# (dual-frequency)-dependent dynamic functional connectivity analysis of visual working memory capacity

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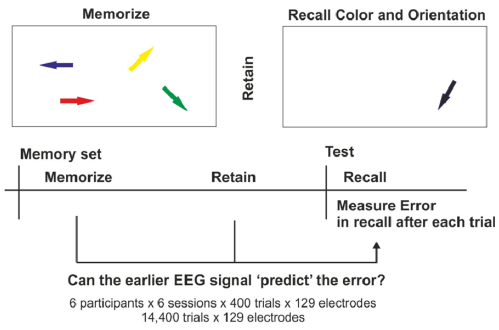
In this companion document we give in the first two sections complementary information on the experiment and on an assumption used to derived our theoretical results. In Section 3 we give all the proofs of the results appearing in the paper. Finally, in Section 4, for pedagogical purposes, we provide the detailed proofs in the case of spatial time-harmonizable process.

# 1 Illustration of the experiment designed in order to assess the VWM capacity




# 2 Remark on the conditions $(L)$ of Section 3.3

In Section 3.3 we assume the condition $(L)$ on the spectrum. It could be replaced by a condition on the covariance:

**(*LC*)** There exists a function $C : [0,1]^6 \times \mathbb{Z}^2 \to \mathbb{R}$ which is Lipschitz-continuous with respect to the space and the time components uniformly on the frequency components,

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that is there exists some constant $L > 0$ such that for each $\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4, \in [0,1]^2$, $\tau_1, \tau_2 \in [0,1]$ and each $k_1, k_2 \in \mathbb{Z}$,

$$\left|C_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(k_1,k_2) - C_{\tau_3,\tau_4\underline{u}_3,\underline{u}_4}(k_1,k_2)\right| \leq L\left(|\tau_1-\tau_3| + |\tau_2-\tau_4| + \|\underline{u}_1-\underline{u}_3\| + \|\underline{u}_2-\underline{u}_4\|\right), \tag{1}$$

and there exists some constant $Q > 0$ such that

$$\left|C^{T,\underline{S},N}_{\underline{s}_1,\underline{s}_2}(t_1+k_1,t_2+k_2) - C_{\ddot{t}_1,\ddot{t}_2,\ddot{\underline{s}}_1,\ddot{\underline{s}}_2}(k_1,k_2)\right| \leq \frac{Q}{S_1S_2T}, \tag{2}$$

for any $k_1, k_2 = 0, \pm 1, \ldots, \pm N$, where $\underline{s}_i = (s_{i,1}, s_{i,2})$, $\ddot{\underline{s}}_i = (s_{i,1}/S_1, s_{i,2}/S_2), \ddot{t}_i = t_i/T, i = 1,2$.

$N$ fixed or going to $\infty$ with some rate.

# 3 Proofs for the location dependent time-varying case

We start with properties of the kernel that are used later in the document to prove properties of our estimators.
Let us recall that the kernel $w(\cdot)$ is bounded and piecewise Lipschitz and $supp(w(\cdot) \subset [-1,1]$. Thus, there are $k = k_w \in \mathbb{N}$, $0 = u_1 < \cdots < u_k$ such that $w(\cdot)$ is Lipschitz on each interval $(u_j, u_{j+1})$. This includes the rectangular kernel, as well as the triangular kernel.

## 3.1

**Lemma 1** (1)**.** *For* $S = 1, 2, \ldots$

$$\frac{1}{Sh}\sum_{s=1}^{S} w\left(\frac{u-s/S}{h}\right) = 1 + \mathcal{O}\left(\frac{1}{Sh^2}\right)$$

*uniformly with respect to* $u$ *such that* $h \leq u \leq 1-h$.

*Proof.* Notice that

$$\sum_{s=1}^{S}\frac{1}{Sh}w\left(\frac{u-s/S}{h}\right) = \frac{1}{h}\int_0^1 w\left(\frac{u-v}{h}\right)dv + \frac{1}{h}\sum_{s=1}^{S}\int_{\frac{s-1}{S}}^{\frac{s}{S}}\left(w\left(\frac{u-s/S}{h}\right) - w\left(\frac{u-v}{h}\right)\right)dv.$$

Note that for $h \leq u \leq 1-h$

$$\frac{1}{h}\int_0^1 w\left(\frac{u-v}{h}\right)\,dv = 1.$$

Next for each $v$, $(s-1)/S \leq v \leq s/S$

$$\frac{1}{Sh}\left|w\left(\frac{u-s/S}{h}\right) - w\left(\frac{u-v}{h}\right)\right| \leq \frac{1}{S^2h^2}$$

except for a finite number of $s$ that is bounded by $k_w$ (which is defined above Lemma 1) and which does not depend on $u$ and on $S$. In any case

$$\frac{1}{Sh}\left|w\left(\frac{u-\frac{s}{S}}{h}\right) - w\left(\frac{u-v}{h}\right)\right| \leq \frac{2\sup_x w(x)}{Sh} = o\left(\frac{1}{Sh^2}\right)$$

(because $h \to 0$). Hence

$$\frac{1}{h}\sum_{s=1}^{S}\int_{\frac{s-1}{S}}^{\frac{s}{S}}\left(w\left(\frac{u-s/S}{h}\right) - w\left(\frac{u-v}{h}\right)\right)dv \leq \frac{1}{Sh^2} + k_w \times \frac{2\sup_x w(x)}{Sh} = \mathcal{O}\left(\frac{1}{Sh^2}\right),$$

which completes the proof. □

From Lemma 1, we deduce

$$\sum_{\underline{s}} w_{\underline{u}}(\underline{s}) = \left(1+\mathcal{O}\left(\frac{1}{S_1h^2}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{S_2h^2}\right)\right) = 1 + \frac{1}{h^2}\mathcal{O}\left(\frac{1}{S_2}+\frac{1}{S_2}\right)$$

uniformly with respect to $\underline{h} \preceq \underline{u} \preceq \underline{1-h}$, $\qquad S_1h^2, S_2h^2 \to \infty$.

To get the uniformity with respect to $\underline{u}$ varying in $[0,1]\times[0,1]$, it suffices choose the kernel function $w(\cdot)$ with a compact support which is a subset of the open interval $(0,1)$.

## 3.2 Proof of Theorem 3.1

From Lemma 2 and Lemma 3 hereafter we readily deduce Theorem 3.1.

**Lemma 2.** *(Limit of the expectation)*
*Under conditions (F) and (LC), the expectation* $\mathrm{E}\left(\widetilde{f}^{T,\underline{S},N,R}_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)$ *does not depend on* $R$ *and converges to* $f^{(\omega_1,\omega_2)}_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}$*, for* $\omega_i = \frac{\pi k_i}{M}$*,* $k_1 = -M,\ldots,M-1$*,* $i = 1,2$*, provided that* $N = nM$*, as well as* $(S_1+S_2)h^2, T\hbar^2 \to \infty$*,* $n^2(S_1S_2T)^{-1} \to 0$ *and* $n^2(h+\hbar), n^3T^{-1} \to 0$ *as* $T, S_1, S_2 \to \infty$*,* $h, \hbar \to 0$ *whatever is the behaviour of* $n > 1$*.*

*Proof.*
Let $\tau_1,\tau_2,\underline{u}_1,\underline{u}_2,\omega_1,\omega_2$ be fixed. From the definitions of $\widetilde{f}^R_{1,2}(\omega_1,\omega_2)$, $\widetilde{d}^r_1(\omega_1)$ and $\widetilde{d}^r_2(\omega_2)$, we have

$$\begin{aligned}4\pi^2\mathrm{E}\left(\widetilde{f}^R_{1,2}(\omega_1,\omega_2)\right) &= \mathrm{E}\left(\widetilde{d}^r_1(\omega_1)\,\overline{\widetilde{d}^r_2(\omega_2)}\right)\\ &= \sum_{t_1}\sum_{t_2}\sum_{\underline{s}_1}\sum_{\underline{s}_2} W_{\tau_1}(t_1)W_{\tau_2}(t_2)w_{\underline{u}_1}(\underline{s}_1)w_{\underline{u}_2}(\underline{s}_2)\\ &\quad\times\sum_{k_1=t_1-N}^{t_1+N-1}\sum_{k_2=t_2-N}^{t_2+N-1}\mathrm{E}\left(X^{T,\underline{S},r}_{k_1}(\underline{s}_1)\,X^{T,\underline{S},r}_{k_2}(\underline{s}_2)\right)e^{-i(\omega_1k_1-\omega_2k_2)}.\end{aligned}$$

Note that

$$C^{T,\underline{S}}_{\underline{s}_1,\underline{s}_2}(k_1,k_2) = \mathrm{E}\left(X^{T,\underline{S},r}_{k_1}(\underline{s}_1)\,X^{T,\underline{S},r}_{k_2}(\underline{s}_2)\right)$$

does not depend on $r$. Recall that

$$C^{T,\underline{S}}_{\underline{s}_1,\underline{s}_2}(t_1,t_2) = \frac{\pi^2}{N^2}\sum_{j_1=-N}^{N-1}\sum_{j_2=-N}^{N-1} f^{T,\underline{S},N}_{t_1,t_2,\underline{s}_1,\underline{s}_2}\left(\omega^{(N)}_{j_1},\omega^{(N)}_{j_2}\right)e^{i\left(\omega^{(N)}_{j_1}t_1-\omega^{(N)}_{j_2}t_2\right)}. \tag{3}$$

This entails

$$\begin{aligned}\mathrm{E}\left(\widetilde{f}^R_{1,2}(\omega_1,\omega_2)\right) &= \frac{1}{4N^2}\sum_{t_1}\sum_{t_2}\sum_{\underline{s}_1}\sum_{\underline{s}_2} W_{\tau_1}(t_1)W_{\tau_2}(t_2)w_{\underline{u}_1}(\underline{s}_1)w_{\underline{u}_2}(\underline{s}_2)\\ &\quad\times\sum_{k_1=t_1-N}^{t_1+N-1}\sum_{k_2=t_2-N}^{t_2+N-1}\sum_{j_1=-N}^{N-1}\sum_{j_2=-N}^{N-1} f^{T,\underline{S},N}_{k_1,k_2,\underline{s}_1,\underline{s}_2}\left(\omega^{(N)}_{j_1},\omega^{(N)}_{j_2}\right)e^{i\left(\omega^{(N)}_{j_1}k_1-\omega^{(N)}_{j_2}k_2\right)}e^{-i(\omega_1k_1-\omega_2k_2)}\end{aligned} \tag{4}$$

Since

$$\left|f^{(\omega_1,\omega_2)}_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2} - f^{(\omega_1,\omega_2)}_{\tau_3,\tau_4\underline{u}_3,\underline{u}_4}\right| \le L\left(|\tau_1-\tau_3|+|\tau_2-\tau_4|+\|\underline{u}_1-\underline{u}_3\|+\|\underline{u}_2-\underline{u}_4\|\right), \tag{5}$$

and there exists some constant $Q > 0$ such that

$$\left| f^{T,\underline{S},N}_{t_1,t_2,\underline{s}_1,\underline{s}_2}(\omega_1,\omega_2) - f_{\ddot{t}_1,\ddot{t}_2,\ddot{\underline{s}}_1,\ddot{\underline{s}}_2}(\omega_1,\omega_2) \right| \leq \frac{Q}{S_1 S_2 T}, \tag{6}$$

we have

$$\begin{aligned} f^{T,\underline{S},N}_{k_1,k_2,\underline{s}_1,\underline{s}_2}\left(\omega^{(N)}_{j_1},\omega^{(N)}_{j_2}\right) &= f_{\ddot{k}_1,\ddot{k}_2,\ddot{\underline{s}}_1,\ddot{\underline{s}}_2}(\omega^{(N)}_{j_1},\omega^{(N)}_{j_2}) + \mathcal{O}\left(\frac{1}{S_1 S_2 T}\right) \\ &= f_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega^{(N)}_{j_1},\omega^{(N)}_{j_2}) + \mathcal{O}\left(\frac{1}{S_1 S_2 T}\right) + \mathcal{O}(h) + \mathcal{O}(\hbar) + \mathcal{O}\left(\frac{N}{T}\right) \end{aligned}$$

for $|\tau_i - \ddot{t}_i| \leq \hbar$, $|t_i - k_i| \leq N$ and $\|\underline{u}_i - \ddot{\underline{s}}_i\| \leq h$, $i = 1, 2$. Hence

$$\begin{aligned} \mathrm{E}\left(\widetilde{f}^{R}_{1,2}(\omega_1,\omega_2)\right) &= \sum_{t_1}\sum_{t_2}\sum_{\underline{s}_1}\sum_{\underline{s}_2} W_{\tau_1}(t_1) W_{\tau_2}(t_2) w_{\underline{u}_1}(\underline{s}_1) w_{\underline{u}_2}(\underline{s}_2) \\ &\quad \times \frac{1}{4N^2} \sum_{k_1=t_1-N}^{t_1+N-1} \sum_{k_2=t_2-N}^{t_2+N-1} \sum_{j_1=-N}^{N-1} \sum_{j_2=-N}^{N-1} \left( f_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega^{(N)}_{j_1},\omega^{(N)}_{j_2}) + \mathcal{O}\left(\frac{1}{S_1 S_2 T}\right) \right. \\ &\quad \left. + \mathcal{O}(h) + \mathcal{O}(\hbar) + \mathcal{O}\left(\frac{N}{T}\right) \right) e^{i\left(\omega^{(N)}_{j_1} - \omega_1\right)k_1} e^{-i\left(\omega^{(N)}_{j_2} - \omega_2\right)k_2} \\ &= \left(1 + \mathcal{O}\left(\frac{1}{S_1 h^2}\right)\right)\left(1 + \mathcal{O}\left(\frac{1}{S_2 h^2}\right)\right)\left(1 + \mathcal{O}\left(\frac{1}{T\hbar^2}\right)\right)^2 \\ &\quad \times \left( f_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) + \frac{16N^4}{4N^2}\left( \mathcal{O}\left(\frac{1}{S_1 S_2 T}\right) + \mathcal{O}(h) + \mathcal{O}(\hbar) + \mathcal{O}\left(\frac{N}{T}\right)\right)\right) \end{aligned}$$

Above we used the fact that for $k \neq 0$

$$\sum_{j=-N}^{N-1} e^{\frac{i\pi jk}{N}} = 0. \tag{7}$$

□

Then we can refine Lemma 2 stating the rate of convergence.

**Corollary 1.** *(Rate of convergence the expectation)*
*Under conditions (F) and (L),*

$$\lim_{R\to\infty} \sqrt{R}\left(\mathrm{E}\left(\widetilde{f}^{T,\underline{S},N,R}_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) - f_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) = 0$$

*for* $\omega_i = \frac{\pi k_i}{M}$, $k_i = -M, \ldots, M-1$, $i = 1, 2$, *provided that* $N = nM$, *as well as* $R(S_1^{-2} + S_2^{-2})h^{-4}$, $RT^{-2}\hbar^{-4}$, $Rn^4(S_1S_2T)^{-2}, Rn^4(h^2+\hbar^2), Rn^6T^{-2} \to 0$, *as* $T, S_1, S_2 \to \infty$, $h, \hbar \to 0$, *independently of the behavior of* $n > 1$.

