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# A new approach of coherent risk-measure pricing

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**Abstract:** In this paper, we revisit the discrete-time partial hedging problem of contingent claims with respect to a dynamic risk-measure defined by its acceptance sets. A natural and sufficient weak no-arbitrage condition is studied to characterize the minimal risk-hedging prices. The method relies only on conditional optimization techniques. In particular, we do not need robust representation of the risk-measure and we do not suppose the existence of a risk-neutral probability measure. Numerical experiments illustrate the efficiency of the method.

**Keywords and phrases:** Risk-hedging prices, Dynamic risk-measures, Absence of immediate profit, Random sets, Conditional essential infimum.

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JEL Classification: C02, C61, G13.

## 1. Introduction

The problem of super-hedging a contingent claim  $\xi_T$  at time  $T > 0$  by a self-financing portfolio process  $V$  such that  $V_T \geq \xi_T$  is very classical in mathematical finance. For frictionless markets, we consider a no-arbitrage condition (NA) characterized by the fundamental theorem of asset pricing

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(FTAP), see [9, 18, 19] in discrete time and [10, 11] in continuous time. For discrete-time models, the Dalang-Morton-Willinger theorem [9] characterizes the NA condition as equivalent to the existence of a martingale probability measure under which the discounted price process is a martingale. Moreover, under NA, the set of all super-hedging prices is closed and the minimal price is the supremum of the expected discounted payoff under the equivalent probability measures, see [24] and [20, Theorem 2.1.11].

Contrarily to the classical approach, where the inequality  $V_T \geq \xi_T$  holds with probability 1, Cherny [8] proposes to relax this constraint which is difficult to achieve in practice. He assumes that the portfolio manager accepts to take a (reasonable) risk for the portfolio not to super-hedge the payoff. To do so, a risk-measure  $\rho_T$  is considered and the hedging error  $\epsilon_T = V_T - \xi_T$  is only supposed to be acceptable at the maturity date, i.e.  $\rho_T(\epsilon_T) \leq 0$ . The classical case of the literature coincides with the specific risk-measure defined by  $\rho_T(X) = -\text{ess inf}(X)$ , where the essential infimum  $\text{ess inf}(X)$  is the minimal element of the support of  $X$ , see [2]. In the setting of coherent risk-measures, the classical notion of arbitrage opportunity is replaced by the concept of good deal, i.e. a non negative claim attainable from a negative risk. Cherny [8] formulates a version of the FTAP theorem, i.e. characterizes the absence of good deals (NGD). Moreover, he provides upper and lower bounds for the prices of super-hedging and sub-hedging strategies [7] in the case of discrete time coherent risk-measures.

It is well known that risk-measures are usually defined on  $L^\infty$  and the spaces  $L^p$ ,  $p \in [1, \infty)$  allow natural extensions [23]. Actually, the choice of  $L^\infty$  and, more generally  $L^p$ , is mainly motivated by the dual representations of risk-measures. However, the space  $L^0$  is more adapted to financial and actuarial problems such as hedging, pricing, portfolio choice, equilibrium and optimal reinsurance with respect to risk-measures. Delbaen [12, 13] extends the coherent risk-measures to  $L^0$  by enlarging their range to  $\mathbf{R} \cup \{+\infty\}$  as there is no real-valued coherent risk-measure on  $L^0$  when the probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  is atomless (see [13, Theorem 5.1]).

Actually, it seems to be hopeless to axiomatize the notion of a coherent risk-measure on  $L^0$  (see [13, Definition 5.2]) and then to deduce a robust representation. Motivated by the representation theorem in  $L^\infty$ , Delbaen constructs a support functional associated with a set of probability measures and proves that it is a coherent risk-measure on  $L^0$  under some conditions (see [13, Theorem 5.4]). In detail, he truncates random variables from above, i.e. only considers possible future wealth up to some threshold as the space

$L^0$  contains non integrable random variables. It is then possible to compute a risk-measure from the robust representation in  $L^\infty$  as a supremum of expected loss with respect to some absolutely continuous probability measures (see [13, Definition 5.3]). Therefore, the robust representation on  $L^\infty$  appears to be the key point to extend coherent risk-measures to  $L^0$ , see [6, 7, 8]. This allows to formulate a FTAP with respect to NGD and solve super-replication problems. In this approach, coherent risk-measures remain characterised through families of probability measures which are not necessarily convenient to handle in practice, see e.g. the explicit representation of this family for the Weighted VaR risk-measure [6].

In this paper, we consider dynamic coherent risk-measures directly defined on the space  $L^0$  with values in  $\bar{\mathbf{R}} = [-\infty, +\infty]$ . They are naturally defined from acceptable sets, i.e. a risk-measure is seen as the minimal capital requirement added to the financial position for it to be acceptable. As such a capital may be infinite, we first define a risk-measure on its *effective domain* of all positions which are acceptable when adding a finite capital and then, we extend it to the whole space  $L^0$ .

As in [8], the aim of this paper is to solve the risk-hedging problem with respect to a coherent risk-measure but the novelty is that we do not use any dual representation. In particular, we do not make the assumption that there exists a risk-neutral probability measure for the price process. This is an approach which is rarely explored, see also the paper [16] where the classical semi-martingale setting is not supposed. Therefore, a backward dynamic principle is established. This means that we first solve the problem in the one-time step model. Precisely, we characterize the set of risk-hedging prices of a non negative contingent claim by a jointly measurable random function  $g(\omega, x)$  so that it is possible to compute the minimal risk-hedging price as the  $\omega$ -wise infimum of  $g(\omega, \cdot)$  on  $\bar{\mathbf{R}}$ . Then, in the discrete-time setting, the minimal risk-hedging prices are defined recursively in the spirit of [2].

Our analysis reveals possible immediate profits with respect to the risk-measure, as introduced in [2]. Such an arbitrage opportunity allows to super-replicate the zero contingent claim from a negative price. A weak version of the fundamental theorem of asset pricing is therefore formulated to characterize the absence of immediate profits (AIP). Actually, this weak no arbitrage condition AIP is not formulated for the whole market as it depends on the risk-measure or, equivalently, depends on the acceptable set chosen by the portfolio manager. Actually, we also consider a slightly stronger condition we call Strict Absence of Immediate Profit (SAIP) under which the sets of

risk-hedging prices are closed for payoff functions of linear growth. Moreover, the risk-hedging prices are time-consistent if the dynamic risk-measure is.

The paper is organized as follows. Section 2 defines the dynamic risk-measures on the space  $L^0$  with values in  $\overline{\mathbf{R}} = [-\infty, +\infty]$ . Section 3 introduces the problem of risk-hedging with respect to acceptable sets. We characterize the essential infimum of the risk-hedging prices as a (deterministic)  $\omega$ -wise infimum. A weak version of fundamental theorem of asset pricing is then formulated. Section 4 provides characterisations of minimal risk-hedging prices while lower and upper bounds are obtained in Section 5. At last, in Section 6, a numerical illustration computes explicit minimal risk-hedging prices and their strategies for an European call option in the Chinese market. The method appears to perform well on real data.

## 2. Dynamic risk-measures

In discrete-time, we consider a stochastic basis  $(\Omega, \mathcal{F} := (\mathcal{F}_t)_{t=0, \dots, T}, \mathbf{P})$  where the complete  $\sigma$ -algebra  $\mathcal{F}_t$  represents the information of the market available at time  $t$ . For any  $t \leq T$ ,  $L^0(\mathbf{R}, \mathcal{F}_t)$  is the metric space of all  $\mathbf{R}$ -valued random variables which are  $\mathcal{F}_t$ -measurable. Similarly,  $L^p(\mathbf{R}, \mathcal{F}_t, \mathbf{P})$ ,  $p \in [1, \infty)$  (resp.  $p = \infty$ ), is the normed space of all  $\mathbf{R}$ -valued random variables which are  $\mathcal{F}_t$ -measurable and admit a moment of order  $p$  under the probability measure  $\mathbf{P}$  (resp. bounded). Without any confusions, we omit the notation  $\mathbf{P}$  and just denote  $L^p(\mathbf{R}, \mathcal{F}_t)$ . In particular,  $L^p(\mathbf{R}_+, \mathcal{F}_t) = \{X \in L^p(\mathbf{R}, \mathcal{F}_t) | X \geq 0\}$  and  $L^p(\mathbf{R}_-, \mathcal{F}_t) = \{X \in L^p(\mathbf{R}, \mathcal{F}_t) | X \leq 0\}$  when  $p = 0$  or  $p \in [1, \infty]$ . All equalities and inequalities between random variables are understood to hold everywhere on  $\Omega$  up to a negligible set.

### 2.1. Construction from the acceptance set.

The dynamic risk-measure  $X \mapsto (\rho_t(X))_{t \leq T}$  considered in this work is defined on  $L^0$ . It is constructed from its acceptance sets defined as follows:

**Definition 2.1.** *A dynamic acceptable set is a family  $(\mathcal{A}_t)_{t \leq T}$  of non empty subsets of  $L^0(\mathbf{R}, \mathcal{F}_T)$  satisfying the following conditions at time  $t \leq T$ :*

- 1)  $X + Y \in \mathcal{A}_t$  for all  $X, Y \in \mathcal{A}_t$ ;
- 2)  $Y \in \mathcal{A}_t$  whenever  $Y \geq X$  and  $X \in \mathcal{A}_t$ ;
- 3)  $\mathcal{A}_t \cap L^0(\mathbf{R}, \mathcal{F}_t) = L^0(\mathbf{R}_+, \mathcal{F}_t)$ ;

4)  $k_t X \in \mathcal{A}_t$  for any  $X \in \mathcal{A}_t$  and  $k_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ .

Any element of  $\mathcal{A}_t$  is said acceptable at time  $t$ . Note that  $\mathcal{A}_t$  is a convex cone. For any  $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ , we denote by  $\mathcal{A}_t^X$  the set of all  $C_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  such that  $X + C_t \in \mathcal{A}_t$ , i.e,

$$\mathcal{A}_t^X := \{C_t \in L^0(\mathbf{R}, \mathcal{F}_t) | X + C_t \in \mathcal{A}_t\}.$$

Note that  $\mathcal{A}_t^X$  may be empty. We denote by  $\text{Dom } \mathcal{A}_t$  the set of all  $X \in L^0(\mathbf{R}, \mathcal{F}_T)$  such that  $\mathcal{A}_t^X \neq \emptyset$ , i.e.

$$\text{Dom } \mathcal{A}_t := \{X \in L^0(\mathbf{R}, \mathcal{F}_T) | \mathcal{A}_t^X \neq \emptyset\}.$$

In the following, we use the notation  $\overline{\mathbf{R}} = [-\infty, \infty]$ .

**Definition 2.2.** Let  $(\mathcal{A}_t)_{t \leq T}$  be a dynamic acceptance set. The coherent risk-measure associated to  $(\mathcal{A}_t)_{t \leq T}$  is defined at any time  $t$  by the mapping  $\rho_t : \text{Dom } \mathcal{A}_t \rightarrow L^0(\overline{\mathbf{R}}, \mathcal{F}_t)$  such that:

$$\rho_t(X) := \text{ess inf } \mathcal{A}_t^X, \quad X \in \text{Dom } \mathcal{A}_t. \quad (2.1)$$

Observe that  $\rho_t(X)$  is the minimal  $\mathcal{F}_t$ -measurable capital requirement we add to the position  $X$  for it to be acceptable at time  $t$ . It is clear that  $\rho_t(X) < \infty$  when  $\mathcal{A}_t^X \neq \emptyset$ .

**Definition 2.3.** A system  $(\rho_t)_{0 \leq t \leq T}$  is called dynamic coherent risk-measure if  $\rho_t$  is a coherent risk-measure defined by (2.1) for each  $t \leq T$ .

In the following, we formulate some properties satisfied by the coherent risk-measures  $\rho_t$  as defined in Definition 2.2. The proofs are postponed in Appendix B.1.

**Proposition 2.4.** The risk-measure  $\rho_t$  defined by (2.1) satisfies the following properties on  $\text{Dom } \mathcal{A}_t$ :

*Normalization:*  $\rho_t(0) = 0$ ;

*Monotonicity:*  $\rho_t(X) \geq \rho_t(X')$  whatever  $X, X' \in \text{Dom } \mathcal{A}_t$  s.t.  $X \leq X'$ ;

*Cash invariance:*  $\rho_t(X + m_t) = \rho_t(X) - m_t$  if  $m_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ ,  $X \in \text{Dom } \mathcal{A}_t$ ;

*Subadditivity:*  $\rho_t(X + X') \leq \rho_t(X) + \rho_t(X')$  if  $X, X' \in \text{Dom } \mathcal{A}_t$ ;

*Positive homogeneity:*  $\rho_t(k_t X) = k_t \rho_t(X)$  if  $k_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ ,  $X \in \text{Dom } \mathcal{A}_t$ .

Moreover, if the acceptable set  $\mathcal{A}_t$  is closed, then  $\rho_t$  is lower semi-continuous, satisfies  $\rho_t(X) > -\infty$  a.s. for all  $X \in \text{Dom } \mathcal{A}_t$ , and  $\mathcal{A}_t$  can be represented by  $\rho_t$  as

$$\mathcal{A}_t = \{X \in \text{Dom } \mathcal{A}_t \mid \rho_t(X) \leq 0\}. \quad (2.2)$$

## 2.2. Normalizations of the sets $\mathcal{A}_t^X$ , $X \in \text{Dom } \mathcal{A}_t$ .

Recall that  $\rho_t$  is lower semi-continuous means the following: If  $X_n \rightarrow X$ , then  $\rho_t(X) \leq \liminf_n \rho_t(X_n)$  a.s. We may suppose that the inequality holds everywhere on  $\Omega$  as  $\rho_t(X)$  may be modified on a negligible set since  $\mathcal{A}_t^X$  is  $\mathcal{F}_t$ -decomposable, see the proof of Proposition 2.4. Actually, this means that  $\rho_t(X)$  is not uniquely defined except if we consider  $\rho_t(X)$  as a class of equivalent random variables. In the following, we suppose that  $\mathcal{A}_t$  is closed for every  $t \leq T$  and we propose to *normalize* the sets  $\mathcal{A}_t^X$  (up to negligible sets) for  $X \in \text{Dom } \mathcal{A}_t$  so that it is possible to uniquely define  $\rho_t(X)$  on the whole space  $\Omega$  whatever  $X \in \text{Dom } \mathcal{A}_t$ .

**Lemma 2.5.** *Suppose that  $\mathcal{A}_t$ ,  $t \leq T$ , is closed and let  $X \in \text{Dom } \mathcal{A}_t$ . Then, there exists a  $\mathcal{F}_t$ -measurable closed and convex set  $A_t^X$  such that  $\mathcal{A}_t^X = L^0(A_t^X, \mathcal{F}_t)$ . Moreover,  $A_t^X(\omega) = \text{cl} \{C_n(\omega) : n \geq 1\}$  for every  $\omega \in \Omega$ , where  $(C_n)_{n \geq 1}$  is a countable family of  $\mathcal{A}_t^X$  <sup>1</sup>.*

*Proof.* Observe that  $\mathcal{A}_t^X$  is  $\mathcal{F}_t$ -decomposable and closed by the assumption on  $\mathcal{A}_t$ . Therefore, it suffices to apply [25, Corollary 2.5 and Proposition 2.7].  $\square$

Let us consider for every  $X \in \text{Dom } \mathcal{A}_t$ , the subset  $\tilde{\mathcal{A}}_t^X \subseteq \mathcal{A}_t^X$  of all elements  $C \in \mathcal{A}_t^X$  such that  $C(\omega) \in A_t^X(\omega)$  for every  $\omega \in \Omega$ . Note that  $\tilde{\mathcal{A}}_t^X \neq \emptyset$  by Lemma 2.5. Moreover, each  $C \in \mathcal{A}_t^X$  satisfies  $C \in A_t^X$  a.s., i.e.  $C$  coincides with an element of  $\tilde{\mathcal{A}}_t^X$  up to a negligible set. It is then natural to replace  $\mathcal{A}_t^X$  by  $\tilde{\mathcal{A}}_t^X$  as the elements of these two sets admit the same equivalence classes, up to a negligible set. Note that  $\tilde{\mathcal{A}}_t^X$  is still  $\mathcal{F}_t$ -decomposable, closed for the convergence everywhere, and it is an upper set as  $\mathcal{A}_t^X$ .

At last, for  $X \in \text{Dom } \mathcal{A}_t$ , it is worth noticing that  $\text{ess inf } \tilde{\mathcal{A}}_t^X$  coincides with  $\text{ess inf } \mathcal{A}_t^X$  up to a negligible set. Our goal is then to choose a specific element representing the equivalence class of  $\text{ess inf } \tilde{\mathcal{A}}_t^X$ . To do so, we use the following:

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<sup>1</sup>  $(C_n)_{n \geq 1}$  is called a Castaing representation of  $\mathcal{A}_t^X$ .

