

VARIATIONS ON THE PETERSEN COLOURING CONJECTURE

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ABSTRACT. The Petersen colouring conjecture states that every bridgeless cubic graph admits an edge-colouring with 5 colours such that for every edge e , the set of colours assigned to the edges adjacent to e has cardinality either 2 or 4, but not 3. We prove that every bridgeless cubic graph G admits an edge-colouring with 4 colours such that at most $\frac{4}{5} \cdot |V(G)|$ edges do not satisfy the above condition. This bound is tight and the Petersen graph is the only connected graph for which the bound cannot be decreased. We obtain such a 4-edge-colouring by using a carefully chosen subset of edges of a perfect matching, and the analysis relies on a simple discharging procedure with essentially no reductions and very few rules.

1. INTRODUCTION

At the ninth British Combinatorial Conference in 1983, Fouquet and Jolivet [4] introduced strong edge-colourings of cubic graphs. This notion was further studied by Jaeger, who formulated a conjecture which is, arguably, one of the most challenging conjecture in graph theory. Proving Jaeger's conjecture to be true would have tremendous consequences, such as confirming the Cycle double cover conjecture, the Berge-Fulkerson conjecture and the nowhere-zero 5-flow conjecture.

For any integer $k \geq 3$, consider a k -edge-colouring of a cubic graph $G = (V, E)$, that is, a mapping $f: E \rightarrow \{1, \dots, k\}$ such that $f(e) \neq f(e')$ for every two edges e and e' that share a vertex. For an edge $e \in E$, let $\mathcal{E}(e)$ be the set of four edges adjacent to e . The edge e is *rich* if $|f(\mathcal{E}(e))| = 4$, while it is *poor* if $|f(\mathcal{E}(e))| = 2$. The edge-colouring f is *normal* if every edge is either rich or poor. The Petersen colouring conjecture reads as follows.

Conjecture 1.1 (The Petersen coloring conjecture—Jaeger, 1985). *Every cubic bridgeless graph admits a normal 5-edge-colouring.*

Now it is maybe a good time to explain the links with the ubiquitous Petersen graph \mathcal{P} . A *Petersen colouring* of a cubic graph $G = (V, E)$ is a mapping g that associates to each edge of G an edge of \mathcal{P} such that if two edges e and e' of G share a vertex, then so do the edges $g(e)$ and $g(e')$ of \mathcal{P} . As observed by Jaeger [8], normal colourings and Petersen colourings of cubic graphs are in one-to-one correspondence.

Indeed, as is well known, the Petersen graph \mathcal{P} can be seen as the Kneser graph with parameters 5 and 2, defined as follows: the vertices are in one-to-one correspondence with the 2-element subsets of $\{1, \dots, 5\}$ and two vertices are adjacent if and only if the corresponding subset are disjoint. With this definition in mind, we can label every edge uv of \mathcal{P} by the unique integer $\ell(uv) \in \{1, \dots, 5\}$ that does not belong to $X_u \cup X_v$, where X_w is the 2-element subset of $\{1, \dots, 5\}$ that corresponds to w , for every vertex w of \mathcal{P} . Notice that if g is a Petersen colouring of a cubic graph $G = (V, E)$, then $f: E \rightarrow \{1, \dots, 5\}$ defined by $f(e) := \ell(g(e))$ is a normal colouring of G .

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Conversely, assume that f is a normal 5-edge-colouring of a cubic graph $G = (V, E)$. Keeping in mind the labelling ℓ of the edges of \mathcal{P} given above, we define the mapping $g: E \rightarrow E(\mathcal{P})$ as follows. For each edge $e = uv \in E$, we define $g(e)$ to be the edge e' of \mathcal{P} such that first $\ell(e') = f(e)$, and second e' is incident to the vertex $w \in V(\mathcal{P})$ such that the three colours assigned by f to the edges of G incident to u are the elements of $X_w \cup \{f(e)\}$. A straightforward checking ensures that g is a Petersen colouring of G . We just proved the following equivalence, which was first established by Jaeger [8].

Proposition 1.2. *Let G be a cubic graph. Then, G admits a normal colouring if and only if G admits a Petersen colouring.*

Notice that a 3-edge-colouring of a (connected) cubic graph G is precisely a normal colouring in which every edge is poor. Fouquet and Jolivet [4] coined the term *strong colouring* to define an edge-colouring in which every edge is rich. This corresponds precisely to an edge-colouring “at distance 2” or, in other words, to a vertex-colouring of the square of the line graph of G . The unique normal colouring of the Petersen graph is a strong colouring.