**Lemma 3.** *Assume that the mixing and boundedness assumptions (M) and (B) are satisfied. Then,*

$$\mathrm{Var}\left(\widetilde{f}^R_{1,2}(\omega_1, \omega_2)\right) \leq \frac{c\, N^4}{R}.$$

*for all* $\omega_1, \omega_2 \in [-\pi, \pi]$. *Here* $c$ *does not depend on* $R, T, \underline{S}, N, \tau_1, \tau_2, \underline{u}_1\underline{u}_2, \omega_1, \omega_2$.

*Proof.* First

$$\mathrm{Var}\left(\widetilde{f}^R_{1,2}(\omega_1, \omega_2)\right) = \frac{1}{R^2}\sum_{r_1=1}^{R}\sum_{r_2=1}^{R}\mathrm{Cov}\left(\widetilde{I}^{r_1}_{1,2}(\omega_1, \omega_2), \widetilde{I}^{r_2}_{1,2}(\omega_1, \omega_2)\right).$$

From (M) and (B) we have

$$\begin{aligned}
16\pi^4\left|\mathrm{Cov}\left(\widetilde{I}^{r_1}_{1,2}(\omega_1, \omega_2), \widetilde{I}^{r_2}_{1,2}(\omega_1, \omega_2)\right)\right| &= \left|\mathrm{Cov}\left(\widetilde{d}^{r_1}_1(\omega_1)\,\overline{\widetilde{d}^{r_1}_2(\omega_2)}, \widetilde{d}^{r_2}_1(\omega_1)\,\overline{\widetilde{d}^{r_2}_2(\omega_2)}\right)\right| \\
&\leq \sum_{t_1}\sum_{t_2}\sum_{t_3}\sum_{t_4}\sum_{\underline{s}_1}\sum_{\underline{s}_2}\sum_{\underline{s}_3}\sum_{\underline{s}_4} W_{\tau_1}(t_1)W_{\tau_2}(t_2)W_{\tau_3}(t_3)W_{\tau_4}(t_4) \\
&\quad \times w_{\underline{u}_1}(\underline{s}_1)w_{\underline{u}_2}(\underline{s}_2)w_{\underline{u}_3}(\underline{s}_3)w_{\underline{u}_4}(\underline{s}_4) \\
&\quad \times \sum_{k_1=-N}^{N-1}\sum_{k_2=-N}^{N-1}\sum_{k_3=-N}^{N-1}\sum_{k_4=-N}^{N-1}\left|\mathrm{Cov}\left(X^{T,\underline{S},r_1}_{k_1+t_1}X^{T,\underline{S},r_1}_{k_2+t_2}, X^{T,\underline{S},r_2}_{k_3+t_3}X^{T,\underline{S},r_2}_{k_4+t_4}\right)\right| \\
&\leq c\sum_{t_1}\sum_{t_2}\sum_{t_3}\sum_{t_4}\sum_{\underline{s}_1}\sum_{\underline{s}_2}\sum_{\underline{s}_3}\sum_{\underline{s}_4} W_{\tau_1}(t_1)W_{\tau_2}(t_2)W_{\tau_3}(t_3)W_{\tau_4}(t_4) \\
&\quad \times w_{\underline{u}_1}(\underline{s}_1)w_{\underline{u}_2}(\underline{s}_2)w_{\underline{u}_3}(\underline{s}_3)w_{\underline{u}_4}(\underline{s}_4)\, N^4\alpha_X^{\frac{\delta}{4+\delta}}(|r_1 - r_2|).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{Var}\left(\widetilde{f}^R_{1,2}(\omega_1, \omega_2)\right) &\leq \frac{1}{R^2}\sum_{r_1=1}^{R}\sum_{r_2=1}^{R} c\, N^4\alpha_X^{\frac{\delta}{4+\delta}}(|r_1 - r_2|) \\
&\leq \frac{c\, N^4}{R}\sum_{k=-R+1}^{R-1}\left(1 - \frac{|k|}{R}\right)\alpha_X^{\frac{\delta}{4+\delta}}(|k|).
\end{aligned}$$

The lemma is proved. □

## 3.3 Proof of Lemma 3.1

*Proof.*

$$R^2\mathrm{Cov}\left(\widetilde{f}^R_{1,2}\left(\omega_1,\omega_2\right),\widetilde{f}^R_{3,4}\left(\omega_3,\omega_4\right)\right)=\sum_{r_1=1}^{R}\sum_{r_2=1}^{R}\mathrm{Cov}\left(\widetilde{I}^{r_1}_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2),\widetilde{I}^{r_2}_{\tau_3,\tau_4,\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4)\right)$$

$$=\frac{1}{16\pi^4}\sum_{r_1=1}^{R}\sum_{r_2=1}^{R}\mathrm{Cov}\left(\widetilde{d}^{r_1}_1\left(\omega_1\right)\overline{\widetilde{d}^{r_1}_2\left(\omega_2\right)},\widetilde{d}^{r_2}_3\left(\omega_3\right)\overline{\widetilde{d}^{r_2}_4\left(\omega_4\right)}\right).$$

Moreover,

$$\mathrm{Cov}\left(\widetilde{d}^{r_1}_1\left(\omega_1\right)\overline{\widetilde{d}^{r_1}_2\left(\omega_2\right)},\widetilde{d}^{r_2}_3\left(\omega_3\right)\overline{\widetilde{d}^{r_2}_4\left(\omega_4\right)}\right)=\mathrm{E}\left(\widetilde{d}^{r_1}_1\left(\omega_1\right)\overline{\widetilde{d}^{r_1}_2\left(\omega_2\right)\widetilde{d}^{r_2}_3\left(\omega_3\right)}\widetilde{d}^{r_2}_4\left(\omega_4\right)\right)$$
$$-\mathrm{E}\left(\widetilde{d}^{r_1}_1\left(\omega_1\right)\overline{\widetilde{d}^{r_1}_2\left(\omega_2\right)}\right)\mathrm{E}\left(\overline{\widetilde{d}^{r_2}_3\left(\omega_3\right)}\widetilde{d}^{r_2}_4\left(\omega_4\right)\right).$$

Using the fact that observations are Gaussian we obtain

$$\mathrm{Cov}\left(\widetilde{d}^{r_1}_1\left(\omega_1\right)\overline{\widetilde{d}^{r_1}_2\left(\omega_2\right)},\widetilde{d}^{r_2}_3\left(\omega_3\right)\overline{\widetilde{d}^{r_2}_4\left(\omega_4\right)}\right)=\mathrm{E}\left(\widetilde{d}^{r_1}_1\left(\omega_1\right)\overline{\widetilde{d}^{r_2}_3\left(\omega_3\right)}\right)\mathrm{E}\left(\overline{\widetilde{d}^{r_1}_2\left(\omega_2\right)}\widetilde{d}^{r_2}_4\left(\omega_4\right)\right)$$
$$+\mathrm{E}\left(\widetilde{d}^{r_1}_1\left(\omega_1\right)\widetilde{d}^{r_2}_4\left(\omega_4\right)\right)\mathrm{E}\left(\overline{\widetilde{d}^{r_1}_2\left(\omega_2\right)\widetilde{d}^{r_2}_3\left(\omega_3\right)}\right).$$

Additionally, because of the finite number $M$ of Fourier frequencies and the fact that we choose $N=nM$.

$$\mathrm{E}\left(\widetilde{d}^{r_1}_1\left(\omega_1\right)\overline{\widetilde{d}^{r_2}_3\left(\omega_3\right)}\right)$$
$$=\sum_{t_1}\sum_{t_3}\sum_{\underline{s}_1}\sum_{\underline{s}_3}W_{\tau_1}(t_1)W_{\tau_3}(t_3)w_{\underline{u}_1}(\underline{s}_1)w_{\underline{u}_3}(\underline{s}_3)$$
$$\times\sum_{k_1=t_1-N}^{t_1+N-1}\sum_{k_3=t_3-N}^{t_3+N-1}C^{T,\underline{S},r_1,r_2}_{\underline{s}_1,\underline{s}_3}(k_1,k_3)\,e^{-i(\omega_1k_1-\omega_3k_3)}$$
$$=\sum_{t_1}\sum_{t_3}\sum_{\underline{s}_1}\sum_{\underline{s}_3}W_{\tau_1}(t_1)W_{\tau_3}(t_3)w_{\underline{u}_1}(\underline{s}_1)w_{\underline{u}_3}(\underline{s}_3)$$
$$\times\frac{1}{4N^2}\sum_{k_1=t_1-N}^{t_1+N-1}\sum_{k_3=t_3-N}^{t_3+N-1}\sum_{j_1=-N}^{N-1}\sum_{j_3=-N}^{N-1}f^{T,\underline{S},r_1,r_2}_{k_1,k_3,\underline{s}_1,\underline{s}_3}\left(\omega^{(N)}_{j_1},\omega^{(N)}_{j_3}\right)\,e^{i\left(\omega^{(N)}_{j_1}-\omega_1\right)k_1}e^{-i\left(\omega^{(N)}_{j_3}-\omega_3\right)k_3}.$$

Under condition (LR) we have

$$f^{T,\underline{S},r_1,r_2}_{k_1,k_2,\underline{s}_1,\underline{s}_2}(\omega_{j_1},\omega_{j_2}) = f^{r_1,r_2}_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega_{j_1},\omega_{j_2}) + \mathcal{O}\left(\frac{1}{S_1S_2T}\right) + \mathcal{O}(h) + \mathcal{O}(\hbar) + \mathcal{O}\left(\frac{N}{T}\right)$$

We get that

$$\begin{aligned}
&\mathrm{E}\left(\widetilde{d}^{r_1}_1(\omega_1)\,\overline{\widetilde{d}^{r_2}_3(\omega_3)}\right) \\
&\quad= \sum_{t_1}\sum_{t_3}\sum_{\underline{s}_1}\sum_{\underline{s_3}} W_{\tau_1}(t_1)W_{\tau_3}(t_3)w_{\underline{u}_1}(\underline{s}_1)w_{\underline{u}_3}(\underline{s}_3) \\
&\qquad\times \frac{1}{4N^2}\sum_{k_1=t_1-N}^{t_1+N-1}\sum_{k_3=t_3-N}^{t_3+N-1}\sum_{j_1=-N}^{N-1}\sum_{j_3=-N}^{N-1}\left(f^{r_1,r_2}_{\tau_1,\tau_3,\underline{u}_1,\underline{u}_3}(\omega^{(N)}_{j_1},\omega^{(N)}_{j_3}) + \mathcal{O}\left(\frac{1}{S_1S_2T}\right)\right. \\
&\qquad\qquad\left. + \mathcal{O}(h) + \mathcal{O}(\hbar) + \mathcal{O}\left(\frac{N}{T}\right)\right) e^{i\left(\omega^{(N)}_{j_1}-\omega_1\right)k_1} e^{-i\left(\omega^{(N)}_{j_3}-\omega_3\right)k_3} \\
&\quad= \sum_{t_1}\sum_{t_3}\sum_{\underline{s}_1}\sum_{\underline{s_3}} W_{\tau_1}(t_1)W_{\tau_3}(t_3)w_{\underline{u}_1}(\underline{s}_1)w_{\underline{u}_3}(\underline{s}_3) \\
&\qquad\times \left(f^{r_1,r_2}_{\tau_1,\tau_3,\underline{u}_1,\underline{u}_3}(\omega_1,\omega_3) + N^2\left(\mathcal{O}\left(\frac{1}{S_1S_2T}\right) + \mathcal{O}(h) + \mathcal{O}(\hbar) + \mathcal{O}\left(\frac{N}{T}\right)\right)\right) \\
&\quad= \left(1+\mathcal{O}\left(\frac{1}{S_1h^2}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{S_2h^2}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{T\hbar^2}\right)\right)^2 \\
&\qquad\times \left(f^{r_1,r_2}_{\tau_1,\tau_3,\underline{u}_1,\underline{u}_3}(\omega_1,\omega_3) + N^2\left(\mathcal{O}\left(\frac{1}{S_1S_2T}\right) + \mathcal{O}(h) + \mathcal{O}(\hbar) + \mathcal{O}\left(\frac{N}{T}\right)\right)\right),
\end{aligned}$$

where $\omega_1$ and $\omega_3$ are Fourier frequencies.

Under (SR) we get

$$\begin{aligned}
&\sum_{r_1=1}^{R}\sum_{r_2=1}^{R}\mathrm{E}\left(\widetilde{d}_1^{r_1}(\omega_1)\,\overline{\widetilde{d}_3^{r_2}(\omega_3)}\right)\times\mathrm{E}\left(\overline{\widetilde{d}_2^{r_1}(\omega_2)}\widetilde{d}_4^{r_2}(\omega_4)\right)\\
&\quad=\left(1+\mathcal{O}\Big(\frac{1}{S_1h^2}\Big)\right)^2\left(1+\mathcal{O}\Big(\frac{1}{S_2h^2}\Big)\right)^2\left(1+\mathcal{O}\Big(\frac{1}{T\hbar^2}\Big)\right)^4\\
&\qquad\times\sum_{k=-R+1}^{R-1}(R-|k|)\\
&\qquad\times\left(f^k_{\tau_1,\tau_3,\underline{u}_1,\underline{u}_3}(\omega_1,\omega_3)+N^2\left(\mathcal{O}\left(\frac{1}{S_1S_2T}\right)+\mathcal{O}(h)+\mathcal{O}(\hbar)+\mathcal{O}\Big(\frac{N}{T}\Big)\right)\right)\\
&\qquad\times\left(\overline{f^k_{\tau_1,\tau_3,\underline{u}_2,\underline{u}_4}(\omega_2,\omega_4)}+N^2\left(\mathcal{O}\left(\frac{1}{S_1S_2T}\right)+\mathcal{O}(h)+\mathcal{O}(\hbar)+\mathcal{O}\Big(\frac{N}{T}\Big)\right)\right).
\end{aligned}$$

This ends the proof. □

## 3.4 Proof of Theorem 3.2

Theorem 3.2 is a direct consequence of application of the Cramér-Wold device to the univariate asymptotic normality result that we deduce from the Corollary 1 and from the Proposition 1 proved below.