**Proposition 2.6.** *For all  $X \in \text{Dom } \mathcal{A}_t$ ,*

$$\inf \tilde{\mathcal{A}}_t^X = \inf_n C_n \in A_t^X,$$

*everywhere on  $\Omega$ , where  $(C_n)_{n \geq 1}$  is a Castaing representation of  $\mathcal{A}_t^X$ .*

*Proof.* Since  $A_t^X = \text{cl}\{C_n : n \geq 1\}$  on  $\Omega$ , it is clear that  $\inf_{C \in \tilde{\mathcal{A}}_t^X} C = \inf_n C_n$  is  $\mathcal{F}_t$ -measurable. The conclusion follows.  $\square$

**Corollary 2.7.** *For all  $X \in \text{Dom } \mathcal{A}_t$ ,  $\inf \tilde{\mathcal{A}}_t^X$  is  $\mathcal{F}_t$ -measurable. We deduce that  $\rho_t(X) := \inf \tilde{\mathcal{A}}_t^X$  is a version of the essential infimum  $\text{ess inf } \mathcal{A}_t^X$ . Moreover,  $\mathcal{A}_t^X = L^0([\rho_t(X), \infty))$ .*

*Proof.* As  $A_t^X(\omega) = \text{cl}\{C_n(\omega) : n \geq 1\}$  and  $\mathcal{A}_t^X = L^0(A_t^X, \mathcal{F}_t)$ , it follows that  $C \geq \inf \tilde{\mathcal{A}}_t^X$  a.s. for any  $C \in \mathcal{A}_t^X$ . Thus,  $\inf \tilde{\mathcal{A}}_t^X$  is a version of  $\text{ess inf } \mathcal{A}_t^X = \rho_t(X)$  and finally  $\mathcal{A}_t^X = L^0([\rho_t(X), \infty))$ .  $\square$

### 2.3. Extension of the risk-measure to $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ .

For any  $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ , consider the set

$$\Lambda_t(X) := \{F_t \in \mathcal{F}_t : X1_{F_t} \in \text{Dom } \mathcal{A}_t\}.$$

**Lemma 2.8.** *For any  $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ ,  $\Lambda_t(X)$  is directed-upward and admits a maximal element  $F_t(X)$ .*

*Proof.* We need to show that  $F_t^{(1)} \cup F_t^{(2)} \in \Lambda_t(X)$  for any  $F_t^{(1)}, F_t^{(2)} \in \Lambda_t(X)$ . To see it recall that, by Definition 2.1,  $X_1 + X_2 \in \text{Dom } \mathcal{A}_t$  for any  $X_1, X_2 \in \text{Dom } \mathcal{A}_t$  and  $k_t X \in \text{Dom } \mathcal{A}_t$  for any  $X \in \text{Dom } \mathcal{A}_t$  and  $k_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ . Then, for any  $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$  and  $F_t^{(1)}, F_t^{(2)} \in \Lambda_t(X)$ , we get that

$$X1_{F_t^{(1)} \cup F_t^{(2)}} = X(1_{F_t^{(1)}} + 1_{F_t^{(2)}} - 1_{F_t^{(1)}}1_{F_t^{(2)}}) = X1_{F_t^{(1)}}(1 - 1_{F_t^{(2)}}) + X1_{F_t^{(2)}} \in \text{Dom } \mathcal{A}_t,$$

i.e.  $F_t^{(1)} \cup F_t^{(2)} \in \Lambda_t(X)$ . Consider  $\gamma := \text{ess sup}\{1_{F_t} : F_t \in \Lambda_t(X)\}$ . As  $\Lambda_t(X)$  is directed-upward, there exists an increasing sequence  $(F_t^n)_{n \geq 1} \in \Lambda_t(X)$  such that  $\gamma = 1_{F_t(X)}$  where  $F_t(X) = \cup_n F_t^n$ . Let us define the sequence  $(E_t^n)_{n \geq 1}$  by  $E_t^n = F_t^n \setminus F_t^{n-1}$ ,  $n \geq 1$ , where  $F_t^0 = \emptyset$ . Then, any two sets of  $\{E_t^n, n \geq 1\}$  are disjoint and  $\cup_n F_t^n = \cup_n E_t^n$ . Moreover, for every  $n \geq 1$ , there exists  $C_t^n \in L^0(\mathbf{R}, \mathcal{F}_t)$  such that  $X1_{F_t^n} + C_t^n \in \mathcal{A}_t$ . We may suppose w.l.o.g. that  $F_t^n = E_t^n$ . Summing up, we deduce that  $X1_{F_t(X)} + C_t \in \mathcal{A}_t$  where



$C_t = \sum_n C_t^n 1_{E_t^n}$  belongs to  $\mathcal{A}_t$  since the later is closed and  $\mathcal{F}_t$ -decomposable. We deduce that  $F_t(X)$  is the maximal element of  $\Lambda_t(X)$ .  $\square$

By Lemma 2.8, it is possible to extend  $\rho_t$  on the whole space  $L^0(\overline{\mathbf{R}}, \mathcal{F}_t)$ . Precisely, we set

$$\rho_t(X) := \rho_t(X 1_{F_t(X)}), \text{ on } F_t(X), \text{ and } +\infty, \text{ otherwise.}$$

We claim that Proposition 2.4 may be extended to  $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ . The proofs can be found in Appendix B.2.

**Proposition 2.9.** *Proposition 2.4 holds on  $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$  with the conventions  $0 \times (+\infty) = 0$ ,  $(0, \infty) \times (+\infty) = \{+\infty\}$  and  $\mathbf{R} + (\infty) = +\infty$ .*

**Example 2.10.** *For the classical one-step super-hedging problem, a contingent claim  $h_T$  may be super-replicated at time  $T - 1$  if there exist a price  $P_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  and a strategy  $\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  such that  $P_{T-1} + \theta_{T-1} \Delta S_T - h_T \geq 0$  a.s. This means that the acceptable set  $\mathcal{A}_{T-1}$  is*

$$\begin{aligned} \mathcal{A}_{T-1} &= \{X \in L^0(\mathbf{R}, \mathcal{F}_T) | X \geq 0\} \\ &= \{X \in L^0(\mathbf{R}, \mathcal{F}_T) | \text{ess inf}_{\mathcal{F}_{T-1}} X \geq 0\} \\ &= \{X \in L^0(\mathbf{R}, \mathcal{F}_T) | -\text{ess inf}_{\mathcal{F}_{T-1}} X \leq 0\}. \end{aligned}$$

In particular, we have  $\rho_{T-1}(X) = -\text{ess inf}_{\mathcal{F}_{T-1}} X$ .

**Remark 2.11.** *Since  $X + \text{ess sup}_{\mathcal{F}_{T-1}}(-X) \geq 0 \in \mathcal{A}_{T-1}$  and  $\text{ess sup}_{\mathcal{F}_{T-1}}(-X)$  is  $\mathcal{F}_{T-1}$ -measurable, we get that*

$$\rho_{T-1}(X) \leq \text{ess sup}_{\mathcal{F}_{T-1}}(-X) = -\text{ess inf}_{\mathcal{F}_{T-1}}(X).$$

### 3. Risk-hedging problem and absence of immediate profit

In discrete-time, let  $(S_t)_{t \leq T}$  be the discounted price process of a risky asset such that  $S_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$  for any  $t \geq 0$ . Let  $(\rho_t)_{t \leq T}$  be a dynamic risk-measure as in Definition 2.3. A contingent claim with maturity  $T$  is defined by a real-valued  $\mathcal{F}_T$ -measurable random variable  $h_T$ . The goal is to find a self-financing strategy process  $(\theta_t)_{t \leq T}$  to super-hedge the contingent claim  $h_T$ . Here, super-hedging needs to be understood with respect to an acceptable set.

### 3.1. Minimal risk-hedging prices in the one step model

Let us start with the one time-step model between time  $T - 1$  and time  $T$ .

**Definition 3.1.** *The contingent claim  $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$  is said to be risk-hedged at time  $T - 1$  if there exists a risk-hedging price  $P_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  and a strategy  $\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  such that  $P_{T-1} + \theta_{T-1}\Delta S_T - h_T$  is acceptable at time  $T - 1$ .*

Let  $\mathcal{P}_{T-1}(h_T)$  be the set of all risk-hedging prices  $P_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  at time  $T - 1$  as in Definition 3.1. In the following, we suppose that  $h_T$  is non negative and  $\mathcal{P}_{T-1}(h_T) \neq \emptyset$ . This is the case if there exist  $a_{T-1}, b_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  such that  $h_T \leq a_{T-1}S_T + b_{T-1}$  for all  $S_T \in L^0(\mathbf{R}_+, \mathcal{F}_T)$ . This property trivially holds for European call and put options.

By Proposition 2.9, we may extend the set of acceptable positions (2.2) to  $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ :

$$\mathcal{A}_t = \{X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T) | \rho_t(X) \leq 0\}.$$

By cash invariance, we have  $P_{T-1} + \theta_{T-1}\Delta S_T - h_T \in \mathcal{A}_{T-1}$  if and only if  $P_{T-1} \geq \theta_{T-1}S_{T-1} + \rho_{T-1}(\theta_{T-1}S_T - h_T)$ . Therefore, the set of risk-hedging prices is:

$$\begin{aligned} \mathcal{P}_{T-1}(h_T) = & \{ \theta_{T-1}S_{T-1} + \rho_{T-1}(\theta_{T-1}S_T - h_T) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1}) \} \\ & + L^0(\mathbf{R}_+, \mathcal{F}_{T-1}). \end{aligned}$$

The next step is to construct a jointly measurable version of the random function

$$g_{T-1}(\omega, x) := xS_{T-1} + \rho_{T-1}(xS_T - h_T). \quad (3.3)$$

To do so, we consider the family  $\mathcal{G}_{T-1}$  of all  $Z = (X, Y) \in L^0(\mathbf{R}^2, \mathcal{F}_{T-1})$  such that  $Y \geq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$  a.s. Since we suppose that  $\mathcal{P}_{T-1}(h_T) \neq \emptyset$ ,  $\mathcal{G}_{T-1}$  is not empty.

**Lemma 3.2.** *Suppose that  $\mathcal{A}_{T-1}$  is closed. Then,  $\mathcal{G}_{T-1}$  is a closed convex subset of  $L^0(\mathbf{R}^2, \mathcal{F}_{T-1})$ . Moreover,  $\mathcal{G}_{T-1}$  is  $\mathcal{F}_{T-1}$ -decomposable and, for fixed  $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ , the section  $\{Y : (X, Y) \in \mathcal{G}_{T-1}\}$  is an upper set.*

*Proof.* Note that

$$\mathcal{G}_{T-1} = \{(X, Y) \in L^0(\mathbf{R}^2, \mathcal{F}_{T-1}) : X\Delta S_T - h_T + Y \in \mathcal{A}_{T-1}\}.$$

We deduce that  $\mathcal{G}_{T-1}$  is closed and convex since  $\mathcal{A}_{T-1}$  is closed by assumption and is a convex cone. Moreover,  $\mathcal{A}_{T-1}$  is  $\mathcal{F}_{T-1}$ -decomposable and so  $\mathcal{G}_{T-1}$  is. At last, for fixed  $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ , since  $\mathcal{A}_{T-1}$  is an upper set, so is the  $X$ -section of  $\mathcal{G}_{T-1}$ .  $\square$

**Lemma 3.3.** *Suppose that  $\mathcal{A}_{T-1}$  is closed. Then, there exists a non empty  $\mathcal{F}_{T-1}$ -measurable random closed set  $G_{T-1}$  such that  $\mathcal{G}_{T-1} = L^0(G_{T-1}, \mathcal{F}_{T-1})$ . Moreover,  $G_{T-1}$  is convex and  $G_{T-1}(\omega) = \text{cl}\{Z^n(\omega) : n \geq 1\}$  for every  $\omega \in \Omega$ , where  $(Z^n)_{n \geq 1}$  is a countable family of  $\mathcal{G}_{T-1}$ .*

*Proof.* By Lemma 3.2, we deduce that  $\mathcal{G}_{T-1} = L^0(G_{T-1}, \mathcal{F}_{T-1})$  for some  $\mathcal{F}_{T-1}$ -measurable random closed set  $G_{T-1}$ , see [20, Proposition 5.4.3]. As  $\mathcal{G}_{T-1}$  is not empty, we deduce that  $G_{T-1} \neq \emptyset$  a.s. Moreover, there exists a Castaing representation of  $G_{T-1}$  such that  $G_{T-1}(\omega) = \text{cl}\{Z^n(\omega) : n \geq 1\}$  for every  $\omega \in \Omega$ , where  $(Z^n)_{n \geq 1}$  is a countable family of  $\mathcal{G}_{T-1}$ , see [25, Proposition 2.7]. Then, by a contradiction argument and using a measurable selection argument, we may show that  $G_{T-1}$  is convex as  $\mathcal{G}_{T-1}$ .  $\square$

The following propositions provide a jointly measurable function  $g_{T-1}(\omega, x)$  such that  $g_{T-1}(X) = XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ .

**Proposition 3.4.** *Suppose that  $\mathcal{A}_{T-1}$  is closed. There exists a  $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ -measurable function  $g_{T-1}$  such that  $G_{T-1} = \{(x, y) : y \geq g_{T-1}(\omega, x)\}$  and, with  $X, Y \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ , we have  $Y \geq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$  if and only if  $Y \geq g_{T-1}(X)$ . Moreover,  $x \mapsto g_{T-1}(\omega, x)$  is a.s. convex and lower semi-continuous.*

*Proof.* Let us define the random function  $g_{T-1}$  as follows:

$$g_{T-1}(\omega, x) := \inf\{\alpha \in \mathbf{R} : (x, \alpha) \in G_{T-1}(\omega)\} \in [-\infty, \infty]. \quad (3.4)$$

We first show that  $g_{T-1}$  is  $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ -measurable. To see it, since the  $x$ -sections of  $G_{T-1}$  are upper sets, we get that

$$g_{T-1}(\omega, x) := \inf\{\alpha \in \mathbf{Q} : (x, \alpha) \in G_{T-1}(\omega)\}$$

where  $\mathbf{Q}$  is the set of all rational numbers of  $\mathbf{R}$ . Let us define the  $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ -measurable function  $I(\omega, x) = 1$  if  $(\omega, x) \in G_{T-1}$  and  $I(\omega, x) = +\infty$  if  $(\omega, x) \notin G_{T-1}$ . Then, define, for each  $\alpha \in \mathbf{Q}$ ,  $\theta^\alpha(\omega, x) = \alpha I(\omega, x)$  with the convention  $\mathbf{R} \times (+\infty) = +\infty$ . As  $\theta^\alpha$  is  $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ -measurable, we deduce that  $g_{T-1}(\omega, x) = \inf_{\alpha \in \mathbf{Q}} \theta^\alpha(\omega, x)$  is also  $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ -measurable.

Since  $G_{T-1}$  is closed, it is clear that  $(x, g_{T-1}(\omega, x)) \in G_{T-1}(\omega)$  a.s. when  $g_{T-1}(\omega, x) < \infty$  and, moreover,  $g_{T-1}(\omega, x) > -\infty$  by Proposition 2.4. Therefore,  $G_{T-1}(\omega)$  is the epigraph of the random function  $x \mapsto g_{T-1}(\omega, x)$ . As  $Y \geq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$  if and only if  $(X, Y) \in \mathcal{G}_{T-1}$ , or equivalently  $(X, Y) \in G_{T-1}$  a.s., we deduce that it is equivalent to  $Y \geq g_{T-1}(X)$ .

Moreover, as  $G_{T-1}$  is convex, we deduce that  $x \mapsto g_{T-1}(\omega, x)$  is a.s. convex. Let us show that  $x \mapsto g_{T-1}(\omega, x)$  is a.s. lower-semi continuous. Consider a sequence  $x^n \in \mathbf{R}$  which converges to  $x_0 \in \mathbf{R}$ . Let us denote  $\beta^n := g_{T-1}(x^n)$ . We have  $(x^n, \beta^n) \in G_{T-1}$  from the above discussion. In the case where  $\inf_n \beta^n = -\infty$ ,  $g_{T-1}(\omega, x) - 1 > \beta_n$  for  $n$  large enough (up to a subsequence) hence  $(x^n, g_{T-1}(\omega, x) - 1) \in G_{T-1}(\omega)$  since the  $x^n$ -sections of  $G_{T-1}$  are upper sets. As  $n \rightarrow \infty$ , we deduce that  $(x, g_{T-1}(\omega, x) - 1) \in G_{T-1}(\omega)$ . This contradicts the definition of  $g_{T-1}$ . Moreover, the inequality  $g_{T-1}(x) \leq \liminf_n \beta^n$  is trivial when the r.h.s. is  $+\infty$ . Otherwise,  $\beta^\infty := \liminf_n \beta^n < \infty$  and  $(x_0, \beta^\infty) \in G_{T-1}$  as  $G_{T-1}$  is closed. It follows by definition of  $g_{T-1}$  that  $g_{T-1}(x_0) \leq \liminf_n g_{T-1}(x^n)$ , i.e.  $g_{T-1}$  is lower-semi continuous.  $\square$

Observe that, on the set  $\{(\omega, x) : g_{T-1}(\omega, x) = +\infty\}$ ,  $G_{T-1}$  have empty  $x$ -sections.

**Corollary 3.5.** *We have  $g_{T-1}(X) = XS_{T-1} + \rho_{T-1}(XS_T - h_T)$  a.s. whatever  $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ .*

*Proof.* Consider a measurable selection  $(x_{T-1}, y_{T-1}) \in \mathcal{G}_{T-1} \neq \emptyset$ . We have  $y_{T-1} \geq g_{T-1}(x_{T-1})$  by definition hence  $g_{T-1}(x_{T-1}) < \infty$  a.s. Let us define  $X_{T-1} = x_{T-1}1_{g_{T-1}(X)=\infty} + X1_{g_{T-1}(X)<\infty}$ . Since we have

$$g_{T-1}(X_{T-1}) = g_{T-1}(x_{T-1})1_{g_{T-1}(X)=\infty} + g_{T-1}(X)1_{g_{T-1}(X)<\infty},$$

is a.s. finite,  $(X_{T-1}, g_{T-1}(X_{T-1})) \in G_{T-1}$  a.s. We deduce that

$$g_{T-1}(X_{T-1}) \geq X_{T-1}S_{T-1} + \rho_{T-1}(X_{T-1}S_T - h_T)$$

as  $\mathcal{G}_{T-1} = L^0(G_{T-1}, \mathcal{F}_{T-1})$ . Therefore,  $g_{T-1}(X) \geq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$  on the set  $\{g_{T-1}(X) < \infty\}$ . Moreover, the inequality trivially holds when  $g_{T-1}(X) = +\infty$ . Similarly, let us define

$$\begin{aligned} Y_{T-1} &= (XS_{T-1} + \rho_{T-1}(XS_T - h_T))1_{XS_{T-1} + \rho_{T-1}(XS_T - h_T) < \infty} \\ &+ y_{T-1}1_{XS_{T-1} + \rho_{T-1}(XS_T - h_T) = +\infty}. \end{aligned}$$

We have  $(X_{T-1}, Y_{T-1}) \in \mathcal{G}_{T-1}$  a.s. hence, by definition of  $g$ ,  $g(X_{T-1}) \leq Y_{T-1}$ . Then,  $g(X) \leq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$  on  $\{XS_{T-1} + \rho_{T-1}(XS_T - h_T) < \infty\}$ .

The inequality being trivial on the complementary set, we finally conclude that the equality holds a.s.  $\square$

From above, we rewrite  $\mathcal{P}_{T-1}(h_T)$  as

$$\mathcal{P}_{T-1}(h_T) = \{g_{T-1}(\theta_{T-1}) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1}) + L^0(\mathbf{R}_+, \mathcal{F}_{T-1})\}. \quad (3.5)$$

**Remark 3.6.** When  $\mathcal{A}_{T-1}$  is closed,  $\rho_{T-1}(xS_T - h_T)$  is distinct from  $-\infty$  a.s. By construction of  $g_{T-1}$ , we also deduce a lower semi-continuous and jointly measurable version of

$$\hat{g}_{T-1}(\omega, x) := \rho_{T-1}(xS_T - h_T)(\omega, x) := g_{T-1}(\omega, x) - xS_{T-1}(\omega)$$

which is  $+\infty$  if and only if  $g_{T-1}(\omega, x) = +\infty$ .

