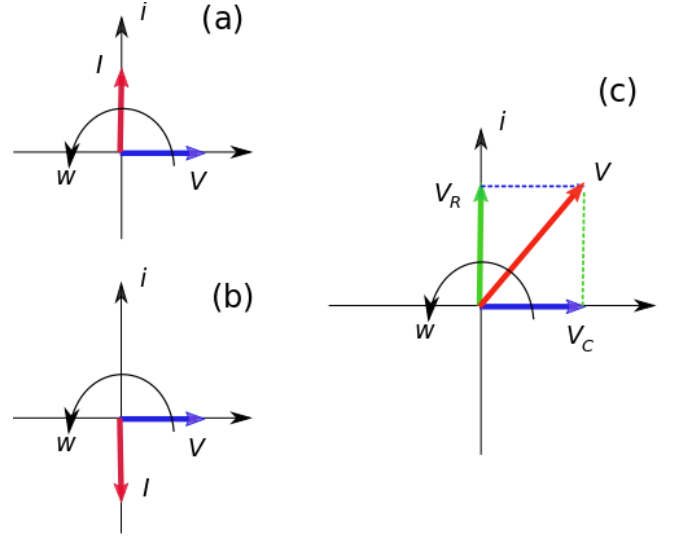


Figure 2: (a): In a *capacitor*, the phasor notation helps to visualize that the voltage  $V$  lags the current  $I$  by  $\pi/2$ . (b): In an *inductor*, the current  $I$  lags the voltage  $V$  by  $\pi/2$ . (c): The voltage phasors with respect to the example in the text, consisting of voltage source  $V$  in series with a resistance  $R$  (voltage  $V_C$ ) and a capacitor  $C$  (voltage  $V_L$ ). The current phasor  $I$ , not shown, is parallel to  $V_R$ .

## 5 Linear Filters

AC circuits such as that in the RC example can be understood as *filters* (e.g. [1, 9]), in the sense that they will attenuate the input signal (voltage or current) as a function of the signal frequencies  $\omega$ . Filters are very important in several applications ranging from electronics to neurosciences. In this section we illustrate how the phasor approach can be applied to better represent and understand filter operation. Generally speaking, linear filters involve the passive components resistors, capacitors and inductors. A filter is often studied by considering its operation, i.e. how it changes the magnitude and phase of the input signal  $V$  in terms of the respective frequency  $f$ , which can be expressed as:

$$\frac{V_{out}}{V_{in}} = \frac{A_{out}}{A_{in}} \frac{e^{i\theta_{out}}}{e^{i\theta_{in}}} = \alpha_V e^{i(\theta_{out}-\theta_{in})} = \alpha_V e^{i\phi_V}, \quad (12)$$

where  $\alpha_V$  is the *voltage gain* and  $\phi_V$  is the *relative phase* implied by the filter. Observe that both these gains are functions of the frequency  $f$ .

An example of a linear filter corresponds to the series configuration of  $V$ ,  $R$  and  $C$  addressed in section 4. By allowing  $V$  to vary in terms of  $f$ , instead of fixating it at  $f = 1/(2\pi RC)$ , we have that the impedance of the capacitor will also change with  $f$  as  $Z_C = -i/(\omega C)$ , so that  $I = V/Z = V/(R - i/(\omega C))$ , implying  $V_C = Z_C I = V/(1 + i\omega RC)$ , so that

$$\frac{V_C}{V} = \frac{1}{1 + i\omega RC}. \quad (13)$$

Figure 3 shows the phasors  $V$ ,  $V_R$  and  $V_C$  as the frequency  $f$  increases, in steps, from  $f_1 = 500Hz$  to  $f_9 = 50kHz$  along the sense indicated by the respective integer indices. Observe that, for any frequency, we can obtain the phasor associated to the voltage source as  $V = V_R + V_C$ , as illustrated by the dashed black arrows in this figure for  $f_9$ . It can be readily verified from the obtained phasor representation that, for small values of  $f$ ,  $V_R$  tends to align with  $V$ ; while as  $f$  increases,  $V_C$  will be substantially reduced while aligning with  $V$  as a consequence of the reduction of the capacitive reactance verified for higher frequencies.