**Proposition 1.**
*Assume that the time-spatial random arrays* $\left\{\boldsymbol{X}_t^{\underline{S},r}:t=0,\ldots,T,r=1,\ldots,R\right\}$, $R\geq 0$, *are Gaussian (condition (GR)), stationary with respect to r (condition (SR)) and satisfy condition (M) as well as*

$$\sup_{t,\underline{s},\underline{S}}\mathrm{E}\big(\big|X_t^{\underline{S}}(\underline{s})\big|^{4+2\delta}\big)<\infty\quad\textit{and}\quad\sum_k\alpha_X(k)^{\delta/(2+\delta)}<\infty$$

*for some* $\delta>0$. *Then under the assumptions of Lemma 3.1, we have*

$$\lim_{R\to\infty}\mathcal{L}\left(\sqrt{R}\left(\widetilde{f}^R_{1,2}(\omega_1,\omega_2)-\mathrm{E}\left(\widetilde{f}^R_{1,2}(\omega_1,\omega_2)\right)\right)\right)=\mathcal{N}_2\left(0,\Sigma_2\right),$$

*for any Fourier frequencies* $\omega_1,\omega_2$. *Here* $\Sigma_2=\boldsymbol{\Sigma}^{(\omega_1,\omega_2)}_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}$ *is the limit variance* $(2\times2)$*-matrix.*

*Proof.* To prove the convergence in distribution of the two-dimensional random vector

$$\sqrt{R}\left(\widetilde{f}_{1,2}^{R}\left(\omega_1,\omega_2\right)-\mathrm{E}\left(\widetilde{f}_{1,2}^{R}\left(\omega_1,\omega_2\right)\right)\right)$$

we use Cramér-Rao device. We show the asymptotic normality of $\sqrt{R}\left(\eta_R-\mathrm{E}\left(\eta_R\right)\right)$, where

$$\zeta_R=a_1\Re\widetilde{f}_{1,2}^{R}\left(\omega_1,\omega_2\right)+a_2\Im\widetilde{f}_{1,2}^{R}\left(\omega_1,\omega_2\right)$$

for all real numbers $a_1$ and $a_2$. For the sake of simplicity, we take $a_1=1$ and $a_2=0$. Thus,

$$\zeta_R=\Re\widetilde{f}_{1,2}^{R}\left(\omega_1,\omega_2\right)=\frac{1}{R}\sum_{r=1}^{R}W_{R,r},$$

where the triangular random array $W_{R,r}:=(2\pi)^{-2}\Re\left(\widetilde{d}_1^r\left(\omega_1\right)\overline{\widetilde{d}_2^r\left(\omega_2\right)}\right)$, for $r=1,\ldots,R$, $R=1,2,\ldots$. Recall that $T$, $\underline{S}$, $\hbar$ and $h$ depend on $R$, and $\tau_1,\tau_2$, $\underline{u}_1$, $\underline{u}_2$, $\omega_1$, and $\omega_2$ are fixed. As in Proposition 2 to get the asymptotic normality of $\sqrt{R}\left(\zeta_R-E\left(\zeta_R\right)\right)$, we use Theorem 3.3.1 from Guyon (1995) and hence we verify if the following conditions hold:

(i) $\sup_R\max_{r=1,\ldots,R}\mathrm{E}\left(\left|W_{R,r}\right|^{2+\delta}\right)<\infty$;

(ii) $\limsup_{R\to\infty}\sum_{k=-R+1}^{R-1}\alpha_{W_R}(k)^{\delta/(2+\delta)}<\infty$;

(iii) $\frac{1}{R}\mathrm{Var}\left(\sum_{r=1}^{R}W_{R,r}\right)\longrightarrow\sigma^2$ as $R\to\infty$, where $\sigma^2$ is the (1,1)-component of the variance $2\times 2$-matrix $\Sigma_2$.

Since

$$\widetilde{d}_i^T\left(\omega\right)\;=\;\sum_{t_i}\sum_{\underline{s}_i}W_{\tau_i}(t_i)w_{\underline{u}_i}(\underline{s}_i)\frac{1}{\sqrt{N}}\sum_{k_i=t_i-N}^{t_i+N-1}X_{k_i}^{T,\underline{S},r}\left(\underline{s}_i\right)e^{-i\omega k_i},$$

and $N$ is fixed, the condition $(i)$ is a direct consequence of Cauchy-Schwarz inequality, the triangle inequality for metric and the hypothesis $\sup_{t,\underline{s},R}\mathrm{E}\left(\left|X_t^{\underline{S},r}(\underline{s})\right|^{4+2\delta}\right)<\infty$. Furthermore, for each $R$ we have that $\alpha_{W_R}(k)\leq\alpha_X(k)$ and hence the condition (ii) is a consequence of the hypothesis $\sum_k\alpha_X(k)^{\delta/(2+\delta)}<\infty$. Finally, condition $(iii)$ is obtained from Lemma 3.1. □

**Theorem 1** (Asymptotic normality)**.**
*Assume that the time-spatial random arrays* $\left\{ \boldsymbol{X}_t^{\underline{S},r} : t = 0, \ldots, T, r = 1, \ldots, R \right\}$, $R \geq 1$, *are Gaussian (condition (GR)), stationary with respect to r (condition (SR)). Assume also Assume that conditions (LR), (F) and (M) are satisfied as well as the conditions*

$$\sup_{t,\underline{s},\underline{S}} \mathrm{E}\left( \left| X_t^{\underline{S}}(\underline{s}) \right|^{4+2\delta} \right) < \infty \quad \textit{and} \quad \sum_k \alpha_X(k)^{\delta/(2+\delta)} < \infty$$

*for some* $\delta > 0$, *and*

$$\sum_{k \in \mathbb{Z}} \left| f^k_{\tau_j, \tau_{j'}, \underline{u}_j, \underline{u}_{j'}}(\omega_j, \omega_{j'}) \right| < \infty.$$

*Then*

$$\lim_{R \to \infty} \mathcal{L}\left( \sqrt{R} \left( \tilde{f}^R_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega_1, \omega_2) - f_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega_1, \omega_2) \right) \right) = \mathcal{N}_2\left( 0, \boldsymbol{\Sigma}^{(\omega_1,\omega_2)}_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2} \right),$$

*for any Fourier frequencies* $\omega_i = \frac{2\pi k_i}{M}$, $k_i = -M, \ldots, M-1$, *provided that* $R(S_1^{-2} + S_2^{-2})h^{-4}, RT^{-2}\hbar^{-4}, Rn^2(S_1S_2T)^{-1}, Rn^2(h+\hbar), Rn^3T^{-1} \to 0$ *as* $R, T, S_1, S_2 \to \infty$ *and* $h, \hbar \to 0$, *whatever is the behaviour of* $N = nM$, $n > 1$.

*Proof.* Theorem is a direct consequence of Corollary 1 and Proposition 1. □

## 3.5 Proof of Corollary 3.1

*Proof.* We apply the delta method (see e.g. van der Vaart (1998)) with the function $\phi : ((0,\infty] \times \mathbb{R})^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\phi(x_1, y_1, x_2, y_3, x_3, y_3) := \frac{x_3^2 + y_3^2}{x_1 x_2}$. Indeed,

$$\tilde{\rho}^{T,\underline{S},N,R}_{\tau_1,\tau_2,\underline{u}_1,\underline{u}_2}(\omega_1, \omega_2) = \phi\left( \boldsymbol{f}^{T,\underline{S},N,R} \right)$$

where

$$\boldsymbol{f}^{T,\underline{S},N,R} = \left( \left( \tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_1}(\omega_1, \omega_1) \right)', \left( \tilde{f}^{T,\underline{S},R}_{\underline{u}_2,\underline{u}_2}(\omega_2, \omega_2) \right)', \left( \tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1, \omega_2) \right)' \right)'.$$

Notice that as complex numbers, $\tilde{f}^{T,\underline{S},R}_{\underline{u}_j,\underline{u}_j}(\omega_j, \omega_j) = \Re \tilde{f}^{T,\underline{S},R}_{\underline{u}_j,\underline{u}_j}(\omega_j, \omega_j)$ are real and non negative.
Since the gradient of the function $\phi$ is equal to

$$\nabla\phi(x_1, y_1, x_2, y_2, x_3, y_3) = \left( \frac{-2(x_3^2 + y_3^2)}{x_1^2 x_2}, 0, \frac{-2(x_3^2 + y_3^2)}{x_1 x_2^2}, 0, \frac{2x_3}{x_1 x_2}, \frac{2y_3}{x_1 x_2} \right)',$$

the convergence result given by the Corollary 3.1 is a direct consequence of Theorem 3.1 in van der Vaart (1998) and of Theorem 1. □

## 3.6 Proof of Theorem 3.2

Hereafter we state the bootstrap consistency and its multivariate version which are direct application of Theorem 3.2 from Lahiri (2003).

We present first the notation for the multidimensional counterparts of $\tilde{\boldsymbol{f}}^{T,\underline{S},R}$ and its expected value. By $P^*$ and $E^*$ we denote the conditional probability and conditional expectation given the sample. Let

$$\tilde{\boldsymbol{f}}^{*,T,\underline{S},N,R} = \left(\tilde{f}^{*,T,\underline{S},N,R}_{\tau_{11},\tau_{21},\underline{u}_{11},\underline{u}_{21}}(\omega_{11},\omega_{21}),\ldots,\tilde{f}^{*,T,\underline{S},N,R}_{\tau_{1r},\tau_{2r},\underline{u}_{1r},\underline{u}_{2r}}(\omega_{1r},\omega_{2r})\right)'$$

and

$$\mathrm{E}^*\big(\tilde{\boldsymbol{f}}^{*,T,\underline{S},N,R}\big) = \left(\mathrm{E}^*\big(\tilde{f}^{*,T,\underline{S},N,R}_{\tau_{11},\tau_{21},\underline{u}_{11},\underline{u}_{21}}(\omega_{11},\omega_{21})\big),\ldots,\mathrm{E}^*\big(\tilde{f}^{*,T,\underline{S},N,R}_{\tau_{1r},\tau_{2r},\underline{u}_{1r},\underline{u}_{2r}}(\omega_{1r},\omega_{2r})\big)\right)'.$$

Note that the bootstrap versions $\tilde{f}^{*,T,\underline{S},N,R}_{\tau_{11},\tau_{21},\underline{u}_{11},\underline{u}_{21}}(\omega_{11},\omega_{21}),\ldots,\tilde{f}^{*,T,\underline{S},N,R}_{\tau_{1r},\tau_{2r},\underline{u}_{1r},\underline{u}_{2r}}(\omega_{1r},\omega_{2r})$ are constructed using the same bootstrap blocks (see step 2 of the CBB algorithm).

**Theorem 2.** *Under assumptions of Theorem 3.2 and assuming that $b^{-1}+R^{-1}b=o(1)$ the CBB is consistent i.e.,*

$$\sup_{x\in\mathbb{R}^2}\Big|P^*\left(\sqrt{R}\left(\tilde{f}^{*,R}_{1,2}(\omega_1,\omega_2)-\mathrm{E}^*\big(\tilde{f}^{*,R}_{1,2}(\omega_1,\omega_2)\big)\right)\le x\right) - P\left(\sqrt{R}\left(\tilde{f}^{R}_{1,2}(\omega_1,\omega_2)-\mathrm{E}\big(\tilde{f}^{R}_{1,2}(\omega_1,\omega_2)\big)\right)\le x\right)\Big| \xrightarrow{p} 0 \quad as \quad R\longrightarrow\infty.$$

**Theorem 3.** *Under conditions of Theorem 2*

$$\sup_{x\in\mathbb{R}^{2r}}\Big|P^*\left(\sqrt{R}\left(\tilde{\boldsymbol{f}}^{*,T,\underline{S},N,R}-\mathrm{E}^*\big(\tilde{\boldsymbol{f}}^{*,T,\underline{S},N,R}\big)\right)\le x\right) - P\left(\sqrt{R}\left(\tilde{\boldsymbol{f}}^{T,\underline{S},N,R}-\boldsymbol{f}\right)\le x\right)\Big| \xrightarrow{p} 0 \quad as \quad R\longrightarrow\infty.$$

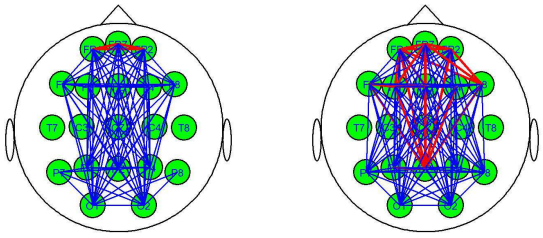
By $P^*$ and $E^*$ we denote the conditional probability and conditional expectation given the sample.

Then the Theorem 3.2 is an almost direct application of Theorem 4.1 in Lahiri (2003) for the smooth function $\phi(x_1,y_1,x_2,y_3,x_3,y_3)=\frac{x_3^2+y_3^2}{x_1x_2}$ and the sequence $1/(2\pi)^2\tilde{d}^{T,\underline{S},N,r}_{\underline{u}_1}(\omega_1)\,\overline{\tilde{d}^{T,\underline{S},N,r}_{\underline{u}_2}(\omega_2)}, r=1,\ldots,R$. We should just be aware that the mentioned Theorem cannot be applied directly and requires a small adjustment. Indeed, Lahiri (2003) assumes that the considered estimator is unbiased. In our problem this condition does not hold (see equation 4 in the proof the Lemma 3.2 in the companion document), but one can easily show that Theorem 4.1 holds in our problem.

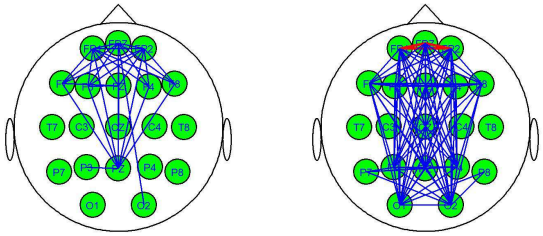
# 4 Connectivity graphs

Estimated functional connections between $\alpha$ and $\beta$ frequency bands associated to poor VWM results (left-side graph), to good VWM results (right-side graph). Blue lines correspond to coherence values in $[0.3, 0.7)$; Red lines correspond to coherence values in $[0.7, 1]$.

