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# Space of 2D elastic materials: a geometric journey

B. Desmorat · N. Auffray

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**Abstract** In this paper, we describe geometrically the domain of elastic materials in terms of invariants of the integrity basis (including the positive definiteness condition), prove a theoretical link between those polynomial invariants and the Kelvin invariants of the elasticity tensor, to finally introduce the concept of design transformation which leads to subsets of elastic materials having identical Kelvin invariants. As an example of this approach, the set of 2D pentamode materials is fully characterized.

**Keywords** 2D elasticity · Invariants · Integrity basis · Eigenvalues

## 1 Introduction

In linear elasticity, it is customary to describe an elastic material by its tensor specified with respect to some basis. Such a way to describe elastic materials is not satisfactory since rotating the matter produces, with respect to the same basis, another elasticity tensor describing the same elastic material. Hence an elastic material is not defined by an unique tensor but rather by the collection of all elasticity tensors related by orthogonal transformations. This idea can be *condensed* by describing elastic materials by a collection of quantities which are invariant with respect to orthogonal transformations. These quantities are often simply referred to as *the invariants of the elasticity tensor* [9, 42, 5, 32]. A basis of invariants that are polynomial in terms of tensor components is called

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an *Integrity Basis* ( $\mathcal{IB}$ ). Elements of  $\mathcal{IB}$ , also referred to as Boehler invariants [9] in the mechanical community, satisfy polynomial relations defining the set of elastic materials as a domain within a higher dimensional space. Any point of this domain is a uniquely defined elastic material, and its symmetry class is encoded in the topology of the domain. In the case of planar elasticity the domain of elastic materials lives in  $\mathbb{R}^5$  (the situation is much more complicated for 3D elasticity and will not be discussed here).

On the other hand, the Kelvin representation, originally due to Kelvin [38] and appearing again in the literature of the 80's [35, 28], allows for the definition of 3 polynomial invariants (the elementary symmetric functions), called Kelvin invariants, which are uniquely related to the 3 eigenvalues of such tensorial representation. Such invariants are used in mechanics with different purposes: dissymmetric elastic behaviour in tension and in compression [16], yield and strength criteria [1, 20], continuum damage mechanics [6, 22, 18, 19, 23, 27], characterization of elastic materials [35, 10, 11], material design [14]... Moreover, the link between the eigenvalues of the Kelvin decomposition of 2D elasticity tensor and the polar formalism [41, 40] was described in [17].

In this paper, devoted to the 2D case, we will describe geometrically the elastic material domain in terms of invariants of the integrity basis (including the positive definiteness condition), prove a theoretical link between the two non-equivalent sets of polynomial Kelvin and Boehler invariants, to finally introduce the concept of design transformation which leads to elastic materials subdomains with identical Kelvin invariants. Using design transformation, an explicit example is constructed showing that symmetry classes identification cannot be achieved by studying the eigenvalue multiplicity of an elasticity tensor. Finally, as a concluding example, the set of 2D pentamode materials (bimode materials) initially introduced by [29] and well-investigated in the field of meta-material studies is fully characterized.

## Organization of the paper

The [section 2](#) is devoted to a brief description of the space of 2D linear elastic materials. In [section 3](#), the decomposition of second-order symmetric tensor space into deviatoric and spheric subspaces is introduced. In [section 4](#), the Clebsh-Gordan harmonic decomposition of elasticity tensors is detailed and the integrity basis defined accordingly. In [section 5](#) the positive definiteness condition expressed in terms of the invariants of the integrity basis is provided, and a geometrical description of the elastic material domain is proposed. In [section 6](#), an explicit link between Boehler and Kelvin invariants, that makes use of the spheric direction introduced in [section 3](#) is shown. And, finally, we introduce in [section 7](#) the concept of design transformation which transforms elastic materials while preserving the eigenvalues of the elasticity tensor. An explicit characterization of the geometric domains obtained by all possible design transformations for an initially isotropic material is studied, with an application to the specific case of pentamode materials.

## Notations

Throughout this paper, the physical space is modeled on the Euclidean space  $\mathcal{E}^2$  with  $\mathbb{E}^2$  its associated vector space. Once an arbitrary reference point chosen, those spaces can be associated and  $\mathcal{P} = \{\underline{e}_1, \underline{e}_2\}$  will denote an orthonormal basis of  $\mathbb{E}^2$ . For forthcoming needs, let also defined  $\mathcal{K} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$  the orthonormal canonical basis of  $\mathbb{R}^3$ ,  $\mathcal{K}$  will be referred to as the Kelvin basis.

### Tensor spaces:

- $\mathbb{T}$  denotes a tensor space;
- $S^2(\mathbb{R}^n)$  is the space of symmetric second-order tensors in  $n$ -D;
- $\mathbb{K}^n$  is the space of  $n$ -th order completely symmetric and traceless tensors on  $\mathbb{R}^2$ , called *harmonic tensors*.

Tensors of order 0, 1, 2 and 4 are denoted respectively by  $\alpha$ ,  $\underline{v}$ ,  $\underline{a}$ ,  $\underline{\underline{A}}$ . The simple, double and fourth-order contractions are written  $\cdot$ ,  $:$ ,  $::$  respectively. In components, with respect to  $\mathcal{P}$ , these notations correspond to

$$\begin{aligned} \underline{u} \cdot \underline{v} &= u_i v_i, & \underline{a} : \underline{b} &= a_{ij} b_{ij}, \\ (\underline{\underline{A}} : \underline{\underline{B}})_{ijkl} &= A_{ijpq} B_{pqkl}, & (\underline{\underline{A}} :: \underline{\underline{B}}) &= A_{pqrs} B_{pqrs}. \end{aligned}$$

where the Einstein summation on repeated indices is used. When needed, index symmetries of both spaces and their elements are expressed as follows:  $(\dots)$  indicates invariance under permutations of the indices in parentheses,  $\dots$  indicates symmetry with respect to permutations of the underlined blocks.

### Tensor products:

- $\otimes$  stands for the classical product, and  $\otimes^n$  indicates its  $n$ -th power;
- $S^2$  denotes its the completely symmetrized product, and  $S^n$  its extension to product of  $n$  elements;
- $\overline{\otimes}$  indicates the twisted tensor product defined by

$$(\underline{a} \overline{\otimes} \underline{b})_{ijkl} = \frac{1}{2}(a_{ik} b_{jl} + a_{il} b_{jk}).$$

### Special tensors:

- $\underline{\underline{1}}$  the second-order identity tensor;
- $\underline{\underline{\underline{1}}} = \underline{\underline{1}} \overline{\otimes} \underline{\underline{1}}$  the fourth-order identity tensor of  $S^2(\mathbb{R}^n)$ ;
- $\underline{\underline{\underline{K}}} = \frac{1}{2} \underline{\underline{1}} \otimes \underline{\underline{1}}$  the spheric projector;
- $\underline{\underline{\underline{J}}} = \underline{\underline{\underline{1}}} - \underline{\underline{\underline{K}}}$  the deviatoric projector.

*Matrix spaces:*

- $\mathcal{M}_n$  is the space of  $n \times n$  dimensional square matrices;
- $\mathcal{M}_n^S$  is the space of  $n \times n$  dimensional symmetric square matrices;
- $\mathcal{M}_{p,q}$  is the space of  $p \times q$  rectangular matrices.

*Groups:*

The following matrix groups are considered in the paper:

- $\text{GL}(d)$ : the group of all linear invertible transformations of  $\mathbb{R}^d$ , i.e.  $F \in \text{GL}(d)$  iff  $\det(F) \neq 0$ ;
- $\text{O}(d)$ : the orthogonal group, that is the group of all isometries of  $\mathbb{R}^d$  i.e.  $Q \in \text{O}(d)$  iff  $Q \in \text{GL}(d)$  and  $Q^{-1} = Q^T$ , where the superscript  $T$  denotes the transposition;
- $\text{SO}(d)$ : the special orthogonal group, i.e. the subgroup of  $\text{O}(d)$  consisting of transformations satisfying  $\det(Q) = 1$ .

Let detail the case  $d = 2$ . As a matrix group,  $\text{O}(2)$  can be generated by:

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad 0 \leq \theta < 2\pi, \quad \text{and} \quad P(\underline{e}_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

in which  $R(\theta)$  is a rotation by an angle  $\theta$  and  $P(\underline{n})$  is the reflection across the line normal to  $\underline{n}$ .  $\text{SO}(2)$  corresponds to the group of plane rotations generated by  $R(\theta)$ . The following finite subgroups of  $\text{O}(2)$  will be used:

- $\text{Id}$ , the identity group;
- $Z_k$ , the cyclic group with  $k$  elements generated by  $R(2\pi/k)$ ;
- $D_k$ , the dihedral group with  $2k$  elements generated by  $R(2\pi/k)$  and  $P(\underline{e}_2)$ .

*Miscellaneous notations:*

- $\mathcal{IB}$  means *Integrity Basis*;
- $\simeq$  denotes hereafter an isomorphism;
- $\mathcal{L}(E, F)$  indicates the space of linear applications from  $E$  to  $F$ ;
- $\mathcal{L}(E)$  indicates the space of linear applications from  $E$  to  $E$ ;
- $\mathcal{L}^s(E)$  indicates the space of self-adjoint linear applications on  $E$ .

## 2 Space of linear elastic materials

### 2.1 The space elasticity tensors

In the field of linear elasticity, the constitutive law is a local linear relation between the second-order symmetric Cauchy stress tensor  $\underline{\sigma}$  and the second-order symmetric infinitesimal strain tensor  $\underline{\varepsilon}$ :

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}. \quad (1)$$

$\underline{\sigma}$  and  $\underline{\varepsilon}$  belong to  $S^2(\mathbb{R}^2)$ , the space of bi-dimensional symmetric second-order tensors. As a consequence elasticity tensors possess *minor* index symmetries

$$C_{ijkl} = C_{jikl} = C_{ijlk},$$

which are condensed in the notation:  $C_{(ij)(kl)}$ . Due to the potential energy associated to the elastic behaviour another index symmetry has to be taken into account:

$$C_{ijkl} = C_{klij}.$$

This so-called *major* symmetry is encoded in the notation  $C_{ij \ \underline{kl}}$ . Hence, combined with the minor ones, we obtain the elastic index symmetries:  $C_{\underline{(ij)} \ \underline{(kl)}}$ . The vector space of 2D elasticity tensors is defined as<sup>1</sup>:

$$\mathbb{E}la := \{ \underline{C} \in \otimes^4 \mathbb{R}^2 \mid C_{\underline{(ij)} \ \underline{(kl)}} \}, \quad \dim \mathbb{E}la = 6,$$

and can also be viewed as

$$\mathbb{E}la = \mathcal{L}^s(S^2(\mathbb{R}^2)).$$

For being admissible, an elasticity tensor, considered as a quadratic form on  $S^2(\mathbb{R}^2)$ , should further be positive definite, meaning that its eigenvalues  $\lambda_i$  should verify

$$\exists M \in \mathbb{R}^{*+}, \quad 0 < \lambda_i \leq M.$$

## 2.2 From active physical rotations to elastic materials

Consider  $O(2)$  the set of 2D isometric transformations. Its action on an element  $\underline{T}$  of  $\mathbb{E}la$  gives a new element  $\overline{\underline{T}}$  of  $\mathbb{E}la$ ,

$$\overline{\underline{T}} = \underline{Q} \star \underline{T},$$

in which the star product  $\star$  stands for the standard tensorial action. In components, with respect to  $\mathcal{P}$ , this action can be detailed:

$$\overline{T}_{ijkl} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} T_{pqrs}.$$

The nature of a material does not change when it is subjected to a rotation or a flip (mirror isometry though a line). On the contrary the elasticity tensor will change with respect to a given reference frame, for instance  $\mathcal{P}$ , so that multiple tensors can be associated to one elastic material.

