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# SUBLINEAR QUASICONFORMALITY AND THE LARGE-SCALE GEOMETRY OF HEINTZE GROUPS

GABRIEL PALLIER

ABSTRACT. This article analyzes sublinearly quasisymmetric homeomorphisms (generalized quasisymmetric mappings), and draws applications to the sublinear large-scale geometry of negatively curved groups and spaces. It is proven that those homeomorphisms lack analytical properties but preserve a conformal dimension and appropriate function spaces, distinguishing certain (nonsymmetric) Riemannian negatively curved homogeneous spaces, and Fuchsian buildings, up to sublinearly biLipschitz equivalence (generalized quasiisometry).

## INTRODUCTION

An embedding  $f$  between metric spaces is quasisymmetric if there is an increasing homeomorphism  $\eta : [0, +\infty) \rightarrow [0, +\infty)$  such that for any  $x, y, z$  in the source space and positive real  $t$ ,

$$(1) \quad d(x, y) \leq td(x, z) \implies d(f(x), f(y)) \leq \eta(t)d(f(x), f(z)).$$

The properties of sufficiently well-behaved compact metric spaces that are invariant under quasisymmetric homeomorphisms are known to be counterparts of the coarse (or quasiisometrically invariant) properties of proper geodesic Gromov-hyperbolic spaces, the two categories being related by the Gromov boundary and hyperbolic cone functors ([BS00], [Roe03, 2.5]). Instances are the conformal dimension [Pan89] and the  $\ell_p$  or  $L_p$  cohomology [BP03].

This article is part of our aim to transpose this equivalence by replacing quasiisometries with sublinearly biLipschitz equivalences, which originated from the work of Cornulier on the asymptotic cones of connected Lie groups<sup>1</sup> [Cor08]. Here the sublinear feature is described by an asymptotic class  $O(u)$ , where  $u$  is a strictly sublinear nondecreasing positive function on the half line such that  $\limsup_r u(2r)/u(r) < +\infty$ , e.g.  $u(r) = \log r$  (we call such a function admissible).

In previous work the Gromov-boundary behavior of sublinearly biLipschitz equivalences between Gromov-hyperbolic spaces was characterized [Pal18,

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<sup>1</sup>Beware that we use the terminology of [Cor17].

Theorem 1]. It differs from that of quasimetric homeomorphisms sublinearly in a certain sense; we shall indicate how in 1.2. The purpose of the present paper is to push further the analysis of those boundary mappings and identify the structure preserved on the boundary. A numerical invariant is derived. It is denoted by  $\text{Cdim}_{O(u)}$ ; Pansu's conformal dimension introduced in [Pan89, 3] and usually denoted  $\text{Cdim}$  corresponds to  $\text{Cdim}_{O(1)}$ . We compute this invariant and prove that it equals  $\text{Cdim}$  on the examples originally studied by Pansu and Bourdon (that we recall below). Function spaces of locally bounded energy, that are carried by sublinearly quasimetric mappings up to shifts in parameters, are also constructed. These form algebras reminiscent of the Sobolev spaces  $W_{\text{loc}}^{1,p}$ . One can sometimes have access to the topological dimension of the spectrum of these algebras; its dependence over  $p$  measures the degree of energy needed to break invariance along certain foliations on the boundary, and provides further invariants. This later approach is inspired from Bourdon [Bou07, p.248], Bourdon-Kleiner [BK13, Section 10] and Carrasco Piaggio [CP17, p.465].

A purely real Heintze group is a simply connected solvable group which splits as an extension of  $\mathbf{R}$  by its nilradical  $N$ , associated to  $\rho : \mathbf{R} \rightarrow \text{Aut}(\text{Lie}(N))$  with positive real roots. From such a group  $H$  one can make another one  $H_\Sigma$  by forgetting the unipotent part of  $\rho$ . Since the nilradical of  $H$  is strictly exponentially distorted, following Cornuier one can prove that this does not alter the logarithmic sublinear large-scale structure (see [Cor11, Th 1.2] recalled here in 3.1.1). We prove a partial converse.

**Theorem.** *Let  $H$  and  $H'$  be purely real Heintze groups with abelian nilradicals. Let  $u$  be any sublinear, admissible function. If  $H$  and  $H'$  are  $O(u)$ -sublinearly biLipschitz equivalent then  $H_\Sigma$  and  $H'_\Sigma$  are isomorphic.*

This answers positively to Cornuier [Cor17, 1.16(1)] who raised the question for  $\dim H = 3$ . For comparison, it is known that two purely real Heintze groups with abelian nilradicals are quasiisometric if and only if they are isomorphic by the work of Xie [Xie14]; this was also obtained by Carrasco Piaggio [CP17, 1.10]. In the vein of conjecture [Cor18, 6C2], we ask the following question.

**Question.** *Let  $H$  and  $H'$  be purely real Heintze group. Assume that  $H$  and  $H'$  are sublinearly biLipschitz equivalent. Are  $H_\Sigma$  and  $H'_\Sigma$  isomorphic?*

A positive answer would imply the previous theorem as well as [Pal18, Theorem 2]. The classification problem can be motivated beyond Lie groups by the fact that the purely real Heintze groups are known to parametrize other objects:

- The commability<sup>2</sup> classes of compactly generated locally compact groups that are hyperbolic with a topological sphere at infinity [Cor15, 5.16].
- Together with orbits of scalar products, the connected Riemannian negatively curved homogeneous spaces [Hei74] [GW88, Corollary 5.3].

Unlike Heintze groups, hyperbolic buildings become rare in large dimension [GP01]. The two-dimensional case displays a vast subfamily with local finiteness properties, that of Fuchsian buildings, for which the dimension at infinity  $\text{Cdim } \partial_\infty$  is known: it was computed by Bourdon in 1997 [Bou97] for some of them and 2000 in full generality [Bou00]. We check that  $\text{Cdim}_{O(u)} \partial_\infty$  equals the former in this case, distinguishing pairs of Fuchsian buildings up to sublinear biLipschitz equivalence. Here is the statement for the Bourdon buildings.

**Proposition** (Strengthening of [Bou97, Théorème 1.1]). *Let  $p, q \in \mathbf{Z}$  with  $p \geq 5$  and  $q \geq 2$ . Let  $I_{pq}$  be a Bourdon building (right-angled Fuchsian, with constant thickness  $q$ ). For all strictly sublinear admissible  $u$ ,*

$$(2) \quad \text{Cdim}_{O(u)} \partial_\infty I_{pq} = \text{Cdim}_{O(1)} \partial_\infty I_{pq} = 1 + \frac{\log(q-1)}{\text{argch}((p-2)/2)}.$$

*Convention, notation.* Through all the text,  $u : \mathbf{R}_{\geq 0} \rightarrow \mathbf{R}_{\geq 1}$  is a nondecreasing, strictly sublinear, doubling function, i.e.  $u(r) \ll r$  as  $r \rightarrow +\infty$  and  $\sup_r u(2r)/u(r) < +\infty$ . Examples are:  $u(r) = \sup(1, r^\gamma)$  with  $0 \leq \gamma < 1$  and  $u(r) = \sup(1, \log(r))$ . The combinatorial moduli and certain associated measures are multiply parametrized; we stick to Pansu’s and Tyson’s notation [Tys98] [Pan89], but in order to emphasize certain monotonicities with respect to the parameters the following convention will be applied: when  $m$  in a poset  $M$  is parametrized over  $p$  in a poset  $P$ , we write  $(m_p)_{p \in P}$  if  $p \leq p' \implies m_{p'} \leq m_p$ , and  $(m^p)_{p \in P}$  if  $p \leq p' \implies m^p \leq m^{p'}$ .

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<sup>2</sup>Namely, to such a group  $G$  one can associate the purely real core of the unique focal-universal group commable to  $G$ . Commability is a variant of weak commensurability adapted to the locally compact setting, see [Cor15]. For the definition of a hyperbolic locally compact group see [CC<sup>+</sup>15].

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## 1. SUBLINEAR QUASICONFORMALITY

1.1.  **$O(u)$ -quasisymmetric structures.** The notion of a quasisymmetric structure is a reformulation of that of a space with a quasidistance, where the emphasis is made on balls, their inclusion relations and relative sizes, rather than on a given quasidistance function. Related notions are:  $b$ -metric topological spaces [Mar91, IV.1], Margulis structures [GP91, p.62].

1.1.1. *Definition.*

**Definition 1.1** (Compare<sup>3</sup> [Pan89, 1.1 and 2.7] for  $u = 1$ ). Let  $Z$  be a set. A  $O(u)$ -quasisymmetric structure on  $Z$  is a set  $\beta$  of abstract balls<sup>4</sup> together with a realization map  $\beta \rightarrow \mathcal{P}(Z) \setminus \{\emptyset\}$ ,  $b \mapsto \widehat{b}$ , a map  $\delta : \beta \rightarrow \mathbf{Z}$  and a shift map  $\mathbf{Z}_{\geq 0} \times \beta \rightarrow \beta$ ,  $(k, b) \mapsto k.b$  such that

- (SC0) The shift is an action and  $\delta$  is equivariant with respect to the shift:  $\forall k, k' \in \mathbf{Z}_{\geq 0}, k'.k.b = (k' + k).b$  and  $\forall k \in \mathbf{Z}, \forall b \in \beta, \delta(k.b) = \delta(b) - k$ .
- (SC1)  $\forall k \in \mathbf{Z}_{\geq 0}, \forall b, b' \in \beta$ ,
  - (i)  $\widehat{k.b} \supseteq \widehat{b}$
  - (ii) if  $\widehat{b} \subseteq \widehat{b'}$  then  $\widehat{k.b} \subseteq \widehat{k.b'}$
  - (iii) if  $\delta(b) < \delta(b')$  then  $\widehat{b} \not\subseteq \widehat{b'}$ .
- (SC2) There exists  $n_0 \in \mathbf{Z}_{\geq 0}$  and a function  $q : \mathbf{Z}_{\geq n_0} \rightarrow \mathbf{Z}_{\geq 0}$ ,  $q = O(u)$  and such that

$$\forall b, b' \in \beta, \left( n_0 \leq \delta(b) \leq \delta(b'), \widehat{b} \cap \widehat{b'} \neq \emptyset \right) \implies q(\widehat{\delta(b)}) . b \supset \widehat{b'}.$$

- (SC3)  $\forall x \in Z, \forall y \in Z \setminus \{x\}, \forall n \in \mathbf{Z}, \exists b \in \beta : \delta(b) \geq n, x \in \widehat{b}, y \notin \widehat{b}$ .

<sup>3</sup>In [Pan89, 1.1 and 2.7] they are called “bonnes structures quasiconformes”. The “bonne” axiom is a disguised form of the quasi-triangle inequality, here (SC2).

<sup>4</sup>This formalism is here to avoid referring directly to centers and radii, which are preferable to diameters, but may not be uniquely defined. The notion of a constituent (see [Edg01, Definition 2]) circumvents the problem of radii, but it still makes use of centers.

**Example 1.1** (Space with a quasidistance). Recall that a quasidistance on a set  $\mathcal{Z}$  is a kernel  $\varrho : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathbf{R}$  with the axioms of a distance, the triangle inequality being replaced by

$$(\Delta_K) \quad \forall (x, y, z) \in \mathcal{Z}^3, \varrho(x, z) \leq K (\varrho(x, y) \vee \varrho(y, z))$$

where  $K \in \mathbf{R}_{\geq 1}$  is a constant. Given a dense<sup>5</sup> subspace (to be thought of as a set of centers)  $X \subseteq \mathcal{Z}$ , a quasidistance gives to  $\mathcal{Z}$  a  $O(1)$ -quasisymmetric structure in which  $\beta = X \times \mathbf{Z}$  and for  $b = (x, n)$  in  $\beta$  and  $k \in \mathbf{Z}$ ,  $\delta(b) = n$ ,  $k.b = (x, n - k)$  and  $\widehat{b} = \{z \in \mathcal{Z} : \varrho(x, z) \leq e^{-n}\}$ .  $(\Delta_K)$  is responsible for (SC2) with  $q = K^2$ , the separation axiom for (SC3).

**Subexample 1.1.1.**  $Z = \mathbf{R}$  and  $\beta$  is  $\mathbf{R} \times \mathbf{Z}$ ; for  $b = (x, n)$ ,  $\delta(b) = -n$ . For all  $b = (s, n)$  in  $\beta$ ,  $\widehat{b} = s + e^{-n}[0, 1]$  (One can replace  $[0, 1]$  by any bounded closed interval). One can take  $q = 3/2$  in (SC2). The shift is such that  $\widehat{k.b} = e^k \widehat{b}$ .

**Proposition 1.2.** *Let  $(Z, \beta, q, \delta)$  be a  $O(u)$ -quasisymmetric structure. For all  $n \in \mathbf{Z}$ , define*

$$E_n = \bigcup_{b \in \beta : \delta(b) \geq n} \widehat{b} \times \widehat{b}.$$

*Then  $E_n$  forms a fundamental system of entourages, endowing  $Z$  with a uniform structure.*

*Proof.* Let  $\mathcal{E} \subset \mathcal{P}(Z \times Z)$  be the set of subsets containing one of the  $E_n$ . It follows from the definition that  $\mathcal{E}$  is a filter. Let  $E \in \mathcal{E}$  and let  $n \in \mathbf{Z}$  be such that  $E \supset E_n$ . To check Weil's axiom (U'III) [Wei37, p.8] one needs find  $m \in \mathbf{Z}$  such that

$$(3) \quad \forall x, y, z \in E, \{(x, y)\} \cup \{(y, z)\} \subset E_m \implies (x, z) \in E_n.$$

This can be rephrased as follows: for any pair of distinct  $x, y \in Z$ , set

$$\varrho(x, y) = \exp(-\inf\{\delta(b) : b \in \beta, \{x\} \cup \{y\} \subset b\})$$

and  $\varrho(x, x) = 0$ . Especially  $(x, y) \in E_n \iff -\log \varrho(x, y) \geq n$ . Then for all  $x, y, z \in Z^3$

$$(\Delta_{O(u)}) \quad \varrho(x, z) \leq e^{v(\inf\{-\log \varrho(x, y), -\log \varrho(y, z)\})} [\varrho(x, y) \vee \varrho(y, z)]$$

where  $v = O(u)$  (one may take  $v(n) = q(n)$  at least for  $n \geq n_0$ ). Set  $m_0 = 2n \vee 2 \sup\{m' \in \mathbf{Z}_{\geq 0} : v(m') \geq \frac{m'}{2}\}$ . Then for every  $m \geq m_0$ ,  $m - v(m) \geq m - m/2 = m/2 \geq n$ . (3) is achieved.  $\square$

*Remark 1.3.* An open subspace  $\Omega$  of a  $O(u)$ -quasisymmetric structure  $(Z, \beta)$  inherits a  $O(u)$ -quasisymmetric structure  $(\Omega, \beta|_{\Omega})$  where

$$\beta|_{\Omega} = \left\{ b \in \beta : \forall k \in \mathbf{Z}_{\geq 0}, \widehat{k.b} \cap \Omega \neq \emptyset \right\},$$

the shift is restricted to  $\beta|_{\Omega}$ , and the realization is  $\widehat{b}|_{\Omega} = \widehat{b} \cap \Omega$ .

<sup>5</sup>A quasidistance induces a topology, see [PR18, 1.99].

1.1.2. *Hyperbolic cones and sublinear large-scale geometry.* The boundary of a Gromov-hyperbolic space has a Margulis structure, see e.g. [GP91]; further, the boundary construction can be reversed as suggested by M. Gromov [Gro87, 1.8.A(b)] and elaborated by M. Bonk and O. Schramm ([BS00, § 7], see also [Pau96]), so that in the current formalism any  $O(1)$ -quasisymmetric structure occurs at the boundary of a Gromov-hyperbolic space<sup>6</sup>. It is a classical fact that quasiisometries between Gromov hyperbolic groups extend to biHölder, quasisymmetric homeomorphism between their boundaries, i.e. they do so in a way that preserves the features of the  $O(1)$ -quasisymmetric structure. This paper is rather concerned with sublinearly biLipschitz maps, for which we recall the definition:

**Definition 1.2** (Cornulier, [Cor17]). Let  $(Y, o)$  and  $(Y', o')$  be metric spaces. A  $O(u)$ -sublinearly biLipschitz equivalence (SBE) is a map  $f : Y \rightarrow Y'$  for which there exists  $\lambda \in \mathbf{R}_{\geq 1}$  and  $v = O(u)$  such that

- (1)  $\forall y_1, y_2 \in Y, \frac{1}{\lambda}d(y_1, y_2) - v(\sup\{d(o, y_1), d(o, y_2)\}) \leq d(f(y_1), f(y_2))$
- (2)  $\forall y_1, y_2 \in Y, \lambda d(y_1, y_2) + v(\sup\{d(o, y_1), d(o, y_2)\}) \geq d(f(y_1), f(y_2))$
- (3)  $\forall y' \in Y', \exists y \in Y, d(y', f(y)) \leq v(d(y, o))$ .