Despite original approaches [10–12], few progress has been made on the Petersen colouring conjecture: ways to infirm it remain elusive as possible counter-examples must be snarks, that is bridgeless cubic graphs that are not 3-edge-colourable (the ones we know are usually obtained from well-structured graph operations, for which the Petersen colouring conjecture can be verified [2, 6]), and confirming the conjecture is expected to be a difficult task since as reported earlier this would confirm several difficult and most researched graph conjectures.

In view of the difficulty of the question, it is natural to ask for weaker versions of the conjecture. Because a strong colouring is normal, we know that every cubic graph admits a normal colouring using at most 10 colours: indeed Andersen [1] and, independently, Horák, He and Totter [7] established this statement (which confirms, for the particular case of graphs with maximum degree 3, a conjecture of Erdős and Nešetřil formulated in 1985 during a seminar in Prague). Further, it has been noted before [2] that Seymour’s 8-flow theorem provides, for any bridgeless cubic graph, a normal 7-edge-colouring. To the best of our knowledge, whether a normal 6-edge-colouring can be found for any such graph is still an open question. A line of study is then to find an edge-colouring that is normal on a large proportion of the graph, which we formalise in the next subsection. Before that, we end this part with a remark.

As mentioned earlier, a 3-edge-colourable graph G always has a Petersen colouring: let e_1 , e_2 and e_3 be the three edges incident to an arbitrary vertex v of the Petersen graph. Label e_i by $\ell(e_i) := i$ for each $i \in \{1, 2, 3\}$. If c is a 3-edge-colouring of G , then defining $f: E(G) \rightarrow E(\mathcal{P})$ by $f(e') := e_{c(e')}$ yields a Petersen colouring of G . This Petersen colouring is, in some sense, trivial: it only uses the incidences at a single vertex of the Petersen graph. This is equivalent to saying that the original graph admits a 3-edge-colouring. A fact that seems worth pointing out, however, is that Petersen colourings of bridgeless cubic graphs are either “trivial” (and hence the graph admits a 3-edge-colouring) or surjective.

1.1. The rich, the poor and the medium. Given an edge-colouring of a cubic graph G , define an edge e to be *medium* if it is neither rich nor poor. Since the Petersen colouring conjecture states that every bridgeless cubic graphs admits a 5-edge-colouring such that no edge is medium, it seems interesting to investigate the minimum number of medium edges in edge-colourings of bridgeless cubic graphs. As observed by Bílková [2, p. 9], Petersen’s perfect matching theorem combined with Vizing’s edge-colouring theorem (and some further analysis if the graph has cycles

of length less than 5) directly yield for every bridgeless cubic graph a 5-edge-colouring such that at least one third of the edges are rich or poor. This lower bound was improved [2, Theorem 3.2] to two thirds of the edges for cubic graphs having a 2-factor consisting of two cycles of the same length — the class of “generalised prisms” — and to roughly half the edges in graphs with no short cycles [2, Theorem 3.6].

Ways how poor, rich and medium edges combine in edge-colourings looks intriguing. We already pointed out that an edge-colouring with poor edges only is actually a 3-edge-colouring. Oppositely, an edge-colouring with rich edges only is a strong colouring. Notice that every normal 5-edge-colouring of the Petersen graph actually contains no poor edge: it is thus a strong colouring.

We consider 4-edge-colourings and prove the following result.

Theorem 1.3. *Every connected cubic bridgeless graph G admits a 4-edge-colouring such that at most $\frac{4}{5} \cdot |V(G)|$ edges are neither rich nor poor. Furthermore, no 4-edge-colouring of G yields less medium edges if and only if G is the Petersen graph.*

Since an n -vertex cubic graph has $\frac{3n}{2}$ edges, Theorem 1.3 ensures that every bridgeless cubic graph G admits a 4-edge-colouring containing at most $\frac{8}{15} \cdot |E(G)|$ medium edges. This bound cannot be improved in general, since the Petersen graph has 15 edges and each of its 4-edge-colourings yields at least 8 medium edges.

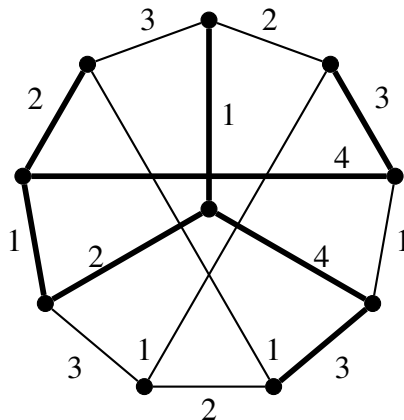


FIGURE 1. A 4-edge-colouring of the Petersen graph with exactly 8 medium edges, drawn thicker.

