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# MODELS OF NONLINEAR ACOUSTICS VIEWED AS AN APPROXIMATION OF THE KUZNETSOV EQUATION

Adrien Dekkers\*, Vladimir Khodygo<sup>†</sup> and Anna Rozanova-Pierrat<sup>‡</sup>

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## Abstract

We relate together different models of non linear acoustic in thermo-elastic media as the Kuznetsov equation, the Westervelt equation, the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation and the Nonlinear Progressive wave Equation (NPE) and estimate the time during which the solutions of these models keep closed in the  $L^2$  norm. The KZK and NPE equations are considered as paraxial approximations of the Kuznetsov equation. The Westervelt equation is obtained as a nonlinear approximation of the Kuznetsov equation. In the aim to compare the solutions of the exact and approximated systems in found approximation domains the well-posedness results (for the Kuznetsov equation in a half-space with periodic in time initial and boundary data) were obtained.

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# 1 Introduction.

One of the most general model to describe an acoustic wave propagation in an homogeneous thermo-elastic medium is the compressible Navier-Stokes system in  $\mathbb{R}^n$

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (1)$$

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla p + \eta \Delta \mathbf{v} + \left( \zeta + \frac{\eta}{3} \right) \nabla \cdot \operatorname{div}(\mathbf{v}), \quad (2)$$

$$\begin{aligned} \rho T[\partial_t S + (\mathbf{v} \cdot \nabla) S] &= \kappa \Delta T + \zeta (\operatorname{div} \mathbf{v})^2 \\ &+ \frac{\eta}{2} \left( \partial_{x_k} v_i + \partial_{x_i} v_k - \frac{2}{3} \delta_{ik} \partial_{x_i} v_i \right)^2, \end{aligned} \quad (3)$$

$$p = p(\rho, S), \quad (4)$$

where the pressure  $p$  is given by the state law  $p = p(\rho, S)$ . The density  $\rho$ , the velocity  $\mathbf{v}$ , the temperature  $T$  and the entropy  $S$  are unknown functions in system (1)–(4). The coefficients  $\beta$ ,  $\kappa$  and  $\eta$  are constant viscosity coefficients. For the acoustical framework the wave motion is supposed to be potential and the viscosity coefficients are supposed to be small in terms of a dimensionless small parameter  $\varepsilon > 0$ , which also characterizes the size of the perturbations near the constant state  $(\rho_0, 0, S_0, T_0)$ . Here the velocity  $\mathbf{v}_0$  is taken equal to 0 just using a Galilean transformation.

Actually,  $\varepsilon$  is the Mach number, which is supposed to be small [5] ( $\varepsilon = 10^{-5}$  for the propagation in water with an initial power of the order of  $0.3 \text{ W/cm}^2$ ):

$$\frac{\rho - \rho_0}{\rho_0} \sim \frac{T - T_0}{T_0} \sim \frac{|\mathbf{v}|}{c} \sim \varepsilon,$$

where  $c = \sqrt{p'(\rho_0)}$  is the speed of sound in the unperturbed media.

Hence as in [13, 41], system (1)–(4) becomes an isentropic Navier-Stokes system

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = 0, \quad (5)$$

$$\rho_\varepsilon[\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] = -\nabla p(\rho_\varepsilon) + \varepsilon \nu \Delta \mathbf{v}_\varepsilon, \quad (6)$$

with the approximate state equation  $p(\rho, S) = p(\rho_\varepsilon) + O(\varepsilon^3)$ :

$$p(\rho_\varepsilon) = p_0 + c^2(\rho_\varepsilon - \rho_0) + \frac{(\gamma - 1)c^2}{2\rho_0}(\rho_\varepsilon - \rho_0)^2, \quad (7)$$

where  $\gamma = C_p/C_V$  denotes the ratio of the heat capacities at constant pressure and at constant volume respectively and with a small enough and positive viscosity coefficient:

$$\varepsilon \nu = \beta + \kappa \left( \frac{1}{C_V} - \frac{1}{C_p} \right).$$

If we go on on physical assumptions of the wave motion [5, 19, 29, 48] for the perturbations of the density or of the velocity or of the pressure the isentropic system (5)–(6) brings us

1. the Westervelt equation for the potential of the velocity, derived initially by Westervelt [48] and later by other authors [1, 47]:

$$\partial_t^2 \Pi - c^2 \Delta \Pi = \varepsilon \partial_t \left( \frac{\nu}{\rho_0} \Delta \Pi + \frac{\gamma + 1}{2c^2} (\partial_t \Pi)^2 \right), \quad (8)$$

with the same constant introduced for the Navier-Stokes system.

2. the Kuznetsov equation also for the potential of the velocity, firstly introduced by Kuznetsov [29] for the velocity potential, see also Refs. [19, 24, 26, 31] for other different methods of its derivation:

$$\partial_t^2 u - c^2 \Delta u = \partial_t \left( (\nabla u)^2 + \frac{\gamma - 1}{2c^2} (\partial_t u)^2 + \frac{\varepsilon \nu}{\rho_0} \Delta u \right). \quad (9)$$

3. the Khokhlov-Zabolotskaya-Kuznetsov (KZK) [5, 40] for the density:

$$c \partial_{\tau z}^2 I - \frac{(\gamma + 1)}{4\rho_0} \partial_\tau^2 I^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0. \quad (10)$$

4. the Nonlinear Progressive wave Equation (NPE) derived in Ref. [37] also for the density:

$$\partial_{\tau z}^2 \xi + \frac{(\gamma + 1)c}{4\rho_0} \partial_z^2 [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0. \quad (11)$$

In [13] it is shown that the Kuznetsov equation comes from the Navier-Stokes or Euler system only by small perturbations, but to obtain the KZK and the NPE equations we also need to perform in addition to the small perturbations a paraxial change of variables. In this article we show that the KZK and the NPE equations can be also obtained from the Kuznetsov equation just performing the corresponding paraxial change of variables and that the Westervelt equation can be also viewed as an approximation of the Kuznetsov equation by a nonlinear perturbation. Formally if we consider the differential operators of the Kuznetsov (9) and the Westervelt (8) equations, the only difference between these two models is that the Westervelt equation keeps only one of two non-linear terms of the Kuznetsov equation, producing cumulative effects in a progressive wave propagation [1]. Let us also notice that the Kuznetsov equation (9) (and also the Westervelt equation (8)) is a non-linear wave equation containing the terms of different order on  $\varepsilon$ . But the KZK- and NPE-paraxial approximations allow to have the approximate equations with all terms of the same order, *i.e.* the KZK and NPE equations. For the well-posedness of the Cauchy problem for the Kuznetsov equation we cite [12] and for boundary valued problems in regular bounded domains see Refs. [25, 27, 38].

The physical context and the physical using of the KZK and the NPE equations are different: the NPE equation is usually used to describe short-time pulses and a long-range propagation, for instance, in an ocean wave-guide, where the refraction phenomena are important [8, 36], while the KZK equation typically models the ultrasonic propagation with strong diffraction phenomena, combining with finite amplitude effects (see Ref. [40] and the references therein). But in the same time [13], between the variables of these two

models and they can be presented by the same type differential operator with constant positive coefficients:

$$Lu = 0, \quad L = \partial_{tx}^2 - c_1 \partial_x (\partial_x \cdot)^2 - c_2 \partial_x^3 \pm c_3 \Delta_y \quad \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}, y \in \mathbb{R}^{n-1}.$$

Therefore, the results on the solutions of the KZK equation from Ref. [39] are valid for the NPE equation. See also Ref. [22] for the exponential decay of the solutions of these models in the viscous case.

To keep a physical sense of the approximation problems, we consider especially the two or three dimensional cases, *i.e.*  $\mathbb{R}^n$  with  $n = 2$  or  $3$ , and in the following we use the notation  $x = (x_1, x') \in \mathbb{R}^n$  with one propagative axis  $x_1 \in \mathbb{R}$  and the traversal variable  $x' \in \mathbb{R}^{n-1}$ .

To be able to consider the approximation of the Kuznetsov equation by the KZK equation (see Section 2), we firstly establish global well-posedness results for the Kuznetsov equation in the half space similar to the previous framework for the KZK and the Navier-Stokes system considered in [13]. We study two cases: the purely time periodic boundary problem in the *ansatz* variables  $(z, \tau, y)$  moving with the wave and the initial boundary-value problem for the Kuznetsov equation in the initial variables  $(t, x_1, x')$  with data coming from the solution of the KZK equation. We validate these two types approximations in Subsection 2.3 for the viscous and inviscid cases. Finally in Sections 3 and 4 we validate the approximation between the Kuznetsov and NPE equation and the Kuznetsov and Westervelt equations respectively (see Table 1). We can summarize the approximation results of the Kuznetsov equation in the following way: if  $u$  is a solution of the Kuznetsov equation and  $\bar{u}$  is a solution of the NPE or of the the KZK (for the initial boundary value problem) or of the Westervelt equations found for rather closed initial data

$$\|\nabla_{t,x}(u(0) - \bar{u}(0))\|_{L^2(\Omega)} \leq \delta \leq \varepsilon,$$

then there exist constants  $K > 0$ ,  $C_1 > 0$ ,  $C_2 > 0$  and  $C > 0$  independent on  $\varepsilon$ ,  $\delta$  and on time, such that for all  $t \leq \frac{C}{\varepsilon}$  it holds

$$\|\nabla_{t,x}(u - \bar{u})\|_{L^2(\Omega)} \leq C_1(\varepsilon^2 t + \delta)e^{C_2 \varepsilon t} \leq K\varepsilon.$$

## 2 The Kuznetsov equation and the KZK equation.

### 2.1 Derivation of the KZK equation from the Kuznetsov equation.

If the velocity potential is given [29] by

$$u(x, t) = \Phi(t - x_1/c, \varepsilon x_1, \sqrt{\varepsilon} x') = \Phi(\tau, z, y), \quad (12)$$

we directly obtain from the Kuznetsov equation (9) with the paraxial change of variable

$$\tau = t - \frac{x_1}{c}, \quad z = \varepsilon x_1, \quad y = \sqrt{\varepsilon} x', \quad (13)$$

that

$$\begin{aligned} & \partial_t^2 u - c^2 \Delta u - \varepsilon \partial_t \left( (\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \\ &= \varepsilon \left[ 2c \partial_{\tau z}^2 \Phi - \frac{\gamma+1}{2c^2} \partial_\tau (\partial_\tau \Phi)^2 - \frac{\nu}{\rho_0 c^2} \partial_\tau^3 \Phi - c^2 \Delta_y \Phi \right] + \varepsilon^2 R^{Kuz-KZK} \end{aligned} \quad (14)$$

with

$$\begin{aligned} \varepsilon^2 R^{Kuz-KZK} &= \varepsilon^2 \left( -c^2 \partial_z^2 \Phi + \frac{2}{c} \partial_\tau (\partial_\tau \Phi \partial_z \Phi) - \partial_\tau (\nabla_y \Phi)^2 + \frac{2\nu}{c\rho_0} \partial_\tau^2 \partial_z \Phi - \frac{\nu}{\rho_0} \partial_\tau \Delta_y \Phi \right) \\ &+ \varepsilon^3 \left( -\partial_\tau (\partial_z \Phi)^2 - \frac{\nu}{\rho_0} \partial_\tau \partial_z^2 \Phi \right). \end{aligned} \quad (15)$$

Therefore, we find that the right-hand side  $\varepsilon$ -order terms in Eq. (14) is exactly the KZK equation (10). Thanks to [39, 22] we have the following well posedness result for the KZK equation in the half space with periodic boundary conditions:

**Theorem 1** [39] *For  $\nu \geq 0$ , let us consider the Cauchy problem for the integrated KZK equation given by*

$$\begin{cases} c \partial_z I - \frac{(\gamma+1)}{4\rho_0} \partial_\tau I^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^2 I - \frac{c^2}{2} \partial_\tau^{-1} \Delta_y I = 0 \text{ on } \mathbb{T}_\tau \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ I(\tau, 0, y) = I_0(\tau, y) \text{ on } \mathbb{T}_\tau \times \mathbb{R}^{n-1}, \end{cases} \quad (16)$$

where  $I_0$  is periodic with a period  $L$  on  $\tau$

$$I_0(\tau + L, y) = I_0(\tau, y)$$

and of mean value zero

$$\int_0^L I_0(\ell, y) d\ell = 0.$$

According to Eq. (14),

$$I = \frac{\rho_0}{c^2} \partial_\tau \Phi \quad (17)$$

and the operator  $\partial_\tau^{-1}$  is defined by formula

$$\partial_\tau^{-1} I(\tau, z, y) := \int_0^\tau I(\ell, z, y) d\ell + \int_0^L \frac{\ell}{L} I(\ell, z, y) d\ell. \quad (18)$$

Then the following results hold true

1. (Local existence) For  $s > \left[\frac{n}{2}\right] + 1$  there exists a constant  $C(s, L)$  such that for any initial data  $I_0 \in H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$  on an interval  $[0, T[$  with

$$T \geq \frac{1}{C(s, L) \|I_0\|_{H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}}$$

problem (16) has a unique solution  $I$  such that

$$I \in C([0, T[, H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})) \cap C^1([0, T[, H^{s-2}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})),$$

which satisfies the zero mean value condition

$$\int_0^L I(\ell, z, y) d\ell = 0. \quad (19)$$

2. (Shock formation) Let  $T^*$  be the largest time on which such a solution is defined, then we have

$$\int_0^{T^*} \sup_{\tau, y} (|\partial_\tau I(\tau, t, y)| + |\nabla_y I(\tau, t, y)|) dt = +\infty.$$

3. (Global existence) If  $\nu > 0$  we have the global existence for small enough initial data: there exists a constant  $C_1 > 0$  such that

$$\|I_0\|_{H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})} \leq C_1 \Rightarrow T^* = +\infty.$$

4. (Exponential decay) [22] If  $\nu > 0$ ,  $s \in \mathbb{N}$  and  $s \geq \lceil \frac{n+1}{2} \rceil$ , then there exists a constant  $C_2 > 0$  such that  $\|I_0\|_{H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})} \leq C_2$  implies for all  $z \geq 0$

$$\|I(z)\|_{H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})} \leq C \|I_0\|_{H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})} e^{-\ell z},$$

where  $C > 0$  and  $\ell \in ]0, 1[$  are constants.