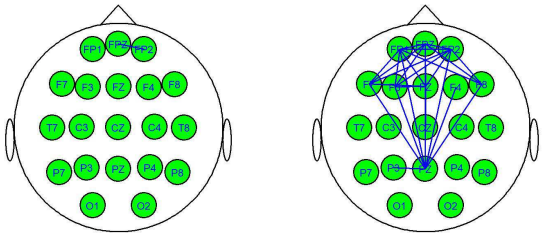
set stays on screen ; time (in s),  0.002 ; left comp_1; right comp_2



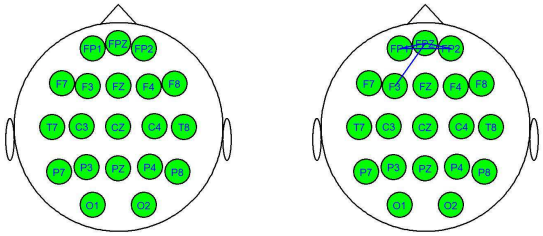
set stays on screen ; time (in s),  0.062 ; left comp_1; right comp_2



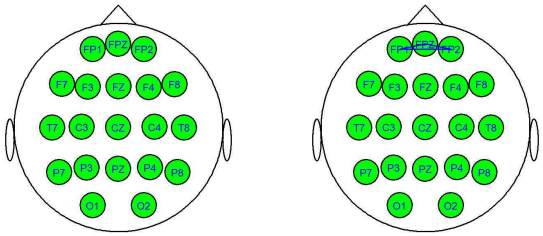
set stays on screen ; time (in s),  0.122 ; left comp_1; right comp_2



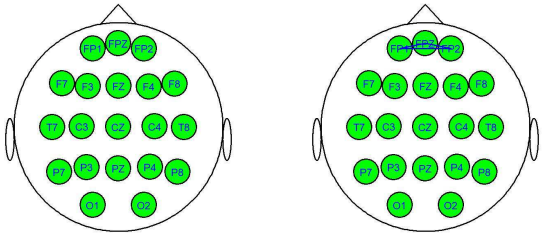

set stays on screen ; time (in s),  0.182 ; left comp_1; right comp_2



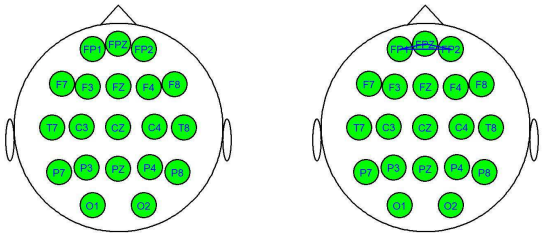
set stays on screen ; time (in s),  0.242 ; left comp_1; right comp_2



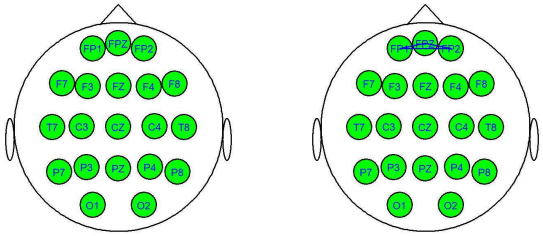

set stays on screen ; time (in s),  0.302 ; left comp_1; right comp_2



set stays on screen ; time (in s),  0.362 ; left comp_1; right comp_2




set stays on screen ; time (in s),  0.422 ; left comp_1; right comp_2



# 5 Location dependent case

In this section, we define the spatial Loève spectrum and an estimation procedure which smooths dual-frequency periodograms over the space using tensor kernels. This allows for estimation of the spatial dual-frequency coherence.
Let $\left\{\boldsymbol{X}_t^{\underline{S}}\right\} = \left\{\boldsymbol{X}_t^{\underline{S}}, t \in \mathbb{Z}\right\} := \left\{X_t^{\underline{S}}(\underline{s}), t \in \mathbb{Z}, \underline{s} \in \{1,\dots,S_1\} \times \{1,\dots,S_2\}\right\}$, $\underline{S} := (S_1, S_2) \in \{1,\dots\} \times \{1,\dots\}$, be a family of time-harmonizable spatial (location-dependent) processes

$$X_t^{\underline{S}}(\underline{s}) = \int_{-\pi}^{\pi} e^{-i\omega t} dZ_{\underline{s}}^{\underline{S}}(\omega)\,,$$

such that

$$\mathrm{Cov}\left(dZ_{\underline{s}_1}^{\underline{S}}(\omega_1)\,, dZ_{\underline{s}_2}^{\underline{S}}(\omega_2)\right) = f_{\underline{s}_1,\underline{s}_2}^{\underline{S}}(\omega_1,\omega_2)\, d\omega_1 d\omega_2,$$

where $f_{\underline{s}_1,\underline{s}_2}^{\underline{S}}(\omega_1,\omega_2)$ is the location-dependent Loève spectrum. We observe the process $X_t^{\underline{S}}(\underline{s})$ for $t = 1,\dots,T$ in $S_1 \times S_2$ different locations, i.e. $s_1$ and $s_2$ can take $S_1$ and $S_2$ different values, respectively: $s_i = 1,\dots,S_i$ for $i = 1, 2$.

Notice that the condition for time-harmonizability and existence of a two-dimensional spectral density for second order spatial random processes is of the form

$$\sum_{(t_1,t_2)\in\mathbb{Z}^2} \left|C_{\underline{s}_1,\underline{s}_2}^{\underline{S}}(t_1,t_2)\right| < \infty,$$

where $C_{\underline{s}_1,\underline{s}_2}^{\underline{S}}(t_1,t_2)$ designes the covariance between $X_{t_1}^{\underline{S}}(\underline{s}_1)$ and $X_{t_2}^{\underline{S}}(\underline{s}_2)$. Then the Loève spectrum is a continuous function and coincides with

$$f_{\underline{s}_1,\underline{s}_2}^{\underline{S}}(\omega_1,\omega_2) = \frac{1}{4\pi^2} \sum_{(t_1,t_2)\in\mathbb{Z}^2} C_{\underline{s}_1,\underline{s}_2}^{\underline{S}}(t_1,t_2)\, e^{-i(\omega_1 t_1 - \omega_2 t_2)}.$$

We observe the time series on $[0,T]$ and hence we are going to estimate

$$f_{\underline{s}_1,\underline{s}_2}^{+,\underline{S}}(\omega_1,\omega_2) := \frac{1}{4\pi^2} \sum_{(t_1,t_2)\in\mathbb{N}^2} C_{\underline{s}_1,\underline{s}_2}^{\underline{S}}(t_1,t_2)\, e^{-i(\omega_1 t_1 - \omega_2 t_2)}.$$

Then for $t_1, t_2 \geq 0$,

$$C_{\underline{s}_1,\underline{s}_2}^{\underline{S}}(t_1,t_2) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{\underline{s}_1,\underline{s}_2}^{+,\underline{S}}(\omega_1,\omega_2)\, e^{i(\omega_1 t_1 - \omega_2 t_2)} d\omega_1 d\omega_2.$$

Let $C_{\underline{s}_1,\underline{s}_2}^{+,\underline{S}}(t_1,t_2) := C_{\underline{s}_1,\underline{s}_2}(t_1,t_2)$ for $t_1, t_2 \geq 0$, and $C_{\underline{s}_1,\underline{s}_2}^{+,\underline{S}}(t_1,t_2) = 0$ otherwise.

## 5.1 Rescaling the space localisation

Following Gorrostieta et al. (2019) we make the assumption that $f^{\underline{S}}_{\underline{s}_1,\underline{s}_2}(\omega_1', \omega_2')$ can be approximated by some function which is smooth with respect to the space components.

**Notation** Let $S_1, S_2 = 1, \ldots$ be fixed. For $\underline{s} = (s_1, s_2) \in \{1, \ldots, S_1\} \times \{1, \ldots, S_2\}$ define $\ddot{\underline{s}} = (\ddot{s}_1, \ddot{s}_2) := \left(\frac{s_1}{S_1}, \frac{s_2}{S_2}\right)$. For $\underline{u} \in [0,1]^2$ denote $\underline{u} \sim \ddot{\underline{s}}$ when $\underline{s} = (\lfloor u_1 S_1 \rfloor, \lfloor u_2 S_2 \rfloor)$. Hence $s_1 \leq u_1 S_1 < s_1 + 1$ and $s_2 \leq u_2 S_2 < s_2 + 1$ or equivalently $\ddot{s}_1 \leq u_1 < \ddot{s}_1 + 1/S_1$ and $\ddot{s}_2 \leq u_2 < \ddot{s}_2 + 1/S_2$.

**(L)** There exists a function $f^{(+)} : [0,1]^4 \times [-\pi, \pi)^2 \to \mathbb{C}$, which is Lipschitz-continuous with respect to the space components uniformly on the frequency components, that is there exist some constant $L > 0$ such that for each $\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4 \in [0,1]^2$ and each $\omega_1, \omega_2 \in [-\pi, \pi)$.

$$|f^{(+)}_{\underline{u}_1,\underline{u}_2}(\omega_1, \omega_2) - f^{(+)}_{\underline{u}_3,\underline{u}_4}(\omega_1, \omega_2)| \leq L\left(||\underline{u}_1 - \underline{u}_3|| + ||\underline{u}_2 - \underline{u}_4||\right) \tag{8}$$

and there exist some constant $Q > 0$ such that

$$\left| f^{(+),\underline{S}}_{\underline{s}_1,\underline{s}_2}(\omega_1, \omega_2) - f^{(+)}_{\ddot{\underline{s}}_1,\ddot{\underline{s}}_2}(\omega_1, \omega_2) \right| \leq \frac{Q}{S_1 S_2}, \tag{9}$$

where $\underline{s}_i = (s_{i,1}, s_{i,2})$ and $\ddot{\underline{s}}_i = (s_{i,1}/S_1, s_{i,2}/S_2), i = 1, 2$.

In the following, the function $f_{\underline{u}_1,\underline{u}_2}(\omega_1, \omega_2)$ will be called the **rescaled space varying Loève spectrum function**.

We can consider the spatial localization of the covariance function and therefore we consider the following condition.

(**LC**) There exists a function $C^{(+)} : [0,1]^4 \times \mathbb{Z}^2 \to \mathbb{R}$ which is Lipschitz-continuous with respect to the space components uniformly on the time components, that is, there exist some constant values $L' > 0$ and $Q' > 0$ such that for each $\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4 \in [0,1]^2$, and each $t_1, t_2 \in \mathbb{Z}$

$$\left| C^{(+)}_{\underline{u}_1,\underline{u}_2}(t_1, t_2) - C^{(+)}_{\underline{s}_1,\underline{s}_2}(t_1, t_2) \right| \leq L'\left(\|\underline{u}_1 - \ddot{\underline{s}}_1\| + \|\underline{u}_2 - \ddot{\underline{s}}_2\|\right) \tag{10}$$

and

$$\left| C^{(+),\underline{S}}_{\underline{s}_1,\underline{s}_2}(t_1, t_2) - C^{(+)}_{\ddot{\underline{s}}_1,\ddot{\underline{s}}_2}(t_1, t_2) \right| \leq \frac{Q'}{S_1 S_2}. \tag{11}$$

Hence

$$\left|C^{(+)}_{\underline{u}_1,\underline{u}_2}(t_1,t_2) - C^{(+),\underline{S}}_{\underline{s}_1,\underline{s}_2}(t_1,t_2)\right| \leq \frac{Q'}{S_1 S_2} + L'\left(\|\underline{u}_1 - \ddot{\underline{s}}_1\| + \|\underline{u}_2 - \ddot{\underline{s}}_2\|\right)$$

where $\ddot{\underline{s}}_i = (\ddot{s}_{i,1}, \ddot{s}_{1,2})$, and $\ddot{s}_{i,j} = s_{i,j}/S_j$. Then we can deduce that we have covariance properties for $C_{\underline{u}_1,\underline{u}_2}(t_1,t_2)$.
Notice that when condition $(L)$ is satisfied then condition (LC) is also satified with $L' = 4\pi^2 L$, $Q' = 4\pi^2 Q$, and

$$C^{(+)}_{\underline{u}_1,\underline{u}_2}(t_1,t_2) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f^{(+)}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\, e^{i(\omega_1 t_1 - \omega_2 t_2)} d\omega_1 d\omega_2. \tag{12}$$

Furthermore, when $\sum_{t_1,t_2}\left|C^{(+)}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)\right| < \infty$, then

$$f^{(+)}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) = \frac{1}{4\pi^2}\sum_{(t_1,t_2)\in\mathbb{Z}^2} C^{(+)}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)\, e^{-i(\omega_1 t_1 - \omega_2 t_2)}.$$

## 5.2 Construction of the estimator

Moreover, from now on we assume that we have $R$ replicates of $X^{\underline{S}}_t(\underline{s})$, which we denote by $X^{\underline{S},r}_t(\underline{s}), r = 1,\ldots,R$. The covariance $C^{\underline{S},r}_{\underline{s}_1,\underline{s}_2}(t_1,t_2) = \mathrm{Cov}\left(X^{\underline{S},r}_{t_1}(\underline{s}_1), X^{\underline{S},r}_{t_2}(\underline{s}_2)\right)$ does not depend on the $r$-replicate.
To obtain our results we assume that $S_1, S_2, R \to \infty$.
We define the Fourier transform of the $r$-th replicate at location $\underline{s}$

$$d^{T,\underline{S},r}_{\underline{s}}(\omega) := \sum_{t=0}^{T-1} X^{\underline{S},r}_t(\underline{s})\, e^{-i\omega t}$$

and the dual-frequency periodogram of the $r$-th replicate

$$I^{T,\underline{S},r}_{\underline{s}_1,\underline{s}_2}(\omega_1,\omega_2) := \frac{1}{4\pi^2} d^{T,\underline{S},r}_{\underline{s}_1}(\omega_1)\,\overline{d^{T,\underline{S},r}_{\underline{s}_2}(\omega_2)}.$$

Notice that the dual-frequency periodograms $I^{T,\underline{S},r}_{\underline{s}_1,\underline{s}_2}(\omega_1,\omega_2)$, $r = 1,\ldots,R$ are identically distributed.
We average over the replicates to get a consistent estimate of the bispectrum. Having many replicates, there is no need to smooth over frequencies

$$\hat{f}^{T,\underline{S},R}_{\underline{s}_1,\underline{s}_2}(\omega_1,\omega_2) := \frac{1}{R}\sum_{r=1}^{R} I^{T,\underline{S},r}_{\underline{s}_1,\underline{s}_2}(\omega_1,\omega_2).$$

Then smoothing and rescaling spatially, we define the Loève spectrum estimator $\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ by relation (13). Finally, we compute the estimate of the dual-frequency spatial coherence

$$\widetilde{\rho}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2), := \frac{\left|\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right|^2}{\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1)\,\widetilde{f}^{T,\underline{S},R}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)},$$

which is defined as

$$\rho^{(+)}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) := \frac{\left|f^{(+)}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right|^2}{f^{(+)}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1)\,f^{(+)}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)}.$$

**Rescaling the space localisation : space localisation kernel function**

Let us define the two-dimensional kernel function which allows us to rescale in space the bidimensional spectrum. For simplification of presentation, we consider an isotropic kernel of the form

$$w_{\underline{u}}(\underline{s}) := w_{u_1}(s_1)w_{u_2}(s_2)$$

where $\underline{u} = (u_1, u_2)$ and $\underline{s} = (s_1, s_2)$ and

$$w_u(s_i) := \frac{1}{S_i h} w\left(\frac{u_i - s_i/S_i}{h}\right),$$

for $u \in (0,1)$, $h \to 0$ as $S_1, S_2 \to \infty$. We assume that the kernel function $w(\cdot) : \mathbb{R} \to [0,\infty)$ is symmetric non-negative with support contained in $[-1,1]$ and such that $\int_{-1}^{1} w(u)du = 1$. Moreover, we assume that $w(\cdot)$ is piecewise Lipschitz-continuous in the sense that there exist $k \in \mathbb{N}$, $u_1, \ldots, u_k \in [-1,1]$ such that $w(\cdot)$ is Lipschitz-continuous on each interval $(u_j, u_{j+1})$. We deduce that $w(\cdot)$ is bounded. This definition includes the rectangular kernel, as well as the triangular kernel.
Moreover, we assume that $S_1 h$ and $S_2 h$ tend to $\infty$ as $S_1, S_2 \to \infty$.