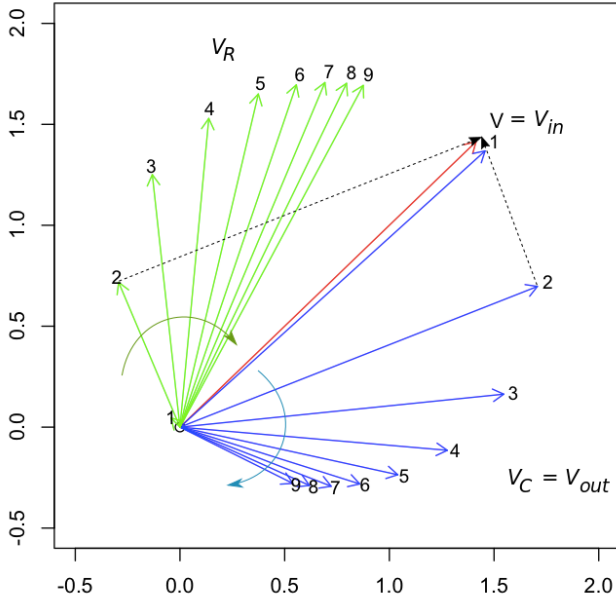


Figure 3: The behavior of the RC filter example from Section 4 ( $R = 1000\Omega$ ,  $C = 0.01\mu F$  and  $V = 2e^{i\pi/4}$  can be intuitively represented in phasor notation.

As the magnitude of  $V_C$  attenuates with  $f$ , it becomes interesting to define some critical point attenuation point. The standard reference value is the *cutoff frequency*,  $f_c$ , corresponding to where the signal power is halved. This implies an attenuation by  $\alpha_V = \sqrt{2}/2 \approx 0.707$  of the signal. It can be found that

$$f_c = \frac{1}{2\pi RC}. \quad (14)$$

For  $f = f_c$ , we also have that the signal corresponding to  $V$  undergoes an attenuation of  $20\text{Log}(\sqrt{2}/2) \approx -3\text{dB}$  and the output signal lags  $V$  by  $-\pi/4$  (as is the case with the previous example, see Figure 2). The *passing band* of this RC filter can be understood as the frequency interval  $0 \leq f \leq f_c$ , implying this circuit to operate as a *lowpass* filter.

## 6 Harmonic Oscillators

Let's now briefly consider an important physical phenomenon, namely harmonic oscillations. Consider the pendulum (e.g. [10]) in Figure 4(a). Here, a sphere (or bob) with mass  $m$  is attached to a pivot through a weightless rod of length  $L$  which can rotate freely around its pivot. For simplicity's sake, we will assume that all the mass  $m$  is concentrated at the center of mass of the sphere. Also, it is assumed that the pendulum is in the vacuum and that the pivot is frictionless, so that there is no energy dissipation. The angular position  $\beta(t)$  of the sphere is taken with respect to the vertical, and the initial angular position is taken as  $\beta(0) = \beta_0$ . The initial angular velocity is assumed as  $\dot{\beta}(0) = 0$ , i.e. the sphere is initially at rest.

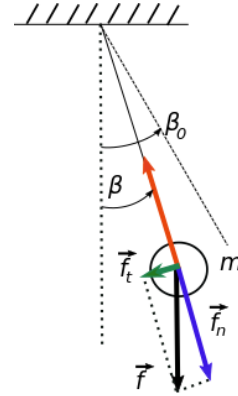


Figure 4: Modeling the angular movement of a pendulum. A sphere with mass  $m$  is attached to a pivot through a weightless rod. The angular position  $\beta(t)$  of the sphere is taken with respect to the horizontal axis.

The mass is under the force of gravity  $\vec{f} = m\vec{g}$ , which can be decomposed into two orthogonal components, one along the rod orientation,  $\vec{f}_n$ , and another tangential to the sphere displacement,  $\vec{f}_t$ . The action of  $\vec{f}_n$  is counteracted by a respective reaction  $\vec{f}_r$ , as there is no possibility of the rod to expand or contract, so that  $\vec{f}_n + \vec{f}_r = 0$  and then  $\vec{f}_r = -\vec{f}_n$ . On the other hand, the tangential component induces an angular displacement of the sphere. By trigonometric decomposition of  $\vec{f}$ , we derive the magnitudes  $f_t = f \sin(\beta(t))$  and  $f_n = f \cos(\beta(t))$ . Recall that the angular acceleration  $\ddot{\beta}(t)$  is related to the linear counterpart  $a(t)$  as  $a(t) = L\ddot{\beta}(t)$ . By Applying Newton's second law, we derive:

$$f_t = m g \sin(\beta(t)) = -m a(t) \approx -m L \ddot{\beta}(t) \Rightarrow \quad (15)$$

$$\ddot{\beta}(t) = -\frac{g}{L} \sin(\beta(t)).$$

Though this differential equation completely specifies the angular displacement  $\beta(t)$ , it is difficult to solve as

a consequence of its nonlinearity. For small angular displacements  $\beta(t) \ll 1$  rad, we can adopt the approximation  $\sin(\beta) \approx \beta$ , implying the simplified linear differential model

$$\ddot{\beta}(t) = -\frac{g}{L}\beta(t). \quad (16)$$

We observe that the name *harmonic motion* traditionally refers to the fact that the restoring force is directly proportional to the opposite displacement, as in the previous equation.