2. PROOF OF THEOREM 1.3

We demonstrate the upper bound: every bridgeless cubic graph G admits a 4-edge-colouring such that at most $\frac{4}{5} \cdot |V(G)|$ edges are medium. While developing the proof, we shall see that the only case where the bound must be attained is if G is the Petersen graph. We proceed by induction on the number of vertices of G . The induction yields the conclusion in a standard way if G has a triangle, and we thus first deal with this case. To make the argument smoother, we actually prove the result for (loopless) bridgeless cubic multi-graphs. For instance any 4-edge-colouring of a triple edge between two vertices yields three poor edges. Similarly, if a cubic graph G contains two vertices with exactly two parallel edges between them, then in any 4-edge-colouring of G either both edges are poor or both edges are medium.

As reported earlier, every 3-edge-colouring of a cubic graph contains only poor edges, and hence the statement of Theorem 1.3 is correct if G admits a 3-edge-colouring, and hence in particular if $|V(G)| = 2$. We hence consider a connected bridgeless cubic multi-graph G that admits no 3-edge-colouring, and we set $n := |V(G)|$ (so $n \geq 10$). Our first two arguments are standard and well known to people used to graph colouring but they are included for completeness.

We use induction to prove the statement if G contains a multi-edge. Indeed, suppose that e_1 and e_2 are two different edges with end-vertices v_1 and v_2 . For each $i \in \{1, 2\}$, let u_i be the neighbour of v_i different from v_{3-i} . Since $n > 2$ and because G is bridgeless, $u_1 \neq u_2$. Let G' be the bridgeless cubic multi-graph obtained from $G - \{v_1, v_2\}$ by adding a new edge e' between u_1 and u_2 . The induction hypothesis ensures that G' admits a 4-edge-colouring c' yielding at most $\frac{4}{5} \cdot (n - 2)$ medium edges. It is straightforward to deduce from c' a 4-edge-colouring c of G with no more medium edges, which thus prove the statement (including “the furthermore part”) if G contains a multi-edge. One obtains a 4-edge-colouring c of G by setting $c(e) = c'(e)$ if $e \in E(G) \cap E(G')$ and $c(v_i u_i) := c'(e')$ for each $i \in \{1, 2\}$, and letting $c(e_1)$ and $c(e_2)$ be the two colours of the two edges of G' incident to u_1 that are different from e' . The obtained 4-edge-colouring c of G yields at most $\frac{4}{5} \cdot (n - 2)$ medium edges, which is less than $\frac{4}{5} \cdot n$. Consequently, we may assume that G is simple.

We now use induction to prove the statement if G contains a triangle $v_0 v_1 v_2$. Indeed, we then define G' to be the multi-graph obtained from G by contracting v_0, v_1 and v_2 into a single vertex x . It follows that G' is a bridgeless and cubic multi-graph with $n - 2$ vertices, and therefore the induction hypothesis yields that G' admits a 4-edge-colouring c' yielding at most $\frac{4}{5} \cdot (n - 2)$ medium edges. This colouring c' can be extended into a 4-edge-colouring c of G by setting $c(e) := c'(e)$ if $e \in E(G) \cap E(G')$ and $c(v_i v_{i+1}) := c'(v_{i+2} u_{i+2})$ where $i \in \{0, 1, 2\}$ is considered modulo 3 and u_i is the neighbour of v_i not in $\{v_{i+1}, v_{i+2}\}$. The obtained 4-edge-colouring c yields no more medium edges in G than c' does in G' . Consequently, we may assume that G is a simple bridgeless cubic graph with no triangle.

It remains to deal with the case where G is a bridgeless cubic graph with no triangles. Let F a 2-factor of G and M the perfect matching such that $F = G - M$. Because G admits no 3-edge-colouring, we know that F contains at least two odd cycles. In particular, there exists an edge in M that is not a chord of a cycle in F . Note also that cycles of G with length (at most) 5 have no chord, as G has no triangle.

If $v \in V(G)$, we define C_v to be the cycle in F to which v belongs and v' to be the unique neighbour of v in G such that $vv' \in M$. For every cycle $C \in F$, there is a cyclic ordering φ_C of the vertices in $V(C)$, which we extend to the edges in M that are incident to a vertex in C : two edges e and e' in M that have each exactly one end-vertex on a given cycle $C \in F$ are *consecutive* if their end-vertices in C are consecutive with respect to φ_C . (Notice that chords are purposely excluded from this definition.)