<sup>1</sup> Even if obvious, it is worth mentioning that both the stiffness tensor and its inverse, the compliance tensor, belong to the same vector space. Hence in the following no physical interpretation (stiffness or compliance) will be given to elements of  $\mathbb{E}la$ .

From a physical point of view, the necessary and sufficient conditions for two elasticity tensors  $\mathbb{T}_1, \mathbb{T}_2 \in \mathbb{Ela}$  to represent the same elastic material, denoted  $\mathbb{T}_1 \underset{\approx}{\sim} \mathbb{T}_2$ , writes

$$\mathbb{T}_1 \underset{\approx}{\sim} \mathbb{T}_2 \quad \Leftrightarrow \quad \exists \mathbb{Q} \in O(2) \mid \mathbb{T}_1 \underset{\approx}{=} \mathbb{Q} \star \mathbb{T}_2 \underset{\approx}{=}.$$

The collection of all elasticity tensors that describe the same elastic material is a geometric object called the orbit of  $\mathbb{T}_1$  and defined by

$$\mathcal{O}(\mathbb{T}_1) = \{\mathbb{T} \in \mathbb{Ela}, \exists \mathbb{Q} \in O(2) \mid \mathbb{T} \underset{\approx}{=} \mathbb{Q} \star \mathbb{T}_1 \underset{\approx}{=}\}.$$

For some transformation, the resulting tensor is identical to the original one. The set of orthogonal transformations letting  $\mathbb{T}_1$  invariant constitutes its symmetry group

$$\mathbb{G}_{\mathbb{T}_1} := \{\mathbb{Q} \in O(2), \mid \mathbb{T}_1 \underset{\approx}{=} \mathbb{Q} \star \mathbb{T}_1 \underset{\approx}{=}\}.$$

Let consider the following equivalence relation among elements of  $\mathbb{Ela}$ ,

$$\mathbb{T}_1 \underset{\approx}{\approx} \mathbb{T}_2 \quad \Leftrightarrow \quad \exists \mathbb{Q} \in O(2) \mid \mathbb{G}_{\mathbb{T}_1} \underset{\approx}{=} \mathbb{Q} \mathbb{G}_{\mathbb{T}_2} \mathbb{Q}^{-1}.$$

This relation indicates that two tensors are equivalent if their symmetry groups are conjugate. The equivalence classes for this relation are called strata. More specifically, in what follows  $\Sigma_{[G]}$  will denote the equivalence class of elasticity tensors having their symmetry group conjugate to  $G$ . In other words,  $[G]$  is the symmetry class of the elements of the stratum  $\Sigma_{[G]}$  [4, 3]. The space of 2D elasticity tensors is divided into 4 strata [25, 42, 3]:

$$\mathbb{Ela} = \Sigma_{[Z_2]} \cup \Sigma_{[D_2]} \cup \Sigma_{[D_4]} \cup \Sigma_{[O(2)]}.$$

In mechanical terms,  $\Sigma_{[Z_2]}$  corresponds to the set of biclinic materials,  $\Sigma_{[D_2]}$  to the set of orthotropic materials,  $\Sigma_{[D_4]}$  to the set of tetragonal materials and  $\Sigma_{[O(2)]}$  to isotropic materials [45].

### 3 Decomposition of second-order symmetric tensors space

Since elasticity tensors are linear symmetric applications on  $S^2(\mathbb{R}^2)$ , a first step to describe  $\mathbb{Ela}$  is to understand the structure of  $S^2(\mathbb{R}^2)$ . This is the aim of the present section.

### 3.1 The harmonic basis

With respect to  $\mathcal{P} = (\underline{e}_1, \underline{e}_2)$ ,  $\underline{t} \in S^2(\mathbb{R}^2)$  can be represented by

$$\underline{t} = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix}_{\mathcal{P}}.$$

It is possible to turn this second-order symmetric tensor into a genuine vector of  $\mathbb{R}^3$  by defining the following linear application  $\phi : S^2(\mathbb{R}^2) \rightarrow \mathbb{R}^3$ :

$$\begin{aligned} \hat{\underline{e}}_1 &= \phi(\underline{e}_1 \otimes \underline{e}_1), \\ \hat{\underline{e}}_2 &= \phi(\underline{e}_2 \otimes \underline{e}_2), \\ \hat{\underline{e}}_3 &= \phi\left(\frac{\sqrt{2}}{2}(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1)\right). \end{aligned} \quad (2)$$

The canonical basis  $\mathcal{K} = \{\hat{\underline{e}}_i\}$  of  $\mathbb{R}^3$  will be referred to as the Kelvin basis. With respect to  $\mathcal{K}$ , we can define the vector  $\hat{\underline{t}}$  image of  $\underline{t}$  by  $\phi$

$$\hat{\underline{t}} = \phi(\underline{t}) = \begin{pmatrix} t_{11} \\ t_{22} \\ \sqrt{2}t_{12} \end{pmatrix}_{\mathcal{K}}.$$

These two representations are isometric since  $\underline{t} : \underline{t} = \hat{\underline{t}} \cdot \hat{\underline{t}}$ . A third representation is possible. This representation is associated to the decomposition of  $S^2(\mathbb{R}^2)$  into a deviatoric space ( $\mathbb{K}^2$ ) and a spheric one ( $\mathbb{K}^0$ ):

$$S^2(\mathbb{R}^2) \simeq \mathbb{K}^2 \oplus \mathbb{K}^0. \quad (3)$$

Hence any  $\underline{t} \in S^2(\mathbb{R}^2)$  can be decomposed accordingly. This decomposition, which is unique, is given by the well-known formula:

$$\underline{t} = \underline{d} + \alpha \underline{1}, \quad \text{with} \quad \alpha = \frac{1}{2} \underline{t} : \underline{1}, \quad \underline{d} = \underline{t} - \alpha \underline{1}. \quad (4)$$

with  $\underline{d} \in \mathbb{K}^2$  and  $\alpha \in \mathbb{K}^0$ . Associated to this decomposition, we define a new basis  $\mathcal{H} = \{\hat{\underline{f}}_1, \hat{\underline{f}}_2, \hat{\underline{f}}_3\}$ , with

$$\hat{\underline{f}}_1 = \frac{\sqrt{2}}{2}(\hat{\underline{e}}_1 - \hat{\underline{e}}_2), \quad \hat{\underline{f}}_2 = \hat{\underline{e}}_3, \quad \hat{\underline{f}}_3 = \frac{\sqrt{2}}{2}(\hat{\underline{e}}_1 + \hat{\underline{e}}_2).$$

This new basis  $\mathcal{H}$  will be referred to as the *harmonic basis*. The passage matrix  $P$  from  $\mathcal{K}$  to  $\mathcal{H}$  is given by

$$P_{ij} = \hat{\underline{e}}_i \cdot \hat{\underline{f}}_j, \quad P = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}_{\mathcal{K}}.$$



and can be shown to be an element of  $O(3)\backslash SO(3)$ . With respect to  $\mathcal{H}$ ,  $\hat{\underline{t}}$  is expressed as

$$\hat{\underline{t}} = \sqrt{2} \begin{pmatrix} \frac{t_{11}-t_{22}}{2} \\ t_{12} \\ \frac{t_{11}+t_{22}}{2} \end{pmatrix}_{\mathcal{H}}.$$

$(\frac{t_{11}-t_{22}}{2}, t_{12})^T$  and  $\frac{t_{11}+t_{22}}{2}$  corresponds respectively to the deviatoric and spheric parts of  $\hat{\underline{t}}$ .

The relations between these different bases are summed up in the following diagram:

$$\begin{array}{ccc} \text{Physical Space: } \mathbb{R}^2, & \mathcal{P} = \{\underline{e}_i\} & \xrightarrow{\otimes} \mathcal{P}^{\otimes n} = \{\underline{e}_{i_1} \otimes \dots \otimes \underline{e}_{i_n}\} \\ & & \downarrow \phi \\ \text{Strain-Stress Space (Kelvin basis): } \mathbb{R}^3, & \mathcal{K} = \{\hat{\underline{e}}_i\} & \xrightarrow{\otimes} \mathcal{K}^{\otimes m} = \{\hat{\underline{e}}_{i_1} \otimes \dots \otimes \hat{\underline{e}}_{i_m}\} \\ & \downarrow \text{P} & \downarrow \text{P} \\ \text{Strain-Stress Space (Harmonic basis): } \mathbb{R}^3, & \mathcal{H} = \{\hat{\underline{f}}_i\} & \xrightarrow{\otimes} \mathcal{H}^{\otimes m} = \{\hat{\underline{f}}_{i_1} \otimes \dots \otimes \hat{\underline{f}}_{i_m}\} \end{array}$$

### 3.2 Representation of physical rotations

The interest of the different bases introduced is revealed when studying how tensors transform with respect to active rotations, that is to elements of the rotation group  $SO(2)$ . An element of  $SO(2)$  is parametrized by

$$\underset{\sim}{\mathbb{R}} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}_{\mathcal{P}}.$$

Its action on  $\underline{t}$  of  $S^2(\mathbb{R}^2)$  gives a new element  $\bar{\underline{t}}$  of  $S^2(\mathbb{R}^2)$ , the components of which are related to those of  $\underline{t}$  in the following manner

$$\bar{t}_{ij} = R_{ip}R_{jq}t_{pq}.$$

By defining  $\hat{\underset{\sim}{\mathbb{R}}} = \phi(\underset{\sim}{\mathbb{R}} \otimes \underset{\sim}{\mathbb{R}})$ , the former action expressed in  $\mathbb{R}^2$ , can be reformulated directly in  $\mathbb{R}^3$ :

$$\hat{\underline{t}} = \hat{\underset{\sim}{\mathbb{R}}} \hat{\underline{t}}.$$

In the same way, a rotated elasticity tensor expressed as a second-order tensor in  $\mathbb{R}^3$  is

$$\hat{\underset{\sim}{\mathbb{T}}} = \hat{\underset{\sim}{\mathbb{R}}} \hat{\underset{\sim}{\mathbb{T}}} \hat{\underset{\sim}{\mathbb{R}}}^T.$$

$\hat{\mathbb{R}}_{\sim}$  has the following matrix expression in  $\mathcal{K}$ :

$$\hat{\mathbb{R}}_{\sim} = \frac{1}{2} \begin{pmatrix} 1 + \cos 2\theta & 1 - \cos 2\theta & -\sqrt{2} \sin 2\theta \\ 1 - \cos 2\theta & 1 + \cos 2\theta & \sqrt{2} \sin 2\theta \\ \sqrt{2} \sin 2\theta & -\sqrt{2} \sin 2\theta & 2 \cos 2\theta \end{pmatrix}_{\mathcal{K}}.$$

It can be checked that  $\hat{\mathbb{R}}_{\sim} \in \text{SO}(3)$  and corresponds to a rotation of  $2\theta$  around the axis  $\hat{\mathbf{f}}_3$ . Expressed in  $\mathcal{H}$ , the matrix of  $\hat{\mathbb{R}}_{\sim}$  reads

$$\hat{\mathbb{R}}_{\sim} = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) & 0 \\ \sin(2\theta) & \cos(2\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\mathcal{H}}. \quad (5)$$

Physical rotation matrices are well-structured in the harmonic basis. The geometric content is clear, when an element of  $S^2(\mathbb{R}^2)$  is rotated by an angle  $\theta$ , its spherical part is invariant, while its deviatoric part is turned by an angle  $2\theta$ .