The solution of Equation 16 yields  $\beta(t) = \beta_0 \cos(\sqrt{g/L}t) = \beta_0 \sin(\sqrt{g/L}t + \pi/2)$ . Thus, we find that the considered pendulum undergoes a sinusoidal oscillation around the vertical axis with angular amplitude of the oscillation equal to  $\beta_0$ . By considering that  $\sqrt{g/L} = \omega(t)$ , we have  $f = 1/(2\pi)\sqrt{g/L}$ . By making  $\theta = \pi/2$ , the solution can now be expressed as

$$\beta(t) = \beta_0 \sin(\omega t + \theta) = \beta_0 \sin(\Phi(t)), \quad (17)$$

which is in the same form as Equation 2.

Therefore, we obtain that a linear pendulum produces a signal that can be described by a phasor. We also have that  $\Phi(t)$  corresponds to the *angular displacement* of the phasor associated to the oscillator, i.e.  $e^{i\Phi(t)}$ .

## 7 Independent Harmonic Oscillators

Let's now consider not one, but  $N$  oscillators that are completely disconnected, and therefore, independent one another. Each of these oscillators  $j$  will have respective angular frequency  $\omega_j$  and phase  $\theta_j$ . Because the phase is  $2\pi$ -periodical, it is convenient to visualize the phase in the complex plane. Considering the Euler equation, we have:

$$\sin(\omega_j t + \theta_j) = \text{Im}\left(e^{i(\omega_j t + \theta_j)}\right) = \text{Im}\left(e^{i(\omega_j t)} \cdot e^{i\theta_j}\right). \quad (18)$$

In case all the angular frequencies *are equal*, meaning that  $\omega_j = \omega$  for any oscillator  $j$ , the phase difference between any pair of oscillating signals  $x_i(t)$  is time-invariant, so we can conveniently visualize the oscillations only in terms of the respective quantities  $e^{i\theta_j}$  (i.e. a phasor) defined by the respective phases  $\theta_j$ .

Figure 5 shows, in the complex plane, the distribution of  $N = 100$  phases following a uniform distribution in the interval  $[-\pi, \pi)$ . In order not to clutter the visualization of the phasors, each of them is represented only in terms of small circles of different colors.

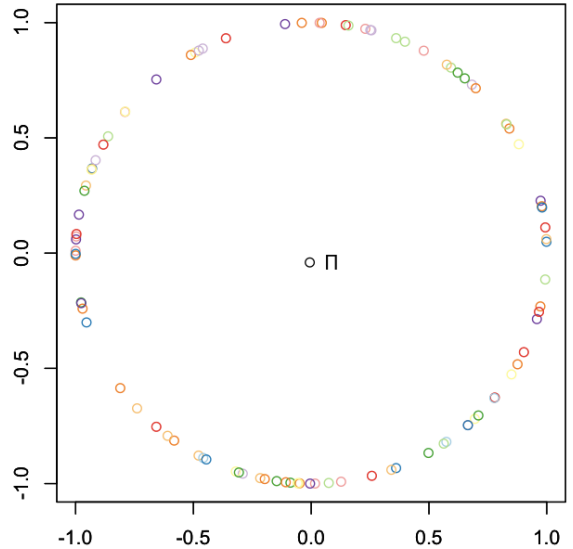


Figure 5: Phasor representation of the phases of  $N = 100$  oscillators of the type  $x_j(t) = \sin(\omega t + \theta_j)$ , where the phases  $\theta_i$  were drawn with uniformly random probability in the interval  $[-\pi, \pi)$ . The black dot near the center of the complex plane corresponds to the parameter  $\Pi$  in Equation 19. Note that  $\Pi \rightarrow 0$  as  $N \rightarrow \infty$ .

This type of phase distribution presents an intrinsic ‘order’ that can be evaluated in terms of the parameter:

$$\Pi = r e^{i\psi} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}, \quad (19)$$

where  $r$  gives the *phase coherence* and  $\psi$  is the average phase. Observe also that  $\Pi$  corresponds to the phasors center of mass. For uniformly random distributions such as that in Figure 5, it can be shown that  $r \rightarrow 0$  as  $N \rightarrow \infty$ .