Unlike quasiisometries (which are the  $O(u)$ -SBE with  $u = 1$ ), SBEs are not coarse equivalences in general. However they do preserve certain coarse sublinear structures in the sense of Dranishnikov and Smith [DS07, 2], or large-scale sublinear structures in the sense of Dydak and Hoffland [DH08, p.1014]. Sublinear-conformal structures are boundary analogs of the former, in a more specific way where  $u$  is explicit. In all our applications  $Y$  and  $Y'$  will be Gromov-hyperbolic, proper geodesic metric spaces, and when the boundaries come under consideration the function  $d(o, \cdot)$  could be replaced by the positive part of a fixed Busemann function  $h$  on  $Y$  (the resulting requirements are weaker and fulfilled by SBE maps, even if function  $v$  may be changed). Boundary maps of sublinearly biLipschitz equivalences are still homeomorphisms, however a notion more general than quasiconformality needs to be defined.

## 1.2. $O(u)$ -quasisymmetric homeomorphisms.

1.2.1. *Definition and comparison with quasisymmetric mappings.* Denote by  $\mathcal{O}^+(u)$  the semigroup of germs of functions  $v$  valued in  $\mathbf{Z}_{\geq 0}$ , defined on large enough integers, such that  $v = O(u)$ , with the composition law  $(v_1 \dot{+} v_2)(n) = v_2(n) + v_1(n - v_2(n))$ .  $\mathbf{Z}_{\geq 0}$  embeds in  $\mathcal{O}^+(u)$  as the commutative<sup>7</sup> subsemigroup of constant functions.  $\mathcal{O}^+(u)$  acts on small enough abstract

<sup>6</sup>Namely a certain quotient space of  $\beta$ , two abstract balls being close if close for  $\delta$  and if their realizations intersect, compare e.g. [Roe03, chapter 2]. Abstract, resp. concrete balls are turned into geodesic segments, resp. their endpoints. The metric hyperbolicity is implied by (SC1) and (SC2).

<sup>7</sup>Noncommutativity of  $\mathcal{O}^+(u)$  should not be a concern: the defect of commutativity is analogous to a higher-order term in an asymptotic expansion; we prefer not to formalize this here.

balls: for every  $v$  in  $\mathcal{O}^+(u)$  there exists  $n_0 \in \mathbf{Z}$  such that  $\mathbf{Z}_{\geq n_0}$  lies in the domain of  $v$  and if  $\delta(b) \geq n_0$  then  $v.b$  is defined as  $v.b = v(\delta(b)).b$ .

**Definition 1.3** (rounds sets and rings, compare [Tys98, 3.4]). Let  $\beta \rightarrow \mathcal{P}(Z)$  be a  $O(u)$ -quasisymmetric structure. Given  $k \in O(u)_{\geq 0}$  and  $n \in \mathbf{Z}$ , a subset  $a \in \mathcal{P}(X)$  is a  $(k, n)$ -round set (or simply a  $k$ -round set) if there exists  $b \in \beta$  such that  $\delta(b) = n$  and  $\widehat{b} \subseteq a \subseteq \widehat{k.b}$ . A couple of subsets  $(a^-, a^+) \in \mathcal{P}(X)^2$  is a  $(k, n)$ -ring if there exists  $b \in \beta$  such that  $\delta(b) \geq n$  and  $\widehat{b} \subseteq a^- \subseteq a^+ \subseteq \widehat{k.b}$ . Denote by  $\mathcal{B}_n^k(\beta)$  resp.  $\mathcal{R}_n^k(\beta)$  the collection of  $(k, n)$ -round sets, resp. of  $(k, n)$ -rings, and  $\mathcal{B}^k(\beta)$  resp.  $\mathcal{R}^k(\beta)$  their union over  $n \in \text{domain}(k)$ .

**Definition 1.4** (outer rings). Let  $\beta \rightarrow \mathcal{P}(Z)$  be a  $O(u)$ -quasisymmetric structure. Given  $j \in \mathcal{O}^+(u)$ , a pair of subsets  $(a^-, a^+) \in \mathcal{P}(X)^2$  is a  $(j, n)$ -outer ring if there exists  $n \in \mathbf{Z}$  and  $b \in \beta$  such that  $a^- \subseteq \widehat{b} \subseteq \widehat{j.b} \subseteq a^+$  and  $\delta(b) \geq n$ . Denote by  $\mathcal{O}_{j;n}(\beta)$  the collection of  $(j, n)$  outer rings.

The reader may think of  $k$  as a parameter of asphericity<sup>8</sup> (akin to  $\log t$  in (1)) that depends on the scale. Whereas quasisymmetric mappings preserve bounded asphericities,  $O(u)$ -conformal homeomorphisms will be asked to preserve asphericities within the  $O(u)$  class. We define them in two steps.

**Definition 1.5** (Equivalent  $O(u)$ -quasisymmetric structures). Let  $\beta$  and  $\beta'$  be two  $O(u)$ -quasisymmetric structures on a set  $Z$ .  $\beta'$  is finer than  $\beta$  if there exists  $\lambda \in \mathbf{R}_{>0}$  and  $n_0 \in \mathbf{Z}$  such that

$$(4) \quad \forall k \in \mathcal{O}^+(u), \exists k' \in \mathcal{O}^+(u) : \forall n \in \mathbf{Z}_{\geq n_0}, \mathcal{R}_n^k(\beta) \subseteq \mathcal{R}_{\lfloor \lambda n \rfloor}^{k'}(\beta')$$

$$(5) \quad \forall j' \in \mathcal{O}^+(u), \exists j \in \mathcal{O}^+(u) : \forall n \in \mathbf{Z}_{\geq n_0}, \mathcal{O}_{j;n}(\beta) \subseteq \mathcal{O}_{j'; \lfloor \lambda n \rfloor}(\beta')$$

$\beta$  and  $\beta'$  are said equivalent if both finer than each other. Up to taking logarithms  $k'$  plays with respect to  $k$  in (4) the rôle of  $\eta(t)$  with respect to  $t$  in (1), so that we will still denote  $\eta : \mathcal{O}^+(u) \rightarrow \mathcal{O}^+(u)$  a map such that one may take  $k' = \eta(k)$  in (4). Similarly, denote  $\bar{\eta} : \mathcal{O}^+(u) \rightarrow \mathcal{O}^+(u)$  a map such that one may take  $j = \bar{\eta}(j')$  in (5).  $\lambda$  is analogous to a Hölder exponent comparing snowflake-equivalent metrics.

**Definition 1.6** ( $O(u)$ -quasisymmetric homeomorphism). Let  $\varphi : Z \rightarrow Z'$  be a bijection between two sets endowed with  $O(u)$ -quasisymmetric structures  $\beta$  and  $\beta'$ . One can pull-back  $\beta'$  to  $Z$  by means of  $\varphi$ . The map  $\varphi$  is a  $O(u)$ -quasisymmetric homeomorphism if  $\beta$  and  $\varphi^*\beta'$  are  $O(u)$ -equivalent.

Two  $O(u)$ -equivalent structures on  $Z$  define the same uniform structure on  $Z$  so that  $O(u)$ -conformal homeomorphisms are uniform homeomorphisms. This can be made more quantitative: they are biHölder continuous when

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<sup>8</sup>We borrow the term “asphericity” to the survey [GP91, p.88]. Another choice is “modulus” adopted in [Pal18] but it would be misleading here since for our purposes in section 2, moduli are global rather than infinitesimal conformal invariants. Still another term found in the modern literature is “eccentricity”. We prefer not to define asphericity for subsets since we would face the same issues as with radii and centers.



this makes sense [Pal18, 4.4]. Not every quasi-symmetric homeomorphism is  $O(1)$ -quasisymmetric, but every power-quasisymmetric homeomorphisms<sup>9</sup> is. Note that a consequence of Definition 1.6 is that

$$\forall k \in \mathcal{O}^+(u), \exists k' \in \mathcal{O}^+(u), \mathcal{B}^k(\beta) \subseteq \mathcal{B}^{k'}(\varphi^* \beta')$$

since  $k$ -balls may be identified with the  $k$ -annuli  $(a^-, a^+)$  for which there is equality  $a^- = a^+$ . This does not suffice for all our needs, nevertheless it is simpler and we shall use it when possible.

*Remark 1.4.* A reformulation of (4) and (5) is

$$\forall K \in [1, +\infty), \exists K' \in [1, +\infty), \mathcal{R}_n^{[Ku]}(\beta) \subseteq \mathcal{R}_{[\lambda n]}^{[K'u]}(\beta').$$

$$\forall J' \in [1, +\infty), \exists J \in [1, +\infty), \mathcal{O}_n^{[Ju]}(\beta) \subseteq \mathcal{O}_{[\lambda n]}^{[J'u]}(\beta').$$

*Remark 1.5.* The requirement (5) will be needed only when we deal with packings.

### 1.2.2. $O(u)$ -conformal homeomorphisms as boundary mappings.

**Theorem 1.6** (Consequence of [Pal18, Theorem 4.2 and Proposition 4.9]). *Let  $Y$  and  $Y'$  be Gromov-hyperbolic, geodesic, proper metric spaces with uniformly perfect Gromov boundaries. Let  $f : Y \rightarrow Y'$  be a  $O(u)$ -sublinearly biLipschitz equivalence (Definition 1.2). Let  $\beta$  and  $\beta'$  be the  $O(u)$ -quasisymmetric structures on the Gromov boundaries of  $Z$  and  $Z'$  associated to visual kernels. Then  $\partial_\infty f : Z \rightarrow Z'$  is a  $O(u)$ -quasisymmetric homeomorphism.*

Since the original statement is not this one, we will give details on how to deduce it from [Pal18]. Here is an illustration:

**Example 1.7** (The plane and the twisted plane). Let  $Y = \mathbf{R}^2 \rtimes_\alpha \mathbf{R}$  and  $Y' = \mathbf{R}^2 \rtimes_{\alpha'} \mathbf{R}$  where

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the semi-direct products are formed with  $t \in \mathbf{R}$  acting on  $\mathbf{R}^2$  as  $e^{t\alpha}$  and  $e^{t\alpha'}$  respectively. Equip  $Y$  and  $Y'$  with left invariant metrics; they are Gromov-hyperbolic and  $-t$  is a Busemann function. Identify both Gromov boundaries  $\partial_\infty^* Y = \partial_\infty Y \setminus [-t]$  and  $\partial_\infty^* Y' = \partial_\infty Y' \setminus [-t]$  to  $\mathbf{R}^2$ . The map  $\iota : Y \rightarrow Y'$  which is the identity in coordinates is a  $O(\log)$ -sublinearly biLipschitz equivalence [Cor11]. The identity map  $\partial_\infty^* \iota$  of  $\mathbf{R}^2$  is a  $O(\log)$ -quasisymmetric homeomorphism. See Figure 1.

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<sup>9</sup>A power quasisymmetric embedding is an embedding for which one can take  $\eta(t) = \sup\{t^\alpha, t^{1/\alpha}\}$  for some  $\alpha \in (0, +\infty)$  in (1); this is not restrictive between uniformly perfect metric spaces (called ‘‘homogeneously dense’’ by Tukia and Väisälä) [TV80, 3.12] [Hei01, 11.3].

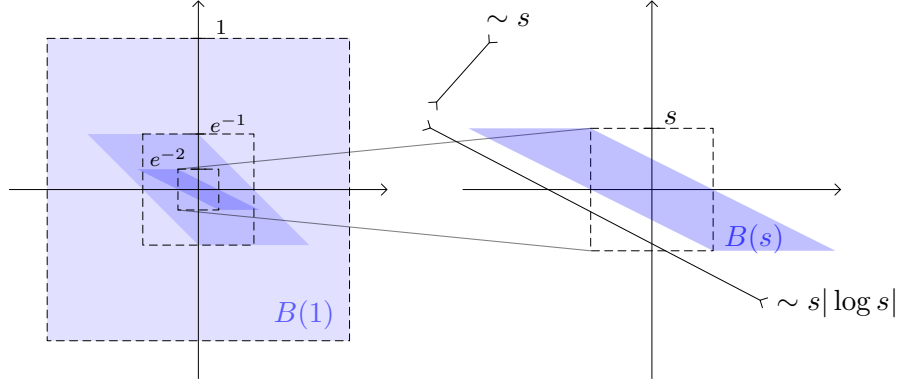


FIGURE 1. Concentric balls of a quasidistance on  $\mathbf{R}^2$  that is invariant under translation and dilation by  $\exp(t\alpha')$  with unipotent, non identity  $\alpha'$ , and coincides with the  $\ell^\infty$  distance for pairs of points at distance 1. For comparison, dashed  $\ell^\infty$  spheres of equal radii. Compare Figure 2.

*How to deduce Theorem 1.6 from [Pal18].* Fix Gromov kernels  $d$  on  $Z$  and  $Z'$ , start assuming for simplicity that every metric sphere of positive radius in  $Z$  and  $Z'$  has at least one point, denote  $\varphi = \partial_\infty f$ ; note that  $\varphi$  and  $\varphi^{-1}$  are both sublinearly quasiMöbius ([Pal18, Definition 4.1 and Theorem 4.2]), especially they are biHölder; up to snowflaking  $Z$  or  $Z'$  let  $\gamma \in (0, 1)$  be a Hölder exponent for both. By [Pal18, 4.9]<sup>10</sup> sufficiently small rings of inner radius  $r$  and asphericity  $\log(R/r)$  are sent by  $\varphi$  to rings with asphericity  $\log R/r + O(u(-\log r))$  and inner radii greater than  $r^{1/\gamma}$ ; this implies (4) translating  $-\log r$  into  $n$ ,  $\log R/r$  into  $k$  and noting that  $u(\gamma n) = O(u(n))$  since  $u$  is doubling. Let us prove (5). Fix  $\ell' \in O(u)$  a positive function. We need  $\ell$  such that if  $A$  contains a  $\ell$ -outer ring then  $f(A)$  will contain an  $\ell'$ -outer ring. Fix  $\zeta \in Z$  and  $r > 0$ . Let  $r' = \sup \{d(\varphi(\zeta), \varphi(\xi)) : d(\zeta, \xi) \leq r\}$ . Let  $\xi_0 \in Z$  be such that  $d(\varphi(\xi_0), \varphi(\zeta)) = r'$ . Let  $\xi_1 \in Z$  be such that  $\varphi(\xi_1) \in B(\varphi(\zeta), r' \exp(\ell'(-\log r')))$ . By quasiMöbiusness of  $\varphi^{-1}$ , there exists  $\lambda \in \mathbf{R}_{>0}$  and  $v \in O(u)$  a positive function such that

$$\begin{aligned} \log^+ \frac{d(\zeta, \xi_1)}{d(\zeta, \xi_0)} &\geq \lambda \log^+ \frac{d(\varphi(\zeta), \varphi(\xi_1))}{d(\varphi(\zeta), \varphi(\xi_0))} \\ &\quad - v(-\log \inf \{d(|\varphi(\zeta), \varphi(\xi_0)), d(|\varphi(\zeta), \varphi(\xi_1))\}) \\ &\geq \ell'(\lfloor -\log r' \rfloor) - v(-\log r'). \end{aligned}$$

Setting  $\ell(n) = \ell'(n/\gamma) + v(n/\gamma)$  this proves (5) for the quasisymmetric structure  $\beta$  and the pullback  $\varphi^* \beta'$  on  $Z$ . Finally, uniform perfectness of  $Z$  and  $Z'$  allows to carry the proof up to bounded approximations should certain points not exist.  $\square$

<sup>10</sup>Beware that one must translate “annulus” into “ring” and “modulus” into “asphericity” to conform to our current terminology.

### 1.2.3. Examples and non-properties of $O(u)$ -quasisymmetric homeomorphisms.

The following indicates a way to produce  $O(u)$ -quasisymmetric homeomorphisms of the Euclidean plane starting from the observation that products of biLipschitz homeomorphisms are quasisymmetric homeomorphisms.

The first step of the construction is to build a homeomorphism of the circle with controlled (almost Lipschitz in a precise sense) modulus of continuity. Let  $T$  be a rooted infinite binary tree, whose set of vertices  $V$  is identified with the set of finite words over the alphabet  $\{0, 1\}$ . Let  $(\epsilon_j) \in (0, 1/2)^{\mathbf{N}}$  be a decreasing sequence with limit 0. To every  $\eta \in \{-1, 1\}^V$  we associate a homeomorphism  $\Phi_\eta$  of the circle as follows:

- (1) for each  $v$  of length  $|v|$  one associates a real number  $\tau^v$  with the binary expansion  $v : \tau^v = \sum_{i=1}^{|v|} v_i 2^{-i}$ .
- (2) Let  $M_\eta(v)$  be the uniform measure on  $[0, 2^{-|v|}]$  with total mass

$$\|M_\eta(v)\| = \prod_{w \in \text{Pref}(v) \setminus \{v\}} \left( \frac{1}{2} + \eta(w) \epsilon_{|w|} \right),$$

where  $\text{Pref}(v)$  denotes the set of prefixes of  $v$  (including the empty one).

- (3) For any nonnegative integer  $\ell$ ,  $M_\eta^t := \sum_{v \in V: |v|=t} \tau_*^v M(v)$ , where  $\tau_*^v$  is the pushforward by the translation  $x \mapsto x + \tau^v$ .
- (4)  $M_\eta$  is the limit of  $M_\eta^t$  as  $t \rightarrow +\infty$ , and let  $\Phi \in \text{Homeo}^+([0, 1])$  be such that  $\Phi' = M$  (in the distributional sense). Realizing  $S^1$  as  $[0, 1]/\sim$  where  $0 \sim 1$  one may view  $\Phi$  as a random homeomorphism of the circle.