#### 4 Invariant description of $\mathbb{E}la$

To construct a geometric description of the space of 2D elastic materials, the use of invariant functions is required. For our needs, those functions will mostly be considered as polynomial. To construct the polynomial invariants of a tensor, the first step is to decompose this tensor into elementary parts. This decomposition is the object of [subsection 4.1](#). The construction of polynomial invariants and their geometric interpretation will be the object of [subsection 4.2](#) and [subsection 4.3](#).

##### 4.1 Decompositions of fourth-order elasticity tensors

Let's go back to  $\mathbb{E}la$ , this space is shown to have the following isotypic structure [\[3\]](#)

$$\mathbb{E}la \simeq \mathbb{K}^4 \oplus \mathbb{K}^2 \oplus 2\mathbb{K}^0,$$

in which  $\mathbb{K}^4$  is the space of 2D fourth-order harmonic tensors, that is of complete symmetric and traceless fourth-order tensors. Since more than one copy of  $\mathbb{K}^0$  are involved there are multiple explicit harmonic decompositions [\[24, 3\]](#). This leads to multiple couples of isotropic parameters. A specific choice is the Clebsch-Gordan interpretation of the harmonic decomposition [\[2\]](#)

$$\mathbb{T}_{\approx} = \mathbb{D}_{\approx} + \frac{1}{2}(\mathbb{1}_{\sim} \otimes \mathbb{d}_{\sim} + \mathbb{d}_{\sim} \otimes \mathbb{1}_{\sim}) + \kappa(\mathbb{1}_{\sim} \otimes \mathbb{1}_{\sim}) + \gamma \mathbb{J}_{\approx}, \quad (6)$$

where  $\mathbb{D}_{\approx} \in \mathbb{K}^4$ ,  $\mathbb{d}_{\sim} \in \mathbb{K}^2$  and  $\{\kappa, \gamma\} \in \mathbb{K}^0$ .

Using the Kelvin representation, a fourth-order elasticity tensor is represented as the following matrix in basis  $\mathcal{K}$

$$\hat{\mathbb{T}}_{\sim} = \begin{pmatrix} T_{1111} & T_{1122} & \sqrt{2}T_{1112} \\ T_{1122} & T_{2222} & \sqrt{2}T_{1222} \\ \sqrt{2}T_{1112} & \sqrt{2}T_{1222} & 2T_{1212} \end{pmatrix}_{\mathcal{K}}. \quad (7)$$

$\mathbb{d} \in \mathbb{K}^2$  and  $\mathbb{D} \in \mathbb{K}^4$  can be parametrized in the Kelvin basis by

$$\hat{\mathbb{d}}_{\sim} = \begin{pmatrix} d_1 \\ -d_1 \\ \sqrt{2}d_2 \end{pmatrix}_{\mathcal{K}}, \quad \hat{\mathbb{D}}_{\sim} = \begin{pmatrix} D_1 & -D_1 & \sqrt{2}D_2 \\ -D_1 & D_1 & -\sqrt{2}D_2 \\ \sqrt{2}D_2 & -\sqrt{2}D_2 & -2D_1 \end{pmatrix}_{\mathcal{K}}. \quad (8)$$

A structure is made apparent by writing the harmonic decomposition of  $\mathbb{E}la$  in the basis<sup>2</sup>  $\mathcal{H}$ :

$$\hat{\mathbb{T}}_{\sim} = \begin{pmatrix} 2D_1 + \gamma & 2D_2 & d_1 \\ 2D_2 & -2D_1 + \gamma & d_2 \\ d_1 & d_2 & 2\kappa \end{pmatrix}_{\mathcal{H}}. \quad (9)$$

The previous matrix is well-structured which means,

$$\begin{aligned} \sigma^d &= \mathbb{T}^{dd} : \varepsilon^d + \mathbb{T}^{ds} : \varepsilon^s, \\ \sigma^s &= \mathbb{T}^{sd} : \varepsilon^d + \mathbb{T}^{ss} : \varepsilon^s, \end{aligned} \quad (10)$$

with

$$\mathbb{T}^{dd}_{\sim} = \mathbb{D}_{\sim} + \gamma \mathbb{J}_{\sim}, \quad \mathbb{T}^{ds}_{\sim} = \frac{1}{2} \mathbb{d}_{\sim} \otimes \mathbb{1}_{\sim}, \quad \mathbb{T}^{ss}_{\sim} = \kappa (\mathbb{1}_{\sim} \otimes \mathbb{1}_{\sim}).$$

This particular block form for writing the elasticity tensor will be referred to as the *Clebsch-Gordan* representation. This form consists in writing the elasticity tensor as a symmetric linear application on  $\mathbb{K}^2 \oplus \mathbb{K}^0$ :

$$\mathbb{E}la \simeq \mathcal{L}^s(\mathbb{K}^2 \oplus \mathbb{K}^0) = \mathcal{L}^s(\mathbb{K}^2) \oplus \mathcal{L}^s(\mathbb{K}^0) \oplus \mathcal{L}(\mathbb{K}^0, \mathbb{K}^2).$$

This decomposition is induced on the linear operator and the elasticity tensor appears to be structured by blocks:

$$\mathbb{T}_{\sim} = \begin{bmatrix} \mathbb{T}^{dd} & \mathbb{T}^{ds} \\ \mathbb{T}^{sd} & \mathbb{T}^{ss} \end{bmatrix}_{\sim}. \quad (11)$$

<sup>2</sup> The matrix normal forms of an elasticity tensor  $\mathbb{T}$  for each symmetry classes are provided in the basis  $\mathcal{H}$  in Appendix A.

## 4.2 Integrity basis for elasticity tensors

Integrity basis is, for a given space  $\mathbb{V}$  and for a given group action  $G$ , a set of fundamental polynomial invariants such that any  $G$ -invariant polynomial on  $\mathbb{V}$  is a polynomial in the elements of the integrity basis [43]. Integrity basis will be denoted  $\mathcal{IB}(\mathbb{V}, G)$ .

In the present situation, that is for  $\mathbb{V} = \mathbb{E}la$  and  $G = O(2)$ , the integrity basis:

1. is finitely generated, i.e.  $\#\mathcal{IB}(\mathbb{E}la, O(2)) < +\infty$ ;
2. separates the orbits

$$\mathcal{IB}(\mathbb{E}la, O(2))(\underset{\sim}{\mathbb{T}}_1) = \mathcal{IB}(\mathbb{E}la, O(2))(\underset{\sim}{\mathbb{T}}_2) \Leftrightarrow \underset{\sim}{\mathbb{T}}_1 \underset{\sim}{\sim} \underset{\sim}{\mathbb{T}}_2.$$

In other terms, the invariants of the integrity basis define an application from the space of elasticity tensors to  $\mathbb{E}la/O(2)$ , the space of elastic materials [5]. From now on, the notation will be shortened simply to  $\mathcal{IB}$ . Integrity bases for  $O(2)$ -action on the space of plane elasticity tensors are known since the second-half of the 90' [7, 42] and are constructed from the harmonic decomposition of  $\mathbb{E}la$  [9, 31, 5]. Let consider the following quantities:

$$I_1 = \kappa, \quad J_1 = \gamma, \quad I_2 = \underset{\sim}{d} : \underset{\sim}{d}, \quad J_2 = \underset{\sim}{D} :: \underset{\sim}{D}, \quad I_3 = \underset{\sim}{d} : \underset{\sim}{D} : \underset{\sim}{d}. \quad (12)$$

Those elements are  $O(2)$ -invariant and, in the chosen notation, the subscript indicates the degree of the polynomial invariants in the elasticity tensor.

We have the following result [42]:

**Theorem 4.1.** *A minimal integrity basis for  $O(2)$ -action on  $\mathbb{E}la$  is*

$$\mathcal{IB} = (I_1, J_1, I_2, J_2, I_3).$$

Those elements are free meaning that they are not related by any polynomial relation. They satisfy, however, the following inequality

$$I_2^2 J_2 - 2I_3^2 \geq 0. \quad (13)$$

This Cauchy-Schwarz type inequality comes from the fact we are dealing with real valued tensors [2]. This inequality is fundamental and its geometric meaning explicited in the next subsection.

We define the following application from  $\mathbb{E}la$  to  $\mathbb{E}la/O(2)$  which associates to a tensor its (uniquely defined) elastic material:

$$\mathcal{IB}(\underset{\sim}{\mathbb{T}}) := \left( I_1(\underset{\sim}{\mathbb{T}}), J_1(\underset{\sim}{\mathbb{T}}), I_2(\underset{\sim}{\mathbb{T}}), J_2(\underset{\sim}{\mathbb{T}}), I_3(\underset{\sim}{\mathbb{T}}) \right).$$

This set of polynomial invariants will be referred to as Boehler invariants.

*Remark 4.2.* The invariants  $I_2$  and  $J_2$  are, by definition, such that

$$I_2 \geq 0, \quad J_2 \geq 0. \quad (14)$$

*Remark 4.3.* Using the parametrization (8), the invariants  $I_2$ ,  $J_2$ ,  $I_3$  of the integrity basis read

$$I_2 = 2(d_1^2 + d_2^2), \quad J_2 = 8(D_1^2 + D_2^2), \quad I_3 = 4D_1(d_1^2 - d_2^2) + 8D_2d_1d_2. \quad (15)$$

*Remark 4.4.* In the polar formalism [41, 40], the elasticity tensor is parametrized by the polar components  $(t_0, t_1, r_0, r_1, \varphi_0, \varphi_1)$ . Their link with the integrity basis is given by

$$I_1 = 2t_1, \quad J_1 = 2t_0, \quad I_2 = 32r_1^2, \quad J_2 = 8r_0^2, \quad I_3 = 64r_0r_1^2 \cos 4(\varphi_1 - \varphi_0).$$

### 4.3 A geometric representation of elastic materials (vector space $\mathbb{E}la$ )

As introduced in section 2,  $\mathbb{E}la$  is divided into 4 strata:

$$\mathbb{E}la = \Sigma_{[Z_2]} \cup \Sigma_{[D_2]} \cup \Sigma_{[D_4]} \cup \Sigma_{[O(2)]}.$$

For the least symmetric class, that is for biclinic elastic materials, the polynomial invariants of  $\mathcal{IB}$  are algebraically independent. A biclinic material is described by five independent quantities, that is by a point in  $\mathbb{R}^5$ . The location of this point is not any, since constrained by the relations (13) and (14).

In Figure 1 are summed-up the different transitions from a symmetry class to another expressed in terms of polynomial relations between invariants. While  $\Sigma_{[D_2]} = \Sigma_{[D_2]}^{ord} \cup \Sigma_{[D_2]}^{spec}$ , the set  $\Sigma_{[D_2]}^{ord}$  and  $\Sigma_{[D_2]}^{spec}$  have been considered in this figure. Elements of the set  $\Sigma_{[D_2]}^{ord}$  are *ordinary* orthotropic elasticity tensors. It only contains elements obtained just by imposing orthotropic invariance to generic anisotropic tensors. Elements of the set  $\Sigma_{[D_2]}^{spec}$  are *special* orthotropic tensors, they are not ordinary since extra restriction, i.e. other than invariance properties, are needed to define them (those tensors correspond to  $R_0$ -orthotropic tensors [39]).