Due to the well posedness domain of the KZK equation, to validate the approximation between the solutions of the KZK and the Kuznetsov equations, we need to study the well posedness of the Kuznetsov equation on the half space with boundary conditions coming from the initial condition for the KZK equation.

## 2.2 Well posedness of the Kuznetsov equation in the half space.

### 2.2.1 Periodic boundary problem.

Let us consider the following periodic in time problem for the Kuznetsov equation in the half space  $\mathbb{R}_+ \times \mathbb{R}^{n-1}$  with periodic in time Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \cdot \nabla u_t & \text{on } \mathbb{T}_t \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ u|_{x_1=0} = g & \text{on } \mathbb{T}_t \times \mathbb{R}^{n-1}, \end{cases} \quad (20)$$

where  $g$  is a  $L$ -periodic in time and of mean value zero function. To show the well-posedness of problem (20) we study the maximal regularity of the associated linear operator and then use an equivalent to the fixed point theorem. Using [10], we directly obtain the following result of maximal regularity:

**Theorem 2** Let  $n = 3$ ,  $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$  and  $p \in ]1, +\infty[$ . Then there exists a unique solution  $u \in W_p^2(\mathbb{T}_t; L^p(\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^2(\Omega))$  with the mean value zero

$$\forall x \in \Omega \quad \int_{\mathbb{T}_t} u(s, x) ds = 0 \quad (21)$$

of the following system

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = f & \text{on } \mathbb{T}_t \times \Omega, \\ u = g & \text{on } \mathbb{T}_t \times \partial\Omega \end{cases} \quad (22)$$

if and only if the functions

$$f \in L^p(\mathbb{T}_t; L^p(\Omega)) \text{ and } g \in W_p^{2-\frac{1}{2p}}(\mathbb{T}_t; L^p(\partial\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^{2-\frac{1}{p}}(\partial\Omega)) \quad (23)$$

and are of mean value zero:

$$\forall x \in \Omega \quad \int_{\mathbb{T}_t} f(l, x) dl = 0 \text{ and } \forall x' \in \partial\Omega \quad \int_{\mathbb{T}_t} g(l, x') dl = 0. \quad (24)$$

Moreover, we have the following stability estimate

$$\|u\|_{W_p^2(\mathbb{T}_t; L^p(\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^2(\Omega))} \leq C \left( \|f\|_{L^p(\mathbb{T}_t; L^p(\Omega))} + \|g\|_{W_p^{2-\frac{1}{2p}}(\mathbb{T}_t; L^p(\partial\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^{2-\frac{1}{p}}(\partial\Omega))} \right).$$

**Proof :** On one hand, if  $f$  and  $g$  satisfy (23)–(24), the necessity of the conditions is shown in Ref. [10]. On the other hand, the conditions (23)–(24) are sufficient by a direct application of the trace theorems recalled in Ref. [10] and proved in Ref. [14] for example.  $\square$

The results of Ref. [10] allow to see that Theorem 2 does not depend on  $n$ , moreover if we look at the case  $p = 2$  the linearity of the operator  $\partial_t^2 - c^2 \Delta - \nu \Delta \partial_t$  from Eq. (22) implies that we can work with  $H^s(\Omega)$  instead of  $L^2(\Omega)$  :

**Lemma 1** *Let  $n \in \mathbb{N}^*$ ,  $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ ,  $s \geq 0$  then there exists a unique solution*

$$u \in X = \left\{ u \in H^2(\mathbb{T}_t; H^s(\Omega)) \cap H^1(\mathbb{T}_t; H^{s+2}(\Omega)) \mid \forall x \in \Omega \quad \int_{\mathbb{T}_t} u(s, x) ds = 0 \right\} \quad (25)$$

of system (22) for the linear strongly damped wave equation if and only if

$$f \in L^2(\mathbb{T}_t; H^s(\Omega)) \text{ and } g \in \mathbb{F}_{\mathbb{T}} = H^{\frac{7}{4}}(\mathbb{T}_t; H^s(\partial\Omega)) \cap H^1(\mathbb{T}_t; H^{s+\frac{3}{2}}(\partial\Omega)) \quad (26)$$

both satisfying (24).

Moreover we have the following stability estimate

$$\|u\|_X \leq C(\|f\|_{L^2(\mathbb{T}_t; H^s(\Omega))} + \|g\|_{\mathbb{F}_{\mathbb{T}}}).$$

Here  $H^2(\mathbb{T}_t; H^s(\Omega)) \cap H^1(\mathbb{T}_t; H^{s+2}(\Omega))$  is endowed with its usual norm denoted here and in the sequel by  $\|\cdot\|_X$ .

To prove the global well-posedness of problem (20) for the Kuznetsov equation we use the following theorem [46]:

**Theorem 3** [46] *Let  $X$  be a Banach space, let  $Y$  be a separable topological vector space, let  $L : X \rightarrow Y$  be a linear continuous operator, let  $U$  be the open unit ball in  $X$ , let  $P_{LU} : LX \rightarrow [0, \infty[$  be the Minkowski functional of the set  $LU$ , and let  $\Phi : X \rightarrow LX$  be a mapping satisfying the condition*

$$P_{LU}(\Phi(x) - \Phi(\bar{x})) \leq \Theta(r) \|x - \bar{x}\| \quad \text{for} \quad \|x - x_0\| \leq r, \quad \|\bar{x} - x_0\| \leq r$$



for some  $x_0 \in X$ , where  $\Theta : [0, \infty[ \rightarrow [0, \infty[$  is a monotone non-decreasing function. Set  $b(r) = \max(1 - \Theta(r), 0)$  for  $r \geq 0$ .

Suppose that

$$w = \int_0^\infty b(r) dr \in ]0, \infty], \quad r_* = \sup\{r \geq 0 \mid b(r) > 0\},$$

$$w(r) = \int_0^r b(t) dt \quad (r \geq 0) \quad \text{and} \quad f(x) = Lx + \Phi(x) \quad \text{for } x \in X.$$

Then for any  $r \in [0, r_*[$  and  $y \in f(x_0) + w(r)LU$ , there exists an  $x \in x_0 + rU$  such that  $f(x) = y$ .

Now we can use the maximal regularity result for system (22) with Theorem 3 and the same method as for the Cauchy problem associated with the Kuznetsov equation used in our previous work [12]. We will just have to use the boundary conditions of problem (20) as the initial condition of the corresponding Cauchy problem in  $\mathbb{R}^n$ .

**Theorem 4** *Let  $\nu > 0$ ,  $n \in \mathbb{N}^*$ ,  $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ ,  $s > \frac{n}{2}$ . Let  $X$  be defined by (25) and the boundary condition  $g \in \mathbb{F}_{\mathbb{T}}$  be defined by (26) and in addition, let  $g$  be of mean value zero (see Eq. (24)).*

*Then there exist  $r^* = O(1)$  and  $C_1 = O(1)$  such that for all  $r \in [0, r^*[$ , if  $\|g\|_{\mathbb{F}_{\mathbb{T}}} \leq \frac{\sqrt{\nu\varepsilon}}{C_1}r$ , there exists a unique solution  $u \in X$  of the periodic problem (20) for the Kuznetsov equation such that  $\|u\|_X \leq 2r$ .*

**Proof :** For  $g \in \mathbb{F}_{\mathbb{T}}$  defined in (26) and satisfying (24), let us denote by  $u^* \in X$  the unique solution of the linear problem (22) with  $f = 0$  and  $g \in \mathbb{F}_{\mathbb{T}}$ .

In addition, according to Theorem 1, we take  $X$  defined in (25), this time for  $s > \frac{n}{2}$  (we need this regularity to control the non-linear terms), and introduce the Banach spaces

$$X_0 := \{u \in X \mid u|_{\partial\Omega} = 0 \text{ on } \mathbb{T}_t \times \partial\Omega\} \quad (27)$$

and

$$Y = \left\{ f \in L^2(\mathbb{T}_t; H^s(\Omega)) \mid \forall x \in \Omega \quad \int_{\mathbb{T}_t} f(s, x) ds = 0 \right\}.$$

Then by Lemma 1, the linear operator

$$L : X_0 \rightarrow Y, \quad u \in X_0 \mapsto L(u) := u_{tt} - c^2 \Delta u - \nu\varepsilon \Delta u_t \in Y,$$

is a bi-continuous isomorphism.

Let us now notice that if  $v$  is the unique solution of the non-linear Dirichlet problem

$$\begin{cases} v_{tt} - c^2 \Delta v - \nu\varepsilon \Delta v_t = & \alpha\varepsilon(v + u^*)_t(v + u^*)_{tt} & \text{on } \mathbb{T}_t \times \Omega, \\ & +\beta\varepsilon \nabla(v + u^*) \cdot \nabla(v + u^*)_t & \\ v = 0 \text{ on } \mathbb{T}_t \times \partial\Omega, & & \end{cases} \quad (28)$$

then  $u = v + u^*$  is the unique solution of the periodic problem (20). Let us prove the existence of a such  $v$ , using Theorem 3.

We suppose that  $\|u^*\|_X \leq r$  and define for  $v \in X_0$

$$\Phi(v) := \alpha\varepsilon(v + u^*)_t(v + u^*)_{tt} + \beta\varepsilon\nabla(v + u^*) \cdot \nabla(v + u^*)_t.$$

For  $w$  and  $z$  in  $X_0$  such that  $\|w\|_X \leq r$  and  $\|z\|_X \leq r$ , we estimate the norm  $\|\Phi(w) - \Phi(z)\|_Y$ . By applying the triangular inequality we have

$$\begin{aligned} \|\Phi(w) - \Phi(z)\|_Y &\leq \alpha\varepsilon \left( \|u_t^*(w - z)_{tt}\|_Y + \|(w - z)_t u_{tt}^*\|_Y \right. \\ &\quad \left. + \|w_t(w - z)_{tt}\|_Y + \|(w - z)_t z_{tt}\|_Y \right) \\ &\quad + \beta\varepsilon \left( \|\nabla u^* \nabla(w - z)_t\|_Y + \|\nabla(w - z) \nabla u_t^*\|_Y \right. \\ &\quad \left. + \|\nabla w \nabla(w - z)_t\|_Y + \|\nabla(w - z) \nabla z_t\|_Y \right). \end{aligned}$$

Now, for all  $a$  and  $b$  in  $X$  with  $s \geq s_0 > \frac{n}{2}$  it holds

$$\begin{aligned} \|a_t b_{tt}\|_Y &\leq \|a_t\|_{L^\infty(\mathbb{T}_t \times \Omega)} \|b_{tt}\|_Y \\ &\leq C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)} \|a_t\|_{H^1(\mathbb{T}_t; H^{s_0}(\Omega))} \|b\|_X \\ &\leq C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)} \|a\|_X \|b\|_X, \end{aligned}$$

where  $C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}$  is the embedding constant of  $H^1(\mathbb{T}_t; H^{s_0}(\Omega))$  in  $L^\infty(\mathbb{T}_t \times \Omega)$ , independent on  $s$ , but depending only on the dimension  $n$ . In the same way, for all  $a$  and  $b$  in  $X$  it holds

$$\|\nabla a \nabla b_t\|_Y \leq C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)} \|a\|_X \|b\|_X.$$

Taking  $a$  and  $b$  equal to  $u^*$ ,  $w$ ,  $z$  or  $w - z$ , as  $\|u^*\|_X \leq r$ ,  $\|w\|_X \leq r$  and  $\|z\|_X \leq r$ , we obtain

$$\|\Phi(w) - \Phi(z)\|_Y \leq 4(\alpha + \beta) C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)} \varepsilon r \|w - z\|_X.$$