We then introduce the following random set:

$$\begin{aligned} \text{Dom } g_{T-1}(\omega) &:= \{x \in \mathbf{R} : g_{T-1}(\omega, x) < \infty\} \\ &= \{x \in \mathbf{R} : \rho_{T-1}(xS_T - h_T) < \infty\}. \end{aligned}$$

Observe that  $\text{Dom } g_{T-1}$  is an upper set, i.e. an interval. Moreover, since  $\mathcal{P}_{T-1}(h_T) \neq \emptyset$ , there exists a strategy  $a_{T-1} \in \text{Dom } g_{T-1}$  hence  $\text{Dom } g_{T-1}$  contains the interval  $[a_{T-1}, \infty)$ . We then deduce the following:

**Lemma 3.7.** Suppose that  $\mathcal{A}_{T-1}$  is closed. Then  $\inf_{x \in \mathbf{R}} g_{T-1}(x)$  is  $\mathcal{F}_{T-1}$ -measurable and coincides with  $\inf_{x \in \mathbf{Q}} g_{T-1}(x)$ .

*Proof.* The upper interval  $\text{Dom } g_{T-1}$  admits a non empty interior on which  $g_{T-1}$  is convex hence continuous. It follows that

$$\inf_{x \in \mathbf{R}} g_{T-1}(x) = \inf_{x \in \text{Dom } g_{T-1}} g_{T-1}(x) = \inf_{x \in \text{int } \text{Dom } g_{T-1}} g_{T-1}(x) = \inf_{x \in \mathbf{Q} \cap \text{int } \text{Dom } g_{T-1}} g_{T-1}(x).$$

We deduce that  $\inf_{x \in \mathbf{R}} g_{T-1}(x) \geq \inf_{x \in \mathbf{Q}} g_{T-1}(x)$  so that the equality holds and finally  $\inf_{x \in \mathbf{R}} g_{T-1}(x)$  is  $\mathcal{F}_{T-1}$ -measurable.  $\square$

A generalized concept of conditional essential supremum (resp. conditional essential infimum) of a family of vector-valued random variables with respect to a random partial order is introduced in [21, 22]. In the following, we use the simpler notion with respect to the natural partial order on  $\mathbf{R}$  for a family of real-valued random variables (see Appendix A.1).

**Definition 3.8.** *The minimal risk-hedging price of the contingent claim  $h_T$  at time  $T - 1$  is defined as*

$$P_{T-1}^* := \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}_{T-1}(h_T). \quad (3.6)$$

Note that the minimal risk-hedging price of  $h_T$  is not necessarily a price, i.e. an element of  $\mathcal{P}_{T-1}(h_T)$  but we shall see that  $P_{T-1}^* \in \mathcal{P}_{T-1}(h_T)$  under some extra conditions. If we introduce

$$\mathcal{P}'_{T-1}(h_T) := \{g_{T-1}(\theta_{T-1}) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})\},$$

we easily obtain that

$$P_{T-1}^* = \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}_{T-1}(h_T) = \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}'_{T-1}(h_T).$$

Note that the set  $\mathcal{P}'_{T-1}(h_T)$  consists of all prices making it possible to hedge the contingent claim  $h_T$  up to a zero risk.

**Lemma 3.9.** *Suppose that  $\mathcal{A}_{T-1}$  is closed, then*

$$P_{T-1}^* = \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g_{T-1}(\theta_{T-1}) = \inf_{x \in \mathbf{R}} g_{T-1}(x). \quad (3.7)$$

*Proof.* By Lemma 3.7, we know that  $\inf_{x \in \mathbf{R}} g_{T-1}(x)$  is  $\mathcal{F}_{T-1}$ -measurable. As  $g_{T-1}(\theta_{T-1}) \geq \inf_{x \in \mathbf{R}} g_{T-1}(x)$  for any  $\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ , we deduce that  $\operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g_{T-1}(\theta_{T-1}) \geq \inf_{x \in \mathbf{R}} g_{T-1}(x)$  as  $\inf_{x \in \mathbf{R}} g_{T-1}(x)$  is  $\mathcal{F}_{T-1}$ -measurable. Reciprocally, we have  $g_{T-1}(x) \geq \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g_{T-1}(\theta_{T-1})$  a.s. for all  $x \in \mathbf{Q}$ . Therefore, we deduce that  $\inf_{x \in \mathbf{Q}} g_{T-1}(x) \geq \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g_{T-1}(\theta_{T-1})$  and we get (3.7) by Lemma 3.7.  $\square$

Note that it is unclear in general how to solve an optimization problem defined by the essential infimum. By Lemma 3.9, we have changed the problem into a deterministic one, i.e. it suffices to solve it for each fixed  $\omega \in \Omega$ . Before characterizing the minimal risk-hedging price, we first recall the concept of immediate profit (IP) as introduced in [2]. We then give a weak version of the fundamental theorem of asset pricing allowing for pricing.

### 3.2. Weak fundamental theorem of asset pricing

Let us generalize the definition of acceptable set  $\mathcal{A}_t$  to  $\mathcal{A}_{t,u} \subseteq L^0(\mathbf{R}, \mathcal{F}_u)$  for time between  $t$  and  $u \in [t, T]$  by the same axiomatic conditions as in Definition 2.1. In the sequel, all acceptable sets are supposed to be closed. The risk-measure  $\rho_{t,u}$  is defined on  $\text{Dom } \mathcal{A}_{t,u}$  and then extended to  $L^0(\mathbf{R}, \mathcal{F}_u)$  as above. The risk-measure process satisfies

$$\rho_{t,u}(X) = \text{ess inf}\{Y \in L^0(\mathbf{R}, \mathcal{F}_t) | X + Y \in \mathcal{A}_{t,u}\}, \quad X \in \text{Dom } \mathcal{A}_{t,u}$$

and the corresponding acceptable set can be represented as

$$\mathcal{A}_{t,u} = \{X \in \text{Dom } \mathcal{A}_{t,u} | \rho_{t,u}(X) \leq 0\}.$$

Consider a random variable  $h_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ , which represents a contingent claim at time  $t$ . The general one time-step risk-hedging problem from  $t$  to  $t+1$  for the contingent claim  $h_{t+1}$  aims to characterize the risk-hedging prices  $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  and strategies  $\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  such that  $P_t + \theta_t \Delta S_{t+1} - h_{t+1}$  is acceptable with respect to the acceptable set  $\mathcal{A}_{t,t+1}$ . Equivalently, we may express the set of all risk-hedging prices as

$$\mathcal{P}_t(h_{t+1}) = \{\theta_t S_t + \rho_t(\theta_t S_{t+1} - h_{t+1}) : \theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)\} + L^0(\mathbf{R}_+, \mathcal{F}_t).$$

The minimal risk-hedging price at time  $t$  for this one time-step model is defined as

$$P_t^* := \text{ess inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(h_{t+1}). \quad (3.8)$$

Starting from the contingent claim  $h_T$  (see Section 3.1), we recursively define

$$P_T^* := h_T, \quad P_t^* := \text{ess inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*)$$

where  $P_{t+1}^*$  may be interpreted as a contingent claim  $h_{t+1}$ . The interesting question is whether  $P_{t+1}^*$  is actually a price, i.e. an element of  $\mathcal{P}_t(P_{t+1}^*)$ .

An immediate profit is the possibility to super-replicate the zero contingent claim from a negative price.

**Definition 3.10.** *Absence of Immediate Profit (AIP) holds if, for any  $t \leq T$ ,*

$$\mathcal{P}_t(0) \cap L^0(\mathbf{R}_-, \mathcal{F}_t) = \{0\}. \quad (3.9)$$

It is clear that AIP holds at time  $T$  since  $\mathcal{P}_T(0) = L^0(\mathbf{R}_+, \mathcal{F}_T)$ . Below, we characterize the AIP condition at any time.

**Theorem 3.11** (Weak fundamental theorem of asset pricing). *The AIP condition holds if and only if, for all  $t \leq T - 1$ ,*

$$-\rho_t(S_{t+1}) \leq S_t \leq \rho_t(-S_{t+1}). \quad (3.10)$$

*Proof.* Starting from  $P_T^* = h_T = 0$ , we obtain that the set of risk-hedging prices for the zero contingent claim at time  $T - 1$  is

$$\mathcal{P}_{T-1}(0) = \{\theta_{T-1}S_{T-1} + \rho_{T-1}(\theta_{T-1}S_T) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})\} + L^0(\mathbf{R}_+, \mathcal{F}_{T-1})$$

and the minimal risk-hedging price is  $P_{T-1}^* = \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}_{T-1}(0)$ . Note that AIP holds at time  $T - 1$  means that  $P_{T-1}^* = 0$ . By Lemma 3.9, we have  $P_{T-1}^* = \inf_{x \in \mathbf{R}} g_{T-1}(x)$ , where  $g_{T-1}(x) = xS_{T-1} + \rho_{T-1}(xS_T)$ . We have

$$g_{T-1}(x) = x[S_{T-1} + \rho_{T-1}(S_T)1_{x \geq 0} - \rho_{T-1}(-S_T)1_{x < 0}].$$

With  $\Lambda_{T-1} := \{-\rho_{T-1}(S_T) \leq S_{T-1} \leq \rho_{T-1}(-S_T)\}$ , we can deduce that

$$P_{T-1}^* = (0)1_{\Lambda_{T-1}} + (-\infty)1_{\Omega \setminus \Lambda_{T-1}}.$$

Therefore, condition AIP holds at time  $T - 1$  if and only if  $P(\Omega \setminus \Lambda_{T-1}) = 0$ . At last, observe that, from a price for the zero claim at any instant  $t$ , it is possible to super-hedge a price at time  $t + 1$ . So, if the minimal price at time  $t + 1$  is  $P_{t+1}^* = 0$ , then a price at any instant  $t$  is non negative. So, repeating the procedure backwardly at any time  $t \leq T - 2$ , we finally conclude.  $\square$

**Example 3.12.** *In the classical problem of Example 2.10, recall that the risk-measure at time  $T - 1$  is  $\rho_{T-1}(X) = -\operatorname{ess\,inf}_{\mathcal{F}_{T-1}} X$ ,  $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ . Then, by Theorem 3.11, AIP is equivalent to*

$$\operatorname{ess\,inf}_{\mathcal{F}_{t-1}} S_t \leq S_{t-1} \leq \operatorname{ess\,sup}_{\mathcal{F}_{t-1}} S_t, \quad 1 \leq t \leq T, \quad (3.11)$$

as formulated in [2, Theorem 3.4].

**Remark 3.13.** *The AIP condition for the risk-measure  $\rho_{T-1}$  implies the same for the risk-measure  $\rho_{T-1}(X) = -\operatorname{ess\,inf}_{\mathcal{F}_{T-1}} X$ . Indeed, by Remark 2.11 with  $S_T$  and  $-S_T$ , it holds that  $\rho_{T-1}(-X) \leq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}}(X)$ . Therefore,*

$$\operatorname{ess\,inf}_{\mathcal{F}_{T-1}} S_T \leq -\rho_{T-1}(S_T) \leq S_{T-1} \leq \rho_{T-1}(-S_T) \leq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} S_T.$$



**Interpretation.** Contrarily to classical no-arbitrage conditions, the AIP condition is not a no-arbitrage condition for the whole market, as it depends on the risk-measure or, equivalently, depends on the acceptable sets chosen by the portfolio manager. The AIP condition means that the portfolio manager evaluates the minimal price for the zero claim by 0. By Theorem 3.11, this is equivalent to say that there is no  $\mathcal{F}_t$ -measurable subset of  $\Omega$  at time  $t$  on which the portfolio manager considers  $\Delta S_{t+1}$  or  $-\Delta S_{t+1}$  as strictly acceptable, i.e. such that  $\rho_t(\pm \Delta S_{t+1}) < 0$ . On the contrary case, he could make a positive profit  $-\rho_t(\Delta S_{t+1})$  or  $-\rho_t(-\Delta S_{t+1})$  and obtain at time  $t$  an acceptable position  $\Delta S_{t+1}$  or  $-\Delta S_{t+1}$  by investing in a short or long position.

**Remark 3.14.** Suppose that there exists  $Q \sim P$  such that  $S$  is a  $Q$ -martingale and

$$\mathcal{A}_{T-1} \subseteq \{X \in L^1(\mathbf{R}, \mathcal{F}_T) : E_Q(X|\mathcal{F}_{T-1}) \geq 0\}.$$

Then, AIP holds.

## 4. Characterisation of minimal risk-hedging prices

### 4.1. Conditions under which the minimal risk-hedging price is a price

The main purpose of this section is to obtain the existence of optimal hedging strategies such that it is possible to hedge the contingent claim when starting from the minimal risk-hedging price, i.e. such that the minimal risk-hedging price is actually a price. In that case, the risk-measure of the hedging error vanishes.

#### 4.1.1. One step-time model

Let us consider the one-step model between time  $T - 1$  and  $T$ . We shall see that the minimal risk-hedging price  $P_{T-1}^*$  is actually a price as soon as AIP holds at time  $T - 1$  with some extra conditions, which we call Strict Absence of Immediate Profit (SAIP). To do so, we formulate the the following technical lemmas. The proof of the following is postponed to Appendix B.3.

**Lemma 4.1.** *With the convention  $0/0 = 0$ , the following inequalities holds a.s.:*

$$S_{T-1} - \rho_{T-1}(-S_T) \leq \frac{g_{T-1}(y) - g_{T-1}(x)}{y - x} \leq S_{T-1} + \rho_{T-1}(S_T), \quad x, y \in \mathbf{R}^d.$$

**Corollary 4.2.** *The random function  $x \mapsto g_{T-1}(x)$  is non increasing on the set  $\{S_{T-1} + \rho_{T-1}(S_T) = 0\} \in \mathcal{F}_{T-1}$  and it is non decreasing on the set  $\{S_{T-1} - \rho_{T-1}(-S_T) = 0\} \in \mathcal{F}_{T-1}$ .*

**Lemma 4.3.** *If  $S_{T-1} + \rho_{T-1}(S_T) > 0$ , then  $\lim_{x \rightarrow \infty} g_{T-1}(x) = +\infty$ . If  $S_{T-1} - \rho_{T-1}(-S_T) < 0$ , then  $\lim_{x \rightarrow -\infty} g_{T-1}(x) = +\infty$ .*

*Proof.* For any  $X \in L^0((0, \infty), \mathcal{F}_{T-1})$ , we have

$$g_{T-1}(X) = XS_{T-1} + \rho_{T-1}(XS_T - h_T) = X \left( S_{T-1} + \rho_{T-1} \left( S_T - \frac{h_T}{X} \right) \right).$$

Following the proof of Lemma B.4, we get that

$$\left| \rho_{T-1} \left( S_T - \frac{h_T}{X} \right) - \rho_{T-1}(S_T) \right| \leq \frac{\max(|\rho_{T-1}(-h_T)|, |\rho_{T-1}(h_T)|)}{X}.$$

Therefore, we may choose  $r_{T-1} \in L^0((0, \infty), \mathcal{F}_{T-1})$  large enough so that  $X \geq r_{T-1}$  implies that  $S_{T-1} + \rho_{T-1} \left( S_T - \frac{h_T}{X} \right) > 0$  and finally  $g_{T-1}(X) \geq M_{T-1}$  for a fixed  $M_{T-1} \in L^0((0, \infty), \mathcal{F}_{T-1})$  we arbitrarily choose.

We then deduce that  $\lim_{x \rightarrow \infty} g_{T-1}(x) = +\infty$ . To see it, we argue by contradiction on the (non-null set)  $B = \{\omega : \lim_{x \rightarrow \infty} g_{T-1}(x) \neq +\infty\}$ . As  $g_{T-1}$  is continuous, the set  $B$  may be reformulated as

$$B = \{\omega : \exists M \in \mathbf{Q} \text{ s.t. } \forall r \in \mathbf{Q}, \exists x \in [r, \infty) \cap \mathbf{Q} : g(x) < M\}.$$

This proves that  $B \in \mathcal{F}_{T-1}$ . Moreover, the set

$$G_{T-1} = \{(\omega, M) \in \Omega \times \mathbf{R}_+ : \forall r \in \mathbf{Q}, \exists x \in [r, \infty) \cap \mathbf{Q} : g(x) < M\},$$

is jointly measurable w.r.t.  $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$  and admits non empty  $\omega$ -sections on  $B$ . By measurable selection arguments, there exists  $M_{T-1} \in L^0(\mathbf{R}_+, \mathcal{F}_{T-1})$  such that  $(\omega, M_{T-1}(\omega)) \in G_{T-1}$  on  $B$  and we set  $M_{T-1} = +\infty$  otherwise. Then, still by measurable selection arguments and using the definition of  $G_{T-1}$ , we deduce the existence of  $X \geq r_{T-1}$  such that  $g_{T-1}(X) < M_{T-1}$  on  $B$ . On the other hand, recall that  $g_{T-1}(X)1_B \geq M_{T-1}1_B$  for every  $X \geq r_{T-1}$  large enough. This yields to a contradiction hence  $\lim_{x \rightarrow \infty} g_{T-1}(x) = +\infty$ . Finally, we conclude for the case where  $S_{T-1} - \rho_{T-1}(-S_T) < 0$  by similar arguments.  $\square$

Since  $\lim_{x \rightarrow \infty} g_{T-1}(x) = +\infty$  in the case where  $S_{T-1} + \rho_{T-1}(S_T) > 0$ , we deduce that  $\inf_{x \in \mathbf{R}_+} g_{T-1}(x) = \min_{x \in \mathbf{R}_+} g_{T-1}(x)$  is a minimum. Otherwise, if  $S_{T-1} + \rho_{T-1}(S_T) = 0$  under AIP, then  $g_{T-1}$  is non increasing and  $\inf_{x \in \mathbf{R}_+} g_{T-1}(x) = g_{T-1}(\infty)$ . Therefore, in the case where  $S_{T-1} + \rho_{T-1}(S_T) = 0$ ,  $P_{T-1}^*$  is a price if and only if  $g_{T-1}$  is constant for  $x$  large enough. In the following result, we analyze the case where  $g_{T-1}$  is a constant as  $x \rightarrow \pm\infty$ .