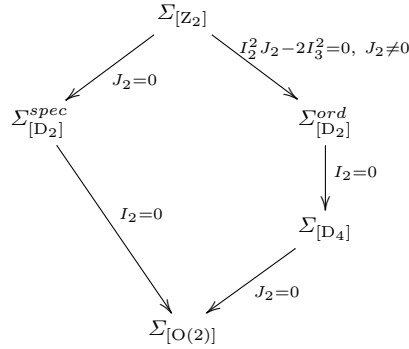


Fig. 1: Breaking symmetry conditions of  $\mathbb{E}la$ .

In Figure 2 is represented the space of elastic materials (vector space  $\mathbb{E}la$ ) with respect to  $(I_2, J_2, I_3)$ , *without taking the positive definiteness condition into account*. The surface, which corresponds to the polynomial equation  $I_2^2 J_2 - 2I_3^2 = 0$  contains all the *at-least-orthotropic* materials (stratum  $\overline{\Sigma_{[D_2]}} = \Sigma_{[D_2]} \cup \Sigma_{[D_4]} \cup \Sigma_{[O(2)]}$ ). The condition  $I_2^2 J_2 - 2I_3^2 > 0$  indicates on which side of the orthotropic surface are the biclinic materials located (strata  $\Sigma_{[Z_2]}$ ).

Finally, we get that, independently of the values of the isotropic invariants  $I_1$  and  $J_1$ :

- biclinic materials (stratum  $\Sigma_{[Z_2]}$ ) are strictly inside the volume defined by the surface;
- point  $O$  corresponds to isotropic materials (stratum  $\Sigma_{[O(2)]}$ );
- open ray  $]OA)$  corresponds to tetragonal materials (stratum  $\Sigma_{[D_4]}$ );
- open ray  $]OB)$  corresponds to  $R_0$ -orthotropic materials (stratum  $\Sigma_{[D_2]}^{spec}$ );
- surface without  $\{O\} \cup ]OA) \cup ]OB)$  corresponds to ordinary orthotropic materials (stratum  $\Sigma_{[D_2]}^{ord}$ ).

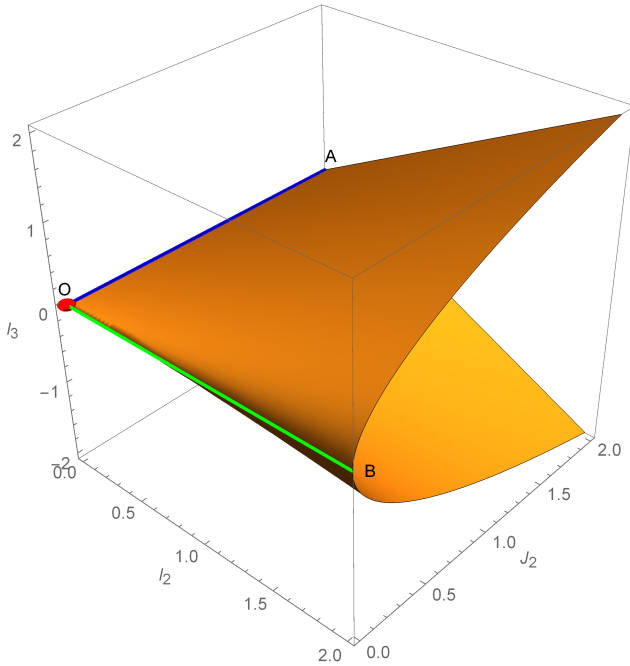


Fig. 2: Algebraic variety of elastic materials with respect to  $(I_2, J_2, I_3)$  (without taking the positive definiteness condition into account).

## 5 Invariant based positive definiteness condition for elastic materials

As previously said in section 2.1, an element  $\mathbb{T}$  of  $\mathbb{E}la$  is admissible if, considered as a linear application on  $S^2(\mathbb{R}^2)$ , its eigenvalues  $\lambda_i$  verifies

$$\exists M \in \mathbb{R}^{*+}, \quad 0 < \lambda_i \leq M.$$

so that only the restriction of  $\mathbb{E}la$  to the cone<sup>3</sup> of symmetric definite elasticity tensors can be associated to elastic materials. Our aim is now to express this condition using the invariants of the integrity basis  $\mathcal{IB}$ .

First, it has to be observed that the eigenvalues of  $\mathbb{T}$  are algebraic (i.e. the roots of a polynomial equation) invariants of  $O(3)$ . We have the following lemma [33]:

**Lemma 5.1.** *A symmetric matrix  $M$  in  $\mathcal{M}_n(\mathbb{R})$  is positive definite if and only if  $\sigma_k(M) > 0$ , with  $\{\sigma_k\}$  the set of elementary symmetric polynomials.*

It can be checked by a direct calculation that

$$\begin{aligned} \sigma_1 &= 2(I_1 + J_1), \\ \sigma_2 &= 4I_1J_1 + J_1^2 - \frac{1}{2}(I_2 + J_2), \\ \sigma_3 &= \frac{1}{2}(I_3 - I_2J_1 + 4I_1J_1^2 - 2I_1J_2). \end{aligned} \quad (16)$$

so that the positive definiteness condition can be expressed in terms of the elements of the integrity basis:

$$\mathbb{T} \text{ is positive definite} \Leftrightarrow \begin{cases} I_1 + J_1 > 0, \\ 4I_1J_1 + J_1^2 - \frac{1}{2}(I_2 + J_2) > 0, \\ I_3 - I_2J_1 + 4I_1J_1^2 - 2I_1J_2 > 0. \end{cases} \quad (17)$$

To obtain simpler expressions, the Clebsch-Gordan form of the elasticity tensor can be exploited. To that aim consider the following lemma [33]:

**Lemma 5.2.** *Consider  $M \in \mathcal{M}_{p+q}^S$  a real symmetric matrix having the following block shape*

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

with  $A \in \mathcal{M}_p^S$ ,  $C \in \mathcal{M}_q^S$ ,  $B \in \mathcal{M}_{p,q}$ . Let  $M/A$  be the Schur complement of  $A$  in  $M$ :

$$M/A = C - B^T A^{-1} B$$

and  $M/C$  be the Schur complement of  $C$  in  $M$ :

$$M/C = A - BC^{-1}B^T$$

Then the following conditions are equivalent:

<sup>3</sup> A subset  $\mathcal{C}$  of a vector space  $\mathbb{V}$  is a cone if for each  $p$  in  $\mathcal{C}$  and positive scalars  $\lambda$ ,  $\lambda p \in \mathcal{C}$ . The cone is said convex provided  $\lambda p + \mu q \in \mathcal{C}$ , for any positive scalars  $\lambda, \mu$  and any  $p, q \in \mathcal{C}$ .

1.  $M$  is positive definite if and only if  $A$  and  $M/A$  are positive definite;
2.  $M$  is positive definite if and only if  $C$  and  $C/A$  are positive definite.

This approach applied respectively to  $\mathbb{T}^{dd}$  and to  $\mathbb{T}^{ss}$  (defined in (10)) provides the following equivalent sets of definite positiveness conditions expressed in terms of the elements of the integrity basis:

$$\begin{aligned} \mathbb{T} \text{ is positive definite} &\Leftrightarrow \begin{cases} J_1 > 0, \\ 2J_1^2 - J_2 > 0, \\ I_3 - 2J_2I_1 - I_2J_1 + 4J_1^2I_1 > 0, \end{cases} \\ &\Leftrightarrow \begin{cases} I_1 > 0, \\ 8I_1J_1 - I_2 > 0, \\ I_3 - 2J_2I_1 - I_2J_1 + 4J_1^2I_1 > 0. \end{cases} \end{aligned}$$

From a physical standpoint, this way of expressing positive definiteness conditions is more satisfactory since we obtain explicit physical conditions:

- the isotropic invariants  $I_1$  and  $J_1$  are strictly positive quantities;
- restriction on the norm of the spheric/deviatoric coupling :  $I_2 < 8I_1J_1$ ;
- restriction on the norm of the anisotropic part of the deviatoric elasticity:  $J_2 < 2J_1^2$ .

The last condition is less straightforward to interpret.

Further, a look at the two sets of conditions suggests to introduce the following reduced anisotropic invariants:

$$i_2 = \frac{I_2}{8I_1J_1}, \quad j_2 = \frac{J_2}{2J_1^2}, \quad i_3 = \frac{I_3}{8I_1J_1^2}. \quad (18)$$

The positive definiteness conditions read

$$\begin{aligned} \mathbb{T} \text{ is positive definite} &\Leftrightarrow \begin{cases} 0 \leq j_2 < 1, \\ 2i_3 - j_2 - 2i_2 + 1 > 0, \end{cases} \\ &\Leftrightarrow \begin{cases} 0 \leq i_2 < 1, \\ 2i_3 - j_2 - 2i_2 + 1 > 0. \end{cases} \end{aligned}$$

The normalization of the inequality (13) shows that  $-1 < i_3 < +1$ . Together with condition (13), the complete set<sup>4</sup> of conditions for positive definiteness of a real valued fourth-order elasticity tensor read

$$\begin{cases} 0 \leq i_2 < 1, \\ 0 \leq j_2 < 1, \\ -1 \leq i_3 \leq +1, \\ \frac{1}{2}(-1 + 2i_2 + j_2) < i_3, \\ i_3^2 \leq i_2^2 j_2. \end{cases} \quad (19)$$

<sup>4</sup> The number of inequations in the presented set is not minimal.



The geometric domain associated to (19) is presented in figure 3 (note that the quantities on the three axes are different from figure 2). The geometric domain of positive definite elasticity tensors is strictly located above the plane and inside the volume defined by the surface. For  $j_2 = 1$ , the plane and the surface are coincident.

*Remark 5.3.* In the framework of the polar method [41, 40], an equivalent set for positive definiteness condition was introduced using the polar components  $(t_0, t_1, r_0, r_1, \varphi_0, \varphi_1)$ :

$$t_0 > r_0 \geq 0 \quad \text{and} \quad t_1(t_0^2 - r_0^2) > 2r_1^2(t_0 - r_0 \cos 4(\varphi_0 - \varphi_1))$$

*Remark 5.4.* By doing the normalization procedure (18), the set of reduced invariants does not design a unique elastic material anymore but rather a family of elastic materials. It is possible to define the following equivalence relation: two elasticity tensors  $\mathbb{T}_1, \mathbb{T}_2 \in \mathbb{Ela}$  are equivalent if and only if

$$\exists \alpha, \beta \in \mathbb{R}, (\alpha, \beta) \neq (0, 0) \text{ such that } \hat{\mathbb{T}}_{\approx 2} = \underset{\sim}{\mathbb{H}}(\alpha, \beta) \hat{\mathbb{T}}_{\approx 1} \underset{\sim}{\mathbb{H}}(\alpha, \beta),$$

with<sup>5</sup>

$$\underset{\sim}{\mathbb{H}}(\alpha, \beta) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}_{\mathcal{H}}$$

For  $\underset{\sim}{\mathbb{T}} = \begin{bmatrix} \underset{\sim}{\mathbb{T}}^{dd} & \underset{\sim}{\mathbb{T}}^{ds} \\ \underset{\sim}{\mathbb{T}}^{sd} & \underset{\sim}{\mathbb{T}}^{ss} \end{bmatrix}$  (using the CG block notation (11)), elasticity tensors of the following form

$$\begin{bmatrix} \alpha^2 \underset{\sim}{\mathbb{T}}^{dd} & \alpha \beta \underset{\sim}{\mathbb{T}}^{ds} \\ \alpha \beta \underset{\sim}{\mathbb{T}}^{sd} & \beta^2 \underset{\sim}{\mathbb{T}}^{ss} \end{bmatrix} \quad \text{with } \alpha, \beta \in \mathbb{R} \text{ and } (\alpha, \beta) \neq (0, 0),$$

possess identical reduced invariants. More details can be found in [Appendix D](#).