**Estimation of the rescaled space varying Loève spectrum function**

Define the estimator of $f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ by space-smoothing $\hat{f}^{T,\underline{S},R}_{\underline{s}_1,\underline{s}_2}(\omega_1,\omega_2)$

$$\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) := \sum_{\underline{s}_1}\sum_{\underline{s}_2} w_{\underline{u}_1}(\underline{s}_1)w_{\underline{u}_2}(\underline{s}_2)\hat{f}^{T,\underline{S},R}_{\underline{s}_1,\underline{s}_2}(\omega_1,\omega_2).$$

Besides, consider the space-smoothed space varying Fourier transform of the replicate observation

$$\widetilde{d}^{T,\underline{S},r}_{\underline{u}_i}(\omega) := \sum_{\underline{s}_i} w_{\underline{u}_i}(\underline{s}_i) \left( \sum_{t=0}^{T-1} X^{\underline{S},r}_t(\underline{s}_i)\, e^{-i\omega t} \right) = \sum_{t=0}^{T-1} \widetilde{X}^{\underline{S},r}_t(\underline{u}_i)\, e^{-i\omega t},$$

where $\widetilde{X}^{\underline{S},r}_t(\underline{u}_i) := \sum_{\underline{s}_i} w_{\underline{u}_i}(\underline{s}_i) X^{\underline{S},r}_t(\underline{s}_i)$. Then the estimator $\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ of $f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ coincides with

$$\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) = \frac{1}{R}\sum_{r=1}^{R} \widetilde{I}^{T,\underline{S},r}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2) = \frac{1}{4\pi^2 R}\sum_{r=1}^{R} \widetilde{d}^{T,\underline{S},r}_{\underline{u}_1}(\omega_1)\, \overline{\widetilde{d}^{T,\underline{S},r}_{\underline{u}_2}(\omega_2)}, \tag{13}$$

where the space-smoothed periodogram $\widetilde{I}^{T,\underline{S},r}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2)$ is defined by

$$\widetilde{I}^{T,\underline{S},r}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2) := \frac{1}{4\pi^2}\widetilde{d}^{T,\underline{S},r}_{\underline{u}_1}(\omega_1)\, \overline{\widetilde{d}^{T,\underline{S},r}_{\underline{u}_2}(\omega_2)}. \tag{14}$$

## 5.3 Theoretical results

We consider a mixing condition across replicates that is allowed to vary across space. It models the training of the brain when the same patient is performing repetitively a task.

### 5.3.1 Bias of the estimator: limit of the mean of the estimator

Let $D_T(0) := T$ and

$$D_T(\omega) := \sum_{t=0}^{T-1} e^{i\omega t} = \frac{1-e^{i\omega T}}{1-e^{i\omega}} \qquad \text{for} \quad \omega \neq 0 \mod 2\pi.$$

Then we can prove that

**Lemma 4** (Limit of the mean)**.**
*Let $\underline{u_1}$ and $\underline{u_2}$ be fixed in $(0,1)^2$, $R$ being fixed or going to $\infty$.*
*(1) Under condition (LC), for every $T > 0$ fixed, $\mathrm{E}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)$ converges as $h \to 0$ and $S_1h^2, S_2h^2 \to \infty$ to*

$$f^{T}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) := \frac{1}{4\pi^2}\sum_{t_1=0}^{T-1}\sum_{t_2=0}^{T-1} C_{\underline{u}_1,\underline{u}_2}(t_1,t_2) e^{-i(\omega_1 t_1 - \omega_2 t_2)}.$$

*(2) If in addition* $\sum_{t_1}\sum_{t_2}\left|C^{+}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)\right| < \infty$ *then*

$$f^{T}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} D_T(\omega_1'-\omega_1)\overline{D_T(\omega_2'-\omega_2)}f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1',\omega_2')\,d\omega_1' d\omega_2'.$$

*Furthermore,* $\mathrm{E}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) \to f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ *provided that* $T^2h, T^2S_1^{-1}h^{-2}, T^2S_2^{-1}h^{-2} \to 0$ *as* $T, S_1, S_2 \to \infty$ *and* $h \to 0$.

*Proof.*
(1) Since the time series $\left\{\boldsymbol{X}^{\underline{S}}_{t}\right\}$ is real-valued, from definition (14) of the space-smoothed periodogram we have

$$4\pi^2\mathrm{E}\left(\widetilde{I}^{T,\underline{S},r}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2)\right) = \sum_{t_1=0}^{T-1}\sum_{t_2=0}^{T-1}\mathrm{E}\left(\widetilde{X}^{\underline{S},r}_{t_1}(\underline{u}_1)\,\widetilde{X}^{\underline{S},r}_{t_2}(\underline{u}_2)\right)e^{-i(\omega_1 t_1-\omega_2 t_2)}$$

and

$$\mathrm{E}\left(\widetilde{X}^{\underline{S},r}_{t_1}(\underline{u}_1)\,\widetilde{X}^{\underline{S},r}_{t_2}(\underline{u}_2)\right) = \sum_{\underline{s}_1}\sum_{\underline{s}_2} w_{\underline{u}_1}(\underline{s}_1)w_{\underline{u}_2}(\underline{s}_2)C^{\underline{S}}_{\underline{s}_1,\underline{s}_2}(t_1,t_2),$$

where $C^{\underline{S}}_{\underline{s}_1,\underline{s}_2}(t_1,t_2) := \mathrm{E}\left(X^{\underline{S}}_{t_1}(\underline{s}_1)\,X^{\underline{S}}_{t_2}(\underline{s}_2)\right) = \mathrm{E}\left(X^{\underline{S},r}_{t_1}(\underline{s}_1)\,X^{\underline{S},r}_{t_2}(\underline{s}_2)\right)$ does not depend on $r$. Using approximation (11) we get that

$$\mathrm{E}\left(X^{\underline{S},r}_{t_1}(\underline{s}_1)\,X^{\underline{S},r}_{t_2}(\underline{s}_2)\right) = C_{\ddot{\underline{s}}_1,\ddot{\underline{s}}_2}(t_1,t_2) + \mathcal{O}\left(\frac{1}{S_1S_2}\right).$$

Then the Lipschitz-continuity property of $C_{\underline{u}_1,\underline{u}_2}(t_1,t_2)$ (inequality (10)) and of the kernel function $w(u)$ as well as the fact that the support of the kernel function $w(u)$ is contained in $[-1,1]$ imply that (Lemma 1)

$$\mathrm{E}\left(\widetilde{X}^{\underline{S},r}_{t_1}(\underline{u}_1)\,\widetilde{X}^{\underline{S},r}_{t_2}(\underline{u}_2)\right) = C_{\underline{u}_1,\underline{u}_2}(t_1,t_2) + \mathcal{O}(h) + \mathcal{O}\left(\frac{1}{S_1h^2}+\frac{1}{S_2h^2}\right)$$

which does not depend on $r$. Finally,

$$\begin{aligned}4\pi^2\mathrm{E}&\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)\\ &= \sum_{t_1=0}^{T-1}\sum_{t_2=0}^{T-1} C_{\underline{u}_1,\underline{u}_2}(t_1,t_2)e^{-i(\omega_1 t_1-\omega_2 t_2)} + \mathcal{O}(hT^2) + \mathcal{O}\left(\frac{T^2}{S_1h^2}+\frac{T^2}{S_2h^2}\right), \quad (15)\end{aligned}$$

as $S_1, S_2 \to \infty$. This proves the first part of the lemma.

(2) Assume now that $\sum_{t_1}\sum_{t_2}\left|C^+_{\underline{u}_1,\underline{u}_2}(t_1,t_2)\right| < \infty$. Then the Loève spectrum $f^+_{\underline{u}_1,\underline{u}_2}(\omega'_1,\omega'_2)$ is well defined by

$$f^+_{\underline{u}_1,\underline{u}_2}(\omega'_1,\omega'_2) = \frac{1}{4\pi^2}\sum_{t_1=0}^{\infty}\sum_{t_2=0}^{\infty} C_{\underline{u}_1,\underline{u}_2}(t_1,t_2)e^{-i(\omega'_1 t_1-\omega'_2 t_2)}.$$

By relation (12), the expression on the right-hand side of the equality above can be rewritten as follows

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} D_T(\omega'_1-\omega_1)\overline{D_T(\omega'_2-\omega_2)}f^+_{\underline{u}_1,\underline{u}_2}(\omega'_1,\omega'_2)\,d\omega'_1 d\omega'_2 + \mathcal{O}(hT^2) + \mathcal{O}\left(\frac{T^2}{S_1h^2}+\frac{T^2}{S_2h^2}\right).$$

Thus

$$\begin{aligned}&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} D_T(\omega'_1-\omega_1)\overline{D_T(\omega'_2-\omega_2)}f^+_{\underline{u}_1,\underline{u}_2}(\omega'_1,\omega'_2)\,d\omega'_1 d\omega'_2\\ &\quad= \frac{1}{4\pi^2}\sum_{t_1=0}^{\infty}\sum_{t_2=0}^{\infty} C_{\underline{u}_1,\underline{u}_2}(t_1,t_2)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} D_T(\omega'_1-\omega_1)\overline{D_T(\omega'_2-\omega_2)}e^{-i(\omega'_1 t_1-\omega'_2 t_2)}\,d\omega'_1 d\omega'_2.\end{aligned}$$

Since

$$\int_{-\pi}^{\pi} D_T(\omega'_1-\omega_1)e^{-i\omega'_1 t_1}\,d\omega'_1 = \begin{cases} 2\pi e^{-i\omega_1 t_1} & \text{for} \quad t_1 = 0,\ldots,T-1\\ 0 & \text{otherwise}\end{cases}$$

we obtain that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} D_T(\omega'_1-\omega_1)\overline{D_T(\omega'_2-\omega_2)}f^+_{\underline{u}_1,\underline{u}_2}(\omega'_1,\omega'_2)\,d\omega'_1 d\omega'_2 \tag{16}$$

$$= \sum_{t_1=0}^{T-1}\sum_{t_2=0}^{T-1} C_{\underline{u}_1,\underline{u}_2}(t_1,t_2)e^{-i(\omega_1 t_1-\omega_2 t_2)}, \tag{17}$$

which converges to $4\pi^2 f^+_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ as $T\to\infty$. This achieves the proof of the lemma. □

**Corollary 2** (Rate of convergence of the bias)**.** *Let $\underline{u_1}$ and $\underline{u_2}$ be fixed in $(0,1)$.*
*(1) Under condition (LC), for every $T>0$ fixed,*

$$\lim_{R\to\infty}\sqrt{R}\left(\mathrm{E}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) - f^T_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) = 0.$$

*provided that* $Rh^2, RS_1^{-2}h^{-4}, RS_2^{-2}h^{-4} \to 0$ *as* $R, S_1, S_2 \to \infty$ *and* $h \to 0$.
*(2) Furthermore*

$$\lim_{R\to\infty} \sqrt{R}\left(\mathrm{E}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) - f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) = 0$$

*provided that* $RT^4h^2, RT^4S_1^{-2}h^{-4}, RT^4S_2^{-2}h^{-4} \to 0$ *as* $R, T, S_1, S_2 \to \infty$ *and* $h \to 0$*, when the rate of convergence of* $\sum_{t_1}\sum_{t_2}\left|C^{+}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)\right|$ *is greater than* $\sqrt{R}$*, that is*

$$\lim_{R\to\infty} \sqrt{R}\sum_{t_1=T}^{\infty}\sum_{t_2=0}^{\infty}\left|C^{+}_{\underline{u}_j,\underline{u}'_j}(t_1,t_2)\right| = 0. \tag{18}$$

*Proof.*
(1) The first part is a direct consequence of relation (15), $T$ being fixed.
(2) First notice that relation (16) implies that $\sum_{t_1}\sum_{t_2}\left|C^{+}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)\right| < \infty$, then the existence of $f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ and

$$4\pi^2\left|\mathrm{E}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) - f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right| \le \sum_{t_1=T}^{\infty}\sum_{t_2=0}^{\infty}\left|C^{+}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)\right| + \sum_{t_1=0}^{T-1}\sum_{t_2=T}^{\infty}\left|C^{+}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)\right|.$$

Thanks to condition (16), the second part of the lemma is verified. □

### 5.3.2 Consistency

We assume $\alpha$-mixing property between replicates, i.e.

**(M)** let $\{X_t^{\underline{S},r}(\underline{s})\}$ be $\alpha$-mixing with respect to $r$ i.e., $\alpha_X(k) \to 0$ as $k \to \infty$ where

$$\alpha_X(k) = \sup_k \sup_{\substack{A\in\mathcal{F}_X(1,r)\\ B\in\mathcal{F}_X(r+k,\infty)}} |P(A\cap B) - P(A)P(B)|$$

and $\mathcal{F}_X(1,r) = \sigma\left(\{X_t^{\underline{S},q}(\underline{s}) : q \le r, t \in \mathbb{Z} \text{ and all locations } \underline{s}\}\right)$,
$\mathcal{F}_X(r+k,\infty) = \sigma\left(\{X_t^{\underline{S},q}(\underline{s}) : q \ge r+k, t \in \mathbb{Z} \text{ and all locations } \underline{s}\}\right)$.

**(B)** Boundedness :

(i) either, there exists some constant value $C > 0$ such that $\sup_{t,r,\underline{s}} \left| X_t^{\underline{S},r}(\underline{s}) \right| \leq c$ and $\sum_r \alpha(r) < \infty$,

(ii) or, for some $\delta > 0$, $\sup_{t,r,\underline{s}} \mathrm{E}\left( \left| X_t^{\underline{S},r}(\underline{s}) \right|^{4+\delta} \right) < \infty$ and $\sum_r \alpha(r)^{\delta/(4+\delta)} < \infty$.

