

Walls Dynamics in Ferromagnetic Nanotubes

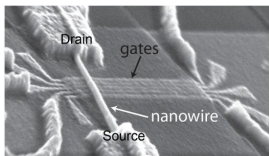
Gilles Carbou

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UMR CNRS 5142

Plan

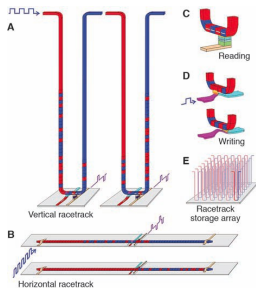
- 1 Physical Context
- 2 Model for Ferromagnetic Nanotubes
- 3 Walls in Nanotubes
- 4 Stability Proof
- 5 Conclusion

Ferromagnetic nanowires



Industrial Applications

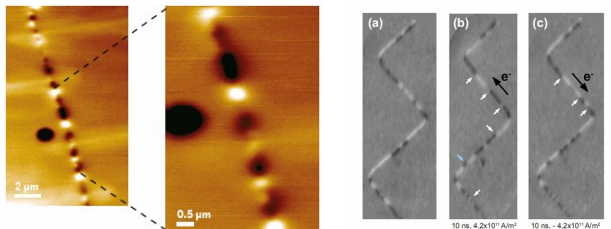
- Mobile phones
- Modern Recording Devices



Stuart S. P. Parkin, Masamitsu Hayashi, Luc Thomas, Magnetic Domain-Wall Racetrack Memory, *Science* **320** (2008)

- 3d storage devices
- Fast access to the information

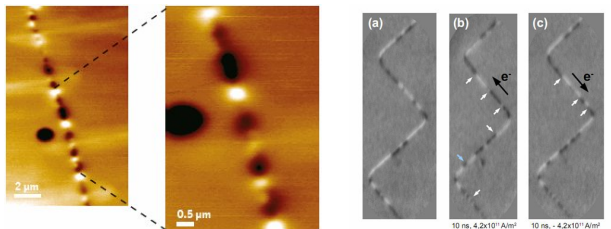
Ferromagnetic nanowires



Al Sayed, Carbou, Jizzini, Labbé, Privat, Takasao, Trélat

- 1d Models
- Description of walls dynamics
- Stability results
- Controllability results

Ferromagnetic nanowires



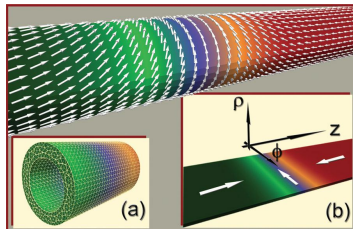
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Alternative : nanotubes

M. Yan, C. Andreas, A. Kákay, F. García-Sánchez and R. Hertel, Fast domain wall dynamics in magnetic nanotubes : Suppression of Walker breakdown and Cherenkov-like spin wave emission, Appl. Phys. Lett. **99** (2011)

- nanotubes instead of nanowires or nano stripes
- avoid boundary destabilizing effects

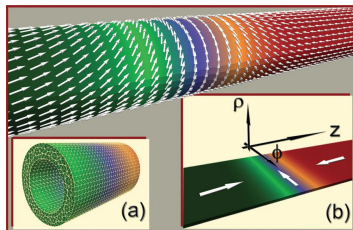


Goal : Modeling of nanotubes, description of walls dynamics

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3d model

Magnetic Moment

- Magnetic material : $\Omega \subset \mathbb{R}^3$
- Magnetization described by the magnetic moment $m : \mathbb{R}_t^+ \times \Omega_x \rightarrow \mathbb{R}^3$

$$|m(t, x)| = 1 \text{ p. p.}$$

- Constitutive Relation : $B = H + m$

Landau-Lifschitz

$$\frac{\partial m}{\partial t} = -m \times H_{\text{eff}} - m \times (m \times H_{\text{eff}})$$

$$H_{\text{eff}} = A\Delta m + H_d(m) + H_a$$

Ferromagnetism Energy

$$E(m) = \frac{A}{2} \int_{\Omega} |\nabla m|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |H_d(m)|^2 dx - \int_{\Omega} H_a \cdot m$$

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Exchange field

- Nonlinear parabolic behaviour
- Homogeneous Neumann boundary condition

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Demagnetizing field

- field generated by the magnetization
- $\text{curl } H_d(m) = 0$
- $\text{div } (H_d(m) + m) = 0$

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Zeeman term

H_a : applied magnetic field

Ferromagnetism Energy

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Model of nanotube

Characteristics : Length \gg diameter \gg thickness

$$\text{Length} = \infty, \quad \text{thickness} = 0$$

Asymptotic model of ferromagnetic thin film (Carbou, Gioia-James)

- Nanotube assimilated to $\mathbb{R} \times \rho S^1$
- Demagnetizing field equivalent to a planar anisotropy

$$\begin{aligned} \bullet \operatorname{curl} H_d(m) &= 0 \\ \bullet \operatorname{div} (H_d(m) + m) &= 0 \end{aligned}$$

 \implies

$$\begin{aligned} \bullet H_d(m) &= -(m \cdot \nu)\nu \\ \bullet \nu &: \text{unitary normal} \\ \bullet E_d(m) &= \int_{\mathbb{R} \times \rho S^1} |m \cdot \nu|^2 \end{aligned}$$