**Lemma 4.4.** *Suppose that the random function  $x \mapsto g_{T-1}(x)$  is constant for  $x$  large enough. Then, under AIP,  $S_{T-1} + \rho_{T-1}(S_T) = 0$ . Suppose that the random function  $x \mapsto g_{T-1}(x)$  is constant for  $-x$  large enough. Then, under AIP,  $S_{T-1} - \rho_{T-1}(-S_T) = 0$ .*

*Proof.* By assumption, we have  $xS_{T-1} + \rho_{T-1}(xS_T - h_T) = c_{T-1}$  for all  $x \geq \alpha_{T-1}$  where  $\alpha_{T-1}, c_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ . We may suppose that  $\alpha_{T-1} > 0$ . Dividing by  $x$  and making  $x \rightarrow \infty$ , we deduce that  $S_{T-1} + \rho_{T-1}(S_T) \leq 0$ . We then conclude by Theorem 3.11.  $\square$

**Corollary 4.5.** *Suppose that AIP holds. Then, the random function  $x \mapsto g(x)$  is constant for  $x$  large enough and is constant for  $-x$  large enough if and only if  $S_{T-1} + \rho_{T-1}(S_T) = S_{T-1} - \rho_{T-1}(-S_T) = 0$ . In that case,  $g_{T-1}$  is a constant function.*

*Proof.* It suffices to apply Lemma 4.4 and Lemma 4.1.  $\square$

**Theorem 4.6.** *Suppose that AIP holds at time  $T-1$  and  $S_{T-1} + \rho_{T-1}(S_T) = 0$  if and only if  $S_{T-1} - \rho_{T-1}(-S_T) = 0$ . Then, the minimal risk-hedging price  $P_{T-1}^*$  is a price.*

*Proof.* When  $S_{T-1} + \rho_{T-1}(S_T) = S_{T-1} - \rho_{T-1}(-S_T) = 0$ , the statement is trivial as  $g_{T-1}$  is a constant function from Corollary 4.5. In particular, the strategy  $\theta_{T-1}^* = 0$  is a candidate to hedge the payoff. Otherwise, we have  $S_{T-1} + \rho_{T-1}(S_T) > 0$  and  $S_{T-1} - \rho_{T-1}(-S_T) < 0$ . By Lemma 4.3,  $\lim_{x \rightarrow \infty} g_{T-1}(x) = \lim_{x \rightarrow -\infty} g_{T-1}(x) = +\infty$  so that the infimum of  $g$  is attained by some strategy  $\theta_{T-1}^* \in \mathbf{R}$ .  $\square$

The theorem above proves the existence of an optimal hedging strategy  $\theta_{T-1}^* \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  such that

$$P_{T-1}^* = g_{T-1}(\theta_{T-1}^*) = \theta_{T-1}^* S_{T-1} + \rho_{T-1}(\theta_{T-1}^* S_T - h_T) \in \mathcal{P}_{T-1}(h_T).$$

#### 4.1.2. Strict absence of immediate profit

Motivated by Theorem 4.6, we introduce the following no-arbitrage condition:

**Definition 4.7.** *We say that Strict Absence of Immediate Profit (SAIP) holds when both AIP holds and, almost surely,  $S_t + \rho_t(S_{t+1}) = 0$  if and only if  $S_t - \rho_t(-S_{t+1}) = 0$ .*

Note that, under SAIP,  $\Delta S_{t+1}$  is risk-neutral if and only if  $-\Delta S_{t+1}$  is risk-neutral. In the following, we characterize the SAIP condition in terms of pricing behaviour satisfied by the portfolio manager. The proof is postponed to Appendix B.4.

**Theorem 4.8.** *Condition SAIP holds at instant  $T-1$  if and only if the minimal risk-hedging price  $P_{T-1}^*$  of each contingent claim  $h_T$  such that  $\rho_{T-1}(-h_T) > 0$  is strictly positive.*

**Corollary 4.9.** *The SAIP condition holds for any  $0 \leq t \leq T-1$  if and only if the minimal risk-hedging price  $P_t^*$  of each contingent claim  $h_{t+1}$  satisfying  $\rho_t(-h_{t+1}) > 0$  is strictly positive.*

#### 4.1.3. Extension to the multi-period model

**Definition 4.10.** *A stochastic process  $(V_t)_{t \leq T}$  adapted to  $(\mathcal{F}_t)_{t \leq T}$ , starting from an initial endowment  $V_0$  is a portfolio process if, for all  $t \leq T-1$ , there exists  $\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  such that  $V_t + \theta_t \Delta S_{t+1} - V_{t+1}$  is acceptable at time  $t$ . Moreover, we say that it hedges the payoff  $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$  if  $V_T \geq h_T$  a.s.*

Note that  $V_{T-1} + \theta_{T-1} \Delta S_T - V_T$  is supposed to be acceptable at time  $T-1$ . Therefore,  $V_T \geq h_T$  implies that  $V_{T-1} + \theta_{T-1} \Delta S_T - h_T$  is acceptable at time  $T-1$ . In the following, we actually set  $V_T = h_T$  where  $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$  is a payoff we compute the risk-hedging prices. Recall that we have defined recursively  $P_T^* = h_T$  and

$$\mathcal{P}_t(P_{t+1}^*) = \{\theta_t S_t + \rho_t(\theta_t S_{t+1} - P_{t+1}^*) : \theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)\} + L^0(\mathbf{R}_+, \mathcal{F}_t)$$

so that

$$P_t^* = \operatorname{ess\,inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*).$$

As in Section 3.1, in order to construct a jointly measurable version of the random function

$$g_t(\omega, x) := x S_t + \rho_t(x S_{t+1} - P_{t+1}^*), \quad (4.12)$$

we consider the family  $\mathcal{G}_t$  of all elements  $Z = (X, Y) \in L^0(\mathbf{R}^2, \mathcal{F}_t)$  such that  $Y \geq XS_t + \rho_t(XS_{t+1} - P_{t+1}^*)$  a.s. We need  $\mathcal{G}_t$  to be non-empty. This is why we suppose in the following that there exists a least one portfolio process  $(V_t)_{t \leq T}$  such that  $V_T$  replicates  $h_T$  at time  $T$  with respect to the risk-measure of consideration. This is the case if there exist two constants  $a, b$  such that  $h_T \leq aS_T + b$ . Thereby, by constructing a jointly measurable version of  $g_t(\omega, x)$ , we may express  $P_t^*$  as

$$P_t^* = \inf_{x \in \mathbf{R}} g_t(x).$$

Since we set  $V_T = h_T$ ,  $\rho_{T-1}(V_{T-1} + \theta_{T-1} \Delta S_T - h_T) \leq 0$ . By definition of the one-step model, we deduce that  $V_{T-1} \geq P_{T-1}^*$ . By induction, we may easily show that  $V_t \geq P_t^*$  for all  $t \leq T$  since  $V_t$  is a price risk-hedging  $V_{t+1} \geq P_{t+1}^*$  at time  $t+1$ . In particular,  $V_t \in \mathcal{P}_t(P_{t+1}^*) \neq \emptyset$  for all  $t \in T-1$ .

So, it is possible to repeat backwardly the procedure developed in the one time step-model. In particular, if we suppose that Condition SAIP holds at any time  $t \leq T$ , we deduce by induction that each minimal risk-hedging price  $P_t^*$  is a price, see Proposition 4.6, and we may finally obtain the minimal risk-hedging price  $P_0^*$  for the claim  $h_T$ .

**Proposition 4.11.** *Suppose that SAIP holds at any time  $t \leq T$ . Then, each minimal risk-hedging price  $P_t^*$  is a price.*

## 4.2. Consistency in time

**Definition 4.12.** *A dynamic risk-measure  $(\rho_t)_{t \leq T}$  is said time-consistent if  $\rho_{t+1}(X) = \rho_{t+1}(Y)$  implies  $\rho_t(X) = \rho_t(Y)$  whatever  $X, Y \in L^0(\mathbf{R}, \mathcal{F}_T)$  and  $t \leq T-1$  (see Section 5 in [14]).*

**Lemma 4.13.** *A dynamic risk-measure  $(\rho_t)_{t \leq T}$  is time-consistent if and only if its family of acceptable sets  $(\mathcal{A}_t)_{t \leq T}$  satisfies*

$$\mathcal{A}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} \tag{4.13}$$

for any  $t \leq T-1$ .

*Proof.* First notice that time-consistency holds if and only if  $\rho_t(-\rho_{t+1}(X)) = \rho_t(X)$  for any  $t \leq T-1$  and  $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ . To see it, observe that  $\rho_{t+1}(-\rho_{t+1}(X))$  coincides with  $\rho_{t+1}(X)$ .

Let us show the “ $\Rightarrow$ ” implication. Let us show that  $\mathcal{A}_t \subseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$ . For any  $X \in \mathcal{A}_t$ ,  $\rho_t(X) \leq 0$ . As  $\rho_t(-\rho_{t+1}(X)) = \rho_t(X)$ , we get that  $-\rho_{t+1}(X) \in$

$\mathcal{A}_{t,t+1}$ . Therefore,  $X = -\rho_{t+1}(X) + \rho_{t+1}(X) + X \in \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$  since  $\rho_{t+1}(\rho_{t+1}(X) + X) \leq 0$  is trivial.

Reciprocally, for any  $X \in \mathcal{A}_{t,t+1}$  and  $Y \in \mathcal{A}_{t+1}$ ,

$$\begin{aligned}\rho_t(X + Y) &= \rho_t(-\rho_{t+1}(X + Y)) \leq \rho_t(-\rho_{t+1}(X) - \rho_{t+1}(Y)) \\ &\leq \rho_t(-\rho_{t+1}(X)) = \rho_t(X) \leq 0\end{aligned}$$

hence  $X + Y \in \mathcal{A}_t$  and, finally,  $\mathcal{A}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$ .

Let us show the “ $\Leftarrow$ ” implication. As  $X + \rho_t(X) \in \mathcal{A}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$ , we deduce  $X_{t,t+1} \in \mathcal{A}_{t,t+1}$  and  $X_{t+1} \in \mathcal{A}_{t+1}$  such that  $X + \rho_t(X) = X_{t,t+1} + X_{t+1}$ . Then,

$$\begin{aligned}-\rho_t(X) + \rho_{t+1}(X) &= \rho_{t+1}(X + \rho_t(X)) = \rho_{t+1}(X_{t,t+1} + X_{t+1}) \\ &= -X_{t,t+1} + \rho_{t+1}(X_{t+1}) \leq -X_{t,t+1}.\end{aligned}$$

It follows that  $\rho_t(X) - \rho_{t+1}(X) \geq X_{t,t+1}$  hence

$$\rho_t(\rho_t(X) - \rho_{t+1}(X)) \leq \rho_t(X_{t,t+1}) \leq 0.$$

We deduce that  $\rho_{t+1}(-\rho_{t+1}(X)) \leq \rho_t(X)$  for all  $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ .

On the other hand,  $-\rho_{t+1}(X) + \rho_t(-\rho_{t+1}(X)) \in \mathcal{A}_{t,t+1}$  and  $X + \rho_{t+1}(X) \in \mathcal{A}_{t+1}$ . Therefore, the sum  $X + \rho_t(-\rho_{t+1}(X)) \in \mathcal{A}_t$  by assumption. We deduce that  $\rho_t(X) - \rho_t(-\rho_{t+1}(X)) \leq 0$ , i.e.  $\rho_t(X) \leq \rho_t(-\rho_{t+1}(X))$ . We deduce that the equality holds by the first part.  $\square$

Observe that, if  $(\rho_t)_{t \leq T}$  is time-consistent, we may show by induction that  $\rho_t(-\rho_{t+s}(\cdot)) = \rho_t(\cdot)$  for any  $t \leq T$  and  $s \geq 0$  such that  $s \leq T - t$ . In the following, we introduce another possible definition for the risk-hedging prices in the multi-period model, where the risk is only measured at time  $t$ .

**Definition 4.14.** *The contingent claim  $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$  is said directly risk-hedged at time  $t \leq T - 1$  if there exists a (direct) price  $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  and a strategy  $(\theta_u)_{t \leq u \leq T-1}$  such that that  $P_t + \sum_{t \leq u \leq T-1} \theta_u \Delta S_{u+1} - h_T$  is acceptable at time  $t$ .*

The set of all direct risk-hedging prices at time  $t$  is then given by

$$\begin{aligned}\bar{\mathcal{P}}_t(h_T) &= \left\{ \rho_t \left( \sum_{t \leq u \leq T-1} \theta_u \Delta S_{u+1} - h_T \right) : (\theta_u)_{t \leq u \leq T-1} \in \Pi_{t \leq u \leq T-1} L^0(\mathbf{R}, \mathcal{F}_u) \right\} \\ &\quad + L^0(\mathbf{R}, \mathcal{F}_t).\end{aligned}$$

and the minimal direct risk-hedging price is

$$\bar{P}_t^* := \operatorname{ess\,inf}_{(\theta_u)_{t \leq u \leq T-1}} \bar{\mathcal{P}}_t(h_T).$$

We propose to prove under some conditions that the direct minimal risk-hedging prices coincide with the minimal ones derived from the step by step backward procedure developed before, i.e. such that  $P_t^* = \operatorname{ess\,inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*)$ , where  $P_T^* = h_T$ .

**Theorem 4.15.** *Suppose that the dynamic risk-measure  $(\rho_t)_{t \leq T}$  is time consistent and Condition SAIP holds. Then,  $\bar{P}_t^* = P_t^*$  for any  $t \leq T-1$  and the direct minimal risk-hedging prices are direct prices.*

*Proof.* It is trivial to see that  $\bar{P}_{T-1}^* = P_{T-1}^*$  as  $\bar{\mathcal{P}}_{T-1}(h_T) = \mathcal{P}_{T-1}(h_T)$ . Let us show the equality at time  $t$  by induction from the hypothesis  $\bar{P}_{t+1}^* = P_{t+1}^*$ . Note that this assumption is equivalent to the equality  $\bar{\mathcal{P}}_{t+1}(h_T) = \mathcal{P}_{t+1}(h_T)$ , since any  $V_t \in \bar{\mathcal{P}}_t$  satisfies  $V_t + \theta_t \Delta S_{t+1} = V_{t+1}$ .

First, we show we that  $\bar{P}_t^* \leq P_t^*$ . By Proposition 4.11, we know that  $P_t^*$  is a price for any  $t \leq T-1$  under SAIP. So, there exists  $\theta_t^* \in L^0(\mathbf{R}, \mathcal{F}_t)$  such that  $P_t^* = \rho_t(\theta_t^* \Delta S_{t+1} - P_{t+1}^*)$ . Moreover, by time consistency, we have:

$$\begin{aligned} P_t^* &= \rho_t(\theta_t^* \Delta S_{t+1} - \rho_{t+1}(\theta_{t+1}^* \Delta S_{t+2} - P_{t+2}^*)) \\ &= \rho_t(-\rho_{t+1}(\theta_t^* \Delta S_{t+1} + \theta_{t+1}^* \Delta S_{t+2} - P_{t+2}^*)) \\ &= \rho_t(\theta_t^* \Delta S_{t+1} + \theta_{t+1}^* \Delta S_{t+2} - P_{t+2}^*) \\ &\dots \\ &= \rho_t(\theta_t^* \Delta S_{t+1} + \theta_{t+1}^* \Delta S_{t+2} + \dots + \theta_{T-1}^* \Delta S_T - h_T). \end{aligned}$$

Therefore,  $P_t^* \geq \bar{P}_t^*$ . Conversely, if  $\bar{P}_t \in \bar{\mathcal{P}}_t(h_T)$ , we also have:

$$\begin{aligned} \bar{P}_t &\geq \rho_t(\theta_t \Delta S_{t+1} + \theta_{t+1} \Delta S_{t+2} + \dots + \theta_{T-1} \Delta S_T - h_T) \\ &\geq \rho_t(-\rho_{t+1}(\theta_t \Delta S_{t+1} + \theta_{t+1} \Delta S_{t+2} + \dots + \theta_{T-1} \Delta S_T - h_T)) \\ &= \rho_t(\theta_t \Delta S_{t+1} - \rho_{t+1}(\theta_{t+1} \Delta S_{t+2} + \dots + \theta_{T-1} \Delta S_T - h_T)) \\ &\geq \rho_t(\theta_t \Delta S_{t+1} - \bar{P}_{t+1}^*). \end{aligned}$$

As  $\bar{P}_{t+1}^* = P_{t+1}^*$ , we deduce that  $\bar{P}_t \geq \rho_t(\theta_t \Delta S_{t+1} - P_{t+1}^*)$  if  $\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  by the induction hypothesis. Therefore,  $\bar{P}_t \geq P_t^*$  and we deduce that  $\bar{P}_t^* \geq P_t^*$  so that the equality finally holds. Moreover, we deduce from above that

$$\bar{P}_t^* \geq P_t^* = \rho_t(\theta_t^* \Delta S_{t+1} + \theta_{t+1}^* \Delta S_{t+2} + \dots + \theta_{T-1}^* \Delta S_T - h_T).$$

As  $\bar{P}_t^*$  is an essential infimum, we deduce that

$$\bar{P}_t^* = \rho_t(\theta_t^* \Delta S_{t+1} + \theta_{t+1}^* \Delta S_{t+2} + \cdots + \theta_{T-1}^* \Delta S_T - h_T),$$

i.e.  $\bar{P}_t^*$  is a direct price.  $\square$

## 5. Lower and upper bounds of the prices

In this section, we show that the minimal risk-hedging price  $P_t^*$  lies in some interval  $[m_t^*, M_t^*]$  for all  $t \leq T - 1$  if and only if the condition AIP holds. In this part, we assume that each asset price  $S_t$ ,  $1 \leq t \leq T$  is strictly positive. The proof of the following theorem is showed in the Appendix B.5.

**Theorem 5.1.** *The condition AIP holds at time  $T - 1$  if and only if the minimal risk-hedging price of non negative contingent claim  $h_T$  satisfies*

$$m_{T-1}^* \leq P_{T-1}^* \leq M_{T-1}^*$$

where  $m_{T-1}^* = S_{T-1} \left( \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right)$  and  $M_{T-1}^* = S_{T-1} \left( \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right)$ .

**Corollary 5.2.** *Suppose that  $h_T$  is an attainable claim. Then AIP, holds if and only if each minimal risk-hedging price  $P_t^*$  of the contingent claim  $h_T$  satisfies*

$$m_t^* \leq P_t^* \leq M_t^* \tag{5.14}$$

where  $m_t^* = S_t \left( \text{ess inf}_{\mathcal{F}_t} \frac{h_T}{S_T} \right)$  and  $M_t^* = S_t \left( \text{ess sup}_{\mathcal{F}_t} \frac{h_T}{S_T} \right)$ .

*Proof.* First, observe that the inequalities (5.14) ensure that AIP holds for all  $t \leq T - 1$ . Indeed,  $m_t^* = M_t^* = 0$  for any  $t \leq T - 1$  when  $h_T = 0$ . So,  $P_t^* = 0$  for all  $t \leq T$ , i.e. AIP holds.