An *edge-selection* is a subset S of M with the following properties:

- (1) every edge in S is incident to two different odd cycles in F (in particular, no edge in S is a chord of a cycle in F);
- (2) every cycle in F is incident to at most two edges in S ; and
- (3) if a cycle $C \in F$ is incident to two edges in S , then these two edges are consecutive.

Given an edge-selection S and a cycle $C \in F$, the *degree* $\deg_S(C)$ of C in S is the number of edges incident to C that belong to S , and hence $\deg_S(C) \in \{0, 1, 2\}$. If in addition $C' \in F$, then C and C' are *S -adjacent* if S contains an edge incident to both C and C' . An *S -component* of G is an

inclusion-wise maximal subset K of F such that for every two distinct cycles C and C' in K , there exists a sequence $(C_i)_{0 \leq i \leq t}$ of cycles in K such that $C_0 = C$, $C_t = C'$ and for every $i \in \{0, \dots, t-1\}$, the cycles C_i and C_{i+1} are S -adjacent. The edges in S joining two cycles in K are said to be *associated with K* . This relation is a surjective mapping from S to the set of all S -components of G . We need a last definition. If K is an S -component, then we let G_K be the multi-graph with vertex set K and x edges between C and C' where $x \in \{0, 1, 2\}$ is the number of edges in S that are incident to both C and C' in G . It follows from the definitions that G_K is either a single vertex, or a path, or a cycle, or two vertices joined by two parallel edges.

Among all edge-selections of maximum order, we choose one such that the number of S -degree-2 cycles is as large as possible. We construct a 4-edge-colouring of G with the following properties:

- every edge in M is coloured 4 and no other edge is coloured 4;
- an edge is coloured 3 only if it belongs to an odd cycle in F ;
- every odd cycle in F has exactly one edge coloured 3;
- every edge in S is adjacent to two edges coloured 3; and
- if an edge e in S is medium, then it is associated with an S -component K such that G_K is an odd cycle and e is the only medium edge associated with K .

To see why such a 4-edge-colouring exists, start by colouring the edges of G that belong to M with 4. Next colour the edges of every even cycle in F using $\{1, 2\}$. By (2) and (3) every odd cycle $C \in F$ has an edge that is incident to all the edges in S incident to C : colour this edge with 3. The remaining uncoloured edges span a vertex-disjoint collection of paths, and we colour them using $\{1, 2\}$, independently for each S -component K . If G_K is not an odd cycle, then one can ensure that no edge associated with K is medium. If G_K is an odd cycle, then one can ensure that exactly one edge S associated with K is medium.

Our goal is to demonstrate that the obtained 4-edge-colouring of G contains at most $4n/5$ medium edges, where n is the number of vertices of G . We use a discharging argument to count the medium edges: we start by assigning a charge of 1 to each medium edge, and thus throughout all the process the total charge in the graph G is precisely the number of medium edges.

We shall define a number of discharging rules: in the first ones, medium edges send charge to cycles in F to which they are incident. Later, some cycles in F will send some charge to other cycles in F . We apply the rules in order and analyse the global state of the charge in the graph after one or more rules have been applied. At the end, we prove that for each S -component K the sum of the charges of the cycles in K is at most $\frac{4}{5}$ times the number of vertices belonging to cycles in K , which implies the sought upper bound on the number of medium edges.

Fact 1. *A cycle in F contains zero medium edge if it is even and 3 medium edges if it is odd.*

Proof. Indeed, if the edge e belongs to an even cycle, then its colour c belongs to $\{1, 2\}$, and each of its end-vertices is incident to an edge coloured 4, which belongs to M , and an edge coloured $3 - c$, which belongs to the even cycle. So e is poor. Let $C = v_0 \dots v_{2k}$ is an odd cycle in F with v_0v_1 being its only edge coloured 3. For each $i \in \{2, \dots, 2k-1\}$, let $c_i \in \{1, 2\}$ be the colour of the edge $v_i v_{i+1}$. Then the edge $v_i v_{i+1}$ is incident to two edges coloured 4 and two edges coloured $3 - c_i$, and hence $v_i v_{i+1}$ is poor. Consequently, the only medium edges on C are v_0v_1 , v_1v_2 and $v_{2k}v_0$. \square

Our first rule reads as follows.

(R0) Every medium edge that belongs to a cycle C in F sends 1 to C .