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$$\partial_t m = -m \times H_{\text{eff}}(m) - m \times (m \times H_{\text{eff}}(m)),$$

$$H_{\text{eff}}(m) = A\Delta m + H_d(m) + H_{\text{app}}$$

Thin film model for the nanotube

$m : (t, x, y) \rightarrow S^2$, 2π -periodic in y

$$\partial_t m = -m \times h(m) - m \times (m \times h(m)),$$

$$h(m) = A \frac{\partial^2 m}{\partial x^2} + \frac{A}{\rho^2} \frac{\partial^2 m}{\partial y^2} - (m \cdot \nu(y))\nu(y) + H_a e_1$$

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$$\nu(y) = \begin{pmatrix} 0 \\ \cos y \\ \sin y \end{pmatrix}$$

Model of nanotube

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Model of nanotube

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2d Model of nanotube

$\mathbf{m} : (t, x, y) \rightarrow S^2$, 2π -periodic in y

$$\partial_t \mathbf{m} = -\mathbf{m} \times \mathbf{h}(\mathbf{m}) - \mathbf{m} \times (\mathbf{m} \times \mathbf{h}(\mathbf{m}))$$

$$\mathbf{h}(\mathbf{m}) = \partial_{xx} \mathbf{m} + \partial_{yy} \mathbf{m} + 2e_1 \times \partial_y \mathbf{m} + \mathbf{m}_1 e_1 - \gamma \mathbf{m}_3 e_3 + h_a e_1, \quad \gamma = \frac{\rho^2}{A}$$

Model of Nanotube

2d Model of nanotube

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- demagnetizing field \implies planar anisotropy
- Curvature \implies uniaxial anisotropy

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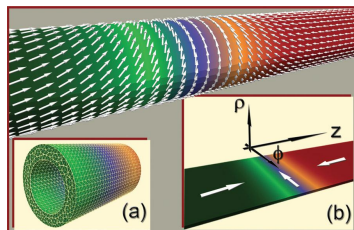
One-Wall profiles

2d-model of nanotube

$\mathbf{m} : (t, x, y) \rightarrow S^2, 2\pi\text{-periodic in } y$

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One-Wall profiles

2d-model of nanotube

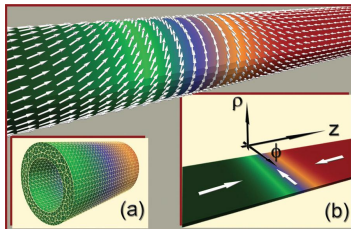
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$h_a = 0$: static case

$$\mathbf{M}_0(x) = \begin{pmatrix} \tanh x \\ \frac{1}{\cosh x} \\ 0 \end{pmatrix}$$



One-Wall profiles

2d-model of nanotube

$\mathbf{m} : (t, x, y) \rightarrow S^2$, 2π -periodic in y

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$h_a \neq 0$: moving wall

$$\mathbf{M}^{h_a}(t, x, y) = \mathbf{R}_\theta(\mathbf{M}_0(\frac{x - ct}{\delta}))$$

$$\mathbf{R}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}, \quad \mathbf{M}_0(x) = \begin{pmatrix} \tanh x \\ 1 \\ \cosh x \\ 0 \end{pmatrix}$$

$$h_a = \sin \theta \cos \theta, \quad \frac{c}{\delta} = -2h_a, \quad \frac{1}{\delta^2} = 1 + \sin^2 \theta$$

Stability Result

2d-model of nanotube

$$\partial_t \mathbf{m} = -\mathbf{m} \times \mathbf{h}(\mathbf{m}) - \mathbf{m} \times (\mathbf{m} \times \mathbf{h}(\mathbf{m}))$$

$$\mathbf{h}(\mathbf{m}) = \Delta \mathbf{m} + 2\mathbf{e}_1 \times \partial_y \mathbf{m} + \mathbf{m}_1 \mathbf{e}_1 - \gamma \mathbf{m}_3 \mathbf{e}_3 + h_a \mathbf{e}_1$$

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Theorem

If $|h_a| \leq h_{max}$, \mathbf{M}^{h_a} is stable and asymptotically stable modulo translations, *i.e.* If $\mathbf{m}(0, x, y)$ is close to $\mathbf{M}^{h_a}(0, x, y)$, then

- $\|\mathbf{m}(t, \cdot) - \mathbf{M}^{h_a}(t, \cdot)\|_{H^2}$ remains small
- there exists σ_∞ such that :

$$\|\mathbf{m}(t, x, y) - \mathbf{M}^{h_a}(t, x - \sigma_\infty, y)\|_{H^2} \rightarrow 0 \quad \text{when} \quad t \rightarrow +\infty.$$

Comparison with other results

2d-model of nanotube

$$\partial_t \mathbf{m} = -\mathbf{m} \times \mathbf{h}(\mathbf{m}) - \mathbf{m} \times (\mathbf{m} \times \mathbf{h}(\mathbf{m}))$$

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Results about 1d models for nanowires

Round-section nanowires

$$\partial_t \mathbf{m} = -\mathbf{m} \times \mathbf{h}(\mathbf{m}) - \mathbf{m} \times (\mathbf{m} \times \mathbf{h}(\mathbf{m}))$$

$$\mathbf{h}(\mathbf{m}) = \partial_{xx} \mathbf{m} + \mathbf{m}_1 \mathbf{e}_1 + h_a \mathbf{e}_1$$

Carbou, Labbé, Jizzini, Privat, Trélat

$$\mathbf{M}(t, x) = \mathbf{R}_{h_a t}(\mathbf{M}_0(x + h_a t))$$

Asymptotic stability modulo translation and rotation

Instability threshold : $|h_a| < 1$

Controlability

Comparison with other results

2d-model of nanotube

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Results about 1d models for nanowires