**Proposition 1.8.** *If  $\epsilon_j \notin \ell^1(\mathbf{N})$  then  $\Phi_\eta$  is not absolutely continuous.*

*Proof.* Let  $\lambda$  be the Haar measure on  $S^1$ , and for  $t \in \mathbf{N}_{\geq 1}$ , let  $\Phi_\eta^t$  be the approximation of  $\Phi_\eta$  at time  $t$  given by  $(\Phi^t)' = M^t$ . Note that whenever  $k$  is an integer with  $0 \leq k \leq 2^t$ , one has  $\Phi(2^{-t}k) = \Phi^t(2^{-t}k)$ . To every  $x \in S^1$  one can associate a geodesic  $\gamma_x \subset T$  representing its base 2 expansion (the finite one for dyadic  $x$ ). Fix  $\rho \in (0, 1)$ . Define  $A_\eta(\rho) = \{x \in [0, 1] : \forall t \in \mathbf{N}, 2^t \|M_\eta(\gamma_x(t))\| \geq \rho\}$ . This is the complementary set in  $[0, 1]$  of

$$\begin{aligned} B_\eta(\rho) &= \{x \in [0, 1] : \exists t \in \mathbf{N}, 2^t \|M_\eta(\gamma_x(t))\| \leq \rho\} \\ &= \bigcup_{v \in V: \|M_\eta(v)\| \leq 2^{-|v|} \rho} [\tau^v, \tau^v + 2^{-|v|}] \\ &= \bigsqcup_{v \in V: \forall w \in \text{Pref}(v) (\|M_\eta(w)\| \leq 2^{-|w|} \rho \implies w=v)} [\tau^v, \tau^v + 2^{-|v|}] \end{aligned}$$

where we used that  $[\tau^w, \tau^w + 2^{-|w|}] \supseteq [\tau^v, \tau^v + 2^{-|v|}]$  if and only if  $w \in \text{Pref}(v)$ , with equality if and only if  $v = w$ . Note that the  $\lambda$ -measure of

$\Phi_\eta(B(\rho))$  is smaller than  $\rho$  for all  $\rho$ , since

$$\begin{aligned} \lambda(\Phi(B_\eta(\rho))) &= \sum_{v \in V: \forall w \in \text{Pref}(v) (\|M(w)\| \leq 2^{-|w|} \rho \implies w=v)} \lambda\left(\Phi^{|v|} \left[ \tau^v, \tau^v + 2^{-|v|} \right]\right) \\ &\leq \sum_{v \in V: \forall w \in \text{Pref}(v) (\|M(w)\| \leq 2^{-|w|} \rho \implies w=v)} 2^{-|v|} \rho \leq \rho. \end{aligned}$$

Indeed the intervals  $[\tau^v, \tau^v + 2^{-|v|}]$  under consideration are disjoint so that the sum of their measures is  $\leq 1$ . On the other hand, if  $\epsilon_j \notin \ell^1(\mathbf{N})$  then

$$\lambda(B_\eta(\rho)) = 1 - \lambda(A_\eta(\rho)) = 1 - 0 = 1,$$

since for almost every  $x$ , the sequence  $(2^t \|M_\eta(\gamma_x(t))\|)$  is not bounded away from 0 : up to a null set (the dyadics) one may identify  $([0, 1], \lambda)$  with the shift space of geodesics rays in  $T$  and consider  $A_\eta(\rho)$  as an event of probability zero. Especially  $\lambda\left(\bigcap_{\rho \downarrow 0} B_\omega(\rho)\right) = 1$ , whereas the image of this set by  $\Phi$  has  $\lambda$ -measure 0.  $\square$

From now on assume that  $\epsilon_j \notin \ell^1(\mathbf{N})$  but decays sufficiently fast so that the partial sums remain controlled by  $u$  :

$$(6) \quad \sum_{j \leq t} \epsilon_j = O(u(t)),$$

where we recall that  $u$  is strictly sublinear. For instance if  $\epsilon_j = (3 + j)^{-\alpha}$  with  $\alpha \in (0, 1)$  one may take  $u(t) = t^{1-\alpha}$ .

**Proposition 1.9.** *Assume that  $\epsilon_j$  decays sufficiently fast so that (6) holds. Then there exists  $v \in O(u)$  such that for all  $\eta \in \{0, 1\}^V$*

$$(7) \quad \log l(\Phi_\eta, s) \leq \log s + v(-\log s)$$

and

$$(8) \quad \log s - v(-\log s) \leq \log L(\Phi_\eta, s)$$

where  $l(\Phi_\eta, s) = \sup \{\Phi_\eta(x) - \Phi_\eta(y) : |x - y| \leq s\}$  and  $L(\Phi_\eta, s) = \inf \{\Phi_\eta(x) - \Phi_\eta(y) : |x - y| \geq s\}$ .

*Proof.* Define  $t = -\lceil \log_2 s \rceil$ . If  $|x - y| \leq s$ , then  $[x, y]$  is contained in the union of two adjacent dyadic intervals of length  $2^{-t}$ . Let  $\gamma$  and  $\gamma'$  be the corresponding geodesic segments in  $T$ . Then

$$|\Phi_\eta(x) - \Phi_\eta(y)| \leq \|M_\eta(\gamma(t))\| + \|M_\eta(\gamma'(t))\| \leq 2 \prod_{j=0}^{t-1} \left( \frac{1}{2} + \epsilon_j \right),$$

Hence  $\log |\Phi_\eta(x) - \Phi_\eta(y)| \leq (1 - t) \log 2 + \sum_{j=0}^{t-1} \log(1 + 2\epsilon_j) \leq \log s + v(-\log s)$  where  $v = O(u)$ . Similarly, if  $|x - y| \geq s$  then  $[x, y]$  contains

a dyadic interval of length  $2^{-1-t}$  with associated geodesic segment  $\gamma$  so that

$$|\Phi_\eta(x) - \Phi_\eta(y)| \geq \|M_\eta(\gamma)\| \geq \prod_{j=0}^{t-1} \left( \frac{1}{2} - \epsilon_j \right),$$

providing (7).  $\square$

*Remark 1.10.* The aim of Proposition 1.9 is only to give a modulus of continuity (and a reverse modulus of continuity) for  $\Phi_\eta$ . However we expect the deviation of  $\log |\Phi_\eta(x) - \Phi_\eta(y)|$  from  $\log |x - y|$  to be typically much lower because of Lindeberg's version of the central limit theorem [Lin22, Satz II].

*Remark 1.11.*  $M_\eta$  is homogeneously multifractal in the sense of Buczolic and Seuret [BS15], and its multifractal spectrum is concentrated at  $\{1\}$ . Especially Proposition 1.9 provides examples for [BS15, Proposition 9].

We can now produce homeomorphisms of  $\mathbf{R}$  in the following way: for every  $k \in \mathbf{Z}$ , choose  $\eta_k \in \{-1, 1\}^V$ , produce a measure  $M_{\eta_k}$  on  $[0, 1]$ , and then set  $\mu = \sum_{k \in \mathbf{Z}} k_* \mu_{\eta_k}$ . Finally  $\psi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  is such that  $\psi(s) = \int_0^s d\mu$ . This may be considered a random process if  $\eta_k$  are considered random variables.

**Proposition 1.12.** *Let  $\Psi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be defined by  $\Psi(x_1, x_2) = (\psi_1(x_1), \psi_2(x_2))$  where  $\psi_1$  and  $\psi_2$  are as above. Then  $\Psi$  is a  $O(u)$ -quasisymmetric<sup>11</sup> homeomorphism.*

*Proof.* Equip  $\mathbf{R}^2$  with the sup norm. Rephrasing Definitions 1.5 and 1.6 we need prove that for every  $K \in \mathbf{R}_{\geq 1}$  and  $k \in O(u)$  there exists  $L \in \mathbf{R}_{\geq 1}$  and  $\ell \in O(u)$  such that for any sequence  $(x^n, y^n, z^n)$  of points in  $\mathbf{R}^2$ ,

$$\begin{cases} K^{-1}n \leq -\log \|y^n - x^n\| \\ -\log \|y^n - x^n\| \leq Kn \\ \left| \log \frac{\|y^n - x^n\|}{\|z^n - x^n\|} \right| \leq k(n), \end{cases} \implies \begin{cases} L^{-1}n \leq -\log (\|\Psi(y^n) - \Psi(x^n)\|) \\ -\log (\|\Psi(y^n) - \Psi(x^n)\|) \leq Ln \\ \left| \log \frac{\|\Psi(y^n) - \Psi(x^n)\|}{\|\Psi(z^n) - \Psi(x^n)\|} \right| \leq \ell(n). \end{cases}$$

Write  $x^n = (x_1^n, x_2^n)$ , similarly for  $y^n$  and  $z^n$ . Let  $v \in O(u)$  be such that (8) holds for every  $\psi_\alpha$ , i.e.

$$(9) \quad \forall \alpha \in \{1, 2\}, |\log |\psi_\alpha(y) - \psi_\alpha(x)| - \log |y - x|| \leq v(-\log |y - x|).$$

Split  $\mathbf{N}$  into three index subsets:

$$\begin{aligned} I_1^y &= \{n \in \mathbf{N} : -\log |y_2^n - x_2^n| > -\log |y_1^n - x_1^n| + 2v(n)\} \\ I_2^y &= \{n \in \mathbf{N} : -\log |y_2^n - x_2^n| < -\log |y_1^n - x_1^n| - 2v(n)\} \\ I_0^y &= \{n \in \mathbf{N} : |\log |y_2^n - x_2^n| - \log |y_1^n - x_1^n|| \leq 2v(n)\}. \end{aligned}$$

Also, define  $I_\alpha^z$  and  $J_\alpha^z$  in the same way for  $\alpha \in \{0, 1, 2\}$ . Note that since  $u$  is non-negative, if  $\alpha \neq 0$

$$(10) \quad \forall n \in I_\alpha^y, \begin{cases} \|y^n - x^n\| = |y_\alpha^n - x_\alpha^n| \\ \|\Psi(y^n) - \Psi(x^n)\| = |\psi_\alpha(y_\alpha^n) - \psi_\alpha(x_\alpha^n)| \end{cases}$$

<sup>11</sup>The  $O(1)$ -quasisymmetric structure, and then the  $O(u)$ -quasisymmetric structure on  $\mathbf{R}^2$ , will not depend on the norm, compare [Hei01, p.78].

and similar equalities hold for  $n$  in  $I_\alpha^z$ , whereas if  $n \in I_0^y$ , resp.  $n \in I_0^z$  then  $\log \|y^n - x^n\| - \log |y_\alpha^n - x_\alpha^n| \leq 2v(Kn + 2v(n))$ , resp.  $\log \|z^n - x^n\| - \log |z_\alpha^n - x_\alpha^n| \leq 2v(Kn + 2v(n))$  for any  $\alpha \in \{1, 2\}$ . By (10), if  $\alpha, \beta \in \{1, 2\}$  then for  $n \in I_\alpha^y \cap I_\beta^z$

$$\frac{\|y^n - x^n\|}{\|z^n - x^n\|} = \frac{|y_\alpha^n - x_\alpha^n|}{|z_\beta^n - x_\beta^n|} \text{ and } \frac{\|\Psi(y^n) - \Psi(x^n)\|}{\|\Psi(z^n) - \Psi(x^n)\|} = \frac{|\psi_\alpha(y_\alpha^n) - \psi_\alpha(x_\alpha^n)|}{|\psi_\beta(z_\beta^n) - \psi_\beta(x_\beta^n)|}$$

so that, taking logarithms and by (10) and (9) and (10) again

$$\begin{aligned} \left| \log \frac{\|\Psi(y^n) - \Psi(x^n)\|}{\|\Psi(z^n) - \Psi(x^n)\|} : \frac{\|y^n - x^n\|}{\|z^n - x^n\|} \right| &\leq \left| \log \frac{\|\Psi(y^n) - \Psi(x^n)\|}{\|\Psi(z^n) - \Psi(x^n)\|} - \log \frac{|y_\alpha^n - x_\alpha^n|}{|z_\beta^n - x_\beta^n|} \right| \\ &\leq v(-\inf\{\log |y_\alpha^n - x_\alpha^n|, \log |z_\beta^n - x_\beta^n|\}) \\ &\leq v(-\inf\{\log \|y^n - x^n\|, \log \|z^n - x^n\|\}) \\ &\leq v(Kn + k(n)). \end{aligned}$$

It remains to treat the case  $n \in I_\alpha^y \cap I_\beta^z$  with  $\inf\{\alpha, \beta\} = 0$ ; in this event define  $\gamma = \sup\{1, \alpha, \beta\}$ . Then

$$\begin{aligned} \left| \log \frac{\|\Psi(y^n) - \Psi(x^n)\|}{\|\Psi(z^n) - \Psi(x^n)\|} : \frac{\|y^n - x^n\|}{\|z^n - x^n\|} \right| &\leq \left| \log \frac{\|\Psi(y^n) - \Psi(x^n)\|}{\|\Psi(z^n) - \Psi(x^n)\|} - \log \frac{|y_\gamma^n - x_\gamma^n|}{|z_\gamma^n - x_\gamma^n|} \right| \\ &\quad + 4v(Kn + 2v(n)) \\ &\leq v(Kn + k(n)) + 2v(Kn + 2v(n)). \end{aligned}$$

Setting  $L = K$  and  $\ell(n) = k(n) + v(Kn + k(n)) + 4v(Kn + 2v(n))$  this finishes the proof.  $\square$

Whereas quasiconformal mappings between open domains of<sup>12</sup>  $\mathbf{R}^2$  have the ACL property (see Väisälä [Väi71, 32.4]; this is instrumental for Mostow rigidity in rank one [Mos71, § 21]), Propositions 1.8 and 1.12 imply that it fails for general  $O(u)$ -quasisymmetric homeomorphisms. This is why our main efforts in this paper are rather directed to global invariants.

### 1.3. Covering and measures.

1.3.1. *Covering lemma: extracting disjoint balls.* Let  $(Z, \beta)$  be a  $O(u)$ -quasisymmetric structure (Definition 1.1) and let  $A \in \mathcal{P}(Z)$  be a subset. Say that countable collection of abstract balls  $\mathcal{B}$  is a covering of  $A$  if the realizations of the members of  $\mathcal{B}$  cover  $A$ . We adapt a classical covering lemma for metric spaces [Fed69, 2.8.4 – 2.8.8], [Mat95, p.24]<sup>13</sup> to  $O(u)$ -quasisymmetric structures; (SC2) may be considered the case with 2 balls. The lemma says that out of any covering  $\mathcal{B}$  one can extract a disjoint subcovering  $\mathcal{C}$  such that  $q \cdot \mathcal{C} = \{q \cdot b : b \in \mathcal{B}\}$  is still a covering, where  $q$  is a positive function in the

<sup>12</sup>Quasisymmetric homeomorphisms of the circle that are not absolutely continuous do exist [Ahl06, IV.B, Remark 2].

<sup>13</sup>We cite both since Federer's statement is more general, but the filtration of balls according to the logarithms of their radii is noticeable in Mattila's proof.

$O(u)$ -class; for metric spaces it is known as the “ $5r$  covering lemma” since one can take 5 as an exponential analog of  $q$ .

**Lemma 1.13.** *Let  $(Z, \beta)$  be a  $O(u)$ -quasisymmetric structure. Let  $F \in \mathcal{P}(Z)$  be a subset and let  $\mathcal{B} \subseteq \beta$  be a countable covering of  $F$ ; assume that  $\inf_{\mathcal{B}} \delta > -\infty$ . There exists  $\mathcal{C} \subset \mathcal{B}$  such that  $q\mathcal{C}$  covers  $A$  and for every  $b, b' \in \mathcal{C}$ ,  $\widehat{b} \cap \widehat{b}' = \emptyset$  unless  $b = b'$ .*

*Proof.* Set  $n_0 = \inf_{b \in \mathcal{B}} \delta(b)$ . For every  $n \in \mathbf{Z}$ , let  $\mathcal{B}_n = \{b \in \mathcal{B} : \delta(b) = n\}$ . By induction on  $n \in \mathbf{Z}_{\geq n_0}$ , choose for each  $n$  (by Zorn’s lemma or Hausdorff’s maximality principle, see [Kel55, 0.24]) a maximal subfamily  $\mathcal{C}_n \subset \mathcal{F}_n$  whose realizations are pairwise disjoint and do not intersect the previously chosen balls, that is:

- $\forall (b, b') \in \mathcal{C}_n \times \mathcal{C}_n, \widehat{b} \cap \widehat{b}' \neq \emptyset \implies b = b'$ .
- $\forall b \in \mathcal{C}_n, \forall m \in \{n_0, \dots, n-1\}, \forall b' \in \mathcal{C}_m, \widehat{b} \cap \widehat{b}' = \emptyset$ .
- $\forall b \in \mathcal{B}_n \setminus \mathcal{C}_n, \exists b' \in \mathcal{C}_n : \widehat{b} \cap \widehat{b}' \neq \emptyset$ .

By construction, the realizations of members of  $\mathcal{C} = \cup_n \mathcal{C}_n$  are disjoint. Let  $x \in A$ ; since  $\mathcal{B}$  covers  $A$  there is  $b' \in \mathcal{B}$  such that  $\widehat{b}' \ni x$ . Either  $b' \in \mathcal{C}$  or, setting  $n = \delta(b)$ ,  $b' \in \mathcal{B}_n$  and there is  $b \in \mathcal{C}_m$  such that  $\widehat{b} \cap \widehat{b}' \neq \emptyset$  with  $m \leq n$ . By (SC2),  $q\widehat{b} \supseteq \widehat{b}'$  so that  $q\widehat{b} \ni x$ .  $\square$

It follows from the lemma that as soon as a  $O(u)$ -quasisymmetric structure has a countable covering, then it also has a countable packing  $\mathcal{C} \subset \beta$  such that  $q\mathcal{C}$  covers. This holds for instance, if the quasisymmetric structure comes from a separable metric space.