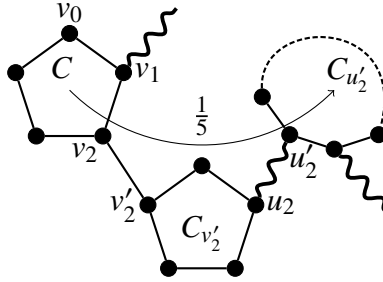


FIGURE 2. Illustration of the discharging rule (R4): C and $C_{v'_2}$ both are cycles of length 5 and S -degree 1 while $C_{u'_2}$ has S -degree 2; waved edges are those in S and one could have $C_{v'_1} = C_{u'_2}$, that is, the two waved edges with only one end-vertex drawn could be the same edge.

After applying rule (R0), an edge has charge 0 except if it is a medium edge that belongs to M , in which case it has charge 1. In addition, a cycle in F has charge 0 if it is even and charge 3 if it is odd. For our next rule, notice that if e is a medium edge that belongs to M , then e is adjacent to an edge coloured 3, which must belong to an odd cycle in F .

(R1) Let e be a medium edge that belongs to M , and let C and C' be the two cycles in F to which e is incident, such that C is odd and the edge coloured 3 on C is adjacent to e . If C' is even, then e sends $1/2$ to C' and $1/2$ to C . If C' is odd its edge coloured 3 is adjacent to e , then e sends $1/2$ to each of C and C' . If C' is odd and its edge coloured 3 is not adjacent to e , then e sends 1 to C and nothing to C' .

After applying rule (R1), every edge has charge 0. If $C \in F$ is an even cycle, then its charge is at most $1/2 \cdot |V(C)|$, which is less than $4|V(C)|/5$. In addition, if $C \in F$ is an odd cycle, then one of the following occurs:

- C has S -degree 0 and charge at most 5;
- C has S -degree 1 and charge at most 4; or
- C has S -degree 2 and charge at most $7/2$.

Indeed, if C is incident to exactly one edge in S , then this edge cannot be medium by the construction of the 4-edge-colouring. If C is incident to exactly two edges in S , then at most one of them is medium, in which case it sends $1/2$ to C . It follows that the charge of C is at most $\frac{4}{5} \cdot |V(C)|$ unless C has length 5 and S -degree 0.

The next step is to apply the three following rules, the third one being illustrated in Figure 2.

(R2) If C is a cycle of length 5 in F with S -degree 0, then C sends $1/5$ to $C_{v'}$ for each $v \in V(C)$.

(R3) Let $C = v_0 \dots v_4$ be a cycle of length 5 in F of S -degree 1, with $v_1v'_1$ being the unique edge in S incident to C . For each $i \in \{0, 2\}$, if $C_{v'_i}$ is not a cycle of length 5 with S -degree 1, then C sends $1/5$ to $C_{v'_i}$ through v_i .

(R4) Let $C = v_0 \dots v_4$ be a cycle of length 5 in F of S -degree 1, with $v_1v'_1$ being the unique edge in S incident to C . Let $i \in \{0, 2\}$. Suppose that $C_{v'_i}$ is a cycle of length 5 of S -degree 1,

written $v'_i u_1 u_2 u_3 u_4$ such that u_2 is its unique vertex incident to an edge in S . If $C_{u'_2}$ is a cycle of S -degree 2 then C sends $1/5$ to $C_{u'_2}$ through v_i .

We now check that after applying (R2)–(R4), for every S -component K , the sum of the charges of the cycles in the component is at most $4/5$ times the number of vertices belonging to cycles in K with equality only if K contains only cycles of length 5. We first analyse the current charge of every cycle $C \in F$. If C is even, then it has charge at most $(1/2 + 1/5) \cdot |V(C)| < \frac{4}{5} \cdot |V(C)|$.

If C is a cycle with S -degree 0 and length $2k + 1$, where $k \geq 2$, then C does not receive any charge. Indeed, there cannot be an edge e in M incident to C and a cycle of length 5 and S -degree 0, as $S \cup \{e\}$ would then contradict the maximality of S . This shows that C does not receive charge by (R2). Moreover, if C would receive charge by (R3) then the definition of (R3) would imply the existence of a cycle $C' = v_0 v_1 v_2 v_3 v_4$ in F of length 5 and S -degree one such that $v_1 v'_1 \in S$ and $C = C_{v'_0}$ (or $C = C_{v'_2}$). Consequently, $S \cup \{v_0 v'_0\}$ would contradict the maximality of S . Therefore, the final charge of C is at most 5 if $k \geq 3$, which is less than $4 \cdot (2k + 1)/5$. If $k = 2$ then by rule (R2) the final charge of C is at most $5 - 5 \cdot \frac{1}{5} = 4 = 4 \cdot (2k + 1)/5$.