Elliptical-section wires

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K. Takasao

$$\mathbf{M}(t, x) = \mathbf{R}_\theta(\mathbf{M}_0(\frac{x-ct}{\delta}))$$

Asymptotic stability modulo translation
 h_a small enough

Comparison with other results

2d-Model

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Result in 3d

3d-model : $\Omega = \mathbb{R}^3$

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Asymptotic Stability

\mathbf{e}_3 easy-axis

Simplified model : $h_d(m) \sim -\mathbf{m}_1 \mathbf{e}_1$

$h_a = 0$ (Only static walls)

$$\mathbf{M}(t, x, y, z) = \begin{pmatrix} 0 \\ 1 \\ \cosh x \\ -\tanh x \end{pmatrix}$$

Comparison with other results

2d-Model

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Difficulties

- Saturation constraint $|\mathbf{m}| = 1$
- Invariances $\implies 0$ is an eigenvalue of the linearized equation
- Sign of the other eigenvalues ? Coercivity ?
- Quasilinear equation - Variational estimates

2d-Model

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$$\mathbf{m}(t, x, y) = \mathbf{R}_\theta \mathbf{v}(t, \frac{x-ct}{\delta}, y), \quad h_a = \sin \theta \cos \theta, \quad \frac{c}{\delta} = -2h_a, \quad \frac{1}{\delta^2} = 1 + \sin^2 \theta$$

Model for \mathbf{v}

$$|\mathbf{v}| = 1$$

$$\partial_t \mathbf{v} + 2h_a \partial_x \mathbf{v} = -\mathbf{v} \times \mathbf{H}(\mathbf{v}) - \mathbf{v} \times (\mathbf{v} \times \mathbf{H}(\mathbf{v}))$$

$$\mathbf{H}(\mathbf{v}) = \frac{1}{\delta^2} \partial_{xx} \mathbf{v} + \partial_{yy} \mathbf{v} + 2\mathbf{e}_1 \times \partial_y \mathbf{v} + \mathbf{v}_1 \mathbf{e}_1 - \gamma(\sin \theta \mathbf{v}_2 + \cos \theta \mathbf{v}_3)(\sin \theta \mathbf{e}_2 + \cos \theta \mathbf{e}_3) + h_a \mathbf{e}_1.$$

\mathbf{M}^{h_a} stable \iff \mathbf{M}_0 stable the new problem

2d-Model

$$\partial_t \mathbf{m} = -\mathbf{m} \times \mathbf{h}(\mathbf{m}) - \mathbf{m} \times (\mathbf{m} \times \mathbf{h}(\mathbf{m}))$$

$$\mathbf{h}(\mathbf{m}) = \Delta \mathbf{m} + 2\mathbf{e}_1 \times \partial_y \mathbf{m} + \mathbf{m}_1 \mathbf{e}_1 - \gamma \mathbf{m}_3 \mathbf{e}_3 + h_a \mathbf{e}_1$$

$$\mathbf{M}^{h_a}(t, x, y) = \mathbf{R}_\theta(\mathbf{M}_0(\frac{x-ct}{\delta}))$$

$$\mathbf{M}_0(x) = \begin{pmatrix} \tanh x \\ 1 \\ \cosh x \\ 0 \end{pmatrix}$$

$$\mathbf{m}(t, x, y) = \mathbf{R}_\theta \mathbf{v}(t, \frac{x-ct}{\delta}, y), \quad h_a = \sin \theta \cos \theta, \quad \frac{c}{\delta} = -2h_a, \quad \frac{1}{\delta^2} = 1 + \sin^2 \theta$$

Model for \mathbf{v}

$$|\mathbf{v}| = 1$$

$$\partial_t \mathbf{v} + 2h_a \partial_x \mathbf{v} = -\mathbf{v} \times \mathbf{H}(\mathbf{v}) - \mathbf{v} \times (\mathbf{v} \times \mathbf{H}(\mathbf{v}))$$

$$\mathbf{H}(\mathbf{v}) = \frac{1}{\delta^2} \partial_{xx} \mathbf{v} + \partial_{yy} \mathbf{v} + 2\mathbf{e}_1 \times \partial_y \mathbf{v} + \mathbf{v}_1 \mathbf{e}_1 - \gamma(\sin \theta \mathbf{v}_2 + \cos \theta \mathbf{v}_3)(\sin \theta \mathbf{e}_2 + \cos \theta \mathbf{e}_3) + h_a \mathbf{e}_1.$$

\mathbf{M}^{h_a} stable $\iff \mathbf{M}_0$ stable the new problem

Saturation constraint $|\mathbf{m}| = 1$ for the perturbations

v-Model : Stability for M_0 ?

$$|\mathbf{v}| = 1$$

$$\partial_t \mathbf{v} + 2h_a \partial_x \mathbf{v} = -\mathbf{v} \times \mathbf{H}(\mathbf{v}) - \mathbf{v} \times (\mathbf{v} \times \mathbf{H}(\mathbf{v}))$$

$$\mathbf{H}(\mathbf{v}) = \frac{1}{\delta^2} \partial_{xx} \mathbf{v} + \partial_{yy} \mathbf{v} + 2\mathbf{e}_1 \times \partial_y \mathbf{v} + \mathbf{v}_1 \mathbf{e}_1 - \gamma(\sin \theta \mathbf{v}_2 + \cos \theta \mathbf{v}_3)(\sin \theta \mathbf{e}_2 + \cos \theta \mathbf{e}_3) + h_a \mathbf{e}_1.$$