By the fact that  $L$  is a bi-continuous isomorphism, there exists a minimal constant  $C_\varepsilon = O\left(\frac{1}{\varepsilon^\nu}\right) > 0$ , coming from the inequality

$$C_0 \varepsilon^\nu \|u\|_X^2 \leq \|f\|_Y \|u\|_X$$

for  $u$ , a solution of the linear problem (22) with homogeneous boundary data (for a maximal constant  $C_0 = O(1) > 0$ ) such that

$$\forall u \in X_0 \quad \|u\|_X \leq C_\varepsilon \|Lu\|_Y.$$

Hence, for all  $f \in Y$

$$P_{LU_{X_0}}(f) \leq C_\varepsilon P_{U_Y}(f) = C_\varepsilon \|f\|_Y.$$

Then we find for  $w$  and  $z$  in  $X_0$ , such that  $\|w\|_X \leq r$ ,  $\|z\|_X \leq r$ , and also for  $\|u^*\|_X \leq r$ , that with the notation

$$\Theta(r) := 4C_\varepsilon(\alpha + \beta) C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)} \varepsilon r$$

it holds

$$P_{LU_{X_0}}(\Phi(w) - \Phi(z)) \leq \Theta(r)\|w - z\|_X.$$

Thus we apply Theorem 3 with  $f(x) = L(x) - \Phi(x)$  and  $x_0 = 0$ . Therefore, knowing that  $C_\varepsilon = \frac{C_0}{\varepsilon\nu}$ , we have, that for all  $r \in [0, r_*[$  with

$$r_* = \frac{\nu}{4C_0(\alpha + \beta)C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}} = O(1), \quad (29)$$

for all  $y \in \Phi(0) + w(r)LU_{X_0} \subset Y$  with

$$w(r) = r - 2\frac{C_0}{\nu}C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}(\alpha + \beta)r^2,$$

there exists a unique  $v \in 0 + rU_{X_0}$  such that  $L(v) - \Phi(v) = y$ . But, if we want that  $v$  be the solution of the non-linear problem (28), then we need to impose  $y = 0$  and thus, to ensure that

$$0 \in \Phi(0) + w(r)LU_{X_0}.$$

Since  $-\frac{1}{w(r)}\Phi(0)$  is an element of  $Y$  and  $LX_0 = Y$ , there exists a unique  $z \in X_0$  such that

$$Lz = -\frac{1}{w(r)}\Phi(0). \quad (30)$$

Let us show that  $\|z\|_X \leq 1$ , what will implies that  $0 \in \Phi(0) + w(r)LU_{X_0}$ . Noticing that

$$\begin{aligned} \|\Phi(0)\|_Y &\leq \alpha\varepsilon\|v_t v_{tt}\|_Y + \beta\varepsilon\|\nabla v \nabla v_t\|_Y \\ &\leq (\alpha + \beta)\varepsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}\|v\|_X^2 \\ &\leq (\alpha + \beta)\varepsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}r^2 \end{aligned}$$

and using (30), we find

$$\begin{aligned} \|z\|_X &\leq C_\varepsilon\|Lz\|_Y = C_\varepsilon\frac{\|\Phi(0)\|_Y}{w(r)} \\ &\leq \frac{C_\varepsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}(\alpha + \beta)\varepsilon r}{(1 - 2C_\varepsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}(\alpha + \beta)\varepsilon r)} < \frac{1}{2}, \end{aligned}$$

as soon as  $r < r^*$ .

Consequently,  $z \in U_{X_0}$  and  $\Phi(0) + w(r)Lz = 0$ . Then we conclude that for all  $r \in [0, r_*[$ , if  $\|u^*\|_X \leq r$ , there exists a unique  $v \in rU_{X_0}$  such that  $L(v) - \Phi(v) = 0$ , *i.e.* the solution of the non-linear problem (28). Thanks to the maximal regularity and a priori estimate following from Theorem 1 with  $f = 0$ , there exists a constant  $C_1 = O(\varepsilon^0) > 0$ , such that

$$\|u^*\|_X \leq \frac{C_1}{\sqrt{\nu\varepsilon}}\|g\|_{\mathbb{F}_T}.$$

Thus, for all  $r \in [0, r_*[$  and  $\|g\|_{\mathbb{F}_T} \leq \frac{\sqrt{\nu\varepsilon}}{C_1}r$ , the function  $u = u^* + v \in X$  is the unique solution of the time periodic problem for the Kuznetsov equation and  $\|u\|_X \leq 2r$ .  $\square$

### 2.2.2 Initial boundary value problem.

We work on  $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$  and we study the initial boundary value problem for the Kuznetsov equation on this space, *i.e.* the perturbation of an imposed initial condition by a source on the boundary, which will later be determined by the solution of the KZK equation.

**Lemma 2** *Let  $s \geq 0$ ,  $n \in \mathbb{N}$ . There exists a unique solution*

$$u \in \mathbb{E} := H^2(\mathbb{R}_+; H^s(\Omega)) \cap H^1(\mathbb{R}_+; H^{s+2}(\Omega)) \quad (31)$$

*of the linear problem*

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = f & \text{in } \mathbb{R}_+ \times \Omega, \\ u = g & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega \end{cases} \quad (32)$$

*if and only if the data satisfy the following conditions*

- $f \in L^2(\mathbb{R}_+; H^s(\Omega))$ ,
- *for the boundary condition*

$$g \in \mathbb{F}_{\mathbb{R}_+} = H^{7/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap H^1(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)); \quad (33)$$

- $u_0 \in H^{s+2}(\Omega)$  and  $u_1 \in H^{s+1}(\Omega)$ ;
- $g(0) = u_0$  and  $g_t(0) = u_1$  on  $\partial\Omega$  in the trace sense.

*In addition, the solution satisfies the stability estimate*

$$\|u\|_{\mathbb{E}} \leq C(\|f\|_{L^2(\mathbb{R}_+; H^s(\Omega))} + \|g\|_{\mathbb{F}_{\mathbb{R}_+}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^{s+1}}).$$

In order to prove this result we will use the subsequent lemma to remove the inhomogeneity  $g$ .

**Lemma 3** *Let  $s \geq 0$ ,  $n \in \mathbb{N}$  and  $\mathbb{E}$  defined in (31). There exists a unique solution  $w \in \mathbb{E}$  of the following linear problem*

$$\begin{cases} w_{tt} - \nu \varepsilon \Delta w_t = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ w = g & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ w(0) = 0, \quad w_t(0) = 0 & \text{in } \Omega \end{cases} \quad (34)$$

*if and only if*

$g \in \mathbb{F}_{\mathbb{R}_+}$  (the space  $\mathbb{F}_{\mathbb{R}_+}$  is defined in (33)) and it holds the following compatibility conditions:

for all  $x \in \partial\Omega$ ,  $g(0) = 0$  and  $g_t(0) = 0$ .

Moreover, the solution  $w$  satisfies the stability estimate

$$\|w\|_{\mathbb{E}} \leq C\|g\|_{\mathbb{F}_{\mathbb{R}_+}}.$$

**Proof :** First we prove the sufficiency. By assumption (33), we have

$$\partial_t g \in H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)).$$

Thanks to § 3 p. 288 in Ref. [30], we obtain a unique solution

$$v \in H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega))$$

of the parabolic problem

$$v_t - \nu \varepsilon \Delta v = 0 \text{ in } \mathbb{R}_+ \times \Omega, \quad v = \partial_t g \text{ on } \mathbb{R}_+ \times \partial\Omega, \quad v(0) = 0 \text{ in } \Omega.$$

Next we define for  $t \in \mathbb{R}_+$  and  $x \in \Omega$  the function

$$w(t, x) := \int_0^t v(l, x) dl.$$

We have  $w(0) = 0$  and  $w_t(0) = 0$ . Moreover, it satisfies

$$w_{tt} - \nu \varepsilon \Delta w_t = 0, \quad w(t)|_{\partial\Omega} = \int_0^t g_t(l) dl = g(t),$$

as  $g(0) = 0$ . Therefore,  $w$  is a solution of problem (34). The necessity follows from the spatial trace theorem ensuring that the trace operator  $Tr_{\partial\Omega} : u \mapsto u|_{\partial\Omega}$ , considering as a map

$$H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega)) \rightarrow H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)), \quad (35)$$

is bounded and surjective by Lemma 3.5 in Ref. [14]. For the compatibility condition, thanks to Lemma 11 in Ref. [15], we also know that the temporal trace  $Tr_{t=0} : g \mapsto g|_{t=0}$ , considered as a map

$$H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)) \rightarrow H^{s+1/2}(\partial\Omega), \quad (36)$$

is well defined and bounded. Moreover, the spatial trace

$$H^{s+1/2}(\Omega) \rightarrow H^s(\partial\Omega) \quad (37)$$

is bounded by Theorem 1.5.1.1 from Ref. [17].

To obtain uniqueness, let  $w$  be a solution to (34) with  $g = 0$ . Since  $w_t$  solves the heat problem with homogeneous data, we obtain  $w_t = 0$  and therefore also  $w = 0$  by the initial condition  $w(0) = 0$ . The stability estimate follows from the closed graph theorem.  $\square$

Let us prove Lemma 2: **Proof :** We obtain the uniqueness of the solution of the boundary valued problem for the linear strongly damped equation (32) from the fact that in the case  $g = 0$  we can consider  $-\Delta$  as a self-adjoint and non negative operator with homogeneous Dirichlet boundary conditions and we can use [16].

To verify the necessity of the conditions on the data, we suppose that  $u \in \mathbb{E}$  (see Eq. (31) for the definition of  $\mathbb{E}$ ) is a solution of (32). Then

$$u, u_t \in H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega)) \text{ and thus } f \in L^2(\mathbb{R}_+; H^s(\Omega)).$$

Taking as in the previous proof the spatial trace  $Tr_{\partial\Omega}$  as in Eq. (35) we have

$$g, g_t \in H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)), \text{ which implies } g \in \mathbb{F}_{\mathbb{R}_+}.$$

By the Sobolev embedding  $H^1(\mathbb{R}_+; H^{s+2}(\Omega)) \hookrightarrow C(\mathbb{R}_+; H^{s+2}(\Omega))$ , it follows that  $u_0 \in H^{s+2}(\Omega)$  and we also have the temporal trace

$$u \mapsto u|_{t=0} : H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega)) \rightarrow H^{s+1}(\Omega)$$

by Lemma 3.7 in Ref. [14]. Following the proof of Lemma 3, we use Eqs. (36) and (37) to obtain the compatibility conditions.

It remains to prove the sufficiency of the conditions. We extend  $u_0$ ,  $u_1$  and  $f$  in odd functions among  $x_1$  on  $\mathbb{R}^n$  so that we have

$$\tilde{u}_0 \in H^{s+2}(\mathbb{R}^n), \tilde{u}_1 \in H^{s+1}(\mathbb{R}^n) \text{ and } \tilde{f} \in L^2(\mathbb{R}_+; H^s(\mathbb{R}^n)).$$

We consider the problem

$$\begin{cases} \tilde{u}_{tt} - c^2 \Delta \tilde{u} - \nu \varepsilon \Delta \tilde{u}_t = \tilde{f} & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ \tilde{u}(0) = \tilde{u}_0, \quad \tilde{u}_t(0) = \tilde{u}_1 & \text{in } \mathbb{R}^n. \end{cases}$$

By Theorem 4.1 in Ref. [12] we obtain the existence of its unique solution

$$\tilde{u} \in H^2(\mathbb{R}_+; H^s(\mathbb{R}^n)) \cap H^1(\mathbb{R}_+; H^{s+2}(\mathbb{R}^n)).$$

Let  $\bar{u} \in \mathbb{E}$ , denote the restriction of  $\tilde{u}$  to  $\Omega$  and let  $\bar{g} := g - \bar{u}|_{\partial\Omega}$ . By the spatial trace theorem  $\bar{u}|_{\partial\Omega} \in \mathbb{F}_{\mathbb{R}_+}$ , and hence  $\bar{g} \in \mathbb{F}_{\mathbb{R}_+}$ . Then the solution  $u$  of the non homogeneous linear problem (32) is given by  $u = v + \bar{u}$ , where  $v$  solves problem (32) with  $f = u_0 = u_1 = 0$  and  $g = \bar{g}$ . From Lemma 3 we have a unique solution  $\bar{v} \in \mathbb{E}_u$  of problem (34) with  $g = \bar{g}$ . Then the function  $w := v - \bar{v}$  solves the following system

$$\begin{cases} w_{tt} - \Delta w - \nu \varepsilon \Delta w_t = c^2 \Delta \bar{v} & \text{in } \mathbb{R}_+ \times \Omega, \\ w = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ w(0) = 0, \quad w_t(0) = 0 & \text{in } \Omega, \end{cases}$$

which thanks to Theorem 2.6 in Ref. [16] has a unique solution  $w \in \mathbb{E}$  defined in (31). The function  $u := w + \bar{v} + \bar{u}$  is the desired solution of system (32) and the stability estimate follows from the closed graph theorem. This concludes the proof of Lemma 2.  $\square$

The next theorem follows from the maximal regularity result of Lemma 2 and Theorem 3. Its proof is similar to the proof of Theorem 4 and hence is omitted.