Let C be a cycle with S -degree 1. Note that C can receive charge only because of (R2) and (R3). Moreover, C does not receive charge through its vertex incident to an edge in S . If C has length $2k + 1$ with $k \geq 3$, then before applying (R2)–(R4) the charge of C was at most 4 since the edge in S incident to C is not medium, and hence its final charge is at most $4 + 2k/5$, which is less than $4 \cdot (2k + 1)/5$ since $k \geq 3$.

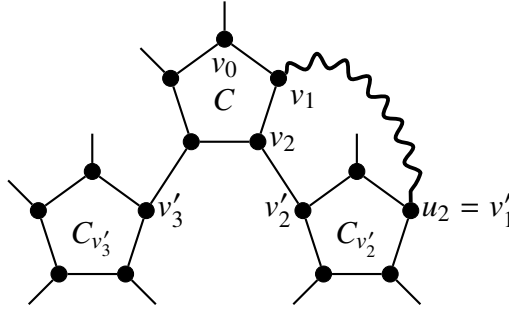


FIGURE 3. If C and $C_{v'_2}$ are S -adjacent cycles of length 5 with S -degree 1, then $C_{v'_3}$ cannot be a cycle of length 5 with S -degree 0, for otherwise $(S \setminus \{v_1 v'_1\}) \cup \{v_2 v'_2, v_3 v'_3\}$ would contradict the maximality of S . Consequently, $C_{v'_3}$ does not send charge to C by (R2).

If C has length 5, then let us write $C = v_0 \dots v_4$ with $v_1 v'_1 \in S$. Recall that $v_1 v'_1$ cannot be medium because the S -component to which C belongs is not an odd cycle. Furthermore, C can receive charge only by (R2). Observe that none of $C_{v'_0}$ and $C_{v'_2}$ is an odd cycle with S -degree 0 for otherwise adding the edge $v_0 v'_0$ or $v_2 v'_2$ to S would contradict its choice. Therefore C can receive charge only through v_3 or v_4 , for a total of at most $2/5$. It follows that if C sends $1/5$ through each of v_0 and v_2 , due to rules (R3) and (R4), then its final charge surely is at most $\frac{4}{5} \cdot |V(G)|$. Let us identify precisely when C sends charge through v_2 , the case for v_0 being identical. If $C_{v'_2}$ is not a cycle of length 5 with S -degree 1, then (R3) applies. So assume that $C_{v'_2}$ is a cycle of length 5 with S -degree 1, written $v'_2 u_1 u_2 u_3 u_4$. None of u_1 and u_4 is incident to an edge in S , for otherwise adding the edge $v_2 v'_2$ to S would contradict its maximality. So we can assume without loss of generality that $u_2 u'_2 \in S$. Now we observe that if $C_{u'_2} \neq C$, then $C_{u'_2}$ cannot be a cycle of S -degree 1,

for otherwise $(S \setminus \{u_2u'_2\}) \cup \{v_2v'_2\}$ would contradict the fact that S , among all edge-selections of maximum order, creates the maximum number of cycles with S -degree 2. Therefore in this case $C_{u'_2}$ is a cycle of S -degree 2 and hence C sends $1/5$ to $C_{u'_2}$ by (R4). It follows that the only case where C does not send charge through v_2 is when $C_{u'_2} = C$ and hence $v'_1 = u_2$ (or $v'_1 = u_3$). In this situation, we argue that C cannot receive charge through v_3 . Indeed C can receive charge through v_3 only by (R2), which applies if and only if $C_{v'_3}$ is a cycle of length 5 and S -degree 0, as illustrated in Figure 3. In this case, $(S \setminus \{v_1v'_1\}) \cup \{v_2v'_2, v_3v'_3\}$ would contradict the maximality of S . Consequently, we proved that either C sends $1/5$ through v_2 or C receives nothing through v_3 . By symmetry of the roles played by v_2 and v_0 , either C sends $1/5$ through v_0 or C receives nothing through v_4 . Since C can receive charge only through v_3 and v_4 , we therefore conclude that the final charge of C is not greater than its charge before applying (R2)–(R4), that is 4.