Mobile Frame

$$M_0(x) = \begin{pmatrix} \tanh x \\ 1 \\ \cosh x \\ 0 \end{pmatrix}, \quad M_1(x) = \begin{pmatrix} 1 \\ -\cosh x \\ \tanh x \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{v}(t, x, y) = M_0(x) + r_1(t, x, y, z)M_1(x) + r_2(t, x, y, z)M_2 + \mu(r)M_0(x)$$

$$\mu(r) = \sqrt{1 - |r|^2} - 1$$

Saturation constraint $|\mathbf{m}| = 1$ for the perturbations

v-Model : Stability for M_0 ?

$$\partial_t \mathbf{v} + 2h_a \partial_x \mathbf{v} = -\mathbf{v} \times \mathbf{H}(\mathbf{v}) - \mathbf{v} \times (\mathbf{v} \times \mathbf{H}(\mathbf{v}))$$

$$\mathbf{H}(\mathbf{v}) = \frac{1}{\delta^2} \partial_{xx} \mathbf{v} + \partial_{yy} \mathbf{v} + 2\mathbf{e}_1 \times \partial_y \mathbf{v} + \mathbf{v}_1 \mathbf{e}_1 - \gamma(\sin \theta \mathbf{v}_2 + \cos \theta \mathbf{v}_3)(\sin \theta \mathbf{e}_2 + \cos \theta \mathbf{e}_3) + h_a \mathbf{e}_1.$$

$$\mathbf{v}(t, x, y) = M_0(x) + r_1(t, x, y, z)M_1(x) + r_2(t, x, y, z)M_2 + \mu(r)M_0(x)$$

Equivalent model : $\partial_t r = \Lambda(r) + 2h_a \mathcal{L}(r) + F,$

$$\Lambda(r) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(r), \quad \mathcal{M} = \begin{pmatrix} \frac{1}{\delta^2} L - \partial_{yy} & 2\text{th } x \partial_y \\ -2\text{th } x \partial_y & \frac{1}{\delta^2} L - \partial_{yy} + \gamma \cos 2\theta \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} \ell & \text{th } x \\ 0 & \partial_x \end{pmatrix} \quad \ell = \partial_x + \text{th } x, \quad L = -\partial_{xx} + (2 \tanh^2 - 1)$$

r takes its values in \mathbb{R}^2 ; Stability of the 0-solution ?

Invariance by translation $\implies 0$ is an eigenvalue of the linearized equation

v-model

$$\partial_t \mathbf{v} + 2h_a \partial_x \mathbf{v} = -\mathbf{v} \times \mathbf{H}(\mathbf{v}) - \mathbf{v} \times (\mathbf{v} \times \mathbf{H}(\mathbf{v}))$$

$$\mathbf{H}(\mathbf{v}) = \frac{1}{\delta^2} \partial_{xx} \mathbf{v} + \partial_{yy} \mathbf{v} + 2\mathbf{e}_1 \times \partial_y \mathbf{v} + \mathbf{v}_1 \mathbf{e}_1 - \gamma(\sin \theta \mathbf{v}_2 + \cos \theta \mathbf{v}_3)(\sin \theta \mathbf{e}_2 + \cos \theta \mathbf{e}_3) + h_a \mathbf{e}_1.$$

Invariance by translation for Landau-Lifschitz

For $\sigma \in \mathbb{R}$, $(t, x, y) \mapsto M_0(x - \sigma)$ is also a solution for LL

One-parameter family of solutions for the r -equation

$$R(s)(x) = \begin{pmatrix} M_0(x - \sigma) \cdot M_1(x) \\ M_0(x - \sigma) \cdot M_2 \end{pmatrix}$$

0 is in the spectrum of $\Lambda(r) + 2h_a \mathcal{L}(r)$ associated to the eigenvector $\partial_\sigma R(0) = \begin{pmatrix} 1 \\ \text{ch } x \\ 0 \end{pmatrix}$

Invariance by translation $\implies 0$ is an eigenvalue of the linearized equation

r-Modele

$$\partial_t r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\delta^2} L - \partial_{yy} & 2\text{th } x \partial_y \\ -2\text{th } x \partial_y & \frac{1}{\delta^2} L - \partial_{yy} + \gamma \cos 2\theta \end{pmatrix} r + 2h_a \begin{pmatrix} \ell & \text{th } x \\ 0 & \partial_x \end{pmatrix} r + F$$

$$\mathcal{L} = \begin{pmatrix} \ell & \text{th } x \\ 0 & \partial_x \end{pmatrix}, \quad \ell = \partial_x + \text{th } x, \quad L = -\partial_{xx} + (2 \tanh^2 - 1)$$

$$\partial_\sigma R(0) = \begin{pmatrix} \frac{1}{\text{ch } x} \\ 0 \end{pmatrix}, \quad \ell\left(\frac{1}{\text{ch } x}\right) = L\left(\frac{1}{\text{ch } x}\right) = 0$$

Invariance by translation $\implies 0$ is an eigenvalue of the linearized equation

r -Modele

$$\partial_t r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\delta^2} L - \partial_{yy} & 2\text{th } x \partial_y \\ -2\text{th } x \partial_y & \frac{1}{\delta^2} L - \partial_{yy} + \gamma \cos 2\theta \end{pmatrix} r + 2h_a \begin{pmatrix} \ell & \text{th } x \\ 0 & \partial_x \end{pmatrix} r + F$$

$$\mathcal{L} = \begin{pmatrix} \ell & \text{th } x \\ 0 & \partial_x \end{pmatrix}, \quad \ell = \partial_x + \text{th } x, \quad L = -\partial_{xx} + (2 \tanh^2 - 1)$$

$$r(t, x, y) = R(\sigma(t)) + w(t, x, y), \quad \text{with} \quad \int_{\mathbb{R} \times [0, 2\pi]} \frac{w_1(t, x, y)}{\text{ch } x} dx dy = 0.$$