1.3.2. *Gauges.* Let  $(Z, \beta)$  be a  $O(u)$ -quasisymmetric structure. Call gauge on  $(Z, \beta)$ , any function  $\phi : \mathcal{P}(Z) \rightarrow [0, +\infty)$ , and denote by  $\mathcal{G}(Z)$  the set of such functions. For every  $\ell \in \mathcal{O}^+(u)$ , define a shifted gauge  $\widetilde{\phi}^\ell : \mathcal{P}(Z) \rightarrow [0, \infty)$  by

$$\widetilde{\phi}^\ell(a) = \sup\{\phi(\widetilde{a}) : (a, \widetilde{a}) \in \mathcal{R}^\ell(\beta)\}.$$

It is important that no restriction is made on  $\phi$ . We define the gauge on  $\mathcal{P}(Z)$  rather than  $\beta$  in order to ease the comparisons when changing structure.

1.3.3. *Caratheodory measures.* Let  $(Z, \beta)$  be a  $O(u)$ -quasisymmetric structure. For all  $k, \ell \in \mathcal{O}^+(u)$ , for all  $A \in \mathcal{P}(Z)$ , define

$$\Phi_{p,k}^n(A) = \inf \left\{ \sum_{b \in \mathcal{F}} \phi(b)^p : \mathcal{F} \subset \mathcal{B}_n^k(\beta), |\mathcal{F}| \leq \aleph_0, \mathcal{F} \text{ covers } A \right\}$$

$$\widetilde{\Phi}_{p,k}^{\ell;n}(A) = \inf \left\{ \sum_{b \in \mathcal{F}} \widetilde{\phi}^\ell(b)^p : \mathcal{F} \subset \mathcal{B}_n^k(\beta), |\mathcal{F}| \leq \aleph_0, \mathcal{F} \text{ covers } A \right\}$$

and  $\Phi_{p,k}(A) = \lim_{n \rightarrow +\infty} \Phi_{p,k}^n(A)$ ,  $\tilde{\Phi}_{p,k}^\ell(A) = \lim_{n \rightarrow +\infty} \tilde{\Phi}_{p,k}^{\ell;n}(A)$ .  $\beta$  is not specified, however if  $\beta$  and  $\beta'$  are two equivalent  $O(u)$ -quasisymmetric structures on  $Z$  and if  $\lambda, \eta, \eta'$  are such that any  $(\ell, n)$ -ring for  $\beta$  (resp. for  $\beta'$ ) is a  $(\eta(\ell), \lfloor \lambda n \rfloor)$ -ring for  $\beta'$  (resp. for  $\beta$ ), then denoting  $\Phi$  and  $\Phi'$  the measures that correspond to  $\phi$  for  $\beta$  and  $\beta'$  then

$$(11) \quad (\Phi')_{p,\eta(k)} \leq \Phi_{p,k} \text{ and } \left( \tilde{\Phi}' \right)_{p,\eta(k)}^\ell \leq \tilde{\Phi}_{p,k}^{\eta'(\ell)}$$

since any covering by  $(k, n)$  round sets with respect to  $\beta$  is a covering by  $(\eta(k), \lfloor \lambda n \rfloor)$  round sets with respect to  $\beta'$ , and any  $\ell$ -ring with respect to  $\beta'$  is a  $\eta'(\ell)$ -ring with respect to  $\beta$  (note that  $\eta$  or  $\eta'$  appears on superscript when on the right of  $\leq$  and on subscript when on the left).

*Remark 1.14* (Comparisons with Hausdorff measures). When the quasisymmetric structure is that of a metric space,  $s \in \mathbf{R}_{>0}$  and  $\phi(\hat{b}) = e^{-s\delta(b)}$ , the Caratheodory measures  $\Phi$  and  $\tilde{\Phi}$  can be compared to Hausdorff measures; namely since  $(k, n)$  round sets contain balls of radii  $e^{-n}$  and have diameter bounded by  $2e^{-n+k(n)}$ , one has for every  $p \in \mathbf{R}_{>0}$ , for every  $\varepsilon \in (0, p)$

$$(12) \quad \mathcal{H}^{sp+\varepsilon} \ll \Phi_{p;k} \leq \tilde{\Phi}_{p;k}^\ell \ll \mathcal{H}^{sp-\varepsilon}$$

for every  $k, \ell \in \mathcal{O}^+(u)$ .

1.3.4. *Packing Pre-measure.* Let  $(Z, \beta)$  be a quasisymmetric structure and let  $A \in \mathcal{P}(Z)$  be a subset. Let  $\mathcal{P}$  be a collection of  $(k, n)$ -outer rings; say that  $\mathcal{P}$  is a  $(k, n)$ -packing centered on  $A$ , denoted  $\mathcal{P} \in \text{Packings}_{k,n}(A)$  if inner sets meet  $A$  and outer sets are disjoint; formally

- For every  $\mathbf{a} = (a^-, a^+)$  in  $\mathcal{P}$ ,  $a^- \cap A \neq \emptyset$ .
- For every  $\mathbf{a}_0, \mathbf{a}_1$  in  $\mathcal{P}$ ,  $a_0^+ \cap a_1^+ \neq \emptyset \implies \mathbf{a}_0 = \mathbf{a}_1$ .

Similarly to the shifted packing measure  $\tilde{\Phi}$ , define a shifted packing pre-measure

$$(13) \quad \text{P}\tilde{\Phi}_{p;k}^\ell(A) = \lim_{n \rightarrow +\infty} \sup \left\{ \sum_{\mathbf{a} \in \mathcal{P}} \tilde{\phi}^\ell(a^-) : \mathcal{P} \in \text{Packings}_{k,n}(A) \right\}$$

or 0 if there exists no packing indexing the sums.

*Remark 1.15.* Let  $\phi = \lambda \cdot^0 \phi + ^1 \phi$  with  $\lambda \in \mathbf{R}_{\geq 0}$  and  $^i \phi \in \mathcal{G}(Z)$  for  $i \in \{0, 1\}$ . Associate  $^i \text{P}\tilde{\Phi}_{p;k}^\ell$  to  $^i \phi$  by (13). Then  $\text{P}\tilde{\Phi}_{p;k}^\ell \leq \lambda \cdot^0 \text{P}\tilde{\Phi}_{p;k}^\ell + ^1 \text{P}\tilde{\Phi}_{p;k}^\ell$ .

*Remark 1.16.* When changing  $O(u)$ -quasisymmetric structure from  $\beta$  to  $\beta'$ , the analogs of the comparisons (11) are

$$(14) \quad (\text{P}\tilde{\Phi}')_{p;k}^{\eta(\ell)} \geq \text{P}\tilde{\Phi}_{p;\eta(k)}^\ell.$$

Indeed (5) implies that  $\text{Packings}_{\eta(k'),n}(\beta) \subset \text{Packings}_{k',\lfloor \lambda n \rfloor}(\beta')$  whereas, every  $\ell$ -ring for  $\beta$  being a  $\eta(\ell)$ -rings with respect to  $\beta'$ , the supremum in (13) is taken over larger sums.



*Remark 1.17.* Pansu uses a notion of packing with bounded multiplicity [Pan18]. However it is not convenient here because even on doubling spaces, if  $b \in \beta$  is such that  $\delta(b) = n$  then  $\widehat{\ell.b}$  cannot be covered by a uniformly bounded number of concrete balls  $\widehat{b'}$  with  $\delta(b') = n$ .

## 2. CONFORMAL INVARIANTS

By conformal invariants we mean real numbers attached to  $O(u)$ -quasi-symmetric structures, possibly parametrized (for instance by asphericities) and respecting invariance under conformal equivalence. This invariance should not be understood too strictly: the vanishing, or infinitude, for some choice of parameters is considered an invariant, though those parameters may vary.

### 2.1. Combinatorial moduli and functions of bounded energy.

2.1.1. *Caratheodory and packing combinatorial moduli.* The modulus is obtained minimizing  $\widetilde{\Phi}$  under a normalization constraint on the gauge functions, compare Pansu [Pan89, 2.4] and Tyson [Tys98, 3.23]: all members of  $\Gamma$  should have measure (to be thought of as a length<sup>14</sup>) greater than 1.

**Definition 2.1.** Let  $\Gamma$  be a family of subsets in a conformal structure  $(Z, \beta)$ ,  $p \in (0, +\infty)$ ,  $k, \ell$  and  $m$  in  $\mathcal{O}^+(u)$ . Define

$$\text{mod}_{p,k}^{\ell;m}(\Gamma, \beta) = \inf \left\{ \widetilde{\Phi}_{p,k}^{\ell}(Z) : \phi \in \mathcal{G}_m(\Gamma, \beta) \right\},$$

where  $\mathcal{G}_m(\Gamma, \beta) = \{\phi \in \mathcal{G}(\beta) : \forall \gamma \in \Gamma, \Phi_{1,m}(\gamma) \geq 1\}$  is called a set of admissible gauges for  $\Gamma$ .

**Definition 2.2** (Packing variant). Let  $\Gamma$  be a family of subsets in a conformal structure  $(Z, \beta)$ ,  $p \in (0, +\infty)$ ,  $\ell$  and  $m$  in  $\mathcal{O}^+(u)$ . Define

$$\text{pmod}_{p,k}^{\ell;m}(\Gamma, \beta) = \inf \left\{ \text{P}\widetilde{\Phi}_{p,k}^{\ell}(Z) : \phi \in \mathcal{G}_m(\Gamma, \beta) \right\}.$$

When changing conformal structure the moduli change accordingly:

**Lemma 2.1.** *Let  $\beta$  and  $\beta'$  be two  $O(u)$ -equivalent structures on  $Z$ . Let  $\Gamma \subset \mathcal{P}(Z)$ . Set  $\eta, \eta'$  and  $\overline{\eta}$  so that  $\mathcal{R}^k(\beta) \subset \mathcal{R}^{\eta(k)}(\beta')$ ,  $\mathcal{R}^k(\beta') \subset \mathcal{R}^{\eta'(k)}(\beta)$  and  $\mathcal{O}_j(\beta) \subset \mathcal{O}_j(\beta')$  for every  $k, j \in \mathcal{O}^+(u)$ . Then*

$$(15) \quad \text{mod}_{p,\eta'(k)}^{\ell;\eta'(m)}(\Gamma, \beta) \leq \text{mod}_{p,k}^{\eta(\ell);m}(\Gamma, \beta') \text{ and}$$

$$(16) \quad \text{pmod}_{p,\overline{\eta}(k)}^{\ell;\eta'(m)}(\Gamma, \beta) \leq \text{pmod}_{p,k}^{\eta(\ell);m}(\Gamma, \beta').$$

---

<sup>14</sup>This is similar in spirit to requiring a Riemannian metric in a given conformal class to confer sufficient length to any curve in a family as in the definition of the classical moduli.

*Proof.* By (11), for all  $\phi \in \mathcal{G}(Z)$ ,  $\left(\tilde{\Phi}'\right)_{p,k}^{\eta(\ell)} \geq \tilde{\Phi}_{p,\eta'(k)}^\ell$ , and  $\mathcal{G}_m(\beta') \subseteq \mathcal{G}_{\eta'(m)}(\beta)$ . Hence, for the modulus computed with respect to  $\beta$  on (15), the infimum in Definition 2.1 is taken over more gauges, while common admissible gauges contribute to lower values. This proves (15). Concerning the packing moduli, using (14) and the same observation:

$$\begin{aligned} \text{pmod}_{p,\overline{\eta}(k)}^{\ell,\eta'(m)}(\Gamma) &= \inf \left\{ \text{P}\tilde{\Phi}_{p,\overline{\eta}(k)}^\ell(Z) : \phi \in \mathcal{G}_{\eta'(m)}(\beta) \right\} \\ &\leq \inf \left\{ \left(\text{P}\tilde{\Phi}'\right)_{p;k}^{\eta(\ell)}(Z) : \phi \in \mathcal{G}_{\eta'(m)}(\beta) \right\} \\ &\leq \inf \left\{ \left(\text{P}\tilde{\Phi}'\right)_{p;k}^{\eta(\ell)}(Z) : \phi \in \mathcal{G}_m(\beta') \right\} = \text{pmod}_{p,k}^{\eta(\ell);m}(\Gamma, \beta'). \quad \square \end{aligned}$$

2.1.2. *Functions of locally bounded energy.* The Dirichlet energies of functions defined on domains of the plane are preserved by conformal mappings; in fact one can prove that they are preserved with multiplicative errors by quasiconformal mappings. We investigate here analogs of the local  $W^{1,p}$ -spaces (without an a priori differentiable structure on domains) that are carried by  $O(u)$ -quasisymmetric homeomorphisms with a shift in an asphericity parameter. The notion of energy we use here is inspired by Pansu's [Pan89, 6.1] but it is actually more closely related to Kleiner and Xie's  $Q$ -variation ([Xie12, Definition 3.2], [Xie14, 4]). For quasimetric spaces they are the same, the reader familiar with  $Q$ -variation may translate  $E_{p,k}^\ell(f)(-)$  into  $V_{Q,K}(f|_-)$  with  $\ell = 0$ ,  $p = Q$  and  $k = \log K$ .

Let  $\beta$  be a  $O(u)$ -quasisymmetric structure on a set  $Z$ , and let  $f : Z \rightarrow \mathbf{C}$  be a continuous function. Given  $p \in [1, +\infty)$  and  $k, \ell \in \mathcal{O}^+(u)$  one can associate to  $f$  a pre-measure on  $Z$  by  $\mathbf{E}_{p;k}^\ell(f) = \text{P}\tilde{\Phi}_{p;k}^\ell$  using the gauge

$$\phi(a) = \text{diam } f(a) =: \text{osc}(f, a).$$

Fix  $k, \ell \in \mathcal{O}^+(u)$ . Say that a continuous function  $f$  has bounded  $(p; k, \ell)$ -energy if  $\mathbf{E}_{p;k}^\ell(f)$  is locally finite. Remark 1.15 noted that the premeasures  $\text{P}\tilde{\Phi}$  are subadditive and sublinear with respect to  $\phi$ , so that functions of bounded  $(p; k, \ell)$ -energy will form vector spaces: if  $\Omega \subset Z$  is an open subset, denote them by

$$\mathcal{W}_{\ell;\text{loc.}}^{p,k}(\Omega) = \left\{ f \in \mathcal{C}^0(\Omega) : \forall K \in \mathcal{P}(\Omega), K \Subset \Omega \implies \mathbf{E}_{p;k}^\ell(f)(K) < +\infty \right\}.$$

For all  $K$  compact in  $\Omega$ ,  $k \in \mathcal{O}^+(u)$ ,  $\ell \in \mathcal{O}^+(u)$  and  $p \in [1, +\infty)$  define a seminorm

$$(17) \quad \|f\|_{p;k}^{K;\ell} := \|f\|_{C^0(K)} + \mathbf{E}_{p;k}^\ell(f)(K)^{1/p}.$$

From now on, in order to define a Fréchet algebra structure on  $\mathcal{W}_{\ell;\text{loc.}}^{p,k}(\Omega)$ , we will need to assume more on the topology associated with  $\beta$ .

**Definition 2.3** (hemicompactness). Let  $X$  be a Hausdorff topological space. An admissible exhaustion of  $X$  is an increasing sequence of compact subspaces  $(K_n)_{n \geq 0}$  of  $X$  such that for every compact  $K$  of  $X$  there exists  $n$  such that  $K \subset K_n$ . A space is hemicompact if it has an admissible exhaustion.

If  $Z$  is a locally compact, second countable topological space, then any open subset of  $Z$  is hemicompact. Indeed by Lindelöf's lemma in a second countable space, every open subset is a Lindelöf space (meaning that any open cover of it has a countable subcover) [Kel55, Chapter 1, Theorem 15], and a locally compact Lindelöf space is hemicompact.