**Theorem 5** *Let  $\nu > 0$ ,  $n \in \mathbb{N}^*$ ,  $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$  and  $s > \frac{n}{2}$ . Considering the initial boundary value problem for the Kuznetsov equation in the half space with the Dirichlet boundary condition*

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \nabla u_t & \text{in } [0, +\infty[ \times \Omega, \\ u = g & \text{on } [0, \infty[ \times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega, \end{cases} \quad (38)$$

*the following results hold: there exist constants  $r^* = O(1)$  and  $C_1 = O(1)$ , such that for all initial data satisfying*

- $g \in \mathbb{F}_{\mathbb{R}^+} := H^{7/4}([0, \infty[; H^s(\partial\Omega)) \cap H^1([0, \infty[; H^{s+3/2}(\partial\Omega))$ ,
- $u_0 \in H^{s+2}(\Omega)$ ,  $u_1 \in H^{s+1}(\Omega)$ ,
- $g(0) = u_0|_{\partial\Omega}$  and  $g_t(0) = u_1|_{\partial\Omega}$ ,

and such that for  $r \in [0, r^*[$

$$\|u_0\|_{H^{s+2}(\Omega)} + \|u_1\|_{H^{s+1}(\Omega)} + \|g\|_{\mathbb{F}_{[0,T]}} \leq \frac{\nu\varepsilon}{C_1}r,$$

there exists a unique solution of problem (38) for the Kuznetsov equation

$$u \in H^2([0, \infty[; H^s(\Omega)) \cap H^1([0, \infty[; H^{s+2}(\Omega)),$$

such that

$$\|u\|_{H^2([0, \infty[; H^s(\Omega)) \cap H^1([0, \infty[; H^{s+2}(\Omega))} \leq 2r.$$

## 2.3 Approximation of the solutions of the Kuznetsov equation by the solutions of the KZK equation.

Given Theorems 1 and 4 for the viscous case, we consider the Cauchy problem associated to the KZK equation (16) for small enough initial data in order to have a time periodic solution  $I$  defined on

$\mathbb{R}_+ \times \mathbb{R}^{n-1}$ . If  $\nu > 0$ , to compare the solutions of the Kuznetsov and the KZK equations we consider two cases. The first case is considered in Sub-subsection 2.3.1, when the Kuznetsov equation can be considered as a time periodic boundary problem coming just from the initial condition  $I_0$  of problem (16) for the KZK equation. In Sub-subsection 2.3.2 we study the second case, when the solution of the KZK equation taken for  $\tau = 0$  gives  $I(0, z, y)$  defined on  $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ , from which we deduce, according to the derivation *ansatz*, both an initial condition for the Kuznetsov equation at  $t = 0$  and a corresponding boundary condition. In this second situation, it also makes sense to consider the inviscid case, briefly commented in the end of Sub-subsection 2.3.2.

### 2.3.1 Approximation problem for the Kuznetsov with periodic boundary conditions.

By Theorem 1 there is a unique solution  $I(\tau, z, y)$  of the Cauchy problem for the KZK equation (16) such that

$$z \mapsto I(\tau, z, y) \in C([0, \infty[, H^s(\Omega_1)) \quad (39)$$

with  $\int_{\mathbb{T}_\tau} I(l, z, y) dl = 0$  and  $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$  where  $\mathbb{T}_\tau$  represents the periodicity in  $\tau$  of period  $L$ . We use the operator  $\partial_\tau^{-1}$  defined in (18). Formula (18), which implies that  $\partial_\tau^{-1}I$  is  $L$ -periodic in  $\tau$  and of mean value zero, gives us the estimate

$$\|\partial_\tau^{-1}I\|_{H^s(\Omega_1)} \leq C\|\partial_\tau\partial_\tau^{-1}I\|_{H^s(\Omega_1)} = C\|I\|_{H^s(\Omega_1)}.$$

So  $\partial_\tau^{-1}I|_{z=0} \in H^s(\Omega_1)$ , and hence by (39)

$$z \mapsto \partial_\tau^{-1}I(\tau, z, y) \in C([0, \infty[, H^s(\Omega_1)),$$

with  $\int_{\mathbb{T}_\tau} \partial_\tau^{-1}I(s, z, y)ds = 0$ .

We define on  $\mathbb{T}_t \times \mathbb{R}_+ \times \mathbb{R}^{n-1}$

$$\bar{u}(t, x_1, x') := \frac{c^2}{\rho_0} \partial_\tau^{-1}I(\tau, z, y) = \frac{c^2}{\rho_0} \partial_\tau^{-1}I\left(t - \frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon}x'\right) \quad (40)$$

with the paraxial change of variable (13) associated to the KZK equation. Thus  $\bar{u}$  is  $L$ -periodic in time and of mean value zero. Now we consider the Kuznetsov problem (20) associated to the following boundary condition, imposed by the initial condition for the KZK equation:

$$g(t, x') := \bar{u}(t, 0, x') = \frac{c^2}{\rho_0} \partial_\tau^{-1}I_0(\tau, y). \quad (41)$$

Taking (see Eqs. (12), (17) and for more details see [13]),

$$\tilde{I} := \frac{\rho_0}{c^2} \partial_\tau \Phi, \quad (42)$$

let  $\tilde{I}$  be the solution of the Kuznetsov equation written in the following form with the remainder  $R^{Kuz-KZK}$  defined in Eq. (15):

$$\begin{cases} c\partial_z \tilde{I} - \frac{(\gamma+1)}{4\rho_0} \partial_\tau \tilde{I}^2 - \frac{\nu}{2c^2\rho_0} \partial_\tau^2 \tilde{I} - \frac{c^2}{2} \Delta_y \partial_\tau^{-1} \tilde{I} + \varepsilon \frac{\rho_0}{2c^2} R^{Kuz-KZK} = 0, \\ \tilde{I}|_{z=0} = I_0, \end{cases} \quad (43)$$

where we can recognize the system associated to the KZK equation (16).

Now we can formulate the following approximation result between the solutions of the KZK and Kuznetsov equations.

**Theorem 6** *Let  $\nu > 0$ . For  $s > \max(\frac{n}{2}, 2)$  and  $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$  small enough in  $H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ , there exists a unique global solution  $I$  of the Cauchy problem for the KZK equation (16) such that*

$$z \mapsto I(\tau, z, y) \in C([0, \infty[, H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})).$$

*In addition, there exists a unique global solution  $\tilde{I}$  of the Kuznetsov problem (43), in the sense  $\tilde{I} := \frac{\rho_0}{c^2} \partial_\tau \Phi$ , with  $\Phi(\tau, z, y) := u(t, x_1, x')$  with the paraxial change of variable (13) and*

$$u \in H^2(\mathbb{T}_t; H^s(\mathbb{R}^+ \times \mathbb{R}^{n-1})) \cap H^1(\mathbb{T}_t; H^{s+2}(\mathbb{R}^+ \times \mathbb{R}^{n-1})),$$

*is the global solution of the periodic problem (20) for the Kuznetsov equation with  $g$  defined by  $I_0$  as in Eq. (41). Moreover there exist  $C_1 > 0$  and  $C_2 > 0$  such that*

$$\frac{1}{2} \frac{d}{dz} \|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}^2 \leq C_1 \|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}^2 + C_2 \varepsilon \|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})},$$



which implies

$$\|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}(z) \leq \frac{C_2}{2} \varepsilon z e^{\frac{c_1}{2}z} \leq \frac{C_2}{C_1} \varepsilon (e^{\frac{c_1}{2}z} - 1)$$

and

$$\|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}(z) \leq K\varepsilon \text{ while } z \leq C$$

with  $K > 0$  and  $C > 0$  independent of  $\varepsilon$ .

**Proof :** For  $s > \max(\frac{n}{2}, 2)$ , the global well-posedness of  $I$  comes from Theorem 1 if  $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$  is small enough. Moreover, since  $g$  is given by Eq. (41), thanks to the definition of  $\partial_\tau^{-1}$  in (18) and the fact that  $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ , we have

$$g \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}^{n-1}) \text{ and } \partial_t g \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}^{n-1}).$$

And thus

$$g \in H^{\frac{7}{4}}(\mathbb{T}_t; H^s(\mathbb{R}^{n-1})) \cap H^1(\mathbb{T}_t; H^{s+2-\frac{1}{2}}(\mathbb{R}^{n-1})).$$

Therefore we can use Theorem 4, which implies the global existence of the periodic in time solution

$$u \in H^2(\mathbb{T}_t; H^s(\mathbb{R}^+ \times \mathbb{R}^{n-1})) \cap H^1(\mathbb{T}_t; H^{s+2}(\mathbb{R}^+ \times \mathbb{R}^{n-1})),$$

of the Kuznetsov periodic boundary value problem (20) as  $I_0$  is small enough in  $H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ . Therefore, it also implies the global existence of  $\tilde{I}$ , defined in (42), which is the solution of the exact Kuznetsov system (43).

Now we subtract the equations in systems (16) and (43) to obtain

$$\begin{aligned} c\partial_z(I - \tilde{I}) - \frac{\gamma+1}{2\rho_0}(I - \tilde{I})\partial_\tau I - \frac{\gamma+1}{2\rho_0}\tilde{I}\partial_\tau(I - \tilde{I}) - \frac{\nu}{2c^2\rho_0}\partial_\tau^2(I - \tilde{I}) \\ - \frac{c^2}{2}\partial_\tau^{-1}\Delta_y(I - \tilde{I}) = \varepsilon \frac{\rho_0}{2c^2} R^{Kuz-KZK}. \end{aligned}$$

Denoting  $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$ , we multiply this equation by  $(I - \tilde{I})$ , integrate over  $\mathbb{T}_\tau \times \mathbb{R}^{n-1}$  and perform a standard integration by parts, which gives

$$\begin{aligned} \frac{c}{2} \frac{d}{dz} \|I - \tilde{I}\|_{L^2(\Omega_1)}^2 - \frac{\gamma+1}{2\rho_0} \int_{\Omega_1} \partial_\tau I (I - \tilde{I})^2 d\tau dy \\ - \frac{\gamma+1}{2\rho_0} \int_{\Omega_1} \tilde{I} (I - \tilde{I}) \partial_\tau (I - \tilde{I}) d\tau dy \\ + \frac{\nu}{2c^2\rho_0} \int_{\Omega_1} (\partial_\tau (I - \tilde{I}))^2 d\tau dy = \varepsilon \frac{\rho_0}{2c^2} \int_{\Omega_1} R^{Kuz-KZK} (I - \tilde{I}) d\tau dy. \end{aligned}$$

Let us notice that

$$\begin{aligned} \int_{\Omega_1} \tilde{I} (I - \tilde{I}) \partial_\tau (I - \tilde{I}) d\tau dy &= \int_{\Omega_1} [(\tilde{I} - I) + I] \frac{1}{2} \partial_\tau (I - \tilde{I})^2 d\tau dy = \\ &= -\frac{1}{2} \int_{\Omega_1} \partial_\tau I (I - \tilde{I})^2 d\tau dy, \end{aligned}$$

and as for  $s > \max(\frac{n}{2}, 2)$  and  $u \in H^2(\mathbb{T}_t; H^s(\Omega)) \cap H^1(\mathbb{T}_t; H^{s+2}(\Omega))$  we also have

$$R^{Kuz-KZK} \in C(\mathbb{R}_+; L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})). \quad (44)$$

This comes from the fact that in system (43) the worst term inside the remainder  $R^{Kuz-KZK}$  (see Eq. (15)) is  $\partial_\tau \partial_z^2 \Phi$  with  $\tilde{I}$  given by Eq. (42). As  $\partial_t^3 u \in L^2(\mathbb{T}_t; H^{s-2}(\Omega))$ , we need to take  $s > \max(\frac{n}{2}, 2)$  to have  $\partial_\tau \partial_z^2 \Phi$  in  $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1}))$ . Therefore, it holds

$$\left| \int_{\Omega_1} R^{Kuz-KZK}(I - \tilde{I}) d\tau dy \right| \leq \|R^{Kuz-KZK}\|_{L^2(\Omega_1)} \|I - \tilde{I}\|_{L^2(\Omega_1)} \leq C \|I - \tilde{I}\|_{L^2(\Omega_1)}$$

with a constant  $C > 0$  independent on  $z$  thanks to (44). It leads to the estimate

$$\frac{1}{2} \frac{d}{dz} \|I - \tilde{I}\|_{L^2(\Omega_1)}^2 \leq K \sup_{\Omega_1} |\partial_\tau I(\tau, z, y)| \|I - \tilde{I}\|_{L^2(\Omega_1)}^2 + C\varepsilon \|I - \tilde{I}\|_{L^2(\Omega_1)},$$

in which, due to the regularity of  $I$  for  $s$  and  $I_0$  (see also Point 1 and 3 of Theorem 1) the term

$$\sup_{\Omega_1} |\partial_\tau I(\tau, z, y)|$$

is bounded by a constant  $C > 0$  independent on  $z$ . Consequently, we have the desired estimate and the other results follow from Gronwall's Lemma.  $\square$

**Remark 1** *Here the regularity*

$$I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$$

for  $s > \max(\frac{n}{2}, 2)$  is the minimal regularity to ensure (44).