## 6 An explicit link between Boehler and Kelvin invariants

The invariants considered in this paper are polynomial functions of the tensor considered as a geometric object in the physical space  $\mathbb{R}^2$ , the group of invariance being  $O(2)$ . As previously discussed ([subsection 4.2](#)) these quantities are referred to as the Boehler invariants [9, 5]. The eigenvalues of symmetric tensors constitute another system of invariant functions. Since eigenvalues are algebraic invariants, we will rather consider the coefficients of the characteristic polynomial which are given by the elementary symmetric polynomials. Those quantities will be referred to as (polynomial) Kelvin invariants. The two sets (with Boehler and Kelvin invariants) coincide only for second-order

<sup>5</sup> The matrices  $\underset{\sim}{\mathbb{H}}(\alpha, \beta)$  and  $\hat{\mathbb{R}}(\theta)$  commute.

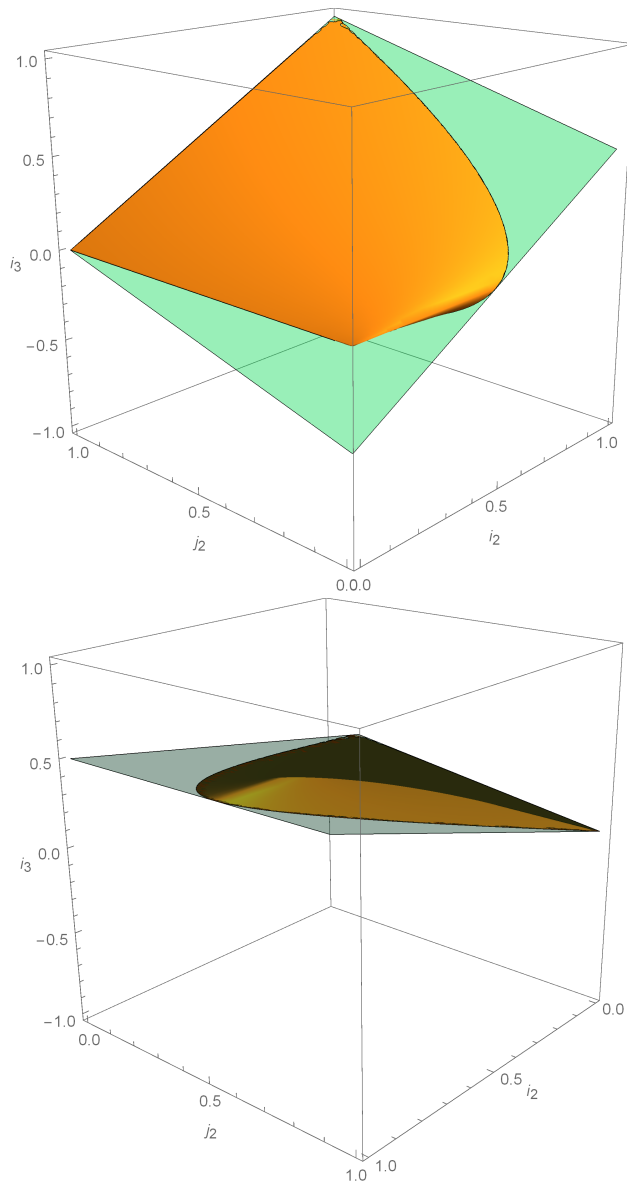


Fig. 3: Positive definiteness elasticity domain in terms of reduced anisotropic invariants (front and back views).

tensors. For the space of fourth-order elasticity tensors  $\mathbb{E}la$ , the Boehler set comprises 5  $O(2)$ -invariants  $(I_1, J_1, I_2, J_2, I_3)$ , while Kelvin set comprises 3  $O(3)$ -invariants  $(\sigma_1, \sigma_2, \sigma_3)$ . To build a bridge between these sets, let consider the following classical result [34, 37, 8, 44]:

**Lemma 6.1.** *Let  $(\underline{a}, \underline{u}) \in \mathbb{V} = S^2(\mathbb{R}^3) \times \mathbb{R}^3$  and set*

$$\begin{aligned} i_{1,0} &:= \operatorname{tr} \underline{a}, & i_{2,0} &:= \operatorname{tr} \underline{a}^2, & i_{3,0} &:= \operatorname{tr} \underline{a}^3, \\ i_{0,2} &:= \underline{u} \cdot \underline{u}, & i_{1,2} &:= \underline{u} \cdot (\underline{a} \cdot \underline{u}), & i_{2,2} &:= \underline{u} \cdot (\underline{a}^2 \cdot \underline{u}). \end{aligned} \quad (20)$$

A minimal integrity basis  $\mathcal{IB}$  for  $\mathbb{V}$  with respect to the standard  $O(3)$ -action is given by the collection

$$\mathcal{IB}(\mathbb{V}) = \{i_{1,0}, i_{2,0}, i_{3,0}, i_{0,2}, i_{1,2}, i_{2,2}\}.$$

In the notation  $i_{p,q}$ ,  $p$  is the degree in  $\underline{a}$  and  $q$  the degree in  $\underline{u}$ .

*Remark 6.2.* For  $\underline{a} \in S^2(\mathbb{R}^3)$ , the Newton sums  $\operatorname{tr} \underline{a}$ ,  $\operatorname{tr} \underline{a}^2$  and  $\operatorname{tr} \underline{a}^3$  are related to the elementary symmetric functions  $\sigma_1, \sigma_2, \sigma_3$  by the uniquely invertible following relations:

$$\begin{cases} \operatorname{tr} \underline{a} = \sigma_1 \\ \operatorname{tr} \underline{a}^2 = \sigma_1^2 - 2\sigma_2 \\ \operatorname{tr} \underline{a}^3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \end{cases} \Leftrightarrow \begin{cases} \sigma_1 = \operatorname{tr} \underline{a} \\ \sigma_2 = \frac{1}{2} \left( (\operatorname{tr} \underline{a})^2 - \operatorname{tr} \underline{a}^2 \right) \\ \sigma_3 = \frac{1}{6} \left( (\operatorname{tr} \underline{a})^3 - 3 \operatorname{tr} \underline{a} \operatorname{tr} \underline{a}^2 + 2 \operatorname{tr} \underline{a}^3 \right) \end{cases}$$

This allows to exchange the first 3 invariants  $i_{1,0}, i_{2,0}, i_{3,0}$  in lemma 6.1 with the elementary symmetric functions.

**Proposition 6.3.** *Consider  $\underline{\mathbb{T}} \in \mathbb{E}la$  and  $\hat{\underline{\mathbb{T}}}$  its representation as a second-order tensor in  $\mathbb{R}^3$ . Consider also the tensor<sup>6</sup>*

$$\underline{s} = \frac{\sqrt{2}}{2} (\underline{e}_1 \otimes \underline{e}_1 + \underline{e}_2 \otimes \underline{e}_2),$$

and  $\hat{\underline{s}} = \phi(\underline{s})$  its image in  $\mathbb{R}^3$ . The set of Boehler invariants of  $\underline{\mathbb{T}}$  for the  $O(2)$ -action is equal to the set of invariants of the pair  $(\hat{\underline{\mathbb{T}}}, \hat{\underline{s}})$  for the  $O(3)$ -action.

*Proof.* The vector  $\hat{\underline{s}}$  is unitary, hence  $i_{0,2} = 1$  and no specific information is given by this quantity. The following relations can be verified by a direct

<sup>6</sup> It is the vector  $\hat{\underline{f}}_3$  of the harmonic basis  $\mathcal{H}$  introduced subsection 3.1.

calculation:

$$\begin{aligned}\sigma_1 &= 2(I_1 + J_1), \\ \sigma_2 &= 4I_1J_1 + J_1^2 - \frac{1}{2}(I_2 + J_2), \\ \sigma_3 &= \frac{1}{2}(I_3 - I_2J_1 + 4I_1J_1^2 - 2I_1J_2), \\ i_{1,2} &= 2I_1, \\ i_{2,2} &= \frac{1}{2}I_2 + 4I_1^2.\end{aligned}$$

This non-linear system is easily uniquely inverted by substitution in the following:

$$\begin{aligned}I_1 &= \frac{1}{2}i_{1,2}, \\ J_1 &= \frac{1}{2}(\sigma_1 - i_{1,2}), \\ I_2 &= 2(i_{2,2} - i_{1,2}^2), \\ J_2 &= \frac{1}{2}(\sigma_1 + i_{1,2})^2 - 2(\sigma_2 + i_{2,2}), \\ I_3 &= 2\sigma_3 - 2\sigma_2i_{1,2} + i_{1,2}^2(\sigma_1 + i_{1,2}) + i_{2,2}(\sigma_1 - 3i_{1,2}).\end{aligned}$$

Again, this system is uniquely invertible (by substitution). This ends the proof.  $\square$

This result shows that Kelvin invariants is the subset of Boehler invariants obtained when the spheric direction  $\hat{\underline{s}}$  is forgot. As a consequence the eigenvalues of the elasticity tensor do not define uniquely an elastic material, but rather a family of elastic materials.

Using the spheric direction  $\hat{\underline{s}}$ , it is possible to define Boehler invariants directly in terms of  $\hat{\underline{T}} = \phi(\underline{T})$ . To that aim, consider  $\hat{\underline{s}}$  the vector direction of the spheric space in  $\mathbb{R}^3$  and construct the projectors

$$\underset{\sim}{\mathbf{P}}^s = \hat{\underline{s}} \otimes \hat{\underline{s}}, \quad \underset{\sim}{\mathbf{P}}^d = \underset{\sim}{\mathbf{1}} - \underset{\sim}{\mathbf{P}}^s.$$

The Clebsch-Gordan elements of the harmonic decomposition are defined as follows

$$\hat{\underset{\sim}{\mathbf{T}}}^{ss} := \underset{\sim}{\mathbf{P}}^s \hat{\underset{\sim}{\mathbf{T}}} \underset{\sim}{\mathbf{P}}^s, \quad \hat{\underset{\sim}{\mathbf{T}}}^{sd} := \underset{\sim}{\mathbf{P}}^s \hat{\underset{\sim}{\mathbf{T}}} \underset{\sim}{\mathbf{P}}^d + \underset{\sim}{\mathbf{P}}^d \hat{\underset{\sim}{\mathbf{T}}} \underset{\sim}{\mathbf{P}}^s, \quad \hat{\underset{\sim}{\mathbf{T}}}^{dd} := \underset{\sim}{\mathbf{P}}^d \hat{\underset{\sim}{\mathbf{T}}} \underset{\sim}{\mathbf{P}}^d.$$

The Boehler invariants of the integrity basis are

$$\begin{aligned}I_1 &= \frac{1}{2} \text{tr} \hat{\underset{\sim}{\mathbf{T}}}^{ss}, & I_2 &= \hat{\underset{\sim}{\mathbf{T}}}^{sd} : \hat{\underset{\sim}{\mathbf{T}}}^{sd}, & I_3 &= 2 \text{tr}(\hat{\underset{\sim}{\mathbf{T}}}^{sd} \hat{\underset{\sim}{\mathbf{T}}}^{dd\star} \hat{\underset{\sim}{\mathbf{T}}}^{sd}), \\ J_1 &= \frac{1}{2} \text{tr} \hat{\underset{\sim}{\mathbf{T}}}^{dd}, & J_2 &= \hat{\underset{\sim}{\mathbf{T}}}^{dd\star} : \hat{\underset{\sim}{\mathbf{T}}}^{dd\star},\end{aligned}$$

in which the notation  $\star$  is defined by

$$\underset{\sim}{\mathbb{T}}^\star := \underset{\sim}{\mathbb{T}} - \frac{1}{2}(\text{tr } \underset{\sim}{\mathbb{T}})\underset{\sim}{\mathbb{P}}^d.$$

## 7 Design transformations

Having two different actions acting on the space of elasticity tensors motivates the introduction of a new concept that we called *design transformation*.