To prove the reverse implication, we first use the tower property satisfied by the conditional essential infimum/supremum operators (see [25, Appendix A]) and we deduce that

$$m_t^* = S_t \left( \text{ess inf}_{\mathcal{F}_t} \frac{m_{t+1}^*}{S_{t+1}} \right) \tag{5.15}$$

and

$$M_t^* = S_t \left( \text{ess sup}_{\mathcal{F}_t} \frac{M_{t+1}^*}{S_{t+1}} \right) \tag{5.16}$$



where  $0 \leq t \leq T-1$  and  $m_T^* = M_T^* = h_T$ . Note that (5.15) and (5.16) trivially hold when  $t = T-1$ . Suppose that (5.15) and (5.16) hold for all  $t+1 \leq u \leq T-1$ , and let us prove them for the time  $u = t$ . Actually, it has

$$\begin{aligned} m_t^* &= S_t \left( \text{ess inf}_{\mathcal{F}_t} \frac{h_T}{S_T} \right) = S_t \left( \text{ess inf}_{\mathcal{F}_t} \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) \\ &= S_t \left( \text{ess inf}_{\mathcal{F}_t} \frac{S_{T-1} \left( \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right)}{S_{T-1}} \right) = S_t \left( \text{ess inf}_{\mathcal{F}_t} \frac{m_{T-1}^*}{S_{T-1}} \right). \end{aligned}$$

We then conclude by induction with the tower property and, by similar arguments, we also get (5.16).

By Theorem 5.1, (5.14) holds at time  $t = T-1$ . Suppose that (5.14) holds at time  $t+1$  and let us show it at time  $t$ . To do so, recall that, for any  $t \leq T-1$ ,  $P_t^* = \text{ess inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*)$ . By Theorem 5.1, we deduce that  $\tilde{m}_t^* \leq P_t^* \leq \tilde{M}_t^*$  where  $\tilde{m}_t^* = S_t(\text{ess inf}_{\mathcal{F}_t} \frac{P_{t+1}^*}{S_{t+1}})$  and  $\tilde{M}_t^* = S_t(\text{ess sup}_{\mathcal{F}_t} \frac{P_{t+1}^*}{S_{t+1}})$ . Thus  $P_t^* \geq \tilde{m}_t^* \geq S_t(\text{ess inf}_{\mathcal{F}_t} \frac{m_{t+1}^*}{S_{t+1}}) = m_t^*$  and  $P_t^* \leq \tilde{M}_t^* \leq S_t(\text{ess sup}_{\mathcal{F}_t} \frac{M_{t+1}^*}{S_{t+1}}) = M_t^*$ .  $\square$

**Example 5.3.** Consider the call option with the payoff  $h_T = (S_T - K)^+$ . We get by simple computations that  $m_t^* = S_t \left( 1 - \frac{K}{\text{ess inf}_{\mathcal{F}_t} S_T} \right) 1_{\text{ess inf}_{\mathcal{F}_t} S_T > K}$  and  $M_t^* = S_t \left( 1 - \frac{K}{\text{ess sup}_{\mathcal{F}_t} S_T} \right) 1_{\text{ess sup}_{\mathcal{F}_t} S_T > K}$ .

## 6. Numerical illustration

In this section, we choose the Conditional Value-at-Risk (CVaR) as the risk-measure to compute the minimal risk-hedging prices of an European call option. Let us recall some well-known properties of the CVaR.

Let  $Z = f(x, y)$  be a loss function associated with the decision vector  $x \in \mathbf{R}^n$  and the random vector  $y \in \mathbf{R}^m$ , where  $y$  stands for the uncertainties in market parameters. The probability of  $y$  is assumed to have a density function  $p(y)$ . Then, the Value-at-Risk (VaR) of the loss  $Z$  associated with a decision  $x$ , denoted as  $\text{VaR}_\beta(x)$ , is defined by

$$\text{VaR}_\beta(x) = \min\{\alpha \in \mathbf{R} : \Psi(x, \alpha) \geq \beta\}$$

where  $\beta \in (0, 1)$  is a confidence level and  $\Psi(x, \alpha) = \int_{f(x, y) \leq \alpha} p(y) dy$  is the probability of loss not exceeding a threshold  $\alpha$ . Furthermore, for a confidence

level  $\beta \in (0, 1)$ , the CVaR for the loss  $Z$  associated with a decision  $x$ , denoted as  $\text{CVaR}_\beta(x)$ , may be defined as the expected value of the loss that exceeds  $\text{VaR}_\beta(x)$ , i.e.,

$$\text{CVaR}_\beta(x) = \frac{1}{1 - \beta} \int_{f(x, y) \geq \text{VaR}_\beta(x)} f(x, y) p(y) dy. \quad (6.17)$$

Let us introduce the function

$$F_\beta(x, \alpha) = \alpha + \frac{1}{1 - \beta} \int_{y \in \mathbf{R}^m} [f(x, y) - \alpha]^+ p(y) dy.$$

By [29, Theorem 1], we have

$$\text{CVaR}_\beta(x) = \min_{\alpha \in \mathbf{R}} F_\beta(x, \alpha).$$

By the law of large numbers, the function  $F_\beta(x, \alpha)$  is approximated by sampling the probability distribution of the random vector  $y$ , that is,

$$F_\beta(x, \alpha) \simeq \tilde{F}_\beta(x, \alpha) = \alpha + \frac{1}{J(1 - \beta)} \sum_{j=1}^J [f(x, y_j) - \alpha]^+$$

where  $J$ , the size of the sample, is large enough. Thus, we approximate

$$\text{CVaR}_\beta(x) \simeq \min_{\alpha \in \mathbf{R}} \left\{ \alpha + \frac{1}{J(1 - \beta)} \sum_{j=1}^J [f(x, y_j) - \alpha]^+ \right\}. \quad (6.18)$$

The proof of the convergence as  $J \rightarrow \infty$  is given in Section A.3. By introducing the auxiliary variables  $z_j$ ,  $j = 1, 2, \dots, J$ , the optimization problem (6.18) can be converted into the following linear programming (LP) problem:

$$\begin{aligned} & \min_{\alpha \in \mathbf{R}} \left\{ \alpha + \frac{1}{J(1 - \beta)} \sum_{j=1}^J z_j \right\} \\ & s.t. \quad z_j \geq f(x, y_j) - \alpha, \quad j = 1, 2, \dots, J, \\ & \quad \quad z_j \geq 0, \quad j = 1, 2, \dots, J. \end{aligned} \quad (6.19)$$

In our model,  $x \in \mathbf{R}$  is the strategy variable and  $y = S_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  is the risky asset price when the loss is considered at time  $t$ . The series of samples

$S_t^j$ ,  $j = 1, 2, \dots, J$  can be simulated by the Monte Carlo method. Note that CVaR is a loss based risk-measure. In what follows, for any  $Z \in L^0(\mathbf{R}, \mathcal{F}_{t+1})$  of the form  $Z = f_{t+1}(x, S_{t+1})$ , we consider the dynamic risk-measure  $\rho_t(Z)$  as the CVaR of the loss  $-Z$ , where  $0 \leq t \leq T - 1$ . By (6.17) to (6.18), we approximate the risk-measure  $\rho_t(Z)$  as

$$\rho_t(Z) \simeq \min_{\alpha \in \mathbf{R}} \left\{ \alpha + \frac{1}{J(1-\beta)} \sum_{j=1}^J [-f_{t+1}(x, S_{t+1}^j) - \alpha]^+ \right\}. \quad (6.20)$$

Furthermore, (6.20) can be converted into the LP problem:

$$\begin{aligned} \min_{\alpha \in \mathbf{R}} & \left\{ \alpha + \frac{1}{J(1-\beta)} \sum_{j=1}^J z_j \right\} \\ \text{s.t. } & z_j \geq -f_{t+1}(x, S_{t+1}^j) - \alpha, \quad j = 1, 2, \dots, J, \\ & z_j \geq 0, \quad j = 1, 2, \dots, J. \end{aligned} \quad (6.21)$$

### 6.1. Model selection and calibration

We consider the SSE 50 Index ETF Option (50 ETF Option) in China. The underlying asset is 50 ETF (stock code:510050). We deal with the daily closing prices of 50ETF from January the 4th of 2016 to December the 29th of 2017 as historical data. For convenience, we perform the time series analysis for the log-return of 50 ETF in every trading day.

As a preliminary, we test whether the series of log-returns is stationary (in the wide sense) or not. The definitions of strictly stationary and weakly stationary (i.e. stationary in the wide sense) can be found in [17, Section 1.2]. The stationarity is the premise of time series analysis in many cases because it guarantees the rationality of the prediction deduced from the historical regression. We adopt the Augmented Dickey-Fuller (ADF) test for a unit root to test the stationarity of log-return series, see [15]. Here the null hypothesis is that the original series has a unit root, i.e. the original series is non-stationary. The test results consist in a Test Statistic and a Critical Value for a given confidence level. If the ‘Test Statistic’ is less than the ‘Critical Value’, we reject the null hypothesis and we claim that the series is stationary. The asserts  $H=1$  or Test Statistic < Critical Value in Table 1 both imply that the null hypothesis is rejected. Thereby, we may claim that the series of log-returns is stationary.

Hypothesis test	H	P-Values	Test Statistics	Critical Values
ADF test	1	0.001	-25.3191	-1.9411
lbqtest of residual	1	$2.0699 \times 10^{-10}$	87.4408	31.4104
lbqtest of residual <sup>2</sup>	1	0	211.0243	31.4104

TABLE 1

*Hypothesis test with a default confidence level 5%.*

Let us assume that  $Y_t = \ln S_t - \ln S_{t-1}$  follows the mean regression equation

$$Y_t = \mu + \varepsilon_t \quad (6.22)$$

where  $\mu$  is the regression mean and  $\varepsilon_t$  is the residual with  $\mathbb{E}(\varepsilon_t) = 0$ . Our aim is to estimate the value of  $\mu$  and model the residual part  $\varepsilon_t$ .

First, let us show the QQ-plot and the distribution of residuals as Figure 1 and Figure 2. We observe that the residuals of log return  $\varepsilon_t$  admit a fat tailed instead of a normal distribution. This motivates us to use a Student's t-distribution in the following process of residual modelling.

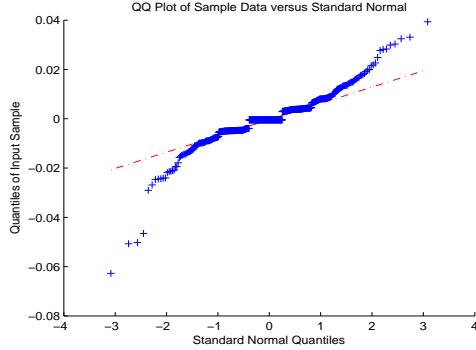


Fig 1: QQ-plot of the residuals

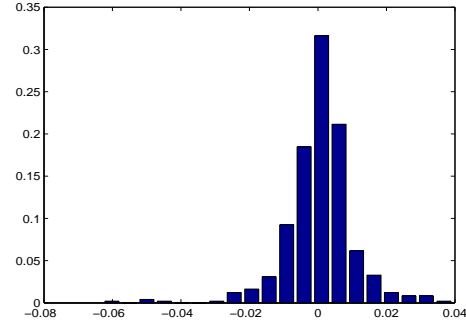


Fig 2: Distribution of the residuals

On the other hand, we may empirically observe the effect of volatility clustering from the residual graph of log-returns, see Figure 3. Large shocks tend to be followed by large shocks and small shocks tend to be followed by small shocks. Precisely, we carry out the Ljung-Box Q-test (lbqtest) for the autocorrelation of residuals and squared residual series. Ljung-Box Q-test is a quantitative way to test for autocorrelation at multiple lags jointly, see [26]. Note that lbqtest does not test directly serial dependencies other than autocorrelation, but it can be used to identify conditional heteroscedasticity

(ARCH effects) by testing squared residuals series, see [27]. Here the null hypothesis is that the tested series is not autocorrelated. The results of hypothesis test are also shown in Table 1. The assertion  $H=1$  implies that the null hypothesis, i.e. the residuals or squared residual series are not autocorrelated, is rejected. Thus, we claim that the residuals, especially the squared residuals are autocorrelated since the p-value of squared residuals is zero.

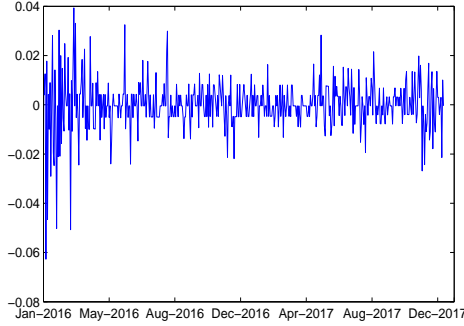


Fig 3: Residuals of log-returns

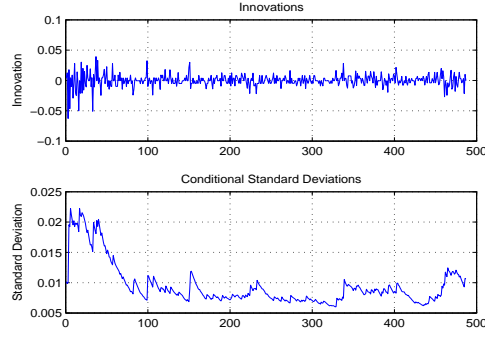


Fig 4: Innovations and standard deviations

Recall that the GARCH (Generalized Autoregressive Conditional Heteroskedastic) process introduced by [3] can better model the effect of volatility clustering or autocorrelation of squared residuals. It is a generalization of ARCH processes, which allows the conditional variance to change over time as a function of past errors. The simplest models of GARCH(1,1) are often very useful because they converge much faster to a local maximum in quasi-maximum likelihood estimation, while delivering very competitive forecasting performance, see [1]. The stochastic process  $(\varepsilon_t)_t$  is said to follow GARCH(1,1)-t model if

$$\varepsilon_t = \sigma_t Z_t, \quad (6.23)$$

$$\sigma_t^2 = \alpha_0 + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2, \quad \alpha_0 > 0, \alpha \geq 0, \beta \geq 0, \quad (6.24)$$

where  $Z_t$  are i.i.d. random variables with a Student's t-distribution of the freedom degree  $v$  and  $\mathbb{E}(Z_t) = 0$ ,  $\text{var}(Z_t) = 1$ , see [4, 5]. Furthermore, the wide-sense stationarity of the GARCH model (6.24) is sufficiently guaranteed by the constraint condition  $\alpha + \beta < 1$ , see [3, Theorem 1]. Later, we will show that the chosen model consisting of (6.22)-(6.24) fits the time series of log-returns quite adequately.

The parameters in mean regression equation (6.22) and conditional variance equation (6.24) are calibrated as shown in Table 2.

Parameter	Value	Standard Error	T Statistic
C	0.00050545	0.00032268	1.5664
K	$2.6379 \times 10^{-6}$	$1.5843 \times 10^{-6}$	1.6651
GARCH(1)	0.90041	0.035308	25.5016
ARCH(1)	0.073249	0.031478	2.3270
DoF	3.9935	0.85359	4.6785

TABLE 2  
*The calibrated parameters*

In detail,  $\mu = C = 0.00050545$ ,  $\alpha_0 = K = 2.6379 \times 10^{-6}$ ,  $\alpha = \text{ARCH}(1) = 0.073249$  and  $\beta = \text{GARCH}(1) = 0.90041$ . The freedom degree of the Student's t-distribution is  $v = \text{DoF} = 3.9935$ . Thus, the log-return of 50 ETF at time  $t$ , i.e.  $Y_t = \ln S_t - \ln S_{t-1}$  can be expressed as

$$Y_t = 0.00050545 + \varepsilon_t, \quad (6.25)$$

$$\sigma_t^2 = 2.6379 \times 10^{-6} + 0.073249\varepsilon_{t-1}^2 + 0.90041\sigma_{t-1}^2 \quad (6.26)$$

where  $\varepsilon_t = \sigma_t Z_t$  and  $Z_t$  follows t-distribution of the freedom degree 3.9935. Notice that  $\alpha + \beta = 0.073249 + 0.90041 < 1$  in the conditional variance equation (6.26), the wide-sense stationarity is satisfied.

The estimated residual (innovation)  $\varepsilon_t$  and conditional standard deviation  $\sigma_t$  are shown in Figure 4. It may be observed that the size of  $\sigma_t$  is consistent with the fluctuations of  $\varepsilon_t$ . Especially, the larger shocks appear at the beginning period where the conditional standard deviation  $\sigma_t$  reaches relatively large values. From this point of view, our estimated volatility  $\sigma_t$  is well-estimated.

Furthermore, let us carry out the calibration by Monte Carlo simulation for the regression models (6.25) and (6.26) using the historical data of the following 255 trading days from January the 8th of 2018 to March the 29th of 2019. The observed prices between all of these trading days are denoted as  $(s_t)_{0 \leq t \leq T-1}$ . The main lines of the simulation are given as follows.

Step 1: Generate a random matrix with t-distribution  $Z = (Z(j, t))_{J \times T}$  where  $j = 1, 2, \dots, J$  and  $t = 0, 1, \dots, T - 1$ . Note that  $\varepsilon_0$  and  $\sigma_0$  are given such that  $Z(j, 0) = Z_0 = \varepsilon_0 / \sigma_0$  in the every simulation  $j$ .

Step 2: Generate a volatility matrix  $\sigma = (\sigma(j, t))_{J \times T}$  by the conditional variance equation (6.26).

Step 3: The price matrix  $S = (S((j, t)))_{J \times T}$  is computed by the mean of equation (6.25) such that  $S(j, t) = S_{t-1} \exp\{\mu + \sigma(j, t)Z(j, t)\}$ , where  $S_{t-1} = s_{t-1}$  for each  $1 \leq t \leq T - 1$ .

Step 4: Finally, the price  $S_t$  is approximated by the average of  $S_t^j = S(j, t)$  with respect to all  $j$ .

The total number of simulated paths is fixed to  $J = 1000$ . The distribution of relative error  $|S_t - s_t|/s_t$  between the estimated price  $S_t$  and the observed one  $s_t$  is shown in Figure 5. In particular, the average of error is 0.0631% and the maximum error is 0.1553%. This implies that the price estimation based on the equations (6.25) and (6.26) is quite satisfactory.

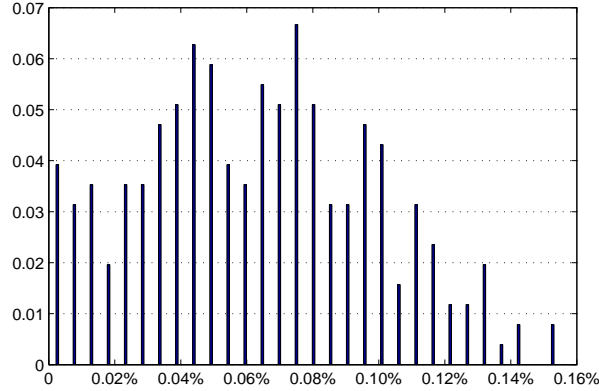


Fig 5: Distribution of the price estimation error.