### 2.3.2 Approximation problem for the Kuznetsov equation with initial-boundary conditions.

Let as previously  $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$ , the period in  $\tau$  being  $L$ , but  $s \geq [\frac{n+1}{2}]$ . Suppose that a function

$$I_0(t, y) = I_0(t, \sqrt{\varepsilon}x')$$

is such that  $I_0 \in H^s(\Omega_1)$  and  $\int_{\mathbb{T}_\tau} I_0(s, y) ds = 0$ . Then by Theorem 1 there is a unique solution  $I(\tau, z, y)$  of the Cauchy problem (16) for the KZK equation such that

$$z \mapsto I(\tau, z, y) \in C([0, \infty[, H^s(\Omega_1)).$$

We define  $\bar{u}$  and  $g$  as in Eqs. (40) and (41) respectively. Thus, for  $R^{Kuz-KZK}$  defined in Eq. (15),  $\bar{u}$  is the solution of the following system

$$\begin{cases} \partial_t^2 \bar{u} - c^2 \Delta \bar{u} - \varepsilon \partial_t \left( (\nabla \bar{u})^2 + \frac{\gamma-1}{2c^2} (\partial_t \bar{u})^2 + \frac{\nu}{\rho_0} \Delta \bar{u} \right) = \varepsilon^2 R^{Kuz-KZK} & \text{in } \mathbb{T}_t \times \Omega, \\ \bar{u} = g & \text{on } \mathbb{T}_t \times \partial\Omega. \end{cases} \quad (45)$$

We study for  $T > 0$  the solution  $u$  of the Dirichlet boundary-value problem (38) for the Kuznetsov equation on  $[0, T] \times \mathbb{R}_+ \times \mathbb{R}^{n-1}$ , taking  $u_0 := \bar{u}(0)$  and  $u_1 := \bar{u}_t(0)$  and considering the time periodic function  $g$  defined by Eq. (41) as a function on  $[0, T]$ . Now we have the following stability result.

**Theorem 7** Let  $T > 0$ ,  $\nu > 0$ ,  $n \geq 2$ ,  $\Omega = \mathbb{R}^+ \times \mathbb{R}^{n-1}$  and  $I_0 \in H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ ,  $s \in \mathbb{R}^+$ . Let  $I$  be the solution of the KZK equation. By  $I$  the solution  $\bar{u}$  of the approximated Kuznetsov problem (45) is constructed using (40) and with  $g$  defined in (41).

Then there hold

1. If  $s \geq 6$  for  $n = 2$  and  $3$ , or else  $[\frac{s}{2}] > \frac{n}{2} + 1$ , there exists  $k > 0$  such that  $\|I_0\|_{H^s} < k$  implies the global well-posedness of the Cauchy problem for the KZK equation. Its solution is denoted for  $0 \leq k \leq [\frac{s}{2}]$  by

$$I \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})),$$

thus

$$\bar{u} \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})), \quad \partial_t \bar{u} \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})),$$

or again

$$\bar{u} \in H^2(\mathbb{T}_t, H^{[\frac{s}{2}]-1}(\Omega)) \cap H^1(\mathbb{T}_t, H^{[\frac{s}{2}]}(\Omega)). \quad (46)$$

The regularity of  $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$  (see Table 1) is minimal to ensure that  $R^{K_{uz}-KZK}$  (see Eq. (15) for the definition) is in  $C([0, +\infty[; L^2(\mathbb{R}_+ \times \mathbb{R}^{n-1}))$ .

2. If  $[\frac{s}{2}] > \frac{n}{2} + 2$ , taking the same initial data for the exact boundary-value problem for the Kuznetsov equation (38) as for  $\bar{u}$ , i.e.

$$\begin{aligned} u(0) = \bar{u}(0) &= \frac{c^2}{\rho_0} \partial_\tau^{-1} I(-\frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon} x') \in H^{[\frac{s}{2}]}(\Omega), \\ u_t(0) = \bar{u}_t(0) &= \frac{c^2}{\rho_0} \partial_\tau I(-\frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon} x') \in H^{[\frac{s}{2}]-1}(\Omega), \end{aligned}$$

there exists  $k > 0$  such that  $\|I_0\|_{H^s} < k$  implies the well-posedness of the exact Kuznetsov equation (38) considered with the Dirichlet boundary condition

$$\begin{aligned} g = \frac{c^2}{\rho_0} \partial_\tau^{-1} I_0 &\in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1}) \subset H^{7/4}([0, T]; H^{[\frac{s}{2}]-2}(\partial\Omega)) \\ &\cap H^1([0, T]; H^{[\frac{s}{2}]-2+3/2}(\partial\Omega)) \end{aligned}$$

ensuring the regularity

$$u \in H^2([0, T], H^{[\frac{s}{2}]-1}(\Omega)) \cap H^1([0, T], H^{[\frac{s}{2}]}(\Omega)). \quad (47)$$

Moreover, there exist constants  $K > 0$ ,  $C > 0$ ,  $C_1 > 0$ ,  $C_2 > 0$ , all independent of  $\varepsilon$ , such that for all  $t \leq \frac{C}{\varepsilon}$

$$\sqrt{\|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2} \leq C_1 \varepsilon^2 t e^{C_2 \varepsilon t} \leq K \varepsilon. \quad (48)$$

3. In addition, let  $u$  be a solution of the Dirichlet boundary-value problem (38) for the Kuznetsov equation, with  $g$  defined by Eq. (41) and  $u_0 \in H^{m+2}(\Omega)$ ,  $u_1 \in H^{m+1}(\Omega)$  with  $m > \frac{n}{2}$  and

$$\|(u - \bar{u})_t(0)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(0)\|_{L^2(\Omega)}^2 \leq \delta^2 \leq \varepsilon^2. \quad (49)$$

There exist constants  $K > 0$ ,  $C > 0$ ,  $C_1 > 0$ ,  $C_2 > 0$ , all independent of  $\varepsilon$ , such that for all  $t \leq \frac{C}{\varepsilon}$

$$\sqrt{\|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2} \leq C_1(\varepsilon^2 t + \delta^2)e^{C_2 \varepsilon t} \leq K\varepsilon. \quad (50)$$

**Proof :** Let  $\bar{u}$  and  $g$  be defined by Eqs. (40) and (41) respectively by the solution  $I$  of the Cauchy problem (16) for the KZK equation with  $I|_{z=0} = I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$  and  $s \geq 6$  for  $n = 2$  and  $3$ , or else  $[\frac{s}{2}] > \frac{n}{2} + 1$ . In this case,  $\bar{u}$  is the global solution of the approximated Kuznetsov system (45), what is a direct consequence of Theorem 1. If  $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$  with the chosen  $s$ , then for  $0 \leq k \leq [\frac{s}{2}]$

$$I(\tau, z, y) \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})).$$

Let us denote  $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$ . Given the equation for  $\bar{u}$  by (40), we have

$$\begin{aligned} \bar{u}(\tau, z, y) \text{ and } \partial_\tau \bar{u}(\tau, z, y) &\in C^k(\{z > 0\}; H^{s-2k}(\Omega_1)), \text{ if } 0 \leq k \leq [\frac{s}{2}], \\ \partial_\tau^2 \bar{u}(\tau, z, y) &\in C^k(\{z > 0\}; H^{s-1-2k}(\Omega_1)), \text{ if } 0 \leq k \leq [\frac{s}{2}] - 1, \end{aligned}$$

but we can also say [22], thanks to Point 4 of Theorem 1, that

$$\begin{aligned} \bar{u}(\tau, z, y) \text{ and } \partial_\tau \bar{u}(\tau, z, y) &\in H^k(\{z > 0\}; H^{s-2k}(\Omega_1)), \\ \partial_\tau^2 \bar{u}(\tau, z, y) &\in H^k(\{z > 0\}; H^{s-1-2k}(\Omega_1)). \end{aligned}$$

This implies as for the chosen  $s$  that

$$\begin{aligned} \bar{u}(t, x_1, x') \text{ and } \partial_t \bar{u}(t, x_1, x') &\in L^2(\mathbb{T}_t; H^{[\frac{s}{2}]}(\Omega)) \cap H^2(\mathbb{T}_t; H^{[\frac{s}{2}]-1}(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in L^2(\mathbb{T}_t; H^{[\frac{s}{2}]-1}(\Omega)) \cap H^2(\mathbb{T}_t; H^{[\frac{s}{2}]-2}(\Omega)). \end{aligned}$$

Therefore

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C^1([0, +\infty[; H^{[\frac{s}{2}]-1}(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^{[\frac{s}{2}]-2}(\Omega)). \end{aligned}$$

With the chosen  $s$ , these regularities of  $\bar{u}(t, x_1, x')$  give us regularity (46) and allow to have all left-hand terms in the approximated Kuznetsov system (45) of the desired regularity, *i.e* in  $C([0, +\infty[; L^2(\Omega))$ . In addition for  $[\frac{s}{2}] > \frac{n}{2} + 2$  with the chosen  $g$ ,  $u_0 = \bar{u}(0)$  and  $u_1 = \bar{u}_t(0)$  in the conditions of the theorem we have

$$u_0 \in H^{[\frac{s}{2}]}(\Omega), \quad u_1 \in H^{[\frac{s}{2}]-1}(\Omega)$$

with

$$g \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1}) \text{ and } \partial_t g \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1}).$$

This implies

$$g \in H^{7/4}([0, T[; H^{[\frac{s}{2}]-2}(\partial\Omega)) \cap H^1([0, T[; H^{[\frac{s}{2}]-2+3/2}(\partial\Omega))$$

with  $\lfloor \frac{s}{2} \rfloor - 2 > \frac{n}{2}$ , as required by Theorem 5 to have the local well-posedness of the solution of the Kuznetsov equation  $u$  associated to system (38). This completes the local well-posedness results and we deduce that  $u$  have the desired regularity (47), announced in the theorem. Moreover, we have  $R^{Kuz-KZK}$  in  $C([0, +\infty[, L^2(\Omega))$ .

To validate the approximation we will only demonstrate the estimate in point (3) as it directly implies the estimate in point (2). We take again  $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$  with  $\lfloor \frac{s}{2} \rfloor > \frac{n}{2} + 2$  to define  $\bar{u}$  and  $g$  and consider  $u$  to be a solution of the Dirichlet boundary-value problem (38) for the Kuznetsov equation under the conditions  $u_0 \in H^{m+2}(\Omega)$ ,  $u_1 \in H^{m+1}(\Omega)$  with  $m > \frac{n}{2}$  satisfying (49). Now we subtract the Kuznetsov equation from the approximated Kuznetsov equation (see system (45)), multiply by  $(u - \bar{u})_t$  and integrate over  $\Omega$  to obtain, as in Ref. [12], the following stability estimate:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} A(t, x) (u - \bar{u})_t^2 + c^2 (\nabla(u - \bar{u}))^2 dx \right) \\ & \leq C\varepsilon \sup(\|u_{tt}\|_{L^\infty(\Omega)}; \|\Delta u\|_{L^\infty(\Omega)}; \|\nabla \bar{u}_t\|_{L^\infty(\Omega)}) \\ & \quad \cdot \left( \|(u - \bar{u})_t\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})\|_{L^2(\Omega)}^2 \right) \\ & \quad + \varepsilon^2 \int_{\Omega} R^{Kuz-KZK} (u - \bar{u})_t dx, \end{aligned}$$

where  $\frac{1}{2} \leq A(t, x) \leq \frac{3}{2}$  for  $0 \leq t \leq T$  and  $x \in \Omega$ . By regularity of the solutions,  $\sup(\|u_{tt}\|_{L^\infty(\Omega)}; \|\Delta u\|_{L^\infty(\Omega)}; \|\nabla \bar{u}_t\|_{L^\infty(\Omega)})$  is bounded in time on  $[0, T]$ . Moreover, we have  $\|R^{Kuz-KZK}(t)\|_{L^2(\Omega)}$  bounded for  $t \in [0, T]$  by the regularity of  $\bar{u}$ , where  $R^{Kuz-KZK}$  is defined in Eq. (15). Then after integration on  $[0, t]$ , we can write

$$\begin{aligned} & \|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2 \\ & \leq 3(\|(u - \bar{u})_t(0)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(0)\|_{L^2(\Omega)}^2) \\ & \quad + C_1 \varepsilon \int_0^t (\|(u - \bar{u})_t(s)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(s)\|_{L^2(\Omega)}^2) ds \\ & \quad + C_2 \varepsilon^2 \int_0^t \sqrt{\|(u - \bar{u})_t(s)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(s)\|_{L^2(\Omega)}^2} ds. \end{aligned}$$