First, the design transformation is defined in section 7.1. Then, we will prove in section 7.2 that any elastic material can be design-transformed into a tetragonal elastic material. Finally, we will study in details in section 7.3 the design transformations of initially isotropic elastic materials. As a result, we will characterize the set of elastic materials for which the associated elasticity tensors possess 2 identical eigenvalues. Since the obtained elastic materials are not necessarily isotropic, this example shows that symmetry classes identification cannot be achieved by studying the eigenvalue multiplicity of an elasticity tensor. As a concluding example, the set of 2D pentamode materials (bimode materials) initially introduced by [29] and well-investigated in the field of metamaterial studies is fully characterized. Mechanical interest in pentamode materials lies in the possibility of realizing acoustic and elastic cloaking device as investigated in the following references [30, 26, 12, 13, 21].

### 7.1 Definition and fundamental properties of a design transformation

**Definition 7.1.** Applications between elastic materials preserving eigenvalues will be called *design transformations*.

The design transformations have the following fundamental properties:

1. the set of design transformations of  $\mathbb{E}la$  is  $SO(3)$ ;
2. any initially positive definite tensor remains positive definite after any design-transformation;
3. trivial design transformations are the physical rotations  $\underset{\sim}{\mathbb{R}}(\theta, \hat{\mathbf{s}}) \in SO(3)$  (c.f. Equation 5).

### 7.2 Tetragonal equivalent elastic material

First of all, the following remarkable property can be observed:

**Proposition 7.2.** *Except for isotropic elasticity tensors having 3 identical eigenvalues, any elasticity tensor  $\underset{\sim}{\mathbb{T}} \in \mathbb{E}la$  is equivalent, up to a design transformation, to at least one tetragonal elasticity tensor.*

*Proof.* Consider an elasticity tensor  $\mathbb{T} \in \mathbb{E}la$  and  $\hat{\mathbb{T}}$  its second-order tensor representation in  $S^2(\mathbb{R}^3)$ . Consider  $[\hat{\mathbb{T}}]_{\mathcal{H}}$ , the matrix representation of  $\hat{\mathbb{T}}$  in the harmonic basis  $\mathcal{H}$ . Diagonalization of  $[\hat{\mathbb{T}}]_{\mathcal{H}}$  allows for defining a 3D rotation which ends up with a diagonal matrix in basis  $\mathcal{H}$ . Let denote  $\{\lambda_i\}_{1 \leq i \leq 3}$  the set of eigenvalues. Suppose first that the  $\{\lambda_i\}_{1 \leq i \leq 3}$  are all distinct (this is the standard situation for a initial tensor that belongs to  $\Sigma_{[D_4]}$ ,  $\Sigma_{[D_2]}$  or  $\Sigma_{[Z_2]}$ ). In this case there are 6 seemingly different permutations of this set. It can be observed that a  $\frac{\pi}{2}$  physical rotation will permute the first two eigenvalues. Hence we end up with 3 distinct  $\Sigma_{[D_4]}$  materials. Suppose now that among  $\{\lambda_i\}_{1 \leq i \leq 3}$  two eigenvalues are equal. Let say that we have  $\{\lambda_1, \lambda_1, \lambda_2\}$  (this is the standard situation for a initial tensor that belongs to  $\Sigma_{[O(2)]}$  and an exceptional situation for tensor in  $\Sigma_{[D_4]}$  or  $\Sigma_{[D_2]}$ ). In this case there are 3 seemingly different permutations of this set. It can be observed that a  $\frac{\pi}{2}$  physical rotation will permute the first two eigenvalues. Hence we end up with 2 distinct situations:  $\{\lambda_1, \lambda_1, \lambda_2\}$  and  $\{\lambda_1, \lambda_2, \lambda_1\}$ . The first one correspond to a tensor in  $\Sigma_{[O(2)]}$  tensor, while the other to a tensor in  $\Sigma_{[D_4]}$ .  $\square$

### 7.3 Study of design-transformed initially isotropic material

Let consider an isotropic elastic material

$$\hat{\mathbb{T}}_{\sim} = \begin{pmatrix} \gamma_0 & 0 & 0 \\ 0 & \gamma_0 & 0 \\ 0 & 0 & 2\kappa_0 \end{pmatrix}_{\mathcal{H}}. \quad (21)$$

Using a unit quaternion, any rotation in  $SO(3)$  can be encoded as (see appendix B)

$$\mathbb{R}_{\sim} = \begin{pmatrix} \omega^2 + v_x^2 - v_y^2 - v_z^2 & 2(v_x v_y - \omega v_z) & 2(\omega v_y + v_x v_z) \\ 2(v_x v_y + \omega v_z) & \omega^2 - v_x^2 + v_y^2 - v_z^2 & 2(v_y v_z - \omega v_x) \\ 2(v_x v_z - \omega v_y) & 2(\omega v_x + v_y v_z) & (\omega^2 - v_x^2 - v_y^2 + v_z^2) \end{pmatrix},$$

with

$$\omega^2 + v_x^2 + v_y^2 + v_z^2 = 1.$$

Let introduce the parameter  $\alpha \in [0, 1]$  defined by

$$\alpha := v_x^2 + v_y^2. \quad (22)$$

Using equation (15) and the identity  $\alpha = 1 - v_z^2 - \omega^2$ , the family of elastic materials obtained by carrying out design transformations on the initially

isotropic elastic material (21) is obtained as a parametric set of the following form, with  $\alpha \in [0, 1]$ :

$$\mathcal{IB}(\alpha) = \begin{cases} I_1(\alpha) = \kappa_0 - 2\alpha(1 - \alpha)(2\kappa_0 - \gamma_0), \\ J_1(\alpha) = \gamma_0 + 2\alpha(1 - \alpha)(2\kappa_0 - \gamma_0), \\ I_2(\alpha) = 8\alpha(1 - \alpha)(1 - 2\alpha)^2(2\kappa_0 - \gamma_0)^2, \\ J_2(\alpha) = 8\alpha^2(1 - \alpha)^2(2\kappa_0 - \gamma_0)^2, \\ I_3(\alpha) = 16\alpha^2(1 - \alpha)^2(1 - 2\alpha)^2(2\kappa_0 - \gamma_0)^3. \end{cases} \quad (23)$$

It can be noticed that:

- $\mathcal{IB}(\alpha) = \mathcal{IB}(1 - \alpha) \quad \forall \alpha \in [0, 1]$  so that the variation of the parameter  $\alpha$  can be reduced to the interval  $[0, \frac{1}{2}]$ ;
- for  $\alpha = 0$  the initial isotropic material is retrieved;
- for  $\alpha = \frac{1}{2}$  a tetragonal material is obtained;
- $I_1(\alpha) + J_1(\alpha) = \kappa_0 + \gamma_0, \quad \forall \alpha \in [0, \frac{1}{2}]$  so that the curve in the  $(I_1, J_1)$  plane is a line segment;
- $2I_3(\alpha)^3 - I_2(\alpha)^2 J_2(\alpha) = 0, \quad \forall \alpha \in [0, \frac{1}{2}]$  so that the curve is drawn on the surface of orthotropic elastic materials (in other terms, no biclinic material can be reached by the design transformation of an initially isotropic material).

The parametric set of equations (23) describes a curve in the space of elastic materials. This curve is represented in figure 4 for specific values of  $\kappa_0$  and  $\gamma_0$  (as the material remains positive definite, the positive definiteness condition is not plotted on the graph).

*Remark 7.3.* The characterization in terms of axes and angles of rotation for each value of  $\alpha$  is given in appendix B. Note that the entire curve can be obtained by considering a rotation of any axe orthogonal to  $\hat{\underline{s}}$  and of angle varying in  $[0, \pi]$ .

The variable  $\alpha$  can be eliminated from (23) leading to the following equivalent system of equations, where  $I_1 \in [\frac{\gamma_0}{2}, \kappa_0]$ :

$$\begin{cases} J_1 = \gamma_0 + \kappa_0 - I_1, \\ I_2 = 4(2I_1 - \gamma_0)(\kappa_0 - I_1), \\ J_2 = 2(\kappa_0 - I_1)^2, \\ I_3 = 4(2I_1 - \gamma_0)(\kappa_0 - I_1)^2. \end{cases}$$

Moreover, the variables  $(\gamma_0, \kappa_0)$  can be eliminated from the previous system, leading to the following implicit algebraic system of equations, where  $I_1 > 0$  and  $J_1 > 0$ :

$$\begin{cases} I_2^2 - 32I_1^2 J_2 - 4I_2 J_2 + 32I_1 J_1 J_2 - 8J_1^2 J_2 + 4J_2^2 = 0, \\ I_2^2 - 8I_1 I_3 + 4I_3 J_1 - 2I_2 J_2 = 0, \\ -I_2 I_3 + 4I_1 I_2 J_2 + 2I_3 J_2 - 2I_2 J_1 J_2 = 0, \\ 2I_3^2 - I_2^2 J_2 = 0. \end{cases} \quad (24)$$

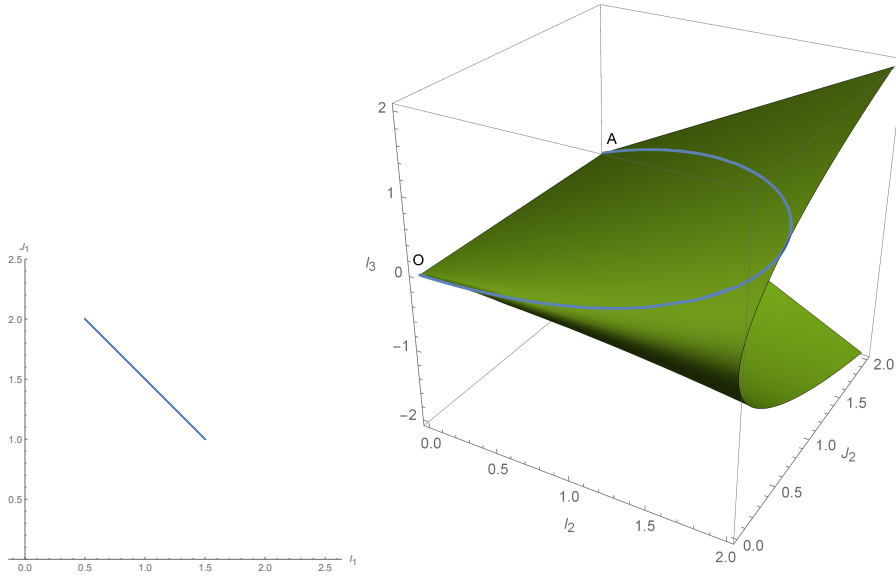


Fig. 4: An iso-eigenvalues curve in the space of elastic materials. The initial material is isotropic (point O ,  $\alpha = 0$ ), with the design transformation it becomes orthotropic (curved line,  $0 < \alpha < \frac{1}{2}$ ) and becomes tetragonal for  $\alpha = \frac{1}{2}$  (point A). Figure plotted with  $\gamma_0 = 1$  and  $\kappa_0 = \frac{3}{2}$ .