**Lemma 5.** *Let the mixing and boundedness assumptions (M) and (B) be satisfied. Then*

$$\mathrm{E}\left( \left| \widetilde{f}_{\underline{u}_1,\underline{u}_2}^{T,\underline{S},R}(\omega_1,\omega_2) - \mathrm{E}\left( \widetilde{f}_{\underline{u}_1,\underline{u}_2}^{T,\underline{S},R}(\omega_1,\omega_2) \right) \right|^2 \right) \leq \frac{cT^4}{R}, \tag{19}$$

*where $c$ is some positive constant which does not depend on $R, T, h, \underline{S}, \underline{u}_1, \underline{u}_2, \omega_1, \omega_2$.*

*Proof.* First remark that

$$\begin{aligned}
\mathrm{E}\left( \left| \widetilde{f}_{\underline{u}_1,\underline{u}_2}^{T,\underline{S},R}(\omega_1,\omega_2) - \mathrm{E}\left( \widetilde{f}_{\underline{u}_1,\underline{u}_2}^{T,\underline{S},R}(\omega_1,\omega_2) \right) \right|^2 \right) &= \mathrm{Var}\left( \widetilde{f}_{\underline{u}_1,\underline{u}_2}^{T,\underline{S},R}(\omega_1,\omega_2) \right) \\
&= \frac{1}{R^2} \sum_{r_1=1}^{R} \sum_{r_2=1}^{R} \mathrm{Cov}\left( \widetilde{I}_{\underline{u}_1\underline{u}_2}^{T,\underline{S},r_1}(\omega_1,\omega_2), \widetilde{I}_{\underline{u}_1\underline{u}_2}^{T,\underline{S},r_2}(\omega_1,\omega_2) \right).
\end{aligned}$$

Moreover, from relation (14) we get

$$\begin{aligned}
&16\pi^4 \left| \mathrm{Cov}\left( \widetilde{I}_{\underline{u}_1\underline{u}_2}^{T,\underline{S},r_1}(\omega_1,\omega_2), \widetilde{I}_{\underline{u}_1\underline{u}_2}^{T,\underline{S},r_2}(\omega_1,\omega_2) \right) \right| \\
&\leq \sum_{t_1=0}^{T-1} \sum_{t_2=0}^{T-1} \sum_{t_3=0}^{T-1} \sum_{t_4=0}^{T-1} \sum_{\underline{s}_1 \in I_{\underline{u}_1}} \sum_{\underline{s}_2 \in I_{\underline{u}_2}} \sum_{\underline{s}_3 \in I_{\underline{u}_1}} \sum_{\underline{s}_4 \in I_{\underline{u}_2}} w_{\underline{u}_1}(\underline{s}_1) w_{\underline{u}_2}(\underline{s}_2) w_{\underline{u}_1}(\underline{s}_3) w_{\underline{u}_2}(\underline{s}_4) \\
&\quad \times \left| \mathrm{Cov}\left( X_{t_1}^{\underline{S},r_1}(\underline{s}_1) X_{t_2}^{\underline{S},r_1}(\underline{s}_2), X_{t_3}^{\underline{S},r_2}(\underline{s}_3) X_{t_4}^{\underline{S},r_2}(\underline{s}_4) \right) \right| \\
&\leq c \sum_{t_1=0}^{T-1} \sum_{t_2=0}^{T-1} \sum_{t_3=0}^{T-1} \sum_{t_4=0}^{T-1} \sum_{\underline{s}_1 \in I_{\underline{u}_1}} \sum_{\underline{s}_2 \in I_{\underline{u}_2}} \sum_{\underline{s}_3 \in I_{\underline{u}_1}} \sum_{\underline{s}_4 \in I_{\underline{u}_2}} w_{\underline{u}_1}(\underline{s}_1) w_{\underline{u}_2}(\underline{s}_2) w_{\underline{u}_3}(\underline{s}_3) w_{\underline{u}_4}(\underline{s}_4) \alpha_X^{\frac{\delta}{4+\delta}}(|r_1 - r_2|) \\
&\leq cT^4 \alpha_X^{\frac{\delta}{4+\delta}}(|r_1 - r_2|) \sum_{\underline{s}_1} \sum_{\underline{s}_2} \sum_{\underline{s}_3} \sum_{\underline{s}_4} w_{\underline{u}_1}(\underline{s}_1) w_{\underline{u}_2}(\underline{s}_2) w_{\underline{u}_1}(\underline{s}_3) w_{\underline{u}_2}(\underline{s}_4),
\end{aligned}$$

where the second inequality is due to covariance inequality for $\alpha$-mixing processes (see i.e. Bolthausen (1982); Guyon (1995)). Here $c$ is some positive constant that may vary

from line to line. Moreover,

$$\sum_{\underline{s}_j} w_{\underline{u}_i}(\underline{s}_j) = \sum_{\underline{s}_j} \frac{1}{S_1 h} w\left(\frac{u_{i,1} - s_{j,1}/S_1}{h}\right) \frac{1}{S_2 h} w\left(\frac{u_{i,2} - s_{j,2}/S_2}{h}\right) \leq 1.$$

Thus,

$$\mathrm{Var}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) \leq \frac{1}{R^2}\sum_{r_1=1}^{R}\sum_{r_2=1}^{R} cT^4 \alpha_X^{\frac{\delta}{4+\delta}}(|r_1 - r_2|) \leq \frac{cT^4}{R}\sum_{k=-R+1}^{R-1}\left(1 - \frac{|k|}{R}\right)\alpha_X^{\frac{\delta}{4+\delta}}(|k|).$$

Assuming that condition (B)(ii) is satisfied, we obtain inequality (19). In the same way we can easily prove inequality (19) under condition (B)(i). □

From Lemma 4 and Lemma 5 we deduce the consistency of the estimator.

**Theorem 4** (quadratic convergence)**.**
*Assume that conditions (LC), (M) and (B) are satisfied, then*
*(1) For every $T > 0$ fixed and $h \to 0$, $S_1 h^2, S_2 h^2 \to \infty$ as $R \to \infty$ we have*

$$\lim_{R\to\infty} \widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)) = f^{T}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) \qquad \textit{in quadratic mean.}$$

*(2) If in addition $\sum_{t_1}\sum_{t_2}\left|C^{+}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)\right| < \infty$, we have*

$$\lim_{R\to\infty} \widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)) = f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) \qquad \textit{in quadratic mean}$$

*provided that $T^4 R^{-1}, T^2 h, T^2 S_1^{-1} h^{-2}, T^2 S_2^{-1} h^{-2} \to 0$ as $R, T, S_1, S_2 \to \infty$ and $h \to 0$.*

### 5.3.3 Limit variance

From relation (13), the covariance of the estimator $\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ satisfies

$$\begin{aligned} &R^2 \mathrm{Cov}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2), \widetilde{f}^{T,\underline{S},R}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4)\right) \\ &\quad = \sum_{r_1=1}^{R}\sum_{r_2=1}^{R} \mathrm{Cov}\left(\widetilde{I}^{T,\underline{S},r_1}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2), \widetilde{I}^{T,\underline{S},r_2}_{\underline{u}_3\underline{u}_4}(\omega_3,\omega_4)\right). \end{aligned} \tag{20}$$

Besides, from definition (14) of the space-smoothed periodogram we have

$$
\begin{aligned}
&16\pi^4 \mathrm{Cov}\left(\widetilde{I}^{T,\underline{S},r_1}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2), \widetilde{I}^{T,\underline{S},r_2}_{\underline{u}_3\underline{u}_4}(\omega_3,\omega_4)\right) \\
&= \sum_{t_1=0}^{T-1}\sum_{t_2=0}^{T-1}\sum_{t_3=0}^{T-1}\sum_{t_4=0}^{T-1} \mathrm{Cov}\left(\widetilde{X}^{\underline{S},r_1}_{t_1}(\underline{u}_1)\,\widetilde{X}^{\underline{S},r_1}_{t_2}(\underline{u}_2), \widetilde{X}^{\underline{S},r_2}_{t_3}(\underline{u}_3)\,\widetilde{X}^{\underline{S},r_2}_{t_4}(\underline{u}_4)\right) e^{-i(\omega_1 t_1-\omega_2 t_2)+i(\omega_3 t_3-\omega_4 t_4)} \\
&= \sum_{t_1=0}^{T-1}\sum_{t_2=0}^{T-1}\sum_{t_3=0}^{T-1}\sum_{t_4=0}^{T-1}\sum_{\underline{s}_1}\sum_{\underline{s}_2}\sum_{\underline{s}_3}\sum_{\underline{s}_4} w_{\underline{u}_1}(\underline{s}_1) w_{\underline{u}_2}(\underline{s}_2) w_{\underline{u}_3}(\underline{s}_3) w_{\underline{u}_4}(\underline{s}_4) \\
&\qquad \times \mathrm{Cov}\left(X^{\underline{S},r_1}_{t_1}(\underline{s}_1)\,X^{\underline{S},r_1}_{t_2}(\underline{s}_2), X^{\underline{S},r_2}_{t_3}(\underline{s}_3)\,X^{\underline{S},r_2}_{t_4}(\underline{s}_4)\right) e^{-i(\omega_1 t_1-\omega_2 t_2)+i(\omega_3 t_3-\omega_4 t_4)}.
\end{aligned}
$$

**Conditions on the replicates** To compute the covariance

$$
\mathrm{Cov}\left(X^{\underline{S},r_1}_{t_1}(\underline{s}_1)\,X^{\underline{S},r_1}_{t_2}(\underline{s}_2), X^{\underline{S},r_2}_{t_3}(\underline{s}_3)\,X^{\underline{S},r_2}_{t_4}(\underline{s}_4)\right),
$$

we need to add some assumptions on the family of processes $\left\{X^{\underline{S},r}_t(\underline{s})\right\}$.
For simplicity we assume the Gaussianity.

(**GR**) The time-spatial random array $\left\{X^{\underline{S},r}_t(\underline{s}) : t\in\mathbb{Z}, \underline{s}\in\{1,\dots,S_1\}\times\{1,\dots,S_2\}, r=1,\dots,R\right\}$ is Gaussian for any $R>0$.

Assume also that

(**LCR**) There is a family of functions $C^{r_1,r_2} : [0,1]^4\times\mathbb{Z}^2\to\mathbb{R}$ such that there are some constants $L'>0$ and $Q'>0$ such that for each $\underline{u}_1,\underline{u}_2,\underline{u}_3,\underline{u}_4\in[0,1]^2$ and for each $t_1,t_2\in\mathbb{Z}$

$$
\left|C^{r_1,r_2}_{\underline{u}_1,\underline{u}_2}(t_1,t_2) - C^{r_1,r_2}_{\underline{u}_3,\underline{u}_4}(t_1,t_2)\right| \le L'\left(\|\underline{u}_1-\underline{u}_3\| + \|\underline{u}_2-\underline{u}_4\|\right)
$$

and

$$
\left|C^{\underline{S},r_1,r_2}_{\underline{s}_1,\underline{s}_2}(t_1,t_2) - C^{r_1,r_2}_{\underline{s}_1,\underline{s}_2}(t_1,t_2)\right| \le \frac{Q'}{S_1S_2}
$$

where $C^{\underline{S},r_1,r_2}_{\underline{s}_j,\underline{s}_{j'}}(t_j,t_{j'}) = \mathrm{E}\left(X^{\underline{S},r_1}_{t_j}(s_j)X^{\underline{S},r_2}_{t_{j'}}(s_{j'})\right)$.

In this case the $2 \times 2$-matrix $\left(C^{r_j,r_{j'}}_{\underline{u}_j,\underline{u}_{j'}}(t_j,t'_j)\right)_{j,j'=1,2}$ is a covariance matrix. Moreover, since the replicates $\left\{X^{\underline{S},r}_t(\underline{s}) : t \in \mathbb{Z}, \underline{s} \in \{1,\dots,S_1\} \times \{1,\dots,S_2\}\right\}$, $r = 1,\dots,R$, are identically distributed, we have that $C^{\underline{S},r,r}_{\underline{s}_1,\underline{s}_2}(t_1,t_2) = C^{\underline{S}}_{\underline{s}_1,\underline{s}_2}(t_1,t_2)$ does not depend on $r$, thus $C^{r,r}_{\underline{u}_j,\underline{u}_{j'}}(t_j,t'_j) = C_{\underline{u}_j,\underline{u}_{j'}}(t_j,t'_j)$ does not depend on $r$.
Then we can calculate the asymptotic covariances for the spatially smoothed periodograms $\widetilde{I}^{T,\underline{S},r_1}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2)$ and $\widetilde{I}^{T,\underline{S},r_2}_{\underline{u}_3\underline{u}_4}(\omega_3,\omega_4)$.

**Lemma 6.** *Assume that conditions (GR) and (LCR) are satisfied. Then*
*(1) For $R$ and $T$ fixed as $h \to 0$ and $S_1h^2, S_2h^2 \to \infty$ we have*

$$\begin{aligned} 16\pi^4 \mathrm{Cov}\left(\widetilde{I}^{T,\underline{S},r_1}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2), \widetilde{I}^{T,\underline{S},r_2}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4)\right) \quad &\longrightarrow \\ \left(\sum_{t_1=0}^{T-1}\sum_{t_3=0}^{T-1} C^{r_1,r_2}_{\underline{u}_1,\underline{u}_3}(t_1,t_3)e^{-i(\omega_1 t_1-\omega_3 t_3)}\right)&\left(\sum_{t_2=0}^{T-1}\sum_{t_4=0}^{T-1} C^{r_1,r_2}_{\underline{u}_2,\underline{u}_4}(t_2,t_4)e^{i(\omega_2 t_2-\omega_4 t_4)}\right) \\ + \left(\sum_{t_1=0}^{T-1}\sum_{t_4=0}^{T-1} C^{r_1,r_2}_{\underline{u}_1,\underline{u}_4}(t_1,t_4)e^{-i(\omega_1 t_1+\omega_4 t_4)}\right)&\left(\sum_{t_2=0}^{T-1}\sum_{t_3=0}^{T-1} C^{r_1,r_2}_{\underline{u}_2,\underline{u}_3}(t_2,t_3)e^{i(\omega_2 t_2+\omega_3 t_3)}\right). \end{aligned} \tag{21}$$

*(2) For $R$ fixed, if $\sum_{t_1}\sum_{t_2}\left|C^{r_1,r_2}_{\underline{u}_j,\underline{u}_{j'}}(t_1,t_2)\right| < \infty$ then*

$$\begin{aligned} 16\pi^4 \mathrm{Cov}\left(\widetilde{I}^{T,\underline{S},r_1}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2), \widetilde{I}^{T,\underline{S},r_2}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4)\right) \quad &\longrightarrow \\ \left(\sum_{t_1=0}^{\infty}\sum_{t_3=0}^{\infty} C^{r_1,r_2}_{\underline{u}_1,\underline{u}_3}(t_1,t_3)e^{-i(\omega_1 t_1-\omega_3 t_3)}\right)&\left(\sum_{t_2=0}^{\infty}\sum_{t_4=0}^{\infty} C^{r_1,r_2}_{\underline{u}_2,\underline{u}_4}(t_2,t_4)e^{i(\omega_2 t_2-\omega_4 t_4)}\right) \\ + \left(\sum_{t_1=0}^{\infty}\sum_{t_4=0}^{\infty} C^{r_1,r_2}_{\underline{u}_1,\underline{u}_4}(t_1,t_4)e^{-i(\omega_1 t_1+\omega_4 t_4)}\right)&\left(\sum_{t_2=0}^{\infty}\sum_{t_3=0}^{\infty} C^{r_1,r_2}_{\underline{u}_2,\underline{u}_3}(t_2,t_3)e^{i(\omega_2 t_2+\omega_3 t_3)}\right) \end{aligned} \tag{22}$$

*provided that $T^2h, T^2S_1^{-1}h^{-2}, T^2S_2^{-1}h^{-2} \to 0$ as $T, S_1, S_2 \to \infty$ and $h \to 0$.*