As

$$\|(u - \bar{u})_t(0)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(0)\|_{L^2(\Omega)}^2 \leq \delta^2 \leq \varepsilon^2,$$

we finally find by the Gronwall Lemma for  $t \leq \frac{C}{\varepsilon}$

$$\sqrt{\|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2} \leq C_1(\varepsilon^2 t + \delta^2) e^{C_2 \varepsilon t} \leq K\varepsilon,$$

what allows us to conclude. □

**Remark 2** *Knowing now for the inviscid case the approximation results between, from one hand, the solutions  $\bar{\mathbf{U}}_{KZK}$  of the KZK equation and  $\mathbf{U}_{Euler}$  of the Euler system [13, 40] (see Theorem 6.8 [13] for the definitions of  $\mathbf{U}_{Euler}$  and  $\bar{\mathbf{U}}_{KZK}$ ) in a cone*

$$C(T) = \{0 < t < T | T < \frac{T_0}{\varepsilon}\} \times Q_\varepsilon(t)$$

with

$$Q_\varepsilon(s) = \{x = (x_1, x') : |x_1| \leq \frac{R}{\varepsilon} - Ms, M \geq c, x' \in \mathbb{R}^{n-1}\}$$

and with

$$\|\nabla \mathbf{U}_{Euler}\|_{L^\infty([0, \frac{T_0}{\varepsilon}]; H^{s-1}(Q_\varepsilon))} < \varepsilon C \text{ for } s > \left[\frac{n}{2}\right] + 1,$$

and, from the other hand, between the solutions  $\mathbf{U}_{Euler}$  of the Euler system and  $\overline{\mathbf{U}}_{Kuzn}$  of the Kuznetsov equation [13] (see Theorem 6.6 [13] for the definitions of  $\mathbf{U}_{Euler}$  and  $\overline{\mathbf{U}}_{Kuzn}$ ) in  $[0, \frac{T_0}{\varepsilon}] \times \mathbb{R}^n$  containing  $C(T)$ , we obtain the approximation result between the solutions  $\overline{\mathbf{U}}_{KZK}$  of the KZK equation and the solutions  $\overline{\mathbf{U}}_{Kuzn}$  of the Kuznetsov equation in  $C(T)$  by the triangular inequality:

$$\|\overline{\mathbf{U}}_{Kuzn} - \overline{\mathbf{U}}_{KZK}\|_{L^2(Q_\varepsilon(t))}^2 \leq K(\varepsilon^3 t + \delta^2) e^{K\varepsilon t} \leq 9\varepsilon^2,$$

as soon as  $\|(\overline{\mathbf{U}}_{Kuzn} - \overline{\mathbf{U}}_{KZK})(0)\|_{L^2(Q_\varepsilon(0))} \leq \delta < \varepsilon$ . The initial data are constructed on the initial data  $I_0$  for the KZK equation. More precisely we take  $I_0 \in H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$  for  $s > \max\{10, [\frac{n}{2}] + 1\}$ , which ensures in the case of the same initial data

$$\overline{\mathbf{U}}_{Kuzn}(0) = \overline{\mathbf{U}}_{KZK}(0) = \mathbf{U}_{Euler}(0)$$

the existence with necessary regularity of all solutions: of the KZK equation, of the Euler system and of the Kuznetsov equation. Otherwise, to ensure the boundness and the minimal regularity  $C([0, \frac{T_0}{\varepsilon}]; L^2(Q_\varepsilon))$  of the remainder terms it sufficient to impose  $s \geq 6$ .

### 3 Approximation of the solutions of the Kuznetsov equation with the solutions of the NPE equation.

Now let us go back to the NPE equation (11) and consider its *ansatz* (see [13] for the derivation of the NPE equation from the isentropic Navier-Stokes system or Euler system). To compare to (12) for the KZK equation, this time the velocity potential is given [41] by

$$u(x, t) = \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon} x') = \Psi(\tau, z, y). \quad (51)$$

Thus we directly obtain from the Kuznetsov equation (9) with the paraxial change of variable

$$\tau = \varepsilon t, \quad z = x_1 - ct, \quad y = \sqrt{\varepsilon} x', \quad (52)$$

that

$$\begin{aligned} & \partial_t^2 u - c^2 \Delta u - \varepsilon \partial_t \left( (\nabla u)^2 + \frac{\gamma - 1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \\ &= \varepsilon \left( -2c \partial_{\tau z}^2 \Psi - c^2 \Delta_y \Psi + \frac{\nu}{\rho_0} c \partial_z^3 \Psi + \frac{\gamma + 1}{2} c \partial_z (\partial_z \Psi)^2 \right) + \varepsilon^2 R^{Kuz-NPE} \end{aligned}$$

with

$$\begin{aligned}
\varepsilon^2 R^{Kuz-NPE} = & \varepsilon^2 \left( \partial_\tau^2 \Psi - \frac{\nu}{\rho_0} \partial_z^2 \partial_\tau \Psi + \frac{\nu}{\rho_0} c \Delta_y \partial_z \Psi - (\gamma - 1) \partial_\tau \Psi \partial_z^2 \Psi \right. \\
& - 2(\gamma - 1) \partial_z \Psi \partial_{\tau z}^2 \Psi - 2 \partial_z \Psi \partial_{\tau z}^2 \Psi + 2c \nabla_y \Psi \nabla_y \partial_z \Psi \\
& + \varepsilon^3 \left( -\frac{\nu}{\rho_0} \Delta_y \partial_\tau \Psi + 2 \frac{\gamma - 1}{c} \partial_\tau \Psi \partial_{\tau z}^2 \Psi + \frac{\gamma - 1}{c} \partial_z \Psi \partial_\tau^2 \Psi \right. \\
& \left. \left. - 2 \nabla_y \Psi \nabla_y \partial_\tau \Psi \right) + \varepsilon^4 \left( -\frac{\gamma - 1}{c^2} \partial_\tau \Psi \partial_\tau^2 \Psi \right). \tag{53}
\end{aligned}$$

We obtain the NPE equation satisfying by  $\partial_z \Psi$  modulo a multiplicative constant:

$$\partial_{\tau z}^2 \Psi - \frac{\gamma + 1}{4} \partial_z (\partial_z \Psi)^2 - \frac{\nu}{2\rho_0} \partial_z^3 \Psi + \frac{c}{2} \Delta_y \Psi = 0.$$

In the sequel we will work with  $\xi$  defined by

$$\xi(\tau, z, y) = -\frac{\rho_0}{c} \partial_z \Psi, \tag{54}$$

which solves the Cauchy problem for the NPE equation

$$\begin{cases} \partial_{\tau z}^2 \xi + \frac{(\gamma+1)c}{4\rho_0} \partial_z^2 [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0 & \text{on } \mathbb{R}_+ \times \mathbb{T}_z \times \mathbb{R}^{n-1}, \\ \xi(0, z, y) = \xi_0(z, y) & \text{on } \mathbb{T}_z \times \mathbb{R}^{n-1}, \end{cases} \tag{55}$$

in the class of  $L$ -periodic functions with respect to the variable  $z$  and with mean value zero along  $z$ . The introduction of the operator  $\partial_z^{-1}$  defined similarly to  $\partial_\tau^{-1}$  in Eq. (18) allows us to consider instead of Eq. (11) the following equivalent equation

$$\partial_\tau \xi + \frac{(\gamma + 1)c}{4\rho_0} \partial_z [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^2 \xi + \frac{c}{2} \Delta_y \partial_z^{-1} \xi = 0 \text{ on } \mathbb{R}_+ \times \mathbb{T}_z \times \mathbb{R}^{n-1}.$$

This time, in comparison with the KZK equation, we use the bijection between this two models (see [13]). We also update our notation for  $\Omega_1 = \mathbb{T}_z \times \mathbb{R}_y^{n-1}$  and  $s > \frac{n}{2} + 1$ . Suppose that

$$\xi_0 \in H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1}) \quad \text{and} \quad \int_{\mathbb{T}_z} \xi_0(z, y) dz = 0.$$

Then there is a constant  $r > 0$  such that if  $\|\xi_0\|_{H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1})} < r$ , then, by Theorem 1, there is a unique solution  $\xi \in C([0, \infty[; H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1}))$  of the NPE Cauchy problem (55) satisfying

$$\int_{\mathbb{T}_z} \xi(\tau, z, y) dz = 0 \quad \text{for any } \tau \geq 0, y \in \mathbb{R}^{n-1}.$$

We define  $\partial_{x_1} \bar{u}(t, x_1, x') := -\frac{c}{\rho_0} \xi(\tau, z, y)$  with the change of variable (52) and

$$\bar{u}(t, x_1, x') = -\frac{c}{\rho_0} \partial_z^{-1} \xi(\tau, z, y) = \left( -\frac{c}{\rho_0} \right) \left( \int_0^z \xi(\tau, s, y) ds + \int_0^L \frac{s}{L} \xi(\tau, s, y) ds \right).$$

We take  $u_1(x_1, x') := \partial_t \bar{u}(0, x_1, x')$  and  $u_0(x_1, x') := -\frac{c}{\rho_0} \partial_z^{-1} \xi_0(z, y)$ , which implies

$$u_0 \in H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}) \text{ and } u_1 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}).$$

Thus for these initial data there exists

$$\bar{u} \in C([0, \infty[; H^{s+1}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})) \cap C^1([0, \infty[; H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})),$$

the unique solution on  $\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}$  of the approximated Kuznetsov system

$$\begin{cases} \bar{u}_{tt} - c^2 \Delta \bar{u} - \nu \varepsilon \Delta \bar{u}_t - \alpha \varepsilon \bar{u}_t \bar{u}_{tt} - \beta \varepsilon \nabla \bar{u} \nabla \bar{u}_t = \varepsilon^2 R^{Kuz-NPE}, \\ \bar{u}(0) = u_0 \in H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}), \quad \bar{u}_t(0) = u_1 \in H^{s+1}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}) \end{cases} \quad (56)$$

with  $R^{Kuz-NPE}$  defined in Eq. (53). If we consider the Cauchy problem

$$\begin{cases} \partial_t^2 u - c^2 \Delta u = \varepsilon \partial_t \left( (\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right), \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases} \quad (57)$$

for the Kuznetsov equation on  $\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}$  with  $u_0$  and  $u_1$  derived from  $\xi_0$ , we have

$$\|u_0\|_{H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})} + \|u_1\|_{H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})} \leq C \|\xi_0\|_{H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1})}.$$

Hence, if  $\|\xi_0\|_{H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1})}$  small enough [12], we have a unique bounded in time solution

$$u \in C([0, \infty[; H^{s+1}(\Omega)) \cap C^1([0, \infty[; H^s(\Omega))$$

of the Kuznetsov equation.

**Theorem 8** *For  $\nu \geq 0$ , we take the defined above solutions  $u$  of the exact Cauchy problem (57) and  $\bar{u}$  of the approximated Cauchy problem (56) for the Kuznetsov equation on  $\Omega = \mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}$  the case  $\nu = 0$  implying just similar results local in time. Then there exist  $K > 0$ ,  $C > 0$ ,  $C_1 > 0$  and  $C_2 > 0$  such that for all  $t < \frac{C}{\varepsilon}$  we have estimate (48) and in addition it holds Point 3 of Theorem 7.*

*Moreover, if for  $n \leq 3$   $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$  with  $s \geq 4$ , then the approximated solution satisfies*

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^4(\Omega)), \quad \partial_t \bar{u}(t, x_1, x') \in C([0, +\infty[; H^2(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; L^2(\Omega)). \end{aligned}$$

*If for  $n \geq 4$   $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$  with  $s \geq \frac{n}{2} + 2$ , then the approximated solution satisfies*

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^s(\Omega)), \quad \partial_t \bar{u}(t, x_1, x') \in C([0, +\infty[; H^{s-2}(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^{s-4}(\Omega)). \end{aligned}$$

*Under these conditions for  $n \geq 1$*

$$R^{Kuz-NPE} \in C([0, +\infty[; L^2(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})).$$

*For  $\nu = 0$  all previous results stay true on a time interval  $[0, T]$ .*



**Proof :** The global existence of  $u$  and  $\bar{u}$  has already been shown. The proof of the approximation estimate follows exactly as in Theorem 7 and is thus omitted. The case  $\nu = 0$  implies the same approximation result except that  $u$  and  $\bar{u}$  are only locally well posed on an interval  $[0, T]$ .