## 6.2. Verification of the AIP condition

In the following, we compute the values of  $-\rho_t(S_{t+1})$  and  $\rho_t(-S_{t+1})$  for each of the 255 trading days from January the 8th of 2018 to March the 29th of 2019. We consider the risk-measure  $\rho_t(Z)$  as the value of CVaR for the loss  $Z = f_{t+1}(x, S_{t+1})$  associated with a decision  $x$ . Recall that  $\rho_t(Z)$  can be

approximated as the optimal value of the LP problem (6.21). That is

$$\begin{aligned} \rho_t(Z) &\simeq \min_{\alpha \in \mathbf{R}} \left\{ \alpha + \frac{1}{J(1-\beta)} \sum_{j=1}^J z_j \right\} \\ \text{s.t. } z_j &\geq -f_{t+1}(x, S_{t+1}^j) - \alpha, \quad j = 1, 2, \dots, J, \\ z_j &\geq 0, \quad j = 1, 2, \dots, J \end{aligned} \quad (6.27)$$

where  $\beta$  is the confidence level and  $S_{t+1}^j$  is the simulated price by Monte Carlo. Recall from Section 6.1, that we consider

$$S_{t+1}^j = S(j, t+1) = S_t \exp\{\mu + \sigma(j, t+1)Z(j, t+1)\},$$

where  $S_t = s_t$  is an observed price. Here,  $Z = f_{t+1}(x, S_{t+1}) = -S_{t+1}$ . Then  $\rho_t(Z) = \rho_t(-S_{t+1})$  is approximated by the optimal value of the LP problem:

$$\begin{aligned} \min_{\alpha \in \mathbf{R}} &\left\{ \alpha + \frac{1}{J(1-\beta)} \sum_{j=1}^J z_j \right\} \\ \text{s.t. } z_j &\geq S_{t+1}^j - \alpha, \quad j = 1, 2, \dots, J, \\ z_j &\geq 0, \quad j = 1, 2, \dots, J. \end{aligned} \quad (6.28)$$

Similarly, with  $Z = f_{t+1}(x, S_{t+1}) = S_{t+1}$ , we approximate  $\rho_t(Z) = \rho_t(S_{t+1})$  by the optimal value of the LP problem:

$$\begin{aligned} \min_{\alpha \in \mathbf{R}} &\left\{ \alpha + \frac{1}{J(1-\beta)} \sum_{j=1}^J z_j \right\} \\ \text{s.t. } z_j &\geq -S_{t+1}^j - \alpha, \quad j = 1, 2, \dots, J, \\ z_j &\geq 0, \quad j = 1, 2, \dots, J. \end{aligned} \quad (6.29)$$

We choose  $\beta = 0.95$  and  $J = 1000$ . Then, the LP problems (6.28) and (6.29) can be solved using the CPLEX optimization software, see [28], with a Matlab interface, which makes possible a relatively fast speed in the large-scaled linear optimization problem.

Comparing the values of  $-\rho_t(S_{t+1})$ ,  $\rho_t(-S_{t+1})$  and  $S_t = s_t$  for every  $0 \leq t \leq T-1$ , we observe that the equivalent characterization of AIP, i.e.  $-\rho_t(S_{t+1}) \leq S_t \leq \rho_t(-S_{t+1})$ , holds for each  $t$ , see Figure 6. Better, the condition SAIP is satisfied on the historical data, i.e.  $-\rho_t(S_{t+1}) < S_t < \rho_t(-S_{t+1})$  holds for



each  $t$ . By Proposition 4.11, each minimal risk-hedging price  $P_t^* = \inf_{x \in \mathbf{R}} g_t(x)$  is a price, where  $g_t(x) = xS_t + \rho_t(xS_{t+1} - P_{t+1}^*)$  and  $P_T^* = h_T$ . In the next part, we compute the minimal risk-hedging prices of European call options with the payoff  $h_T = (S_T - K)^+$ .

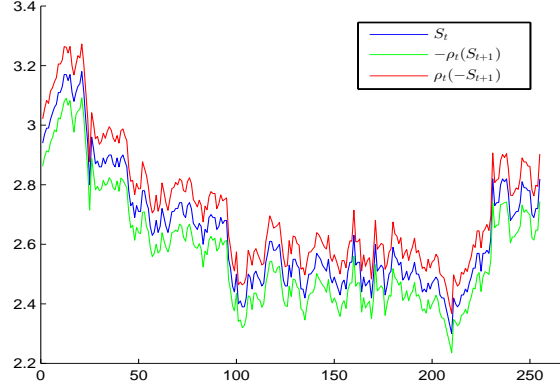


Fig 6: Visualization of the SAIP Condition

### 6.3. Computation of minimal risk-hedging prices

The only option that can be traded in the Chinese market, the 50 ETF option, is observable on February the 9th of 2015. According to the trading rules of 50 ETF options, there are nine strike prices: one at-the-money option, four out-of-the-money options and four in-the-money options. In the following, we focus on computing the risky-hedging price of at-the-money option by considering the case where  $K = S_0$ .

We suppose that the discrete time series is given by  $(t_i)_{0 \leq i \leq n}$  where  $t_i = \frac{iT}{n}$ . In particular,  $t_0 = 0$  and  $t_n = T$ . Note that  $S_{t_i}$  is the price observed in the time interval  $[t_i, t_{i+1})$  for each  $0 \leq i \leq n - 1$ . We choose the interval  $[0, T]$  that corresponds to a period of one week, which is composed of 5 trading days, i.e.  $n = 5$ . The risk-hedging strategy is implemented on every 51 weeks from January the 8th of 2018 to March the 29th of 2019. Starting from  $S_{t_0}$ , the series of prices  $(S_{t_i})_{1 \leq i \leq 4}$  are generated by Monte Carlo simulation of the GARCH model we have calibrated before, i.e.  $S_{t_i}$  may take the values  $(S_{t_i}^j)_{1 \leq j \leq J}$  for each  $1 \leq i \leq 4$ , where  $J$  is the total number of simulation.

At time  $t_i$ , the minimal risk-hedging price  $P_{t_i}^*$  of the 50 ETF call option is given by

$$P_{t_i}^* = \inf_{x \in \mathbf{R}} g_{t_i}(x)$$

where  $g_{t_i}(x) = \rho_{t_i}(xS_{t_{i+1}} - xS_{t_i} - P_{t_{i+1}}^*)$  and  $P_{t_4}^* = (S_{t_4} - K)^+$ . Recall that, if the risk-measure  $\rho_t(Z)$  is the CVaR value of minus the loss  $Z = f_{t+1}(x, S_{t+1})$ , associated with a decision  $x$ , then  $\rho_t(Z)$  can be approximated as (6.27). With the loss function  $Z = f_{t_{i+1}}(x, S_{t_{i+1}}) = xS_{t_{i+1}} - xS_{t_i} - P_{t_{i+1}}^*$ , we then deduce that for any  $0 \leq i \leq 3$

$$\begin{aligned} g_{t_i}(x) &\simeq \min_{\alpha \in \mathbf{R}} \left\{ \alpha + \frac{1}{J(1-\beta)} \sum_{j=1}^J z_j \right\} \\ \text{s.t. } z_j &\geq P_{t_{i+1}}^{*j} - xS_{t_{i+1}}^j + xS_{t_i} - \alpha, \quad j = 1, 2, \dots, J, \\ z_j &\geq 0, \quad j = 1, 2, \dots, J, \end{aligned} \quad (6.30)$$

or equivalently

$$\begin{aligned} g_{t_i}(x) &\simeq \min_{\alpha \in \mathbf{R}} \left\{ \alpha + xS_{t_i} + \frac{1}{J(1-\beta)} \sum_{j=1}^J z_j \right\} \\ \text{s.t. } z_j &\geq P_{t_{i+1}}^{*j} - xS_{t_{i+1}}^j - \alpha, \quad j = 1, 2, \dots, J, \\ z_j &\geq 0, \quad j = 1, 2, \dots, J \end{aligned} \quad (6.31)$$

such that

$$\begin{aligned} P_{t_i}^*(S_{t_i}) &\simeq \min_{x, \alpha \in \mathbf{R}} \left\{ \alpha + xS_{t_i} + \frac{1}{J(1-\beta)} \sum_{j=1}^J z_j \right\} \\ \text{s.t. } z_j &\geq P_{t_{i+1}}^{*j} - xS_{t_{i+1}}^j - \alpha, \quad j = 1, 2, \dots, J, \\ z_j &\geq 0, \quad j = 1, 2, \dots, J, \end{aligned} \quad (6.32)$$

with  $P_{t_4}^{*j} = (S_{t_4}^j - K)^+$  and  $P_{t_{i+1}}^{*j} = P_{t_{i+1}}^*(S_{t_{i+1}}^j)$  for each  $i = 0, 1, 2$ .

Note that (6.32) is a dynamic programming problem. In particular, the objective function and the constraint conditions are all linear although the number of variables is quite large. We may solve it with the help of the professional optimization solver CPLEX, see [28]. As the Section 6.2, we choose  $\beta = 0.95$  and  $J = 1000$ . And the family of price  $(S_{t_{i+1}}^j)_{j \in J}$  is generated by Monte Carlo simulation for each observed price  $S_{t_i} = s_{t_i}$ .

Starting from  $P_{t_4}^* = (S_{t_4} - K)^+$ , we then successively solve the dynamic programming problem (6.32) for  $t_3, \dots, t_0$ . In particular, we get the minimal risk-hedging prices  $P_{t_0}^*$ . The ratios  $P_{t_0}^*/S_{t_0}$  over all 51 weeks are computed, see Figure 7. The empirical average of  $P_{t_0}^*/S_{t_0}$  is 1.74% and its standard deviation is 0.63%.

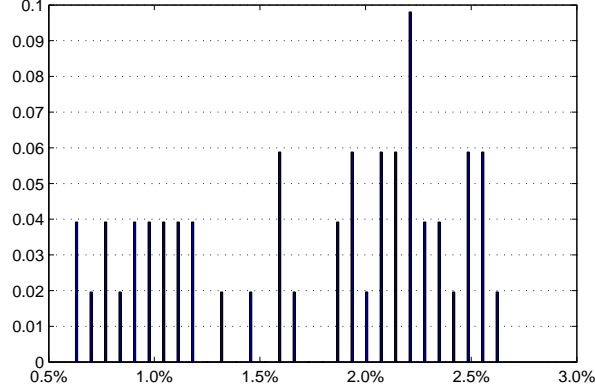


Fig 7: Distribution of the Ratio  $P_{t_0}^*/S_{t_0}$

In the above backward iteration process of solving risk-hedging price  $P_{t_0}^*$ , we may obtain a family of strategies  $(\theta_{t_i})_{0 \leq i \leq 3}$  by considering the observed price  $S_{t_i} = s_{t_i}$  for every  $0 \leq i \leq 3$ . Then, it is possible to express the risky-hedging error as

$$\varepsilon_T^h = P_{t_0}^* + \sum_{i=0}^3 \theta_{t_i} (S_{t_{i+1}} - S_{t_i}) - (S_{t_4} - K)^+$$

where  $S_{t_i} = s_{t_i}$ ,  $0 \leq i \leq 3$ , are all the observed prices of 50 ETF. The computed risky-hedging errors on every 51 weeks are shown in Figure 8. The empirical average of  $\varepsilon_T^h$  is 0.0316 and its standard deviation is 0.0167.

The results are clearly satisfactory despite the possible maximum error of 0.0593 which represents 2.24% of  $\mathbb{E}(S_{t_0}) = 2.65$ . It may be observed in Figure 8 that almost all risky-hedging errors are positive, except for the two weeks between the 8th of October 2018 and the 19th of October 2018. It can be explained by the the falling Chinese stock market by 3.7% the 8th of October 2018.

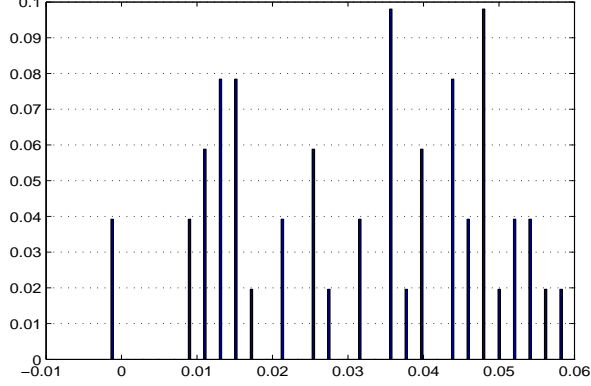


Fig 8: Distribution of the risk-hedging error  $\varepsilon_T^h$

#### 6.4. Comparison to the (exact) super-hedging prices

We compare the ratios  $P_{t_0}^*/S_{t_0}$  and the risk-hedging errors  $\varepsilon_T^h$  with the exact super-hedging prices when the coherent risk-measure is  $\rho_t(X) = -\text{ess inf}_{\mathcal{F}_t} X$ . To do so, we consider the model of [2], i.e.  $\text{ess inf}_{\mathcal{F}_{t-1}} S_t = k_{t-1,t}^d S_{t-1}$  and  $\text{ess sup}_{\mathcal{F}_{t-1}} S_t = k_{t-1,t}^u S_{t-1}$  for any  $t \in \{1, 2, \dots, T\}$ . It has been proved that AIP condition holds at every instant  $t$  if and only if  $k_{t-1,t}^d \in [0, 1]$  and  $k_{t-1,t}^u \in [1, +\infty]$ , see [2, Theorem 3.6]. Implementing Monte Carlo simulations for the equations (6.25) and (6.26) between the 255 trading days from January the 8th of 2018 to March the 29th of 2019, we may estimate the parameters  $k_{t-1,t}^d$  and  $k_{t-1,t}^u$ . The values of  $\text{ess inf}_{\mathcal{F}_{t-1}} S_t$  and  $\text{ess sup}_{\mathcal{F}_{t-1}} S_t$  are estimated respectively as the minimum and maximum simulated prices  $S_t^j$ . We deduced the parameters  $k_{t-1,t}^d$  and  $k_{t-1,t}^u$  as  $S_{t-1} = s_{t-1}$  is an observed price.

Similarly, consider 50 ETF at-the-money option with  $S_0 = K$  and the maturity is one week composed of 5 trading days. Let us implement the super-hedging strategy in [2, Theorem 3.6] on every 51 weeks from January the 8th of 2018 to March the 29th of 2019. The ratios  $\hat{P}_0^*/S_0$  and the super-hedging errors

$$\hat{\varepsilon}_T^h = \hat{P}_{t_0}^* + \sum_{i=0}^3 \hat{\theta}_{t_i} (S_{t_{i+1}} - S_{t_i}) - (S_{t_4} - K)^+$$

are presented in Figure 9 and 10.

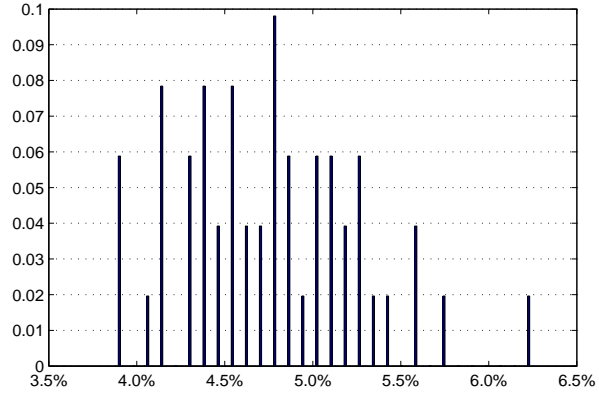


Fig 9: Distribution of the ratio  $\hat{P}_{t_0}^*/S_{t_0}$

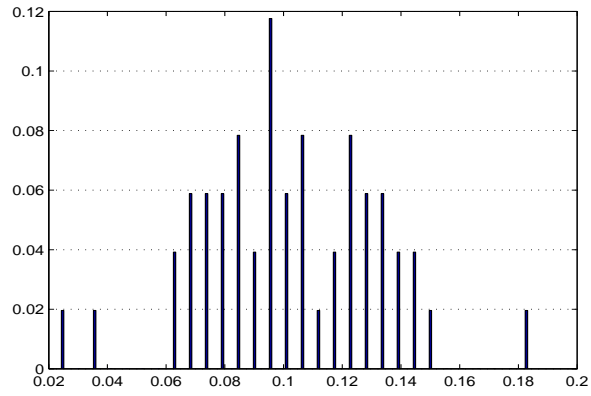


Fig 10: Distribution of the Super-hedging Error  $\hat{\varepsilon}_T^h$

The empirical average of  $\hat{P}_0^*/S_0$  is 4.76% and its standard deviation is 0.51%. As expected, it is larger than the ratio we obtain with the CVaR. The empirical average of  $\hat{\varepsilon}_T^h$  is 0.1019 and its standard deviation is 0.03. The possible maximum error of 0.1854 accounts for 7% of  $\mathbb{E}(S_{t_0}) = 2.65$ . In conclusion, the CVaR strategy provides here better results.

## Appendix A: Auxiliary theory

### A.1. Conditional essential supremum/infimum

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and let  $\mathcal{H}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$ . The concept of generalized conditional essential supremum in  $L^0(\mathbf{R}^d)$  is given in [21, Definition 3.1]. Existence and uniqueness when  $d = 1$  is provided by [21, Lemma 3.9].

**Proposition A.1.** *Let  $\Gamma \neq \emptyset$  be a subset of  $L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{F})$ . There exists a unique  $\mathcal{H}$ -measurable random variable  $\hat{\gamma} \in L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{H})$  denoted as  $\text{ess sup}_{\mathcal{H}} \Gamma$  such that the following conditions hold:*

- (i)  $\hat{\gamma} \geq \gamma$  a.s. for any  $\gamma \in \Gamma$ ;
- (ii) if  $\tilde{\gamma} \in L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{H})$  satisfies  $\tilde{\gamma} \geq \gamma$  for any  $\gamma \in \Gamma$ , then  $\tilde{\gamma} \geq \hat{\gamma}$  a.s.

### A.2. Random sets

A random set is defined as follows. We suggest to read [25, Section 2.1] for more details.

**Definition A.2.** *A set-valued function  $\omega \mapsto X(\omega) \subset \mathbf{R}^d$  from a complete probability space  $(\Omega, \mathcal{F}, P)$  to the family of all subsets of  $\mathbf{R}^d$  is called  $\mathcal{F}$ -measurable random set if its graph*

$$\text{graph } X = \{(\omega, x) \in \Omega \times \mathbf{R}^d : x \in X(\omega)\} \subset \Omega \times \mathbf{R}^d$$

*belongs to the product  $\sigma$ -algebra  $\mathcal{F} \otimes \mathcal{B}(\mathbf{R}^d)$ .*

The random set  $X$  is said to be closed if  $X(\omega)$  is closed for almost all  $\omega$ .