*Proof.* From Gaussian condition (GR) and Isserlis equality

$$\begin{aligned} \mathrm{Cov}\left(X^{\underline{S},r_1}_{t_1}(\underline{s}_1)\, X^{\underline{S},r_1}_{t_2}(\underline{s}_2), X^{\underline{S},r_2}_{t_3}(\underline{s}_3)\, X^{\underline{S},r_2}_{t_4}(\underline{s}_4)\right) \\ = C^{\underline{S},r_1,r_2}_{\underline{s}_1,\underline{s}_3}(t_1,t_3)\, C^{\underline{S},r_1,r_2}_{\underline{s}_2,\underline{s}_4}(t_2,t_4) + C^{\underline{S},r_1,r_2}_{\underline{s}_1,\underline{s}_4}(t_1,t_4)\, C^{\underline{S},r_1,r_2}_{\underline{s}_2,\underline{s}_3}(t_2,t_3). \end{aligned} \tag{23}$$

Then, thanks to relation (14), we obtain

$$
\begin{aligned}
&16\pi^4 \mathrm{Cov}\left(\widetilde{I}^{T,\underline{S},r_1}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2), \widetilde{I}^{T,\underline{S},r_2}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4)\right) \\
&= \left(\sum_{t_1=0}^{T-1}\sum_{t_3=0}^{T-1}\sum_{\underline{s}_1}\sum_{\underline{s}_3} w_{\underline{u}_1}(\underline{s}_1) w_{\underline{u}_3}(\underline{s}_3) C^{\underline{S},r_1,r_2}_{\underline{s}_1,\underline{s}_3}(t_1,t_3) e^{-i(\omega_1 t_1 - \omega_3 t_3)}\right) \\
&\quad \times \left(\sum_{t_2=0}^{T-1}\sum_{t_4=0}^{T-1}\sum_{\underline{s}_2}\sum_{\underline{s}_4} w_{\underline{u}_2}(\underline{s}_2) w_{\underline{u}_4}(\underline{s}_4) C^{\underline{S},r_1,r_2}_{\underline{s}_2,\underline{s}_4}(t_2,t_4) e^{i(\omega_2 t_2 - \omega_4 t_4)}\right) \\
&\quad + \left(\sum_{t_1=0}^{T-1}\sum_{t_4=0}^{T-1}\sum_{\underline{s}_1}\sum_{\underline{s}_4} w_{\underline{u}_1}(\underline{s}_1) w_{\underline{u}_4}(\underline{s}_4) C^{\underline{S},r_1,r_2}_{\underline{s}_1,\underline{s}_4}(t_1,t_4) e^{-i(\omega_1 t_1 + \omega_4 t_4)}\right) \\
&\quad \times \left(\sum_{t_2=0}^{T-1}\sum_{t_3=0}^{T-1}\sum_{\underline{s}_2}\sum_{\underline{s}_3} w_{\underline{u}_2}(\underline{s}_2) w_{\underline{u}_3}(\underline{s}_3) C^{\underline{S},r_1,r_2}_{\underline{s}_2,\underline{s}_3}(t_2,t_3) e^{i(\omega_2 t_2 + \omega_3 t_3)}\right).
\end{aligned}
$$

From condition (LCR)

$$
\begin{aligned}
&16\pi^4 \mathrm{Cov}\left(\widetilde{I}^{T,\underline{S},r_1}_{\underline{u}_1\underline{u}_2}(\omega_1,\omega_2), \widetilde{I}^{T,\underline{S},r_2}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4)\right) \qquad (24) \\
&= \left(\sum_{t_1=0}^{T-1}\sum_{t_3=0}^{T-1} C^{r_1,r_2}_{\underline{u}_1,\underline{u}_3}(t_1,t_3) e^{-i(\omega_1 t_1 - \omega_3 t_3)} + \mathcal{O}(T^2 h) + \mathcal{O}\left(\frac{T^2}{S_1 h^2} + \frac{T^2}{S_2 h^2}\right)\right) \\
&\quad \times \left(\sum_{t_2=0}^{T-1}\sum_{t_4=0}^{T-1} C^{r_1,r_2}_{\underline{u}_2,\underline{u}_4}(t_2,t_4) e^{i(\omega_2 t_2 - \omega_4 t_4)} + \mathcal{O}(T^2 h) + \mathcal{O}\left(\frac{T^2}{S_1 h^2} + \frac{T^2}{S_2 h^2}\right)\right) \\
&\quad + \left(\sum_{t_1=0}^{T-1}\sum_{t_4=0}^{T-1} C^{r_1,r_2}_{\underline{u}_1,\underline{u}_4}(t_1,t_4) e^{-i(\omega_1 t_1 + \omega_4 t_4)} + \mathcal{O}(T^2 h) + \mathcal{O}\left(\frac{T^2}{S_1 h^2} + \frac{T^2}{S_2 h^2}\right)\right) \\
&\quad \times \left(\sum_{t_2=0}^{T-1}\sum_{t_3=0}^{T-1} C^{r_1,r_2}_{\underline{u}_2,\underline{u}_3}(t_2,t_3) e^{i(\omega_2 t_2 + \omega_3 t_3)} + \mathcal{O}(T^2 h) + \mathcal{O}\left(\frac{T^2}{S_1 h^2} + \frac{T^2}{S_2 h^2}\right)\right).
\end{aligned}
$$

We readily deduce convergences (21) and (22). □

In the next step, using (21) and (22) we obtain the form of the limiting covariance $\mathrm{Cov}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2), \widetilde{f}^{T,\underline{S},R}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4)\right)$.
For this purpose we consider the following conditions.

(**HR**) The matrix $\left(C^{r_j,r_{j'}}_{\underline{u}_j,\underline{u}_{j'}}(t_j,t'_j)\right)_{j,j'=1,2}$ is harmonizable with a spectral density matrix : There exists a matrix $\left(f^{r_j,r_{j'}}_{\underline{u}_j,\underline{u}_{j'}}(\omega_1,\omega_2)\right)_{j,j'=1,2}$ of complex valued integrable functions on $(-\pi,\pi]^2$ such that

$$C^{r_j,r_{j'}}_{\underline{u}_j,\underline{u}_{j'}}(t_1,t_2)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f^{r_j,r_{j'}}_{\underline{u}_j,\underline{u}_{j'}}(\omega_1,\omega_2)e^{-i(\omega_1 t_1-\omega_2 t_2)}\,d\omega_1 d\omega_2.$$

(**HRS**) For $j,j'=1,2$ we have

$$\sum_{(t_1,t_2)\in\mathbb{Z}^2}\left|C^{r_j,r_{j'}}_{\underline{u}_j,\underline{u}_{j'}}(t_1,t_2)\right|<\infty.$$

Notice that if condition (HRS) is satisfied then condition (HR) is also satisfied and

$$f^{r_j,r_{j'}}_{\underline{u}_j,\underline{u}_{j'}}(\omega_1,\omega_2)=\frac{1}{4\pi^2}\sum_{(t_1,t_2)\in\mathbb{Z}^2} C^{r_j,r_{j'}}_{\underline{u}_j,\underline{u}_{j'}}(t_1,t_2)\,e^{-i(\omega_1 t_1-\omega_2 t_2)}.$$

Notice also that from Cauchy-Schwarz inequality, condition (HRS) is satisfied when

$$\sum_{t_1\in\mathbb{Z}}\left|C_{\underline{u}_j,\underline{u}_j}(t_1,t_1)\right|^{1/2}<\infty,$$

recalling that $C_{\underline{u}_j,\underline{u}_j}(t_1,t_1)=C^{r,r}_{\underline{u}_j,\underline{u}_j}(t_1,t_1)$ does not depend on $r$.

**Case of a fixed finite number of replications** In this paragraph we suppose that the number $R$ of replications is finite and fixed. Lemma 6 entails the following corollary

**Corollary 3.**
*(1) Assume that conditions (GR) and (LCR) are satisfied. Then for $R$ and $T$ fixed, and $h\to 0$ and $S_1h^2, S_2h^2\to\infty$, we have*

$$\mathrm{Cov}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\,,\widetilde{f}^{T,\underline{S},R}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4)\right)\longrightarrow$$
$$\frac{1}{R^2}\sum_{r_1=1}^{R}\sum_{r_2=1}^{R} f^{T,r_1,r_2}_{\underline{u}_1,\underline{u}_3}(\omega_1,\omega_3))\overline{f^{T,r_1,r_2}_{\underline{u}_2,\underline{u}_4}(\omega_2,\omega_4)}+\frac{1}{R^2}\sum_{r_1=1}^{R}\sum_{r_2=1}^{R} f^{T,r_1,r_2}_{\underline{u}_1,\underline{u}_4}(\omega_1,-\omega_4)\overline{f^{T,r_1,r_2}_{\underline{u}_2,\underline{u}_3}(\omega_2,-\omega_3)},$$

*where*

$$f^{T,r_1,r_2}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) := \frac{1}{4\pi^2}\sum_{t_1=0}^{T-1}\sum_{t_2=0}^{T-1} C^{r_j,r_{j'}}_{\underline{u}_j,\underline{u}_{j'}}(t_1,t_2)\, e^{-i(\omega_1 t_1-\omega_2 t_2)}.$$

*Under condition (HR)*

$$f^{T,r_1,r_2}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} D_T(\omega_1'-\omega_1)\overline{D_T(\omega_2'-\omega_2)} f^{r_1,r_2}_{\underline{u}_1,\underline{u}_2}(\omega_1',\omega_2')\, d\omega_1' d\omega_2'.$$

*(2) Assume that conditions (GR), (LCR) and (HRS) are satisfied. For $R$ fixed we have*

$$\mathrm{Cov}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\,,\widetilde{f}^{T,\underline{S},R}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4)\right) \longrightarrow$$
$$\frac{1}{R^2}\sum_{r_1=1}^{R}\sum_{r_2=1}^{R} f^{+,r_1,r_2}_{\underline{u}_1,\underline{u}_3}(\omega_1,\omega_3))\overline{f^{+,r_1,r_2}_{\underline{u}_2,\underline{u}_4}(\omega_2,\omega_4)} + \frac{1}{R^2}\sum_{r_1=1}^{R}\sum_{r_2=1}^{R} f^{+,r_1,r_2}_{\underline{u}_1,\underline{u}_4}(\omega_1,-\omega_4)\overline{f^{+,r_1,r_2}_{\underline{u}_2,\underline{u}_3}(\omega_2,-\omega_3)}$$

*provided that $hT^2, T^2S_1^{-1}h^{-2}, T^2S_2^{-1}h^{-2} \to 0$ as $T, S_1, S_2 \to \infty$ and $h \to 0$. Here*

$$f^{+,r_1,r_2}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) := \sum_{t_1=0}^{\infty}\sum_{t_1=0}^{\infty} C^{r_1,r_2}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)e^{-i(\omega_1 t_1-\omega_2 t_2)}.$$

*Proof.* Relations (20) and (24) as well as condition (HR) give us the first part of the corollary. Then under condition (HCR) we easily obtain the second part. □

Then we readily deduce the form of the asymptotic covariance.

**Corollary 4.**

*(1) Assume that conditions (GR), (LCR) and (HR) are satisfied. Then for $R$ and $T$ fixed, and $h \to 0$ and $S_1h^2, S_2h^2 \to \infty$, we have*

$$R\mathrm{Var}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) \longrightarrow$$
$$f^{T}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1)\overline{f^{T}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)} + f^{T}_{\underline{u}_1,\underline{u}_2}(\omega_1,-\omega_2)\overline{f^{T}_{\underline{u}_2,\underline{u}_1}(\omega_2,-\omega_1)}$$
$$+\frac{2}{R}\sum_{r_1=2}^{R}\sum_{r_2=1}^{r_1-1}\Re\left(f^{T,r_1,r_2}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1))\overline{f^{T,r_1,r_2}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)}\right) + \frac{2}{R}\sum_{r_1=2}^{R}\sum_{r_2=1}^{r_1-1}\Re\left(f^{T,r_1,r_2}_{\underline{u}_1,\underline{u}_2}(\omega_1,-\omega_2)\overline{f^{T,r_1,r_2}_{\underline{u}_2,\underline{u}_1}(\omega_2,-\omega_1)}\right).$$

*2) Assume that conditions (GR), (LCR) and (HRS) are satisfied. For $R$ fixed,*

$$R\mathrm{Var}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right) \longrightarrow$$
$$f^{+}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1)\overline{f^{+}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)} + f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,-\omega_2)\overline{f^{+}_{\underline{u}_2,\underline{u}_1}(\omega_2,-\omega_1)}$$
$$+\frac{2}{R}\sum_{r_1=2}^{R}\sum_{r_2=1}^{r_1-1}\Re\left(f^{+,r_1,r_2}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1))\overline{f^{+,r_1,r_2}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)}\right) + \frac{2}{R}\sum_{r_1=2}^{R}\sum_{r_2=1}^{r_1-1}\Re\left(f^{+,r_1,r_2}_{\underline{u}_1,\underline{u}_2}(\omega_1,-\omega_2)\overline{f^{+,r_1,r_2}_{\underline{u}_2,\underline{u}_1}(\omega_2,-\omega_1)}\right)$$

*provided that $T^2h, T^2S_1^{-1}h^{-2}, T^2S_2^{-1}h^{-2} \to 0$ as $T, S_1, S_2 \to \infty$ and $h \to 0$, $R$ fixed.*

### 5.3.4 Case when the number of replicates is going to infinity

To study the asymptotic covariance of the estimator $\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ as $R \to \infty$, we need to consider some additional assumptions on replicates.
Thus assume that

(**SR**) Stationarity of the replications (stationarity with respect to $r$). Assume that

$$C^{\underline{S},k}_{\underline{u}_1,\underline{u}_2}(t_1,t_2) := C^{\underline{S},r+k,r}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)$$

for each $k \in \mathbb{Z}$ and each non negative integer $r \geq -k$.

In this case, under condition (LCR), the localized covariance is stationary with respect to $r$, and we can define

$$C^{k}_{\underline{u}_1,\underline{u}_2}(t_1,t_2) := C^{r+k,r}_{\underline{u}_1,\underline{u}_2}(t_1,t_2).$$

Moreover we have $C^{-k}_{\underline{u}_1,\underline{u}_2}(t_1,t_2) = C^{k}_{\underline{u}_2,\underline{u}_1}(t_2,t_1)$ and $C^{0}_{\underline{u}_1,\underline{u}_2}(t_1,t_2) = C_{\underline{u}_1,\underline{u}_2}(t_1,t_2)$ (see condition (L)). Let

$$f^{T,k}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) = \frac{1}{4\pi^2}\sum_{t_1=0}^{T-1}\sum_{t_2=0}^{T-1} C^{k}_{\underline{u}_1,\underline{u}_2}(t_1,t_2)e^{-i(\omega_1 t_1 - \omega_2 t_2)}.$$

With condition (HR) we can define

$$f^{k}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) := f^{r+k,r}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2),$$

which does not depend on $r$, and we have

$$f^{T,k}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} D_T(\omega_1'-\omega_1)\overline{D_T(\omega_2'-\omega_2)}f^{k}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\,d\omega_1' d\omega_2'.$$

Under conditions (SR), (LCR) and (HRS), we can define

$$f^{(+),k}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) := f^{(+),r+k,r}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2),$$

which does not depend on $r$.