We can see for  $n = 2$  or  $3$ , using the previous arguments that the minimum regularity of the initial data (see Table 1) to have the remainder terms

$$R^{Kuz-NPE} \in C([0, +\infty[; L^2(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}))$$

corresponds to  $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$  with  $s \geq 4$ , since then for  $0 \leq k \leq 2$

$$\xi(\tau, z, y) \in C^k([0, +\infty[; H^{s-2k}(\mathbb{T}_z \times \mathbb{R}^{n-2})),$$

which finally implies with formula  $\bar{u} = -\frac{c}{\rho_0} \partial_z^{-1} \xi$  that for  $\Omega = \mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^4(\Omega)), \quad \partial_t \bar{u}(t, x_1, x') \in C([0, +\infty[; H^2(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; L^2(\Omega)). \end{aligned}$$

In the same way for  $n \geq 4$  we find the minimal regularity for  $\xi_0 \in H^s(\Omega)$  with  $s > \frac{n}{2} + 2$  as it implies

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^s(\Omega)), \quad \partial_t \bar{u}(t, x_1, x') \in C([0, +\infty[; H^{s-2}(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^{s-4}(\Omega)). \end{aligned}$$

The optimality of the chosen previously  $s$  also comes from the fact that in Eq. (53) the least regular term in  $R^{Kuz-NPE}$  is  $\partial_\tau \Psi \partial_\tau^2 \Psi$  presenting for both viscous and inviscid cases.  $\square$

## 4 Kuznetsov equation and the Westervelt equation

### 4.1 Derivation of the Westervelt equation from the Kuznetsov equation.

We consider the Kuznetsov equation (9). Similarly as in Ref. [1] we set

$$\Pi = u + \frac{1}{2c^2}\varepsilon\partial_t[u^2] \quad (58)$$

and obtain

$$\partial_t^2\Pi - c^2\Delta\Pi = \varepsilon\partial_t\left(\frac{\nu}{\rho_0}\Delta u + \frac{\gamma+1}{2c^2}(\partial_t u)^2 + \frac{1}{c^2}u(\partial_t^2 - c^2\Delta u)\right).$$

By Definition (58) of  $\Pi$  we have

$$\partial_t^2\Pi - c^2\Delta\Pi = \varepsilon\partial_t\left(\frac{\nu}{\rho_0}\Delta\Pi + \frac{\gamma+1}{2c^2}(\partial_t\Pi)^2\right) + \varepsilon^2 R^{Kuz-Wes},$$

where

$$\begin{aligned} \varepsilon^2 R^{Kuz-Wes} = & \varepsilon^2\partial_t\left[-\frac{1}{2c^2}\frac{\nu}{\rho_0}\Delta(u\partial_t u) - \frac{\gamma+1}{2c^4}\partial_t u\partial_t^2(u^2) \right. \\ & \left. + \frac{1}{c^2}u\partial_t\left((\nabla u)^2 + \frac{\gamma-1}{2c^2}(\partial_t u)^2 + \frac{\nu}{\rho_0}\Delta u\right)\right] \\ & + \varepsilon^3\partial_t\left[-\frac{\gamma+1}{8c^6}[\partial_t^2(u^2)]^2\right]. \end{aligned} \quad (59)$$

We recognize the Westervelt equation (8).

### 4.2 Approximation of the solutions of the Kuznetsov equation by the solutions of the Westervelt equation

The well-posedness of the Westervelt equation directly follows from [12]. For  $u$  solution of the Cauchy problem (57) for the Kuznetsov equation we set

$$\bar{\Pi} = u + \frac{1}{2c^2}\varepsilon\partial_t[u^2], \quad (60)$$

which is the solution of the following Cauchy problem for the Westervelt equation

$$\begin{cases} \partial_t^2\bar{\Pi} - c^2\Delta\bar{\Pi} = \varepsilon\partial_t\left(\frac{\nu}{\rho_0}\Delta\bar{\Pi} + \frac{\gamma+1}{2c^2}(\partial_t\bar{\Pi})^2\right) + \varepsilon^2 R^{Kuz-Wes}, \\ \bar{\Pi}(0) = \Pi_0, \partial_t\bar{\Pi}(0) = \Pi_1, \end{cases} \quad (61)$$

where  $R^{Kuz-Wes}$ , defined by Eq. (59), and in accordance with the definition of  $\bar{\Pi}$

$$\Pi_0 = u_0 + \frac{1}{c^2}\varepsilon u_0 u_1, \quad (62)$$

$$\begin{aligned} \Pi_1 &= u_1 + \frac{1}{c^2}\varepsilon u_1^2 + \frac{1}{c^2}\varepsilon u_0 \partial_t^2 u(0) \\ &= u_1 + \frac{1}{c^2}\varepsilon u_1^2 + \frac{1}{c^2}\varepsilon u_0 \frac{1}{1 - \frac{\gamma-1}{c^2}\varepsilon u_1} \left(c^2\Delta u_0 + \frac{\nu}{\rho_0}\varepsilon\Delta u_1 + 2\varepsilon\nabla u_0\nabla u_1\right) \end{aligned} \quad (63)$$

with  $u_0$  and  $u_1$  initial data of the Cauchy problem (57) for the Kuznetsov equation.

For  $s > \frac{n}{2}$  and  $\nu > 0$ , if we take  $u_0 \in H^{s+3}(\mathbb{R}^n)$  and  $u_1 \in H^{s+3}(\mathbb{R}^3)$ , we have  $\Pi_0 \in H^{s+3}(\mathbb{R}^n) \subset H^{s+2}(\mathbb{R}^n)$  and  $\Pi_1 \in H^{s+1}(\mathbb{R}^n)$  with

$$\|\Pi_0\|_{H^{s+2}(\mathbb{R}^n)} + \|\Pi_1\|_{H^{s+1}(\mathbb{R}^n)} \leq C(\|u_0\|_{H^{s+3}(\mathbb{R}^n)} + \|u_1\|_{H^{s+3}(\mathbb{R}^n)}).$$

For  $s > \frac{n}{2}$  and  $\nu = 0$ , if we take  $u_0 \in H^{s+3}(\mathbb{R}^n)$  and  $u_1 \in H^{s+2}(\mathbb{R}^3)$ , we have  $\Pi_0 \in H^{s+2}(\mathbb{R}^n)$  and  $\Pi_1 \in H^{s+1}(\mathbb{R}^n)$  with

$$\|\Pi_0\|_{H^{s+2}(\mathbb{R}^n)} + \|\Pi_1\|_{H^{s+1}(\mathbb{R}^n)} \leq C(\|u_0\|_{H^{s+3}(\mathbb{R}^n)} + \|u_1\|_{H^{s+2}(\mathbb{R}^n)}).$$

Then, similarly to our previous work [12] we obtain

**Theorem 9** *Let  $n \geq 1$  and  $s > \frac{n}{2}$ .*

1. *If  $\nu > 0$ ,  $u_0 \in H^{s+3}(\mathbb{R}^n)$  and  $u_1 \in H^{s+3}(\mathbb{R}^n)$ , then there exists a constant  $k_2 > 0$  such that for*

$$\|u_0\|_{H^{s+4}(\mathbb{R}^n)} + \|u_1\|_{H^{s+3}(\mathbb{R}^n)} < k_2, \quad (64)$$

*the Cauchy problem for the Westervelt equation*

$$\begin{cases} \partial_t^2 \Pi - c^2 \Delta \Pi = \varepsilon \partial_t \left( \frac{\nu}{\rho_0} \Delta \Pi + \frac{\gamma+1}{2c^2} (\partial_t \Pi)^2 \right), \\ \Pi(0) = \Pi_0, \partial_t \Pi(0) = \Pi_1 \end{cases} \quad (65)$$

*with  $\Pi_0$  and  $\Pi_1$  defined by Eqs. (62) and (63), has a unique global in time solution*

$$\Pi \in H^2([0, +\infty[, H^s(\mathbb{R}^n)) \cap H^1([0, +\infty[, H^{s+2}(\mathbb{R}^n)) \quad (66)$$

*and if  $s \geq 1$*

$$\Pi \in C([0, +\infty[, H^{s+2}(\mathbb{R}^n)) \cap C^1([0, +\infty[, H^{s+1}(\mathbb{R}^n)) \cap C^2([0, +\infty[, H^{s-1}(\mathbb{R}^n)) \quad (67)$$

*Moreover,  $\bar{\Pi}$ , obtained from the solution  $u$  of the Kuznetsov equation with Eq. (60), is the unique global in time solution of the approximated Cauchy problem (61) with the same regularity.*

2. *Let  $\nu = 0$ ,  $u_0 \in H^{s+3}(\mathbb{R}^n)$  and  $u_1 \in H^{s+2}(\mathbb{R}^n)$ . Then there exists a constant  $k_2 > 0$  such that if*

$$\|u_0\|_{H^{s+3}(\mathbb{R}^n)} + \|u_1\|_{H^{s+2}(\mathbb{R}^n)} < k_2, \quad (68)$$

*then the Cauchy problem (65) for the Westervelt equation with  $\Pi_0$  and  $\Pi_1$ , defined by Eqs. (62) and (63), has a unique local in time solution for a  $T > 0$*

$$\Pi \in C([0, T], H^{s+2}(\mathbb{R}^n)) \cap C^1([0, T], H^{s+1}(\mathbb{R}^n)) \cap C^2([0, T], H^s(\mathbb{R}^n)). \quad (69)$$

*Moreover,  $\bar{\Pi}$ , defined by Eq. (60), is the unique local in time solution of the approximated Cauchy problem (61) with the same regularity.*

For  $\Pi$  solution of the Cauchy problem for the Westervelt equation (65) we set  $\bar{u}$  such that  $\Pi = \bar{u} + \frac{\varepsilon}{c^2}\bar{u}\partial_t\bar{u}$  and we obtain

$$\begin{aligned} \partial_t^2\bar{u} - c^2\Delta\bar{u} - \varepsilon\frac{\nu}{\rho_0}\Delta\partial_t\bar{u} - \varepsilon\frac{\gamma-1}{c^2}\partial_t\bar{u}\partial_t^2\bar{u} - 2\varepsilon\nabla\bar{u}\cdot\nabla\partial_t\bar{u} \\ + \varepsilon\left(\frac{1}{c^2}\partial_t\bar{u}\partial_t^2\bar{u} - \partial_t\bar{u}\Delta\bar{u} + \frac{1}{c^2}\bar{u}\partial_t^3\bar{u} - \bar{u}\Delta\partial_t\bar{u}\right) = \varepsilon^2 R_1^{Wes-Kuz} \end{aligned}$$

with

$$\begin{aligned} R_1^{Wes-Kuz} = & \left[ \frac{\nu}{\rho_0 c^2} (2\partial_t\bar{u}\Delta\partial_t\bar{u} + 2(\nabla\partial_t\bar{u})^2 + \partial_t^2\bar{u}\Delta\bar{u} + \bar{u}\Delta\partial_t^2\bar{u} + 2\nabla\bar{u}\cdot\nabla\partial_t^2\bar{u}) \right. \\ & \left. + \frac{\gamma+1}{c^4} ((\partial_t\bar{u})^2 + \bar{u}\partial_t^2\bar{u})\partial_t^2\bar{u} + \frac{\gamma+1}{c^4} (3\partial_t\bar{u}\partial_t^2\bar{u} + \bar{u}\partial_t^3\bar{u})\partial_t\bar{u} \right] \\ & + \varepsilon\frac{\gamma+1}{c^6} ((\partial_t\bar{u})^2 + \bar{u}\partial_t^2\bar{u})(3\partial_t\bar{u}\partial_t^2\bar{u} + \bar{u}\partial_t^3\bar{u}). \end{aligned}$$

And as

$$\partial_t^2\bar{u} - c^2\Delta\bar{u} = O(\varepsilon)$$

if we inject this in the term  $(\frac{1}{c^2}\partial_t\bar{u}\partial_t^2\bar{u} - \partial_t\bar{u}\Delta\bar{u} + \frac{1}{c^2}\bar{u}\partial_t^3\bar{u} - \varepsilon\bar{u}\Delta\partial_t\bar{u})$  we have

$$\partial_t^2\bar{u} - c^2\Delta\bar{u} - \varepsilon\frac{\nu}{\rho_0}\Delta\partial_t\bar{u} - \varepsilon\frac{\gamma-1}{c^2}\partial_t\bar{u}\partial_t^2\bar{u} - 2\varepsilon\nabla\bar{u}\cdot\nabla\partial_t\bar{u} = \varepsilon^2 R^{Wes-Kuz}. \quad (70)$$

Now we can write the following result for the approximation of the Kuznetsov equation by the Westervelt equation.

**Theorem 10** *Let  $n \geq 2$  and  $s > \frac{n}{2}$  with  $s \geq 1$ ,  $\nu \geq 0$ .*

- For  $\bar{u}_0 \in H^{s+3}(\mathbb{R}^n)$  and  $\bar{u}_1 \in H^{s+3}(\mathbb{R}^n)$ , there exists  $k > 0$  such that

$$\|\bar{u}_0\|_{H^{s+3}(\mathbb{R}^n)} + \|\bar{u}_1\|_{H^{s+3}(\mathbb{R}^n)} < k$$

*implies the global (for  $\nu > 0$ ) or local (for  $\nu = 0$ ) existence of  $\Pi$  the solution of the Cauchy problem for the Westervelt equation (65) with  $\Pi_0$  and  $\Pi_1$  defined by Eqs. (62) and (63). The regularities of  $\Pi$  are given in Eqs. (66) and (67), if  $\nu > 0$ , and in Eq. (69), if  $\nu = 0$ .*

- Moreover, for  $u_0 \in H^{s+2}(\mathbb{R}^n)$  and  $u_1 \in H^{s+1}(\mathbb{R}^n)$  there exists the unique exact solution  $u$ , with the same regularity as  $\Pi$ , of the Cauchy problem (57) for the Kuznetsov equation.
- Let  $\bar{u}$  be defined by

$$\Pi = \bar{u} + \frac{\varepsilon}{c^2}\bar{u}\partial_t\bar{u}.$$

*Consequently  $\bar{u}$  is a solution of the approximated Kuznetsov equation (70) with  $\bar{u}(0) = \bar{u}_0$ ,  $\partial_t\bar{u}(0) = \bar{u}_1$ . If the initial data for  $u$  and  $\bar{u}$  satisfy (49):*

$$\|(u - \bar{u})_t(0)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(0)\|_{L^2(\Omega)}^2 \leq \delta^2 \leq \varepsilon^2,$$

*then there exist  $K > 0$ ,  $C > 0$ ,  $C_1 > 0$  and  $C_2 > 0$ , all independent of  $\varepsilon$ , such that for all  $t \leq \frac{C}{\varepsilon}$  it holds estimate (50).*

**Proof :** The existence of  $u$  and  $\bar{u}$  has already been shown by [12] and Theorem 9. The proof of the approximation estimate follows exactly the proof of Theorem 7 and hence it is omitted. The regularity on  $\bar{u}_0$  and  $\bar{u}_1$  (see expressions of  $u_0$  and  $u_1$  in Table 1) is minimal to ensure that  $R^{Wes-Kuz}$  (see Eq. (70)) is in  $C([0, +\infty[; L^2(\mathbb{R}^n))$ . Indeed, with  $\Pi_0$  and  $\Pi_1$  defined by Eqs. (62) and (63) it is necessary in order to have the well-posedness of  $\Pi$  with the same regularity as in Theorem 9.  $\square$

## 5 Conclusion

We summarize all obtained approximation results in a comparative table: Table 1 for the approximations of the Kuznetsov equation.