**Lemma 2.2.** *Let  $(Z, \beta)$  be a  $O(u)$ -quasisymmetric structure with locally compact, second countable topology. For all non-empty open  $\Omega \subset Z$ ,  $\mathscr{W}_{\ell; \text{loc.}}^{p,k}(\Omega)$  defines a unital commutative Fréchet algebra. Further, if  $\varphi : (Z', \beta') \rightarrow (Z, \beta)$  is a  $O(u)$ -quasisymmetric homeomorphism then for every open  $\Omega' \subset Z'$ , letting  $\Omega = \varphi(\Omega')$  there are linear continuous algebra homomorphisms*

$$(18) \quad \mathscr{W}_{\eta \circ \eta'(\ell)}^{p;k}(\Omega') \hookrightarrow \varphi^* \mathscr{W}_{\eta'(\ell)}^{p;\bar{\eta}(k)}(\Omega) \hookrightarrow \mathscr{W}_{\ell}^{p;\bar{\eta}' \circ \bar{\eta}(k)}(\Omega').$$

*Proof.* By the observation above each open subset  $\Omega$  being hemicompact, has an admissible exhaustion  $(K_n)$ . The countable family of seminorms  $(\| - \|_{\ell, \text{loc.}}^{K_n, k})$  defines the Fréchet algebra structure on  $\mathscr{W}_{\text{loc.}}^{p,k}$ ; the hemicompactness ensures that it does not depend on the choice of the sequence  $(K_n)$ . To prove the part about  $O(u)$ -quasisymmetric homeomorphisms we can assume that  $\beta'$  is an  $O(u)$ -equivalent structure on the same set  $Z$ . Denote  $\mathbf{E}$ , resp.  $\mathbf{E}'$  the energies computed with respect to  $\beta$ , resp.  $\beta'$ . By (14),

$$\forall k, \ell \in \mathcal{O}^+(u), \mathbf{E}_{p;k}^{\eta'(\ell)}(f) \geq (\mathbf{E}'_{p;\bar{\eta}'(k)})^{\ell}(f)$$

(this may be compared to Xie [Xie12, Lemma 3.1]) so that  $\mathscr{W}_{\text{loc.}}^{p,k}(\Omega, \beta|_{\Omega})$  continuously embeds in  $\mathscr{W}_{\text{loc.}}^{p,\bar{\eta}'(k)}(\Omega | \beta'_{|\Omega})$ . (18) is obtained by applying this twice and reversing the rôles of  $\beta$  and  $\beta'$ .  $\square$

**Lemma 2.3.** *Let  $\Omega$  be an open set in a  $O(u)$ -quasisymmetric structure  $(Z, \beta)$ . For any compact set  $K \Subset \Omega$ , any  $\ell \geq 1$  and any  $f \in \mathscr{W}_{\ell; \text{loc.}}^{p,k}(\Omega)$ ,*

$$\lim_{j \rightarrow +\infty} \left( \|f^j\|_k^{K, \ell} \right)^{1/j} = \|f\|_{C^0(K)}.$$

*Proof.* In view of (17), for every  $f \in \mathscr{W}_{\ell; \text{loc.}}^{p,k}(\Omega)$ ,

$$\|f^j\|_{C^0(K)} \leq \|f^j\|^{K, \ell} \leq 2 \sup \left\{ \|f^j\|_{C^0(K)}, \mathbf{E}_{p;k}^{\ell}(f^j)(K)^{1/p} \right\},$$

hence it suffices to show that  $\mathbf{E}_{p;k}^{\ell}(f^j)(K)^{1/p} = O\left(\|f\|_{C^0(K)}^j\right)$  as  $j \rightarrow +\infty$ . Precisely this will be implied by the inequality

$$(19) \quad \mathbf{E}_{p;k}^{\ell}(f^j)^{1/p} \leq j \|f\|_{C^0(K)}^{j-1} \mathbf{E}_{p;k}^{\ell}(f)^{1/p}.$$

Let us prove (19). Let  $a \subset \Omega$  be a round set intersecting  $K$ .

$$(20) \quad \text{osc}(f^j, a) \leq j(\sup_a |f|)^{j-1} \text{osc}(f, a) \leq j \|f\|_{C^0(K)}^{j-1} \text{osc}(f, a),$$

where we have used the inequality  $|x^j - y^j| \leq j \sup\{x, y\}^{j-1} |x - y|$  for any positive real numbers  $x$  and  $y$ . Let  $n$  be a large integer. For any  $\mathcal{P} \in \text{Packing}_{k;n}(K)$ ,

$$\sum_{a \in \mathcal{P}} \text{osc}(f^j, a)^p \stackrel{(20)}{\leq} \sum_{a \in \mathcal{P}} j^p \|f\|_{C^0(K)}^{(j-1)p} \text{osc}(f, a)^p.$$

This implies (19) by letting  $n \rightarrow +\infty$ , taking supremum and applying the definition of the energies.  $\square$

Let  $F$  denote a Fréchet  $\mathbf{C}$ -algebra. A character of  $F$  is a continuous non-constant homomorphism to  $\mathbf{C}$ . The space of characters on  $F$  equipped with the weak star topology is denoted by  $\mathbf{M}(F)$ ;  $\mathbf{M}$  stands for “maximal closed ideals”, the equivalence with characters being provided by the Gelfand-Mazur theorem for Fréchet algebras [Gol90, 3.2.11].

**Lemma 2.4.** *Let  $\Omega$  be an open subset of a  $O(u)$ -quasisymmetric structure  $(Z, \beta)$ . A character on  $\mathcal{W}_{\text{loc.}}^{p,k}(\Omega)$  is continuous with respect to the topology induced by  $\mathcal{C}^0(\Omega)$ .*

*Proof.* Let  $\varphi \in \mathbf{M}(\mathcal{W}_{\text{loc.}}^{p,k}(\Omega))$ . For every compact  $K \Subset \Omega$  and  $\ell \geq 1$ , there exists  $C(K, \ell)$  such that for every  $f \in \mathcal{W}_{\text{loc.}}^{p,k}(\Omega)$ ,  $|\varphi(f)| \leq C(K, \ell) \|f\|_k^{K,\ell}$ . Notice that for every integer  $j \geq 0$ , for every  $f \in \mathcal{W}_{\text{loc.}}^{p,k}(\Omega)$ ,  $|\varphi(f)| = |\varphi(f^j)|^{1/j}$  so that applying Lemma 2.3,

$$|\varphi(f)| \leq \lim_{j \rightarrow +\infty} \left( C(K, \ell) \|f^j\|_k^{K,\ell} \right)^{1/j} = \|f\|_{C^0(K)}. \quad \square$$

The lemma ensures that the continuous map  $\mathbf{M}(\mathcal{C}^0(\Omega)) \rightarrow \mathbf{M}(\mathcal{W}_{\text{loc.}}^{p,k}(\Omega))$  obtained by restricting characters is actually surjective. The next proposition uses this to describe the latter spectrum:

**Proposition 2.5.** *Let  $(Z, \beta)$  be a locally compact, second countable  $O(u)$ -quasisymmetric structure. Let  $\Omega$  be an open subspace of  $Z$ . Let  $\mathcal{R}$  be a closed equivalence relation on  $\Omega$ . Denote by  $\Lambda$  the quotient space, and  $\pi : \Omega \rightarrow \Lambda$  the surjection. Assume that  $\mathcal{W}_{\ell;\text{loc.}}^{p,k}(\Omega)$  factors through  $\mathcal{R}$ , and that  $\mathcal{R}$  is maximal for this property so that there is a well-defined continuous embedding of  $\mathcal{W}_{\ell;\text{loc.}}^{p,k}(\Omega)$  as a separating subalgebra of  $C(\Lambda)$ . Then*

$$\begin{aligned} \vartheta : \Lambda &\rightarrow \mathbf{M}(\mathcal{W}_{\text{loc.}}^{p,k}(\Omega)) \\ L &\mapsto \{f \mapsto f(L)\} \end{aligned}$$

*is a homeomorphism.*

*Proof.* We will use the representation of the Fréchet algebras  $\mathscr{W}_{\text{loc.}}^{p,k}(\Omega)$  and  $C(\Lambda)$  as projective limits of Banach algebras and its consequences on the associated spectra, compare the textbook by Goldmann [Gol90, 3.2]. Let  $(K_n)$  be an admissible exhaustion of  $\Omega$ . Introduce a sequence of closed ideals

$$\mathcal{I}_n = \left\{ f \in \mathscr{W}_{\text{loc.}}^{p,k}(\Omega) : f|_{K_n} \equiv 0 \right\}.$$

The quotient  $A^n := \mathscr{W}_{\ell;\text{loc.}}^{p,k}(\Omega)/\mathcal{I}_n$  becomes a Banach algebra when endowed with the norm  $\|f + \mathcal{I}_n\| = \|f\|_k^{K_n,\ell}$ , and embeds in  $B^n := C(\pi(K_n))$ , by mapping the class  $[f] \in A^n$  to the image of  $f|_{K_n}$ . Further  $A^n$  is a  $*$ -invariant algebra in  $B^n$ , separating points by assumption. By the Stone-Weierstrass theorem,  $\iota_n : A^n \rightarrow B^n$  has a dense range so that there is a continuous injective  $\mathbf{M}(\iota_n) : \mathbf{M}(B^n) \rightarrow \mathbf{M}(A^n)$ , which is surjective by Lemma 2.4.  $\mathbf{M}(B^n)$  is compact (it is actually homeomorphic to  $\pi(K_n)$ ) and  $\mathbf{M}(A^n)$  is Hausdorff by Hausdorffness of the weak star topology, so that  $\mathbf{M}(\iota_n)$  is a homeomorphism. Now, under natural identifications

$$\begin{aligned} \mathbf{M}\left(\mathscr{W}_{\ell;\text{loc.}}^{p,k}(\Omega)\right) &= \varinjlim \mathbf{M}(A^n) \\ \mathbf{M}(C(\Lambda)) &= \varinjlim \mathbf{M}(B^n). \end{aligned}$$

The maps  $\mathbf{M}(\iota_n)$  are compatible with the inductive limits. Denote by  $\iota$  their glueing. Since  $C(\Lambda)$  is a uniform Fréchet algebra,  $\mathbf{M}(C(\Lambda))$  is homeomorphic to  $\Lambda$  through the Gelfand map  $\vartheta \circ \mathbf{M}(\iota)$  [Gol90, 4.1.7].  $\square$

*Remark 2.6.* The shifts in parameters  $k, \ell$  defining the algebra when changing  $O(u)$ -quasisymmetric structure (by (18)) are troublesome. One would prefer to define a single algebra and the topological dimension of its spectrum as an invariant. The dependence with respect to  $\ell$  can be removed within the category of Fréchet algebras by taking an additional projective limit (one may restrict to countably many  $\ell$  in  $\mathcal{O}^+(u)$  for the seminorms). However this is not the case with the parameter  $k$  since the seminorms  $\| - \|_k^{K,\ell}$  decrease with respect to  $k$ . Whereas we observe that there remains a Gelfand-Mazur theorem for countable inductive limits of Fréchet algebras [Est97, Theorem 2.2], in our main application in section 3 we will rather keep working with parametrized algebras and use extra information on the spectra (they are to be topological manifolds) to bypass this difficulty.

2.1.3. *Capacitors and capacities.* For  $p \in [1, +\infty)$ ,  $k, \ell \in \mathcal{O}^+(u)$  and  $\Omega$  an open subset in a  $O(u)$ -quasisymmetric structure, denote by  $\mathscr{W}_{\ell;\text{loc.}}^{p,k}(\Omega, \mathbf{R})$  the  $\mathbf{R}$ -subspace of  $\mathscr{W}_{\ell;\text{loc.}}^{p,k}(\Omega)$  of  $\mathbf{R}$ -valued functions.

**Definition 2.4** (Capacitor, capacity). Let  $Z$  be a  $O(u)$ -quasisymmetric structure and let  $\Omega$  be an open subspace. A capacitor in  $\Omega$  is a triple of subspaces  $(C, \partial_0 C, \partial_1 C)$  such that  $C$  is relatively compact,  $\partial_0 C$  and  $\partial_1 C$  are

closed disjoint, and contained in  $\overline{C} \setminus C$ . Its capacity is

$$\text{cap}_{p;k}^\ell(C) = \inf \left\{ \mathbf{E}_{p;k}^\ell(f)(C) : f \in \mathscr{W}_{\ell;\text{loc.}}^{p,k}(\Omega, \mathbf{R}), f|_{\partial_0 C} \leq 0, f|_{\partial_1 C} \geq 1 \right\}.$$

**Lemma 2.7.** *Let  $(C, \partial_0 C, \partial_1 C)$  be a capacitor in  $\Omega$ , open subset of a  $O(u)$ -quasisymmetric structure  $\beta$ . For all  $k, \ell, m \in \mathcal{O}^+(u)$ , if  $\Gamma$  is any family of curves joining  $\partial_0 C$  and  $\partial_1 C$  in  $C$  then*

$$(21) \quad \text{pmod}_{p;k}^{\ell,m}(\Gamma) \leq \text{cap}_{p;k}^\ell(C).$$

*Proof.* Let  $\varepsilon > 0$ . Let  $f \in \mathscr{W}_{\ell;\text{loc.}}^{p,k}(\Omega, \mathbf{R})$  be such that  $f|_{\partial_0 C} \leq 0, f|_{\partial_1 C} \geq 1$ . Let us prove that the gauge  $\phi : a \mapsto \text{diam} f(a)$  is in  $\mathcal{G}_m(\Gamma, \beta)$ ; the conclusion will follow by applying the definition of capacities and energies. By the intermediate value theorem, for every  $\gamma$  in  $\Gamma$ ,  $f(\gamma)$  contains  $[0, 1]$ . Consequently, whenever  $\mathcal{F}$  is a covering of  $\gamma$  by  $m$ -round sets, by countable subadditivity of the outer measure  $\mathcal{H}^1$  on  $\mathbf{R}$

$$\sum_{a \in \mathcal{F}} \phi(a) = \sum_{a \in \mathcal{F}} \text{diam} f(a) \geq \sum_{a \in \mathcal{F}} \mathcal{H}^1 f(a) \geq \mathcal{H}^1 \left( \bigcup_{a \in \mathcal{F}} f(a) \right) \geq 1. \quad \square$$

**2.2. Diffusivity.** The following is a central result in conformal dimension theory [MT10, 4.1.3]. The guiding principle is a length-volume estimate for a Riemannian parallelootope [Pan89, 2.2]; in order to transpose this to the combinatorial moduli, one has to retain a diffusivity condition expressing that a family of curves is sufficiently spread out in the space,  $(\mathbf{D}(p, r))$  below. We give two variants: the first is Pansu's original; the second one is a packing variant.

2.2.1. *Caratheodory variant.*

**Proposition 2.8.** *Let  $(Z, \beta, \delta, q)$  be a  $O(u)$ -quasisymmetric structure. Let  $\Gamma$  be a collection of subsets in  $Z$ , endowed with a positive measure  $d\gamma$  such that for any  $b \in \beta$ ,  $\{\gamma \in \Gamma : \gamma \cap \widehat{b} \neq \emptyset\}$  is measurable. For each  $\gamma \in \Gamma$ , let  $m_\gamma$  be a probability Borel measure on  $\gamma$ . Let  $p \in (1, +\infty)$ . Assume that there exists a constant  $\tau \in (0, +\infty)$  and  $r \in \mathcal{O}^+(u)$  such that*

$$(\mathbf{D}(p, r)) \quad \limsup_{n \rightarrow +\infty} \sup_{\underline{b} \in \beta : \delta(\underline{b}) \geq n} \int_{\Gamma} m_\gamma(\gamma \cap r \cdot \widehat{\underline{b}})^{1-p} [\gamma \cap \widehat{\underline{b}} \neq \emptyset] d\gamma \leq \tau.$$

Then for every  $k, m \in \mathcal{O}^+(u)$ ,

$$(22) \quad \text{mod}_{p;k}^{\ell,m}(\Gamma) \geq \frac{1}{\tau} \int_{\Gamma} d\gamma,$$

where<sup>15</sup>  $\ell = q \dot{+} r \dot{+} k$ .

<sup>15</sup>The conclusion of the lemma (as the assumption  $(\mathbf{D}(p, r))$  is all the more weaker that  $r$  is large. In subsection 3.1 we can arrange the quasisymmetric structure so that  $r$  can be assumed 0, however in subsection 3.2 it is really necessary.