The algebraic system of equations (24) characterizes the set of *at-least-orthotropic* elastic materials that can be design-transformed into an isotropic material. Moreover, we have the following property (proof given in Appendix C):

**Proposition 7.4.** *The algebraic system (24) corresponds exactly to the subset of at-least-orthotropic materials having two identical eigenvalues.*

This result shows that symmetry classes identification cannot be achieved by studying the eigenvalue multiplicity of an elasticity tensor.

#### 7.4 Particular case of pentamode materials

Let consider an isotropic elastic material:

$$\hat{\mathbb{T}} \underset{\sim}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\kappa_0 \end{pmatrix}_{\mathcal{H}}.$$



The integrity basis reads then, with  $\alpha \in [0, \frac{1}{2}]$ :

$$\mathcal{IB}(\alpha) = \begin{cases} I_1(\alpha) = (1 - 2\alpha)^2 \kappa_0, \\ J_1(\alpha) = 4\alpha(1 - \alpha)\kappa_0, \\ I_2(\alpha) = 32\alpha(1 - \alpha)(1 - 2\alpha)^2 \kappa_0^2, \\ J_2(\alpha) = 32\alpha^2(1 - \alpha)^2 \kappa_0^2, \\ I_3(\alpha) = 128\alpha^2(1 - \alpha)^2(1 - 2\alpha)^2 \kappa_0^3. \end{cases} \quad (25)$$

When the variable  $\alpha$  is eliminated, this leads to the following equivalent system of equations, where  $I_1 \in [0, \kappa_0]$ :

$$\begin{cases} J_1 = \kappa_0 - I_1, \\ I_2 = 8I_1(\kappa_0 - I_1), \\ J_2 = 2(\kappa_0 - I_1)^2, \\ I_3 = 8I_1(\kappa_0 - I_1)^2. \end{cases}$$

The parameter  $\kappa_0$  can be eliminated from the previous system of equations, with  $I_1 \geq 0$  and  $J_1 \geq 0$ :

$$\begin{cases} I_2 = 8I_1J_1, \\ J_2 = 2J_1^2, \\ I_3 = 8I_1J_1^2. \end{cases} \quad (26)$$

(26) is an algebraic system of equations that defines the entire set of *at-least-orthotropic* elastic materials that can be design-transformed into an isotropic material with two eigenvalues equal to zero. Moreover, we have the following property (proof given in Appendix C):

**Proposition 7.5.** *The algebraic system (26) corresponds to the set of at-least-orthotropic materials having 2 eigenvalues equal to zero.*

## 8 Conclusion

We proposed in this paper a geometric interpretation of what is a 2D elastic material. The polynomial invariants of the integrity basis are used, and this leads to a straightforward proof of positive definiteness condition, which uses the harmonic decomposition and the associated harmonic basis for the matrix representation of the elasticity tensor. The explicit link between the three elementary symmetric polynomials (that are uniquely related to the three eigenvalues of the elasticity tensor) and the polynomial invariants of the integrity basis is provided with the help of a particular eigenvector of the elasticity tensor, the so-called spherical direction. This theoretical link explicated, the concept of design transformation (that leads to distinct elastic materials with identical eigenvalues) is introduced and thoroughly studied in the particular case of a double eigenvalue. By using design transformation, the existence of non standard orthotropic and tetragonal elastic materials possessing a double

eigenvalue (instead of three different ones) have been evidenced. This explicit construction shows that symmetry classes identification cannot be achieved by studying the eigenvalue multiplicity of an elasticity tensor. Finally, as a concluding example, the set of 2D pentamode materials (bimode materials) initially introduced by [29] and well-investigated in the field of metamaterial studies is fully characterized.

Other examples of metamaterials could be treated in further contributions, as for example linear micropolar materials or pantographic structures [15,36] that are usually modeled using strain gradient elasticity. More precisely, the methodology presented in this paper in the particular case of the elasticity tensor could possibly be used with 4th order tensors with other indicial symmetries and/or with 5th and 6th order elasticity tensors involved in strain gradient elasticity. The two steps of such study would be: first, to obtain the integrity basis of such tensors, and second, to perform the analysis with such integrity basis that may contain much more polynomial invariants than the elasticity case.

## A Normal forms

In this appendix, normal forms for tensors of different symmetry classes are provided with respect to the harmonic basis<sup>7</sup>  $\mathcal{H}$ . Normal forms are provided for generic tensors within each symmetry class. In the following,  $\bullet$  corresponds to independent coefficients while  $\times$  indicates equal coefficients.

–  $\Sigma_{[Z_2]}$

$$\begin{pmatrix} \bullet & 0 & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}_{\mathcal{H}} \quad \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}_{\mathcal{H}} \quad \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & 0 & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}_{\mathcal{H}}$$

*Remark: A physical rotation of  $\frac{\pi}{2}$  switches normal form 2 to normal form 3.*

–  $\Sigma_{[D_2]}$

$$\begin{pmatrix} \bullet & 0 & 0 \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}_{\mathcal{H}} \quad \begin{pmatrix} \bullet & 0 & \bullet \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{pmatrix}_{\mathcal{H}}$$

*Remark: A physical rotation of  $\frac{\pi}{2}$  switches the normal form one to another.*

–  $\Sigma_{[D_4]}$

$$\begin{pmatrix} \bullet & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{pmatrix}_{\mathcal{H}} \tag{27}$$

–  $\Sigma_{[O(2)]}$

$$\begin{pmatrix} \times & 0 & 0 \\ \times & 0 & 0 \\ \bullet & \bullet & \bullet \end{pmatrix}_{\mathcal{H}} \tag{28}$$

<sup>7</sup> It is worth emphasizing that usually normal forms are provided with respect to the Voigt or the Kelvin representation.

## B Quaternionic parametrization of rotations

Let denote  $\mathcal{S}^d := \{x \in \mathbb{R}^{d+1} : \|x\| = r\}$  the unit sphere in dimension  $d$ . Consider  $q$  a pure quaternion:

$$q \in \text{Im}(\mathbb{H}) \Leftrightarrow q = q_x \mathbf{i} + q_y \mathbf{j} + q_z \mathbf{k},$$

with the Hamilton relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

As  $q \in \text{Im}(\mathbb{H})$ ,  $q$  will be denoted by  $\underline{q}$  from now on.  $\text{SO}(3)$  action on  $\text{Im}(\mathbb{H})$  is

$$\underline{q}^* = h \underline{q} h^{-1},$$

with  $h \in \mathcal{S}^3$  the space of unit quaternions:

$$h \in \mathcal{S}^3 \Leftrightarrow h = \omega + v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} \quad \text{with} \quad \omega^2 + v_x^2 + v_y^2 + v_z^2 = 1.$$

The expression of  $h$  can be shortened to

$$h = \omega + \underline{v},$$

in which  $\omega$  is the scalar/real part of  $h$  and  $\underline{v}$  is the vector/imaginary part of  $h$ . Unit quaternions can also be expressed under the so called polar form

$$h = \cos\left(\frac{\theta}{2}\right) + \underline{n} \sin\left(\frac{\theta}{2}\right), \quad \text{with} \quad \underline{n} \in \mathcal{S}^2. \quad (29)$$

By identification

$$\omega = \cos\left(\frac{\theta}{2}\right), \quad v_x = \sin\left(\frac{\theta}{2}\right) n_x, \quad v_y = \sin\left(\frac{\theta}{2}\right) n_y, \quad v_z = \sin\left(\frac{\theta}{2}\right) n_z. \quad (30)$$

Using the following spherical coordinate for  $\underline{n}$

$$(\phi, \psi) \in ]0, \pi] \times [0, 2\pi], \quad \underline{n} = \begin{pmatrix} n_x = \sin(\phi) \cos(\psi) \\ n_y = \sin(\phi) \sin(\psi) \\ n_z = \cos(\phi) \end{pmatrix},$$

we define the following hyperspherical coordinates

$$\begin{pmatrix} x_t = R \cos\left(\frac{\theta}{2}\right) \\ x_z = R \sin\left(\frac{\theta}{2}\right) \cos(\phi) \\ x_x = R \sin\left(\frac{\theta}{2}\right) \sin(\phi) \cos(\psi) \\ x_y = R \sin\left(\frac{\theta}{2}\right) \sin(\phi) \sin(\psi) \end{pmatrix},$$

with  $R$  the hyper radius. We define the radius of the projected sphere

$$r = R \sin\left(\frac{\theta}{2}\right),$$

and the radius of the projected circle

$$\rho = r \sin(\phi) = R \sin\left(\frac{\theta}{2}\right) \sin(\phi),$$

in such a way that

$$\begin{pmatrix} x_t = R \cos\left(\frac{\theta}{2}\right) \\ x_z = r \cos(\phi) \\ x_x = \rho \cos(\psi) \\ x_y = \rho \sin(\psi) \end{pmatrix}.$$

For points on the unit hypersphere:

$$\begin{cases} R = 1 \\ r = \sin\left(\frac{\theta}{2}\right) \\ \rho = r \sin(\phi) = \sin\left(\frac{\theta}{2}\right) \sin(\phi) \end{cases}$$

Let's go back to the rotation formula

$$\underline{q}^* = h\underline{q}h^{-1} = (\omega + \underline{v})\underline{q}(\omega - \underline{v}) = \omega^2\underline{q} + \omega(\underline{v}\underline{q} - \underline{q}\underline{v}) - \underline{v}\underline{q}\underline{v}.$$

The multiplication of two pure quaternions gives

$$\underline{v}\underline{q} = -\underline{v} \cdot \underline{q} + \underline{v} \wedge \underline{q},$$

so that

$$\underline{q}^* = h\underline{q}h^{-1} = (\omega^2 + \underline{v} \cdot \underline{v})\underline{q} + 2\omega(\underline{v} \wedge \underline{q}) + 2\underline{v} \wedge (\underline{v} \wedge \underline{q}) = \underline{q} + 2\omega(\underline{v} \wedge \underline{q}) + 2\underline{v} \wedge (\underline{v} \wedge \underline{q}).$$

In the  $\mathbb{R}^3$  canonical basis

$$Q_{ij} = \underline{q}_j^* \cdot \underline{e}_i = \delta_{ij} + 2\omega(\underline{v} \wedge \underline{e}_j) \cdot \underline{e}_i + 2\underline{v} \wedge (\underline{v} \wedge \underline{e}_j) \cdot \underline{e}_i,$$

so that it finally can be established that (using vectorial identities)

$$\underline{\tilde{Q}} = (\omega^2 - \|\underline{v}\|^2)\underline{I} - 2\omega \underline{\underline{\epsilon}} \cdot \underline{v} + 2\underline{v} \otimes \underline{v}. \quad (31)$$

In matrix form:

$$\underline{\tilde{Q}} = \begin{pmatrix} \omega^2 + v_x^2 - v_y^2 - v_z^2 & 2(v_x v_y - \omega v_z) & 2(\omega v_y + v_x v_z) \\ 2(v_x v_y + \omega v_z) & \omega^2 - v_x^2 + v_y^2 - v_z^2 & 2(v_y v_z - \omega v_x) \\ 2(v_x v_z - \omega v_y) & 2(\omega v_x + v_y v_z) & (\omega^2 - v_x^2 - v_y^2 + v_z^2) \end{pmatrix}. \quad (32)$$

If we use the polar parametrization of quaternion (29) in (31), we obtain

$$\underline{\tilde{Q}} = \cos(\theta)\underline{I} - \sin(\theta)\underline{\underline{\epsilon}} \cdot \underline{n} + (1 - \cos(\theta))\underline{n} \otimes \underline{n},$$

which is the well-known Rodrigues formula for rotations.