(cf. D. Henry (1d) and T. Kapitula (multi-d) for Cahn-Hilliard equation)

Equation for (σ, w)

$$\partial_t w = \Lambda w + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}$$

$$\frac{d\sigma}{dt} = \mathcal{G}(w)$$

To be proved :

- $w \rightarrow 0$ in H^2
- $\sigma \rightarrow \sigma_\infty$

Coercivity for w

$$\partial_t w = \Lambda w + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}, \quad w_1 \in \left(\frac{1}{\text{ch } x}\right)^\perp$$

$$\Lambda w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w), \quad \mathcal{M} = \begin{pmatrix} \frac{1}{\delta^2} L - \partial_{yy} & 2\text{th } x \partial_y \\ -2\text{th } x \partial_y & \frac{1}{\delta^2} L - \partial_{yy} + \gamma \cos 2\theta \end{pmatrix}$$

Fundamental Lemma

If $\langle w_1 | \frac{1}{\text{ch } x} \rangle = 0$,

$$\langle \mathcal{M}w | w \rangle \geq c \cos 2\theta \|w\|_{\mathbf{H}_p^1}^2.$$

$$\langle \mathcal{M}w | w \rangle = \frac{1}{\delta^2} \langle Lw_1 | w_1 \rangle + \|\partial_y w_1\|_{\mathbf{L}_p^2}^2 + \langle Lw_2 | w_2 \rangle + \|\partial_y w_2\|_{\mathbf{L}_p^2}^2 + \gamma \cos 2\theta \|w_2\|_{\mathbf{L}_p^2}^2 + TC$$

Coercivity for w

$$\Lambda w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w), \quad \mathcal{M} = \begin{pmatrix} \frac{1}{\delta^2} L - \partial_{yy} & 2\text{th } x \partial_y \\ -2\text{th } x \partial_y & \frac{1}{\delta^2} L - \partial_{yy} + \gamma \cos 2\theta \end{pmatrix}$$

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$$TC = 2 \int \text{th } x \partial_y w_2 \cdot w_1 - 2 \int \text{th } x \partial_y w_1 \cdot w_2$$

Coercivity for w

Fundamental Lemma

If $\langle w_1 | \frac{1}{\text{ch } x} \rangle = 0$,

$$\langle \mathcal{M}w | w \rangle \geq c \cos 2\theta \|w\|_{\mathbf{H}_p^1}^2.$$

$$\langle \mathcal{M}w | w \rangle = \frac{1}{\delta^2} \langle Lw_1 | w_1 \rangle + \|\partial_y w_1\|_{\mathbf{L}_p^2}^2 + \langle Lw_2 | w_2 \rangle + \|\partial_y w_2\|_{\mathbf{L}_p^2}^2 + \gamma \cos 2\theta \|w_2\|_{\mathbf{L}_p^2}^2 + TC$$

$$TC = 2(\alpha + 1 - \alpha) \int \text{th } x \partial_y w_2 \cdot w_1 - 2(\alpha + 1 - \alpha) \int \text{th } x \partial_y w_1 \cdot w_2$$

$$TC = -4\alpha \int \text{th } x \partial_y w_1 \cdot w_2 + 2(1 - \alpha) \int \partial_y w_2 \cdot \ell w_1 - 2(1 - \alpha) \int \partial_y w_1 \cdot \ell w_2$$

Coercivity for w

Fundamental Lemma

If $\langle w_1 | \frac{1}{\text{ch } x} \rangle = 0$,

$$\langle \mathcal{M}w | w \rangle \geq c \cos 2\theta \|w\|_{\mathbf{H}_p^1}^2.$$

$$\langle \mathcal{M}w | w \rangle = \frac{1}{\delta^2} \langle Lw_1 | w_1 \rangle + \|\partial_y w_1\|_{\mathbf{L}_p^2}^2 + \langle Lw_2 | w_2 \rangle + \|\partial_y w_2\|_{\mathbf{L}_p^2}^2 + \gamma \cos 2\theta \|w_2\|_{\mathbf{L}_p^2}^2 + TC$$

$$TC = 2(\alpha + 1 - \alpha) \int \text{th } x \partial_y w_2 \cdot w_1 - 2(\alpha + 1 - \alpha) \int \text{th } x \partial_y w_1 \cdot w_2$$

$$TC = -4\alpha \int \text{th } x \partial_y w_1 \cdot w_2 + 2(1 - \alpha) \int \partial_y w_2 \cdot \ell w_1 - 2(1 - \alpha) \int \partial_y w_1 \cdot \ell w_2$$

$$\ell = \partial_x + \text{th } x, \quad \int \partial_x w_1 \partial_y w_2 - \int \partial_y w_1 \partial_x w_2 = 0$$

Coercivity for w

Fundamental Lemma

$$\text{If } \langle w_1 | \frac{1}{\text{ch } x} \rangle = 0,$$

$$\langle \mathcal{M}w | w \rangle \geq c \cos 2\theta \|w\|_{\mathbf{H}_p^1}^2.$$

$$\langle \mathcal{M}w | w \rangle = \frac{1}{\delta^2} \langle Lw_1 | w_1 \rangle + \|\partial_y w_1\|_{\mathbf{L}_p^2}^2 + \langle Lw_2 | w_2 \rangle + \|\partial_y w_2\|_{\mathbf{L}_p^2}^2 + \gamma \cos 2\theta \|w_2\|_{\mathbf{L}_p^2}^2 + TC$$

$$TC = -4\alpha \int \text{th } x \partial_y w_1 \cdot w_2 + 2(1 - \alpha) \int \partial_y w_2 \cdot \ell w_1 - 2(1 - \alpha) \int \partial_y w_1 \cdot \ell w_2$$

$$|TC| \leq 4\alpha \|\partial_y w_1\|_{\mathbf{L}_p^2} \|w_2\|_{\mathbf{L}_p^2} + 2(1 - \alpha) \|\partial_y w_2\|_{\mathbf{L}_p^2} \|\ell w_1\|_{\mathbf{L}_p^2} + 2(1 - \alpha) \|\partial_y w_1\|_{\mathbf{L}_p^2} \|\ell w_2\|_{\mathbf{L}_p^2}$$

$$L = \ell^* \circ \ell, \quad \langle Lw | w \rangle = \|\ell w\|_{\mathbf{L}_p^2}^2$$