Then we deduce the asymptotic covariance of the estimator as the number $R \to \infty$.

**Corollary 5.**
*(1) Let $T > 0$ be fixed. Assume that conditions (LCR), (GR) and (SR) holds. If, in addition, conditions (M) and (B)(ii) are satisfied or*

$$\sum_k \left| C^{+,k}_{\underline{u}_j,\underline{u}_{j'}}(t_1,t_2) \right| < \infty, \qquad 0 \le t_1, t_2 \le T, \tag{25}$$

*then*

$$R\mathrm{Cov}\left( \widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2), \widetilde{f}^{T,\underline{S},R}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4) \right) \longrightarrow$$
$$\sum_{k=-\infty}^{\infty} f^{T,k}_{\underline{u}_1,\underline{u}_3}(\omega_1,\omega_3)\overline{f^{T,k}_{\underline{u}_2,\underline{u}_4}(\omega_2,\omega_4)} + \sum_{k=-\infty}^{\infty} f^{T,k}_{\underline{u}_1,\underline{u}_4}(\omega_1,-\omega_4)\overline{f^{T,k}_{\underline{u}_2,\underline{u}_3}(\omega_2,-\omega_3)},$$

*provided that $Rh^2, RS_1^{-2}h^{-4}, RS_2^{-2}h^{-4} \to 0$ as $R, S_1, S_2 \to \infty$ and $h \to 0$.*

*(2) Assume conditions (LCR), (GR), (SR) and*

$$\sum_k \sum_{t_1} \sum_{t_2} \left| C^{+,k}_{\underline{u}_j,\underline{u}_{j'}}(t_1,t_2) \right| < \infty. \tag{26}$$

*Then*

$$R\mathrm{Cov}\left( \widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2), \widetilde{f}^{T,\underline{S},R}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4) \right) \longrightarrow$$
$$\sum_{k=-\infty}^{\infty} f^{+,k}_{\underline{u}_1,\underline{u}_3}(\omega_1,\omega_3)\overline{f^{+,k}_{\underline{u}_2,\underline{u}_4}(\omega_2,\omega_4)} + \sum_{k=-\infty}^{\infty} f^{+,k}_{\underline{u}_1,\underline{u}_4}(\omega_1,-\omega_4)\overline{f^{+,k}_{\underline{u}_2,\underline{u}_3}(\omega_2,-\omega_3)}$$

*provided that $RT^4h^2, RT^4S_1^{-2}h^{-4}, RT^4S_2^{-2}h^{-4} \to 0$ as $R, T, S_1, S_2 \to \infty$ and $h \to 0$.*

*Proof.* Using (20) and under conditions (GR), (LCR) and (SR) we get

$$
\begin{aligned}
&16\pi^4 R\mathrm{Cov}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2),\widetilde{f}^{T,\underline{S},R}_{\underline{u}_3,\underline{u}_4}(\omega_3,\omega_4)\right)\\
&=\sum_{k=-R+1}^{R-1}\left(1-\frac{|k|}{R}\right)\left(\sum_{t_1=0}^{T-1}\sum_{t_3=0}^{T-1}C^k_{\underline{u}_1,\underline{u}_3}(t_1,t_3)e^{-i(\omega_1 t_1-\omega_3 t_3)}+\mathcal{O}(T^2h)+\mathcal{O}\left(\frac{T^2}{S_1h^2}+\frac{T^2}{S_2h^2}\right)\right)\\
&\quad\times\left(\sum_{t_2=0}^{T-1}\sum_{t_4=0}^{T-1}C^k_{\underline{u}_2,\underline{u}_4}(t_2,t_4)e^{i(\omega_2 t_2-\omega_4 t_4)}+\mathcal{O}(T^2h)+\mathcal{O}\left(\frac{T^2}{S_1h^2}+\frac{T^2}{S_2h^2}\right)\right)\\
&\quad+\sum_{k=-R+1}^{R-1}\left(1-\frac{|k|}{R}\right)\left(\sum_{t_1=0}^{T-1}\sum_{t_4=0}^{T-1}C^k_{\underline{u}_1,\underline{u}_4}(t_1,t_4)e^{-i(\omega_1 t_1+\omega_4 t_4)}+\mathcal{O}(T^2h)+\mathcal{O}\left(\frac{T^2}{S_1h^2}+\frac{T^2}{S_2h^2}\right)\right)\\
&\quad\times\left(\sum_{t_2=0}^{T-1}\sum_{t_3=0}^{T-1}C^k_{\underline{u}_2,\underline{u}_3}(t_2,t_3)e^{i(\omega_2 t_2+\omega_3 t_3)}+\mathcal{O}(T^2h)+\mathcal{O}\left(\frac{T^2}{S_1h^2}+\frac{T^2}{S_2h^2}\right)\right).
\end{aligned}
$$

Then we readily deduce the results of the corollary. Notice that conditions (M) and (B)(ii) implies relation (25). Notice also that under condition (26), condition (HRS) is fulfilled and $f^{+,k}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ is well defined. □

From the equality (13) we have that

$$
\begin{aligned}
\Re\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) &= \frac{1}{2}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)+\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(-\omega_1,-\omega_2)\right),\\
\Im\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) &= \frac{1}{2i}\left(\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)-\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(-\omega_1,-\omega_2)\right),
\end{aligned}
$$

since the conjugate $\overline{\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)}$ coincides with $\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(-\omega_1,-\omega_2)$ and the observed process $\left\{\boldsymbol{X}^{\underline{S}}_t\right\}$ is real valued. Then we can compute the limiting covariance matrix of $\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)$ of the estimator. We get

$$
\begin{aligned}
&\mathrm{Var}\left(\Re\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)=\frac{1}{2}\left(\mathrm{Var}\left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)+\Re\mathrm{Cov}\left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2),\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(-\omega_1,-\omega_2)\right)\right),\\
&\mathrm{Cov}\left(\Re\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2),\Im\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)=\frac{1}{2}\Im\mathrm{Cov}\left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2),\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(-\omega_1,-\omega_2)\right),\\
&\mathrm{Var}\left(\Im\widetilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)=\frac{1}{2}\left(\mathrm{Var}\left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)-\Re\mathrm{Cov}\left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2),\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(-\omega_1,-\omega_2)\right)\right).
\end{aligned}
$$

### 5.3.5 Central limit theorem and bootstrap

**Asymptotic normality** First we establish the central limit theorem for $\big(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2} - \mathrm{E}\big(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}\big)\big)$ as $\underline{S}, R \to \infty$, ($T$ fixed or $\to \infty$).

Recall that for each $R > 0$, the observations $\left\{\boldsymbol{X}_t^{\underline{S},r}\right\}$, $r = 1, \ldots, R$, are replications of the same time-spatial process, so they have the same law.

**Proposition 2** (Central Limit Theorem)**.**
*Assume that the time-spatial random arrays* $\left\{\boldsymbol{X}_t^{\underline{S},r} : t = 0, \ldots, T, r = 1, \ldots, R\right\}$, $R \geq 0$, *are Gaussian (condition (GR)), stationary with respect to r (condition (SR)) and satisfy condition (M) as well as*

$$\sup_{t,\underline{s},\underline{S}} \mathrm{E}\big(\big|X_t^{\underline{S}}(\underline{s})\big|^{4+2\delta}\big) < \infty \quad \textit{and} \quad \sum_k \alpha_X(k)^{\delta/(2+\delta)} < \infty$$

*for some* $\delta > 0$. *Then under the assumptions of Corollary 5 we have*

$$\lim_{R\to\infty} \mathcal{L}\left(\sqrt{R}\left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) - \mathrm{E}\left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)\right)\right) = \mathcal{N}_2\left(0, \Sigma_2\right),$$

*where* $\Sigma_2 = \boldsymbol{\Sigma}^{(\omega_1,\omega_2)}_{\underline{u}_1,\underline{u}_2}$ *is the limit covariance* $(2\times 2)$*-matrix. The expression of this matrix can be deduced from the Corollary 4. Nevertheless, it depends on the unknown Loève spectrum and it is therefore not interesting for us since we are going to use bootstrap method to approximate the limit law and to define confidence intervals. respectively.*

The proof follows exactly the same steps as proof of Proposition 1.

**Theorem 5** (Asymptotic normality)**.**
*Assume that the time-spatial random arrays* $\left\{\boldsymbol{X}_t^{\underline{S},r} : t = 0, \ldots, T, r = 1, \ldots, R\right\}$, $R \geq 0$, *are Gaussian families (condition (GR)) which are stationary with respect to r (condition (SR)). Assume also conditions (LCR) and (M) as well as*

$$\sup_{t,\underline{s},\underline{S}} \mathrm{E}\left(\big|X_t^{\underline{S}}(\underline{s})\big|^{4+2\delta}\right) < \infty \quad \textit{and} \quad \sum_k \alpha_X(k)^{\delta/(2+\delta)} < \infty$$

*for some* $\delta > 0$. *Then*
*(1) for T fixed,*

$$\lim_{R\to\infty} \mathcal{L}\left(\sqrt{R}\left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) - f^{T}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)\right) = \mathcal{N}_2\left(0, \boldsymbol{V}^{T,(\omega_1,\omega_2)}_{\underline{u}_1,\underline{u}_2}\right),$$

*provided that* $Rh^2, RS_1^{-2}h^4, RS_2^{-2}h^4 \to 0$ *as* $R, S_1, S_2 \to \infty$ *and* $h \to 0$.
*(2) Furthermore*

$$\lim_{R\to\infty} \mathcal{L}\left(\sqrt{R}\left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2) - f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)\right) = \mathcal{N}_2\left(0, \boldsymbol{V}^{+,(\omega_1,\omega_2)}_{\underline{u}_1,\underline{u}_2}\right),$$

*provided that* $RT^4h^2, RT^4S_1^{-2}h^4, RT^4S_2^{-2}h^4 \to 0$ *as* $R, T, S_1, S_2 \to \infty$ *and* $h \to 0$ *when*

$$\lim_{R\to\infty} \sqrt{R}\sum_{t_1=T}^{\infty}\sum_{t_2=0}^{\infty}\left|C^{+}_{\underline{u}_j,\underline{u}_{j'}}(t_1,t_2)\right| = 0 \quad \textit{and} \quad \sum_{k}\sum_{t_1}\sum_{t_2}\left|C^{+,k}_{\underline{u}_j,\underline{u}_{j'}}(t_1,t_2)\right| < \infty.$$

*Proof.* This theorem is a direct consequence of Corollary 2 and Proposition 2. □

Below we present the multidimensional version of Theorem 5. Let

$$\tilde{\boldsymbol{f}}^{T,\underline{S},R} := \left(\left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_{1,1},\underline{u}_{2,1}}(\omega_{1,1},\omega_{2,1})\right)', \ldots, \left(\tilde{f}^{T,\underline{S},R}_{\underline{u}_{1,k},\underline{u}_{2,k}}(\omega_{1,k},\omega_{2,k})\right)'\right)',$$

$$\boldsymbol{f}^{T} := \left(\left(f^{T}_{\underline{u}_{1,1},\underline{u}_{2,1}}(\omega_{1,1},\omega_{2,1})\right)', \ldots, \left(f^{T}_{\underline{u}_{1,k},\underline{u}_{2,k}}(\omega_{1,k},\omega_{2,k})\right)'\right)',$$

and

$$\boldsymbol{f}^{+} := \left(\left(f^{+}_{\underline{u}_{1,1},\underline{u}_{2,1}}(\omega_{1,1},\omega_{2,1})\right)', \ldots, \left(f^{+}_{\underline{u}_{1,k},\underline{u}_{2,k}}(\omega_{1,k},\omega_{2,k})\right)'\right)',$$

where $(\cdot)'$ denotes transposition, $k$ is some positive integer and $\underline{u}_{j,l} \in [0,1]^2, \omega_{j,l} \in (\pi,\pi]$, $j = 1,2, l = 1,\ldots,k$.

**Theorem 6.** *Under conditions of Theorem 5*

$$\lim_{R\to\infty} \mathcal{L}\left(\sqrt{R}\left(\tilde{\boldsymbol{f}}^{T,\underline{S},R} - \boldsymbol{f}^{(+),(T)}\right)\right) = \mathcal{N}_{2k}\left(0, \boldsymbol{\Sigma}_{2k}\right),$$

*where the components of the variance* $(2k \times 2k)$*-matrix* $\boldsymbol{\Sigma}_{2k} = \boldsymbol{\Sigma}^{(\omega_{1,1},\omega_{2,1},\ldots,\omega_{1,k},\omega_{2,k})}_{\underline{u}_{1,1},\underline{u}_{2,1},\ldots,\underline{u}_{1,k},\underline{u}_{2,k}}$ *can be computed using Corollary 5.*

Since the proof is a direct consequence of the Cramér-Wold device, we skip the technical details.

Finally, using the delta method one may formulate the following corollary

**Corollary 6** (Estimation of the coherence)**.** *Under conditions of Theorem 5*

$$\sqrt{R}\left(\tilde{\rho}^{T,\underline{S},R}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)-\rho^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right)\Longrightarrow\mathcal{N}(0,\gamma^2), \tag{27}$$

*provided that* $f^{+}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1)\,f^{+}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)\neq 0$.
*Here* $\gamma^2=\nabla^{(\omega_1,\omega_2)}_{\underline{u}_1,\underline{u}_2}\mathbf{\Sigma}_6\nabla^{(\omega_1,\omega_2)\prime}_{\underline{u}_1,\underline{u}_2}$. *The variance* $6\times 6$*-matrix* $\mathbf{\Sigma}_6$ *is given in Theorem 6, with* $k=3$, $\underline{u}_{1,1}=\underline{u}_{2,1}=\underline{u}_{1,3}=\underline{u}_1$, $\underline{u}_{1,2}=\underline{u}_{2,2}=\underline{u}_{2,3}=\underline{u}_2$, $\omega_{1,1}=\omega_{2,1}=\omega_{1,3}=\omega_1$, *and* $\omega_{1,2}=\omega_{2,2}=\omega_{2,3}=\omega_2$. *Moreover*

$$\begin{aligned}\nabla^{(\omega_1,\omega_2)}_{\underline{u}_1,\underline{u}_2} \;=\; \Bigg(&\frac{-\left|f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right|^2}{\left(f^{+}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1)\right)^2 f^{+}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)},\,0,\,\frac{-\left|f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)\right|^2}{f^{+}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1)\left(f^{+}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)\right)^2},\\ &0,\,\frac{2\Re f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)}{f^{+}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1)\,f^{+}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)},\,\frac{2\Im f^{+}_{\underline{u}_1,\underline{u}_2}(\omega_1,\omega_2)}{f^{+}_{\underline{u}_1,\underline{u}_1}(\omega_1,\omega_1)\,f^{+}_{\underline{u}_2,\underline{u}_2}(\omega_2,\omega_2)}\Bigg)'.\end{aligned}$$

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