### A.3. Technical results

Let  $X$  be an integrable random variable on the probability space  $(\Omega, \mathcal{F}, P)$ . Let  $K \in \mathbf{R}$  and  $k > 1$ . We define the function

$$g(\alpha) = \alpha + k\gamma(\alpha), \quad \gamma(\alpha) = \mathbb{E}(X - \alpha)^+.$$

As  $k > 1$ , we may show that  $\lim_{|\alpha| \rightarrow +\infty} g(\alpha) = +\infty$ . Therefore, the continuous and convex function  $g$  admits a minimal element  $\min(g)$  reached at some point  $\alpha^* \in \mathbf{R}$ . We consider a sequence  $(X_i)_{i \geq 1}$  of i.i.d. random variables distributed as  $X$ . Let us introduce

$$g^J(\alpha) = \alpha + \frac{k}{J} \sum_{i=1}^J (X_i - \alpha)^+, \quad J \geq 1.$$

We may also show that  $\lim_{|\alpha| \rightarrow +\infty} g^J(\alpha) = +\infty$  and, by a measurable selection argument, there exists  $\alpha_J^* \in L^0(\mathbf{R}, \mathcal{F})$  such that  $\min(g^J) = g^J(\alpha_J^*)$ .

**Theorem A.3.** *We have  $\min(g^J) \rightarrow \min(g)$  a.s. as  $J \rightarrow \infty$ .*

*Proof.* Let us fix  $\epsilon > 0$  arbitrarily small. By the law of large numbers, there exists a.s.  $J_1(\omega)$  such that  $J \geq J_1(\omega)$  implies that  $\min(g) = g(\alpha^*) \geq g^J(\alpha^*) - \epsilon \geq \min(g^J) - \epsilon$ . On the other hand, for every rational number  $\alpha \in \mathbb{Q}$ , there exists  $J_0(\alpha, \omega)$  such that  $J \geq J_0(\alpha, \omega)$  implies  $g(\alpha) \leq g^J(\alpha) + \epsilon/2$ . We may suppose that this property holds with probability one for all  $\alpha \in \mathbb{Q}$  as  $\mathbb{Q}$  is countable. By continuity of  $g^J$ , let us define  $\tilde{\alpha}^J \in L^0(\mathbb{Q}, \mathcal{F})$  such that  $g^J(\tilde{\alpha}^J) \leq \min(g^J) + \epsilon/2$ . We then define  $J_0(\omega) = J_1(\omega) \vee J_0(\tilde{\alpha}^J(\omega), \omega)$ . For  $J \geq J_0(\omega)$ , we have  $\min(g) \leq g^J(\tilde{\alpha}^J(\omega)) + \epsilon/2 \leq \min(g^J) + \epsilon$ . The conclusion follows.  $\square$

**Theorem A.4.** *The sequence  $\alpha_J^* \in L^0(\mathbf{R}, \mathcal{F})$  is almost surely convergent, at least for a random subsequence, to  $\alpha_\infty^* \in L^0(\mathbf{R}, \mathcal{F})$  such that  $g(\alpha_\infty^*) = \min(g)$ .*

*Proof.* By definition, we have

$$\alpha_J^* = \min(g^J) - \frac{k}{J} \sum_{i=1}^J (X_i - \alpha_J^*)^+.$$

By Theorem A.3, the sequence  $(\min(g^J))_{J \geq 1}$  is a.s. convergent. Moreover,

$$\frac{1}{J} \sum_{i=1}^J (X_i - \alpha_J^*)^+ \leq \frac{1}{J} \sum_{i=1}^J |X_i|,$$

and the r.h.s. of the inequality above is a.s. convergent by the law of large numbers. We deduce that the sequence  $(\alpha_J^*)_{J \geq 1}$  is a.s. bounded. By a compactness argument, we deduce a random subsequence  $J_m$  such that  $\alpha_{J_m}^* \rightarrow \alpha_\infty^*$ . Moreover, we observe  $\alpha \mapsto g^J$  is a Lipschitz function with Lipschitz coefficient  $k + 1$ . We deduce that  $\lim_{J \rightarrow \infty} g^J(\alpha_{J_m}^*) = \lim_{J \rightarrow \infty} g^J(\alpha_\infty^*)$  such that  $\min(g) = g(\alpha_\infty^*)$ .  $\square$

**Lemma A.5.** *The function  $g$  admits a unique  $\alpha^*$  such that  $\min g = g(\alpha^*)$ .*

*Proof.* Suppose that there exists  $\alpha_1 < \alpha_2$  such that  $\min g = g(\alpha_1) = g(\alpha_2)$ . Let us define  $\alpha = (\alpha_1 + \alpha_2)/2$ . By convexity,  $g(\alpha) \leq \min g$ . Therefore, we get that  $g(\alpha) = \min g = (g(\alpha_1) + g(\alpha_2))/2$  and we deduce that

$$E(X - (\alpha_1 + \alpha_2)/2)^+ = \frac{1}{2}E(X - \alpha_1)^+ + \frac{1}{2}E(X - \alpha_2)^+.$$

This implies that

$$(X - (\alpha_1 + \alpha_2)/2)^+ = \frac{1}{2}(X - \alpha_1)^+ + \frac{1}{2}(X - \alpha_2)^+, \text{ a.s.}$$

Therefore  $X \geq \alpha_2$  or  $X \leq \alpha_1$  a.s. Recall that  $g(\alpha) = Eh(\alpha)$  where  $h(\alpha) = \alpha + k(X - \alpha)^+$ . By above, we get that

$$\begin{aligned} h(\alpha_i) &= \alpha_i 1_{X \leq \alpha_1} + (\alpha_i(1 - k) + kX) 1_{X \geq \alpha_2}, \\ g(\alpha_i) &= \alpha_i(1 - kP(X \geq \alpha_2)) + kE(X 1_{X \geq \alpha_2}), \quad i = 1, 2. \end{aligned}$$

We claim that  $1 - kP(X \geq \alpha_2) \neq 0$ . Indeed, otherwise, we deduce that  $g$  is a constant function, hence  $\lim_{\alpha \rightarrow \infty} g(\alpha) \neq +\infty$ . Therefore, it must have  $\alpha_1 = \alpha_2$  as  $g(\alpha_1) - g(\alpha_2) = (\alpha_1 - \alpha_2)(1 - kP(X \geq \alpha_2)) = 0$ , which contradicts our assumption. On the other hand, the assumption that there exists  $\alpha_1 > \alpha_2$  such that  $\min g = g(\alpha_1) = g(\alpha_2)$  can be excluded by the similar arguments. Finally, we may conclude.  $\square$

The following corollary is a directed consequence of Lemma A.5 and Theorem A.4.

**Corollary A.6.** *We have  $\alpha_J^* \rightarrow \alpha^*$  a.s.*

## Appendix B: Main proofs

### B.1. Proof of Proposition 2.4

First we give the following two preliminary lemmas.

**Lemma B.1.** *For any  $X \in \text{Dom } \mathcal{A}_t$ , there exists a sequence  $C_n \in \mathcal{A}_t^X$  such that  $\rho_t(X) = \lim_{n \rightarrow \infty} \downarrow C_n$  everywhere on  $\Omega$ .*

*Proof.* We first observe that the set  $\mathcal{A}_t^X$  is  $\mathcal{F}_t$ -decomposable, i.e. if  $\Lambda_t \in \mathcal{F}_t$  and  $C_1, C_2 \in \mathcal{A}_t^X$ , then  $C_1 1_{\Lambda_t} + C_2 1_{\Omega \setminus \Lambda_t} \in \mathcal{A}_t^X$ . To see it, we use conditions 1)



and 4) of Definition 2.1. We then deduce that  $\mathcal{A}_t^X$  is directed downward, i.e. if  $C_1, C_2 \in \mathcal{A}_t^X$ , then  $C_1 \wedge C_2 \in \mathcal{A}_t^X$ . Indeed,  $C_1 \wedge C_2 = C_1 1_{C_1 \leq C_2} + C_2 1_{C_1 > C_2}$  with  $\{C_1 \leq C_2\} \in \mathcal{F}_t$ . Therefore, there exists a sequence  $C_n \in \mathcal{A}_t^X$  such that  $\rho_t(X) = \lim_{n \rightarrow \infty} \downarrow C_n$  everywhere on  $\Omega$ , see [20, Section 5.3.1.].  $\square$

**Lemma B.2.** *Let  $X \in \text{Dom } \mathcal{A}_t$  and  $F_t \in \mathcal{F}_t$ . Then,  $X 1_{F_t} \in \text{Dom } \mathcal{A}_t$  and  $\rho_t(X 1_{F_t}) = \rho_t(X) 1_{F_t}$ .*

*Proof.* Observe that  $C_t \in \mathcal{A}_t^X$  implies that  $C_t 1_{F_t} \in \mathcal{A}_t^{X 1_{F_t}}$ . We deduce that  $X 1_{F_t} \in \text{Dom } \mathcal{A}_t$ . In particular, we have  $1_{F_t} \mathcal{A}_t^X \subseteq \mathcal{A}_t^{X 1_{F_t}}$ . Reciprocally, consider  $C_t \in \mathcal{A}_t^{X 1_{F_t}}$ , then  $C_t 1_{F_t} \in \mathcal{A}_t^{X 1_{F_t}}$  and finally  $X + Z_t \in \mathcal{A}_t$  where  $Z_t = C_t 1_{F_t} + \rho_t(X) 1_{\Omega \setminus F_t}$ . Therefore,  $Z_t \in \mathcal{A}_t^X$  and  $C_t 1_{F_t} = Z_t 1_{F_t} \in 1_{F_t} \mathcal{A}_t^X$ , i.e.  $1_{F_t} \mathcal{A}_t^{X 1_{F_t}} \subseteq 1_{F_t} \mathcal{A}_t^X$ . We finally deduce that  $1_{F_t} \mathcal{A}_t^{X 1_{F_t}} = 1_{F_t} \mathcal{A}_t^X$ . Therefore,

$$\begin{aligned} 1_{F_t} \rho_t(X) &= 1_{F_t} \text{ess inf } \mathcal{A}_t^X = \text{ess inf } (1_{F_t} \mathcal{A}_t^X) \\ &= \text{ess inf } (1_{F_t} \mathcal{A}_t^{X 1_{F_t}}) = 1_{F_t} \text{ess inf } (\mathcal{A}_t^{X 1_{F_t}}) \\ &= 1_{F_t} \rho_t(X 1_{F_t}). \end{aligned}$$

Since  $1_{F_t} \mathcal{A}_t^{X 1_{F_t}} \subseteq \mathcal{A}_t^{X 1_{F_t}}$ ,  $\rho_t(X 1_{F_t}) \leq C_t 1_{F_t}$  if  $C_t \in \mathcal{A}_t^{X 1_{F_t}}$  hence  $\rho_t(X 1_{F_t}) \leq 0$  on  $\Omega \setminus F_t$ . At last, since  $X 1_{F_t} + \rho_t(X 1_{F_t}) \in \mathcal{A}_t$ , multiplying this property by  $1_{\Omega \setminus F_t}$ , we deduce that  $\rho_t(X 1_{F_t}) 1_{\Omega \setminus F_t} \in \mathcal{A}_t$  hence  $\rho_t(X 1_{F_t}) 1_{\Omega \setminus F_t} \geq 0$  and finally  $\rho_t(X 1_{F_t}) 1_{\Omega \setminus F_t} = 0$ . The conclusion follows.  $\square$

*Proof of Proposition 2.4:* The first four statements are directly deduced from the definition of  $\rho_t$  in (2.1). The positive homogeneity is easily seen in the case where  $k_t \in L^0((0, \infty), \mathcal{F}_t)$  as  $\mathcal{A}_t^{k_t X} = k_t \mathcal{A}_t^X$  when  $k_t > 0$  a.s. In the general case, i.e.  $k_t \geq 0$ , consider  $\tilde{k}_t = k_t 1_{k_t > 0} + 1_{k_t = 0} \in L^0((0, \infty), \mathcal{F}_t)$ . Then,  $\rho_t(\tilde{k}_t X) = \tilde{k}_t \rho_t(X)$  for any  $X \in \text{Dom } \mathcal{A}_t$ . Replacing  $X$  by  $X 1_{k_t \neq 0}$ , we get by Lemma B.2 that

$$\rho_t(k_t X) = \rho_t(k_t 1_{k_t \neq 0} X) = \tilde{k}_t \rho_t(X 1_{k_t \neq 0}) = k_t \rho_t(X).$$

By Lemma B.1, we know that  $\rho_t(X) = \lim_{n \rightarrow \infty} \downarrow C_n$  a.s. where  $C_n \in \mathcal{A}_t^X$  a.s. As the set  $\mathcal{A}_t$  is closed,  $\mathcal{F}_t$ -decomposable and contains 0, we deduce that  $X + \rho_t(X) \in \mathcal{A}_t$  on the set  $\rho_t(X) \in \mathbf{R}$ . Actually, we shall see below that  $\rho_t(X) > -\infty$  holds with probability one.

Let us consider a sequence  $X_n \in \text{Dom } \mathcal{A}_t$  which converges to  $X_0$ . With  $\alpha_n := \rho_t(X_n)$ ,  $X_n + \alpha_n \in \mathcal{A}_t$ . Notice that we may assume w.l.o.g. that

$\liminf \alpha_n < \infty$ . Indeed, otherwise, the inequality  $\rho_t(X_n) \leq \liminf \alpha_n = \infty$  trivially holds. Suppose that  $\alpha_\infty := \liminf \alpha_n = -\infty$ . Then, using the normalization procedure  $\tilde{\alpha}_n := \frac{\alpha_n}{|\alpha_n|}$ , we get that  $\frac{X_n}{|\alpha_n|} + \tilde{\alpha}_n \in \mathcal{A}_t$ . As  $|\tilde{\alpha}_n| = 1$  and  $\mathcal{A}_t$  is closed,  $\mathcal{F}_t$ -decomposable, we may assume by [20, Lemma 2.1.2], up to some  $\mathcal{F}_t$ -measurable random subsequence, that  $\tilde{\alpha}_n$  converges to some  $\tilde{\alpha}$  with  $|\tilde{\alpha}| = 1$ . Actually,  $\tilde{\alpha} = -1$  a.s. as  $\tilde{\alpha}_n < 0$  for  $n$  large enough. Then  $\lim_n(\frac{X_n}{|\alpha_n|} + \tilde{\alpha}_n) = \tilde{\alpha} = -1 \in \mathcal{A}_t$  if  $\mathcal{A}_t$  is closed, which contradicts the third condition in Definition 2.1. So, we may assume w.l.o.g. that  $\alpha_\infty \in L^0(\mathbf{R}, \mathcal{F}_t)$  and  $X_0 + \alpha_\infty \in \mathcal{A}_t$ . It follows that  $\alpha_\infty \in \mathcal{A}_t^{X_0}$  and  $\rho_t(X_0) \leq \alpha_\infty = \liminf \rho_t(X_n)$  a.s. Note that on the negligible set  $\{\rho_t(X_0) > \alpha_\infty\}$ , we may replace  $\rho_t(X_0)$  by  $\alpha_\infty$  so that we may suppose that the inequality holds everywhere, i.e.  $\rho_t$  satisfies the l.s.c. inequality on  $\Omega$ . The constraint  $\rho_t(X) > -\infty$  for all  $X \in \text{Dom } \mathcal{A}_t$  is proven similarly using Lemma B.1.

At last, if the set  $\mathcal{A}_t$  is closed,  $X + \rho_t(X) \in \mathcal{A}_t$  on  $\rho_t(X) < \infty$  for all  $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ . Then,  $\mathcal{A}_t = \{X \in \text{Dom } \mathcal{A}_t | \rho_t(X) \leq 0\}$ . Indeed, it is clear that  $\rho_t(X) \leq 0$  for all  $X \in \mathcal{A}_t$ . Reciprocally, if  $\rho_t(X) \leq 0$  we get that  $X = -\rho_t(X) + a_t$  where  $a_t \in \mathcal{A}_t$ . Finally, we deduce that  $X \in \mathcal{A}_t$  since  $0 \leq -\rho_t(X) \in \mathcal{A}_t$  and  $\mathcal{A}_t + \mathcal{A}_t \subseteq \mathcal{A}_t$ .  $\square$

## B.2. Proof of Proposition 2.9

**Lemma B.3.** *Let  $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$  and  $F_t \in \mathcal{F}_t$ . Then,  $\rho_t(X1_{F_t}) = \rho_t(X)1_{F_t}$ .*

*Proof.* First, observe that

$$\Lambda_t(X1_{F_t}) = \left\{ \tilde{F}_t \in \mathcal{F}_t : X1_{F_t}1_{\tilde{F}_t} \in \text{Dom } \mathcal{A}_t \right\} = \left\{ \tilde{F}_t \in \mathcal{F}_t : F_t \cap \tilde{F}_t \in \Lambda_t(X) \right\}.$$

It follows that  $F_t(X1_{F_t}) = (F_t(X) \cap F_t) \cup (\Omega \setminus F_t)$ . Therefore,

$$\begin{aligned} \rho_t(X1_{F_t}) &= \rho_t(X1_{F_t}1_{F_t(X1_{F_t})})1_{F_t(X1_{F_t})} + \infty 1_{\Omega \setminus F_t(X1_{F_t})} \\ &= \rho_t(X1_{F_t(X) \cap F_t})1_{F_t(X1_{F_t})} + \infty 1_{\Omega \setminus F_t(X1_{F_t})}. \end{aligned}$$

By Lemma B.2 applied to  $X1_{F_t(X)} \in \text{Dom } \mathcal{A}_t$ , we deduce that

$$\rho_t(X1_{F_t(X) \cap F_t})1_{F_t(X1_{F_t})} = \rho_t(X)1_{F_t(X) \cap F_t}.$$

On the other hand,  $\Omega \setminus F_t(X1_{F_t}) = (\Omega \setminus F_t(X)) \cap F_t$  hence

$$\infty 1_{\Omega \setminus F_t(X1_{F_t})} = \rho_t(X)1_{\Omega \setminus F_t(X)}1_{F_t}.$$

The conclusion follows.  $\square$

*Proof of Proposition 2.9:* Recall that, for every  $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ ,

$$\rho_t(X) = \rho_t(X1_{F_t(X)})1_{F_t(X)} + (+\infty)1_{\Omega \setminus F_t(X)}.$$

Let us verify the statements of Proposition 2.4. Normalization is trivial as  $F_t(0) = \Omega$ . If  $X, X' \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$  with  $X \leq X'$ , we deduce by Definition 2.1, Condition 2), that  $\mathcal{A}_t^X \subseteq \mathcal{A}_t^{X'}$ . Therefore,  $\Lambda_t(X) \subseteq \Lambda_t(X')$  and, finally,  $F_t(X) \subseteq F_t(X')$ . We deduce that

$$\rho_t(X') = \rho_t(X'1_{F_t(X)})1_{F_t(X)} + \rho_t(X'1_{F_t(X') \setminus F_t(X)})1_{F_t(X') \setminus F_t(X)} + (+\infty)1_{\Omega \setminus F_t(X')}$$

where  $X'1_{F_t(X)} \in \text{Dom } \mathcal{A}_t$ . Since  $X1_{F_t(X)} \leq X'1_{F_t(X)}$ , we deduce that  $\rho_t(X1_{F_t(X)}) \geq \rho_t(X'1_{F_t(X)})$  so that  $\rho_t(X) \geq \rho_t(X')$  on  $F_t(X)$ , and the inequality still holds on  $\Omega$  as  $\rho_t(X) = +\infty$  on  $\Omega \setminus F_t(X)$ .