*Proof.* Up to the formalism, the proof is due to Pansu [Pan89, 2.9] and we do not depart from it. Inequality (22) will actually be obtained through a stronger one: for any 0-admissible gauge  $\phi$ ,

$$(23) \quad \tilde{\Phi}_{p,k}^\ell(\Gamma) \geq \tau^{-1} \int_{\Gamma} \Phi_{1,0}(\gamma)^p d\gamma.$$

(To see why (23) implies (22) with  $m = 0$  note that since  $p \geq 1$  and  $\phi$  is admissible the right-hand side is greater than  $\int_{\Gamma} d\gamma$ ; finally  $\text{mod}_{p,k}^{\ell,m}$  increases with  $m$ ). Set an admissible gauge  $\phi$ . Define, for all  $n$ ,

$$\tau_n := \sup_{\underline{b} \in \beta: \delta(\underline{b}) \geq n} \int_{\Gamma} m_{\gamma}(\gamma \cap r.\widehat{\underline{b}})^{1-p} [\gamma \cap \widehat{\underline{b}} \neq \emptyset] d\gamma.$$

Fix  $n \in \mathbf{Z}$ . Let  $k \in \mathcal{O}^+(u)$ . Let  $\mathcal{F}$  be a countable covering of  $Z$  by  $(k, n)$ -round sets of  $\beta$ ; taking inner ball  $\underline{b} \in \beta$  for each round set  $b \in \mathcal{F}$  gives a countable  $\mathcal{B} \subset \beta$  such that  $k.\mathcal{B}$  covers  $Z$ . For  $\gamma \in \Gamma$  define  $\mathcal{B}_{\gamma} = \{\underline{b} \in \mathcal{B} : b \cap \gamma \neq \emptyset\}$ . For every  $\gamma$ ,  $k.\mathcal{B}_{\gamma}$  is a covering of  $\gamma$ , since every  $x \in \gamma$  is contained in a  $b \in \mathcal{F}$  such that  $\underline{b}$  has been selected in  $\mathcal{B}_{\gamma}$ . All the more,  $r.k.\mathcal{B}_{\gamma}$  is a covering of  $\gamma$  and by Lemma 1.13 one can extract  $\mathcal{C}_{\gamma}$  from  $\mathcal{B}_{\gamma}$  such that  $q.r.k.\mathcal{C}_{\gamma}$  covers  $\gamma$  and have disjoint realizations. Note that  $(b, q.r.k.\underline{b}) \in \mathcal{R}^{q+r+k}(\beta)$ , hence

$$\phi(q.r.k.\widehat{\underline{b}}) \leq \sup \left\{ \phi(\tilde{b}) : (b, \tilde{b}) \in \mathcal{R}^{\ell}(\beta) \right\} = \tilde{\phi}^{\ell}(b).$$

Recall that  $q.r.k.\mathcal{B}_{\gamma}$  covers  $\gamma$ . Thus

$$(24) \quad \Phi_{1,0}^{2n-\ell(n)}(\gamma) \leq \sum_{\underline{b} \in \mathcal{C}_{\gamma}} \phi(q.r.k.\widehat{\underline{b}}) \leq \sum_{\underline{b} \in \mathcal{C}_{\gamma}} \tilde{\phi}^{\ell}(b).$$

Next, apply Hölder's inequality to  $\alpha, \zeta : \mathcal{C}_{\gamma} \rightarrow \mathbf{R}$  defined by

$$\alpha(\underline{b}) = \tilde{\phi}^{\ell}(\widehat{\underline{b}}) m_{\gamma}(r.k.\widehat{\underline{b}} \cap \gamma)^{(1-p)/p} \text{ and } \zeta(\underline{b}) = m_{\gamma}(r.k.\widehat{\underline{b}} \cap \gamma)^{(p-1)/p}$$

so that

$$(25) \quad \begin{aligned} \Phi_{1,0}^{n-\ell(n)}(\gamma)^p &\stackrel{(24)}{\leq} \left( \sum_{\underline{b} \in \mathcal{C}_{\gamma}} \alpha(\underline{b})^p \right) \left( \sum_{\underline{b} \in \mathcal{C}_{\gamma}} \zeta(\underline{b})^{p/(p-1)} \right)^{p-1} \\ &\leq \left( \sum_{\underline{b} \in \mathcal{C}_{\gamma}} \tilde{\phi}^{\ell}(b)^p m_{\gamma}(r.k.\widehat{\underline{b}} \cap \gamma)^{1-p} \right) \left( \sum_{\underline{b} \in \mathcal{C}_{\gamma}} m_{\gamma}(r.k.\widehat{\underline{b}} \cap \gamma) \right)^{p-1} \\ &\leq \left( \sum_{\underline{b} \in \mathcal{C}_{\gamma}} \tilde{\phi}^{\ell}(b)^p m_{\gamma}(r.k.\widehat{\underline{b}} \cap \gamma)^{1-p} \right) (m_{\gamma}(\gamma))^{p-1}. \end{aligned}$$

The last inequality comes from the fact that the  $r.k.\widehat{\underline{b}}$  for  $b \in \mathcal{C}_{\gamma}$  are disjoint by construction, hence their intersections with  $\gamma$  are disjoint, and  $m_{\gamma}$  is

subadditive. Further, since  $m_\gamma$  is a probability measure, (25) rewrites

$$\Phi_{1,0}^{n-\ell(n)}(\gamma)^p \leq \sum_{b \in \mathcal{C}_\gamma} \tilde{\phi}^\ell(b)^p m_\gamma(\widehat{r.k.\underline{b}} \cap \gamma)^{1-p}.$$

Integrating over  $\Gamma$  yields

$$\begin{aligned} \int_\Gamma \Phi_{1,0}^{n-\ell(n)}(\gamma)^p d\gamma &\leq \int_\Gamma \sum_{b \in \mathcal{C}_\gamma} \tilde{\phi}^\ell(b)^p m_\gamma(\widehat{r.k.\underline{b}} \cap \gamma)^{1-p} d\gamma \\ &\leq \sum_{\underline{b} \in \mathcal{C}} \tilde{\phi}^\ell(b)^p \int_\Gamma [\underline{b} \in \mathcal{C}_\gamma] m_\gamma(\widehat{r.k.\underline{b}} \cap \gamma)^{1-p} d\gamma \leq \tau_n \sum_{b \in \mathcal{F}} \tilde{\phi}^\ell(b)^p. \end{aligned}$$

Infimizing over every countable  $\mathcal{F} \subset \mathcal{B}_n^k(\beta)$  that covers  $X$  one obtains:

$$(26) \quad \tilde{\Phi}_{p,k}^{\ell;n-\ell(n)}(X) \geq \tau_n^{-1} \int_\Gamma \Phi_{1,0}^{n-\ell(n)}(\gamma)^p d\gamma.$$

By monotone convergence, if  $\phi \in \mathcal{G}_m(\beta)$  then

$$\lim_{n \rightarrow +\infty} \int_\Gamma \Phi_{1,0}^{n-\ell(n)}(\gamma)^p d\gamma = \int_\Gamma \Phi_{1,0}(\gamma)^p \geq \int_\Gamma d\gamma > 0.$$

Since  $\ell$  is sublinear,  $n-\ell(n)$  goes to  $+\infty$  as  $n \rightarrow +\infty$ . Especially,  $\tilde{\Phi}_p^{k,\ell;n}(X)$  is bounded below by  $(\mathbf{D}(p, r))$ . The conclusion is reached by applying the Definition 2.1 of the modulus.  $\square$

### 2.2.2. Packing variant.

**Proposition 2.9.** *Same assumptions as in Proposition 2.8. Assume in addition that the quasisymmetric structure is that of a separable quasimetric space. For every  $k, m \in \mathcal{O}^+(u)$ , setting  $\ell = q \dot{+} r \dot{+} k$ ,*

$$(27) \quad \text{pmod}_{p;k}^{\ell,m}(\Gamma) \geq \frac{1}{\tau} \int_\Gamma d\gamma.$$

*Proof.* Fix  $n$ , pick a countable  $(k \dot{+} r, n)$  packing  $\mathcal{P}$  of  $Z$  with the following conditions: for every  $\mathbf{a} \in \mathcal{P}$  write  $\mathbf{a} = (a^-, a^+)$ , enclosing  $(\widehat{\underline{b}}, \widehat{k.r.\underline{b}})$  in  $\mathbf{a}$  the  $q.r.k.\underline{b}$  cover. Such packings exist by 1.13. This gives a countable  $\mathcal{B} \subset \beta$  (the collection of  $k.r.\underline{b}$ ) such that the realizations of  $k.r.\mathcal{B}$  are disjoint. Define  $\mathcal{Q}_\gamma = \{\underline{b} \in \mathcal{B} : \widehat{\underline{b}} \cap \gamma \neq \emptyset\}$ . The realization of  $\ell.\mathcal{Q}_\gamma$  will cover  $\gamma$  if  $\ell \geq q \dot{+} r \dot{+} k$  and then, by definition of the Caratheodory measure,  $\Phi_{1,0}^{n-\ell(n)}(\gamma) \leq \sum_{b \in \mathcal{Q}_\gamma} \tilde{\phi}^\ell(b)$ . This gives an inequality equivalent to (24) with  $\mathcal{Q}_\gamma$  instead of  $\mathcal{C}_\gamma$ . The rest of the proof follows the same lines as for Proposition 2.8 but instead of (26) one obtains:

$$(28) \quad \tau_n \text{P}\tilde{\Phi}_{p,k}^{\ell;n-\ell(n)}(X) \geq \tau_n \sum_{b \in \cup_\gamma \mathcal{Q}_\gamma} \tilde{\phi}^\ell(b)^p \geq \int_\Gamma \Phi_{1,0}^{n-\ell(n)}(\gamma)^p d\gamma,$$

before infimizing over every admissible gauge, which gives a lower bound on  $\text{pmod}_{p;k}^{\ell;0}$  and then on  $\text{pmod}_{p;k}^{\ell;m}$  for every  $m$ .  $\square$



### 2.3. Conformal dimensions.

**Definition 2.5.** Let  $(Z, \beta)$  be a  $O(u)$ -quasisymmetric structure, and let  $\Gamma$  be a family of subsets in  $Z$ . The  $O(u)$ -conformal dimension of  $\beta$  with respect to  $\Gamma$  is

$$\text{Cdim}_{O(u)}^\Gamma(\beta) = \sup \left\{ p \in \mathbf{R}_{>0} : \forall k \in \mathcal{O}^+(u), \exists \ell \in \mathcal{O}^+(u) \right. \\ \left. \forall m \in \mathcal{O}^+(u), \text{mod}_{p;k}^{\ell,m}(\Gamma, \beta) = +\infty \right\}$$

or 0 if this set is empty. Similarly, define

$$\text{PCdim}_{O(u)}^\Gamma(\beta) = \sup \left\{ p \in \mathbf{R}_{>0} : \forall k \in \mathcal{O}^+(u), \exists \ell \in \mathcal{O}^+(u) \right. \\ \left. \forall m \in \mathcal{O}^+(u), \text{pmod}_{p;k}^{\ell,m}(\Gamma, \beta) = +\infty \right\}$$

or 0 if this set is empty.

*Remark 2.10.* Given that moduli decrease with respect to  $p$ , the conformal dimension  $\text{Cdim}_{O(u)}^\Gamma(\beta)$  can be bounded above by

$$\inf \left\{ p \in \mathbf{R}_{>0} : \exists k \in \mathcal{O}^+(u), \forall \ell, m \in \mathcal{O}^+(u), \text{mod}_{p;k}^{\ell,m}(\Gamma, \beta) = 0 \right\}$$

or  $+\infty$  if this set is empty, and similarly,  $\text{PCdim}_{O(u)}^\Gamma(\beta)$  by

$$\inf \left\{ p \in \mathbf{R}_{>0} : \exists k \in \mathcal{O}^+(u), \forall \ell, m \in \mathcal{O}^+(u), \text{pmod}_{p;k}^{\ell,m}(\Gamma, \beta) = 0 \right\}.$$

**Proposition 2.11** (Conformal invariance of the conformal dimensions). *Let  $\varphi : (Z, \beta) \rightarrow (Z', \beta')$  be a  $O(u)$ -quasisymmetric homeomorphism and let  $\Gamma$ , resp.  $\Gamma'$  be a family of subsets in  $Z$ , resp.  $Z'$ , such that for all  $\gamma \in \Gamma$  there exists a unique  $\gamma' \in \Gamma'$  such that  $\varphi(\gamma) = \gamma'$ . Then*

$$(29) \quad \text{Cdim}_{O(u)}^\Gamma(\beta) = \text{Cdim}_{O(u)}^{\Gamma'}(\beta')$$

$$(30) \quad \text{PCdim}_{O(u)}^\Gamma(\beta) = \text{PCdim}_{O(u)}^{\Gamma'}(\beta').$$

*Proof.* One can assume  $Z = Z'$ ,  $\Gamma = \Gamma'$  and that  $\varphi$  is the identity map. Let us start with (29). By symmetry we need only prove  $\text{Cdim}_{O(u)}^\Gamma(\beta) \leq \text{Cdim}_{O(u)}^\Gamma(\beta')$  and  $\text{PCdim}_{O(u)}^{\Gamma;N}(\beta) \leq \text{PCdim}_{O(u)}^{\Gamma;N}(\beta')$ . The conformal dimension  $\text{Cdim}_{O(u)}^\Gamma(\beta)$  may be rewritten

$$\text{Cdim}_{O(u)}^\Gamma(\beta) = \sup \left\{ p \in \mathbf{R}_{>0} : \exists L : \mathcal{O}^+(u) \rightarrow \mathcal{O}^+(u) \right. \\ \left. \forall k, m \in \mathcal{O}^+(u), \text{mod}_{p,k}^{L(k),m}(\Gamma, \beta) = +\infty \right\}.$$

Now assume that a real number  $p$  is in the set defined on the right and let  $L$  be the corresponding map from  $\mathcal{O}^+(u)$  to itself. Define  $L' = \eta \circ L \circ \eta'$ . By Lemma 2.1, for every  $k$  and  $m$  in  $\mathcal{O}^+(u)$ ,

$$0 < \text{mod}_{p,\eta'(k)}^{L(\eta'(k)),\eta'(m)}(\Gamma, \beta) \leq \text{mod}_{p,k}^{L'(k),m}(\Gamma, \beta').$$

and the left-hand side is infinite, thus  $\text{Cdim}_{\mathcal{O}(u)}^\Gamma(\beta') > p$ , finishing the proof. (30) is obtained in the same way.  $\square$

2.4. **Upper bound on  $\text{Cdim}_{\mathcal{O}(u)}$ .**

**Lemma 2.12** (Conformal dimension is less or equal than Hausdorff dimension). *Let  $Z$  be a metric space with Hausdorff dimension  $q$ . Let  $\Gamma$  be the family of nonconstant curves in  $Z$ . Then  $\text{Cdim}_{\mathcal{O}(u)}^\Gamma Z \leq q$ .*

*Proof.* In view of remark 2.10 this will be proved if we can show that for every  $\varepsilon \in (0, q)$ ,

$$(31) \quad \exists k \in \mathcal{O}^+(u), \forall \ell, m \in \mathcal{O}^+(u), \text{mod}_{q+\varepsilon; k}^{\ell, m}(\Gamma) = 0.$$

For  $s \in (0, 1)$  consider  $\phi_s \in \mathcal{G}(\beta)$  such that  $\phi(\widehat{b}) = e^{-s\delta(b)}$  on concrete balls. By comparison with the Hausdorff measures (12),  $\Phi_{1; m} \gg \mathcal{H}^1$ . The nonconstant curves have positive  $\mathcal{H}^1$  measure by the triangle inequality, so  $\phi_s \in \mathcal{G}_m(\Gamma)$  for all  $s$ . On the other hand, again by (12),  $(\widehat{\Phi}_s)_{q+\varepsilon; k}^\ell \ll \mathcal{H}_{qs+\varepsilon s}$  for every  $\varepsilon' \in (0, qs)$ . For  $s$  sufficiently close to 1,  $qs + \varepsilon s > q$ , so (31) is attained.  $\square$

3. APPLICATIONS TO LARGE-SCALE GEOMETRY

Here two metric spaces  $Y$  and  $Y'$  are said sublinearly biLipschitz equivalent if there exists a sublinearly biLipschitz equivalence  $f : Y \rightarrow Y'$  (Definition 1.2).

3.1. **Heintze groups.**

3.1.1. *Definition.*

**Definition 3.1.** A connected solvable group  $S$  is a purely real Heintze group if its Lie algebra sits in a split extension

$$(32) \quad 1 \rightarrow \mathfrak{n} \rightarrow \mathfrak{s} \rightarrow \mathfrak{a} \rightarrow 1$$

where  $\mathfrak{n}$  is the nilradical of  $\mathfrak{s}$ ,  $\dim \mathfrak{a} = 1$  and the roots associated to  $\mathfrak{a} \rightarrow \text{Der}(\mathfrak{n})$  are real and positive multiples of each other. In addition, we say it is of diagonalizable type if  $\text{ad}_{\mathfrak{a}}$  is  $\mathbf{R}$ -diagonalizable.

It is convenient to encode a purely real Heintze group type as a pair  $(N, \alpha)$  where  $N$  is a nilpotent Lie group and  $\alpha$  is a derivation of its Lie algebra with real spectrum and lowest eigenvalue 1, realizing  $\mathfrak{a} \rightarrow \text{Der}(\mathfrak{n})$  once an infinitesimal generator  $\partial_t \in \mathfrak{a}$  has been fixed. Such an  $\alpha$  being nonsingular,  $N$  is the derived subgroup and  $(N, \alpha)$  is metabelian if and only if  $N$  is abelian. Every Heintze group admits left-invariant negatively curved Riemannian metrics<sup>16</sup> and hence is Gromov-hyperbolic.

The nilradical of a connected solvable group contains an other characteristic subgroup  $\text{Exprad}(S)$ , defined as the set of exponentially distorted

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<sup>16</sup>Though all left-invariant metrics may not be negatively curved.

elements (which does not depend on the choice of a left-invariant proper metric) together with 1. For purely real Heintze groups both are equal<sup>17</sup>.

**Theorem 3.1** (Implied by Cornulier, [Cor11, Th 1.2]). *Let  $H$  be a purely real Heintze group with data  $(N, \alpha)$ . Decompose  $\alpha = \sigma + \nu$  where  $\sigma$  is semisimple and  $\nu$  is a nilpotent derivation of  $\mathfrak{n}$  such that  $[\sigma, \nu] = 0$ . Denote by  $H_\Sigma$  the purely real Heintze group of diagonalizable type with data  $(N, \sigma)$ . Then  $H$  and  $H_\Sigma$  are  $O(\log)$ -SBE.*

3.1.2. *Punctured boundary.* From now on, under the auspices of Theorem 3.1 we work with a purely real Heintze group of diagonalizable type  $S$  with data  $(N, \alpha)$ , that is  $S = N \rtimes \mathbf{R}$  where, denoting by  $t$  the  $\mathbf{R}$  coordinate,  $t.x = e^{t\alpha}(x)$  for  $x \in N$  and we recall that  $\alpha$  is diagonalisable with real positive eigenvalues. It is known that this eases the computation of conformal dimension. Because of Bourdon's reformulation of the diffusivity lemma the latter is attained, indeed by an Ahlfors regular metric (whereas for the twisted plane of Example 1.7 it is not [BK05, 6]; also, one can prove elementarily that no distance has this scaling [DG19, 5.4]).