Using the parametrization (29), equation (22) reads, with  $n_x^2 + n_y^2 = \sin^2(\phi)$  and  $\rho^2 = \alpha$ ,

$$\rho^2 = \sin^2\left(\frac{\theta}{2}\right) \sin^2(\phi). \quad (33)$$

Moreover, as  $\underline{n} \in \mathcal{S}^2$ , we get

$$n_z^2 = \cos^2(\phi). \quad (34)$$

Consider first the case  $\rho \in ]0, 1[$  and suppose that  $\phi \neq 0$ . In this case the system (33)-(34) can be rewritten as

$$\begin{cases} \theta = \pm 2 \arcsin\left(\frac{\rho}{\sin(\phi)}\right), \\ n_z = \pm \cos(\phi), \end{cases} \quad (35)$$

and the set of rotations associated to the parameter  $\rho^2$  is

$$\Phi^{-1}(\rho^2) = \left\{ \underline{n}^\pm = \begin{pmatrix} \sin(\phi) \cos(\psi) \\ \sin(\phi) \sin(\psi) \\ \pm \cos(\phi) \end{pmatrix}, \theta = \pm 2 \arcsin\left(\frac{\rho}{\sin(\phi)}\right), \forall (\phi, \psi) \in ]0, \pi] \times [0, 2\pi] \right\}.$$

Consider now the case  $\rho = 1$ . In such a case, system (33)-(34) implies

$$\sin^2\left(\frac{\theta}{2}\right) = 1, \quad \sin(\phi)^2 = 1,$$

hence

$$\Phi^{-1}(1) = \left\{ \underline{n}^\pm = \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix}, \theta = (2k+1)\pi, \forall k \in \mathbb{Z}, \psi \in [0, 2\pi] \right\}.$$

Consider now the case  $\rho = 0$ . The system (33)-(34) reads

$$\sin^2\left(\frac{\theta}{2}\right) \sin(\phi)^2 = 0,$$

which implies that either  $\sin(\phi)^2 = 0$  or  $\sin^2\left(\frac{\theta}{2}\right) = 0$ . The case  $\sin(\phi)^2 = 0$  corresponds to a physical rotation

$$\Phi^{-1}(0) = \left\{ \underline{n}^\pm = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}, \theta \in [0, 2\pi] \right\},$$

while the case  $\sin^2\left(\frac{\theta}{2}\right) = 0$  corresponds to a trivial rotation around any direction:

$$\Phi^{-1}(0) = \{ \underline{n}^\pm \in S^2, \theta = 2k\pi, \forall k \in \mathbb{Z} \}.$$

## C Proofs of propositions 7.4 and 7.5

First, recall that:

- the tensors considered are real valued, so that the Boehler invariants  $\in \mathbb{R}$ ;
- $I_2 \geq 0$  and  $J_2 \geq 0$  by definition;
- the elementary symmetric functions can be expressed in terms of Boehler invariants, see (16).

*Proof.* Proposition 7.4

The condition for an elastic material to have at least 2 equal eigenvalues reads in terms of elementary symmetric functions:

$$-\sigma_1^2 \sigma_2^2 + 4\sigma_2^3 + 4\sigma_1^3 \sigma_3 - 18\sigma_1 \sigma_2 \sigma_3 + 27\sigma_3^2 = 0. \quad (36)$$

Using (16), the subset of *at-least-orthotropic* materials with two identical eigenvalues is defined by the algebraic system:

$$\begin{cases} 2I_3^2 - I_2^2 J_2 = 0, \\ - (I_1 + J_1)^2 (I_2 - 2J_1(4I_1 + J_1) + J_2)^2 - \frac{1}{2}(I_2 - 2J_1(4I_1 + J_1) + J_2)^3 + \dots \\ + 16(I_1 + J_1)^3 (I_3 - I_2 J_1 + 4I_1 J_1^2 - 2I_1 J_2) - \dots \\ - 9(I_1 + J_1)(-I_2 + 2J_1(4I_1 + J_1) - J_2)(I_3 - I_2 J_1 + 4I_1 J_1^2 - 2I_1 J_2) + \dots \\ + \frac{27}{4}(I_3 - I_2 J_1 + 4I_1 J_1^2 - 2I_1 J_2)^2 = 0. \end{cases} \quad (37)$$

1. For  $I_2 = 0$ , (24) and (37) reduce to the same polynomial conditions

$$\begin{cases} I_3 = 0, \\ J_2(-8I_1^2 + 8I_1J_1 - 2J_1^2 + J_2)^2 = 0. \end{cases}$$

2. The case  $I_2 \neq 0$  and  $J_2 = 0$  cannot occur as in this case the material is  $R_0$ -orthotropic and have three distinct eigenvalues:

$$\begin{cases} \lambda_1 = J_1, \\ \lambda_2 = \frac{1}{2} \left( 2I_1 + J_1 - \sqrt{2I_2 + (2I_1 - J_1)^2} \right), \\ \lambda_3 = \frac{1}{2} \left( 2I_1 + J_1 + \sqrt{2I_2 + (2I_1 - J_1)^2} \right). \end{cases}$$

3. With  $I_2 \neq 0$  and  $J_2 \neq 0$ ,  $I_3 = \pm \frac{1}{\sqrt{2}} I_2 \sqrt{J_2}$  (recall that by definition  $J_2 \geq 0$ ).

- (a) Let consider the case  $I_3 = +\frac{1}{\sqrt{2}} I_2 \sqrt{J_2}$ . Solutions  $J_1^{a,b,c}$  of (37) are

$$J_1^a = 2I_1 - \frac{I_2 - 2J_2}{2\sqrt{2}\sqrt{J_2}}, \quad J_1^{b,c} = 2I_1 - \frac{\sqrt{J_2}}{\sqrt{2}} \pm 2i \frac{\sqrt{I_2}}{\sqrt{2}},$$

so that  $J_1^a$  is the only real valued solution. On the other hand, the Groebner basis of (24) reads in this case

$$-I_2 \frac{\sqrt{J_2}}{\sqrt{2}} + 4I_1 J_2 - 2J_1 J_2 + \sqrt{2} J_2^{\frac{3}{2}} = 0, \quad (38)$$

for which the unique solution with respect to  $J_1$  is  $J_1^a$  again.

- (b) The case  $I_3 = -\frac{1}{\sqrt{2}} I_2 \sqrt{J_2}$  is similar to the previous case up to some sign changes.

We have thus proved that the real valued solution of (37) are identical to the (real valued) solution of (24). This ends the proof.  $\square$

*Proof.* Proposition 7.5

The necessary and sufficient conditions for an elastic material to have at least 2 eigenvalues equal to zero reads in terms of elementary symmetric functions:

$$\sigma_2 = 0 \quad \text{and} \quad \sigma_3 = 0. \quad (39)$$

Using (16), the subset of *at-least-orthotropic* material with two zero eigenvalues is defined by the algebraic system:

$$\begin{cases} 2I_3 - I_2^2 J_2 = 0, \\ 4I_1 J_1 + J_1^2 - \frac{1}{2}(I_2 + J_2) = 0, \\ I_3 - I_2 J_1 + 4I_1 J_1^2 - 2I_1 J_2 = 0. \end{cases} \quad (40)$$

The Groebner basis of the previous algebraic system reads (with lexicographic order  $(I_3, J_2, I_2, J_1, I_1)$ ):

$$\begin{cases} (8I_1^2 + I_2)(I_2 - 8I_1 J_1)^2 = 0, \\ I_2 - 8I_1 J_1 - 2J_1^2 + J_2 = 0, \\ 2I_1 I_2 + I_3 - 16I_1^2 J_1 - I_2 J_1 = 0. \end{cases}$$

The case  $I_1 = 0$  implies that  $I_2 = 0$ , so that  $J_2 = 2J_1^2$  and  $I_3 = 0$ .

The case  $I_1 \neq 0$  implies that  $8I_1^2 + I_2 \neq 0$  because  $I_2 \geq 0$ , so that  $I_2 = 8I_1 J_1$ , and finally  $J_2 = 2J_1^2$  and  $I_3 = 8I_1 J_1^2$ . We have thus proved that (40) implies (26).

It is trivial to check that (26) implies (40). This ends the proof.  $\square$

## D On reduced anisotropic invariants

Let consider the harmonic basis  $\mathcal{H} = \{\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \hat{\mathbf{f}}_3\}$  introduced [subsection 3.1](#), and define the following second-order tensors:

$$\underset{\sim}{\mathbf{d}} := \hat{\mathbf{f}}_1 \otimes \hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_2 \otimes \hat{\mathbf{f}}_2, \quad \underset{\sim}{\mathbf{s}} := \hat{\mathbf{f}}_3 \otimes \hat{\mathbf{f}}_3.$$

We can specify the following group of transformations

$$\text{Dil} := \{\underset{\sim}{\mathbf{H}} \in \text{GL}(3), |\underset{\sim}{\mathbf{H}} = \alpha \underset{\sim}{\mathbf{d}} + \beta \underset{\sim}{\mathbf{s}}\},$$

which elements have the following matrix form with respect to  $\mathcal{H}$

$$\underset{\sim}{\mathbf{H}}(\alpha, \beta) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}_{\mathcal{H}}.$$

Consider the following relation among elasticity tensors

$$\underset{\sim}{\mathbf{T}}_1 \sim \underset{\sim}{\mathbf{T}}_2 \Leftrightarrow \exists \underset{\sim}{\mathbf{H}} \in \text{Dil} \mid \underset{\sim}{\mathbf{T}}_2 = \underset{\sim}{\mathbf{H}} \star \underset{\sim}{\mathbf{T}}_1.$$

This relation can be shown to be: reflexive, symmetric and transitive. In other words, this is an equivalence relation. The equivalence class of an element  $\underset{\sim}{\mathbf{T}} \in \mathbb{E}\text{la}$  is defined as

$$[\underset{\sim}{\mathbf{T}}_1] = \{\underset{\sim}{\mathbf{T}} \in \mathbb{E}\text{la}, \exists \underset{\sim}{\mathbf{H}} \in \text{Dil} \mid \underset{\sim}{\mathbf{T}} = \underset{\sim}{\mathbf{H}} \star \underset{\sim}{\mathbf{T}}_1\}.$$

The equivalence class can also be defined as the (non-compact) orbit of  $\underset{\sim}{\mathbf{T}}_1$  with respect to Dil. As well-known the space  $\mathbb{E}\text{la}$  can be partitioned into disjoint sets of equivalence classes. It can be observed that the reduced anisotropic invariants

$$i_2 = \frac{I_2}{8I_1J_1}, \quad j_2 = \frac{J_2}{2J_1^2}, \quad i_3 = \frac{I_3}{8I_1J_1^2}, \quad (41)$$

introduced in [section 5](#) are constant for the equivalence classes introduced above. For example, let consider a tensor  $\underset{\sim}{\mathbf{T}}_1$  having  $(i_2, j_2, i_3)$  as reduced invariants and that the associated elastic material is defined by  $(I_1, J_2, I_2, J_2, I_3)$ . A tensor  $\underset{\sim}{\mathbf{T}}_2$  in the same equivalence class is associated to the elastic material  $(\beta^2 I_1, \alpha^2 J_1, \alpha^2 \beta^2 I_2, \alpha^4 J_2, \alpha^4 \beta^2 I_3)$ , a direct computation shows that the associated reduced invariants will remain unchanged. The converse is not true, since different tensors (and not elastic materials) can have the same set of reduced invariants without being related by the equivalence relation. For example, consider  $\underset{\sim}{\mathbf{T}}_1$  and  $\underset{\sim}{\mathbf{T}}_2$  two tensors which are conjugated module  $\text{SO}(2)$ . Those tensors possess the same set of reduced invariants but are not related by any element of Dil.

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