For all  $m_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ , it has  $F_t(X + m_t) = F_t(X)$  as  $\mathcal{A}_t^{X+m_t} \neq \emptyset$  if and only if  $\mathcal{A}_t^X \neq \emptyset$ . Therefore, by Proposition 2.4,

$$\begin{aligned} \rho_t(X) - m_t &= \rho_t(X1_{F_t(X)})1_{F_t(X)} + (+\infty)1_{\Omega \setminus F_t(X)} - m_t1_{F_t(X)} - m_t1_{\Omega \setminus F_t(X)} \\ &= (\rho_t(X1_{F_t(X)}) - m_t1_{F_t(X)})1_{F_t(X)} + (+\infty)1_{\Omega \setminus F_t(X)} \\ &= \rho_t((X + m_t)1_{F_t(X)})1_{F_t(X)} + (+\infty)1_{\Omega \setminus F_t(X)} = \rho_t(X + m_t) \end{aligned}$$

For any  $X, X' \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ , we have  $\mathcal{A}_t^X + \mathcal{A}_t^{X'} \subseteq \mathcal{A}_t^{X+X'}$ . We deduce that  $F_t(X) \cap F_t(X') \subseteq F_t(X+X')$  and  $\rho_t(X+X') \leq \rho_t(X) + \rho_t(X')$  by Proposition 2.4 on the set  $F_t(X) \cap F_t(X')$ . Otherwise, we see that  $\rho_t(X) + \rho_t(X') = +\infty$  on the set  $\Omega \setminus (F_t(X) \cap F_t(X')) = (\Omega \setminus F_t(X)) \cup (\Omega \setminus F_t(X'))$ . Thus, the inequality  $\rho_t(X+X') \leq \rho_t(X) + \rho_t(X')$  holds on  $\Omega$ .

Note that, if  $k_t \in L^0((0, \infty), \mathcal{F}_t)$ ,  $\mathcal{A}_t^{k_t X} \neq \emptyset$  if and only if  $\mathcal{A}_t^X \neq \emptyset$ . Therefore,  $\Lambda_t(X) = \Lambda_t(k_t X)$  and  $F_t(k_t X) = F_t(X)$ . Therefore, we have

$$\begin{aligned} \rho_t(k_t X) &= \rho_t(k_t X1_{F_t(X)})1_{F_t(X)} + (+\infty)1_{\Omega \setminus F_t(X)}, \\ &= k_t \rho_t(X1_{F_t(X)})1_{F_t(X)} + k_t(+\infty)1_{\Omega \setminus F_t(X)} = k_t \rho_t(X). \end{aligned}$$

Let us now consider the general case  $k_t \geq 0$ . Consider  $\tilde{k}_t = k_t1_{k_t > 0} + 1_{k_t = 0} > 0$ . By the first step, we get that  $\rho_t(\tilde{k}_t X) = \tilde{k}_t \rho_t(X)$  and  $\Lambda_t(X) = \Lambda_t(\tilde{k}_t X)$  whatever  $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ . Replacing  $X$  by  $X1_{k_t \neq 0}$  and using Lemma B.3, we get that

$$\rho_t(k_t X) = \rho_t(k_t1_{k_t \neq 0} X) = \tilde{k}_t \rho_t(X1_{k_t \neq 0}) = k_t \rho_t(X).$$

Finally, l.s.c. and finiteness  $\rho_t(X) > -\infty$  hold for any  $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$  as  $\rho_t(X) = +\infty$  on  $\Omega \setminus F_t(X)$ . Also,  $\mathcal{A}_t = \{X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T) | \rho_t(X) \leq 0\}$ .  $\square$

### B.3. Proof of Lemma 4.1

**Lemma B.4.** For all  $X, Y \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  such that  $X \neq Y$  a.s., we have

$$S_{T-1} - \rho_{T-1}(-S_T) \leq \frac{g_{T-1}(Y) - g_{T-1}(X)}{Y - X} \leq S_{T-1} + \rho_{T-1}(S_T).$$

*Proof.* Suppose that  $X, Y \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ . On the set  $\{Y > X\} \in \mathcal{F}_{T-1}$ , we have:

$$\begin{aligned} \rho_{T-1}(YS_T - h_T) &= \rho_{T-1}(XS_T - h_T + (Y - X)S_T) \\ &\leq \rho_{T-1}(XS_T - h_T) + \rho_{T-1}((Y - X)S_T) \\ &\leq \rho_{T-1}(XS_T - h_T) + (Y - X)\rho_{T-1}(S_T). \end{aligned}$$

We deduce that  $g_{T-1}(Y) \leq g_{T-1}(X) + (Y - X)(S_{T-1} + \rho_{T-1}(S_T))$  and finally

$$\frac{g_{T-1}(Y) - g_{T-1}(X)}{Y - X} \leq S_{T-1} + \rho_{T-1}(S_T). \quad (\text{B.33})$$

By symmetry, we deduce that (B.33) holds also on  $\{Y < X\}$ . Similarly, on the set  $\{Y < X\}$ ,

$$\begin{aligned} \rho_{T-1}(YS_T - h_T) &\leq \rho_{T-1}(XS_T - h_T) + \rho_{T-1}((Y - X)S_T) \\ &\leq \rho_{T-1}(XS_T - h_T) + (X - Y)\rho_{T-1}(-S_T). \end{aligned}$$

Therefore,

$$\frac{g_{T-1}(Y) - g_{T-1}(X)}{Y - X} \geq S_{T-1} - \rho_{T-1}(-S_T). \quad (\text{B.34})$$

By symmetry, we deduce that (B.34) holds also on  $\{Y > X\}$ .  $\square$

**Corollary B.5.** We have  $g_{T-1}(x) \in \mathbf{R}$ , for all  $x \in \mathbf{R}$  a.s. In particular,  $g_{T-1}$  is a.s. continuous.

*Proof.* By Lemma B.4, we have  $|g_{T-1}(X)| \leq k_{T-1}|X| + |\rho_{T-1}(-h_T)|$  for all  $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  so that  $g_{T-1}(X) \in \mathbf{R}$ , where

$$k_{T-1} := \max(|S_{T-1} + \rho_{T-1}(S_T)|, |S_{T-1} - \rho_{T-1}(-S_T)|).$$

Consider the set

$$A := \{\omega \in \Omega : \forall x \in \mathbf{R}, g_{T-1}(\omega, x) < \infty\} = \{\omega \in \Omega : \forall x \in \mathbf{Q}, g_{T-1}(\omega, x) < \infty\}.$$

The equality above holds because  $g_{T-1}$  is convex and any  $x \in \mathbf{R}$  is a convex combination of two rational numbers. We deduce that  $B = \Omega \setminus A \in \mathcal{F}_{T-1}$ . If  $B \neq \emptyset$ , as the jointly measurable set  $\{(\omega, x) \in \Omega \times \mathbf{R} : g_{T-1}(\omega, x) = \infty\}$  admit non empty  $\omega$ -sections, we may use a measurable selection argument. We then deduce  $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  such that  $g_{T-1}(X) = +\infty$  on  $B$ , which yields a contradiction.  $\square$

*Proof of Lemma 4.1:* Consider the set  $B = B_1 \cup B_2$  where

$$\begin{aligned} B_1 &= \bigcup_{y > x} \{\omega \in \Omega : g_{T-1}(y) - g_{T-1}(x) > (S_{T-1} + \rho_{T-1}(S_T))(y - x)\}, \\ B_2 &= \bigcup_{y > x} \{\omega \in \Omega : g_{T-1}(y) - g_{T-1}(x) < (S_{T-1} - \rho_{T-1}(-S_T))(y - x)\}, \end{aligned}$$

By Corollary B.5,  $g$  is a.s. continuous hence  $B_1$  and  $B_2$  may be rewritten as countable unions. Therefore,  $B$  is  $\mathcal{F}_{T-1}$ -measurable. Suppose that  $P(B) > 0$  and let us introduce the jointly measurable set  $G_{T-1} = G_{T-1}^1 \cup G_{T-1}^2$  where

$$\begin{aligned} G_{T-1}^1 &= \{(\omega, x, y) \in \Omega \times \mathbf{R}^2 : g_{T-1}(y) - g_{T-1}(x) > (S_{T-1} + \rho_{T-1}(S_T))(y - x)\}, \\ G_{T-1}^2 &= \{(\omega, x, y) \in \Omega \times \mathbf{R}^2 : g_{T-1}(y) - g_{T-1}(x) < (S_{T-1} - \rho_{T-1}(-S_T))(y - x)\}. \end{aligned}$$

As the  $\omega$ -sections of  $G_{T-1}$  are not empty on  $B$ , we deduce by a measurable selection argument the existence of  $X, Y \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  such that  $(\omega, X(\omega), Y(\omega)) \in G_{T-1}$  a.s. on  $B$  and we set  $X = Y = 0$  otherwise. This leads to a contradiction by Lemma B.4.  $\square$

#### B.4. Proof of theorem 4.8

Suppose that SAIP holds. When  $S_{T-1} + \rho_{T-1}(S_T)$  and  $S_{T-1} - \rho_{T-1}(-S_T)$  are both equal to 0, we know that  $g_{T-1}(x)$  is a constant function by Corollary 4.5 such that  $P_{T-1}^* = g_{T-1}(0) = \rho_{T-1}(-h_T) > 0$ . Otherwise,  $S_{T-1} + \rho_{T-1}(S_T) > 0$  and  $S_{T-1} - \rho_{T-1}(-S_T) < 0$ . Since  $P_{T-1}^*$  is a price by Theorem 4.6, there exists  $\theta_{T-1}^* \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$  such that

$$\begin{aligned} P_{T-1}^* &= \theta_{T-1}^* S_{T-1} + \rho_{T-1}(\theta_{T-1}^* S_T - h_T) \\ &\geq \rho_{T-1}(-h_T) 1_{\theta_{T-1}^* = 0} + (\theta_{T-1}^* S_{T-1} + \rho_{T-1}(\theta_{T-1}^* S_T)) 1_{\theta_{T-1}^* > 0} \\ &\quad + (\theta_{T-1}^* S_{T-1} + \rho_{T-1}(\theta_{T-1}^* S_T)) 1_{\theta_{T-1}^* < 0} \\ &= \rho_{T-1}(-h_T) 1_{\theta_{T-1}^* = 0} + (S_{T-1} + \rho_{T-1}(S_T)) \theta_{T-1}^* 1_{\theta_{T-1}^* > 0} \\ &\quad + (S_{T-1} - \rho_{T-1}(-S_T)) \theta_{T-1}^* 1_{\theta_{T-1}^* < 0} > 0. \end{aligned}$$

For the reverse implication, let us prove first that AIP holds. We fix  $h_T$  such that  $\rho_{T-1}(-h_T) > 0$ . Recall that  $P_{T-1}^* = \inf_{x \in \mathbf{R}} g_{T-1}(x) > 0$  hence  $g_{T-1}(x) > 0$  for all  $x \in \mathbf{R} \cup \{\pm\infty\}$ . Let us show that the sets  $\{S_{T-1} + \rho_{T-1}(S_T) < 0\}$  and  $\{S_{T-1} - \rho_{T-1}(-S_T) > 0\}$  are empty a.s. Otherwise, in the case where  $P(S_{T-1} + \rho_{T-1}(S_T) < 0) > 0$ , we define

$$\theta_{T-1} := -\frac{\rho_{T-1}(-h_T)}{S_{T-1} + \rho_{T-1}(S_T)} 1_{\{S_{T-1} + \rho_{T-1}(S_T) < 0\}} \geq 0.$$

We have

$$\begin{aligned} g_{T-1}(\theta_{T-1}) &= \theta_{T-1} S_{T-1} + \rho_{T-1}(\theta_{T-1} S_T - h_T), \\ &\leq \theta_{T-1} S_{T-1} + \rho_{T-1}(\theta_{T-1} S_T) + \rho_{T-1}(-h_T), \\ &\leq \theta_{T-1} (S_{T-1} + \rho_{T-1}(S_T)) + \rho_{T-1}(-h_T), \\ &\leq \rho_{T-1}(-h_T) 1_{\{S_{T-1} + \rho_{T-1}(S_T) \geq 0\}}. \end{aligned}$$

Therefore,  $P_{T-1}^* \leq 0$  on the set  $\{S_{T-1} + \rho_{T-1}(S_T) < 0\}$  in contradiction with the assumption. Similarly if  $P(S_{T-1} - \rho_{T-1}(-S_T) > 0) > 0$  we define

$$\theta_{T-1} := -\frac{\rho_{T-1}(-h_T)}{S_{T-1} - \rho_{T-1}(-S_T)} 1_{\{S_{T-1} - \rho_{T-1}(-S_T) > 0\}} \leq 0.$$

We also get that  $P_{T-1}^* \leq 0$  when  $S_{T-1} - \rho_{T-1}(-S_T) > 0$ , i.e. a contradiction.

Let us show that  $S_{T-1} - \rho_{T-1}(-S_T) = 0$  if  $S_{T-1} + \rho_{T-1}(S_T) = 0$ . Otherwise, suppose that  $\Lambda_{T-1} := \{S_{T-1} + \rho_{T-1}(S_T) = 0\} \cap \{S_{T-1} - \rho_{T-1}(-S_T) < 0\}$  satisfies  $P(\Lambda_{T-1}) > 0$ . If  $h_T = \Delta S_T$ , then  $\rho_{T-1}(-h_T) = \rho_{T-1}(-\Delta S_T) > 0$  on  $\Lambda_{T-1}$ . On the complimentary set, we fix  $h_T = \gamma_{T-1} > 0$ ,  $\gamma_{T-1} \in L^0((0, \infty), \mathcal{F}_{T-1})$ . It follows that  $\rho_{T-1}(-h_T) > 0$  hence  $P_{T-1}^*(h_T) > 0$ . On the other hand, if  $\theta_{T-1} \geq 1$ ,  $P_{T-1}^*(h_T) \leq \rho_{T-1}(\theta_{T-1} \Delta S_T - \Delta S_T) = (\theta_{T-1} - 1) \rho_{T-1}(\Delta S_T) = 0$  on  $\Lambda_{T-1}$ . It follows that  $P_{T-1}^*(h_T) \leq 0$  on  $\Lambda_{T-1}$ , i.e. a contradiction. Similarly, we show that  $\Gamma_{T-1} := \{S_{T-1} - \rho_{T-1}(-S_T) = 0\} \cap \{S_{T-1} + \rho_{T-1}(S_T) > 0\}$  is empty a.s. Finally, we conclude that  $S_{T-1} + \rho_{T-1}(S_T) = 0$  if and only if  $S_{T-1} - \rho_{T-1}(-S_T) = 0$ .  $\square$

### B.5. Proof of theorem 5.1

Notice that  $m_{T-1}^* \geq 0$  when  $h_T \geq 0$  hence AIP holds with  $h_T = 0$ . Let us show the reverse implication. To do so, we observe that

$$g_{T-1}(x) = x S_{T-1} + \rho_{T-1} \left( \left( x - \frac{h_T}{S_T} \right) S_T \right).$$

Since  $\text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \leq \frac{h_T}{S_T} \leq \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}$ , we deduce that

$$\left(x - \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}\right) S_T \leq \left(x - \frac{h_T}{S_T}\right) S_T \leq \left(x - \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}\right) S_T.$$

Therefore,

$$\rho_{T-1} \left( \left(x - \frac{h_T}{S_T}\right) S_T \right) \geq \rho_{T-1} \left( \left(x - \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}\right) S_T \right)$$

and

$$\rho_{T-1} \left( \left(x - \frac{h_T}{S_T}\right) S_T \right) \leq \rho_{T-1} \left( \left(x - \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}\right) S_T \right).$$

This implies that

$$g_{T-1}(x) \geq x S_{T-1} + \rho_{T-1} \left( \left(x - \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}\right) S_T \right) := m_{T-1}(x) \quad (\text{B.35})$$

and

$$g_{T-1}(x) \leq x S_{T-1} + \rho_{T-1} \left( \left(x - \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}\right) S_T \right) := M_{T-1}(x), \quad (\text{B.36})$$

where the r.h.s.  $m_{T-1}(x)$  and  $M_{T-1}(x)$  of the inequalities (B.35) and (B.36) need to be understood as jointly measurable and l.s.c. random functions w.r.t.  $(\omega, x)$ . To see it, it suffices to repeat the construction of  $g_{T-1}$ . As we have  $m_{T-1}(x) \leq g_{T-1}(x) \leq M_{T-1}(x)$  for all  $x$ , we get by Lemma 3.9 that

$$m_{T-1}^* := \inf_{x \in \mathbf{R}} m_{T-1}(x) \leq P_{T-1}^* \leq \inf_{x \in \mathbf{R}} M_{T-1}(x) =: M_{T-1}^*.$$

Notice that

$$m_{T-1}(x) = \begin{cases} x(S_{T-1} + \rho_{T-1}(S_T)) - \left(\text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}\right) \rho_{T-1}(S_T), & \text{if } x \geq \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \\ x(S_{T-1} - \rho_{T-1}(-S_T)) + \left(\text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}\right) \rho_{T-1}(-S_T), & \text{if } x < \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}. \end{cases}$$

By Theorem 3.11,  $S_{T-1} + \rho_{T-1}(S_T) \geq 0$  and  $S_{T-1} - \rho_{T-1}(-S_T) \leq 0$  under AIP. Thus, the infimum of  $m_{T-1}(x)$  is reached at point  $\underline{x}^* := \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}$  and

$$m_{T-1}^* = S_{T-1} \left( \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right).$$

Similarly, the upper bound function  $M_{T-1}$  may be expressed as

$$M_{T-1}(x) = \begin{cases} x(S_{T-1} + \rho_{T-1}(S_T)) - \left( \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) \rho_{T-1}(S_T), & \text{if } x \geq \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \\ x(S_{T-1} - \rho_{T-1}(-S_T)) + \left( \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) \rho_{T-1}(-S_T), & \text{if } x < \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}. \end{cases}$$

Similarly, under AIP, the infimum of  $M_{T-1}(x)$  is reached at  $\bar{x}^* := \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}$  and

$$M_{T-1}^* = S_{T-1} \left( \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right).$$

The conclusion follows.  $\square$

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