It remains to deal with cycles with S -degree 2. Let us write $C = v_0 \dots v_{2k}$, where $k \geq 2$, with $v_1v'_1 \in S$ and $v_2v'_2 \in S$. Suppose first that G_K is not an odd cycle, where K is the S -component to which C belongs. In particular, none of $v_1v'_1$ and $v_2v'_2$ is medium. In this case, we show that the charge of C is at most $\frac{4}{5} \cdot (2k + 1)$, with equality only if $k = 2$. Observe that, for each $i \in \{1, 2\}$, the cycle C can receive some charge through v_i only if $C_{v'_i}$ is a cycle of length 5 and S -degree 1. Further, according to (R4), the cycle C can receive at most $2/5$ through v_i , because only the two vertices at distance two from v'_i on $C_{v'_i}$ can be involved in an application of (R4). As a result, it is enough to prove that if C receives $2/5$ through v_i , then C does not receive any charge through v_{6-3i} . More explicitly, if C receives $2/5$ through v_1 then C receives nothing through v_3 ; and if C receives $2/5$ through v_2 then C receives nothing through v_0 . In total, the final charge of C would then be at most $3 + (2k + 1)/5$, which is at most $\frac{4}{5} \cdot (2k + 1)$ since $k \geq 2$, with equality if and only if $k = 2$.

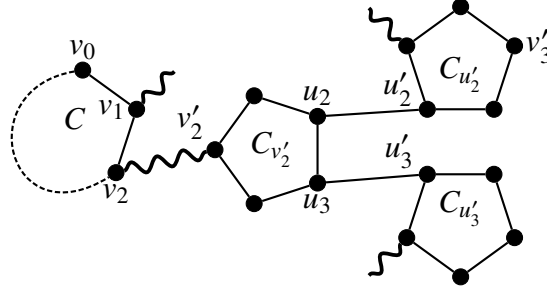


FIGURE 4. If the cycle C receives $2/5$ through v_2 by (R4) and $C_{u'_2} \neq C_{u'_3}$ then $(S \setminus \{v_2v'_2\}) \cup \{u_2u'_2, u_3u'_3\}$ contradicts the maximality of the edge-selection S .

Let us establish the assertion above: assume without loss of generality that C receives $2/5$ through v_2 because of (R4). Writing $C_{v'_2} = v'_2u_1u_2u_3u_4$, we deduce that each of $C_{u'_2}$ and $C_{u'_3}$ is a cycle of length 5 and S -degree 1. In addition, by the definition of (R4) for each $i \in \{2, 3\}$ the vertex of $C_{u'_i}$ incident to an edge in S is a neighbour of u'_i on $C_{u'_i}$. Suppose first that $C_{u'_2} \neq C_{u'_3}$, as illustrated in Figure 4. Then $(S \setminus \{v_2v'_2\}) \cup \{u_2u'_2, u_3u'_3\}$ contradicts the maximality of S . If, on the contrary, $C_{u'_2} = C_{u'_3}$, then without loss of generality we may write $C_{u'_2} = w_0u'_2w_1w_2u'_3$ with $w_0w'_0 \in S$, as illustrated in Figure 5. Now, if C receives charge through v_0 then $C_{v'_0}$ is a cycle of length 5 and S -degree 0 or 1. In the latter case, we notice that the vertex of $C_{v'_0}$ incident to an edge in S is consecutive to v'_0 on $C_{v'_0}$ and, consequently in both cases $C_{v'_0} \neq C_{u'_2}$. From this and the fact that $v'_0 \notin \{u'_2, u'_3\}$, we deduce that in any case $(S \setminus \{v_2v'_2\}) \cup \{v_0v'_0, u_2u'_2\}$ contradicts the choice of S .

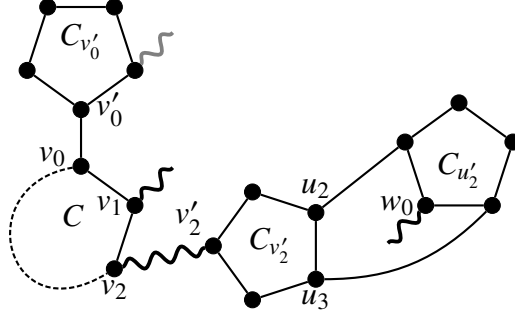


FIGURE 5. If the cycle C receives $2/5$ through v_2 by (R4) and $C_{u'_2} = C_{u'_3}$, then C cannot receive $1/5$ through v_0 by (R2) or (R3), for otherwise $(S \setminus \{v_2v'_2\}) \cup \{v_0v'_0, u_2u'_2\}$ would contradict the maximality of the edge-selection S . (The gray wavy edge belongs to S only if $C_{v'_0}$ has S -degree 1.)