Coercivity for w

Fundamental Lemma

If $\langle w_1 | \frac{1}{\text{ch } x} \rangle = 0$,

$$\langle \mathcal{M}w | w \rangle \geq c \cos 2\theta \|w\|_{\mathbf{H}_p^1}^2.$$

$$\langle \mathcal{M}w | w \rangle = \frac{1}{\delta^2} \langle Lw_1 | w_1 \rangle + \|\partial_y w_1\|_{L_p^2}^2 + \langle Lw_2 | w_2 \rangle + \|\partial_y w_2\|_{L_p^2}^2 + \gamma \cos 2\theta \|w_2\|_{L_p^2}^2 + TC$$

$$|TC| \leq 4\alpha \|\partial_y w_1\|_{L_p^2} \|w_2\|_{L_p^2} + 2(1 - \alpha) \|\partial_y w_2\|_{L_p^2} \|\ell w_1\|_{L_p^2} + 2(1 - \alpha) \|\partial_y w_1\|_{L_p^2} \|\ell w_2\|_{L_p^2}$$

$$L = \ell^* \circ \ell, \quad \langle Lw | w \rangle = \|\ell w\|_{L_p^2}^2$$

$$\alpha = \frac{\gamma \cos 2\theta}{16}$$

Coercivity for w

Fundamental Lemma

If $\langle w_1 | \frac{1}{\text{ch } x} \rangle = 0$,

$$\langle \mathcal{M}w | w \rangle \geq c \cos 2\theta \|w\|_{\mathbf{H}_p^1}^2.$$

$$\langle \mathcal{M}w | w \rangle = \frac{1}{\delta^2} \langle Lw_1 | w_1 \rangle + \|\partial_y w_1\|_{L_p^2}^2 + \langle Lw_2 | w_2 \rangle + \|\partial_y w_2\|_{L_p^2}^2 + \gamma \cos 2\theta \|w_2\|_{L_p^2}^2 + TC$$

$$\langle \mathcal{M}w | w \rangle \geq \frac{\gamma \cos 2\theta}{32} \left(\|\ell w_1\|_{L_p^2}^2 + \|\partial_y w_1\|_{L_p^2}^2 + \|\ell w_2\|_{L_p^2}^2 + \|\partial_y w_2\|_{L_p^2}^2 + \|w_2\|_{L_p^2}^2 \right)$$

In $(\frac{1}{\text{ch } x})^\perp$, $L \geq 1$ so $\|\ell w_1\|_{L_p^2}$ and $\|\partial_y w_1\|_{L_p^2}$ control $\|w_1\|_{\mathbf{H}_p^1}$

Variational Estimates

$$\partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w) + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}$$

Inner product with $\mathcal{M}(w)$:

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{L_p^2}^2 \leq 2|h_a| |\langle \mathcal{L}^\perp(w) | \mathcal{M}(w) \rangle| + |\langle \mathcal{F} | \mathcal{M}(w) \rangle|$$

$$\mathcal{L} = \begin{pmatrix} \ell & \text{th } x \\ 0 & \partial_x \end{pmatrix}, \quad \|\mathcal{L}w\|_{L_p^2} \leq c \|w\|_{H_p^1} \leq k \|\mathcal{M}(w)\|_{L_p^2}$$

$$\|\mathcal{F}\|_{L_p^2} \leq K(|\sigma| + \|w\|_{L^\infty}) \|w\|_{H_p^2}$$

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{L_p^2}^2 \leq c \|w\|_{H_p^2}^2 (|h_a| + |\sigma| + \|w\|_{L^\infty})$$

Variational Estimates

$$\partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w) + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}$$

Inner product with $\mathcal{M}(w)$:

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{\mathbf{L}_p^2}^2 \leq 2|h_a| |\langle \mathcal{L}^\perp(w) | \mathcal{M}(w) \rangle| + |\langle \mathcal{F} | \mathcal{M}(w) \rangle|$$

$$\mathcal{L} = \begin{pmatrix} \ell & \text{th } x \\ 0 & \partial_x \end{pmatrix}, \quad \|\mathcal{L}w\|_{\mathbf{L}_p^2} \leq c \|w\|_{\mathbf{H}_p^1} \leq k \|\mathcal{M}(w)\|_{\mathbf{L}_p^2}$$

$$\|\mathcal{F}\|_{\mathbf{L}_p^2} \leq K(|\sigma| + \|w\|_{L^\infty}) \|w\|_{\mathbf{H}_p^2}$$

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{\mathbf{L}_p^2}^2 \leq c \|w\|_{\mathbf{H}_p^2}^2 (|h_a| + |\sigma| + \|w\|_{L^\infty})$$

Variational Estimates

$$\partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w) + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}$$

Inner product with $\mathcal{M}(w)$:

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{\mathbf{L}_p^2}^2 \leq 2|h_a| |\langle \mathcal{L}^\perp(w) | \mathcal{M}(w) \rangle| + |\langle \mathcal{F} | \mathcal{M}(w) \rangle|$$

$$\mathcal{L} = \begin{pmatrix} \ell & \text{th } x \\ 0 & \partial_x \end{pmatrix}, \quad \|\mathcal{L}w\|_{\mathbf{L}_p^2} \leq c \|w\|_{\mathbf{H}_p^1} \leq k \|\mathcal{M}(w)\|_{\mathbf{L}_p^2}$$

$$\|\mathcal{F}\|_{\mathbf{L}_p^2} \leq K(|\sigma| + \|w\|_{L^\infty}) \|w\|_{\mathbf{H}_p^2}$$

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{\mathbf{L}_p^2}^2 \leq c \|w\|_{\mathbf{H}_p^2}^2 (|h_a| + |\sigma| + \|w\|_{L^\infty})$$

Variational Estimates

$$\partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w) + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}$$

Inner product with $\mathcal{M}(w)$:

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{L^2}^2 (1 - c(|h_a| + |\sigma| + \|w\|_{L^\infty})) \leq 0$$

While $|h_a|$, $|\sigma|$ and $\|w\|_{L^\infty}$ remain small, $\|w\|_{H_p^1}$ is exponentially decreasing

- $|h_a|$ small : we assume that !
- $|\sigma|$ small : we will see that with the equation for σ
- $\|w\|_{L^\infty}$ small : H^2 -estimate obtained by taking the inner product with $\mathcal{M}^2 w$

$$\frac{d}{dt} \|w\|_{H_p^2}^2 + k \|w\|_{H_p^3}^2 \left(1 - c(|h_a| + |\sigma| + \|w\|_{H_p^2}) \right) \leq 0.$$

Assumption : $|h_a| \leq \frac{1}{4c}$. While $|\sigma| < \frac{1}{4c}$ and $\|w\|_{H_p^2} < \frac{1}{4c}$,

$$\frac{d}{dt} \|w\|_{H_p^2}^2 + \frac{k}{4} \|w\|_{H_p^3}^2 \leq 0 \implies \|w\|_{H_p^2}^2(t) \leq e^{-\frac{kt}{4}} \|w(0)\|_{H_p^2}^2$$

Variational Estimates

$$\partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w) + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}$$

Inner product with $\mathcal{M}(w)$:

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{\mathbf{L}^2_p}^2 (1 - c(|h_a| + |\sigma| + \|w\|_{L^\infty})) \leq 0$$

While $|h_a|$, $|\sigma|$ and $\|w\|_{L^\infty}$ remain small, $\|w\|_{\mathbf{H}^1_p}$ is exponentially decreasing

- $|h_a|$ small : we assume that !
- $|\sigma|$ small : we will see that with the equation for σ
- $\|w\|_{L^\infty}$ small : H^2 -estimate obtained by taking the inner product with $\mathcal{M}^2 w$

$$\frac{d}{dt} \|w\|_{\mathbf{H}^2_p}^2 + k \|w\|_{\mathbf{H}^3_p}^2 \left(1 - c(|h_a| + |\sigma| + \|w\|_{\mathbf{H}^2_p})\right) \leq 0.$$

Assumption : $|h_a| \leq \frac{1}{4c}$. While $|\sigma| < \frac{1}{4c}$ and $\|w\|_{\mathbf{H}^2_p} < \frac{1}{4c}$,

$$\frac{d}{dt} \|w\|_{\mathbf{H}^2_p}^2 + \frac{k}{4} \|w\|_{\mathbf{H}^3_p}^2 \leq 0 \implies \|w\|_{\mathbf{H}^2_p}^2(t) \leq e^{-\frac{kt}{4}} \|w(0)\|_{\mathbf{H}^2_p}^2$$

Variational Estimates

$$\partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w) + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}$$

Inner product with $\mathcal{M}(w)$:

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{L^2}^2 (1 - c(|h_a| + |\sigma| + \|w\|_{L^\infty})) \leq 0$$

While $|h_a|$, $|\sigma|$ and $\|w\|_{L^\infty}$ remain small, $\|w\|_{H^1_p}$ is exponentially decreasing

- $|h_a|$ small : we assume that !
- $|\sigma|$ small : we will see that with the equation for σ
- $\|w\|_{L^\infty}$ small : H^2 -estimate obtained by taking the inner product with $\mathcal{M}^2 w$

$$\frac{d}{dt} \|w\|_{H^2_p}^2 + k \|w\|_{H^3_p}^2 (1 - c(|h_a| + |\sigma| + \|w\|_{H^2_p})) \leq 0.$$

Assumption : $|h_a| \leq \frac{1}{4c}$. While $|\sigma| < \frac{1}{4c}$ and $\|w\|_{H^2_p} < \frac{1}{4c}$,

$$\frac{d}{dt} \|w\|_{H^2_p}^2 + \frac{k}{4} \|w\|_{H^3_p}^2 \leq 0 \implies \|w\|_{H^2_p}^2(t) \leq e^{-\frac{kt}{4}} \|w(0)\|_{H^2_p}^2$$

Variational Estimates

$$\partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w) + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}$$

Inner product with $\mathcal{M}(w)$:

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{L^2}^2 (1 - c(|h_a| + |\sigma| + \|w\|_{L^\infty})) \leq 0$$

While $|h_a|$, $|\sigma|$ and $\|w\|_{L^\infty}$ remain small, $\|w\|_{H_p^1}$ is exponentially decreasing