## 8 The Kuramoto Model

The Kuramoto model was developed by Yoshiki Kuramoto along the 70's as a simplified version of Winfree's approach [2] to the synchronization of several coupled oscillators. Kuramoto was then able to solve this non-linear dynamical problem in analytical fashion.

In the Kuramoto model (e.g.[3, 4]), the dynamics performed by each oscillators is assumed to be governed by the following equation

$$\dot{\Phi}_i(t) = \omega_i + \sum_{i=1}^N K_{i,j} \sin(\Phi_j(t) - \Phi_i(t)) = \omega_i + \dot{\theta}_i(t). \quad (20)$$

Here,  $K$  is a coupling matrix and  $\Phi_i(t)$  is the angular displacement of oscillator  $i$ , with  $i = 1, \dots, N$ . Each oscillator has its own natural angular velocity  $\omega_i$ . Observe that when  $K = 0$ , i.e. the oscillators are uncoupled, we

have  $\dot{\Phi}_i = \omega_i$ , and each oscillator  $i$  will acquire an independent angular velocity  $\omega_i$ . Also important to keep in mind that the individual oscillators in the Kuramoto model are non-harmonic and non-linear.

For each oscillator, we can write

$$\dot{\Phi}_i(t) = \omega_i + \dot{\theta}_i(t) \Rightarrow \Phi_i(t) = \omega_i t + \theta(t) + \theta_0, \quad (21)$$

for  $i = 1, \dots, N$ , and where  $\theta_0$  corresponds to a reference phase. This formulation allows us to associate each oscillator to a respective phasor  $e^{i(\omega_i t + \theta_i(t))}$ , while disregarding the reference phase  $\theta_0$ . In this way, we can summarize the dynamics performed by each oscillator to the function  $\sin(\Phi_i(t))$ .

Let's take into account the situation arising when each oscillator has a slightly different frequency  $\omega_i$ . More specifically, it is assumed that these values are drawn from a normal distribution with average  $\mu$  and relatively small standard deviation  $\sigma$ . For simplicity's sake, but without loss of generality, we make  $\mu = 0$ . Let's also assume that all coupling weights are given as  $K_{i,j} = K/N$ , and that  $K$  is relatively small, e.g.  $K = 1$ . As all oscillators in the Kuramoto model have the same unitary amplitude, all phasors will lie along the circle of radius 1 centered at the origin of the complex plane.

Figure 6 illustrates the unfolding of the angular displacements  $\Phi_i(t)$  (calculated numerically) represented in terms of the respective phasors. Here,  $K = 1$ ,  $\mu = 0$  and  $\sigma = 0.1$  rad.

Recall that the angular velocities  $\omega_i$  of each oscillator are slightly different one another, remaining fixed along time. So, how can these oscillators synchronize in frequency even so? This takes place because the phase variations  $\dot{\theta}_i(t)$  implied by the governing Equation 20 effectively act so as to achieve similar effective angular velocities  $\omega_i + \dot{\theta}_i(t)$ .

An important aspect of the Kuramoto dynamics regards its specific *stability*. Depending on  $r$  and  $K$ , the synchronization can be *stable* or *unstable*, defining respective *bifurcation* diagrams (e.g. [3, 4]). The Kuramoto model has motivated several extensions and variations, including the consideration of complex interconnections (e.g. [11]) on the respective dynamics (e.g. [4]).

## 9 Concluding Remarks

Despite their relative simplicity, phasors allows us to visualize and identify relationships between phases in several types of problems, including circuits, oscillators and complex systems. In this text, we presented the phasor notation at an introductory level and illustrated its potential for applications with respect to an RC circuit, the

harmonic oscillations of a linear pendulum, a system of independent oscillators, and the Kuramoto model. Several other applications exist, and others will continue to be devised.

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### Costa's Didactic Texts – CDTs

This is a *Costa's Didactic Text* (CDT). CDTs intend to be a halfway point between a formal scientific article and a dissemination text in the sense that they: (i) explain and illustrate concepts in a more informal, graphical and accessible way than the typical scientific article; and, at the same time, (ii) provide more in-depth mathematical developments than a more traditional dissemination work.

It is hoped that CDTs can also provide integration and new insights and analogies concerning the reported concepts and methods. We hope these characteristics will contribute to making CDTs interesting both to beginners as well as to more senior researchers.