It remains to deal with the case where C belongs to an S -component K such that G_K is an odd cycle. In this case, we prove the sum of the charges of all cycles in K to be less than $\frac{4}{5}$ times the number of vertices belonging to cycles in K . Every cycle in K is incident to exactly two edges in S . It follows that none of these cycles receive charge by (R4). Moreover, exactly one edge in S associated with K is medium. Consequently, if $C \in K$ is not incident to a medium edge that belongs to S , then the final charge of C is at most $3 + \frac{1}{5} \cdot (|V(C)| - 2)$. If C is one of the two cycles incident to the medium edge in S associated with K , then the final charge of C is at most $\frac{7}{2} + \frac{1}{5} \cdot (|V(C)| - 2)$. It now suffices to sum these quantities over the cycles in K : let us write $K = \{C_1, \dots, C_t\}$ where t is an odd number at least 3. Setting $\ell_i := |V(C_i)|$ for each $i \in \{1, \dots, t\}$, the sum of the final charges of the cycles in K is at most

$$3t + 1 + \sum_{i=1}^t \frac{\ell_i - 2}{5} = 1 + \frac{1}{5} \left(13t + \sum_{i=1}^t \ell_i \right).$$

It only remains to show that this last quantity is less than $\frac{4}{5} \cdot \sum_{i=1}^t \ell_i$, that is,

$$(1) \quad 5 + 13t < 3 \cdot \sum_{i=1}^t \ell_i.$$

Since each cycle in F has length at least 5, one has $\sum_{i=1}^t \ell_i \geq 5t$, and hence (1) holds because $t \geq 3$.

Looking at the above inequalities, we observe that as soon as F contains a cycle C of length different from 5, then the number of medium edges is less than $\frac{4}{5} \cdot |V(G)|$. Therefore, the number of medium edges obtained is strictly less than $\frac{4}{5} \cdot |V(G)|$ unless F contains only cycles of length 5. As it turns out, it has been proved [3] that every connected bridgeless cubic graph different from the Petersen graph admits a 2-factor containing a cycle of length different from 5.¹ This concludes the proof of Theorem 1.3.

¹See also [9] for a different and short argument.

3. FURTHER WORK

We point out that, using more involved discharging rules and a lengthier analysis, one can show that there exists a positive ε (which we did not try to optimise) such that for every connected bridgeless cubic graph G different from the Petersen graph, there exists a 4-edge-colouring yielding at most $(4/5 - \varepsilon) |V(G)|$ medium edges.

It seems stimulating to try and obtain upper bounds for the least possible number of medium edges in a k -edge-colouring of a bridgeless cubic graph. As we saw, this number is 0 if $k \geq 7$ and at most $\frac{4}{5} \cdot |V(G)|$ if $k = 4$. Since the Petersen colouring conjecture states that this number should be 0 when $k = 5$, can one obtain at least a sub-linear (in the number of vertices) upper bound in this case? What can be proved when $k = 6$?

REFERENCES

- [1] L. D. Andersen, *The strong chromatic index of a cubic graph is at most 10*, Discrete Math. **108** (1992), no. 1–3, 231–252.
- [2] H. Bílková, *Variants of Petersen coloring for some graph classes*, Charles University in Prague, 2015. Master Thesis.
- [3] M. DeVos, V. V. Mkrtychyan, and S. S. Petrosyan, *5-cycles and the Petersen graph*, [arXiv:0801.3714v1](https://arxiv.org/abs/0801.3714v1), unpublished.
- [4] J.-L. Fouquet and J.-L. Jolivet, *Strong edge-colorings of graphs and applications to multi- k -gons*, Ars Combin. **16** (1983), no. A, 141–150.
- [5] T. Kaiser and J.-S. Sereni, unpublished.
- [6] J. Hägglund and E. Steffen, *Petersen-colorings and some families of snarks*, Ars Math. Contemp. **7** (2014), 161–173.
- [7] P. Horák, Q. He, and W. T. Trotter, *Induced matchings in cubic graphs*, J. Graph Theory **17** (1993), no. 2, 151–160.
- [8] F. Jaeger, *On five-edge-colorings of cubic graphs and nowhere-zero flow problems*, Ars Combin. **20** (1985), no. B, 229–244.
- [9] D. Král'. Personal communication to the second author; see http://lbgi.fr/~sereni/Lectures/AG_Fall09/ag09_3.pdf.
- [10] E. Máčajová and M. Škoviera, *Fano colourings of cubic graphs and the Fulkerson conjecture*, Theoret. Comput. Sci. **349** (2005), no. 1, 112–120.
- [11] J. Nešetřil and R. Šámal, *On tension continuous mappings*, European J. Combin. **29** (2008), no. 4, 1025–1054.
- [12] R. Šámal, *New approach to Petersen coloring*, Electr. Notes Discrete Math. **38** (2011), 755–760.

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