## References

- [1] S. I. AANONSEN, T. BARKVE, J. N. TJO/TTA, AND S. TJO/TTA, *Distortion and harmonic generation in the nearfield of a finite amplitude sound beam*, The Journal of the Acoustical Society of America, 75 (1984), pp. 749–768.
- [2] R. A. ADAMS, *Sobolev spaces*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.
- [3] S. ALINHAC, *Temps de vie des solutions régulières des équations d’Euler compressibles axisymétriques en dimension deux*, Invent. Math., 111 (1993), pp. 627–670.
- [4] ———, *A minicourse on global existence and blowup of classical solutions to multidimensional quasilinear wave equations*, in Journées “Équations aux Dérivées Partielles” (Forges-les-Eaux, 2002), Univ. Nantes, Nantes, 2002, pp. Exp. No. I, 33.
- [5] N. S. BAKHVALOV, Y. M. ZHILEĬ KIN, AND E. A. ZABOLOTSKAYA, *Nonlinear theory of sound beams*, American Institute of Physics Translation Series, American Institute of Physics, New York, 1987. Translated from the Russian by Robert T. Beyer.
- [6] ———, *Nonlinear theory of sound beams*, American Institute of Physics Translation Series, American Institute of Physics, New York, 1987. Translated from the Russian by Robert T. Beyer.
- [7] L. BERS, F. JOHN, AND M. SCHECHTER, *Partial differential equations*, American Mathematical Society, Providence, R.I., 1979. With supplements by Lars Gårding and A. N. Milgram, With a preface by A. S. Householder, Reprint of the 1964 original, Lectures in Applied Mathematics, 3A.
- [8] P. CAINE AND M. WEST, *A tutorial on the non-linear progressive wave equation (NPE). Part 2. Derivation of the three-dimensional cartesian version without use of perturbation expansions*, Applied Acoustics, 45 (1995), pp. 155–165.

Table 1: Approximation results for models derived from the Kuznetsov equation

	KZK		NPE	Westervelt	
	periodic boundary condition problem	initial boundary value problem	viscous and inviscid case	viscous case	inviscid case
Theorem	Theorem 6	Theorem 7	Theorem 8	Theorem 10	
Derivation	paraxial approximation $u = \Phi(t - \frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon} \mathbf{x}')$		paraxial approximation $u = \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon} \mathbf{x}')$	$\Pi = u + \frac{1}{c^2} \varepsilon u \partial_t u$	
Approximation domain	the half space $\{x_1 > 0, x' \in \mathbb{R}^{n-1}\}$		$\mathbb{T}_{x_1} \times \mathbb{R}^2$	$\mathbb{R}^3$	
Approximation order	$O(\varepsilon)$		$O(\varepsilon)$	$O(\varepsilon^2)$	
Estimation	$\ I - I_{approx}\ _{L^2(\mathbb{T}_t \times \mathbb{R}^{n-1})} \leq \varepsilon$ $z \leq K$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+\ \nabla(u - \bar{u})(t)\ _{L^2}$ $\leq K\varepsilon.$ $t < \frac{T}{\varepsilon}$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+\ \nabla(u - \bar{u})(t)\ _{L^2}$ $\leq K\varepsilon$ $t < \frac{T}{\varepsilon}$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+\ \nabla(u - \bar{u})(t)\ _{L^2}$ $\leq K\varepsilon$ $t < \frac{T}{\varepsilon}$	
Initial data regularity	$I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $s > \max(\frac{n}{2}, 2)$	$I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $[\frac{s}{2}] > \frac{n}{2} + 2$	$\xi_0 \in H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$ for $s > \frac{n}{2} + 1$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+3}(\mathbb{R}^3)$ for $s > \frac{n}{2}$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+2}(\mathbb{R}^3)$ for $s > \frac{n}{2}$
Data regularity for remainder boundness	$I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $s > \max(\frac{n}{2}, 2)$	$I_0 \in H^6(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $n = 2, 3,$ $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $[\frac{s}{2}] > \frac{n}{2} + 1, n \geq 4$	$\xi_0 \in H^4(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$ for $n = 2, 3.$ $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$ for $s > \frac{n}{2} + 2, n \geq 4.$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+3}(\mathbb{R}^3)$ for $s > \frac{n}{2}$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+2}(\mathbb{R}^3)$ for $s > \frac{n}{2}$

- [9] Z. CAO, H. YIN, L. ZHANG, AND L. ZHU, *Large time asymptotic behavior of the compressible Navier-Stokes equations in partial space-periodic domains*, Acta Math. Sci. Ser. B (Engl. Ed.), 36 (2016), pp. 1167–1191.
- [10] A. CELIK AND M. KYED, *Nonlinear wave equation with damping: Periodic forcing and non-resonant solutions to the Kuznetsov equation*, (2016).
- [11] C. M. DAFERMOS, *Hyperbolic conservation laws in continuum physics*, vol. 325 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, fourth ed., 2016.
- [12] A. DEKKERS AND A. ROZANOVA-PIERRAT, *Cauchy problem for the Kuznetsov equation*, Discrete Contin. Dyn. Syst. Ser. A, 39 (2016), pp. 277–307.
- [13] A. DEKKERS AND A. ROZANOVA-PIERRAT, *Models of nonlinear acoustics viewed as an approximation of the Navier-Stokes and Euler compressible isentropic systems*, submitted,hal-01935515v2.
- [14] R. DENK, M. HIEBER, AND J. PRÜSS, *Optimal  $L^p - L^q$ -estimates for parabolic boundary value problems with inhomogeneous data*, Math. Z., 257 (2007), pp. 193–224.
- [15] G. DI BLASIO, *Linear parabolic evolution equations in  $L^p$ -spaces*, Ann. Mat. Pura Appl. (4), 138 (1984), pp. 55–104.
- [16] M. GHISI, M. GOBBINO, AND A. HARAUX, *Local and global smoothing effects for some linear hyperbolic equations with a strong dissipation*, Trans. Amer. Math. Soc., 368 (2016), pp. 2039–2079.
- [17] P. GRISVARD, *Elliptic problems in nonsmooth domains*, vol. 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [18] B. GUSTAFSSON AND A. SUNDSTRÖM, *Incompletely parabolic problems in fluid dynamics*, SIAM J. Appl. Math., 35 (1978), pp. 343–357.
- [19] M. HAMILTON AND D. BLACKSTOCK, *Nonlinear Acoustics*, Academic Press, 1998.
- [20] D. HOFF, *Strong Convergence to Global Solutions for Multidimensional Flows of Compressible, Viscous Fluids with Polytopic Equations of State and Discontinuous Initial Data*, Arch. Rational Mech. Anal. 132 (1995) 1-14.
- [21] D. HOFF, *Discontinuous Solutions of the Navier-Stokes Equations for Multidimensional Flows of Heat-Conducting Fluids*, Arch. Rational Mech. Anal. 139 (1997), pp. 303–354.
- [22] K. ITO, *Smooth global solutions of the two-dimensional Burgers equation*, Canad. Appl. Math. Quart., 2 (1994), pp. 283–323.

- [23] F. JOHN, *Nonlinear wave equations, formation of singularities*, vol. 2 of University Lecture Series, American Mathematical Society, Providence, RI, 1990. Seventh Annual Pitcher Lectures delivered at Lehigh University, Bethlehem, Pennsylvania, April 1989.
- [24] P. M. JORDAN, *An analytical study of Kuznetsov's equation: diffusive solitons, shock formation, and solution bifurcation*, Phys. Lett. A, 326 (2004), pp. 77–84.
- [25] B. KALTENBACHER AND I. LASIECKA, *Well-posedness of the Westervelt and the Kuznetsov equation with nonhomogeneous Neumann boundary conditions*, Discrete Contin. Dyn. Syst., (2011), pp. 763–773.
- [26] B. KALTENBACHER AND M. THALHAMMER, *Fundamental Models in Nonlinear Acoustics - Part I. Analytical Comparison* Math. Models Methods Appl. Sci., 28 (2018), pp. 2403–2455.
- [27] ———, *An analysis of nonhomogeneous Kuznetsov's equation: local and global well-posedness; exponential decay*, Math. Nachr., 285 (2012), pp. 295–321.
- [28] T. KATO, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Rational Mech. Anal., 58 (1975), pp. 181–205.
- [29] V. KUZNETSOV, *Equations of nonlinear acoustics*, Soviet Phys. Acoust., 16 (1971), pp. 467–470.
- [30] O. A. LADYŽENSKAJA, V. A. SOLONNIKOV, AND N. N. URAL'CEVA, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968.
- [31] M. LESSER AND R. SEEBASS, *The structure of a weak shock wave undergoing reflection from a wall*, J. Fluid Mech., 31 (1968), pp. 501–528.
- [32] T. LUO, C. XIE AND Z. XIN, *Non-uniqueness of admissible weak solutions to compressible Euler systems with source terms*, Adv. Math., 291 (2016), pp. 542–583.
- [33] S. MAKAROV AND O. M., *Nonlinear and thermoviscous phenomena in acoustics, part ii*, Acustica, 83 (1997), pp. 197–222.
- [34] A. MATSUMURA AND T. NISHIDA, *Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Comm. Math. Phys., 89 (1983), pp. 445–464
- [35] A. MATSUMURA AND T. NISHIDA, *The initial value problem for the equations of motion of viscous and heat-conductive gases*, J. Math. Kyoto Univ., 20 (1980), pp. 67–104.
- [36] B.E. McDONALD, P. CAINE AND M. WEST, *A tutorial on the nonlinear progressive wave equation (NPE)–Part 1*. Applied Acoustics, 43 (1994), pp. 159–167.



- [37] B. E. McDONALD AND W. A. KUPERMAN, *Time-domain solution of the parabolic equation including nonlinearity*, Comput. Math. Appl., 11 (1985), pp. 843–851. Computational ocean acoustics (New Haven, Conn., 1984).
- [38] S. MEYER AND M. WILKE, *Global well-posedness and exponential stability for Kuznetsov’s equation in  $L_p$ -spaces*, Evol. Equ. Control Theory, 2 (2013), pp. 365–378.
- [39] A. ROZANOVA-PIERRAT, *Qualitative analysis of the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation*, Math. Models Methods Appl. Sci., 18 (2008), pp. 781–812.
- [40] ———, *On the derivation of the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation and validation of the KZK-approximation for viscous and non-viscous thermo-elastic media*, Commun. Math. Sci., 7 (2009), pp. 679–718.
- [41] A. ROZANOVA-PIERRAT *Approximation of a compressible Navier-Stokes system by non-linear acoustical models*, Proceedings of the International Conference DAYS on DIFFRACTION, 2015 May 25–29, 2015, St. Petersburg, Russia, pp. 270–276.
- [42] T. C. SIDERIS, *Formation of singularities in three-dimensional compressible fluids*, Comm. Math. Phys., 101 (1985), pp. 475–485.
- [43] ———, *The lifespan of smooth solutions to the three-dimensional compressible Euler equations and the incompressible limit*, Indiana Univ. Math. J., 40 (1991), pp. 535–550.
- [44] ———, *The lifespan of 3D compressible flow*, in Séminaire sur les Équations aux Dérivées Partielles, 1991–1992, École Polytech., Palaiseau, 1992, pp. Exp. No. V, 12.
- [45] ———, *Delayed singularity formation in 2D compressible flow*, Amer. J. Math., 119 (1997), pp. 371–422.
- [46] M. F. SUKHININ, *On the solvability of the nonlinear stationary transport equation*, Teoret. Mat. Fiz., 103 (1995), pp. 23–31.
- [47] J. N. TJO/TTA AND S. TJO/TTA, *Nonlinear equations of acoustics, with application to parametric acoustic arrays*, The Journal of the Acoustical Society of America, 69 (1981), pp. 1644–1652.
- [48] P. J. WESTERVELT, *Parametric acoustic array*, The Journal of the Acoustical Society of America, 35 (1963), pp. 535–537.
- [49] H. YIN AND Q. QIU, *The lifespan for 3-D spherically symmetric compressible Euler equations*, Acta Math. Sinica (N.S.), 14 (1998), pp. 527–534.