The vertical geodesics with tangent vector  $\partial_t$  all end at time  $+\infty$  at a distinguished point  $\omega$ , and at time  $-\infty$  on the punctured boundary  $\partial_\infty^* S$  so that we can identify the punctured boundary with  $N$ ; through this identification the one-parameter subgroup generated by  $\alpha$  is the dilation subgroup of  $\partial_\infty^* S$ . Note that if  $\rho$  and  $\rho'$  are any two proper left-invariant continuous real-valued kernels on  $\partial_\infty^* S$  such that  $\rho(\xi, \eta) = 0 \iff \xi = \eta$  and  $\rho(e^{t\alpha}\xi, e^{t\alpha}\eta) = e^t \rho(\xi, \eta)$  for all  $t, \xi, \eta$  and similarly for  $\rho'$ , then  $\rho$  and  $\rho'$  will only differ by multiplicative constants<sup>18</sup>. There are several ways to construct such kernels; one is the Euclid-Cygan kernel of Paulin and Hersensky [HP97, appendix] which depends<sup>19</sup> on a negatively curved metric on  $S$ . Another one is Hamenstädt's [Ham89, p.456] (see Dymarz-Peng for its use on boundaries of Heintze groups [DP11, 2]). Given the formalism developed in 1.1 we will rather use  $O(1)$ -quasisymmetric structures on the punctured boundary of the form below, which may vary according to our needs.

**Definition 3.2.** Let  $B$  be a compact subset of  $N$ . We say that a  $O(1)$ -quasisymmetric structure is generated by  $B$  if  $\beta = N \times \mathbf{Z}$  and for all  $b = (x, n) \in \beta$  in this product decomposition,  $\widehat{b} = xe^{-\alpha n}(B)$  (note that  $\widehat{k.b} = xe^{\alpha k}x^{-1}\widehat{b}$ ).

We do not fix  $B$ , nevertheless the resulting structures for  $B, B'$  are equivalent as soon as  $e^{-t\alpha}(B') \subseteq B \subseteq e^{t\alpha}(B')$  for some  $t$ . Especially when

<sup>17</sup>One reason for this is that  $\alpha$  is nonsingular, compare Peng [Pen11, 2.1] keeping in mind that the Cartan subgroup has rank one here.

<sup>18</sup>This follows from the same compactness argument which proves that all norm topologies on a finite-dimensional vector space are uniformly equivalent.

<sup>19</sup>This kernel was originally made for boundaries of CAT(-1) spaces and might not always be a distance in our setting, but its quasimetric constant will be bounded by  $2^{\lambda/\kappa}$  for any pair of positive numbers  $(\lambda, \kappa)$  such that  $\text{sect.}(g_\lambda) \leq -\kappa^2$ , where  $g_\lambda$  is the 1-parameter family of metrics described by Heintze just before stating his theorem 2 [Hei74].

computing the conformal dimension, we write  $\text{Cdim}_{O(u)}^\Gamma(\partial_\infty^* S)$  and do not specify a  $B$ .

*Eigencurves.* For any nonzero eigenvector  $v$  of  $\alpha$ , let  $\Gamma_v$  denote the collection of smooth curves in  $N$  everywhere tangent to the eigenspace generated by  $v$ . A curve  $\gamma \in \Gamma_v$  can be parametrized by  $\gamma(s) = \gamma(0)e^{sv}$ , and thus  $\Gamma_v$  is the space of left cosets  $N/\{e^{vt}\}$ . The homogeneous space  $\Gamma_v$  has a  $N$ -invariant,  $\alpha$ -equivariant measure  $\omega_v$  [Wei40, § 9]: for any  $\lambda$  and nonzero  $v \in \ker(\alpha - \lambda)$ , for any Borel subset  $A$  of  $\Gamma_v$ ,

$$(33) \quad \omega_v(e^{\alpha t} A) = e^{\text{tr}(\alpha) - \lambda} \omega_v(A).$$

3.1.3. *Moduli of families of eigencurves and conformal dimension.* Let  $S$  be a purely real Heintze group of diagonalizable type with data  $(N, \alpha)$ . The following lemma corresponds to [Pan89, 2.10].

**Lemma 3.2** (Lower bound). *Let  $\lambda \in \mathbf{R}_{>0}$ . Let  $v \in \ker(\alpha - \lambda)$  be nonzero. Let  $W$  be a  $\alpha$ -invariant subspace such that  $W \oplus \mathbf{R}v = \mathfrak{n}$ . Let  $\beta$  be the  $O(1)$ -quasisymmetric structure generated by  $B_0 = \{\exp P \exp sv\}_{s \in [0,1]}$ , where  $P \subset W$  is a compact convex subset. For every  $\varepsilon > 0$ , for every  $k \in \mathcal{O}^+(u)$ , there exists  $\ell \in \mathcal{O}^+(u)$ , for every  $m \in \mathcal{O}^+(u)$ ,*

$$(34) \quad \text{mod}_{(\text{tr}(\alpha)/\lambda) - \varepsilon, k}^{\ell, m}(\Gamma_v, \beta) = +\infty$$

$$(35) \quad \text{pmod}_{(\text{tr}(\alpha)/\lambda) - \varepsilon, k}^{\ell, m}(\Gamma_v, \beta) = +\infty.$$

*Proof.* Set  $p = (\text{tr}(\alpha)/\lambda) - \varepsilon$ . Let  $m_\gamma$  be the Lebesgue measure supported on  $\gamma$  with mass 1 on intersections with left translates of  $\exp B_0$ . For every  $\underline{b} \in \beta$ , denoting by  $n = \delta(\underline{b})$ , by (33)  $\omega_v\{\gamma \in \Gamma_v : \gamma \cap \widehat{b} \neq \emptyset\} = \exp\{-n(\text{tr} \alpha - \lambda)\}$  while for every  $\gamma \in \Gamma_v$ ,  $m_\gamma(\gamma \cap \widehat{b}) = [\gamma \cap \widehat{b} \neq \emptyset]e^{-\lambda n}$ . Consequently,

$$\begin{aligned} \log \int_{\Gamma_v} m_\gamma(\gamma \cap \widehat{b})^{1-p} [\gamma \cap \widehat{b} \neq \emptyset] d\gamma &\leq -n(\text{tr} \alpha - \lambda) - (1-p)\lambda n \\ &= -\varepsilon \lambda n. \end{aligned}$$

Thus (D( $p, r$ )) is fulfilled for  $r = 0$  and for every  $\tau \in (0, +\infty)$ ; Propositions 2.8 and 2.9 then yield (34) and (35) respectively.  $\square$

**Proposition 3.3.** *Let  $S$  be a purely real Heintze group of diagonalizable type with data  $(N, \alpha)$ . Let  $\Gamma$  be the collection of nonconstant curves in  $\partial_\infty^* S$ . Then*

$$(36) \quad \text{Cdim}_{O(u)}^\Gamma(\partial_\infty^* S) = \text{tr}(\alpha).$$

*Proof.* Lemma 3.2 provides one inequality: choosing  $v$  in  $\ker(\alpha - 1)$  and using that  $\Gamma_v \subseteq \Gamma$ ,  $\text{tr}(\alpha) \leq \text{Cdim}_{O(u)}^{\Gamma_v}(\beta_{v,W}) \leq \text{Cdim}_{O(u)}^\Gamma(\partial_\infty^* S)$ . As for the reverse inequality, we need find a gauge that confers nonzero  $\Phi_{1,m}$ -measure

to members of  $\Gamma$  for all  $m$ , and then evaluate  $\tilde{\Phi}$ . We will use the quasisymmetric structure generated by the exponential of a unit measure polytope in  $\mathbf{n}$  adapted to a diagonalization basis of  $\alpha$ . Observe that for every  $n \in \mathbf{Z}_{>0}$ ,

$$(37) \quad B_n \subset e^{-n}(B_0)$$

since  $\alpha$  is diagonalizable and 1 is its lowest eigenvalue. Let  $d$  be a Riemannian left-invariant distance on  $N$  giving a diameter smaller than 1 to  $B_0$ . By (37) and since any left translate of  $e^{-n}B_0$  has diameter  $\leq e^{-n}$  ( $d$  being Riemannian),  $\{\Phi_s\}_1^m \geq \mathcal{H}_d^{1/s}$ . Let  $q > \text{tr}(\alpha)$ . Then  $\phi_s^{qs}(\exp(B_n)) = e^{-qn} \ll_{n \rightarrow +\infty} e^{-\text{tr}(\alpha)n} = \mu(\exp(B_n))$ , where  $\mu$  is a Haar measure on  $N$ . Especially,  $\mu$  is locally finite, so  $\Phi_{qs,0} = 0$ . Since  $qs + s - 1 > qs$ , for every  $k, \ell \in \mathcal{O}^+(u)$ ,  $\tilde{\Phi}_{qs+(s-1),k}^\ell \leq \Phi_{qs,0}$ , so that

$$\forall s > 1, \text{mod}_{qs+(s-1),k}^{\ell;0} = 0 :$$

the moduli vanish in degree  $> q$ . Applying Definition 1.5 of the conformal dimension, one obtains  $\text{Cdim}_{\mathcal{O}(u)}^\Gamma(\beta) \leq qs$  for every  $q > \text{tr}(\alpha)$  and  $s > 1$ , hence  $\text{Cdim}_{\mathcal{O}(u)}^\Gamma(\beta) \leq \text{tr}(\alpha)$ .  $\square$

**Proposition 3.4** (Generalization of [Pal18, Prop 5.9]). *Let  $S$  and  $S'$  be purely real Heintze groups, write  $S = N \rtimes_\alpha \mathbf{R}$  and  $S' = N \rtimes_{\alpha'} \mathbf{R}$  with normalized  $\alpha$  and  $\alpha'$ . If  $S$  and  $S'$  are SBE then  $\text{tr}(\alpha) = \text{tr}(\alpha')$ .*

*Proof.* By the previously stated theorem 3.1 of Cornulier we may assume that  $S$  and  $S'$  are of diagonalizable type. If  $\varphi : \partial_\infty S \rightarrow \partial_\infty S'$  is the boundary mapping of the sublinearly biLipschitz equivalence, one can also assume without loss of generality that  $\varphi$  preserves the focal points [Cor18, 6D1] (this is stated for quasisymmetric mappings but the proof applies without change). Then by Lemma 2.11, Theorem 1.6 and Lemma 3.3, letting  $\beta$  and  $\beta'$  be the quasisymmetric structures,

$$\text{tr}(\alpha) = \text{Cdim}_{\mathcal{O}(u)}(\beta, \Gamma) = \text{Cdim}_{\mathcal{O}(u)}(\beta, \Gamma') = \text{tr}(\alpha'). \quad \square$$

3.1.4. *Proof of the main theorem.* We will prove the main theorem stated in the introduction through the following formula: for a metabelian purely real Heintze group with derivation  $\alpha$ , for  $\beta$  a quasisymmetric structure on  $\partial_\infty^* S$ , for every  $k \in \mathcal{O}^+(u)$ , for every  $\ell$  large enough with respect to  $k$  (namely  $\ell \geq q + k$  where  $q$  is attached to  $\beta$  via Definition 1.1),

$$(38) \quad \dim \mathbf{M}_{\ell, \text{loc.}}^{p,k}(\partial_\infty^* S, \beta) = \sum_{\mu \geq \text{tr}(\alpha)/p} \dim \ker(\alpha - \mu)$$

which expresses that, from the point of view of a function of bounded  $p$ -energy, the tangent space to the boundary appears split in a subbundle where any variations has infinite cost and the remaining directions where they are allowed. Since the nilradical is abelian the subbundle of forbidden variation is always integrable, and the dimension of the resulting leaf space is recovered in the spectrum through Proposition 2.5; when  $p$  increases, the

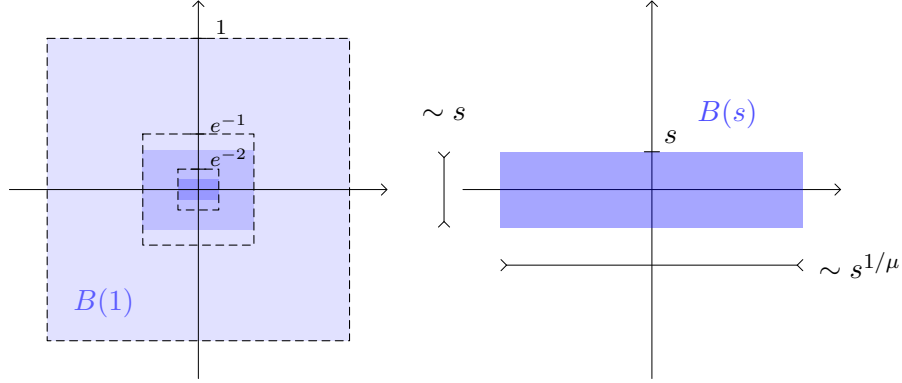


FIGURE 2. Concentric balls of a quasidistance on  $\mathbf{R}^2$  that is invariant under translation and dilation by  $\exp(t \operatorname{diag}(1, 4/3))_{t \in \mathbf{R}}$ , and coincides with the  $\ell^\infty$  distance for pairs of points at distance 1. Compare Figure 1.

right-hand side of (38) increases reflecting the fact that functions of bounded  $p$ -energy become less and less invariant, and the gaps are recorded on the left-hand side. The latter is not exactly an invariant (cf. remark 2.6) but thanks to the homeomorphism of  $\mathbf{M}\mathscr{W}_{q+k; \text{loc.}}^{p,k}(\beta)$  with a  $\mathbf{R}^d$  space we can prove that it is carried up to shifts in parameters by  $O(u)$ -quasisymmetric homeomorphisms; we provide details below.

**Lemma 3.5** (Compare [Pan89, 6.1] for  $u = 1$ ). *Let  $S$  be a Heintze group of diagonalizable type with data  $(N, \alpha)$ . Let  $\Omega$  be an open subspace of  $\partial_\infty^* S$  identified with  $N$  and equipped with a quasisymmetric structure  $(\beta, \delta, q)$ . Let  $k \in \mathcal{O}^+(u)$ . For every  $p \in [1, +\infty)$ , if  $f \in \mathscr{W}_{\ell; \text{loc.}}^{p,k}(\Omega)$  with  $\ell \geq q + k$  then  $f$  is locally invariant along the left cosets of  $H$ , where*

$$(39) \quad \mathfrak{h} = \operatorname{Liespan} \left\{ \ker(\alpha - \mu) : \mu < \frac{\operatorname{tr}(\alpha)}{p} \right\}.$$

*Proof.* Start assuming  $f$  is in  $\mathscr{W}_{\ell; \text{loc.}}^{p,k}(\Omega, \mathbf{R})$  with  $\ell \geq q + k$ . Let  $\mu \in (0, \operatorname{tr}(\alpha)/p)$  and let  $v \in \ker(\alpha - \mu)$ ; up to pre-composing  $f$  with dilations and translations assume by contradiction that  $f(\exp(\varepsilon v)) \neq f(1)$  for arbitrarily small  $\varepsilon$  and that  $1 \in \Omega$ . Up to post-composing  $f$  by translations and dilations of  $\mathbf{R}$  one can further assume  $f(1) = 0$  and  $f(\exp(\varepsilon v)) > 1$ . Construct a capacitor  $(C, \partial_0 C, \partial_1 C)$  in  $\Omega$  as follows:  $W$  is a supplementary  $\alpha$ -invariant subspace of  $v$  in  $\mathfrak{n}$ ,  $F$  is a Borel subset of  $\exp W$ ,  $C = \{we^{sX} : s \in (0, \varepsilon)\}$  and  $\partial_i C = \{we^{i\varepsilon X}\}$ . By Lemma 2.7, for every  $\ell \in \mathcal{O}^+(u)$ ,  $\operatorname{pmod}_{p;k}^{\ell,0}(\Gamma) \leq \operatorname{cap}_{p;k}^\ell(C)$ , where  $\Gamma$  is the family of curves between  $\partial_0 C$  and  $\partial_1 C$ , which includes  $\Gamma_v$ . By Lemma 3.2,  $\operatorname{pmod}_{p;k}^{\ell,0}(\Gamma_v) = +\infty$  if  $\ell \geq q + k$ , and then  $\mathbf{E}_{p;k}^\ell(f)(C) = +\infty$ , a contradiction. So  $f$  was indeed  $\langle v \rangle$ -invariant, and then locally invariant on the left cosets of  $H$ . Finally, allow  $f$  to take complex

values. Note that  $f$  is in  $\mathcal{W}_{\ell;\text{loc.}}^{p,k}(\Omega)$  if and only if  $\Re f, \Im f \in \mathcal{W}_{\ell;\text{loc.}}^{p,k}(\Omega, \mathbf{R})$  as  $\text{osc}(\Re f, a)^p \vee \text{osc}(\Im f, a)^p \leq \text{osc}(f, a)^p \leq 2^p \sup\{\text{osc}(\Re f, a)^p, \text{osc}(\Im f, a)^p\}$  for every  $a \in \mathcal{P}(\Omega)$ , which brings the argument back to the previous case.  $\square$

We assume from now on that  $N$  is abelian, identify it (as well as  $\mathfrak{n}$ ) with  $\mathbf{R}^d$  and decompose  $\mathbf{R}^d = \bigoplus_{i=1}^r \ker(\alpha - \mu_i) = \bigoplus_{i=1}^r \langle e_i^1 \dots e_i^{d_i} \rangle$ . Let  $f_i^j \in (\mathbf{R}^d)^\vee$  denote the dual basis of linear forms.