Though CDTs are intended primarily for those who have some preliminary experience in the covered concepts, they can also be useful as summary of main topics and concepts to be learnt by other readers interested in the respective CDT theme.

Each CDT focuses on a few interrelated concepts. Though attempting to be relatively self-contained, CDTs also aim at being relatively short. Links to related material are provided in order to complement the covered subjects.

The complete set of CDTs can be found at: <https://www.researchgate.net/project/Costas-Didactic-Texts-CDTs>.



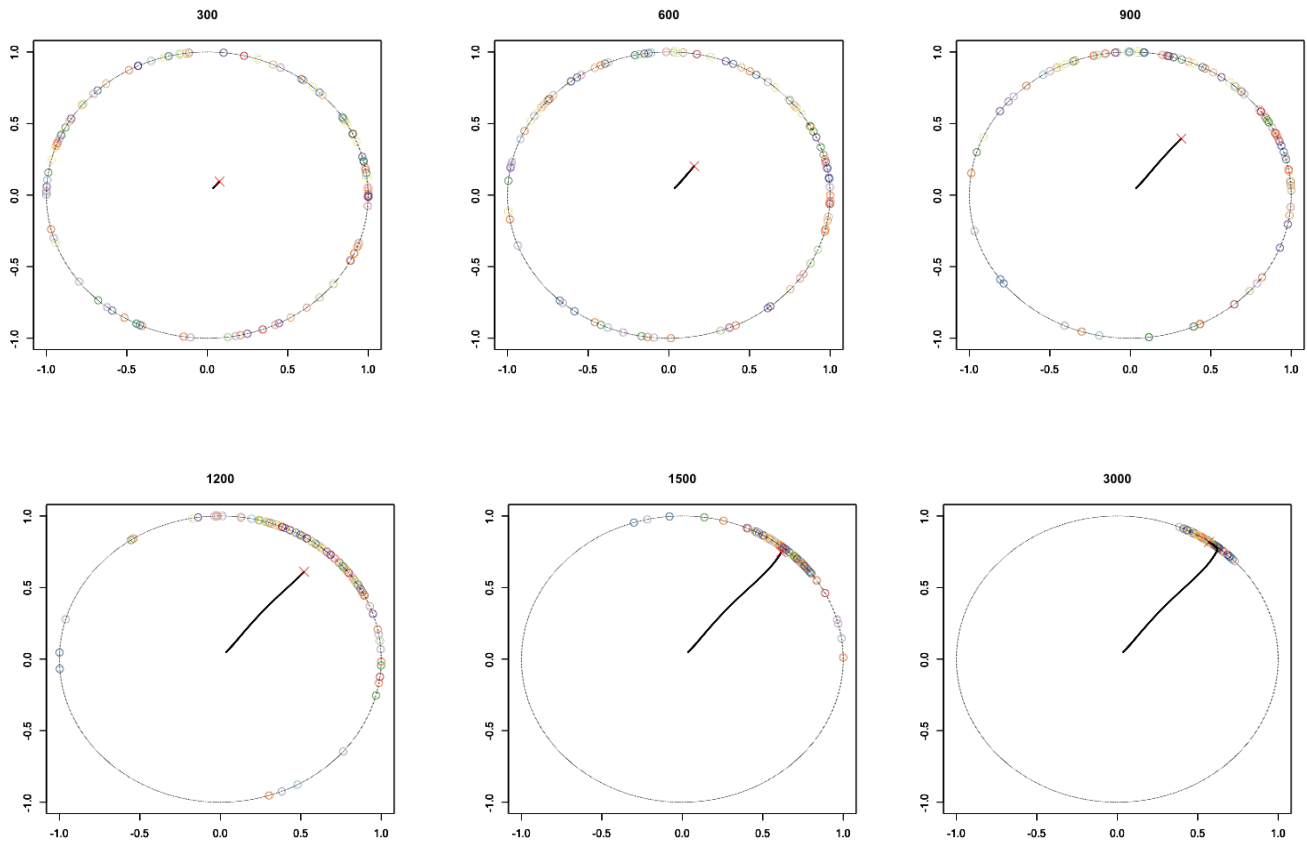


Figure 6: Time unfolding of phasors obtained from the phases of  $N$  oscillators in the Kuramoto model assuming  $K = 1$ ,  $\mu = 0$  and  $\sigma = 0.1$  rad. The angular velocities were drawn from a normal distribution with null mean and unit variance, and are remain fixed along time. The plot titles indicate the respective time interaction, while the red  $X$  identifies the phasors center of mass, leaving the trajectory in black.

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