- $|h_a|$ small : we assume that !
- $|\sigma|$ small : we will see that with the equation for σ
- $\|w\|_{L^\infty}$ small : H^2 -estimate obtained by taking the inner product with $\mathcal{M}^2 w$

$$\frac{d}{dt} \|w\|_{H_p^2}^2 + k \|w\|_{H_p^3}^2 (1 - c(|h_a| + |\sigma| + \|w\|_{H_p^2})) \leq 0.$$

Assumption : $|h_a| \leq \frac{1}{4c}$. While $|\sigma| < \frac{1}{4c}$ and $\|w\|_{H_p^2} < \frac{1}{4c}$,

$$\frac{d}{dt} \|w\|_{H_p^2}^2 + \frac{k}{4} \|w\|_{H_p^3}^2 \leq 0 \implies \|w\|_{H_p^2}^2(t) \leq e^{-\frac{kt}{4}} \|w(0)\|_{H_p^2}^2$$

Variational Estimates

$$\partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w) + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}$$

Inner product with $\mathcal{M}(w)$:

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{L^2}^2 (1 - c(|h_a| + |\sigma| + \|w\|_{L^\infty})) \leq 0$$

While $|h_a|$, $|\sigma|$ and $\|w\|_{L^\infty}$ remain small, $\|w\|_{H_p^1}$ is exponentially decreasing

- $|h_a|$ small : we assume that !
- $|\sigma|$ small : we will see that with the equation for σ
- $\|w\|_{L^\infty}$ small : H^2 -estimate obtained by taking the inner product with $\mathcal{M}^2 w$

$$\frac{d}{dt} \|w\|_{H_p^2}^2 + k \|w\|_{H_p^3}^2 (1 - c(|h_a| + |\sigma| + \|w\|_{H_p^2})) \leq 0.$$

Assumption : $|h_a| \leq \frac{1}{4c}$. While $|\sigma| < \frac{1}{4c}$ and $\|w\|_{H_p^2} < \frac{1}{4c}$,

$$\frac{d}{dt} \|w\|_{H_p^2}^2 + \frac{k}{4} \|w\|_{H_p^3}^2 \leq 0 \implies \|w\|_{H_p^2}^2(t) \leq e^{-\frac{kt}{4}} \|w(0)\|_{H_p^2}^2$$

Variational Estimates

$$\partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}(w) + 2h_a \mathcal{L}^\perp(w) + \mathcal{F}$$

Inner product with $\mathcal{M}(w)$:

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{M}w | w \rangle + \|\mathcal{M}w\|_{\mathbf{L}_p^2}^2 (1 - c(|h_a| + |\sigma| + \|w\|_{L^\infty})) \leq 0$$

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Control of σ

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Equation for σ

$$\frac{d\sigma}{dt} = \mathcal{G}(\sigma, w)$$

$$|\mathcal{G}(\sigma, w)| \leq K \|w\|_{\mathbf{H}_p^2}$$

$$\left| \frac{d\sigma}{dt} \right| \leq |\mathcal{G}| \leq K \|w\|_{\mathbf{H}_p^2}, \quad |\sigma(t)| \leq |\sigma(0)| + K \|w(0)\|_{\mathbf{H}_p^2}$$

If $|\sigma(0)|$ and $\|w(0)\|_{\mathbf{H}_p^2}$ are small enough, $|\sigma| < \frac{1}{4c}$, and the previous estimates remains valid for all times.

$$\frac{d\sigma}{dt} \text{ integrable} \implies \sigma \rightarrow \sigma_\infty \text{ when } t \rightarrow +\infty.$$

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Plan

- 1 Physical Context
- 2 Model for Ferromagnetic Nanotubes
- 3 Walls in Nanotubes
- 4 Stability Proof
- 5 Conclusion

Conclusion

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- The 2d-model is a bit more realistic than the 3d-model used for the stability of static walls in 3d
- multi-d model without boundary
- Large applied field ?

Stability threshold for $|h_a| < \frac{1}{2}$?

Behavior for $h_a > \frac{1}{2}$? Precessional Regime ?

André Thiaville and Yoshinobu Nakatani
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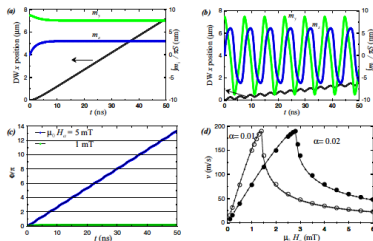


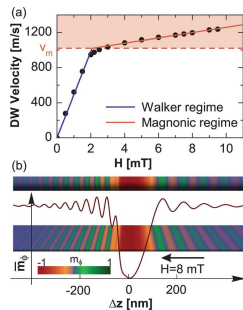
Fig. 11. Transverse-wall dynamics in a rectangular nanowire (edge sizes 10 nm and 5 nm, mesh size 5 nm). The field H_a applied along the wire axis is (a) 10 Oe ($\mu_0 H_a = 1$ mT) and (b) 50 Oe ($\mu_0 H_a = 5$ mT), the damping constant is $\alpha = 0.01$. The DW position is plotted as well as the integrated transverse magnetization components. The DW velocity is high at the low field (a), with no steady DW magnetization precession. At the higher field (b) this precession appears and the DW velocity drops and displays oscillations. The angle ϕ of the DW magnetization orientation is displayed in (c) and the DW velocity in (d), versus applied field H_a , with the damping coefficient α as a parameter. The analytical results (full thin lines and small dots) are drawn in (c) and (d), using the parameters of Table 1

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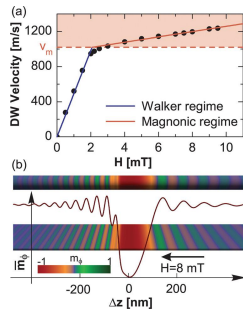
Better approximation for the demagnetizing field ?

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