**Lemma 3.6.** *Let  $\beta$  be the quasisymmetric structure on  $\mathbf{R}^d$  generated by  $B = [-1/2, 1/2]^d$ . For all  $i \in \{1, \dots, r\}$ , for all  $j \in \{1, \dots, d_i\}$ , for all  $k, \ell \in \mathcal{O}^+(u)$ ,  $f_i^j \in \mathcal{W}_{\ell;\text{loc.}}^{p,k}(\beta, \mathbf{R})$  for  $p > \text{tr}(\alpha)/\mu_i$ .*

*Proof.* Let  $\nu$  be a Haar measure on  $N$ , normalized so that  $\nu(B) = 1$ . Set  $p = (1 + \epsilon) \text{tr}(\alpha)/\mu_i$  with  $\epsilon > 0$ . We need prove that  $\mathbf{E}_{p;k}^\ell(f_i^j)$  is locally finite for every  $\epsilon$  and  $\ell \in \mathcal{O}^+(u)$ . We may as well prove that  $\mathbf{E}_{p;k}^\ell(f_i^j)(B) < +\infty$ . Let  $n \in \mathbf{Z}_{\geq 0}$ . Recall that by definition  $\mathbf{E}_{p;k}^\ell(f_i^j)(B)$  is  $\mathbf{P}\tilde{\Phi}_{p;k}^\ell(B)$  for  $\phi(b) = \text{osc}(f|_b)$ , so that  $\phi(e^{-\alpha n} B)^p = (e^{-\mu_i n})^p = e^{-\text{tr}(\alpha)(1+\epsilon)n}$  and  $\phi$  increases with respect to inclusion. If  $\mathcal{P} \in \text{Packings}_{k,n}(B)$ , enclose into each  $(a^-, a^+)$  of  $\mathcal{P}$  a pair  $(\widehat{b}, \widehat{k} \cdot \widehat{b})$  and note that the  $\widehat{b}$  are disjoint; for  $n$  large enough they are also contained in  $[-1, 1]^d$  (since the  $\widehat{b}$  all intersect  $B$ ) so

$$\sum_{\mathbf{a} \in \mathcal{P}} \nu(\widehat{b}) = \nu\left(\bigcup_{\mathbf{a} \in \mathcal{P}} \widehat{k} \cdot \widehat{b}\right) \leq \nu([-1, 1]^d) = 2^d.$$

From there, and using that  $\nu(\widehat{b}) = e^{-\text{tr}(\alpha)\delta(b)}$  for every  $b \in \beta$ , and that  $\ell$  is sublinear, for  $n$  large enough

$$(40) \quad \sum_{\mathbf{a} \in \mathcal{P}} \tilde{\phi}^\ell(a^-)^p \leq e^{p\ell(n)} \sum_{\mathbf{a} \in \mathcal{P}} \phi(\widehat{b})^p \leq \sum_{\mathbf{a} \in \mathcal{P}} \nu(\widehat{b}) \leq 2^d.$$

This is a uniform bound for all packings so  $\mathbf{E}_{p;k}^\ell(f_i^j)(B) < +\infty$ .  $\square$

*Remark 3.7.* Actually, the  $p$ -energy of coordinates (or even Lipschitz) functions in the corresponding directions is zero, as can be obtained by replacing  $\nu$  with  $\mathcal{H}^d$  with  $d$  slightly greater than  $\text{tr}(\alpha)$  in the previous proof. To get functions with nonzero yet finite energy one should form linear combinations of the examples constructed in 1.2.3 composed with coordinates.

*Remark 3.8.* The lower bound on energies obtained in the proof of Lemma 3.5, resp. the upper bound given by Lemma 3.6 can be compared to Xie's [Xie14, Lemma 4.2] resp. [Xie14, Lemma 4.5]. Xie's technique for the lower bound is essentially different.

Now let  $S$  and  $S'$  be two purely real Heintze groups of diagonalizable type,  $\varphi : \partial_\infty^* S \rightarrow \partial_\infty^* S'$  a  $\mathcal{O}(u)$ -quasisymmetric homeomorphism. Fix  $p \in [1, +\infty)$  and  $k \in \mathcal{O}^+(u)$ . Taking the sequence of maps between spectra associated with (18), combining Proposition 2.5 with Lemma 3.5 and Lemma 3.6 and denoting  $\iota_0$  resp.  $\iota_1$  the inclusion of  $\mathcal{W}_\ell^{p;\overline{\eta} \circ \overline{\eta}^{(k)}}(\partial_\infty^* S, \beta)$  resp. of  $\mathcal{W}_{\eta \circ \eta'(\ell)}^{p;k}(\partial_\infty^* S, \beta)$

in the space of continuous functions over the spaces of leaves one obtains for  $\ell$  large enough the following diagram, the leftwards arrows start existing as homeomorphisms when  $\ell$  is large enough so that

$$(41) \quad \inf \{ \ell, \eta'(\ell), \eta \circ \eta'(\ell) \} \geq q + \sup \{ k, \bar{\eta}(k), \bar{\eta}' \circ \bar{\eta}(k) \}.$$

$$\begin{array}{ccc}
 & \mathbf{M} \mathscr{W}_{\ell}^{p; \bar{\eta}' \circ \bar{\eta}(k)}(\partial_{\infty}^* S, \beta) \xleftarrow[\vartheta]{\text{Prop. 2.5}} \Lambda = \frac{\mathbf{R}^d}{\text{span}_{\mu < \text{tr}(\alpha)/p} \{ \ker(\alpha - \mu) \}} & \\
 & \downarrow \psi & \\
 \mathbf{M}(\mathcal{C}(\Lambda)) & \mathbf{M} \mathscr{W}_{\eta'(\ell)}^{p; \bar{\eta}(k)}(\partial_{\infty}^* S', \beta') \xleftarrow[\vartheta']{\text{Prop. 2.5}} \Lambda' = \frac{\mathbf{R}^d}{\text{span}_{\mu < \text{tr}(\alpha')/p} \{ \ker(\alpha' - \mu) \}} & \\
 & \downarrow \psi' & \\
 & \mathbf{M} \mathscr{W}_{\eta \circ \eta'(\ell)}^{p; k}(\partial_{\infty}^* S, \beta) \xleftarrow[\vartheta'']{\text{Prop. 2.5}} \Lambda = \frac{\mathbf{R}^d}{\text{span}_{\mu < \text{tr}(\alpha)/p} \{ \ker(\alpha - \mu) \}} & \\
 & \swarrow \mathbf{M}(\iota_0)^{-1} \quad \searrow \mathbf{M}(\iota_1) & \\
 & & 
 \end{array}$$

The downward arrows  $\psi$  and  $\psi'$  can be thought of as  $\mathbf{M}(\varphi_*)$  and  $\mathbf{M}(\varphi^*)$  respectively; this is not specific enough since the parameters vary and these are not a priori homeomorphisms. Nevertheless they are continuous, thanks to (18) and [Gol90, 3.2.5]. Further, the existence of the set-theoretic bijection  $\mathbf{M}(\iota_1) \circ \mathbf{M}(\iota_0)^{-1}$  (Lemma 2.4), and the commutativity of the corresponding square shows that  $\psi$  is injective. By Brouwer's invariance of domain,  $\dim \Lambda \leq \dim \Lambda'$ . The reverse inequality follows from the same argument applied with  $\varphi^{-1}$ .

**3.1.5. Comparisons.** There are other algebras on the boundary of hyperbolic spaces, the extensions (modulo  $\mathbf{R}$ ) of representatives of  $\ell^p H^1(X)$  to  $\partial_{\infty} X$ . Bourdon and Kleiner have studied the corresponding equivalence relations, called the  $\ell^p$ -equivalence relations see e.g. [BK13, 10]. For Heintze groups of diagonalizable type, comparing our result with that provided by Carrasco Piaggio [CP17], the  $\ell^p$ -equivalence relations coincides with those we obtain for  $\mathscr{W}_{\ell; \text{loc.}}^{p, k}$  algebras for adequate  $k$  and  $\ell$ , except perhaps at the critical degrees.

**3.2. Fuchsian buildings.** The point here is to show that  $\text{Cdim}_{\mathcal{O}(u)}$  equals  $\text{Cdim}$  in this case, following Bourdon's proof; we provide a few details of this proof.

**3.2.1. Fuchsian buildings.** We recall below a definition according to Bourdon [Bou00, 2]. Let  $r \geq 3$  be an integer, let  $R$  be a polygon in  $\mathbb{H}^2$  with  $r$  vertices labeled by  $\mathbf{Z}/r\mathbf{Z}$  and angles  $\pi/m_i$  where  $m_i \geq 2$  for every  $i \in \mathbf{Z}/r\mathbf{Z}$ .  $R$  is the fundamental domain for a cocompact Fuchsian representation of the Coxeter group

$$W = \langle s_i \mid s_i^2, (s_i s_{i+1})^{m_i} \rangle,$$



where  $\langle s_i \rangle$  stabilizes the edge between vertices  $i$  and  $i+1$ . For every  $i \in \mathbf{Z}/r\mathbf{Z}$ , let  $q_i \geq 2$  be an integer. Let  $\mathbf{m}, \mathbf{q} : \mathbf{Z}/r\mathbf{Z} \rightarrow \mathbf{Z}_{\geq 0}$  be the corresponding data. A cell 2-complex  $\Delta$  is a Fuchsian building

- (FB1) Each 2-cell is isomorphic to the labelled  $R$ , and each 1-cell with label  $i$  lies in exactly  $(1 + q_i)$  2-cells, those are called chambers.
- (FB2) Each pair of distinct 2-chambers is contained in a subcomplex isomorphic (as a labelled cell complex) to the Coxeter complex of  $(W, \{s_i\})$ , those are called apartments.
- (FB3) Given two apartments  $A$  and  $A'$  with at least one common 2-cell  $C$ , the identity map of  $C$  extends to an isomorphism of labelled complexes  $A \rightarrow A'$ .

The Bourdon buildings are those for which  $m = 2$  (they are called right-angled) and  $q_i$  are constants. A building of such type always exists provided  $p \geq 5$ , and is uniquely defined<sup>20</sup>; it is usually denoted by  $I_{pq}$ , where the thickness  $q$  designates the constant<sup>21</sup>  $q_i + 1$  and  $p$  designates  $r$ . Once the chambers are equipped with the hyperbolic metric, Fuchsian building are CAT(-1) spaces in view of the description of their links and Ballmann's criterion, we refer to [Bou00] and reference therein for these facts as well as many examples.

*Weighted combinatorial distance.* Starting from a Fuchsian building  $\Delta$  one can associate to it a dual graph  $\mathcal{G}(\Delta)$  whose vertices are the chambers of  $\Delta$ , edges record adjacency, and they are assigned length  $\log q$  for edges of type  $q$ . Choosing any embedding of the Cayley graph of  $W$  with respect to the  $\{s_i\}$  as a subgraph of  $\mathcal{G}(\Delta)$  yields a distance on  $W$ ; for  $w \in W$ ,  $|w|_{\mathbf{q}}$  denotes the length of  $w$  for this distance. The growth rate of  $W$  with respect to  $\mathbf{q}$  is  $\mathcal{S} := \limsup_n \frac{1}{n} \log \# \{w \in W : |w|_{\mathbf{q}} \leq n\}$ ; this can be made more explicit [Bou00, 3.1.1] (for the Bourdon building the growth rate with no weight is  $\operatorname{argch}((p-2)/2)$ ) so that  $\mathcal{S} = \operatorname{argch}((p-2)/2)/\log(q-1)$  for  $I_{pq}$ . The distance between two chambers  $d, d'$  in  $\Delta$  is denoted by  $|d - d'|_{\mathbf{q}}$ , this is  $|w|_{\mathbf{q}}$  for  $w$  such that  $d = w.d'$  in any common apartment. The distance  $|\cdot - \cdot|_{\mathbf{q}}$  on  $\mathcal{G}(\Delta)$  is quasiisometric to the CAT(-1) metric on  $\Delta$ , especially it is Gromov-hyperbolic.

*Measure on spaces of marked apartments.* Given a chamber  $c$  in  $\Delta$ , let  $\mathcal{F}_c$  denote the space of embeddings of the Coxeter complex marked at  $c$  into  $\Delta$ . There is a unique probability measure  $\nu$  on  $\mathcal{F}_c$  such that for any chamber  $d$ ,  $\nu[\pi \in \mathcal{F}_c : \pi \ni d] = e^{-|d-c|_{\mathbf{q}}}$  [Bou00, 2.2.4].

<sup>20</sup>In general a building of type  $(r, \mathbf{m}, \mathbf{q})$  may or may not exist, and may or may not be unique up to isomorphism of labelled complexes.

<sup>21</sup>The shift between  $q$  and  $q_i$  is here to conform with the building of  $\operatorname{SL}(3, \mathbf{Q}_\ell)$  where links are projective planes over the residue field so that edges are incident to  $1 + \ell$  cells.

*Geodesic metric on the boundary.* The Gromov product on  $\partial_\infty\Delta$  associated to  $|\cdot|_{\mathbf{q}}$  is denoted by  $(\xi, \eta) \mapsto \{\xi, \eta\}_c$ . For  $\xi, \eta$  in  $\partial_\infty\Delta$ ,  $\varrho(\xi, \eta) = \exp(-\mathcal{T}\{\xi, \eta\}_c)$  and then  $\delta(\xi, \eta) = \inf \sum \varrho(\xi_i, \xi_{i+1})$  over chains  $\xi = \xi_0 \dots \xi_s = \eta$  in  $\partial_\infty\Delta$ . Bourdon proves that  $\delta$  and  $\varrho$  are comparable (this is the most involved part of the proof; the details for this point are given in [Bou97, p.362]), and that  $\text{Hdim}(\partial_\infty\Delta)$  equals  $1 + 1/\mathcal{T}$  [Bou00, 2.2.7]. Once this is proven,  $\delta$  induces the same quasisymmetric structure on the boundary, and by Lemma 2.12,  $\text{Cdim}_{\mathcal{O}(u)}\partial_\infty\Delta \leq 1 + 1/\mathcal{T}$ .

### 3.2.2. Diffusivity condition and lower bound.

**Lemma 3.9** (After Bourdon [Bou00, 2.2.2]). *Let  $(Z, d)$  be an Ahlfors-regular metric space. Let  $\beta$  be the associated quasisymmetric structure. Let  $\Gamma$  be a family of rectifiable curves in  $Z$  whose lengths are bounded below. Let  $d\gamma$  be a measure on  $\Gamma$ . Let  $p'$  be greater than 1. If there exists  $\eta < +\infty$  such that*

$$(D'(p')) \quad \forall b \in \beta, \log \int_{\Gamma} [\gamma \cap \widehat{b} \neq \emptyset] d\gamma - (1 - p')\delta(b) \leq \eta,$$

then  $\text{Cdim}_{\mathcal{O}(u)}^\Gamma(\beta) \geq p'$ .

Let us check that  $(D'(p'))$  implies  $(D(p, r))$  provided  $p > p'$  and  $r \in \mathcal{O}^+(u)$  is nonzero. Since the  $\gamma \in \Gamma$  have been assumed rectifiable, they bear normalized arclength measures  $m_\gamma$ . Moreover, by the reverse triangle inequality, for every  $\gamma \in \Gamma$ ,  $m_\gamma(\gamma \cap \widehat{r.b}) \geq [\gamma \cap \widehat{b} \neq \emptyset] (e^{r(\delta(b))} - 1) e^{-\delta(b)} \text{length}(\gamma)$ , so that

$$\begin{aligned} m_\gamma(\gamma \cap \widehat{r.b})^{1-p} [\gamma \cap \widehat{b} \neq \emptyset] &\leq (e^{r(\delta(b))} - 1)^{1-p} e^{(p-1)\delta(b)} \text{length}(\gamma)^{1-p} \\ &\leq C(1 - 1/e)^{1-p} \exp((p-1)(\delta(b) - r(\delta(b)))). \end{aligned}$$

Now, using  $(D'(p'))$  with  $p' < p$ ,

$$\int_{\Gamma} m_\gamma(\gamma \cap \widehat{r.b})^{1-p} [\gamma \cap \widehat{b} \neq \emptyset] d\gamma \leq C e^\eta e^{(q-p)\delta(b) + r(\delta(b))}.$$

The right-hand side goes to 0 because  $r$  is sublinear, so  $(D(p, r))$  holds for every  $\tau \in \mathbf{R}_{>0}$ .

Going back to Fuchsian buildings it remains to specify  $\Gamma$ ,  $d\gamma$  and  $p'$ . Following Bourdon, given a reference chamber in  $\Delta$ ,  $\Gamma$  is the collection of boundaries of apartments containing the reference chamber  $c$  :

$$\Gamma = \{\partial_\infty \text{im}(\pi) : \pi \in \mathcal{F}_c\},$$

$d\gamma$  is the measure on  $\Gamma$  corresponding to  $\nu$  on  $\mathcal{F}_c$ . The fact that the  $\gamma \in \Gamma$  are rectifiable follows from [Bou00, 2.2.6(ii)]. The condition  $(D'(p'))$  for  $p' = 1 + 1/\mathcal{T}$  is checked by Bourdon [Bou00, 2.3.8]. By Lemma 2.8,  $\text{Cdim}_{\mathcal{O}(u)}(\partial_\infty\Delta) > 1 + 1/\mathcal{T} - \varepsilon$  for every positive real  $\varepsilon$  arbitrarily small. This finishes the proof that  $\text{Cdim}_{\mathcal{O}(u)}\partial_\infty\Delta = 1 + 1/\mathcal{T}$ .

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